Mathematical Proof: Problem Set 3

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Problem.1

(a)

Some element of U cannot be expressed as $x + y$, where any $x \in S$ and any $y \in T$.

(b)

For some $x \in S$ and $y \in S$, $xy \notin S$.

(c)

For some element $x \in S$, there is not an element $y \in T$ such that $y > x$.

Problem.2

(a)

$$x^2 - x = 0 \Leftrightarrow x(x-1) = 0 \ \therefore x = 0, 1 \in \mathbb{R}$$

Thus the truth value is T.

(b)

$$1 + 1 \geq 2, \text{ and for any other } n \in \mathbb{N}, n + 1 \geq 1 + 1 \geq 2.$$

Thus the truth value is T.

(c)

$$\text{Counterexample: } x = -1 \in \mathbb{R} \Rightarrow \sqrt{(-1)^2} = 1 \neq -1$$

Thus the truth value is F.

(d)

$$3x^2 - 27 = 0 \Leftrightarrow x^2 = 9 \ \therefore x = \pm 3 \in \mathbb{Q}$$

Thus the truth value is T.

(e)

$$\text{Example: } x = 3 \in \mathbb{R}, y = 2 \in \mathbb{R} \Rightarrow 3 + 2 + 3 = 8$$

Thus the truth value is T.

(f)

$$x^2 + y^2 = 9 \Leftrightarrow x^2 = 9 - y^2$$

Therefore, for given $y > 3$, $x^2 = 9 - y^2 < 0$ and there is no such $x \in \mathbb{R}$ $s.t.$ $x^2 < 0$. Thus the truth value is F.

(g) The same argument is applicable as in (f), since x and y is defined over the same domain and the open sentence $x^2 + y^2 = 9$ is symmetric under exchange of variables x and y. Thus the truth value is F.

Problem.3

My answer is only (d). The below is the reasoning for each question.
The De Morgan's law,

$$\sim (P(x) \wedge Q(x)) \equiv (\sim P(x)) \vee (\sim Q(x)), \sim (P(x) \vee Q(x)) \equiv (\sim P(x)) \wedge (\sim Q(x))$$

Also, we know that

$$\sim (P(x) \Rightarrow Q(x)) \equiv P(x) \wedge (\sim Q(x))$$

Therefore,

$$\sim ((\sim P(x)) \Rightarrow Q(x)) \equiv (\sim P(x)) \wedge (\sim Q(x))$$

Considering the result above, here we define the statement K as follows:

$$K : \exists x \in S, (\sim P(x)) \wedge (\sim Q(x))$$

Suppose the quantified statement R, and we define the compound statement K' as follows:

$$K' : R \Rightarrow K$$

For each statement R, if we could sufficiently show that K' is always true, the answer is YES. However, if we could find the counterexample of the statement K', or could not sufficiently show that K' is always true, the answer is NO.

(a) By the De Morgan's law,

$$\sim (P(x) \wedge Q(x)) \equiv (\sim P(x)) \vee (\sim Q(x))$$

Therefore, here we define the statement R as follows:

$$R : \forall x \in S, (\sim P(x)) \vee (\sim Q(x))$$

Therefore, K' is not necessarily true, since either $\sim P(x)$ or $\sim Q(x)$ could be false. So the answer is NO.

(b) Here, we define the statement R as follows:

$$R : \forall x \in S, P(x)$$

The answer is obviously NO because $\forall x \in S, P(x)$ does not satisfy $(\sim P(x)) \wedge (\sim Q(x))$, thus K' is false.

(c) Here, we define the statement R as follows:

$$R : \forall x \in S, Q(x)$$

The answer is obviously NO because $\forall x \in S, Q(x)$ does not satisfy $(\sim P(x)) \wedge (\sim Q(x))$, thus K' is false.

(d) By the De Morgan's law,

$$\sim (P(x) \vee Q(x)) \equiv (\sim P(x)) \wedge (\sim Q(x))$$

Therefore, here we define the statement R as follows:

$$R : \exists x \in S, (\sim P(x)) \wedge (\sim Q(x))$$

The statement R is identical to the statement K, so K' is always true. Thus, the answer is YES.

(e) By the De Morgan's law,

$$\sim (P(x) \wedge (\sim Q(x)) \equiv (\sim P(x)) \vee (Q(x))$$

Therefore, here we define the statement R as follows:

$$R : \forall x \in S, (\sim P(x)) \vee (Q(x))$$

Therefore, K' is not necessarily true, since either $\sim P(x)$ or $\sim Q(x)$ could be false. So the answer is NO.

Problem.4

Proof: Assume a, b, and c are odd integers such that $a+b+c=0$.
By definition,

$$\exists k, l, m \in \mathbb{Z},\ s.t.\ a=2k+1, b=2l+1, c=2m+1$$

Therefore,

$$a+b+c=(2k+1)+(2l+1)+(2m+1)=2(k+l+m+1)+1$$

Thus,

$$\exists n \in \mathbb{Z},\ s.t.\ a+b+c=2n+1 \neq 0$$

which contradicts the assumption. So $\forall a, b, c \in \mathbb{Z}, a+b+c=0$ is false, therefore the implication is true. ■ (vacuous proof)

Problem.5

Proof: Assume x is an even integer, by definition,

$$\exists k \in \mathbb{Z},\ s.t.\ x=2k$$

Therefore,

$$7x-3=7 \cdot 2k-3=2(7k-2)+1$$

Thus,

$$\exists l \in \mathbb{Z},\ s.t.\ 7x-3=2l+1$$

and $7x-3$ is an odd integer.
For the converse, prove by contrapositive. Assume x is an odd integer, by definition,

$$\exists k \in \mathbb{Z},\ s.t.\ x=2k+1$$

Therefore,

$$7x-3=7(2k+1)-3=14k+4=2(7k+2)$$

Thus,

$$\exists l \in \mathbb{Z},\ s.t.\ 7x-3=2l$$

and $7x-3$ is an even integer.
From the above, the statement is true. ■

Problem.6

<u>*Proof*</u>: Let $x \in \mathbb{Z}$. Assume $3x-1$ is even, by definition,

$$\exists k \in \mathbb{Z},\ s.t.\ 3x-1=2k$$

Therefore,

$$5x+2=(3x-1)+2(x+1)+1=2k+2(x+1)+1=2(k+x+1)+1$$

Thus,

$$\exists l \in \mathbb{Z},\ s.t.\ 5x+2=2l+1$$

and $5x+2$ is odd.
Next, assume $5x+2$ is odd, by definition,

$$\exists m \in \mathbb{Z},\ s.t.\ 5x+2=2m+1$$

Therefore,

$$3x-1=(5x+2)-2x-3=(2m+1)+2(-x-2)+1=2(m-x-1)$$

Thus,

$$\exists n \in \mathbb{Z},\ s.t.\ 3x-1=2n$$

and $3x-1$ is even.
From the above, the statement is true. ■

Problem.7

Recall $\mathbb{E}, \mathbb{O}$ are sets of even and odd integers respectively.
First, we prove a lemma below.

$$\text{Lemma 1: Let } x \in \mathbb{Z}.\ x^2 \in \mathbb{E} \Rightarrow x \in \mathbb{E}$$

<u>*Proof*</u>: We prove by contrapositive. Let $x \in \mathbb{O}$. By definition,

$$\exists k \in \mathbb{Z},\ s.t.\ x=2k+1$$

Therefore,

$$x^2=(2k+1)^2=4k^2+4k+1=2(2k^2+2k)+1 \in \mathbb{O} \quad \blacksquare$$

Next we prove another lemma below.

Lemma 2: Let $a, b, c \in \mathbb{Z}$. $a^2 + b^2 = c^2 \Rightarrow (abc)^2 \in \mathbb{E}$

Proof: Let $a, b, c \in \mathbb{Z}$. Assume $a^2 + b^2 = c^2$.
Since a and b are symmetric variable in this statement, and also since $\{\mathbb{E}, \mathbb{O}\}$ is a partition of $\mathbb{Z}$, we proceed by three cases, according to whether a and b are even or odd.

Case 1: $a, b \in \mathbb{E}$.
By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 2k, b = 2l$$

Therefore, by the assumption

$$c^2 = a^2 + b^2 = (2k)^2 + (2l)^2 = 4(k^2 + l^2) \in \mathbb{Z}$$

$$\Rightarrow (abc)^2 = a^2b^2c^2 = 2(2a^2b^2(k^2 + l^2)) \in \mathbb{E}.$$

Case 2: $a, b \in \mathbb{O}$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 2k + 1, b = 2l + 1$$

Therefore, by the assumption

$$c^2 = a^2 + b^2 = (2k+1)^2 + (2l+1)^2 = 4(k^2 + l^2) + 4(k + l) + 2 = 2(2(k^2 + l^2) + 2(k + l) + 1) \in \mathbb{Z}$$

$$\Rightarrow (abc)^2 = a^2b^2c^2 = 2(a^2b^2(2(k^2 + l^2) + 2(k + l) + 1)) \in \mathbb{E}.$$

Case 3: $a \in \mathbb{E}, b \in \mathbb{O}$. By definition,

$$\exists k, l \in \mathbb{Z},\ s.t.\ a = 2k, b = 2l + 1$$

Therefore, by the assumption

$$c^2 = a^2 + b^2 = (2k)^2 + (2l+1)^2 = 4(k^2 + l^2) + 4l + 1 \in \mathbb{Z}$$

$$\Rightarrow (abc)^2 = a^2b^2c^2 = (2k)^2b^2c^2 = 2(2k^2b^2c^2) \in \mathbb{E}. \ \blacksquare$$

From the Lemma 1 and Lemma 2 above, we immediately prove the statement:

Let $a, b, c \in \mathbb{Z}$. $a^2 + b^2 = c^2 \Rightarrow abc \in \mathbb{E}$

Proof: Let $a, b, c \in \mathbb{Z}$. Assume $a^2 + b^2 = c^2$.
By using the Lemma 2, $(abc)^2 \in \mathbb{E}$. By using the Lemma 1, $abc \in \mathbb{E}$. $\blacksquare$

Problem.8

Proof: Let $a, b, c \in \mathbb{R}$ be the sides of a triangle $\mathcal{T}$ where $a \leq b \leq c$. Assume $\mathcal{T}$ is a right triangle.
By $a \leq b \leq c$ and the Pythagorean theorem,

$$c^2 = a^2 + b^2$$

Therefore,

$$\begin{aligned}
3(abc)^2 - (c^6 - a^6 - b^6) &= 3a^2b^2(a^2 + b^2) - ((a^2 + b^2)^3 - a^6 - b^6) \\
&= 3a^4b^2 + 3a^2b^4 + -(a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - a^6 - b^6) \\
&= 3a^4b^2 + 3a^2b^4 + -3a^4b^2 - 3a^2b^4 \\
&= 0
\end{aligned}$$

$$\therefore (abc)^2 = \frac{c^6 - a^6 - b^6}{3} \quad \blacksquare$$