Math 531: Problem Set 7

Due: Friday, 19 April 2024

Directions: Work all the problems below. Give complete answers and provide your reasoning when you think it is appropriate or useful to do so. In any case, follow any problem-specific instructions that are given. Note that these problems are all proofs, in which both the mathematics and the valid, correct composition and presentation are being graded. So, be careful and thorough in both of these aspects. Please do not submit first drafts of arguments, since these will often contain superfluous material. Edit with an eye for conciseness; write only what is necessary to establish the result. Use complete sentences and the appropriate balance of verbal and non-verbal language. There are numerous examples of such in the textbook – do not ignore these! Partial credit will be given where appropriate.

1. Find a formula for
$$1 + 4 + 7 + \cdots (3n - 2)$$
for positive integers n and then verify your formula by mathematical induction.

2. Prove the following inequality for every positive integer n:
$$2! \cdot 4! \cdot 6! \cdots (2n)! \geq ((n+1)!)^n$$

3. Prove that for every real number $x > -1$ and every positive integer n,
$$(1+x)^n \geq 1 + nx$$

4. Prove that $81 \mid (10^{n+1} - 9n - 10)$ for every positive integer n.

5. A sequence $\{a_n\}$ is defined recursively by
$$a_1 = 1, a_2 = 2; a_n = a_{n-1} + 2a_{n-2},$$
for $n \geq 3$. Conjecture a formula for a_n and verify that your conjecture is correct.

6. Consider the sequence of *Fibonacci numbers* $\{F_n\}$, where
$$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2},$$
for $n \geq 3$.
   (a) Prove: $2 \mid F_n$ if and only if $3 \mid n$.
   (b) For $n \geq 1$, prove by induction that $2^{n-1}F_n \equiv n \bmod 5$.

7. Use the method of minimum counterexample to prove Result 6.23: if r is a non-zero real number such that $r + \frac{1}{r}$ is an integer, then $r^n + \frac{1}{r^n}$ is an integer for every positive integer n