Mathematical Proof: Problem Set 1

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Problem.1

(a) is NOT a set, since the curly brackets are not properly matched and does not satisfy the definition of a set.

(b) is a set, since the curly brackets are properly matched and satisfies the definition of a set.

(c) is a set, since the curly brackets are properly matched and satisfies the definition of a set.

(d) is NOT a set, since the objects are not enclosed in curly brackets and does not satisfy the definition of a set.

Problem.2

(a)

$$A = \{-2, -1, 0, 1, 2, 3, 4, 5\}$$

(b)

$$B = \{1, 4, 9, 16, 25, 36, 49, 64, 81\}$$

(c)

$$C = \{-64, -27, -8, -1, 0, 1, 8, 27, 64\}$$

(d)

$$D = \emptyset$$

(e)

$$E = \{0, 1\}$$

Problem.3

(a)

$$A = \{0\},\ B = \{\{0\}, 1\},\ C = \{\{\{0\}, 1\}, 2\}$$

(b)

$$A = \{0\},\ B = \{\{0\}, 1\},\ C = \{0, 1\}$$

(c)

$$A = \{0, 1\},\ B = \{1, 2\},\ C = \{1, \{0, 1\}, \{1, 2\}\}$$

$$A \cap B = \{1\} \subset C,\ A \cup B = \{0, 1, 2\} \notin C$$

Problem.4

$$A = \{-1, 0, 1\},\ B = \{-1, 0, 1\},\ C = \{0, 1\},\ D = \{-1, 0, 1\},\ E = \{-1, 0, 1\}$$

$$\therefore A = B = D = E$$

Problem.5

Here, $i \in \mathbb{Z}, B_i = \{i-1, i, i+1\}$.

(a)

$$\begin{aligned}\cap_{i=1}^{3}(B_{2i-1} \cup B_{2i+1}) &= (B_1 \cup B_3) \cap (B_3 \cup B_5) \cap (B_5 \cup B_7) \\ &= (\{0,1,2\} \cup \{2,3,4\}) \cap (\{2,3,4\} \cup \{4,5,6\}) \cap (\{4,5,6\} \cup \{6,7,8\}) \\ &= \{0,1,2,3,4\} \cap \{2,3,4,5,6\} \cap \{4,5,6,7,8\} \\ &= \{4\}\end{aligned}$$

(b)

$$B_i \cap B_{i+1} = \{i-1, i, i+1\} \cap \{i, i+1, i+2\} = \{i, i+1\}$$

$$\begin{aligned}\therefore \cup_{i=1}^{n}(B_i \cap B_{i+1}) &= \cup_{i=1}^{n}\{i, i+1\} = \{1,2\} \cup \{2,3\} \cup \cdots \cup \{n, n+1\} \\ &= \{1, 2, \ldots, n, n+1\} = \{l \in \mathbb{N} \mid 1 \leq l \leq n+1\} \quad (n \in \mathbb{N})\end{aligned}$$

(c)

$$B_{3i} \cup B_{-3i} = \{3i-1, 3i, 3i+1\} \cup \{-3i-1, -3i, -3i+1\} = \{-3i-1, -3i, -3i+1, 3i-1, 3i, 3i+1\}$$

$$\begin{aligned} \therefore \cup_{i=0}^{\infty} (B_{3i} \cup B_{-3i}) &= \cup_{i=0}^{\infty}\{-3i-1, -3i, -3i+1, 3i-1, 3i, 3i+1\} \\ &= \{-1, 0, 1\} \cup \{-4, -3, -2, 2, 3, 4\} \cup \{-7, -6, -5, 5, 6, 7\} \cup \ldots \\ &= \mathbb{Z} \end{aligned}$$

$$(\because 3(i+1) - 1 = 3i + 2 > 3i + 1, -3(i+1) + 1 = -3i - 2 < -3i - 1)$$

Problem.6

Here, $A = \{1, 2, 3, 4, \ldots, 10\}$.

(a) A partition that meets the requirements of this problem S is

$$S = \{\{1\}, \{2, 3, 4\}, \{5, 6, 7, 8, 9, 10\}\}.$$

(b) Let A a set, if no two cells in that partition have the same cardinality, the greatest number of cells that can be taken in such a partition $|S|_{max}$ is the number of elements of natural numbers added from 1 in order and become larger than $|A|$, minus one. In this case, $|A| = 10$, and $1+2+3+4 = 10, 1+2+3+4+5 = 15 > 10$.

$$\therefore |S|_{max} = 4$$

(c) Given that the cells of a partition are nonempty and pairwise disjoint, the smallest cardinality a set B can have and still possess a partition with exactly three cells, no two of which have the same cardinality $|B|_{min}$, is the number of natural numbers added from 1 to 3.

$$\therefore |B|_{min} = 1 + 2 + 3 = 6$$

Problem.7

Here, $r \in \mathbb{R},\ r \geq 0,\ C_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = r^2\}$, and $\mathcal{D} = \{C_r \mid r \in \mathbb{R} \text{ and } r > 0\}$. If $r > 0 \Rightarrow r^2 > 0$, therefore

$$0 < r^2 = x^2 + y^2 \Rightarrow x^2 > 0 \text{ or } y^2 > 0 \Rightarrow x \neq 0 \text{ or } y \neq 0.$$

Thus, even though $(0, 0) \in \mathbb{R} \times \mathbb{R}$,

$$(0, 0) \notin \cup_{r \in \mathbb{R}, r > 0} C_r \neq \mathbb{R} \times \mathbb{R}$$

and $\mathcal{D}$ is NOT a partition of the Cartesian plane $\mathbb{R} \times \mathbb{R}$.