Mathematical Proof: Problem Set 6

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Problem.1

Proof: Let $a \in (\mathbb{R} - \mathbb{Q}),\ b \in \mathbb{Q}\ s.t.\ b \neq 0$.

Assume, to the contrary, that $ab \in \mathbb{Q}$. By definition,

$$\exists p, q \in \mathbb{Z}\ s.t.\ q \neq 0,\ ab = \frac{p}{q}$$

Also, since b is a nonzero rational number,

$$\exists r, s \in \mathbb{Z}\ s.t.\ r \neq 0,\ s \neq 0,\ b = \frac{r}{s}$$

Therefore,

$$ab = \frac{p}{q} = a\frac{r}{s}$$

$\Leftrightarrow$

$$a = \frac{ps}{qr} \in \mathbb{Q}$$

which is a contradiction since $a \notin \mathbb{Q}$.

From the above, the statement is true. ■

Problem.2

Proof: We know that $\sqrt{2}, \sqrt{3} \in (\mathbb{R} - \mathbb{Q})$.

Assume, to the contrary, that $\sqrt{2} + \sqrt{3} \in \mathbb{Q}$. By definition,

$$\exists r \in \mathbb{Q}\ s.t. \quad \sqrt{2} + \sqrt{3} = r$$

So,

$$\sqrt{2} = r - \sqrt{3}$$

By squaring the both hand sides,

$$2 = r^2 - 2\sqrt{3}r + 3$$

$\Leftrightarrow$

$$\sqrt{3} \cdot 2r = r^2 + 1$$

Here, $r > 0$, so

$$\sqrt{3} = \frac{(r^2+1)}{2r}$$

Since $r^2 + 1, 2r \in \mathbb{Q}$,

$$\exists p, q, s, l \in Z \ s.t. \ q \neq 0, \ l \neq 0, \ r^2 + 1 = \frac{p}{q}, \ 2r = \frac{s}{l}$$

Again, $r > 0$, so $s \neq 0$. Therefore,

$$\sqrt{3} = \frac{(r^2+1)}{2r} = \frac{pl}{qs} \in \mathbb{Q}$$

which is a contradiction since $\sqrt{3} \notin \mathbb{Q}$.

From the above, the statement is true. ■

Problem.3

Proof: Let $a, b, c \in \mathbb{R}$.

Assume, to the contrary, that $\exists a, b, c \ s.t. \ a + b + c = ab = ac = bc = abc, a \neq b, b \neq c, c \neq a$.

Since, a, b, and c are symmetric, and also $a \neq b, b \neq c, c \neq a$, we can split by cases if $a = 0, b \neq 0, c \neq 0$ or $a \neq 0, b \neq 0, c \neq 0$.

Case 1: $a = 0, b \neq 0, c \neq 0$.

Then $ab = ac = 0$, but $bc \neq 0$, which is a contradiction since $ab = bc$.

Case 2: $a \neq 0, b \neq 0, c \neq 0$.

Then $abc = bc \Leftrightarrow a = 1$, and $abc = ca \Leftrightarrow b = 1$. Thus, $a = b$ which is a contradiction since $a \neq b$.

From the above, the statement is true. ■.

Problem.4

Proof: Let $a, b, c, d \in \mathbb{R}$.

We know that the product of two real numbers is positive if and only if both numbers are positive or both are negative.

Assume, to the contrary, that $\exists a, b, c, d\ s.t.\ ab, ac, ad, bc, bd, cd$ are all negative, or the five numbers among ab, ac, ad, bc, bd, cd are negative.

Case 1: ab, ac, ad, bc, bd, cd are all negative.

Since a, b, c, d are symmetric, without loss of generality, we can assume that $a < 0$ and $b > 0$, since $ab < 0$. Therefore, since $bc < 0$, $c < 0$. Then $ac > 0$, which is a contradiction since $ac < 0$.

Case 2: The five numbers among ab, ac, ad, bc, bd, cd are negative.

Since a, b, c, d are symmetric, without loss of generality, we can assume that $cd > 0$. Also, we can assume that $a < 0$ and $b > 0$, since $ab < 0$. Therefore, since $bc < 0$, $c < 0$. Then $ac > 0$, which is a contradiction since $ac < 0$.

From the above, the statement is true. ■

Problem.5

Proof: Let $a, n \in \mathbb{Z}\ s.t.\ a \geq 2, n \geq 1$.

Assume, to the contrary, that $\exists a, n\ s.t.\ a^2 + 1 = 2^n$.

Case 1: $a \in \mathbb{O}$.

Then

$$\exists k \in \mathbb{Z}\ s.t.\ a = 2k + 1$$

Note that $a \geq 2$, so that $k \geq 1$. Therefore,

$$a^2 + 1 = (2k + 1)^2 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1) = 2^n$$

$\Leftrightarrow$

$$2(k^2 + k) + 1 = 2^{n-1}$$

Note that $n \geq 1$.

When $n = 1$, $2^{n-1} = 1 = 2(k^2 + k) + 1 > 1$, which is a contradiction.

When $n > 1$, then $2^{n-1} = 2(k^2 + k) + 1 \in \mathbb{O}$ which is a contradiction since $2^{n-1} \in \mathbb{E}$ and $2^{n-1} \notin \mathbb{O}$.

Case 2: $a \in \mathbb{E}$.

Then

$$\exists l \in \mathbb{Z} \; s.t. \; a = 2l$$

Note that $a \geq 2$, so that $l \geq 1$. Therefore,

$$a^2 + 1 = (2l)^2 + 1 = 2(2l^2) + 1 = 2^n$$

When $n = 1$, $2^{n-1} = 1 = 2(2l^2) + 1 > 1$, which is a contradiction.

When $n > 1$, then $2^{n-1} = 2(2l^2) + 1 \in \mathbb{O}$ which is a contradiction since $2^{n-1} \in \mathbb{E}$ and $2^{n-1} \notin \mathbb{O}$.

From the above, the statement is true. ■

Problem.6

Proof: Let $a, b \in \mathbb{R}$.

Assume, to the contrary, that $\exists a, b \in (0,1) \; s.t. \; 4a(1-b) > 1$ and $4b(1-a) > 1$.

Then, $a > 0$, $b > 0$, so

$$\begin{cases} 4a(1-b) > 1 \\ 4b(1-a) > 1 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} 1 - b > \frac{1}{4a} \\ 1 - a > \frac{1}{4b} \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} b < 1 - \frac{1}{4a} \\ a < 1 - \frac{1}{4b} \end{cases}$$

Also, since $0 < a < 1$, $0 < b < 1$,

$$\begin{cases} 4a > \frac{1}{1-b} \\ 4b > \frac{1}{1-a} \end{cases}$$

$$\begin{cases} a > \frac{1}{4(1-b)} \\ b > \frac{1}{4(1-a)} \end{cases}$$

Therefore,

$$\begin{cases} \frac{1}{4(1-b)} < 1 - \frac{1}{4b} \\ \frac{1}{4(1-a)} < 1 - \frac{1}{4a} \end{cases}$$

By transforming one of the equations above (note that $1 - a > 0$, $a > 0$),

$$\begin{aligned} \frac{1}{4(1-a)} < 1 - \frac{1}{4a} &\Leftrightarrow \frac{1}{4} < 1 - a - \frac{1-a}{4a} \\ &\Leftrightarrow \frac{a}{4} < a - a^2 - \frac{1}{4} + \frac{a}{4} \\ &\Leftrightarrow a^2 - a + \frac{1}{4} < 0 \\ &\Leftrightarrow \left(a - \frac{1}{2}\right)^2 < 0 \end{aligned}$$

which is a contradiction since $\forall x \in \mathbb{R}, x^2 \geq 0$, so that $\left(a - \frac{1}{2}\right)^2 \in \mathbb{R}, \left(a - \frac{1}{2}\right)^2 \geq 0$.

From the above, the statement is true. $\blacksquare$