Mathematical Proof: Problem Set 5

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We know the triangle inequality from the textbook(or our class):

Theorem 4.17: Let $a, b \in \mathbb{R}$. $|a+b| \leq |a| + |b|$.

Problem.1

Proof: Let $a, b, c, d \in \mathbb{R}$. Here,

$$(ac+bd)^2 = a^2c^2 + b^2d^2 + 2abcd$$

$$(ab-cd)^2 = a^2b^2 + c^2d^2 - 2abcd$$

Since the square of any real number is non-negative, observe

$$(ab-cd)^2 \geq 0$$

$\Leftrightarrow$

$$a^2b^2 + c^2d^2 - 2abcd \geq 0$$

$\Leftrightarrow$

$$2abcd \leq a^2b^2 + c^2d^2$$

By adding the same terms on both hand sides,

$$a^2c^2 + b^2d^2 + 2abcd \leq a^2b^2 + a^2c^2 + b^2d^2 + c^2d^2$$

$\Leftrightarrow$

$$(ac+bd)^2 \leq (a^2+c^2)(b^2+d^2)$$

$\Rightarrow$

$$ac+bd \leq |ac+bd| \leq \sqrt{(a^2+c^2)(b^2+d^2)} = \sqrt{(a^2+c^2)} \cdot \sqrt{(b^2+d^2)}$$

From the above, the statement is true. ■

Problem.2

<u>*Proof*</u>: Let $x, y, z \in \mathbb{Z}$. Here, $(x-y)+(y-z) = x-z$, also $x-z, x-y, y-z \in \mathbb{R}$. By applying the triangle inequality above as $a = x-y,\ b = y-z$, we get

$$|x-z| \leq |x-y| + |y-z|. \blacksquare$$

Problem.3

<u>*Proof*</u>: Let $a, b \in \mathbb{R}$ $s.t.$ $a > 0,\ b > 0$.

Since the square of any real number is non-negative, observe

$$(a-b)^2 = a^2 + b^2 - 2ab \geq 0$$

$\Leftrightarrow$

$$a^2 + b^2 \geq 2ab$$

$\Leftrightarrow$

$$\frac{a^2+b^2}{ab} \geq 2 \quad (\because ab > 0)$$

$\Leftrightarrow$

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

From the above, the statement is true. $\blacksquare$

By following the proof above backwards as a equation, immediately we see that

$$\frac{a}{b} + \frac{b}{a} = 2 \Leftrightarrow (a-b)^2 = 0 \Leftrightarrow a = b$$

Therefore, the complete solution set $U = \{(a, b) \mid \forall (a, b) \in \mathbb{R}^2\ s.t.\ a > 0,\ b > 0,\ a = b\}$.

Problem.4

<u>*Proof*</u>: Let A and B be sets.

$(\Leftarrow)$

Suppose $A = B$, by definition, $\forall x \in A,\ x \in B$ and $\forall x \in B,\ x \in A$.
We prove $(A \cup B) \subseteq (A \cap B)$ and $(A \cup B) \supseteq (A \cap B)$.

($\subseteq$) Let $x \in (A \cup B)$.
Case 1: $x \in A$. Then $x \in B$ ($\because A = B$). Therefore, $x \in (A \cap B)$.
Case 2: $x \in B$. Then $x \in A$ ($\because A = B$). Therefore, $x \in (A \cap B)$.
Thus, $(A \cup B) \subseteq (A \cap B)$.

($\supseteq$) Let $x \in (A \cap B)$. By definition, $x \in A$ and $x \in B$. Therefore, $x \in (A \cup B)$.
Thus, $(A \cup B) \supseteq (A \cap B)$.

So, $(A \cup B) = (A \cap B)$.

($\Rightarrow$)

Suppose $(A \cup B) = (A \cap B)$. We prove $A \subseteq B$ and $A \supseteq B$.

($\subseteq$) $\forall x \in A \subseteq (A \cup B) = (A \cap B)$. Therefore, $x \in B$, $A \subseteq B$.

($\supseteq$) $\forall x \in B \subseteq (A \cup B) = (A \cap B)$. Therefore, $x \in A$, $A \supseteq B$.

So, $A = B$.

From the above, the statement is true. ■

Problem.5

We prove that

$$(A \times B) \cap (B \times A) = \emptyset \Leftrightarrow A \cap B = \emptyset$$

Proof: Let A and B be sets.

($\Leftarrow$)

Suppose $A \cap B = \emptyset$, by definition, $\forall x \in A$, $x \notin B$, and $\forall y \in B$, $y \notin A$.
Let $a \in (A \times B)$, then $\exists x \in A, \exists y \in B,\ s.t.\ a = (x, y)$, but $a \notin (B \times A)$, since $x \notin B$, and $y \notin A$.
Similarly, let $b \in (B \times A)$, then $\exists y \in B, \exists x \in A,\ s.t.\ b = (y, x)$, but $b \notin (A \times B)$, since $y \notin A$, and $x \notin B$.
Therefore, $(A \times B) \cap (B \times A) = \emptyset$.

($\Rightarrow$)

Suppose $(A \times B) \cap (B \times A) = \emptyset$. Let $x \in A, y \in B$, then $\exists a \in (A \times B)\ s.t.\ a = (x, y)$, and also, $\exists b \in (B \times A)\ s.t.\ b = (y, x)$. However, since $(A \times B) \cap (B \times A) = \emptyset$, $a \neq b \Rightarrow x \neq y$.
So $x \notin B, y \notin A$, $A \cap B = \emptyset$.

From the above, the statement is true. ■

Problem.6

Proof: Let A, B, C and D be sets.

We prove $(A\times B)\cap(C\times D) \subseteq (A\cap C)\times(B\cap D)$ and $(A\times B)\cap(C\times D) \supseteq (A\cap C)\times(B\cap D)$.

$(\subseteq)$

Let $w \in (A \times B) \cap (C \times D)$, by definition, $w \in (A \times B)$, and $w \in (C \times D)$. By definition, $\exists x \in A, \exists y \in B$ $s.t.$ $w = (x, y)$, but also $x \in C, y \in D$ ($\because$ $w \in (C \times D)$). Therefore, $x \in A$ and $x \in C$, also $y \in B$ and $y \in D$. Thus, $w = (x, y) \in (A \cap C) \times (B \cap D)$, $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.

$(\supseteq)$

Let $w \in (A \cap C) \times (B \cap D)$, by definition, $\exists x \in (A \cap C), \exists y \in (B \cap D)$ $s.t.$ $w = (x, y)$. By definition, $x \in A$ and $x \in C$, also $y \in B$ and $y \in D$. Therefore, $w = (x, y) \in (A \times B)$ and $w = (x, y) \in (C \times D)$. Thus, $w \in (A\times B)\cap(C\times D)$, $(A\times B)\cap(C\times D) \supseteq (A\cap C)\times(B\cap D)$.

From the above, the statement is true. ■