Mathematical Proof: Problem Set 9

Koichiro Takahashi

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The definition of function is given below.

Definition: Let A, B be sets. A function $f : A \longrightarrow B$ is a relation from A to B, *s.t.*

$$(i)\ (a,b),(a,c) \in f \Rightarrow b = c$$

$$(ii)\ \forall a \in A, \exists b \in B\ s.t.\ (a,b) \in f$$

Also, two properties of specific function are defined below.

Definition: A function $f : A \longrightarrow B$ is one-to-one, or injective,

if $f(x) = f(y)$, where $x, y \in A$, then $x = y$.

Definition: A function $f : A \longrightarrow B$ is onto, or surjective, if $f(A) = B$.

We define for two sets to have the same cardinality as following.

Definition: Two sets A and B have the same cardinality, denoted by $|A| = |B|$,

if $A = B = \emptyset$, or $\exists f : A \longrightarrow B$ $s.t.$ f is a bijection.

For the use in Problem.2, we define the floor function $f : \mathbb{R} \longrightarrow \mathbb{Z}$ by

$$f(x) = \lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$$

Problem.1

(a)
Here,

$$A_1 = \mathbb{R},\ R_1 = \{(x,y) \mid x \in A_2, y = 4x - 3\}$$

Proof: Let $a \in A_1$. We prove that R_1 is a function by showing that it satisfies the condition (i) and (ii).

(i) We prove by contradiction. Assume, to the contrary, that $\exists b, c \in \mathbb{R},\ s.t.\ b = R_1(a), c = R_1(a),\ b \neq c$. Then, since $b \neq c$, $4a-3 \neq 4a-3 \Leftrightarrow 4a \neq 4a \Leftrightarrow a \neq a$, which is a contradiction since $a = a$. Therefore, $b = c$, R_1 satisfies the condition (i).

(ii) Obviously, $\forall x \in A_1, \exists y \in \mathbb{R},\ s.t.\ (x, y) \in R_1$, since $R_1(x) = 4x - 3 \in \mathbb{R}$. Therefore, R_1 also satisfies the condition (ii).

From the above, R_1 is a function. ■

(b)
Here,

$$A_2 = [0, \infty),\ R_2 = \{(x, y) \mid x \in A_2, (y + 2)^2 = x\}$$

Disproof: We disprove that R_2 is a function by showing that there exists a counterexample of the condition (i).
Let $a \in A_2$ $s.t.$ $a \neq 0$. Here,

$$(y + 2)^2 = x \Leftrightarrow y = -2 \pm \sqrt{x}.$$

We choose $b, c \in \mathbb{R}$ $s.t.$ $b = -2 + \sqrt{a},\ c = -2 - \sqrt{a}$. Then, since $a \neq 0$, $\sqrt{a} \neq -\sqrt{a} \Leftrightarrow -2 + \sqrt{a} \neq -2 - \sqrt{a} \Leftrightarrow b \neq c$. However, $(b + 2)^2 = (-2 + \sqrt{a} + 2)^2 = (\sqrt{a})^2 = a$, $(c + 2)^2 = (-2 - \sqrt{a} + 2)^2 = (-\sqrt{a})^2 = a$. Therefore, $\exists a, b, c,\ s.t.\ (a, b), (a, c) \in R_2,\ b \neq c$, which violates the condition (i).

From the above, R_2 is not a function. ♦

(c)
Here,

$$A_3 = \mathbb{R},\ R_3 = \{(x, y) \mid x \in A_3, (x + y)^2 = 4\}$$

Disproof: We disprove that R_3 is a function by showing that there exists a counterexample of the condition (i).
Let $a \in A_3$. Here,

$$(x + y)^2 = 4 \Leftrightarrow y = \pm 2 - x.$$

We choose $b, c \in \mathbb{R}$ $s.t.$ $b = 2 - a,\ c = -2 - a$. Then, since $2 \neq -2$, $2 - a \neq -2 - a \Leftrightarrow b \neq c$. However, $(a + b)^2 = (a + 2 - a)^2 = (2)^2 = 4$, $(a + c)^2 = (a - 2 - a)^2 = (-2)^2 = 4$. Therefore, $\exists a, b, c,\ s.t.\ (a, b), (a, c) \in R_3,\ b \neq c$, which violates the condition (i).

From the above, R_3 is not a function. ♦

Problem.2

Example for each problem, $f : \mathbb{N} \longrightarrow \mathbb{N}$ is denoted as $f(n)$ for $n \in \mathbb{N}$.

(a)

$$f(n) = n$$

(b)

$$f(n) = 2n$$

(c)

$$f(n) = \left\lfloor \frac{n}{2} + 1 \right\rfloor$$

(d)

$$f(n) = (n-1)^2 + 1$$

Problem.3

Here, $A = \{2, 4, 6\}, B = \{1, 3, 4, 7, 9\}$. Therefore,

$$\mathcal{F} = \{(a, b) \mid a \in A, b \in B,\ s.t.\ 5 \mid (ab+1)\}$$

$$= \{(2, 7), (4, 1), (6, 4), (6, 9)\}$$

Thus, $(6, 4), (6, 9) \in \mathcal{F}$, but $4 \neq 9$. This violates the condition (i) for $\mathcal{F}$ to be a function. Therefore, $\mathcal{F}$ is neither one-to-one nor a well-defined function.

Problem.4

I also excluded $\{1\}$ from A for f to be a well-defined function.

Here, $A = \mathbb{R} - \{0, 1\}$, and $f : A \longrightarrow A$ defined by $f(x) = 1 - \frac{1}{x}$.

(a)

Let $x \in A$. Since $x \neq 0, x \neq 1$,

$$(f \circ f \circ f)(x) = (f \circ f)\left(1 - \frac{1}{x}\right) = f\left(1 - \frac{1}{\left(1 - \frac{1}{x}\right)}\right) = f\left(1 - \frac{x}{x-1}\right) = f\left(-\frac{1}{x-1}\right)$$

$$= f\left(\frac{1}{1-x}\right) = \left(1 - \frac{1}{\left(\frac{1}{1-x}\right)}\right) = 1 - (1-x) = x = i_A(x)$$

(b)

$$f^{-1}(x) = (f \circ f)(x) = \frac{1}{1-x}$$

Problem.5

Here, $F : \mathbb{N} \longrightarrow (\mathbb{N} \cup \{0\})$ is defined by $F(n) = m$, where $n \in \mathbb{N}, \exists k \in \mathbb{O}\ s.t.\ k > 0, 3n+1 = 2^m k$.

(a)
Disproof: We disprove that F is one-to-one by showing that there exists a counterexample.

We choose $2, 4 \in \mathbb{N}$. Then, $3 \cdot 2 + 1 = 7 = 2^0 \cdot 7$, so that $F(2) = 0$ $(\because 7 \in \mathbb{O}, 7 > 0)$. Also, $3 \cdot 4 + 1 = 13 = 2^0 \cdot 13$, so that $F(4) = 0$ $(\because 3 \in \mathbb{O}, 3 > 0)$. Therefore, $(2,0), (4,0) \in F$, $2 \neq 4$, which is a counterexample. From the above, F is not one-to-one. ♦

(b)
We prove that F is onto, so that prove

$$\forall m \in (\mathbb{N} \cup \{0\}), \exists n \in \mathbb{N}\ s.t.\ F(n) = m.$$

Here,

$$F(n) = m \Leftrightarrow \exists k \in \mathbb{O}\ s.t.\ k > 0, 3n+1 = 2^m k$$

Proof: We employ induction.

For each non-negative integer m, let $P(m)$ be a statement, defined by below:

$$P(m) : \exists n \in \mathbb{N}\ s.t.\ (n, m) \in F$$

First, we prove $P(0)$ and $P(1)$ are true.

Note that $2 \in \mathbb{N}$. Since $3 \cdot 2 + 1 = 7 = 2^0 \cdot 7$ and $7 \in \mathbb{O}, 7 > 0$, $F(2) = 0$. Therefore, $(2,0) \in F$, so that $P(0)$ is true.

Note that $3 \in \mathbb{N}$. Since $3 \cdot 3 + 1 = 10 = 2^1 \cdot 5$ and $5 \in \mathbb{O}, 5 > 0$, $F(3) = 1$. Therefore, $(3, 1) \in F$, so that $P(1)$ is true.

Let $i \in (\mathbb{N} \cup \{0\})$, assume that $P(i)$ is true. Then,

$$\exists n_i \in \mathbb{N}, k_i \in \mathbb{O} \ s.t. \ k_i > 0, 3n_i + 1 = 2^i k_i$$

Also, observe

$$3(n_i + 2^i k_i) + 1 = 2^i k_i + 2^i k_i \cdot 3 = 2^i \left(k_i + 3k_i\right) = 2^i \cdot 4k_i = 2^{i+2} k_i$$

Therefore, by choosing $n_{i+2} = n_i + 2^i k_i, k_{i+2} = k_i$,

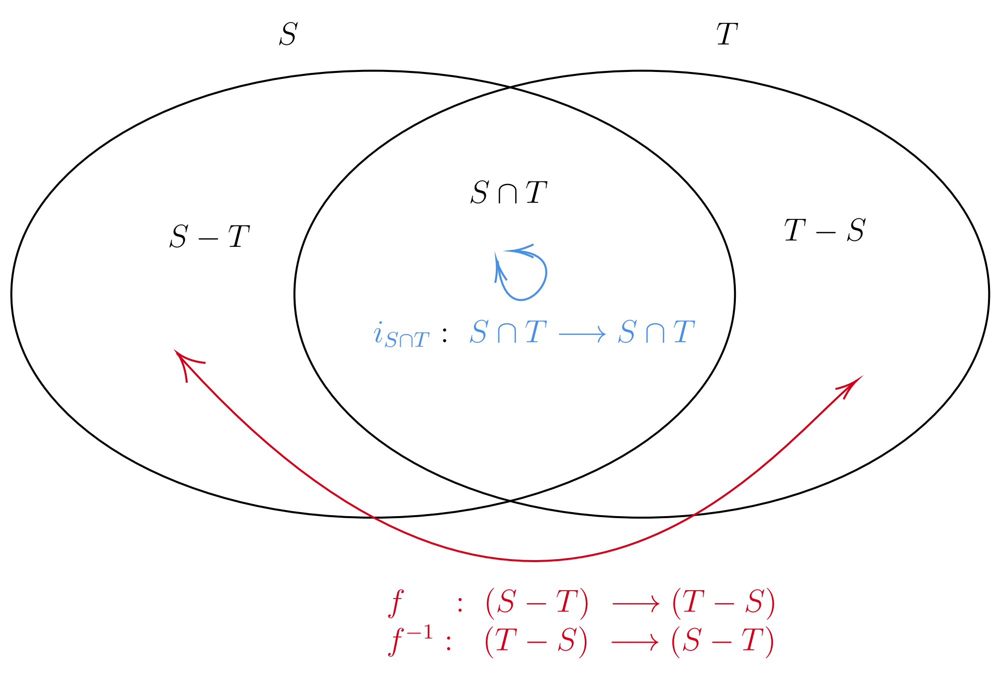
$$3n_{i+2} + 1 = 2^{i+2} k_{i+2}$$

Note that $k_{i+2} \in \mathbb{O}, k_{i+2} > 0$, thus $n_{i+2} \in \mathbb{N}$. Therefore, $\exists n_{i+2} \in \mathbb{N} \ s.t. \ (n_{i+2}, i+2) \in F$, so that $P(i+2)$ is true.

By the Principle of Mathematical Induction, $P(m)$ is true for every $m \in (\mathbb{N} \cup \{0\})$.

From the above, F is onto. ■

Problem.6



First, we show a lemma below:
Lemma.1: Let S, T be non-empty sets $s.t.$ $T - S \neq \emptyset, S - T \neq \emptyset$.

$$\mathcal{G} = \{\{S - T\}, \{S \cap T\}\} \text{ is a partition of } S$$

Proof: Here, the set identity, (Note that $((S - T) = (S \cap \bar{T}))$

$$S = S \cap (T \cup \bar{T}) = (S \cap \bar{T}) \cup (S \cap T) = (S - T) \cup (S \cap T)$$

Also, $T \cap \bar{T} = \emptyset$, so that

$$(S - T) \cap (S \cap T) = \emptyset$$

From the above, $\mathcal{G} = \{\{S - T\}, \{S \cap T\}\}$ is a partition of S. ■

Now, we show the statement in the problem.

Proof: Let S, T be sets. Suppose $|S - T| = |T - S|$. By definition, $\exists f : (T - S) \longrightarrow (S - T)$ $s.t.$ f is a bijection.

Case 1: $S - T = T - S = \emptyset$.

Then, $S = T$. Therefore, since $\exists g : S \longrightarrow S$ defined by $g = i_S = g^{-1}$, $\exists g : S \longrightarrow T$ $s.t.$ g is a bijection. So, $|S| = |T|$.

Case 2: $T - S \neq \emptyset, S - T \neq \emptyset$.

Let $x \in (T - S)$. Then, $\exists y \in (S - T)$ $s.t.$ $(x, y) \in f$ and $(y, x) \in f^{-1}$. By definition, $x \in T, x \notin S$. Also, $y \in T, y \notin S$.

Here, we show that $h : S \longrightarrow T$, defined by

$$h(x) = \begin{cases} f(x) = y \in (T - S) \subseteq T \text{ if } x \in (S - T) \subseteq S \\ \\ i_{(S \cap T)}(x) = x \in (S \cap T) \subseteq T \text{ if } x \in (S \cap T) \subseteq S \end{cases}$$

is a bijection.

By Lemma.1, h is at least a function, and can be split by cases.
Subcase.1: $x \in (S - T)$, then $f(x)$ is a bijection.
Subcase.2: $x \in (S \cap T)$, then obviously the identity function $i_{(S \cap T)}(x)$ is a bijection.

Therefore, $\exists h : S \longrightarrow T$ $s.t.$ h is a bijection. So, $|S| = |T|$.

From the above, the statement is true. ■