Mathematical Proof: Problem Set 4

Koichiro Takahashi

March 11, 2024

Recall $\mathbb{E}, \mathbb{O}$ are sets of even and odd integers respectively.
Also, we know the theorems from the textbook:

$$\text{Theorem 3.16: Let } x, y \in \mathbb{Z}. \text{ Then, } x \text{ and } y \text{ are of the same parity}$$

$$\text{if and only if } x + y \text{ is even.}$$

$$\text{Theorem 3.17: Let } x, y \in \mathbb{Z}. \text{ Then } xy \text{ is even if and only if } x \text{ is even or } y \text{ is even.}$$

For the use in Problem.6, we introduce the corollaries of the Theorem 3.17:

$$\text{Corollary 1: Let } x \in \mathbb{Z}.\ x^2 \in \mathbb{O} \Rightarrow x \in \mathbb{O}$$

$$\text{Corollary 2: Let } x \in \mathbb{Z}.\ x^2 \in \mathbb{E} \Rightarrow x \in \mathbb{E}$$

Problem.1

Proof: Let $x, y, z \in \mathbb{Z}$.

Case 1: $x, y \in \mathbb{E},\ z \in \mathbb{O}$.
Therefore, $3x \in \mathbb{E}$, $5y \in \mathbb{E}$, $7z \in \mathbb{O}$ ($\because$ Theorem 3.17). And immediately, $3x + 5y \in \mathbb{E}$, $3x + 5y + 7z \in \mathbb{O}$ ($\because$ Theorem 3.16).

Case 2: $x, z \in \mathbb{E},\ y \in \mathbb{O}$.
Therefore, $3x \in \mathbb{E}$, $5y \in \mathbb{O}$, $7z \in \mathbb{E}$ ($\because$ Theorem 3.17). And immediately, $3x + 5y \in \mathbb{O}$, $3x + 5y + 7z \in \mathbb{O}$ ($\because$ Theorem 3.16).

Case 3: $y, z \in \mathbb{E},\ x \in \mathbb{O}$.
Therefore, $3x \in \mathbb{O}$, $5y \in \mathbb{E}$, $7z \in \mathbb{E}$ ($\because$ Theorem 3.17). And immediately, $3x + 5y \in \mathbb{O}$, $3x + 5y + 7z \in \mathbb{O}$ ($\because$ Theorem 3.16).

From the above, the statement is true. ■

Problem.2

Proof: Let $a, b \in \mathbb{Z}$. Suppose $ab = 4$.

$$\begin{aligned}(a-b)^3 - 9(a-b) &= (a-b)\{(a-b)^2 - 9\} \\ &= (a-b)(a^2 + b^2 - 2ab - 9) \\ &= (a-b)(a^2 + b^2 - 17) \quad (\because ab = 4)\end{aligned}$$

Therefore,

$$(a-b)^3 - 9(a-b) = 0 \Leftrightarrow a - b = 0, \text{ or, } a^2 + b^2 - 17 = 0$$

Case 1: $a = \pm 4,\ b = \pm 1$ (double sign in the same order).
Then,

$$a^2 + b^2 - 17 = 16 + 1 - 17 = 0$$

Case 2: $a = \pm 2, b = \pm 2$ (double sign in the same order).
Then,

$$a - b = \pm(2 - 2) = 0$$

Case 3: $a = \pm 1, b = \pm 4$ (double sign in the same order).
Then,

$$a^2 + b^2 - 17 = 1 + 16 - 17 = 0$$

From the above, the statement is true. ■

Problem.3

Proof: Let $a \in \mathbb{Z}$. We prove by contrapositive.
Suppose $3 \nmid a$. By definition,

$$\exists k \in \mathbb{Z},\ s.t.\ a = 3k + 1 \text{ or } 3a = k + 2$$

Case 1: $a = 3k + 1$. Then,

$$2a = 2(3k + 1) = 3(2k) + 2$$

Since $2k \in \mathbb{Z},\ 3 \nmid 2a$.

Case 2: $a = 3k + 2$. Then,

$$2a = 2(3k + 2) = 3(2k + 1) + 1$$

Since $2k + 1 \in \mathbb{Z},\ 3 \nmid 2a$.

From the above, the statement is true. ■

Problem.4

Proof: Let $x, y \in \mathbb{Z}$. Suppose $3 \nmid x$ and $3 \nmid y$. Here,

$$x^2 - y^2 = (x+y)(x-y)$$

Case 1: $\exists k, l \in \mathbb{Z},\ s.t.\ x = 3k+1$ and $y = 3l+1$. Then,

$$(x+y)(x-y) = (3k+1+3l+1)\{3k+1-(3l+1)\} = 3(3(k+l)+2)(k-l)$$

Since $(3(k+l)+2)(k-l) \in \mathbb{Z},\ 3 \mid (x^2-y^2)$.

Case 2: $\exists k, l \in \mathbb{Z},\ s.t.\ x = 3k+1$ and $y = 3l+2$. Then,

$$(x+y)(x-y) = (3k+1+3l+2)\{3k+1-(3l+2)\} = 3(k+l+1)(3(k-l)-1)$$

Since $(k+l+1)(3(k-l)-1) \in \mathbb{Z},\ 3 \mid (x^2-y^2)$.

Case 3: $\exists k, l \in \mathbb{Z},\ s.t.\ x = 3k+2$ and $y = 3l+1$. Then,

$$(x+y)(x-y) = (3k+2+3l+1)\{3k+2-(3l+1)\} = 3(k+l+1)(3(k-l)+1)$$

Since $(k+l+1)(3(k-l)+1) \in \mathbb{Z},\ 3 \mid (x^2-y^2)$.

Case 4: $\exists k, l \in \mathbb{Z},\ s.t.\ x = 3k+2$ and $y = 3l+2$. Then,

$$(x+y)(x-y) = (3k+2+3l+2)\{3k+2-(3l+2)\} = 3(3(k+l+1)+1)(k-l)$$

Since $(3(k+l+1)+1)(k-l) \in \mathbb{Z},\ 3 \mid (x^2-y^2)$.

From the above, the statement is true. ■

Problem.5

Proof: Let $m, n \in \mathbb{N}\ s.t.\ m \mid n$. Suppose $a, b \in \mathbb{Z}\ s.t.\ a \equiv b \pmod{n}$. By definition,

$$\exists k \in \mathbb{Z},\ s.t.\ n = m \cdot k$$

and since $n \mid (b-a)$,

$$\exists l \in \mathbb{Z},\ s.t.\ b - a = n \cdot l$$

Thus,

$$b - a = m \cdot k \cdot l = m(kl)$$

where $kl \in \mathbb{Z}$. Therefore,

$$n \mid (b-a) \Leftrightarrow a \equiv b \pmod{m}\ ■.$$

Problem.6

<u>*Proof*</u>: Let $n \in \mathbb{Z}$.

$(\Rightarrow)$

Suppose $2 \mid (n^4 - 3)$. By definition, $n^4 - 3 \in \mathbb{E}$. Therefore, $n^4 \in \mathbb{O}$ ($\because$ Theorem 3.16). Thus, $n^2 \in \mathbb{O}$, $n \in \mathbb{O}$ ($\because$ Corollary 1). By definition,

$$\exists k \in \mathbb{Z},\ s.t.\ n = 2k + 1$$

So,

$$n^2 + 3 = (2k+1)^2 + 3 = 4k^2 + 4k + 4 = 4(k^2 + k + 1)$$

Since $k^2 + k + 1 \in \mathbb{Z}$, $4 \mid n^2 + 3$.

$(\Leftarrow)$

We prove by contrapositive. Suppose $2 \nmid (n^4 - 3)$. By definition, $n^4 - 3 \in \mathbb{O}$. Therefore, $n^4 \in \mathbb{E}$ ($\because$ Theorem 3.16). Thus, $n^2 \in \mathbb{E}$, $n \in \mathbb{E}$ ($\because$ Corollary 2). By definition,

$$\exists k \in \mathbb{Z},\ s.t.\ n = 2k$$

So,

$$n^2 + 3 = (2k)^2 + 3 = 4(k^2) + 3$$

Since $k^2 \in \mathbb{Z}$, $4 \nmid n^2 + 3$.

From the above, the statement is true. ■

Problem.7

<u>*Proof*</u>: Let $a, b \in \mathbb{Z}$.

$(\Leftarrow)$

<u>*Case 1*</u>: $a \equiv b \equiv 0 \pmod 3$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 3k,\ b = 3l$$

Therefore,

$$a^2 + 2b^2 - 0 = 9k^2 + 18l^2 = 3(3k^2 + 6l^2)$$

where $3k^2 + 6l^2 \in \mathbb{Z}$. Thus,

$$3 \mid (a^2 + 2b^2 - 0) \Leftrightarrow a^2 + 2b^2 \equiv 0 \pmod 3.$$

Case 2: $a \not\equiv 0, b \not\equiv 0 \pmod 3$.

Subcase 2.1: $a \equiv 1, b \equiv 1 \pmod 3$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 3k + 1,\ b = 3l + 1.$$

Therefore,

$$\begin{aligned} a^2 + 2b^2 - 0 &= (3k+1)^2 + 2(3l+1)^2 \\ &= 9k^2 + 6k + 1 + 2(9l^2 + 6l + 1) \\ &= 3(3(k^2 + 2l^2) + 2(k + 2l) + 1) \end{aligned}$$

where $3(k^2 + 2l^2) + 2(k + 2l) + 1 \in \mathbb{Z}$. Thus,

$$3 \mid (a^2 + 2b^2 - 0) \Leftrightarrow a^2 + 2b^2 \equiv 0 \pmod 3.$$

Subcase 2.2: $a \equiv 1, b \equiv 2 \pmod 3$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 3k + 1,\ b = 3l + 2.$$

Therefore,

$$\begin{aligned} a^2 + 2b^2 - 0 &= (3k+1)^2 + 2(3l+2)^2 \\ &= 9k^2 + 6k + 1 + 2(9l^2 + 12l + 4) \\ &= 3(3(k^2 + 2l^2) + 2(k + 4l) + 3) \end{aligned}$$

where $3(k^2 + 2l^2) + 2(k + 4l) + 3 \in \mathbb{Z}$. Thus,

$$3 \mid (a^2 + 2b^2 - 0) \Leftrightarrow a^2 + 2b^2 \equiv 0 \pmod 3.$$

Subcase 2.3: $a \equiv 2, b \equiv 2 \pmod 3$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 3k + 2,\ b = 3l + 2.$$

Therefore,

$$\begin{aligned} a^2 + 2b^2 - 0 &= (3k+2)^2 + 2(3l+2)^2 \\ &= 9k^2 + 12k + 4 + 2(9l^2 + 12l + 4) \\ &= 3(3(k^2 + 2l^2) + 4(k + 2l) + 4) \end{aligned}$$

where $3(k^2+2l^2)+4(k+2l)+4 \in \mathbb{Z}$. Thus,

$$3 \mid (a^2+2b^2-0) \Leftrightarrow a^2+2b^2 \equiv 0 \pmod 3.$$

($\Rightarrow$)

Prove by contrapositive. Let the statements:

$$A\colon\ a \equiv 0 \text{ and } b \equiv 0 \pmod 3$$

$$B\colon\ a \not\equiv 0 \text{ and } b \not\equiv 0 \pmod 3$$

and the compound statement:

$$R \equiv A \vee B$$

Then

$$\sim R \equiv \sim (A \vee B) \equiv (\sim A) \wedge (\sim B) \quad (\because \text{ De Morgan's law})$$

Therefore, suppose

$$a \not\equiv 0 \text{ and } b \equiv 0 \pmod 3,\ \text{ or, }\ a \equiv 0 \text{ and } b \not\equiv 0 \pmod 3$$

Case 1: $a \not\equiv 0$ and $b \equiv 0 \pmod 3$.

Subcase 1.1: $a \equiv 1 \pmod 3$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 3k+1,\ b = 3l.$$

Therefore,

$$\begin{aligned} a^2+2b^2-0 &= (3k+1)^2+2(3l)^2 \\ &= 9k^2+6k+1+2(9l^2) \\ &= 3(3(k^2+2l^2)+2k)+1 \end{aligned}$$

where $3(k^2+2l^2)+2k \in \mathbb{Z}$. Thus,

$$3 \nmid (a^2+2b^2-0) \Leftrightarrow a^2+2b^2 \not\equiv 0 \pmod 3.$$

Subcase 1.2: $a \equiv 2 \pmod 3$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 3k+2,\ b = 3l.$$

Therefore,

$$a^2 + 2b^2 - 0 = (3k+2)^2 + 2(3l)^2$$
$$= 9k^2 + 12k + 4 + 2(9l^2)$$
$$= 3(3(k^2 + 2l^2) + 4k + 1) + 1$$

where $3(k^2 + 2l^2) + 4k + 1 \in \mathbb{Z}$. Thus,

$$3 \nmid (a^2 + 2b^2 - 0) \Leftrightarrow a^2 + 2b^2 \not\equiv 0 \pmod 3.$$

<u>*Case* 2</u>: $a \equiv 0$ and $b \not\equiv 0 \pmod 3$.

<u>*Subcase* 2.1</u>: $b \equiv 1 \pmod 3$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 3k,\ b = 3l + 1.$$

Therefore,

$$a^2 + 2b^2 - 0 = (3k)^2 + 2(3l+1)^2$$
$$= 9k^2 + 2(9l^2 + 6l + 1)$$
$$= 3(3(k^2 + 2l^2) + 4l) + 2$$

where $3(k^2 + 2l^2) + 4l \in \mathbb{Z}$. Thus,

$$3 \nmid (a^2 + 2b^2 - 0) \Leftrightarrow a^2 + 2b^2 \not\equiv 0 \pmod 3.$$

<u>*Subcase* 2.2</u>: $b \equiv 2 \pmod 3$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 3k,\ b = 3l + 2.$$

Therefore,

$$a^2 + 2b^2 - 0 = (3k)^2 + 2(3l+2)^2$$
$$= 9k^2 + 2(9l^2 + 12l + 4)$$
$$= 3(3(k^2 + 2l^2) + 8l + 2) + 2$$

where $3(k^2 + 2l^2) + 8l + 2 \in \mathbb{Z}$. Thus,

$$3 \nmid (a^2 + 2b^2 - 0) \Leftrightarrow a^2 + 2b^2 \not\equiv 0 \pmod 3.$$

From the above, the statement is true. ■