Mathematical Proof: Problem Set 7

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Problem.1

A formula

$$1 + 4 + 7 + \cdots + (3n-2) = \sum_{k=1}^{n}(3k-2) = \frac{n}{2}(1+3n-2) = \frac{n}{2}(3n-1)$$

Therefore, we define the open sentence

$$P(n):\ 1 + 4 + 7 + \cdots + (3n-2) = \frac{n}{2}(3n-1)$$

<u>*Proof*</u>: We employ induction.

Since

$$\frac{1}{2}(3 \cdot 1 - 1) = 1$$

the statement $P(1)$ is true.
For $k \in \mathbb{N}$, assume that $P(k)$ is true, so that

$$1 + 4 + 7 + \cdots + (3k-2) = \sum_{l=1}^{k}(3l-2) = \frac{k}{2}(3k-1)$$

is true. Then,

$$\begin{aligned}
1 + 4 + 7 + \cdots + (3k-2) + (3(k+1)-2) &= \sum_{l=1}^{k}(3l-2) + (3k+1) \\
&= \frac{k}{2}(3k-1) + 3k + 1
\end{aligned}$$

Here,

$$\begin{aligned}\frac{k}{2}(3k-1)+3k+1 &= \frac{3k^2}{2}-\frac{k}{2}+3k+1\\ &= \frac{3k^2}{2}+\frac{5k}{2}+1\\ &= \frac{(k+1)}{2}(3k+2)\end{aligned}$$

Therefore,

$$1+4+7+\cdots+(3k-2)+(3(k+1)-2)=\frac{(k+1)}{2}(3k+2)$$

so that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer.$\blacksquare$

Problem.2

The open sentence

$$P(n):\ 2!\cdot 4!\cdot\cdot 6!\ldots(2n)!\geq((n+1)!)^n$$

<u>*Proof*</u>: We employ induction.

Since

$$(2\cdot 1)!\geq((1+1)!)^1=2$$

the statement $P(1)$ is true.
For $k\in\mathbb{N}$, assume that $P(k)$ is true, so that

$$2!\cdot 4!\cdot\cdot 6!\ldots(2k)!=\prod_{l=1}^{k}(2l)!\geq((k+1)!)^k$$

is true. Then,

$$\begin{aligned}2!\cdot 4!\cdot\cdot 6!\ldots(2k)!\cdot(2(k+1))! &= \left[\prod_{l=1}^{k}(2l)!\right](2(k+1))!\\ &\geq((k+1)!)^k\cdot(2(k+1))!\quad(\because (2(k+1))!>0)\end{aligned}$$

Here,

$$(2(k+1))! = (2k+2)\cdot(2k+1)\dots(k+2)\cdot(k+1)\cdot k\cdot(k-1)\dots 1$$

$$> (k+2)^k\cdot(k+2)\cdot(k+1)\cdot k\cdot(k-1)\dots 1 \quad (\because 2k+2 > 2k+1 > \cdots > k+3 > k+2 > 1)$$

$$= (k+2)^k\cdot(k+2)!$$

Therefore,

$$\begin{aligned} 2!\cdot 4!\cdot\cdot 6!\dots(2k)!\cdot(2(k+1))! &\geq ((k+1)!)^k\cdot(2(k+1))! \\ &> ((k+1)!)^k\cdot(k+2)^k\cdot(k+2)! \\ &= ((k+2)!)^k\cdot(k+2)! \\ &= ((k+2)!)^{k+1} = (((k+1)+1)!)^{k+1} \end{aligned}$$

so that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer.■

Problem.3

For every real number $x > -1$, the open sentence

$$P(n):\ (1+x)^n \geq 1+nx$$

Proof: We employ induction.

Since

$$1+x \geq 1+x$$

the statement $P(1)$ is true.
For $k \in \mathbb{N}$, assume that $P(k)$ is true, so that

$$(1+x)^k \geq 1+kx$$

is true. Then,

$$(1+x)^k(1+x) \geq (1+kx)(1+x) \quad (\because 1+x > 0)$$

$\Rightarrow$

$$(1+x)^{k+1} \geq (1+kx)(1+x) = 1+(k+1)x+kx^2 \geq 1+(k+1)x \quad \left(\because kx^2 \geq 0\right)$$

Therefore,

$$(1+x)^{k+1} \geq 1+(k+1)x$$

so that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer.■

Problem.4

The open sentence

$$P(n):\ 81 \mid \left(10^{n+1} - 9n - 10\right)$$

<u>*Proof*</u>: We employ induction.

Since

$$10^{1+1} - 9 \cdot 1 - 10 = 100 - 9 - 10 = 81 \cdot 1$$

So, note that $1 \in \mathbb{Z}$,

$$81 \mid \left(10^{1+1} - 9 \cdot 1 - 10\right)$$

the statement $P(1)$ is true.
For $k \in \mathbb{N}$, assume that $P(k)$ is true, so that

$$81 \mid \left(10^{k+1} - 9k - 10\right)$$

is true. Then,

$$\exists m \in \mathbb{Z} \ s.t.\ 10^{k+1} - 9k - 10 = 81 \cdot m$$

$\Rightarrow$

$$\begin{aligned}
10^{(k+1)+1} - 9(k+1) - 10 &= 10 \cdot 10^{k+1} - 9k - 9 - 10 \\
&= \left(10^{k+1} - 9k - 10\right) + 9 \cdot 10^{k+1} - 9 \\
&= 81 \cdot m + 9\left(10^{k+1} - 1\right) + (-81k - 81) - (-81k - 81) \\
&= 81 \cdot m + 9\left(10^{k+1} - 9k - 10\right) + 81(k+1) \\
&= 81 \cdot m + 9 \cdot 81 \cdot m + 81(k+1) \\
&= 81\left(10m + k + 1\right)
\end{aligned}$$

Therefore, since $(10m + k + 1) \in \mathbb{Z}$

$$81 \mid \left(10^{(k+1)+1} - 9(k+1) - 10\right)$$

so that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer.■

Problem.5

A sequence $\{a_n\}$ is given by

$$a_1 = 1, a_2 = 2; a_n = a_{n-1} + 2a_{n-2}$$

The experiment

$$a_1 = 1,\ a_2 = 2,\ a_3 = 4,\ a_4 = 8,\ a_5 = 16,\ \ldots$$

So the conjecture

$$P(n):\ a_n = 2^{n-1}$$

Proof: We employ induction.

Since

$$2^{1-1} = 1$$

the statement $P(1)$ is true.
Also,

$$2^{2-1} = 2$$

the statement $P(2)$ is true.
For $k \in \mathbb{N}$ *s.t.* $k \geq 2$, assume that $P(k-1)$ and $P(k)$ is true, so that

$$a_{k-1} = 2^{(k-1)-1} = 2^{k-2},\ a_k = 2^{k-1}$$

is true. Then,

$$a_{k+1} = a_k + a_{k-1} = 2^{k-1} + 2 \cdot 2^{k-2} = 2 \cdot 2^{k-1} = 2^{(k+1)-1}$$

so that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer.$\blacksquare$

Problem.6

The sequence of *Fibonacci numbers*, $\{F_n\}$ is given by

$$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$$

(a) The open sentence

$$P(n) : 2 \mid F_n \Leftrightarrow 3 \mid n$$

$\underline{Proof}$: We employ induction. Let $n \in \mathbb{N}$.

Since $F_1 = 1,\ 2 \nmid 1$, and $3 \nmid 1$, so the statement $P(1)$ is true.($\because$ the Law of Hypothesis)
Similarly, since $F_2 = 1,\ 2 \nmid 1$, and $3 \nmid 2$, so the statement $P(2)$ is true.($\because$ the Law of Hypothesis)

Similarly, since $F_3 = F_1 + F_2 = 2,\ 2 \mid 2$, and $3 \nmid 3$, so the statement $P(2)$ is true.

For $k \in \mathbb{N}$ $s.t.$ $k \geq 3$, assume that $P(k-2)$, $P(k-1)$, and $P(k)$ is true, so that

$$2 \mid F_{k-2} \Leftrightarrow 3 \mid (k-2)$$

$$2 \mid F_{k-1} \Leftrightarrow 3 \mid (k-1)$$

$$2 \mid F_k \Leftrightarrow 3 \mid k$$

Note that $F_{k+1} = F_k + F_{k-1} \in \mathbb{Z} \quad (\because F_k, F_{k-1} \in \mathbb{Z})$.
Here,

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} \\ &= 2F_{k-1} + F_{k-2} \end{aligned}$$

$\Leftrightarrow$

$$2F_{k-1} = F_{k+1} - F_{k-2} \in \mathbb{E}$$

By Theorem 3.16, F_{k+1} and F_{k-2} are of the same parity, i.e. $2 \mid F_{k+1} \Leftrightarrow 2 \mid F_{k-2}$.
Now, observe that

$$\begin{aligned} 2 \mid F_{k+1} &\Leftrightarrow 2 \mid F_{k-2} \\ &\Leftrightarrow 3 \mid (k-2) \quad (\because \text{assumption}) \\ &\Leftrightarrow 3 \mid (k-2+3) \Leftrightarrow 3 \mid (k+1) \end{aligned}$$

so that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer.■

(b) $\underline{Proof}$: We employ induction.

Since

$$2^{1-1}F_1 = 1 \cdot 1 \equiv 1 \pmod 5$$

the statement $P(1)$ is true.
Also,

$$2^{2-1}F_2 = 2 \cdot 1 \equiv 2 \pmod 5$$

the statement $P(2)$ is true.
For $k \in \mathbb{N}$ *s.t.* $k \geq 3$, assume that $P(k-1)$ and $P(k)$ is true, so that

$$2^{(k-1)-1}F_{k-1} = 2^{k-2}F_{k-1} \equiv k-1 \pmod 5$$

and

$$2^{k-1}F_k \equiv k \pmod 5$$

is true. Then,

$$2^{(k+1)-1}F_{k+1} = 2^k\left(F_{k-1} + F_k\right) = 4 \cdot 2^{k-2}F_{k-1} + 2 \cdot 2^{k-1}F_k$$

$$\equiv 4 \cdot (k-1) + 2 \cdot k \equiv 6k - 4 \equiv 5k - 5 + k + 1 \equiv k+1 \pmod 5$$

so that $P(k+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer.■

Problem.7

Proof: Use the method of minimum counterexample. Let $r \in \mathbb{R}$ *s.t.* $r \neq 0, r + \frac{1}{r} \in \mathbb{Z}$, so that

$$\exists m_1 \in \mathbb{Z} \ s.t. \ r + \frac{1}{r} = m_1$$

Then, we immediately show that

$$r^2 + \frac{1}{r^2} = \left(r + \frac{1}{r}\right)\left(r + \frac{1}{r}\right) - 2 = m_1^2 - 2 \in \mathbb{Z}$$

Assume to the contrary that

$$S = \{m \in \mathbb{N} \mid r^m + \frac{1}{r^m} \notin \mathbb{Z}\} \neq \emptyset$$

By the Well-Ordering Principle, $\exists \mu \in S$ *s.t.* $\forall x \in S, 2 < \mu \leq x$. By the definition, (and $3 \leq \mu$,) μ satisfies $r^\mu + \frac{1}{r^\mu} \notin \mathbb{Z}$, and

$$\exists m_{\mu-1} \in \mathbb{Z} \ s.t. \ r^{\mu-1} + \frac{1}{r^{\mu-1}} = m_{\mu-1}$$

also

$$\exists m_{\mu-2} \in \mathbb{Z} \ s.t. \ r^{\mu-2} + \frac{1}{r^{\mu-2}} = m_{\mu-2}$$

Then,

$$\left(r^{\mu-1} + \frac{1}{r^{\mu-1}}\right)\left(r + \frac{1}{r}\right) = r^{\mu} + \frac{1}{r^{\mu}} + r^{\mu-2} + \frac{1}{r^{\mu-2}}$$

$\Leftrightarrow$

$$r^{\mu} + \frac{1}{r^{\mu}} = \left(r^{\mu-1} + \frac{1}{r^{\mu-1}}\right)\left(r + \frac{1}{r}\right) - \left(r^{\mu-2} + \frac{1}{r^{\mu-2}}\right)$$

$$= m_{\mu-1}m_1 - m_{\mu-2} \in \mathbb{Z}$$

which is a contradiction since $r^{\mu} + \frac{1}{r^{\mu}} \notin \mathbb{Z}$. $\blacksquare$