Mathematical Proof: Problem Set 2

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Problem.1

Here, $U = \{1, 2, 3\}, A = \{1, 2\}, B = \{2, 3\}$ and $C = \{1, 3\}$.

(a)

$$\begin{aligned}(A \cup B) - (B \cap C) &= (\{1, 2\} \cup \{2, 3\}) - (\{2, 3\} \cap \{1, 3\}) \\ &= \{1, 2, 3\} - \{3\} \\ &= \{1, 2\}\end{aligned}$$

(b)

$$\overline{A} = U - A = \{1, 2, 3\} - \{1, 2\} = \{3\}$$

(c)

$$\begin{aligned}\overline{B \cup C} &= U - (B \cup C) \\ &= \{1, 2, 3\} - (\{2, 3\} \cup \{1, 3\}) \\ &= \{1, 2, 3\} - \{1, 2, 3\} \\ &= \emptyset\end{aligned}$$

(d)

$$\begin{aligned}A \times B &= \{1, 2\} \times \{2, 3\} \\ &= \{(1, 2), (1, 3), (2, 2), (2, 3)\}\end{aligned}$$

(e) By using the result in (b),

$$\begin{aligned}\overline{A} \times C &= \{3\} \times \{1,3\} \\ &= \{(3,1),(3,3)\}\end{aligned}$$

Problem.2

Here, $A = \{1\}$ and $C = \{1,2\}$.

$$\mathcal{P}(A) = \{\emptyset, \{1\}\}$$

$$\mathcal{P}(C) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

$\therefore$ an example of a set B $s.t.$ $\mathcal{P}(A) \subset B \subset \mathcal{P}(C)$ is

$$B = \{\emptyset, \{1\}, \{1,2\}\}$$

Problem.3

Here, $R = \{(1,1),(1,2),(1,3),(2,1),(4,2),(4,3)\}$.

Now, to find the subsets $A, B, C, D \subset \{1,2,3,4\}$ $s.t.$ $R = ((A \times B) \cup (C \times D)) - (D \times D)$, first one can think about D. In R, since only $(1,1)$ is the ordered pair by the same element, we take $D = \{2,3\}$. Then we should take $C = \{1,4\}$ to generate the appropriate ordered pairs in R. Considering the rest elements in R that cannot be generated by C and D, you take $A = \{1,2\}, B = \{1\}$. Therefore, if you take the subsets $A, B, C, D \subset \{1,2,3,4\}$ as

$$A = \{1,2\},\ B = \{1\},\ C = \{1,4\},\ D = \{2,3\},$$

certainly this can be shown to satisfy the condition as follows.

$$\begin{aligned}&((A \times B) \cup (C \times D)) - (D \times D) \\ &= ((\{1,2\} \times \{1\}) \cup (\{1,4\} \times \{2,3\})) - (\{2,3\} \times \{2,3\}) \\ &= (\{(1,1),(2,1)\} \cup \{(1,2),(1,3),(4,2),(4,3)\}) - \{(2,2),(2,3),(3,2),(3,3)\} \\ &= \{(1,1),(1,2),(1,3),(2,1),(4,2),(4,3)\} - \{(2,2),(2,3),(3,2),(3,3)\} \\ &= \{(1,1),(1,2),(1,3),(2,1),(4,2),(4,3)\} = R\end{aligned}$$

※ The impossibility of finding the subsets

$$A, B, C, D \subset \{1, 2, 3, 4\} \ s.t. \ R = ((A \times B) \cup (C \times D)) - (D \times D),$$

where

$$R = \{(1, 1), (2, 1), (3, 2), (3, 3), (4, 2), (4, 3)\}$$

Problem.4

Here, the statements;

P: I pass the test.

Q: I pass the course.

R: I make the dean's list.

(a)(i)

$P \vee (\sim Q)$: I pass the test or don't pass the course.

(ii)

$(\sim P) \Rightarrow (\sim Q)$: If I don't pass the test, then I don't pass the course.

(iii)

$Q \Rightarrow P$: If I pass the course, then I pass the test.

(iv)

$(\sim P) \Rightarrow (\sim R)$: If I don't pass the test, then I don't make the dean's list.

(v)

$(\sim P) \wedge Q$: I don't pass the test and pass the course.

(vi)

$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$: If I pass the test, then I pass the course, and if I pass the course, then I make the dean's list.

(vii)

$Q \wedge (\sim R)$: I pass the course and don't make the dean's list.

(viii)

$$Q \Rightarrow R\text{: If I pass the course, then I make the dean's list.}$$

(ix)

$$(P \Rightarrow R) \wedge (R \Rightarrow P) \equiv (R \Leftrightarrow P)\text{: I pass the test if and only if I make the dean's list.}$$

(b)(i)

$$P \Rightarrow R$$

(ii)

$$Q \vee (\sim R)$$

(iii)

$$(\sim Q) \Rightarrow (\sim R)$$

(iv)

$$P \Leftarrow Q$$

(v)

$$Q \wedge (\sim R)$$

(vi)

$$P \Leftrightarrow Q$$

Problem.5

(a)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(b)

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \vee P$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

(c)

P	Q	$\sim P$	$\sim Q$	$(\sim P) \vee (\sim Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(d)

P	Q	$P \Rightarrow Q$	$\sim P$	$(P \Rightarrow Q) \Rightarrow (\sim P)$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

(e)

P	Q	$P \Rightarrow Q$	$\sim (P \Rightarrow Q)$	$P \Rightarrow (\sim (P \Rightarrow Q))$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

(※(d)&(e) is contraposition)

Problem.6

Here, $A = \{4, 5, 6, 7, 8, 9\}$, and the open sentences

$$P(B) : B \cap \{4, 6, 8\} = \emptyset$$

$$Q(B) : B \neq \emptyset$$

over the domain $\mathcal{P}(A)$.

(a) For $P(B) \wedge Q(B)$ to be true, B is nonempty and is disjoint with $\{4, 6, 8\}$. Therefore, let C be a set given by

$$C = \mathcal{P}(A - \{4, 6, 8\}) - \{\emptyset\}$$

$$= \mathcal{P}(\{5, 7, 9\}) - \{\emptyset\}$$

$$= \{\{5\}, \{7\}, \{9\}, \{5, 7\}, \{5, 9\}, \{7, 9\}, \{5, 7, 9\}\}$$

and $\forall B \in C \subset \mathcal{P}(A)$ satisfy $P(B) \wedge Q(B)$ is true, and that is all.

(b) For $P(B) \vee (\sim Q(B))$ to be true, B has to be an empty set, or has to be disjoint with {4,6,8}. So let D be a set given by

$$D = \mathcal{P}(A - \{4, 6, 8\})$$

$$= \mathcal{P}(\{5, 7, 9\})$$

$$= \{\emptyset, \{5\}, \{7\}, \{9\}, \{5, 7\}, \{5, 9\}, \{7, 9\}, \{5, 7, 9\}\}$$

and $\forall B \in D \subset \mathcal{P}(A)$ satisfy $P(B) \vee (\sim Q(B))$ is true, and that is all.

(c) For $(\sim P(B)) \wedge (\sim Q(B))$ to be true, first the only element in $\mathcal{P}(A)$ satisfy that $\sim\ Q(B)$ is true is an empty set, but $(\sim\ P(\emptyset))$ is false. Therefore, $B \in \mathcal{P}(A)$ which satisfy $(\sim P(B)) \wedge (\sim Q(B))$ does not exist.

Problem.7

Suppose the set S cardinality of 2 exists, given by

$$S = \{a, b\}$$

where $a \neq b$, $P(a) \Rightarrow Q(a)$, $Q(a) \Rightarrow R(a)$, and $R(b) \Rightarrow P(b)$ are true, and also $P(a) \Leftarrow Q(a)$, $Q(a) \Leftarrow R(a)$, and $R(b) \Leftarrow P(b)$ are false. ($\because$ one can always define $a,\ b$ in this way without loss of generality, because the open sentences are cyclic, and over the domain, the pigeonhole principle of dividing three open sentences into two elements applies.)

In general, to be the implication $P(n) \Rightarrow Q(n)$ to be false, $P(n)$ is true and $Q(n)$ is false are necessary.
Thus, $a \in S$ satisfies that $Q(a) \Leftarrow R(a)$ is false, therefore $Q(a)$ is false.

However, this contradicts that $P(a) \Leftarrow Q(a)$ is false because of the law of false hypothesis.

Therefore, such a set S does NOT exist.