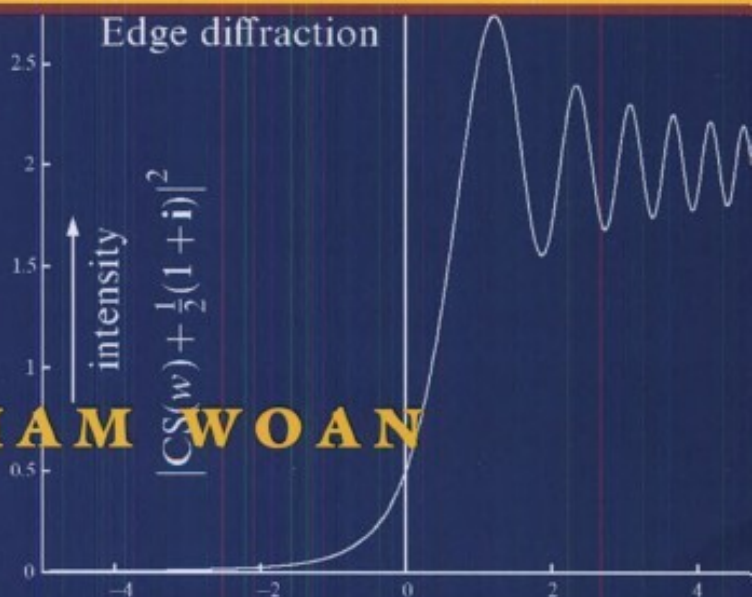
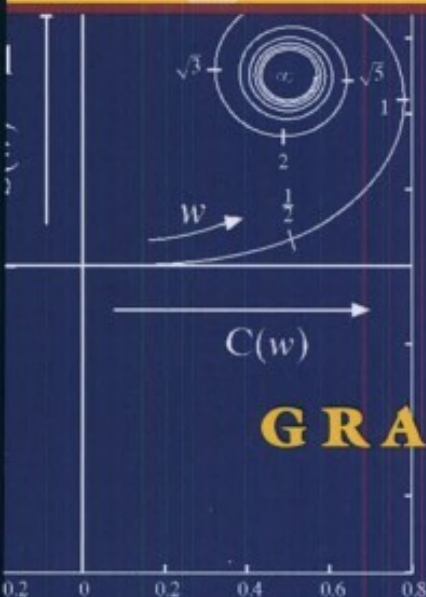


THE CAMBRIDGE HANDBOOK OF PHYSICS FORMULAS

$$\left(\frac{\alpha}{\omega} \right)$$



GRAHAM WOAN

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The Cambridge Handbook of Physics Formulas

The Cambridge Handbook of Physics Formulas is a quick-reference aid for students and professionals in the physical sciences and engineering. It contains more than 2000 of the most useful formulas and equations found in undergraduate physics courses, covering mathematics, dynamics and mechanics, quantum physics, thermodynamics, solid state physics, electromagnetism, optics, and astrophysics. An exhaustive index allows the required formulas to be located swiftly and simply, and the unique tabular format crisply identifies all the variables involved.

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The Cambridge Handbook of Physics Formulas

2003 Edition

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University of Glasgow*

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Preface

In *A Brief History of Time*, Stephen Hawking relates that he was warned against including equations in the book because “each equation... would halve the sales.” Despite this dire prediction there is, for a scientific audience, some attraction in doing the exact opposite.

The reader should not be misled by this exercise. Although the equations and formulas contained here underpin a good deal of physical science they are useless unless the reader *understands* them. Learning physics is not about remembering equations, it is about appreciating the natural structures they express. Although its format should help make some topics clearer, this book is not designed to teach new physics; there are many excellent textbooks to help with that. It is intended to be useful rather than pedagogically complete, so that students can use it for revision and for structuring their knowledge *once they understand the physics*. More advanced users will benefit from having a compact, internally consistent, source of equations that can quickly deliver the relationship they require in a format that avoids the need to sift through pages of rubric.

Some difficult decisions have had to be made to achieve this. First, to be short the book only includes ideas that can be expressed succinctly in equations, without resorting to lengthy explanation. A small number of important topics are therefore absent. For example, Liouville’s theorem can be algebraically succinct ($\dot{q} = 0$) but is meaningless unless \dot{q} is thoroughly (and carefully) explained. Anyone who already understands what \dot{q} represents will probably not need reminding that it equals zero. Second, empirical equations with numerical coefficients have been largely omitted, as have topics significantly more advanced than are found at undergraduate level. There are simply too many of these to be sensibly and confidently edited into a short handbook. Third, physical data are largely absent, although a periodic table, tables of physical constants, and data on the solar system are all included. Just a sighting of the marvellous (but dimensionally misnamed) *CRC Handbook of Chemistry and Physics* should be enough to convince the reader that a good science data book is thick.

Inevitably there is personal choice in what should or should not be included, and you may feel that an equation that meets the above criteria is missing. If this is the case, I would be delighted to hear from you so it can be considered for a subsequent edition. Contact details are at the end of this preface. Likewise, if you spot an error or an inconsistency then please let me know and I will post an erratum on the web page.

Acknowledgments This venture is founded on the generosity of colleagues in Glasgow and Cambridge whose inputs have strongly influenced the final product. The expertise of Dave Clarke, Declan Diver, Peter Duffett-Smith, Wolf-Gerrit Früh, Martin Hendry, Rico Ignace, David Ireland, John Simmons, and Harry Ward have been central to its production, as have the linguistic skills of Katie Lowe. I would also like to thank Richard Barrett, Matthew Cartmell, Steve Gull, Martin Hendry, Jim Hough, Darren McDonald, and Ken Riley who all agreed to field-test the book and gave invaluable feedback.

My greatest thanks though are to John Shakeshaft who, with remarkable knowledge and skill, worked through the entire manuscript more than once during its production and whose legendary red pen hovered over (or descended upon) every equation in the book. What errors remain are, of course, my own, but I take comfort from the fact that without John they would be much more numerous.

Contact information A website containing up-to-date information on this handbook and contact details can be found through the Cambridge University Press web pages at us.cambridge.org (North America) or uk.cambridge.org (United Kingdom), or directly at radio.astro.gla.ac.uk/hbhome.html.

Production notes This book was typeset by the author in $\text{\LaTeX 2}_{\epsilon}$ using the CUP Times fonts. The software packages used were *WinEdt*, *MiKTeX*, *Mayura Draw*, *Gnuplot*, *Ghostscript*, *Ghostview*, and *Maple V*.

Comments on the 2002 edition I am grateful to all those who have suggested improvements, in particular Martin Hendry, Wolfgang Jitschin, and Joseph Katz. Although this edition contains only minor revisions to the original its production was also an opportunity to update the physical constants and periodic table entries and to reflect recent developments in cosmology.

How to use this book

The format is largely self-explanatory, but a few comments may be helpful. Although it is very tempting to flick through the pages to find what you are looking for, the best starting point is the index. I have tried to make this as extensive as possible, and many equations are indexed more than once. Equations are listed both with their equation number (in square brackets) and the page on which they can be found. The equations themselves are grouped into self-contained and boxed “panels” on the pages. Each panel represents a separate topic, and you will find descriptions of all the variables used at the right-hand side of the panel, usually adjacent to the first equation in which they are used. You should therefore not need to stray outside the panel to understand the notation. Both the panel as a whole and its individual entries may have footnotes, shown below the panel. Be aware of these, as they contain important additional information and conditions relevant to the topic.

Although the panels are self-contained they may use concepts defined elsewhere in the handbook. Often these are cross-referenced, but again the index will help you to locate them if necessary. Notations and definitions are uniform over subject areas unless stated otherwise.

Chapter 1 Units, constants, and conversions

1.1 Introduction

The determination of physical constants and the definition of the units with which they are measured is a specialised and, to many, hidden branch of science.

A quantity with dimensions is one whose value must be expressed relative to one or more standard units. In the spirit of the rest of the book, this section is based around the International System of units (SI). This system uses seven base units¹ (the number is somewhat arbitrary), such as the kilogram and the second, and defines their magnitudes in terms of physical laws or, in the case of the kilogram, an object called the “international prototype of the kilogram” kept in Paris. For convenience there are also a number of derived standards, such as the volt, which are defined as set combinations of the basic seven. Most of the physical observables we regard as being in some sense fundamental, such as the charge on an electron, are now known to a relative standard uncertainty,² u_r , of less than 10^{-7} . The least well determined is the Newtonian constant of gravitation, presently standing at a rather lamentable u_r of 1.5×10^{-3} , and the best is the Rydberg constant ($u_r = 7.6 \times 10^{-12}$). The dimensionless electron g -factor, representing twice the magnetic moment of an electron measured in Bohr magnetons, is now known to a relative uncertainty of only 4.1×10^{-12} .

No matter which base units are used, physical quantities are expressed as the product of a numerical value and a unit. These two components have more-or-less equal standing and can be manipulated by following the usual rules of algebra. So, if $1 \cdot \text{eV} = 160.218 \times 10^{-21} \cdot \text{J}$ then $1 \cdot \text{J} = [1/(160.218 \times 10^{-21})] \cdot \text{eV}$. A measurement of energy, U , with joule as the unit has a numerical value of U/J . The same measurement with electron volt as the unit has a numerical value of $U/\text{eV} = (U/\text{J}) \cdot (\text{J}/\text{eV})$ and so on.

¹The **metre** is the length of the path travelled by light in vacuum during a time interval of $1/299\,792\,458$ of a second. The **kilogram** is the unit of mass; it is equal to the mass of the international prototype of the kilogram. The **second** is the duration of $9\,192\,631\,770$ periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom. The **ampere** is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per metre of length. The **kelvin**, unit of thermodynamic temperature, is the fraction $1/273.16$ of the thermodynamic temperature of the triple point of water. The **mole** is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is “mol.” When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles. The **candela** is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of $1/683$ watt per steradian.

²The relative standard uncertainty in x is defined as the estimated standard deviation in x divided by the modulus of x ($x \neq 0$).

1.2 SI units

SI base units

<i>physical quantity</i>	<i>name</i>	<i>symbol</i>
length	metre ^a	m
mass	kilogram	kg
time interval	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

^aOr “meter”.

SI derived units

<i>physical quantity</i>	<i>name</i>	<i>symbol</i>	<i>equivalent units</i>
catalytic activity	katal	kat	mol s^{-1}
electric capacitance	farad	F	C V^{-1}
electric charge	coulomb	C	A s
electric conductance	siemens	S	Ω^{-1}
electric potential difference	volt	V	J C^{-1}
electric resistance	ohm	Ω	V A^{-1}
energy, work, heat	joule	J	N m
force	newton	N	m kg s^{-2}
frequency	hertz	Hz	s^{-1}
illuminance	lux	lx	cd sr m^{-2}
inductance	henry	H	$\text{V A}^{-1} \text{ s}$
luminous flux	lumen	lm	cd sr
magnetic flux	weber	Wb	V s
magnetic flux density	tesla	T	V s m^{-2}
plane angle	radian	rad	m m^{-1}
power, radiant flux	watt	W	J s^{-1}
pressure, stress	pascal	Pa	N m^{-2}
radiation absorbed dose	gray	Gy	J kg^{-1}
radiation dose equivalent ^a	sievert	Sv	$[\text{J kg}^{-1}]$
radioactive activity	becquerel	Bq	s^{-1}
solid angle	steradian	sr	$\text{m}^2 \text{ m}^{-2}$
temperature ^b	degree Celsius	$^{\circ}\text{C}$	K

^aTo distinguish it from the gray, units of J kg^{-1} should not be used for the sievert in practice.

^bThe Celsius temperature, T_{C} , is defined from the temperature in kelvin, T_{K} , by $T_{\text{C}} = T_{\text{K}} - 273.15$.

SI prefixes^a

<i>factor</i>	<i>prefix</i>	<i>symbol</i>	<i>factor</i>	<i>prefix</i>	<i>symbol</i>
10 ²⁴	yotta	Y	10 ⁻²⁴	yocto	y
10 ²¹	zetta	Z	10 ⁻²¹	zepto	z
10 ¹⁸	exa	E	10 ⁻¹⁸	atto	a
10 ¹⁵	peta	P	10 ⁻¹⁵	femto	f
10 ¹²	tera	T	10 ⁻¹²	pico	p
10 ⁹	giga	G	10 ⁻⁹	nano	n
10 ⁶	mega	M	10 ⁻⁶	micro	μ
10 ³	kilo	k	10 ⁻³	milli	m
10 ²	hecto	h	10 ⁻²	centi	c
10 ¹	deca ^b	da	10 ⁻¹	deci	d

^aThe kilogram is the only SI unit with a prefix embedded in its name and symbol. For mass, the unit name “gram” and unit symbol “g” should be used with these prefixes, hence 10⁻⁶ kg can be written as 1 mg. Otherwise, any prefix can be applied to any SI unit.

^bOr “deka”.

Recognised non-SI units

<i>physical quantity</i>	<i>name</i>	<i>symbol</i>	<i>SI value</i>
area	barn	b	10 ⁻²⁸ m ²
energy	electron volt	eV	≈ 1.602 18 × 10 ⁻¹⁹ J
length	ångström	Å	10 ⁻¹⁰ m
	fermi ^a	fm	10 ⁻¹⁵ m
	micron ^a	μm	10 ⁻⁶ m
plane angle	degree	°	(π/180) rad
	arcminute	'	(π/10 800) rad
	arcsecond	"	(π/648 000) rad
pressure	bar	bar	10 ⁵ N m ⁻²
time	minute	min	60 s
	hour	h	3 600 s
	day	d	86 400 s
mass	unified atomic mass unit	u	≈ 1.660 54 × 10 ⁻²⁷ kg
	tonne ^{a,b}	t	10 ³ kg
volume	litre ^c	l, L	10 ⁻³ m ³

^aThese are non-SI names for SI quantities.

^bOr “metric ton.”

^cOr “liter”. The symbol “l” should be avoided.

1.3 Physical constants

The following 1998 CODATA recommended values for the fundamental physical constants can also be found on the Web at physics.nist.gov/constants. Detailed background information is available in *Reviews of Modern Physics*, Vol. 72, No. 2, pp. 351–495, April 2000.

The digits in parentheses represent the 1σ uncertainty in the previous two quoted digits. For example, $G = (6.673 \pm 0.010) \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$. It is important to note that the uncertainties for many of the listed quantities are correlated, so that the uncertainty in any expression using them in combination cannot necessarily be computed from the data presented. Suitable covariance values are available in the above references.

Summary of physical constants

speed of light in vacuum ^a	c	2.997 924 58	$\times 10^8 \text{ m s}^{-1}$
permeability of vacuum ^b	μ_0	4π $= 12.566 370 614 \dots$	$\times 10^{-7} \text{ H m}^{-1}$ $\times 10^{-7} \text{ H m}^{-1}$
permittivity of vacuum	ϵ_0	$1/(\mu_0 c^2)$ $= 8.854 187 817 \dots$	F m^{-1} $\times 10^{-12} \text{ F m}^{-1}$
constant of gravitation ^c	G	6.673(10)	$\times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Planck constant	h	6.626 068 76(52)	$\times 10^{-34} \text{ J s}$
$h/(2\pi)$	\hbar	1.054 571 596(82)	$\times 10^{-34} \text{ J s}$
elementary charge	e	1.602 176 462(63)	$\times 10^{-19} \text{ C}$
magnetic flux quantum, $h/(2e)$	Φ_0	2.067 833 636(81)	$\times 10^{-15} \text{ Wb}$
electron volt	eV	1.602 176 462(63)	$\times 10^{-19} \text{ J}$
electron mass	m_e	9.109 381 88(72)	$\times 10^{-31} \text{ kg}$
proton mass	m_p	1.672 621 58(13)	$\times 10^{-27} \text{ kg}$
proton/electron mass ratio	m_p/m_e	1 836.152 667 5(39)	
unified atomic mass unit	u	1.660 538 73(13)	$\times 10^{-27} \text{ kg}$
fine-structure constant, $\mu_0 c e^2/(2h)$	α	7.297 352 533(27)	$\times 10^{-3}$
inverse	$1/\alpha$	137.035 999 76(50)	
Rydberg constant, $m_e c \alpha^2/(2h)$	R_∞	1.097 373 156 854 9(83)	$\times 10^7 \text{ m}^{-1}$
Avogadro constant	N_A	6.022 141 99(47)	$\times 10^{23} \text{ mol}^{-1}$
Faraday constant, $N_A e$	F	9.648 534 15(39)	$\times 10^4 \text{ C mol}^{-1}$
molar gas constant	R	8.314 472(15)	$\text{J mol}^{-1} \text{ K}^{-1}$
Boltzmann constant, R/N_A	k	1.380 650 3(24)	$\times 10^{-23} \text{ J K}^{-1}$
Stefan–Boltzmann constant, $\pi^2 k^4/(60 \hbar^3 c^2)$	σ	5.670 400(40)	$\times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Bohr magneton, $e \hbar/(2m_e)$	μ_B	9.274 008 99(37)	$\times 10^{-24} \text{ J T}^{-1}$

^aBy definition, the speed of light is exact.

^bAlso exact, by definition. Alternative units are N A^{-2} .

^cThe standard acceleration due to gravity, g , is defined as exactly $9.806 65 \text{ m s}^{-2}$.

General constants

speed of light in vacuum	c	2.997 924 58	$\times 10^8 \text{ m s}^{-1}$
permeability of vacuum	μ_0	4π $=12.566\,370\,614\dots$	$\times 10^{-7} \text{ H m}^{-1}$ $\times 10^{-7} \text{ H m}^{-1}$
permittivity of vacuum	ϵ_0	$1/(\mu_0 c^2)$ $=8.854\,187\,817\dots$	F m^{-1} $\times 10^{-12} \text{ F m}^{-1}$
impedance of free space	Z_0	$\mu_0 c$ $=376.730\,313\,461\dots$	Ω Ω
constant of gravitation	G	6.673(10)	$\times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Planck constant	h	6.626 068 76(52)	$\times 10^{-34} \text{ J s}$
in eV s		4.135 667 27(16)	$\times 10^{-15} \text{ eV s}$
$\hbar/(2\pi)$	\hbar	1.054 571 596(82)	$\times 10^{-34} \text{ J s}$
in eV s		6.582 118 89(26)	$\times 10^{-16} \text{ eV s}$
Planck mass, $(\hbar c/G)^{1/2}$	m_{Pl}	2.176 7(16)	$\times 10^{-8} \text{ kg}$
Planck length, $\hbar/(m_{\text{Pl}} c) = (\hbar G/c^3)^{1/2}$	l_{Pl}	1.616 0(12)	$\times 10^{-35} \text{ m}$
Planck time, $l_{\text{Pl}}/c = (\hbar G/c^5)^{1/2}$	t_{Pl}	5.390 6(40)	$\times 10^{-44} \text{ s}$
elementary charge	e	1.602 176 462(63)	$\times 10^{-19} \text{ C}$
magnetic flux quantum, $h/(2e)$	Φ_0	2.067 833 636(81)	$\times 10^{-15} \text{ Wb}$
Josephson frequency/voltage ratio	$2e/h$	4.835 978 98(19)	$\times 10^{14} \text{ Hz V}^{-1}$
Bohr magneton, $e\hbar/(2m_e)$	μ_{B}	9.274 008 99(37)	$\times 10^{-24} \text{ J T}^{-1}$
in eV T ⁻¹		5.788 381 749(43)	$\times 10^{-5} \text{ eV T}^{-1}$
μ_{B}/k		0.671 713 1(12)	K T^{-1}
nuclear magneton, $e\hbar/(2m_{\text{p}})$	μ_{N}	5.050 783 17(20)	$\times 10^{-27} \text{ J T}^{-1}$
in eV T ⁻¹		3.152 451 238(24)	$\times 10^{-8} \text{ eV T}^{-1}$
μ_{N}/k		3.658 263 8(64)	$\times 10^{-4} \text{ K T}^{-1}$
Zeeman splitting constant	$\mu_{\text{B}}/(hc)$	46.686 452 1(19)	$\text{m}^{-1} \text{ T}^{-1}$

Atomic constants^a

fine-structure constant, $\mu_0 c e^2/(2h)$	α	7.297 352 533(27)	$\times 10^{-3}$
inverse	$1/\alpha$	137.035 999 76(50)	
Rydberg constant, $m_e c \alpha^2/(2h)$	R_∞	1.097 373 156 854 9(83)	$\times 10^7 \text{ m}^{-1}$
$R_\infty c$		3.289 841 960 368(25)	$\times 10^{15} \text{ Hz}$
$R_\infty hc$		2.179 871 90(17)	$\times 10^{-18} \text{ J}$
$R_\infty hc/e$		13.605 691 72(53)	eV
Bohr radius ^b , $\alpha/(4\pi R_\infty)$	a_0	5.291 772 083(19)	$\times 10^{-11} \text{ m}$

^aSee also the Bohr model on page 95.^bFixed nucleus.

Electron constants

electron mass	m_e	9.109 381 88(72)	$\times 10^{-31}$ kg
in MeV		0.510 998 902(21)	MeV
electron/proton mass ratio	m_e/m_p	5.446 170 232(12)	$\times 10^{-4}$
electron charge	$-e$	-1.602 176 462(63)	$\times 10^{-19}$ C
electron specific charge	$-e/m_e$	-1.758 820 174(71)	$\times 10^{11}$ C kg $^{-1}$
electron molar mass, $N_A m_e$	M_e	5.485 799 110(12)	$\times 10^{-7}$ kg mol $^{-1}$
Compton wavelength, $h/(m_e c)$	λ_C	2.426 310 215(18)	$\times 10^{-12}$ m
classical electron radius, $\alpha^2 a_0$	r_e	2.817 940 285(31)	$\times 10^{-15}$ m
Thomson cross section, $(8\pi/3)r_e^2$	σ_T	6.652 458 54(15)	$\times 10^{-29}$ m 2
electron magnetic moment	μ_e	-9.284 763 62(37)	$\times 10^{-24}$ J T $^{-1}$
in Bohr magnetons, μ_e/μ_B		-1.001 159 652 186 9(41)	
in nuclear magnetons, μ_e/μ_N		-1 838.281 966 0(39)	
electron gyromagnetic ratio, $2 \mu_e /\hbar$	γ_e	1.760 859 794(71)	$\times 10^{11}$ s $^{-1}$ T $^{-1}$
electron g-factor, $2\mu_e/\mu_B$	g_e	-2.002 319 304 3737(82)	

Proton constants

proton mass	m_p	1.672 621 58(13)	$\times 10^{-27}$ kg
in MeV		938.271 998(38)	MeV
proton/electron mass ratio	m_p/m_e	1 836.152 667 5(39)	
proton charge	e	1.602 176 462(63)	$\times 10^{-19}$ C
proton specific charge	e/m_p	9.578 834 08(38)	$\times 10^7$ C kg $^{-1}$
proton molar mass, $N_A m_p$	M_p	1.007 276 466 88(13)	$\times 10^{-3}$ kg mol $^{-1}$
proton Compton wavelength, $h/(m_p c)$	$\lambda_{C,p}$	1.321 409 847(10)	$\times 10^{-15}$ m
proton magnetic moment	μ_p	1.410 606 633(58)	$\times 10^{-26}$ J T $^{-1}$
in Bohr magnetons, μ_p/μ_B		1.521 032 203(15)	$\times 10^{-3}$
in nuclear magnetons, μ_p/μ_N		2.792 847 337(29)	
proton gyromagnetic ratio, $2\mu_p/\hbar$	γ_p	2.675 222 12(11)	$\times 10^8$ s $^{-1}$ T $^{-1}$

Neutron constants

neutron mass	m_n	1.674 927 16(13)	$\times 10^{-27}$ kg
in MeV		939.565 330(38)	MeV
neutron/electron mass ratio	m_n/m_e	1 838.683 655 0(40)	
neutron/proton mass ratio	m_n/m_p	1.001 378 418 87(58)	
neutron molar mass, $N_A m_n$	M_n	1.008 664 915 78(55)	$\times 10^{-3}$ kg mol $^{-1}$
neutron Compton wavelength, $h/(m_n c)$	$\lambda_{C,n}$	1.319 590 898(10)	$\times 10^{-15}$ m
neutron magnetic moment	μ_n	-9.662 364 0(23)	$\times 10^{-27}$ J T $^{-1}$
in Bohr magnetons	μ_n/μ_B	-1.041 875 63(25)	$\times 10^{-3}$
in nuclear magnetons	μ_n/μ_N	-1.913 042 72(45)	
neutron gyromagnetic ratio, $2 \mu_n /\hbar$	γ_n	1.832 471 88(44)	$\times 10^8$ s $^{-1}$ T $^{-1}$

Muon and tau constants

muon mass	m_μ	1.883 531 09(16)	$\times 10^{-28}$ kg
in MeV		105.658 356 8(52)	MeV
tau mass	m_τ	3.167 88(52)	$\times 10^{-27}$ kg
in MeV		1.777 05(29)	$\times 10^3$ MeV
muon/electron mass ratio	m_μ/m_e	206.768 262(30)	
muon charge	$-e$	-1.602 176 462(63)	$\times 10^{-19}$ C
muon magnetic moment	μ_μ	-4.490 448 13(22)	$\times 10^{-26}$ J T $^{-1}$
in Bohr magnetons, μ_μ/μ_B		4.841 970 85(15)	$\times 10^{-3}$
in nuclear magnetons, μ_μ/μ_N		8.890 597 70(27)	
muon g-factor	g_μ	-2.002 331 832 0(13)	

Bulk physical constants

Avogadro constant	N_A	6.022 141 99(47)	$\times 10^{23}$ mol $^{-1}$
atomic mass constant ^a	m_u	1.660 538 73(13)	$\times 10^{-27}$ kg
in MeV		931.494 013(37)	MeV
Faraday constant	F	9.648 534 15(39)	$\times 10^4$ C mol $^{-1}$
molar gas constant	R	8.314 472(15)	J mol $^{-1}$ K $^{-1}$
Boltzmann constant, R/N_A	k	1.380 650 3(24)	$\times 10^{-23}$ J K $^{-1}$
in eV K $^{-1}$		8.617 342(15)	$\times 10^{-5}$ eV K $^{-1}$
molar volume (ideal gas at stp) ^b	V_m	22.413 996(39)	$\times 10^{-3}$ m 3 mol $^{-1}$
Stefan–Boltzmann constant, $\pi^2 k^4/(60\hbar^3 c^2)$	σ	5.670 400(40)	$\times 10^{-8}$ W m $^{-2}$ K $^{-4}$
Wien’s displacement law constant, ^c $b = \lambda_m T$	b	2.897 768 6(51)	$\times 10^{-3}$ m K

^a= mass of $^{12}\text{C}/12$. Alternative nomenclature for the unified atomic mass unit, u.

^bStandard temperature and pressure (stp) are $T = 273.15$ K (0°C) and $P = 101\,325$ Pa (1 standard atmosphere).

^cSee also page 121.

Mathematical constants

pi (π)	3.141 592 653 589 793 238 462 643 383 279 ...
exponential constant (e)	2.718 281 828 459 045 235 360 287 471 352 ...
Catalan’s constant	0.915 965 594 177 219 015 054 603 514 932 ...
Euler’s constant ^a (γ)	0.577 215 664 901 532 860 606 512 090 082 ...
Feigenbaum’s constant (α)	2.502 907 875 095 892 822 283 902 873 218 ...
Feigenbaum’s constant (δ)	4.669 201 609 102 990 671 853 203 820 466 ...
Gibbs constant	1.851 937 051 982 466 170 361 053 370 157 ...
golden mean	1.618 033 988 749 894 848 204 586 834 370 ...
Madelung constant ^b	1.747 564 594 633 182 190 636 212 035 544 ...

^aSee also Equation (2.119).

^bNaCl structure.

1.4 Converting between units

The following table lists common (and not so common) measures of physical quantities. The numerical values given are the SI equivalent of one unit measure of the non-SI unit. Hence 1 astronomical unit equals 149.5979×10^9 m. Those entries identified with a “*” in the second column represent exact conversions; so 1 abampere equals exactly 10.0 A. Note that individual entries in this list are not recorded in the index, and that values are “international” unless otherwise stated.

There is a separate section on temperature conversions after this table.

<i>unit name</i>	<i>value in SI units</i>	
abampere	10.0*	A
abcoulomb	10.0*	C
abfarad	1.0*	$\times 10^9$ F
abhenry	1.0*	$\times 10^{-9}$ H
abmho	1.0*	$\times 10^9$ S
abohm	1.0*	$\times 10^{-9}$ Ω
abvolt	10.0*	$\times 10^{-9}$ V
acre	4.046 856	$\times 10^3$ m ²
amagat (at stp)	44.614 774	mol m ⁻³
ampere hour	3.6*	$\times 10^3$ C
ångström	100.0*	$\times 10^{-12}$ m
apostilb	1.0*	lm m ⁻²
arcminute	290.888 2	$\times 10^{-6}$ rad
arcsecond	4.848 137	$\times 10^{-6}$ rad
are	100.0*	m ²
astronomical unit	149.597 9	$\times 10^9$ m
atmosphere (standard)	101.325 0*	$\times 10^3$ Pa
atomic mass unit	1.660 540	$\times 10^{-27}$ kg
bar	100.0*	$\times 10^3$ Pa
barn	100.0*	$\times 10^{-30}$ m ²
baromil	750.1	$\times 10^{-6}$ m
barrel (UK)	163.659 2	$\times 10^{-3}$ m ³
barrel (US dry)	115.627 1	$\times 10^{-3}$ m ³
barrel (US liquid)	119.240 5	$\times 10^{-3}$ m ³
barrel (US oil)	158.987 3	$\times 10^{-3}$ m ³
baud	1.0*	s ⁻¹
bayre	100.0*	$\times 10^{-3}$ Pa
biot	10.0	A
bolt (US)	36.576*	m
brewster	1.0*	$\times 10^{-12}$ m ² N ⁻¹
British thermal unit	1.055 056	$\times 10^3$ J
bushel (UK)	36.36 872	$\times 10^{-3}$ m ³
bushel (US)	35.23 907	$\times 10^{-3}$ m ³
butt (UK)	477.339 4	$\times 10^{-3}$ m ³
cable (US)	219.456*	m
calorie	4.186 8*	J

continued on next page ...

<i>unit name</i>	<i>value in SI units</i>	
candle power (spherical)	4π	lm
carat (metric)	200.0*	$\times 10^{-6}$ kg
cental	45.359 237	kg
centare	1.0*	m ²
centimetre of Hg (0 °C)	1.333 222	$\times 10^3$ Pa
centimetre of H ₂ O (4 °C)	98.060 616	Pa
chain (engineers')	30.48*	m
chain (US)	20.116 8*	m
Chu	1.899 101	$\times 10^3$ J
clusec	1.333 224	$\times 10^{-6}$ W
cord	3.624 556	m ³
cubit	457.2*	$\times 10^{-3}$ m
cumec	1.0*	m ³ s ⁻¹
cup (US)	236.588 2	$\times 10^{-6}$ m ³
curie	37.0*	$\times 10^9$ Bq
darcy	986.923 3	$\times 10^{-15}$ m ²
day	86.4*	$\times 10^3$ s
day (sidereal)	86.164 09	$\times 10^3$ s
debye	3.335 641	$\times 10^{-30}$ C m
degree (angle)	17.453 29	$\times 10^{-3}$ rad
denier	111.111 1	$\times 10^{-9}$ kg m ⁻¹
digit	19.05*	$\times 10^{-3}$ m
dioptre	1.0*	m ⁻¹
Dobson unit	10.0*	$\times 10^{-6}$ m
dram (avoirdupois)	1.771 845	$\times 10^{-3}$ kg
dyne	10.0*	$\times 10^{-6}$ N
dyne centimetres	100.0*	$\times 10^{-9}$ J
electron volt	160.217 7	$\times 10^{-21}$ J
ell	1.143*	m
em	4.233 333	$\times 10^{-3}$ m
emu of capacitance	1.0*	$\times 10^9$ F
emu of current	10.0*	A
emu of electric potential	10.0*	$\times 10^{-9}$ V
emu of inductance	1.0*	$\times 10^{-9}$ H
emu of resistance	1.0*	$\times 10^{-9}$ Ω
Eötvös unit	1.0*	$\times 10^{-9}$ m s ⁻² m ⁻¹
esu of capacitance	1.112 650	$\times 10^{-12}$ F
esu of current	333.564 1	$\times 10^{-12}$ A
esu of electric potential	299.792 5	V
esu of inductance	898.755 2	$\times 10^9$ H
esu of resistance	898.755 2	$\times 10^9$ Ω
erg	100.0*	$\times 10^{-9}$ J
faraday	96.485 3	$\times 10^3$ C
fathom	1.828 804	m
fermi	1.0*	$\times 10^{-15}$ m
Finsen unit	10.0*	$\times 10^{-6}$ W m ⁻²
firkin (UK)	40.914 81	$\times 10^{-3}$ m ³

continued on next page ...

<i>unit name</i>	<i>value in SI units</i>	
firkin (US)	34.068 71	$\times 10^{-3} \text{ m}^3$
fluid ounce (UK)	28.413 08	$\times 10^{-6} \text{ m}^3$
fluid ounce (US)	29.573 53	$\times 10^{-6} \text{ m}^3$
foot	304.8*	$\times 10^{-3} \text{ m}$
foot (US survey)	304.800 6	$\times 10^{-3} \text{ m}$
foot of water (4 °C)	2.988 887	$\times 10^3 \text{ Pa}$
footcandle	10.763 91	lx
footlambert	3.426 259	cd m^{-2}
footpoundal	42.140 11	$\times 10^{-3} \text{ J}$
footpounds (force)	1.355 818	J
fresnel	1.0*	$\times 10^{12} \text{ Hz}$
funal	1.0*	$\times 10^3 \text{ N}$
furlong	201.168*	m
g (standard acceleration)	9.806 65*	m s^{-2}
gal	10.0*	$\times 10^{-3} \text{ m s}^{-2}$
gallon (UK)	4.546 09*	$\times 10^{-3} \text{ m}^3$
gallon (US liquid)	3.785 412	$\times 10^{-3} \text{ m}^3$
gamma	1.0*	$\times 10^{-9} \text{ T}$
gauss	100.0*	$\times 10^{-6} \text{ T}$
gilbert	795.774 7	$\times 10^{-3} \text{ A turn}$
gill (UK)	142.065 4	$\times 10^{-6} \text{ m}^3$
gill (US)	118.294 1	$\times 10^{-6} \text{ m}^3$
gon	$\pi/200^*$	rad
grade	15.707 96	$\times 10^{-3} \text{ rad}$
grain	64.798 91*	$\times 10^{-6} \text{ kg}$
gram	1.0*	$\times 10^{-3} \text{ kg}$
gram-rad	100.0*	J kg^{-1}
gray	1.0*	J kg^{-1}
hand	101.6*	$\times 10^{-3} \text{ m}$
hartree	4.359 748	$\times 10^{-18} \text{ J}$
hectare	10.0*	$\times 10^3 \text{ m}^2$
hefner	902	$\times 10^{-3} \text{ cd}$
hogshead	238.669 7	$\times 10^{-3} \text{ m}^3$
horsepower (boiler)	9.809 50	$\times 10^3 \text{ W}$
horsepower (electric)	746*	W
horsepower (metric)	735.498 8	W
horsepower (UK)	745.699 9	W
hour	3.6*	$\times 10^3 \text{ s}$
hour (sidereal)	3.590 170	$\times 10^3 \text{ s}$
Hubble time	440	$\times 10^{15} \text{ s}$
Hubble distance	130	$\times 10^{24} \text{ m}$
hundredweight (UK long)	50.802 35	kg
hundredweight (US short)	45.359 24	kg
inch	25.4*	$\times 10^{-3} \text{ m}$
inch of mercury (0 °C)	3.386 389	$\times 10^3 \text{ Pa}$
inch of water (4 °C)	249.074 0	Pa
jansky	10.0*	$\times 10^{-27} \text{ W m}^{-2} \text{ Hz}^{-1}$

continued on next page ...

<i>unit name</i>	<i>value in SI units</i>	
jar	10/9*	$\times 10^{-9}$ F
kayser	100.0*	m^{-1}
kilocalorie	4.186 8*	$\times 10^3$ J
kilogram-force	9.806 65*	N
kilowatt hour	3.6*	$\times 10^6$ J
knot (international)	514.444 4	$\times 10^{-3}$ m s^{-1}
lambert	10/ π *	$\times 10^3$ cd m^{-2}
langley	41.84*	$\times 10^3$ J m^{-2}
langmuir	133.322 4	$\times 10^{-6}$ Pa s
league (nautical, int.)	5.556*	$\times 10^3$ m
league (nautical, UK)	5.559 552	$\times 10^3$ m
league (statute)	4.828 032	$\times 10^3$ m
light year	9.460 73*	$\times 10^{15}$ m
ligne	2.256*	$\times 10^{-3}$ m
line	2.116 667	$\times 10^{-3}$ m
line (magnetic flux)	10.0*	$\times 10^{-9}$ Wb
link (engineers')	304.8*	$\times 10^{-3}$ m
link (US)	201.168 0	$\times 10^{-3}$ m
litre	1.0*	$\times 10^{-3}$ m^3
lumen (at 555 nm)	1.470 588	$\times 10^{-3}$ W
maxwell	10.0*	$\times 10^{-9}$ Wb
mho	1.0*	S
micron	1.0*	$\times 10^{-6}$ m
mil (length)	25.4*	$\times 10^{-6}$ m
mil (volume)	1.0*	$\times 10^{-6}$ m^3
mile (international)	1.609 344*	$\times 10^3$ m
mile (nautical, int.)	1.852*	$\times 10^3$ m
mile (nautical, UK)	1.853 184*	$\times 10^3$ m
mile per hour	447.04*	$\times 10^{-3}$ m s^{-1}
milliard	1.0*	$\times 10^9$ m^3
millibar	100.0*	Pa
millimetre of Hg (0 °C)	133.322 4	Pa
minim (UK)	59.193 90	$\times 10^{-9}$ m^3
minim (US)	61.611 51	$\times 10^{-9}$ m^3
minute (angle)	290.888 2	$\times 10^{-6}$ rad
minute	60.0*	s
minute (sidereal)	59.836 17	s
month (lunar)	2.551 444	$\times 10^6$ s
nit	1.0*	cd m^{-2}
noggin (UK)	142.065 4	$\times 10^{-6}$ m^3
oersted	1000/(4 π)*	A m^{-1}
ounce (avoirdupois)	28.349 52	$\times 10^{-3}$ kg
ounce (UK fluid)	28.413 07	$\times 10^{-6}$ m^3
ounce (US fluid)	29.573 53	$\times 10^{-6}$ m^3
pace	762.0*	$\times 10^{-3}$ m
parsec	30.856 78	$\times 10^{15}$ m

continued on next page ...

<i>unit name</i>	<i>value in SI units</i>	
peck (UK)	9.092 18*	$\times 10^{-3} \text{ m}^3$
peck (US)	8.809 768	$\times 10^{-3} \text{ m}^3$
pennyweight (troy)	1.555 174	$\times 10^{-3} \text{ kg}$
perch	5.029 2*	m
phot	10.0*	$\times 10^3 \text{ lx}$
pica (printers')	4.217 518	$\times 10^{-3} \text{ m}$
pint (UK)	568.261 2	$\times 10^{-6} \text{ m}^3$
pint (US dry)	550.610 5	$\times 10^{-6} \text{ m}^3$
pint (US liquid)	473.176 5	$\times 10^{-6} \text{ m}^3$
point (printers')	351.459 8*	$\times 10^{-6} \text{ m}$
poise	100.0*	$\times 10^{-3} \text{ Pa s}$
pole	5.029 2*	m
poncelot	980.665*	W
pottle	2.273 045	$\times 10^{-3} \text{ m}^3$
pound (avoirdupois)	453.592 4	$\times 10^{-3} \text{ kg}$
poundal	138.255 0	$\times 10^{-3} \text{ N}$
pound-force	4.448 222	N
promaxwell	1.0*	Wb
psi	6.894 757	$\times 10^3 \text{ Pa}$
puncheon (UK)	317.974 6	$\times 10^{-3} \text{ m}^3$
quad	1.055 056	$\times 10^{18} \text{ J}$
quart (UK)	1.136 522	$\times 10^{-3} \text{ m}^3$
quart (US dry)	1.101 221	$\times 10^{-3} \text{ m}^3$
quart (US liquid)	946.352 9	$\times 10^{-6} \text{ m}^3$
quintal (metric)	100.0*	kg
rad	10.0*	$\times 10^{-3} \text{ Gy}$
rayleigh	10/(4 π)	$\times 10^9 \text{ s}^{-1} \text{ m}^{-2} \text{ sr}^{-1}$
rem	10.0*	$\times 10^{-3} \text{ Sv}$
REN	1/4 000*	S
reyn	689.5	$\times 10^3 \text{ Pa s}$
rhe	10.0*	$\text{Pa}^{-1} \text{ s}^{-1}$
rod	5.029 2*	m
roentgen	258.0	$\times 10^{-6} \text{ C kg}^{-1}$
rood (UK)	1.011 714	$\times 10^3 \text{ m}^2$
rope (UK)	6.096*	m
rutherford	1.0*	$\times 10^6 \text{ Bq}$
rydberg	2.179 874	$\times 10^{-18} \text{ J}$
scruple	1.295 978	$\times 10^{-3} \text{ kg}$
seam	290.949 8	$\times 10^{-3} \text{ m}^3$
second (angle)	4.848 137	$\times 10^{-6} \text{ rad}$
second (sidereal)	997.269 6	$\times 10^{-3} \text{ s}$
shake	100.0*	$\times 10^{-10} \text{ s}$
shed	100.0*	$\times 10^{-54} \text{ m}^2$
slug	14.593 90	kg
square degree	($\pi/180$) ² *	sr
statampere	333.564 1	$\times 10^{-12} \text{ A}$
statcoulomb	333.564 1	$\times 10^{-12} \text{ C}$

continued on next page ...

<i>unit name</i>	<i>value in SI units</i>	
statfarad	1.112 650	$\times 10^{-12}$ F
stathenry	898.755 2	$\times 10^9$ H
statmho	1.112 650	$\times 10^{-12}$ S
statohm	898.755 2	$\times 10^9$ Ω
statvolt	299.792 5	V
stere	1.0*	m ³
sthène	1.0*	$\times 10^3$ N
stilb	10.0*	$\times 10^3$ cd m ⁻²
stokes	100.0*	$\times 10^{-6}$ m ² s ⁻¹
stone	6.350 293	kg
tablespoon (UK)	14.206 53	$\times 10^{-6}$ m ³
tablespoon (US)	14.786 76	$\times 10^{-6}$ m ³
teaspoon (UK)	4.735 513	$\times 10^{-6}$ m ³
teaspoon (US)	4.928 922	$\times 10^{-6}$ m ³
tex	1.0*	$\times 10^{-6}$ kg m ⁻¹
therm (EEC)	105.506*	$\times 10^6$ J
therm (US)	105.480 4*	$\times 10^6$ J
thermie	4.185 407	$\times 10^6$ J
thou	25.4*	$\times 10^{-6}$ m
tog	100.0*	$\times 10^{-3}$ W ⁻¹ m ² K
ton (of TNT)	4.184*	$\times 10^9$ J
ton (UK long)	1.016 047	$\times 10^3$ kg
ton (US short)	907.184 7	kg
tonne (metric ton)	1.0*	$\times 10^3$ kg
torr	133.322 4	Pa
townsend	1.0*	$\times 10^{-21}$ V m ²
troy dram	3.887 935	$\times 10^{-3}$ kg
troy ounce	31.103 48	$\times 10^{-3}$ kg
troy pound	373.241 7	$\times 10^{-3}$ kg
tun	954.678 9	$\times 10^{-3}$ m ³
XU	100.209	$\times 10^{-15}$ m
yard	914.4*	$\times 10^{-3}$ m
year (365 days)	31.536*	$\times 10^6$ s
year (sidereal)	31.558 15	$\times 10^6$ s
year (tropical)	31.556 93	$\times 10^6$ s

Temperature conversions

From degrees Celsius ^a	$T_K = T_C + 273.15$	(1.1)	T_K temperature in kelvin
			T_C temperature in °Celsius
From degrees Fahrenheit	$T_K = \frac{T_F - 32}{1.8} + 273.15$	(1.2)	T_F temperature in °Fahrenheit
From degrees Rankine	$T_K = \frac{T_R}{1.8}$	(1.3)	T_R temperature in °Rankine

^aThe term “centigrade” is not used in SI, to avoid confusion with “10⁻² of a degree”.

1.5 Dimensions

The following table lists the dimensions of common physical quantities, together with their conventional symbols and the SI units in which they are usually quoted. The dimensional basis used is length (L), mass (M), time (T), electric current (I), temperature (Θ), amount of substance (N), and luminous intensity (J).

<i>physical quantity</i>	<i>symbol</i>	<i>dimensions</i>	<i>SI units</i>
acceleration	a	$L T^{-2}$	$m s^{-2}$
action	S	$L^2 M T^{-1}$	J s
angular momentum	L, J	$L^2 M T^{-1}$	$m^2 kg s^{-1}$
angular speed	ω	T^{-1}	$rad s^{-1}$
area	A, S	L^2	m^2
Avogadro constant	N_A	N^{-1}	mol^{-1}
bending moment	G_b	$L^2 M T^{-2}$	N m
Bohr magneton	μ_B	$L^2 I$	$J T^{-1}$
Boltzmann constant	k, k_B	$L^2 M T^{-2} \Theta^{-1}$	$J K^{-1}$
bulk modulus	K	$L^{-1} M T^{-2}$	Pa
capacitance	C	$L^{-2} M^{-1} T^4 I^2$	F
charge (electric)	q	$T I$	C
charge density	ρ	$L^{-3} T I$	$C m^{-3}$
conductance	G	$L^{-2} M^{-1} T^3 I^2$	S
conductivity	σ	$L^{-3} M^{-1} T^3 I^2$	$S m^{-1}$
couple	G, T	$L^2 M T^{-2}$	N m
current	I, i	I	A
current density	J, j	$L^{-2} I$	$A m^{-2}$
density	ρ	$L^{-3} M$	$kg m^{-3}$
electric displacement	D	$L^{-2} T I$	$C m^{-2}$
electric field strength	E	$L M T^{-3} I^{-1}$	$V m^{-1}$
electric polarisability	α	$M^{-1} T^4 I^2$	$C m^2 V^{-1}$
electric polarisation	P	$L^{-2} T I$	$C m^{-2}$
electric potential difference	V	$L^2 M T^{-3} I^{-1}$	V
energy	E, U	$L^2 M T^{-2}$	J
energy density	u	$L^{-1} M T^{-2}$	$J m^{-3}$
entropy	S	$L^2 M T^{-2} \Theta^{-1}$	$J K^{-1}$
Faraday constant	F	$T I N^{-1}$	$C mol^{-1}$
force	F	$L M T^{-2}$	N
frequency	ν, f	T^{-1}	Hz
gravitational constant	G	$L^3 M^{-1} T^{-2}$	$m^3 kg^{-1} s^{-2}$
Hall coefficient	R_H	$L^3 T^{-1} I^{-1}$	$m^3 C^{-1}$
Hamiltonian	H	$L^2 M T^{-2}$	J
heat capacity	C	$L^2 M T^{-2} \Theta^{-1}$	$J K^{-1}$
Hubble constant ¹	H	T^{-1}	s^{-1}
illuminance	E_v	$L^{-2} J$	lx
impedance	Z	$L^2 M T^{-3} I^{-2}$	Ω

continued on next page ...

¹The Hubble constant is almost universally quoted in units of $km s^{-1} Mpc^{-1}$. There are about 3.1×10^{19} kilometres in a megaparsec.

<i>physical quantity</i>	<i>symbol</i>	<i>dimensions</i>	<i>SI units</i>
impulse	I	$L M T^{-1}$	N s
inductance	L	$L^2 M T^{-2} I^{-2}$	H
irradiance	E_e	$M T^{-3}$	$W m^{-2}$
Lagrangian	L	$L^2 M T^{-2}$	J
length	L, l	L	m
luminous intensity	I_v	J	cd
magnetic dipole moment	m, μ	$L^2 I$	$A m^2$
magnetic field strength	H	$L^{-1} I$	$A m^{-1}$
magnetic flux	Φ	$L^2 M T^{-2} I^{-1}$	Wb
magnetic flux density	B	$M T^{-2} I^{-1}$	T
magnetic vector potential	A	$L M T^{-2} I^{-1}$	$Wb m^{-1}$
magnetisation	M	$L^{-1} I$	$A m^{-1}$
mass	m, M	M	kg
mobility	μ	$M^{-1} T^2 I$	$m^2 V^{-1} s^{-1}$
molar gas constant	R	$L^2 M T^{-2} \Theta^{-1} N^{-1}$	$J mol^{-1} K^{-1}$
moment of inertia	I	$L^2 M$	$kg m^2$
momentum	p	$L M T^{-1}$	$kg m s^{-1}$
number density	n	L^{-3}	m^{-3}
permeability	μ	$L M T^{-2} I^{-2}$	$H m^{-1}$
permittivity	ϵ	$L^{-3} M^{-1} T^4 I^2$	$F m^{-1}$
Planck constant	h	$L^2 M T^{-1}$	J s
power	P	$L^2 M T^{-3}$	W
Poynting vector	S	$M T^{-3}$	$W m^{-2}$
pressure	p, P	$L^{-1} M T^{-2}$	Pa
radiant intensity	I_e	$L^2 M T^{-3}$	$W sr^{-1}$
resistance	R	$L^2 M T^{-3} I^{-2}$	Ω
Rydberg constant	R_∞	L^{-1}	m^{-1}
shear modulus	μ, G	$L^{-1} M T^{-2}$	Pa
specific heat capacity	c	$L^2 T^{-2} \Theta^{-1}$	$J kg^{-1} K^{-1}$
speed	u, v, c	$L T^{-1}$	$m s^{-1}$
Stefan–Boltzmann constant	σ	$M T^{-3} \Theta^{-4}$	$W m^{-2} K^{-4}$
stress	σ, τ	$L^{-1} M T^{-2}$	Pa
surface tension	σ, γ	$M T^{-2}$	$N m^{-1}$
temperature	T	Θ	K
thermal conductivity	λ	$L M T^{-3} \Theta^{-1}$	$W m^{-1} K^{-1}$
time	t	T	s
velocity	v, u	$L T^{-1}$	$m s^{-1}$
viscosity (dynamic)	η, μ	$L^{-1} M T^{-1}$	Pa s
viscosity (kinematic)	ν	$L^2 T^{-1}$	$m^2 s^{-1}$
volume	V, v	L^3	m^3
wavevector	k	L^{-1}	m^{-1}
weight	W	$L M T^{-2}$	N
work	W	$L^2 M T^{-2}$	J
Young modulus	E	$L^{-1} M T^{-2}$	Pa

1.6 Miscellaneous

Greek alphabet

<i>A</i>	α	alpha	<i>N</i>	ν	nu
<i>B</i>	β	beta	Ξ	ξ	xi
Γ	γ	gamma	<i>O</i>	<i>o</i>	omicron
Δ	δ	delta	Π	π	ϖ pi
<i>E</i>	ϵ	ε epsilon	<i>P</i>	ρ	ϱ rho
<i>Z</i>	ζ	zeta	Σ	σ	ς sigma
<i>H</i>	η	eta	<i>T</i>	τ	tau
Θ	θ	ϑ theta	Υ	υ	upsilon
<i>I</i>	ι	iota	Φ	ϕ	φ phi
<i>K</i>	κ	kappa	<i>X</i>	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
<i>M</i>	μ	mu	Ω	ω	omega

Pi (π) to 1 000 decimal places

3.1415926535	8979323846	2643383279	5028841971	6939937510	5820974944	5923078164	0628620899	8628034825	3421170679
8214808651	3282306647	0938446095	5058223172	5359408128	4811174502	8410270193	8521105559	6446229489	5493038196
4428810975	6659334461	2847564823	3786783165	2712019091	4564856692	3460348610	4543266482	1339360726	0249141273
7245870066	0631558817	4881520920	9628292540	9171536436	7892590360	0113305305	4882046652	1384146951	9415116094
3305727036	5759591953	0921861173	8193261179	3105118548	0744623799	6274956735	1885752724	8912279381	8301194912
9833673362	4406566430	8602139494	6395224737	1907021798	6094370277	0539217176	2931767523	8467481846	7669405132
0005681271	4526356082	7785771342	7577896091	7363717872	1468440901	2249534301	4654958537	1050792279	6892589235
4201995611	2129021960	8640344181	5981362977	4771309960	5187072113	4999999837	2978049951	0597317328	1609631859
5024459455	3469083026	4252230825	3344685035	2619311881	7101000313	7838752886	5875332083	8142061717	7669147303
5982534904	2875546873	1159562863	8823537875	9375195778	1857780532	1712268066	1300192787	6611195909	2164201989

e to 1 000 decimal places

2.7182818284	5904523536	0287471352	6624977572	4709369995	9574966967	6277240766	3035354759	4571382178	5251664274
2746639193	2003059921	8174135966	2904357290	0334295260	5956307381	3232862794	3490763233	8298807531	9525101901
1573834187	9307021540	8914993488	4167509244	7614606680	8226480016	8477411853	7423454424	3710753907	7744992069
5517027618	3860626133	1384583000	7520449338	2656029760	6737113200	7093287091	2744374704	7230696977	2093101416
9283681902	5515108657	4637721112	5238978442	5056953696	7707854499	6996794686	4454905987	9316368892	3009879312
7736178215	4249992295	7635148220	8269895193	6680331825	2886939849	6465105820	9392398294	8879332036	2509443117
3012381970	6841614039	7019837679	3206832823	7646480429	5311802328	7825098194	5581530175	6717361332	0698112509
9618188159	3041690351	5988885193	4580727386	6738589422	8792284998	9208680582	5749279610	4841984443	6346324496
8487560233	6248270419	7862320900	2160990235	3043699418	4914631409	3431738143	6405462531	5209618369	0888707016
7683964243	7814059271	4563549061	3031072085	1038375051	0115747704	1718986106	8739696552	1267154688	9570350354

Chapter 2 Mathematics

2.1 Notation

Mathematics is, of course, a vast subject, and so here we concentrate on those mathematical methods and relationships that are most often applied in the physical sciences and engineering.

Although there is a high degree of consistency in accepted mathematical notation, there is some variation. For example the spherical harmonics, Y_l^m , can be written Y_{lm} , and there is some freedom with their signs. In general, the conventions chosen here follow common practice as closely as possible, whilst maintaining consistency with the rest of the handbook.

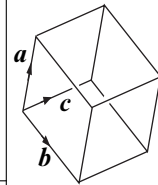
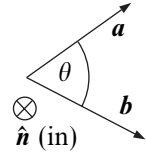
In particular:

scalars	a	general vectors	\mathbf{a}
unit vectors	$\hat{\mathbf{a}}$	scalar product	$\mathbf{a} \cdot \mathbf{b}$
vector cross-product	$\mathbf{a} \times \mathbf{b}$	gradient operator	∇
Laplacian operator	∇^2	derivative	$\frac{df}{dx}$ etc.
partial derivatives	$\frac{\partial f}{\partial x}$ etc.	derivative of r with respect to t	\dot{r}
n th derivative	$\frac{d^n f}{dx^n}$	closed loop integral	$\oint_L dl$
closed surface integral	$\oint_S ds$	matrix	\mathbf{A} or a_{ij}
mean value (of x)	$\langle x \rangle$	binomial coefficient	$\binom{n}{r}$
factorial	$!$	unit imaginary ($\mathbf{i}^2 = -1$)	\mathbf{i}
exponential constant	e	modulus (of x)	$ x $
natural logarithm	\ln	log to base 10	\log_{10}

2.2 Vectors and matrices

Vector algebra

Scalar product ^a	$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \mathbf{b} \cos\theta$	(2.1)
Vector product ^b	$\mathbf{a} \times \mathbf{b} = \mathbf{a} \mathbf{b} \sin\theta \hat{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$	(2.2)
Product rules	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	(2.3)
	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	(2.4)
	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$	(2.5)
	$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$	(2.6)
Lagrange's identity	$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$	(2.7)
Scalar triple product	$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$	(2.8)
	$= (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$	(2.9)
	$= \text{volume of parallelepiped}$	(2.10)
Vector triple product	$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$	(2.11)
	$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$	(2.12)
Reciprocal vectors	$\mathbf{a}' = (\mathbf{b} \times \mathbf{c}) / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(2.13)
	$\mathbf{b}' = (\mathbf{c} \times \mathbf{a}) / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(2.14)
	$\mathbf{c}' = (\mathbf{a} \times \mathbf{b}) / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(2.15)
	$(\mathbf{a}' \cdot \mathbf{a}) = (\mathbf{b}' \cdot \mathbf{b}) = (\mathbf{c}' \cdot \mathbf{c}) = 1$	(2.16)
Vector \mathbf{a} with respect to a nonorthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ ^c	$\mathbf{a} = (\mathbf{e}'_1 \cdot \mathbf{a})\mathbf{e}_1 + (\mathbf{e}'_2 \cdot \mathbf{a})\mathbf{e}_2 + (\mathbf{e}'_3 \cdot \mathbf{a})\mathbf{e}_3$	(2.17)

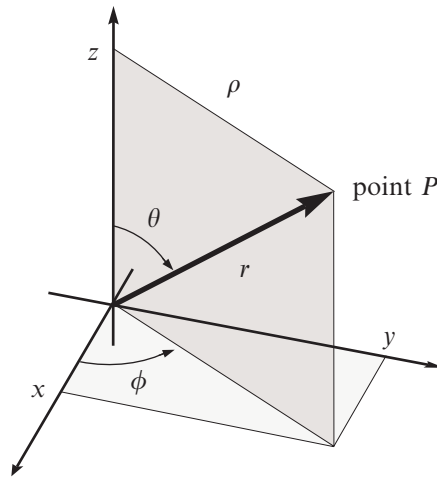


^aAlso known as the “dot product” or the “inner product.”

^bAlso known as the “cross-product.” $\hat{\mathbf{n}}$ is a unit vector making a right-handed set with \mathbf{a} and \mathbf{b} .

^cThe prime (') denotes a reciprocal vector.

Common three-dimensional coordinate systems



$$\begin{aligned} x &= \rho \cos \phi = r \sin \theta \cos \phi & (2.18) \\ y &= \rho \sin \phi = r \sin \theta \sin \phi & (2.19) \\ z &= r \cos \theta & (2.20) \end{aligned} \quad \begin{aligned} \rho &= (x^2 + y^2)^{1/2} & (2.21) \\ r &= (x^2 + y^2 + z^2)^{1/2} & (2.22) \\ \theta &= \arccos(z/r) & (2.23) \\ \phi &= \arctan(y/x) & (2.24) \end{aligned}$$

coordinate system:	rectangular	spherical polar	cylindrical polar
coordinates of P :	(x, y, z)	(r, θ, ϕ)	(ρ, ϕ, z)
volume element:	$dx \, dy \, dz$	$r^2 \sin \theta \, dr \, d\theta \, d\phi$	$\rho \, d\rho \, dz \, d\phi$
metric elements ^a (h_1, h_2, h_3) :	$(1, 1, 1)$	$(1, r, r \sin \theta)$	$(1, \rho, 1)$

^aIn an orthogonal coordinate system (parameterised by coordinates q_1, q_2, q_3), the differential line element dl is obtained from $(dl)^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$.

Gradient

Rectangular coordinates	$\nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}$	(2.25)	f scalar field $\hat{}$ unit vector
Cylindrical coordinates	$\nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}$	(2.26)	ρ distance from the z axis
Spherical polar coordinates	$\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}$	(2.27)	
General orthogonal coordinates	$\nabla f = \frac{\hat{q}_1}{h_1} \frac{\partial f}{\partial q_1} + \frac{\hat{q}_2}{h_2} \frac{\partial f}{\partial q_2} + \frac{\hat{q}_3}{h_3} \frac{\partial f}{\partial q_3}$	(2.28)	q_i basis h_i metric elements

Divergence

Rectangular coordinates	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ (2.29)	\mathbf{A} vector field A_i i th component of \mathbf{A} ρ distance from the z axis
Cylindrical coordinates	$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$ (2.30)	
Spherical polar coordinates	$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$ (2.31)	
General orthogonal coordinates	$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_3 h_1) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right]$ (2.32)	q_i basis h_i metric elements

Curl

Rectangular coordinates	$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$ (2.33)	$\hat{}$ unit vector \mathbf{A} vector field A_i i th component of \mathbf{A} ρ distance from the z axis q_i basis h_i metric elements
Cylindrical coordinates	$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\rho}/\rho & \hat{\phi} & \hat{z}/\rho \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$ (2.34)	
Spherical polar coordinates	$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{r}/(r^2 \sin \theta) & \hat{\theta}/(r \sin \theta) & \hat{\phi}/r \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_r & r A_\theta & r A_\phi \sin \theta \end{vmatrix}$ (2.35)	
General orthogonal coordinates	$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$ (2.36)	

Radial forms^a

$\nabla r = \frac{\mathbf{r}}{r}$ (2.37)	$\nabla(1/r) = \frac{-\mathbf{r}}{r^3}$ (2.41)
$\nabla \cdot \mathbf{r} = 3$ (2.38)	$\nabla \cdot (\mathbf{r}/r^2) = \frac{1}{r^2}$ (2.42)
$\nabla r^2 = 2\mathbf{r}$ (2.39)	$\nabla(1/r^2) = \frac{-2\mathbf{r}}{r^4}$ (2.43)
$\nabla \cdot (r\mathbf{r}) = 4r$ (2.40)	$\nabla \cdot (\mathbf{r}/r^3) = 4\pi\delta(\mathbf{r})$ (2.44)

^aNote that the curl of any purely radial function is zero. $\delta(\mathbf{r})$ is the Dirac delta function.

Laplacian (scalar)

Rectangular coordinates	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	(2.45)	f scalar field ρ distance from the z axis
Cylindrical coordinates	$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$	(2.46)	
Spherical polar coordinates	$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$	(2.47)	
General orthogonal coordinates	$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right]$	(2.48)	q_i basis h_i metric elements

Differential operator identities

$\nabla(fg) \equiv f \nabla g + g \nabla f$	(2.49)	f, g scalar fields \mathbf{A}, \mathbf{B} vector fields
$\nabla \cdot (f \mathbf{A}) \equiv f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$	(2.50)	
$\nabla \times (f \mathbf{A}) \equiv f \nabla \times \mathbf{A} + (\nabla f) \times \mathbf{A}$	(2.51)	
$\nabla(\mathbf{A} \cdot \mathbf{B}) \equiv \mathbf{A} \times (\nabla \times \mathbf{B}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A}$	(2.52)	
$\nabla \cdot (\mathbf{A} \times \mathbf{B}) \equiv \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$	(2.53)	
$\nabla \times (\mathbf{A} \times \mathbf{B}) \equiv \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}$	(2.54)	
$\nabla \cdot (\nabla f) \equiv \nabla^2 f \equiv \triangle f$	(2.55)	
$\nabla \times (\nabla f) \equiv \mathbf{0}$	(2.56)	
$\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$	(2.57)	
$\nabla \times (\nabla \times \mathbf{A}) \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$	(2.58)	

Vector integral transformations

Gauss's (Divergence) theorem	$\int_V (\nabla \cdot \mathbf{A}) dV = \oint_{S_c} \mathbf{A} \cdot d\mathbf{s}$	(2.59)	\mathbf{A} vector field dV volume element S_c closed surface V volume enclosed
Stokes's theorem	$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_L \mathbf{A} \cdot d\mathbf{l}$	(2.60)	S surface $d\mathbf{s}$ surface element L loop bounding S $d\mathbf{l}$ line element
Green's first theorem	$\oint_S (f \nabla g) \cdot d\mathbf{s} = \int_V \nabla \cdot (f \nabla g) dV$	(2.61)	f, g scalar fields
	$= \int_V [f \nabla^2 g + (\nabla f) \cdot (\nabla g)] dV$	(2.62)	
Green's second theorem	$\oint_S [f(\nabla g) - g(\nabla f)] \cdot d\mathbf{s} = \int_V (f \nabla^2 g - g \nabla^2 f) dV$	(2.63)	

Matrix algebra^a

Matrix definition	$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad (2.64)$	\mathbf{A} m by n matrix a_{ij} matrix elements
Matrix addition	$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{if} \quad c_{ij} = a_{ij} + b_{ij} \quad (2.65)$	
Matrix multiplication	$\mathbf{C} = \mathbf{AB} \quad \text{if} \quad c_{ij} = a_{ik} b_{kj} \quad (2.66)$	
	$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \quad (2.67)$	
	$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \quad (2.68)$	
Transpose matrix ^b	$\tilde{a}_{ij} = a_{ji} \quad (2.69)$	\tilde{a}_{ij} transpose matrix (sometimes a_{ij}^T , or a'_{ij})
	$(\mathbf{AB} \dots \mathbf{N}) = \tilde{\mathbf{N}} \dots \tilde{\mathbf{B}} \tilde{\mathbf{A}} \quad (2.70)$	
Adjoint matrix (definition 1) ^c	$\mathbf{A}^\dagger = \tilde{\mathbf{A}}^* \quad (2.71)$	* complex conjugate (of each component)
	$(\mathbf{AB} \dots \mathbf{N})^\dagger = \mathbf{N}^\dagger \dots \mathbf{B}^\dagger \mathbf{A}^\dagger \quad (2.72)$	† adjoint (or Hermitian conjugate)
Hermitian matrix ^d	$\mathbf{H}^\dagger = \mathbf{H} \quad (2.73)$	\mathbf{H} Hermitian (or self-adjoint) matrix
examples:		
$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$		
$\tilde{\mathbf{A}} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \quad \mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{pmatrix}$		
$\mathbf{AB} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$		

^aTerms are implicitly summed over repeated suffices; hence $a_{ik}b_{kj}$ equals $\sum_k a_{ik}b_{kj}$.^bSee also Equation (2.85).^cOr “Hermitian conjugate matrix.” The term “adjoint” is used in quantum physics for the transpose conjugate of a matrix and in linear algebra for the transpose matrix of its cofactors. These definitions are not compatible, but both are widely used [cf. Equation (2.80)].^dHermitian matrices must also be square (see next table).

Square matrices^a

Trace	$\text{tr} \mathbf{A} = a_{ii}$ (2.74)	\mathbf{A}	square matrix
	$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$ (2.75)	a_{ij}	matrix elements
		a_{ii}	implicitly $= \sum_i a_{ii}$
Determinant ^b	$\det \mathbf{A} = \epsilon_{ijk\dots} a_{1i} a_{2j} a_{3k} \dots$ (2.76)	tr	trace
	$= (-1)^{i+1} a_{i1} M_{i1}$ (2.77)	\det	determinant (or $ \mathbf{A} $)
	$= a_{i1} C_{i1}$ (2.78)	M_{ij}	minor of element a_{ij}
	$\det(\mathbf{AB} \dots \mathbf{N}) = \det \mathbf{A} \det \mathbf{B} \dots \det \mathbf{N}$ (2.79)	C_{ij}	cofactor of the element a_{ij}
Adjoint matrix (definition 2) ^c	$\text{adj} \mathbf{A} = \tilde{C}_{ij} = C_{ji}$ (2.80)	adj	adjoint (sometimes written $\hat{\mathbf{A}}$)
		\sim	transpose
Inverse matrix ($\det \mathbf{A} \neq 0$)	$a_{ij}^{-1} = \frac{C_{ji}}{\det \mathbf{A}} = \frac{\text{adj} \mathbf{A}}{\det \mathbf{A}}$ (2.81)	$\mathbf{1}$	unit matrix
	$\mathbf{AA}^{-1} = \mathbf{1}$ (2.82)		
	$(\mathbf{AB} \dots \mathbf{N})^{-1} = \mathbf{N}^{-1} \dots \mathbf{B}^{-1} \mathbf{A}^{-1}$ (2.83)		
Orthogonality condition	$a_{ij} a_{ik} = \delta_{jk}$ (2.84)	δ_{jk}	Kronecker delta ($= 1$ if $i = j$, $= 0$ otherwise)
	i.e., $\tilde{\mathbf{A}} = \mathbf{A}^{-1}$ (2.85)		
Symmetry	If $\mathbf{A} = \tilde{\mathbf{A}}$, \mathbf{A} is symmetric (2.86)		
	If $\mathbf{A} = -\tilde{\mathbf{A}}$, \mathbf{A} is antisymmetric (2.87)		
Unitary matrix	$\mathbf{U}^\dagger = \mathbf{U}^{-1}$ (2.88)	\mathbf{U}	unitary matrix
		\dagger	Hermitian conjugate
examples:			
$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$			
$\text{tr} \mathbf{A} = a_{11} + a_{22} + a_{33} \qquad \text{tr} \mathbf{B} = b_{11} + b_{22}$			
$\det \mathbf{A} = a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{21} a_{12} a_{33} + a_{21} a_{13} a_{32} + a_{31} a_{12} a_{23} - a_{31} a_{13} a_{22}$			
$\det \mathbf{B} = b_{11} b_{22} - b_{12} b_{21}$			
$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22} a_{33} - a_{23} a_{32} & -a_{12} a_{33} + a_{13} a_{32} & a_{12} a_{23} - a_{13} a_{22} \\ -a_{21} a_{33} + a_{23} a_{31} & a_{11} a_{33} - a_{13} a_{31} & -a_{11} a_{23} + a_{13} a_{21} \\ a_{21} a_{32} - a_{22} a_{31} & -a_{11} a_{32} + a_{12} a_{31} & a_{11} a_{22} - a_{12} a_{21} \end{pmatrix}$			
$\mathbf{B}^{-1} = \frac{1}{\det \mathbf{B}} \begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix}$			

^aTerms are implicitly summed over repeated suffices; hence $a_{ik} b_{kj}$ equals $\sum_k a_{ik} b_{kj}$.^b $\epsilon_{ijk\dots}$ is defined as the natural extension of Equation (2.443) to n -dimensions (see page 50). M_{ij} is the determinant of the matrix \mathbf{A} with the i th row and the j th column deleted. The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$.^cOr “adjugate matrix.” See the footnote to Equation (2.71) for a discussion of the term “adjoint.”

Commutators

Commutator definition	$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} = -[\mathbf{B}, \mathbf{A}]$	(2.89)	[.,.] commutator † adjoint
Adjoint	$[\mathbf{A}, \mathbf{B}]^\dagger = [\mathbf{B}^\dagger, \mathbf{A}^\dagger]$	(2.90)	
Distribution	$[\mathbf{A} + \mathbf{B}, \mathbf{C}] = [\mathbf{A}, \mathbf{C}] + [\mathbf{B}, \mathbf{C}]$	(2.91)	
Association	$[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B}$	(2.92)	
Jacobi identity	$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] = [\mathbf{B}, [\mathbf{A}, \mathbf{C}]] - [\mathbf{C}, [\mathbf{A}, \mathbf{B}]]$	(2.93)	

Pauli matrices

Pauli matrices	$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$ $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	(2.94)	$\boldsymbol{\sigma}_i$ Pauli spin matrices $\mathbf{1}$ 2×2 unit matrix \mathbf{i} $\mathbf{i}^2 = -1$
Anticommutation	$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j \boldsymbol{\sigma}_i = 2\delta_{ij} \mathbf{1}$	(2.95)	δ_{ij} Kronecker delta	
Cyclic permutation	$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = \mathbf{i} \boldsymbol{\sigma}_k$ $(\boldsymbol{\sigma}_i)^2 = \mathbf{1}$	(2.96) (2.97)		

Rotation matrices^a

Rotation about x_1	$\mathbf{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$	(2.98)	$\mathbf{R}_i(\theta)$ matrix for rotation about the i th axis θ rotation angle
Rotation about x_2	$\mathbf{R}_2(\theta) = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$	(2.99)	
Rotation about x_3	$\mathbf{R}_3(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(2.100)	α rotation about x_3 β rotation about x'_2 γ rotation about x''_3
Euler angles	$\mathbf{R}(\alpha, \beta, \gamma) = \begin{pmatrix} \cos \gamma \cos \beta \cos \alpha - \sin \gamma \sin \alpha & \cos \gamma \cos \beta \sin \alpha + \sin \gamma \cos \alpha & -\cos \gamma \sin \beta \\ -\sin \gamma \cos \beta \cos \alpha - \cos \gamma \sin \alpha & -\sin \gamma \cos \beta \sin \alpha + \cos \gamma \cos \alpha & \sin \gamma \sin \beta \\ \sin \beta \cos \alpha & \sin \beta \sin \alpha & \cos \beta \end{pmatrix}$		
	(2.101)		\mathbf{R} rotation matrix

^aAngles are in the right-handed sense for rotation of axes, or the left-handed sense for rotation of vectors. i.e., a vector \mathbf{v} is given a right-handed rotation of θ about the x_3 -axis using $\mathbf{R}_3(-\theta)\mathbf{v} \mapsto \mathbf{v}'$. Conventionally, $x_1 \equiv x$, $x_2 \equiv y$, and $x_3 \equiv z$.

2.3 Series, summations, and progressions

Progressions and summations

Arithmetic progression	$S_n = a + (a + d) + (a + 2d) + \cdots$		n number of terms
	$+ [a + (n - 1)d] \quad (2.102)$		S_n sum of n successive terms
	$= \frac{n}{2}[2a + (n - 1)d] \quad (2.103)$		a first term
	$= \frac{n}{2}(a + l) \quad (2.104)$		d common difference l last term
Geometric progression	$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad (2.105)$		r common ratio
	$= a \frac{1 - r^n}{1 - r} \quad (2.106)$		
	$S_\infty = \frac{a}{1 - r} \quad (r < 1) \quad (2.107)$		
Arithmetic mean	$\langle x \rangle_a = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \quad (2.108)$		$\langle \cdot \rangle_a$ arithmetic mean
Geometric mean	$\langle x \rangle_g = (x_1 x_2 x_3 \cdots x_n)^{1/n} \quad (2.109)$		$\langle \cdot \rangle_g$ geometric mean
Harmonic mean	$\langle x \rangle_h = n \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)^{-1} \quad (2.110)$		$\langle \cdot \rangle_h$ harmonic mean
Relative mean magnitudes	$\langle x \rangle_a \geq \langle x \rangle_g \geq \langle x \rangle_h \quad \text{if } x_i > 0 \text{ for all } i \quad (2.111)$		
Summation formulas	$\sum_{i=1}^n i = \frac{n}{2}(n + 1) \quad (2.112)$		i dummy integer
	$\sum_{i=1}^n i^2 = \frac{n}{6}(n + 1)(2n + 1) \quad (2.113)$		
	$\sum_{i=1}^n i^3 = \frac{n^2}{4}(n + 1)^2 \quad (2.114)$		
	$\sum_{i=1}^n i^4 = \frac{n}{30}(n + 1)(2n + 1)(3n^2 + 3n - 1) \quad (2.115)$		
	$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \quad (2.116)$		
	$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i - 1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \quad (2.117)$		
	$\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6} \quad (2.118)$		
Euler's constant ^a	$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \right) \quad (2.119)$		γ Euler's constant

^a $\gamma \simeq 0.577215664 \dots$

Power series

Binomial series ^a	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$	(2.120)
Binomial coefficient ^b	${}^nC_r \equiv \binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$	(2.121)
Binomial theorem	$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$	(2.122)
Taylor series (about a) ^c	$f(a+x) = f(a) + xf^{(1)}(a) + \frac{x^2}{2!}f^{(2)}(a) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(a) + \cdots$	(2.123)
Taylor series (3-D)	$f(\mathbf{a}+\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x} \cdot \nabla)f _a + \frac{(\mathbf{x} \cdot \nabla)^2}{2!}f _a + \frac{(\mathbf{x} \cdot \nabla)^3}{3!}f _a + \cdots$	(2.124)
Maclaurin series	$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \cdots$	(2.125)

^aIf n is a positive integer the series terminates and is valid for all x . Otherwise the (infinite) series is convergent for $|x| < 1$.

^bThe coefficient of x^r in the binomial series.

^c $xf^{(n)}(a)$ is x times the n th derivative of the function $f(x)$ with respect to x evaluated at a , taken as well behaved around a . $(\mathbf{x} \cdot \nabla)^n f|_a$ is its extension to three dimensions.

Limits

$n^c x^n \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{if } x < 1 \quad (\text{for any fixed } c)$	(2.126)
$x^n/n! \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (\text{for any fixed } x)$	(2.127)
$(1+x/n)^n \rightarrow e^x \quad \text{as } n \rightarrow \infty$	(2.128)
$x \ln x \rightarrow 0 \quad \text{as } x \rightarrow 0$	(2.129)
$\frac{\sin x}{x} \rightarrow 1 \quad \text{as } x \rightarrow 0$	(2.130)
If $f(a)=g(a)=0$ or ∞ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f^{(1)}(a)}{g^{(1)}(a)}$ (l'Hôpital's rule)	(2.131)

Series expansions

$\exp(x)$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	(2.132)	(for all x)
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	(2.133)	$(-1 < x \leq 1)$
$\ln\left(\frac{1+x}{1-x}\right)$	$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$	(2.134)	$(x < 1)$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	(2.135)	(for all x)
$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	(2.136)	(for all x)
$\tan(x)$	$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$	(2.137)	$(x < \pi/2)$
$\sec(x)$	$1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$	(2.138)	$(x < \pi/2)$
$\csc(x)$	$\frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots$	(2.139)	$(x < \pi)$
$\cot(x)$	$\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots$	(2.140)	$(x < \pi)$
$\arcsin(x)^a$	$x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} + \cdots$	(2.141)	$(x < 1)$
$\arctan(x)^b$	$\begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & (x \leq 1) \\ \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & (x > 1) \\ -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & (x < -1) \end{cases}$	(2.142)	
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$	(2.143)	(for all x)
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$	(2.144)	(for all x)
$\tanh(x)$	$x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$	(2.145)	$(x < \pi/2)$

^a $\arccos(x) = \pi/2 - \arcsin(x)$. Note that $\arcsin(x) \equiv \sin^{-1}(x)$ etc.^b $\operatorname{arccot}(x) = \pi/2 - \arctan(x)$.

Inequalities

Triangle inequality	$ a_1 - a_2 \leq a_1 + a_2 \leq a_1 + a_2 ;$	(2.146)
	$\left \sum_{i=1}^n a_i \right \leq \sum_{i=1}^n a_i $	(2.147)
Chebyshev inequality	if $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n$	(2.148)
	and $b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$	(2.149)
	then $n \sum_{i=1}^n a_i b_i \geq \left(\sum_{i=1}^n a_i \right) \left(\sum_{i=1}^n b_i \right)$	(2.150)
Cauchy inequality	$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$	(2.151)
Schwarz inequality	$\left[\int_a^b f(x)g(x) \, dx \right]^2 \leq \int_a^b [f(x)]^2 \, dx \int_a^b [g(x)]^2 \, dx$	(2.152)

2.4 Complex variables

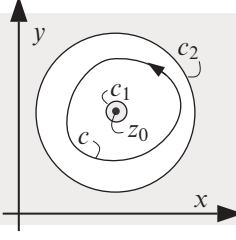
Complex numbers

Cartesian form	$z = x + \mathbf{i}y$	(2.153)	z complex variable \mathbf{i} $\mathbf{i}^2 = -1$ x, y real variables
Polar form	$z = r\mathbf{e}^{i\theta} = r(\cos \theta + \mathbf{i} \sin \theta)$	(2.154)	r amplitude (real) θ phase (real)
Modulus ^a	$ z = r = (x^2 + y^2)^{1/2}$	(2.155)	$ z $ modulus of z
	$ z_1 \cdot z_2 = z_1 \cdot z_2 $	(2.156)	
Argument	$\theta = \arg z = \arctan \frac{y}{x}$	(2.157)	$\arg z$ argument of z
	$\arg(z_1 z_2) = \arg z_1 + \arg z_2$	(2.158)	
Complex conjugate	$z^* = x - \mathbf{i}y = r\mathbf{e}^{-i\theta}$	(2.159)	z^* complex conjugate of $z = r\mathbf{e}^{i\theta}$
	$\arg(z^*) = -\arg z$	(2.160)	
	$z \cdot z^* = z ^2$	(2.161)	
Logarithm ^b	$\ln z = \ln r + \mathbf{i}(\theta + 2\pi n)$	(2.162)	n integer

^aOr “magnitude.”

^bThe principal value of $\ln z$ is given by $n=0$ and $-\pi < \theta \leq \pi$.

Complex analysis^a

Cauchy–Riemann equations ^b	<p>if $f(z) = u(x, y) + \mathbf{i}v(x, y)$</p> <p>then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (2.163)</p> <p>$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (2.164)</p>	<p>z complex variable</p> <p>\mathbf{i} $\mathbf{i}^2 = -1$</p> <p>x, y real variables</p> <p>$f(z)$ function of z</p> <p>u, v real functions</p>
Cauchy–Goursat theorem ^c	$\oint_c f(z) dz = 0$ (2.165)	
Cauchy integral formula ^d	<p>$f(z_0) = \frac{1}{2\pi\mathbf{i}} \oint_c \frac{f(z)}{z - z_0} dz$ (2.166)</p> <p>$f^{(n)}(z_0) = \frac{n!}{2\pi\mathbf{i}} \oint_c \frac{f(z)}{(z - z_0)^{n+1}} dz$ (2.167)</p>	<p>$^{(n)}$ nth derivative</p> <p>a_n Laurent coefficients</p> <p>a_{-1} residue of $f(z)$ at z_0</p> <p>z' dummy variable</p>
Laurent expansion ^e	<p>$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ (2.168)</p> <p>where $a_n = \frac{1}{2\pi\mathbf{i}} \oint_c \frac{f(z')}{(z' - z_0)^{n+1}} dz'$ (2.169)</p>	
Residue theorem	$\oint_c f(z) dz = 2\pi\mathbf{i} \sum \text{enclosed residues}$ (2.170)	

^aClosed contour integrals are taken in the counterclockwise sense, once.

^bNecessary condition for $f(z)$ to be analytic at a given point.

^cIf $f(z)$ is analytic within and on a simple closed curve c . Sometimes called “Cauchy’s theorem.”

^dIf $f(z)$ is analytic within and on a simple closed curve c , encircling z_0 .

^eOf $f(z)$, (analytic) in the annular region between concentric circles, c_1 and c_2 , centred on z_0 . c is any closed curve in this region encircling z_0 .

2.5 Trigonometric and hyperbolic formulas

Trigonometric relationships

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad (2.171)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \quad (2.172)$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (2.173)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)] \quad (2.174)$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)] \quad (2.175)$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \quad (2.176)$$

$$\cos^2 A + \sin^2 A = 1 \quad (2.177)$$

$$\sec^2 A - \tan^2 A = 1 \quad (2.178)$$

$$\csc^2 A - \cot^2 A = 1 \quad (2.179)$$

$$\sin 2A = 2 \sin A \cos A \quad (2.180)$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad (2.181)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad (2.182)$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A \quad (2.183)$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A \quad (2.184)$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \quad (2.185)$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \quad (2.186)$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \quad (2.187)$$

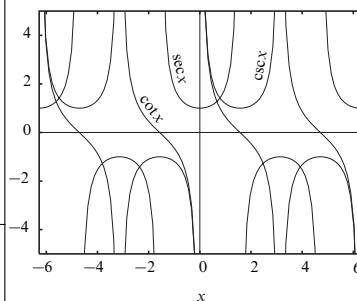
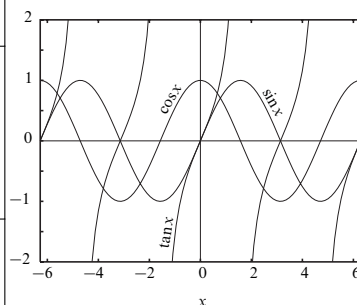
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \quad (2.188)$$

$$\cos^2 A = \frac{1}{2} (1 + \cos 2A) \quad (2.189)$$

$$\sin^2 A = \frac{1}{2} (1 - \cos 2A) \quad (2.190)$$

$$\cos^3 A = \frac{1}{4} (3 \cos A + \cos 3A) \quad (2.191)$$

$$\sin^3 A = \frac{1}{4} (3 \sin A - \sin 3A) \quad (2.192)$$



Hyperbolic relationships^a

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \quad (2.193)$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \quad (2.194)$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \quad (2.195)$$

$$\cosh x \cosh y = \frac{1}{2} [\cosh(x+y) + \cosh(x-y)] \quad (2.196)$$

$$\sinh x \cosh y = \frac{1}{2} [\sinh(x+y) + \sinh(x-y)] \quad (2.197)$$

$$\sinh x \sinh y = \frac{1}{2} [\cosh(x+y) - \cosh(x-y)] \quad (2.198)$$

$$\cosh^2 x - \sinh^2 x = 1 \quad (2.199)$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1 \quad (2.200)$$

$$\coth^2 x - \operatorname{csch}^2 x = 1 \quad (2.201)$$

$$\sinh 2x = 2 \sinh x \cosh x \quad (2.202)$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \quad (2.203)$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \quad (2.204)$$

$$\sinh 3x = 3 \sinh x + 4 \sinh^3 x \quad (2.205)$$

$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x \quad (2.206)$$

$$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2} \quad (2.207)$$

$$\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2} \quad (2.208)$$

$$\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2} \quad (2.209)$$

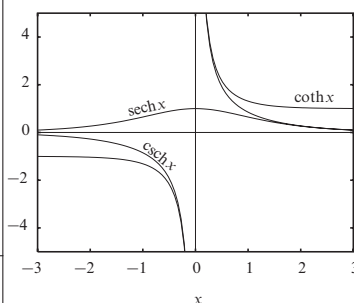
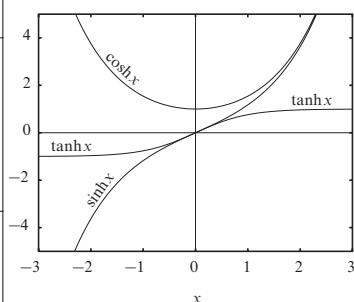
$$\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2} \quad (2.210)$$

$$\cosh^2 x = \frac{1}{2} (\cosh 2x + 1) \quad (2.211)$$

$$\sinh^2 x = \frac{1}{2} (\cosh 2x - 1) \quad (2.212)$$

$$\cosh^3 x = \frac{1}{4} (3 \cosh x + \cosh 3x) \quad (2.213)$$

$$\sinh^3 x = \frac{1}{4} (\sinh 3x - 3 \sinh x) \quad (2.214)$$



^aThese can be derived from trigonometric relationships by using the substitutions $\cos x \mapsto \cosh x$ and $\sin x \mapsto i \sinh x$.

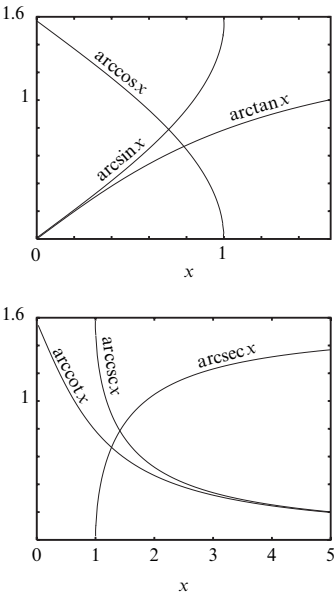
Trigonometric and hyperbolic definitions

de Moivre's theorem	$(\cos x + \mathbf{i} \sin x)^n = \mathbf{e}^{inx} = \cos nx + \mathbf{i} \sin nx$	(2.215)
$\cos x = \frac{1}{2} (\mathbf{e}^{ix} + \mathbf{e}^{-ix})$	(2.216)	$\cosh x = \frac{1}{2} (\mathbf{e}^x + \mathbf{e}^{-x})$ (2.217)
$\sin x = \frac{1}{2\mathbf{i}} (\mathbf{e}^{ix} - \mathbf{e}^{-ix})$	(2.218)	$\sinh x = \frac{1}{2} (\mathbf{e}^x - \mathbf{e}^{-x})$ (2.219)
$\tan x = \frac{\sin x}{\cos x}$	(2.220)	$\tanh x = \frac{\sinh x}{\cosh x}$ (2.221)
$\cos \mathbf{i}x = \cosh x$	(2.222)	$\cosh \mathbf{i}x = \cos x$ (2.223)
$\sin \mathbf{i}x = \mathbf{i} \sinh x$	(2.224)	$\sinh \mathbf{i}x = \mathbf{i} \sin x$ (2.225)
$\cot x = (\tan x)^{-1}$	(2.226)	$\coth x = (\tanh x)^{-1}$ (2.227)
$\sec x = (\cos x)^{-1}$	(2.228)	$\operatorname{sech} x = (\cosh x)^{-1}$ (2.229)
$\csc x = (\sin x)^{-1}$	(2.230)	$\operatorname{csch} x = (\sinh x)^{-1}$ (2.231)

Inverse trigonometric functions^a

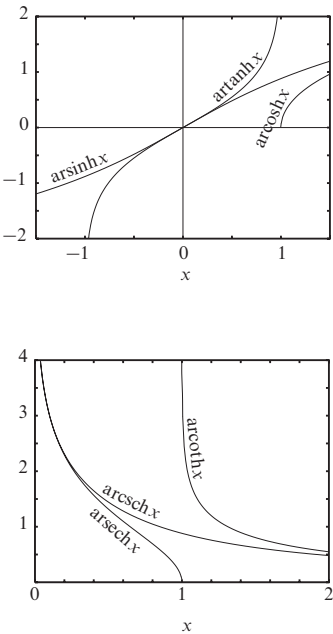
$\arcsin x = \arctan \left[\frac{x}{(1-x^2)^{1/2}} \right]$	(2.232)
$\arccos x = \arctan \left[\frac{(1-x^2)^{1/2}}{x} \right]$	(2.233)
$\operatorname{arccsc} x = \arctan \left[\frac{1}{(x^2-1)^{1/2}} \right]$	(2.234)
$\operatorname{arcsec} x = \arctan \left[(x^2-1)^{1/2} \right]$	(2.235)
$\operatorname{arccot} x = \arctan \left(\frac{1}{x} \right)$	(2.236)
$\arccos x = \frac{\pi}{2} - \arcsin x$	(2.237)

^aValid in the angle range $0 \leq \theta \leq \pi/2$. Note that $\arcsin x \equiv \sin^{-1} x$ etc.



Inverse hyperbolic functions

$\operatorname{arsinh} x \equiv \sinh^{-1} x = \ln \left[x + (x^2 + 1)^{1/2} \right] \quad (2.238)$	for all x
$\operatorname{arcosh} x \equiv \cosh^{-1} x = \ln \left[x + (x^2 - 1)^{1/2} \right] \quad (2.239)$	$x \geq 1$
$\operatorname{artanh} x \equiv \tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \quad (2.240)$	$ x < 1$
$\operatorname{arcoth} x \equiv \coth^{-1} x = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right) \quad (2.241)$	$ x > 1$
$\operatorname{arsech} x \equiv \operatorname{sech}^{-1} x = \ln \left[\frac{1}{x} + \frac{(1-x^2)^{1/2}}{x} \right] \quad (2.242)$	$0 < x \leq 1$
$\operatorname{arcsch} x \equiv \operatorname{csch}^{-1} x = \ln \left[\frac{1}{x} + \frac{(1+x^2)^{1/2}}{x} \right] \quad (2.243)$	$x \neq 0$

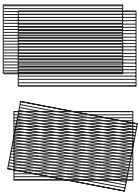


2

2.6 Mensuration

Moiré fringes^a

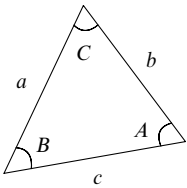
Parallel pattern	$d_M = \left \frac{1}{d_1} - \frac{1}{d_2} \right ^{-1} \quad (2.244)$	d_M Moiré fringe spacing $d_{1,2}$ grating spacings
Rotational pattern ^b	$d_M = \frac{d}{2 \sin(\theta/2) } \quad (2.245)$	d common grating spacing θ relative rotation angle ($ \theta \leq \pi/2$)



^aFrom overlapping linear gratings.
^bFrom identical gratings, spacing d , with a relative rotation θ .

Plane triangles

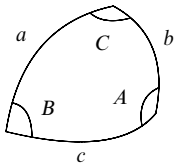
Sine formula ^a	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	(2.246)
Cosine formulas	$a^2 = b^2 + c^2 - 2bc \cos A$	(2.247)
	$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$	(2.248)
	$a = b \cos C + c \cos B$	(2.249)
Tangent formula	$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$	(2.250)
Area	$\text{area} = \frac{1}{2} ab \sin C$	(2.251)
	$= \frac{a^2 \sin B \sin C}{2 \sin A}$	(2.252)
	$= [s(s-a)(s-b)(s-c)]^{1/2}$	(2.253)
	where $s = \frac{1}{2}(a+b+c)$	(2.254)



^aThe diameter of the circumscribed circle equals $a/\sin A$.

Spherical triangles^a

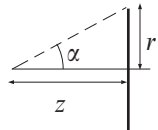
Sine formula	$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$	(2.255)
Cosine formulas	$\cos a = \cos b \cos c + \sin b \sin c \cos A$	(2.256)
	$\cos A = -\cos B \cos C + \sin B \sin C \cos a$	(2.257)
Analogue formula	$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$	(2.258)
Four-parts formula	$\cos a \cos C = \sin a \cot b - \sin C \cot B$	(2.259)
Area ^b	$E = A + B + C - \pi$	(2.260)



^aOn a unit sphere.

^bAlso called the “spherical excess.”

Perimeter, area, and volume

Perimeter of circle	$P = 2\pi r$	(2.261)	P perimeter r radius
Area of circle	$A = \pi r^2$	(2.262)	A area
Surface area of sphere ^a	$A = 4\pi R^2$	(2.263)	R sphere radius
Volume of sphere	$V = \frac{4}{3}\pi R^3$	(2.264)	V volume
Perimeter of ellipse ^b	$P = 4aE(\pi/2, e)$	(2.265)	a semi-major axis b semi-minor axis E elliptic integral of the second kind (p. 45)
	$\simeq 2\pi \left(\frac{a^2 + b^2}{2} \right)^{1/2}$	(2.266)	e eccentricity ($= 1 - b^2/a^2$)
Area of ellipse	$A = \pi ab$	(2.267)	
Volume of ellipsoid ^c	$V = 4\pi \frac{abc}{3}$	(2.268)	c third semi-axis
Surface area of cylinder	$A = 2\pi r(h + r)$	(2.269)	h height
Volume of cylinder	$V = \pi r^2 h$	(2.270)	
Area of circular cone ^d	$A = \pi r l$	(2.271)	l slant height
Volume of cone or pyramid	$V = A_b h/3$	(2.272)	A_b base area
Surface area of torus	$A = \pi^2 (r_1 + r_2)(r_2 - r_1)$	(2.273)	r_1 inner radius r_2 outer radius
Volume of torus	$V = \frac{\pi^2}{4} (r_2^2 - r_1^2)(r_2 - r_1)$	(2.274)	
Area ^d of spherical cap, depth d	$A = 2\pi R d$	(2.275)	d cap depth
Volume of spherical cap, depth d	$V = \pi d^2 \left(R - \frac{d}{3} \right)$	(2.276)	Ω solid angle z distance from centre α half-angle subtended
Solid angle of a circle from a point on its axis, z from centre	$\Omega = 2\pi \left[1 - \frac{z}{(z^2 + r^2)^{1/2}} \right]$	(2.277)	
	$= 2\pi(1 - \cos \alpha)$	(2.278)	

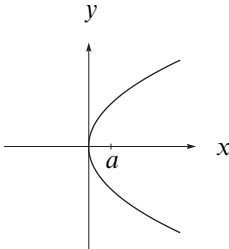
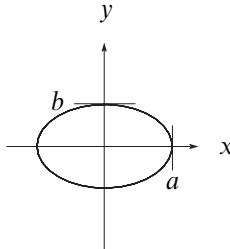
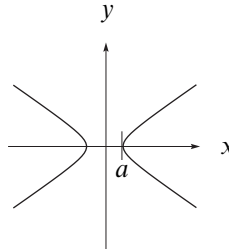
^aSphere defined by $x^2 + y^2 + z^2 = R^2$.

^bThe approximation is exact when $e=0$ and $e \simeq 0.91$, giving a maximum error of 11% at $e=1$.

^cEllipsoid defined by $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

^dCurved surface only.

Conic sections

			
	<i>parabola</i>	<i>ellipse</i>	<i>hyperbola</i>
equation	$y^2 = 4ax$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
parametric form	$x = t^2/(4a)$ $y = t$	$x = a \cos t$ $y = b \sin t$	$x = \pm a \cosh t$ $y = b \sinh t$
foci	$(a, 0)$	$(\pm \sqrt{a^2 - b^2}, 0)$	$(\pm \sqrt{a^2 + b^2}, 0)$
eccentricity	$e = 1$	$e = \frac{\sqrt{a^2 - b^2}}{a}$	$e = \frac{\sqrt{a^2 + b^2}}{a}$
directrices	$x = -a$	$x = \pm \frac{a}{e}$	$x = \pm \frac{a}{e}$

Platonic solids^a

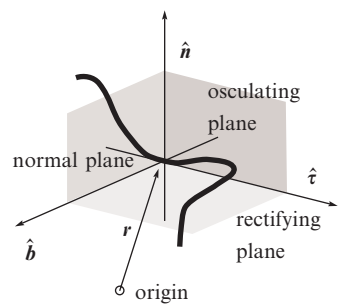
<i>solid</i> <i>(faces, edges, vertices)</i>	<i>volume</i>	<i>surface area</i>	<i>circumradius</i>	<i>inradius</i>
tetrahedron (4, 6, 4)	$\frac{a^3 \sqrt{2}}{12}$	$a^2 \sqrt{3}$	$\frac{a \sqrt{6}}{4}$	$\frac{a \sqrt{6}}{12}$
cube (6, 12, 8)	a^3	$6a^2$	$\frac{a \sqrt{3}}{2}$	$\frac{a}{2}$
octahedron (8, 12, 6)	$\frac{a^3 \sqrt{2}}{3}$	$2a^2 \sqrt{3}$	$\frac{a}{\sqrt{2}}$	$\frac{a}{\sqrt{6}}$
dodecahedron (12, 30, 20)	$\frac{a^3 (15 + 7\sqrt{5})}{4}$	$3a^2 \sqrt{5(5 + 2\sqrt{5})}$	$\frac{a}{4} \sqrt{3}(1 + \sqrt{5})$	$\frac{a}{4} \sqrt{\frac{50 + 22\sqrt{5}}{5}}$
icosahedron (20, 30, 12)	$\frac{5a^3 (3 + \sqrt{5})}{12}$	$5a^2 \sqrt{3}$	$\frac{a}{4} \sqrt{2(5 + \sqrt{5})}$	$\frac{a}{4} \left(\sqrt{3} + \sqrt{\frac{5}{3}} \right)$

^aOf side *a*. Both regular and irregular polyhedra follow the Euler relation, faces – edges + vertices = 2.

Curve measure

Length of plane curve	$l = \int_a^b \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} dx \quad (2.279)$	a start point b end point $y(x)$ plane curve l length
Surface of revolution	$A = 2\pi \int_a^b y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} dx \quad (2.280)$	A surface area
Volume of revolution	$V = \pi \int_a^b y^2 dx \quad (2.281)$	V volume
Radius of curvature	$\rho = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2} \left(\frac{d^2y}{dx^2} \right)^{-1} \quad (2.282)$	ρ radius of curvature

Differential geometry^a

Unit tangent	$\hat{\tau} = \frac{\dot{\mathbf{r}}}{ \dot{\mathbf{r}} } = \frac{\dot{\mathbf{r}}}{v} \quad (2.283)$	τ tangent \mathbf{r} curve parameterised by $\mathbf{r}(t)$ v $ \dot{\mathbf{r}}(t) $
Unit principal normal	$\hat{\mathbf{n}} = \frac{\ddot{\mathbf{r}} - \dot{v}\hat{\tau}}{ \ddot{\mathbf{r}} - \dot{v}\hat{\tau} } \quad (2.284)$	\mathbf{n} principal normal
Unit binormal	$\hat{\mathbf{b}} = \hat{\tau} \times \hat{\mathbf{n}} \quad (2.285)$	\mathbf{b} binormal
Curvature	$\kappa = \frac{ \dot{\mathbf{r}} \times \ddot{\mathbf{r}} }{ \dot{\mathbf{r}} ^3} \quad (2.286)$	κ curvature
Radius of curvature	$\rho = \frac{1}{\kappa} \quad (2.287)$	ρ radius of curvature
Torsion	$\lambda = \frac{\dot{\mathbf{r}} \cdot (\ddot{\mathbf{r}} \times \ddot{\ddot{\mathbf{r}}})}{ \dot{\mathbf{r}} \times \ddot{\mathbf{r}} ^2} \quad (2.288)$	λ torsion
Frenet's formulas	$\dot{\hat{\tau}} = \kappa v \hat{\mathbf{n}} \quad (2.289)$ $\dot{\hat{\mathbf{n}}} = -\kappa v \hat{\tau} + \lambda v \hat{\mathbf{b}} \quad (2.290)$ $\dot{\hat{\mathbf{b}}} = -\lambda v \hat{\mathbf{n}} \quad (2.291)$	

^aFor a continuous curve in three dimensions, traced by the position vector $\mathbf{r}(t)$.

2.7 Differentiation

Derivatives (general)

Power	$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$	(2.292)	n	power index
Product	$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$	(2.293)	u, v	functions of x
Quotient	$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{1}{v} \frac{du}{dx} - \frac{u}{v^2} \frac{dv}{dx}$	(2.294)		
Function of a function ^a	$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx}$	(2.295)	$f(u)$	function of $u(x)$
Leibniz theorem	$\begin{aligned} \frac{d^n}{dx^n}[uv] &= \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1} u}{dx^{n-1}} + \cdots \\ &+ \binom{n}{k} \frac{d^k v}{dx^k} \frac{d^{n-k} u}{dx^{n-k}} + \cdots + \binom{n}{n} u \frac{d^n v}{dx^n} \end{aligned}$	(2.296)	$\binom{n}{k}$	binomial coefficient
Differentiation under the integral sign	$\frac{d}{dq} \left[\int_p^q f(x) dx \right] = f(q) \quad (p \text{ constant})$	(2.297)		
	$\frac{d}{dp} \left[\int_p^q f(x) dx \right] = -f(p) \quad (q \text{ constant})$	(2.298)		
General integral	$\frac{d}{dx} \left[\int_{u(x)}^{v(x)} f(t) dt \right] = f(v) \frac{dv}{dx} - f(u) \frac{du}{dx}$	(2.299)		
Logarithm	$\frac{d}{dx}(\log_b ax) = (x \ln b)^{-1}$	(2.300)	b a	log base constant
Exponential	$\frac{d}{dx}(e^{ax}) = ae^{ax}$	(2.301)		
Inverse functions	$\frac{dx}{dy} = \left(\frac{dy}{dx} \right)^{-1}$	(2.302)		
	$\frac{d^2 x}{dy^2} = -\frac{d^2 y}{dx^2} \left(\frac{dy}{dx} \right)^{-3}$	(2.303)		
	$\frac{d^3 x}{dy^3} = \left[3 \left(\frac{d^2 y}{dx^2} \right)^2 - \frac{dy}{dx} \frac{d^3 y}{dx^3} \right] \left(\frac{dy}{dx} \right)^{-5}$	(2.304)		

^aThe “chain rule.”

Trigonometric derivatives^a

$\frac{d}{dx}(\sin ax) = a \cos ax$	(2.305)	$\frac{d}{dx}(\cos ax) = -a \sin ax$	(2.306)
$\frac{d}{dx}(\tan ax) = a \sec^2 ax$	(2.307)	$\frac{d}{dx}(\csc ax) = -a \csc ax \cdot \cot ax$	(2.308)
$\frac{d}{dx}(\sec ax) = a \sec ax \cdot \tan ax$	(2.309)	$\frac{d}{dx}(\cot ax) = -a \csc^2 ax$	(2.310)
$\frac{d}{dx}(\arcsin ax) = a(1 - a^2 x^2)^{-1/2}$	(2.311)	$\frac{d}{dx}(\arccos ax) = -a(1 - a^2 x^2)^{-1/2}$	(2.312)
$\frac{d}{dx}(\arctan ax) = a(1 + a^2 x^2)^{-1}$	(2.313)	$\frac{d}{dx}(\operatorname{arccsc} ax) = -\frac{a}{ ax }(a^2 x^2 - 1)^{-1/2}$	(2.314)
$\frac{d}{dx}(\operatorname{arcsec} ax) = \frac{a}{ ax }(a^2 x^2 - 1)^{-1/2}$	(2.315)	$\frac{d}{dx}(\operatorname{arccot} ax) = -a(a^2 x^2 + 1)^{-1}$	(2.316)

^a a is a constant.**Hyperbolic derivatives^a**

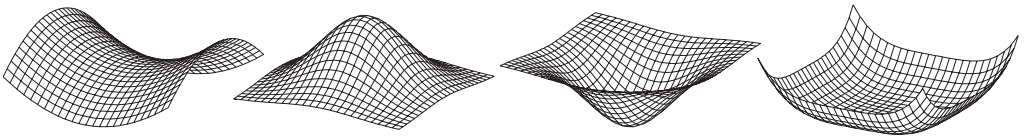
$\frac{d}{dx}(\sinh ax) = a \cosh ax$	(2.317)	$\frac{d}{dx}(\cosh ax) = a \sinh ax$	(2.318)
$\frac{d}{dx}(\tanh ax) = a \operatorname{sech}^2 ax$	(2.319)	$\frac{d}{dx}(\operatorname{csch} ax) = -a \operatorname{csch} ax \cdot \coth ax$	(2.320)
$\frac{d}{dx}(\operatorname{sech} ax) = -a \operatorname{sech} ax \cdot \tanh ax$	(2.321)	$\frac{d}{dx}(\coth ax) = -a \operatorname{csch}^2 ax$	(2.322)
$\frac{d}{dx}(\operatorname{arsinh} ax) = a(a^2 x^2 + 1)^{-1/2}$	(2.323)	$\frac{d}{dx}(\operatorname{arcosh} ax) = a(a^2 x^2 - 1)^{-1/2}$	(2.324)
$\frac{d}{dx}(\operatorname{artanh} ax) = a(1 - a^2 x^2)^{-1}$	(2.325)	$\frac{d}{dx}(\operatorname{arcsch} ax) = -\frac{a}{ ax }(1 + a^2 x^2)^{-1/2}$	(2.326)
$\frac{d}{dx}(\operatorname{arsech} ax) = -\frac{a}{ ax }(1 - a^2 x^2)^{-1/2}$	(2.327)	$\frac{d}{dx}(\operatorname{arcoth} ax) = a(1 - a^2 x^2)^{-1}$	(2.328)

^a a is a constant.

Partial derivatives

Total differential	$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$	(2.329)	f $f(x,y,z)$
Reciprocity	$\left. \frac{\partial g}{\partial x} \right _y \left. \frac{\partial x}{\partial y} \right _g \left. \frac{\partial y}{\partial g} \right _x = -1$	(2.330)	g $g(x,y)$
Chain rule	$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$	(2.331)	
Jacobian	$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$	(2.332)	J Jacobian u $u(x,y,z)$ v $v(x,y,z)$ w $w(x,y,z)$
Change of variable	$\int_V f(x,y,z) dx dy dz = \int_{V'} f(u,v,w) J du dv dw$	(2.333)	V volume in (x,y,z) V' volume in (u,v,w) mapped to by V
Euler–Lagrange equation	if $I = \int_a^b F(x,y,y') dx$ then $\delta I = 0$ when $\frac{\partial F}{\partial y} = \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right)$	(2.334)	y' dy/dx a,b fixed end points

Stationary points^a



saddle point

maximum

minimum

quartic minimum

Stationary point if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at (x_0,y_0) (necessary condition) (2.335)

Additional sufficient conditions

for maximum

$\frac{\partial^2 f}{\partial x^2} < 0,$ and $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$

(2.336)

for minimum

$\frac{\partial^2 f}{\partial x^2} > 0,$ and $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$

(2.337)

for saddle point

$\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} < \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$

(2.338)

^aOf a function $f(x,y)$ at the point (x_0,y_0) . Note that at, for example, a *quartic minimum* $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$.

Differential equations

Laplace	$\nabla^2 f = 0$	(2.339)	f	$f(x, y, z)$
Diffusion ^a	$\frac{\partial f}{\partial t} = D \nabla^2 f$	(2.340)	D	diffusion coefficient
Helmholtz	$\nabla^2 f + \alpha^2 f = 0$	(2.341)	α	constant
Wave	$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$	(2.342)	c	wave speed
Legendre	$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + l(l+1)y = 0$	(2.343)	l	integer
Associated Legendre	$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + \left[l(l+1) - \frac{m^2}{1-x^2} \right] y = 0$	(2.344)	m	integer
Bessel	$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2)y = 0$	(2.345)		
Hermite	$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2\alpha y = 0$	(2.346)		
Laguerre	$x \frac{d^2 y}{dx^2} + (1-x) \frac{dy}{dx} + \alpha y = 0$	(2.347)		
Associated Laguerre	$x \frac{d^2 y}{dx^2} + (1+k-x) \frac{dy}{dx} + \alpha y = 0$	(2.348)	k	integer
Chebyshev	$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0$	(2.349)	n	integer
Euler (or Cauchy)	$x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = f(x)$	(2.350)	a, b	constants
Bernoulli	$\frac{dy}{dx} + p(x)y = q(x)y^a$	(2.351)	p, q	functions of x
Airy	$\frac{d^2 y}{dx^2} = xy$	(2.352)		

^aAlso known as the “conduction equation.” For thermal conduction, $f \equiv T$ and D , the thermal diffusivity, $\equiv \kappa \equiv \lambda/(\rho c_p)$, where T is the temperature distribution, λ the thermal conductivity, ρ the density, and c_p the specific heat capacity of the material.

2.8 Integration

Standard forms^a

$$\int u \, dv = [uv] - \int v \, du \quad (2.353) \quad \int uv \, dx = v \int u \, dx - \int \left(\int u \, dx \right) \frac{dv}{dx} \, dx \quad (2.354)$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1) \quad (2.355) \quad \int \frac{1}{x} \, dx = \ln|x| \quad (2.356)$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} \quad (2.357) \quad \int x e^{ax} \, dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right) \quad (2.358)$$

$$\int \ln ax \, dx = x(\ln ax - 1) \quad (2.359) \quad \int \frac{f'(x)}{f(x)} \, dx = \ln f(x) \quad (2.360)$$

$$\int x \ln ax \, dx = \frac{x^2}{2} \left(\ln ax - \frac{1}{2} \right) \quad (2.361) \quad \int b^{ax} \, dx = \frac{b^{ax}}{a \ln b} \quad (b > 0) \quad (2.362)$$

$$\int \frac{1}{a+bx} \, dx = \frac{1}{b} \ln(a+bx) \quad (2.363) \quad \int \frac{1}{x(a+bx)} \, dx = -\frac{1}{a} \ln \frac{a+bx}{x} \quad (2.364)$$

$$\int \frac{1}{(a+bx)^2} \, dx = \frac{-1}{b(a+bx)} \quad (2.365) \quad \int \frac{1}{a^2+b^2x^2} \, dx = \frac{1}{ab} \arctan \left(\frac{bx}{a} \right) \quad (2.366)$$

$$\int \frac{1}{x(x^n+a)} \, dx = \frac{1}{an} \ln \left| \frac{x^n}{x^n+a} \right| \quad (2.367) \quad \int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| \quad (2.368)$$

$$\int \frac{x}{x^2 \pm a^2} \, dx = \frac{1}{2} \ln|x^2 \pm a^2| \quad (2.369) \quad \int \frac{x}{(x^2 \pm a^2)^n} \, dx = \frac{-1}{2(n-1)(x^2 \pm a^2)^{n-1}} \quad (2.370)$$

$$\int \frac{1}{(a^2-x^2)^{1/2}} \, dx = \arcsin \left(\frac{x}{a} \right) \quad (2.371) \quad \int \frac{1}{(x^2 \pm a^2)^{1/2}} \, dx = \ln|x + (x^2 \pm a^2)^{1/2}| \quad (2.372)$$

$$\int \frac{x}{(x^2 \pm a^2)^{1/2}} \, dx = (x^2 \pm a^2)^{1/2} \quad (2.373) \quad \int \frac{1}{x(x^2-a^2)^{1/2}} \, dx = \frac{1}{a} \operatorname{arcsec} \left(\frac{x}{a} \right) \quad (2.374)$$

^a a and b are non-zero constants.

Trigonometric and hyperbolic integrals

$\int \sin x \, dx = -\cos x$	(2.375)	$\int \sinh x \, dx = \cosh x$	(2.376)
$\int \cos x \, dx = \sin x$	(2.377)	$\int \cosh x \, dx = \sinh x$	(2.378)
$\int \tan x \, dx = -\ln \cos x $	(2.379)	$\int \tanh x \, dx = \ln(\cosh x)$	(2.380)
$\int \csc x \, dx = \ln \left \tan \frac{x}{2} \right $	(2.381)	$\int \operatorname{csch} x \, dx = \ln \left \tanh \frac{x}{2} \right $	(2.382)
$\int \sec x \, dx = \ln \sec x + \tan x $	(2.383)	$\int \operatorname{sech} x \, dx = 2 \arctan(e^x)$	(2.384)
$\int \cot x \, dx = \ln \sin x $	(2.385)	$\int \coth x \, dx = \ln \sinh x $	(2.386)
$\int \sin mx \cdot \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.387)$			
$\int \sin mx \cdot \cos nx \, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.388)$			
$\int \cos mx \cdot \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.389)$			

Named integrals

Error function	$\operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_0^x \exp(-t^2) \, dt$	(2.390)
Complementary error function	$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_x^\infty \exp(-t^2) \, dt$	(2.391)
Fresnel integrals ^a	$C(x) = \int_0^x \cos \frac{\pi t^2}{2} \, dt; \quad S(x) = \int_0^x \sin \frac{\pi t^2}{2} \, dt$	(2.392)
	$C(x) + i S(x) = \frac{1+i}{2} \operatorname{erf} \left[\frac{\pi^{1/2}}{2} (1-i)x \right]$	(2.393)
Exponential integral	$\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} \, dt \quad (x > 0)$	(2.394)
Gamma function	$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad (x > 0)$	(2.395)
Elliptic integrals (trigonometric form)	$F(\phi, k) = \int_0^\phi \frac{1}{(1-k^2 \sin^2 \theta)^{1/2}} \, d\theta \quad (\text{first kind})$	(2.396)
	$E(\phi, k) = \int_0^\phi (1-k^2 \sin^2 \theta)^{1/2} \, d\theta \quad (\text{second kind})$	(2.397)

^aSee also page 167.

Definite integrals

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} \left(\frac{\pi}{a} \right)^{1/2} \quad (a > 0) \quad (2.398)$$

$$\int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a} \quad (a > 0) \quad (2.399)$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \quad (a > 0; n = 0, 1, 2, \dots) \quad (2.400)$$

$$\int_{-\infty}^{\infty} \exp(2bx - ax^2) dx = \left(\frac{\pi}{a} \right)^{1/2} \exp\left(\frac{b^2}{a}\right) \quad (a > 0) \quad (2.401)$$

$$\int_0^{\infty} x^n e^{-ax^2} dx = \begin{cases} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-1)(2a)^{-(n+1)/2} (\pi/2)^{1/2} & n > 0 \text{ and even} \\ 2 \cdot 4 \cdot 6 \cdot \dots \cdot (n-1)(2a)^{-(n+1)/2} & n > 1 \text{ and odd} \end{cases} \quad (2.402)$$

$$\int_0^1 x^p (1-x)^q dx = \frac{p!q!}{(p+q+1)!} \quad (p, q \text{ integers} > 0) \quad (2.403)$$

$$\int_0^{\infty} \cos(ax^2) dx = \int_0^{\infty} \sin(ax^2) dx = \frac{1}{2} \left(\frac{\pi}{2a} \right)^{1/2} \quad (a > 0) \quad (2.404)$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2} \quad (2.405)$$

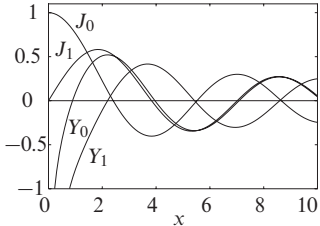
$$\int_0^{\infty} \frac{1}{(1+x)x^a} dx = \frac{\pi}{\sin a\pi} \quad (0 < a < 1) \quad (2.406)$$

2.9 Special functions and polynomials

Gamma function

Definition	$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad [\Re(z) > 0]$	(2.407)
Relations	$n! = \Gamma(n+1) = n\Gamma(n) \quad (n = 0, 1, 2, \dots)$	(2.408)
	$\Gamma(1/2) = \pi^{1/2}$	(2.409)
	$\left(\begin{matrix} z \\ w \end{matrix} \right) = \frac{z!}{w!(z-w)!} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$	(2.410)
Stirling's formulas (for $ z , n \gg 1$)	$\Gamma(z) \simeq e^{-z} z^{z-(1/2)} (2\pi)^{1/2} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \dots \right)$	(2.411)
	$n! \simeq n^{n+(1/2)} e^{-n} (2\pi)^{1/2}$	(2.412)
	$\ln(n!) \simeq n \ln n - n$	(2.413)

Bessel functions

Series expansion	$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k! \Gamma(\nu + k + 1)} \quad (2.414)$ $Y_\nu(x) = \frac{J_\nu(x) \cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)} \quad (2.415)$	$J_\nu(x)$ Bessel function of the first kind $Y_\nu(x)$ Bessel function of the second kind $\Gamma(\nu)$ Gamma function ν order ($\nu \geq 0$)
Approximations	$J_\nu(x) \simeq \begin{cases} \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu & (0 \leq x \ll \nu) \\ \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) & (x \gg \nu) \end{cases} \quad (2.416)$ $Y_\nu(x) \simeq \begin{cases} \frac{-\Gamma(\nu)}{\pi} \left(\frac{x}{2}\right)^{-\nu} & (0 < x \ll \nu) \\ \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) & (x \gg \nu) \end{cases} \quad (2.417)$	
Modified Bessel functions	$I_\nu(x) = (-i)^\nu J_\nu(ix) \quad (2.418)$ $K_\nu(x) = \frac{\pi}{2} i^{\nu+1} [J_\nu(ix) + iY_\nu(ix)] \quad (2.419)$	$I_\nu(x)$ modified Bessel function of the first kind $K_\nu(x)$ modified Bessel function of the second kind
Spherical Bessel function	$j_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{\nu+\frac{1}{2}}(x) \quad (2.420)$	$j_\nu(x)$ spherical Bessel function of the first kind [similarly for $y_\nu(x)$]

Legendre polynomials^a

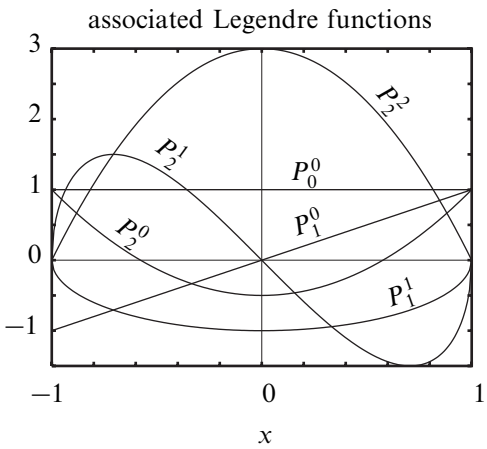
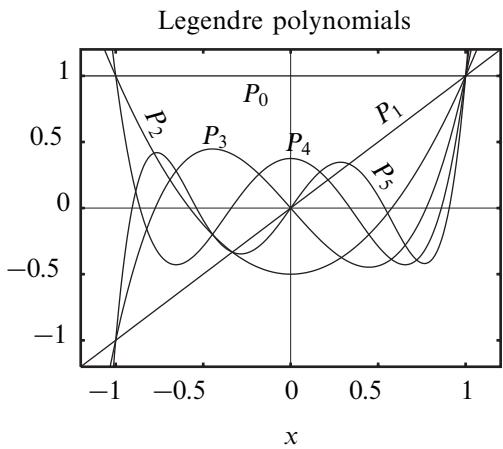
Legendre equation	$(1-x^2) \frac{d^2 P_l(x)}{dx^2} - 2x \frac{dP_l(x)}{dx} + l(l+1)P_l(x) = 0 \quad (2.421)$	P_l Legendre polynomials l order ($l \geq 0$)
Rodrigues' formula	$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \quad (2.422)$	
Recurrence relation	$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x) \quad (2.423)$	
Orthogonality	$\int_{-1}^1 P_l(x) P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'} \quad (2.424)$	
Explicit form	$P_l(x) = 2^{-l} \sum_{m=0}^{l/2} (-1)^m \binom{l}{m} \binom{2l-2m}{l} x^{l-2m} \quad (2.425)$	$\binom{l}{m}$ binomial coefficients
Expansion of plane wave	$\exp(ikz) = \exp(ikr \cos \theta) \quad (2.426)$	k wavenumber z propagation axis $z = r \cos \theta$
	$= \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta) \quad (2.427)$	j_l spherical Bessel function of the first kind (order l)
$P_0(x) = 1$ $P_2(x) = (3x^2 - 1)/2$ $P_4(x) = (35x^4 - 30x^2 + 3)/8$ $P_1(x) = x$ $P_3(x) = (5x^3 - 3x)/2$ $P_5(x) = (63x^5 - 70x^3 + 15x)/8$		

^aOf the first kind.

Associated Legendre functions^a

Associated Legendre equation	$\frac{d}{dx} \left[(1-x^2) \frac{dP_l^m(x)}{dx} \right] + \left[l(l+1) - \frac{m^2}{1-x^2} \right] P_l^m(x) = 0$ <div>(2.428)</div>	P_l^m associated Legendre functions
From Legendre polynomials	$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad 0 \leq m \leq l$ <div>(2.429)</div> $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$ <div>(2.430)</div>	P_l Legendre polynomials
Recurrence relations	$P_{m+1}^m(x) = x(2m+1)P_m^m(x)$ <div>(2.431)</div> $P_m^m(x) = (-1)^m (2m-1)!! (1-x^2)^{m/2}$ <div>(2.432)</div> $(l-m+1)P_{l+1}^m(x) = (2l+1)xP_l^m(x) - (l+m)P_{l-1}^m(x)$ <div>(2.433)</div>	!! $5!! = 5 \cdot 3 \cdot 1$ etc.
Orthogonality	$\int_{-1}^1 P_l^m(x) P_{l'}^m(x) dx = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{ll'}$ <div>(2.434)</div>	$\delta_{ll'}$ Kronecker delta
$P_0^0(x) = 1$ <div></div> $P_1^0(x) = x$ <div></div> $P_2^0(x) = (3x^2 - 1)/2$ <div></div> $P_1^1(x) = x$ <div></div> $P_2^1(x) = -3x(1-x^2)^{1/2}$ <div></div> $P_1^1(x) = -(1-x^2)^{1/2}$ <div></div> $P_2^2(x) = 3(1-x^2)$ <div></div>		

^aOf the first kind. $P_l^m(x)$ can be defined with a $(-1)^m$ factor in Equation (2.429) as well as Equation (2.430).



Spherical harmonics

Differential equation	$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_l^m + l(l+1) Y_l^m = 0$ (2.435)	Y_l^m	spherical harmonics
Definition ^a	$Y_l^m(\theta, \phi) = (-1)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos \theta) e^{im\phi}$ (2.436)	P_l^m	associated Legendre functions
Orthogonality	$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_l^{m*}(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) \sin \theta \, d\theta \, d\phi = \delta_{mm'} \delta_{ll'}$ (2.437)	Y^* $\delta_{ll'}$	complex conjugate Kronecker delta
Laplace series	$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_l^m(\theta, \phi)$ (2.438) where $a_{lm} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_l^{m*}(\theta, \phi) f(\theta, \phi) \sin \theta \, d\theta \, d\phi$ (2.439)	f	continuous function
Solution to Laplace equation	if $\nabla^2 \psi(r, \theta, \phi) = 0$, then $\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_l^m(\theta, \phi) \cdot [a_{lm} r^l + b_{lm} r^{-(l+1)}]$ (2.440)	ψ a, b	continuous function constants
$Y_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$ $Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \qquad Y_2^0(\theta, \phi) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$ $Y_2^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi} \qquad Y_2^{\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$ $Y_3^0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{7}{4\pi}} (5 \cos^2 \theta - 3) \cos \theta \qquad Y_3^{\pm 1}(\theta, \phi) = \mp \frac{1}{4} \sqrt{\frac{21}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{\pm i\phi}$ $Y_3^{\pm 2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{\pm 2i\phi} \qquad Y_3^{\pm 3}(\theta, \phi) = \mp \frac{1}{4} \sqrt{\frac{35}{4\pi}} \sin^3 \theta e^{\pm 3i\phi}$			

^aDefined for $-l \leq m \leq l$, using the sign convention of the Condon–Shortley phase. Other sign conventions are possible.

Delta functions

Kronecker delta	$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (2.441)$	δ_{ij} Kronecker delta i, j, k, \dots indices (= 1, 2 or 3)
	$\delta_{ii} = 3 \quad (2.442)$	
Three-dimensional Levi-Civita symbol (permutation tensor) ^a	$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$	ϵ_{ijk} Levi-Civita symbol (see also page 25)
	$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$	
	$\text{all other } \epsilon_{ijk} = 0$	
	$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \quad (2.444)$	
	$\delta_{ij}\epsilon_{ijk} = 0 \quad (2.445)$	
Dirac delta function	$\epsilon_{ilm}\epsilon_{jlm} = 2\delta_{ij} \quad (2.446)$	$\delta(x)$ Dirac delta function $f(x)$ smooth function of x a, b constants
	$\epsilon_{ijk}\epsilon_{ijk} = 6 \quad (2.447)$	
	$\int_a^b \delta(x) \, dx = \begin{cases} 1 & \text{if } a < 0 < b \\ 0 & \text{otherwise} \end{cases} \quad (2.448)$	
	$\int_a^b f(x)\delta(x-x_0) \, dx = f(x_0) \quad (2.449)$	
	$\delta(x-x_0)f(x) = \delta(x-x_0)f(x_0) \quad (2.450)$	
	$\delta(-x) = \delta(x) \quad (2.451)$	
	$\delta(ax) = a ^{-1}\delta(x) \quad (a \neq 0) \quad (2.452)$	
	$\delta(x) \simeq n\pi^{-1/2}e^{-n^2x^2} \quad (n \gg 1) \quad (2.453)$	

^aThe general symbol $\epsilon_{ijk\dots}$ is defined to be +1 for even permutations of the suffices, −1 for odd permutations, and 0 if a suffix is repeated. The sequence (1, 2, 3, ..., n) is taken to be even. Swapping adjacent suffices an odd (or even) number of times gives an odd (or even) permutation.

2.10 Roots of quadratic and cubic equations

Quadratic equations

Equation	$ax^2 + bx + c = 0 \quad (a \neq 0) \quad (2.454)$	x variable a, b, c real constants
Solutions	$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (2.455)$	x_1, x_2 quadratic roots
	$= \frac{-2c}{b \pm \sqrt{b^2 - 4ac}} \quad (2.456)$	
Solution combinations	$x_1 + x_2 = -b/a \quad (2.457)$	
	$x_1 x_2 = c/a \quad (2.458)$	

Cubic equations

Equation	$ax^3+bx^2+cx+d=0 \quad (a \neq 0)$	(2.459)	x a,b,c,d	variable real constants	
Intermediate definitions	$p=\frac{1}{3}\left(\frac{3c}{a}-\frac{b^2}{a^2}\right)$	(2.460)	D	discriminant	
	$q=\frac{1}{27}\left(\frac{2b^3}{a^3}-\frac{9bc}{a^2}+\frac{27d}{a}\right)$	(2.461)			
	$D=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2$	(2.462)			
If $D \geq 0$, also define:		If $D < 0$, also define:			
$u=\left(\frac{-q}{2}+D^{1/2}\right)^{1/3}$		(2.463)	$\phi=\arccos\left[\frac{-q}{2}\left(\frac{ p }{3}\right)^{-3/2}\right]$		(2.467)
$v=\left(\frac{-q}{2}-D^{1/2}\right)^{1/3}$		(2.464)	$y_1=2\left(\frac{ p }{3}\right)^{1/2}\cos\frac{\phi}{3}$		(2.468)
$y_1=u+v$		(2.465)	$y_{2,3}=-2\left(\frac{ p }{3}\right)^{1/2}\cos\frac{\phi\pm\pi}{3}$		(2.469)
$y_{2,3}=\frac{-(u+v)}{2}\pm\mathbf{i}\frac{u-v}{2}3^{1/2}$		(2.466)			
1 real, 2 complex roots (if $D=0$: 3 real roots, at least 2 equal)		3 distinct real roots			
Solutions ^a	$x_n=y_n-\frac{b}{3a}$	(2.470)	x_n	cubic roots ($n=1,2,3$)	
Solution combinations	$x_1+x_2+x_3=-b/a$	(2.471)			
	$x_1x_2+x_1x_3+x_2x_3=c/a$	(2.472)			
	$x_1x_2x_3=-d/a$	(2.473)			

^a y_n are solutions to the reduced equation $y^3 + py + q = 0$.

2.11 Fourier series and transforms

Fourier series

Real form	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (2.474)$	$f(x)$ periodic function, period $2L$ a_n, b_n Fourier coefficients
	$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad (2.475)$	
	$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad (2.476)$	
Complex form	$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left(\frac{\mathrm{i} n \pi x}{L} \right) \quad (2.477)$	c_n complex Fourier coefficient
	$c_n = \frac{1}{2L} \int_{-L}^L f(x) \exp \left(\frac{-\mathrm{i} n \pi x}{L} \right) dx \quad (2.478)$	
Parseval's theorem	$\frac{1}{2L} \int_{-L}^L f(x) ^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad (2.479)$	modulus
	$= \sum_{n=-\infty}^{\infty} c_n ^2 \quad (2.480)$	

Fourier transform^a

Definition 1	$F(s) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi \mathrm{i} x s} dx \quad (2.481)$	$f(x)$ function of x $F(s)$ Fourier transform of $f(x)$
	$f(x) = \int_{-\infty}^{\infty} F(s) \mathrm{e}^{2\pi \mathrm{i} x s} ds \quad (2.482)$	
Definition 2	$F(s) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i} x s} dx \quad (2.483)$	
	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \mathrm{e}^{\mathrm{i} x s} ds \quad (2.484)$	
Definition 3	$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i} x s} dx \quad (2.485)$	
	$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \mathrm{e}^{\mathrm{i} x s} ds \quad (2.486)$	

^aAll three (and more) definitions are used, but definition 1 is probably the best.

Fourier transform theorems^a

Convolution	$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u) du$	(2.487)	f, g general functions $*$ convolution
Convolution rules	$f * g = g * f$ $f * (g * h) = (f * g) * h$	(2.488) (2.489)	f $f(x) \rightleftharpoons F(s)$ g $g(x) \rightleftharpoons G(s)$
Convolution theorem	$f(x)g(x) \rightleftharpoons F(s) * G(s)$	(2.490)	\rightleftharpoons Fourier transform relation
Autocorrelation	$f^*(x) \star f(x) = \int_{-\infty}^{\infty} f^*(u-x)f(u) du$	(2.491)	\star correlation f^* complex conjugate of f
Wiener–Khinchine theorem	$f^*(x) \star f(x) \rightleftharpoons F(s) ^2$	(2.492)	
Cross-correlation	$f^*(x) \star g(x) = \int_{-\infty}^{\infty} f^*(u-x)g(u) du$	(2.493)	
Correlation theorem	$h(x) \star j(x) \rightleftharpoons H(s)J^*(s)$	(2.494)	h, j real functions H $H(s) \rightleftharpoons h(x)$ J $J(s) \rightleftharpoons j(x)$
Parseval’s relation ^b	$\int_{-\infty}^{\infty} f(x)g^*(x) dx = \int_{-\infty}^{\infty} F(s)G^*(s) ds$	(2.495)	
Parseval’s theorem ^c	$\int_{-\infty}^{\infty} f(x) ^2 dx = \int_{-\infty}^{\infty} F(s) ^2 ds$	(2.496)	
Derivatives	$\frac{df(x)}{dx} \rightleftharpoons 2\pi i s F(s)$ $\frac{d}{dx}[f(x) * g(x)] = \frac{df(x)}{dx} * g(x) = \frac{dg(x)}{dx} * f(x)$	(2.497) (2.498)	

^aDefining the Fourier transform as $F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixs} dx$.

^bAlso called the “power theorem.”

^cAlso called “Rayleigh’s theorem.”

Fourier symmetry relationships

$f(x)$	\rightleftharpoons	$F(s)$	definitions
even	\rightleftharpoons	even	real: $f(x) = f^*(x)$
odd	\rightleftharpoons	odd	imaginary: $f(x) = -f^*(x)$
real, even	\rightleftharpoons	real, even	even: $f(x) = f(-x)$
real, odd	\rightleftharpoons	imaginary, odd	odd: $f(x) = -f(-x)$
imaginary, even	\rightleftharpoons	imaginary, even	Hermitian: $f(x) = f^*(-x)$
complex, even	\rightleftharpoons	complex, even	anti-Hermitian: $f(x) = -f^*(-x)$
complex, odd	\rightleftharpoons	complex, odd	
real, asymmetric	\rightleftharpoons	complex, Hermitian	
imaginary, asymmetric	\rightleftharpoons	complex, anti-Hermitian	

Fourier transform pairs^a

$$f(x) \Rightarrow F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i s x} dx \quad (2.499)$$

$$f(ax) \Rightarrow \frac{1}{|a|} F(s/a) \quad (a \neq 0, \text{ real}) \quad (2.500)$$

$$f(x-a) \Rightarrow e^{-2\pi i a s} F(s) \quad (a \text{ real}) \quad (2.501)$$

$$\frac{d^n}{dx^n} f(x) \Rightarrow (2\pi i s)^n F(s) \quad (2.502)$$

$$\delta(x) \Rightarrow 1 \quad (2.503)$$

$$\delta(x-a) \Rightarrow e^{-2\pi i a s} \quad (2.504)$$

$$e^{-a|x|} \Rightarrow \frac{2a}{a^2 + 4\pi^2 s^2} \quad (a > 0) \quad (2.505)$$

$$xe^{-a|x|} \Rightarrow \frac{8i\pi as}{(a^2 + 4\pi^2 s^2)^2} \quad (a > 0) \quad (2.506)$$

$$e^{-x^2/a^2} \Rightarrow a\sqrt{\pi}e^{-\pi^2 a^2 s^2} \quad (2.507)$$

$$\sin ax \Rightarrow \frac{1}{2i} \left[\delta\left(s - \frac{a}{2\pi}\right) - \delta\left(s + \frac{a}{2\pi}\right) \right] \quad (2.508)$$

$$\cos ax \Rightarrow \frac{1}{2} \left[\delta\left(s - \frac{a}{2\pi}\right) + \delta\left(s + \frac{a}{2\pi}\right) \right] \quad (2.509)$$

$$\sum_{m=-\infty}^{\infty} \delta(x-ma) \Rightarrow \frac{1}{a} \sum_{n=-\infty}^{\infty} \delta\left(s - \frac{n}{a}\right) \quad (2.510)$$

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \quad (\text{"step"}) \Rightarrow \frac{1}{2}\delta(s) - \frac{i}{2\pi s} \quad (2.511)$$

$$f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases} \quad (\text{"top hat"}) \Rightarrow \frac{\sin 2\pi as}{\pi s} = 2a \operatorname{sinc} 2as \quad (2.512)$$

$$f(x) = \begin{cases} \left(1 - \frac{|x|}{a}\right) & |x| \leq a \\ 0 & |x| > a \end{cases} \quad (\text{"triangle"}) \Rightarrow \frac{1}{2\pi^2 a s^2} (1 - \cos 2\pi as) = a \operatorname{sinc}^2 as \quad (2.513)$$

^aEquation (2.499) defines the Fourier transform used for these pairs. Note that $\operatorname{sinc} x \equiv (\sin \pi x)/(\pi x)$.

2.12 Laplace transforms

Laplace transform theorems

Definition ^a	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st} dt \quad (2.514)$	$\mathcal{L}\{\}$	Laplace transform
Convolution ^b	$F(s) \cdot G(s) = \mathcal{L}\left\{\int_0^\infty f(t-z)g(z) dz\right\} \quad (2.515)$ $= \mathcal{L}\{f(t) * g(t)\} \quad (2.516)$	$F(s)$ $G(s)$ $*$	$\mathcal{L}\{f(t)\}$ $\mathcal{L}\{g(t)\}$ convolution
Inverse ^c	$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds \quad (2.517)$ $= \sum \text{residues} \quad (\text{for } t > 0) \quad (2.518)$	γ	constant
Transform of derivative	$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n \mathcal{L}\{f(t)\} - \sum_{r=0}^{n-1} s^{n-r-1} \left.\frac{d^r f(t)}{dt^r}\right _{t=0} \quad (2.519)$	n	integer > 0
Derivative of transform	$\frac{d^n F(s)}{ds^n} = \mathcal{L}\{(-t)^n f(t)\} \quad (2.520)$		
Substitution	$F(s-a) = \mathcal{L}\{e^{at} f(t)\} \quad (2.521)$	a	constant
Translation	$e^{-as} F(s) = \mathcal{L}\{u(t-a)f(t-a)\} \quad (2.522)$ $\text{where } u(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t > 0) \end{cases} \quad (2.523)$	$u(t)$	unit step function

^aIf $|e^{-s_0 t} f(t)|$ is finite for sufficiently large t , the Laplace transform exists for $s > s_0$.

^bAlso known as the “faltung (or folding) theorem.”

^cAlso known as the “Bromwich integral.” γ is chosen so that the singularities in $F(s)$ are left of the integral line.

Laplace transform pairs

$$f(t) \Rightarrow F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt \quad (2.524)$$

$$\delta(t) \Rightarrow 1 \quad (2.525)$$

$$1 \Rightarrow 1/s \quad (s > 0) \quad (2.526)$$

$$t^n \Rightarrow \frac{n!}{s^{n+1}} \quad (s > 0, n > -1) \quad (2.527)$$

$$t^{1/2} \Rightarrow \sqrt{\frac{\pi}{4s^3}} \quad (2.528)$$

$$t^{-1/2} \Rightarrow \sqrt{\frac{\pi}{s}} \quad (2.529)$$

$$e^{at} \Rightarrow \frac{1}{s-a} \quad (s > a) \quad (2.530)$$

$$te^{at} \Rightarrow \frac{1}{(s-a)^2} \quad (s > a) \quad (2.531)$$

$$(1-at)e^{-at} \Rightarrow \frac{s}{(s+a)^2} \quad (2.532)$$

$$t^2e^{-at} \Rightarrow \frac{2}{(s+a)^3} \quad (2.533)$$

$$\sin at \Rightarrow \frac{a}{s^2 + a^2} \quad (s > 0) \quad (2.534)$$

$$\cos at \Rightarrow \frac{s}{s^2 + a^2} \quad (s > 0) \quad (2.535)$$

$$\sinh at \Rightarrow \frac{a}{s^2 - a^2} \quad (s > a) \quad (2.536)$$

$$\cosh at \Rightarrow \frac{s}{s^2 - a^2} \quad (s > a) \quad (2.537)$$

$$e^{-bt} \sin at \Rightarrow \frac{a}{(s+b)^2 + a^2} \quad (2.538)$$

$$e^{-bt} \cos at \Rightarrow \frac{s+b}{(s+b)^2 + a^2} \quad (2.539)$$

$$e^{-at} f(t) \Rightarrow F(s+a) \quad (2.540)$$

2.13 Probability and statistics



Discrete statistics

Mean	$\langle x \rangle = \frac{1}{N} \sum_{i=1}^N x_i$	(2.541)	x_i	data series
			N	series length
			$\langle \cdot \rangle$	mean value
Variance ^a	$\text{var}[x] = \frac{1}{N-1} \sum_{i=1}^N (x_i - \langle x \rangle)^2$	(2.542)	$\text{var}[\cdot]$	unbiased variance
Standard deviation	$\sigma[x] = (\text{var}[x])^{1/2}$	(2.543)	σ	standard deviation
Skewness	$\text{skew}[x] = \frac{N}{(N-1)(N-2)} \sum_{i=1}^N \left(\frac{x_i - \langle x \rangle}{\sigma} \right)^3$	(2.544)		
Kurtosis	$\text{kurt}[x] \simeq \left[\frac{1}{N} \sum_{i=1}^N \left(\frac{x_i - \langle x \rangle}{\sigma} \right)^4 \right] - 3$	(2.545)		
Correlation coefficient ^b	$r = \frac{\sum_{i=1}^N (x_i - \langle x \rangle)(y_i - \langle y \rangle)}{\sqrt{\sum_{i=1}^N (x_i - \langle x \rangle)^2} \sqrt{\sum_{i=1}^N (y_i - \langle y \rangle)^2}}$	(2.546)	x, y	data series to correlate
			r	correlation coefficient

^aIf $\langle x \rangle$ is derived from the data, $\{x_i\}$, the relation is as shown. If $\langle x \rangle$ is known independently, then an unbiased estimate is obtained by dividing the right-hand side by N rather than $N-1$.

^bAlso known as “Pearson’s r .”

Discrete probability distributions

distribution	$\text{pr}(x)$	mean	variance	domain		
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$	$(x=0, 1, \dots, n)$	(2.547)	$\binom{n}{x}$ binomial coefficient
Geometric	$(1-p)^{x-1} p$	$1/p$	$(1-p)/p^2$	$(x=1, 2, 3, \dots)$	(2.548)	
Poisson	$\lambda^x \exp(-\lambda)/x!$	λ	λ	$(x=0, 1, 2, \dots)$	(2.549)	

Continuous probability distributions

distribution	pr(x)	mean	variance	domain	
Uniform	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$(a \leq x \leq b)$	(2.550)
Exponential	$\lambda \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$	$(x \geq 0)$	(2.551)
Normal/ Gaussian	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$	μ	σ^2	$(-\infty < x < \infty)$	(2.552)
Chi-squared ^a	$\frac{e^{-x/2}x^{(r/2)-1}}{2^{r/2}\Gamma(r/2)}$	r	$2r$	$(x \geq 0)$	(2.553)
Rayleigh	$\frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right)$	$\sigma\sqrt{\pi/2}$	$2\sigma^2\left(1-\frac{\pi}{4}\right)$	$(x \geq 0)$	(2.554)
Cauchy/ Lorentzian	$\frac{a}{\pi(a^2+x^2)}$	(none)	(none)	$(-\infty < x < \infty)$	(2.555)

^aWith r degrees of freedom. Γ is the gamma function.

Multivariate normal distribution

Density function	$\text{pr}(\mathbf{x}) = \frac{\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T\right]}{(2\pi)^{k/2}[\det(\mathbf{C})]^{1/2}}$	(2.556)	pr k \mathbf{C} \mathbf{x} $\boldsymbol{\mu}$ T det μ_i σ_{ij} r x_i y_i	probability density number of dimensions covariance matrix variable (k dimensional) vector of means transpose determinant mean of i th variable components of \mathbf{C} correlation coefficient normally distributed deviates deviates distributed uniformly between 0 and 1
Mean	$\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$	(2.557)		
Covariance	$\mathbf{C} = \sigma_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$	(2.558)		
Correlation coefficient	$r = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$	(2.559)		
Box–Muller transformation	$x_1 = (-2 \ln y_1)^{1/2} \cos 2\pi y_2$ $x_2 = (-2 \ln y_1)^{1/2} \sin 2\pi y_2$	(2.560) (2.561)		

Random walk

One-dimensional	$\text{pr}(x) = \frac{1}{(2\pi Nl^2)^{1/2}} \exp\left(\frac{-x^2}{2Nl^2}\right) \quad (2.562)$	<p>x displacement after N steps (can be positive or negative)</p> <p>$\text{pr}(x)$ probability density of x ($\int_{-\infty}^{\infty} \text{pr}(x) dx = 1$)</p> <p>$N$ number of steps</p> <p>l step length (all equal)</p>
rms displacement	$x_{\text{rms}} = N^{1/2}l \quad (2.563)$	x_{rms} root-mean-squared displacement from start point
Three-dimensional	$\text{pr}(r) = \left(\frac{a}{\pi^{1/2}}\right)^3 \exp(-a^2r^2) \quad (2.564)$ <p>where $a = \left(\frac{3}{2Nl^2}\right)^{1/2}$</p>	<p>r radial distance from start point</p> <p>$\text{pr}(r)$ probability density of r ($\int_0^{\infty} 4\pi r^2 \text{pr}(r) dr = 1$)</p> <p>$a$ (most probable distance)⁻¹</p>
Mean distance	$\langle r \rangle = \left(\frac{8}{3\pi}\right)^{1/2} N^{1/2}l \quad (2.565)$	$\langle r \rangle$ mean distance from start point
rms distance	$r_{\text{rms}} = N^{1/2}l \quad (2.566)$	r_{rms} root-mean-squared distance from start point

Bayesian inference

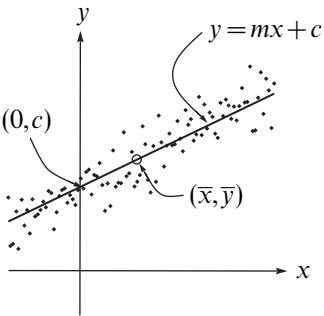
Conditional probability	$\text{pr}(x) = \int \text{pr}(x y') \text{pr}(y') dy' \quad (2.567)$	<p>$\text{pr}(x)$ probability (density) of x</p> <p>$\text{pr}(x y')$ conditional probability of x given y'</p>
Joint probability	$\text{pr}(x, y) = \text{pr}(x) \text{pr}(y x) \quad (2.568)$	$\text{pr}(x, y)$ joint probability of x and y
Bayes' theorem ^a	$\text{pr}(y x) = \frac{\text{pr}(x y) \text{pr}(y)}{\text{pr}(x)} \quad (2.569)$	

^aIn this expression, $\text{pr}(y|x)$ is known as the posterior probability, $\text{pr}(x|y)$ the likelihood, and $\text{pr}(y)$ the prior probability.

2.14 Numerical methods

Straight-line fitting^a

Data	$(\{x_i\}, \{y_i\})$ n points	(2.570)
Weights ^b	$\{w_i\}$	(2.571)
Model	$y = mx + c$	(2.572)
Residuals	$d_i = y_i - mx_i - c$	(2.573)
Weighted centre	$(\bar{x}, \bar{y}) = \frac{1}{\sum w_i} \left(\sum w_i x_i, \sum w_i y_i \right)$	(2.574)
Weighted moment	$D = \sum w_i (x_i - \bar{x})^2$	(2.575)
Gradient	$m = \frac{1}{D} \sum w_i (x_i - \bar{x}) y_i$	(2.576)
	$\text{var}[m] \simeq \frac{1}{D} \frac{\sum w_i d_i^2}{n-2}$	(2.577)
Intercept	$c = \bar{y} - m\bar{x}$	(2.578)
	$\text{var}[c] \simeq \left(\frac{1}{\sum w_i} + \frac{\bar{x}^2}{D} \right) \frac{\sum w_i d_i^2}{n-2}$	(2.579)



^aLeast-squares fit of data to $y = mx + c$. Errors on y -values only.

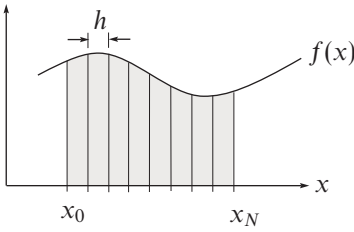
^bIf the errors on y_i are uncorrelated, then $w_i = 1/\text{var}[y_i]$.

Time series analysis^a

Discrete convolution	$(r \star s)_j = \sum_{k=-(M/2)+1}^{M/2} s_{j-k} r_k$	(2.580)	r_i response function s_i time series M response function duration
Bartlett (triangular) window	$w_j = 1 - \left \frac{j - N/2}{N/2} \right $	(2.581)	w_j windowing function N length of time series
Welch (quadratic) window	$w_j = 1 - \left[\frac{j - N/2}{N/2} \right]^2$	(2.582)	
Hanning window	$w_j = \frac{1}{2} \left[1 - \cos \left(\frac{2\pi j}{N} \right) \right]$	(2.583)	
Hamming window	$w_j = 0.54 - 0.46 \cos \left(\frac{2\pi j}{N} \right)$	(2.584)	

^aThe time series runs from $j = 0 \dots (N - 1)$, and the windowing functions peak at $j = N/2$.

Numerical integration

		<div><div>2</div></div> <div>h $= (x_N - x_0)/N$ (subinterval width) f_i $f_i = f(x_i)$ N number of subintervals</div>
Trapezoidal rule	$\int_{x_0}^{x_N} f(x) \, dx \simeq \frac{h}{2} (f_0 + 2f_1 + 2f_2 + \cdots + 2f_{N-1} + f_N) \quad (2.585)$	
Simpson's rule ^a	$\int_{x_0}^{x_N} f(x) \, dx \simeq \frac{h}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 4f_{N-1} + f_N) \quad (2.586)$	

^a N must be even. Simpson's rule is exact for quadratics and cubics.

Numerical differentiation^a

$\frac{df}{dx} \simeq \frac{1}{12h} [-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)] \quad (2.587)$	
$\sim \frac{1}{2h} [f(x+h) - f(x-h)] \quad (2.588)$	
$\frac{d^2f}{dx^2} \simeq \frac{1}{12h^2} [-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)] \quad (2.589)$	
$\sim \frac{1}{h^2} [f(x+h) - 2f(x) + f(x-h)] \quad (2.590)$	
$\frac{d^3f}{dx^3} \sim \frac{1}{2h^3} [f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)] \quad (2.591)$	

^aDerivatives of $f(x)$ at x . h is a small interval in x .
Relations containing “ \simeq ” are $O(h^4)$; those containing “ \sim ” are $O(h^2)$.

Numerical solutions to $f(x)=0$

Secant method	$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \quad (2.592)$	f function of x x_n $f(x_\infty)=0$ f' $= df/dx$
Newton–Raphson method	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (2.593)$	

Numerical solutions to ordinary differential equations^a

Euler’s method	if	$\frac{dy}{dx} = f(x,y)$	(2.594)
	and	$h = x_{n+1} - x_n$	(2.595)
	then	$y_{n+1} = y_n + hf(x_n,y_n) + O(h^2)$	(2.596)
Runge–Kutta method (fourth-order)	if	$\frac{dy}{dx} = f(x,y)$	(2.597)
	and	$h = x_{n+1} - x_n$	(2.598)
		$k_1 = hf(x_n,y_n)$	(2.599)
		$k_2 = hf(x_n + h/2, y_n + k_1/2)$	(2.600)
		$k_3 = hf(x_n + h/2, y_n + k_2/2)$	(2.601)
		$k_4 = hf(x_n + h, y_n + k_3)$	(2.602)
	then	$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)$	(2.603)

^aOrdinary differential equations (ODEs) of the form $\frac{dy}{dx} = f(x,y)$. Higher order equations should be reduced to a set of coupled first-order equations and solved in parallel.

Chapter 3 Dynamics and mechanics

3.1 Introduction

Unusually in physics, there is no pithy phrase that sums up the study of *dynamics* (the way in which forces produce motion), *kinematics* (the motion of matter), *mechanics* (the study of the forces and the motion they produce), and *statics* (the way forces combine to produce equilibrium). We will take the phrase *dynamics and mechanics* to encompass all the above, although it clearly does not!

To some extent this is because the equations governing the motion of matter include some of our oldest insights into the physical world and are consequentially steeped in tradition. One of the more delightful, or for some annoying, facets of this is the occasional use of arcane vocabulary in the description of motion. The epitome must be what Goldstein¹ calls “the jabberwockian sounding statement” *the polhode rolls without slipping on the herpolhode lying in the invariable plane*, describing “Poinsot’s construction” – a method of visualising the free motion of a spinning rigid body. Despite this, dynamics and mechanics, including fluid mechanics, is arguably the most practically applicable of all the branches of physics.

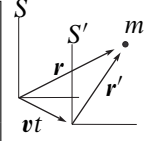
Moreover, and in common with electromagnetism, the study of dynamics and mechanics has spawned a good deal of mathematical apparatus that has found uses in other fields. Most notably, the ideas behind the generalised dynamics of Lagrange and Hamilton lie behind much of quantum mechanics.

¹H. Goldstein, *Classical Mechanics*, 2nd ed., 1980, Addison-Wesley.

3.2 Frames of reference

Galilean transformations

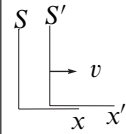
Time and position ^a	$\mathbf{r} = \mathbf{r}' + \mathbf{v}t$	(3.1)	\mathbf{r}, \mathbf{r}'	position in frames S and S'
	$t = t'$	(3.2)	\mathbf{v}	velocity of S' in S
Velocity	$\mathbf{u} = \mathbf{u}' + \mathbf{v}$	(3.3)	t, t'	time in S and S'
Momentum	$\mathbf{p} = \mathbf{p}' + m\mathbf{v}$	(3.4)	\mathbf{u}, \mathbf{u}'	velocity in frames S and S'
Angular momentum	$\mathbf{J} = \mathbf{J}' + m\mathbf{r}' \times \mathbf{v} + \mathbf{v} \times \mathbf{p}'t$	(3.5)	\mathbf{p}, \mathbf{p}'	particle momentum in frames S and S'
Kinetic energy	$T = T' + m\mathbf{u}' \cdot \mathbf{v} + \frac{1}{2}mv^2$	(3.6)	m	particle mass
			\mathbf{J}, \mathbf{J}'	angular momentum in frames S and S'
			T, T'	kinetic energy in frames S and S'



^aFrames coincide at $t=0$.

Lorentz (spacetime) transformations^a

Lorentz factor	$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$	(3.7)	γ	Lorentz factor
Time and position			v	velocity of S' in S
$x = \gamma(x' + vt')$	$x' = \gamma(x - vt)$	(3.8)	c	speed of light
$y = y'$	$y' = y$	(3.9)	x, x'	x-position in frames S and S' (similarly for y and z)
$z = z'$	$z' = z$	(3.10)	t, t'	time in frames S and S'
$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$	$t' = \gamma\left(t - \frac{v}{c^2}x\right)$	(3.11)	X	spacetime four-vector
Differential four-vector ^b	$dX = (cdt, -dx, -dy, -dz)$	(3.12)		

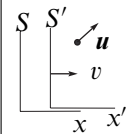


^aFor frames S and S' coincident at $t=0$ in relative motion along x . See page 141 for the transformations of electromagnetic quantities.

^bCovariant components, using the $(1, -1, -1, -1)$ signature.

Velocity transformations^a

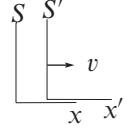
Velocity			γ	Lorentz factor $= [1 - (v/c)^2]^{-1/2}$
$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$	$u'_x = \frac{u_x - v}{1 - u_x v/c^2}$	(3.13)	v	velocity of S' in S
$u_y = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}$	$u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)}$	(3.14)	c	speed of light
$u_z = \frac{u'_z}{\gamma(1 + u'_x v/c^2)}$	$u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)}$	(3.15)	u_i, u'_i	particle velocity components in frames S and S'



^aFor frames S and S' coincident at $t=0$ in relative motion along x .

Momentum and energy transformations^a

Momentum and energy		γ	Lorentz factor = $[1 - (v/c)^2]^{-1/2}$
$p_x = \gamma(p'_x + vE'/c^2);$	$p'_x = \gamma(p_x - vE/c^2)$	v	velocity of S' in S
$p_y = p'_y;$	$p'_y = p_y$	c	speed of light
$p_z = p'_z;$	$p'_z = p_z$	p_x, p'_x	x components of momentum in S and S' (sim. for y and z)
$E = \gamma(E' + vp'_x);$	$E' = \gamma(E - vp_x)$	E, E'	energy in S and S'
$E^2 - p^2c^2 = E'^2 - p'^2c^2 = m_0^2c^4$		m_0	(rest) mass
		p	total momentum in S
Four-vector ^b	$\mathbf{P} = (E/c, -p_x, -p_y, -p_z)$	\mathbf{P}	momentum four-vector

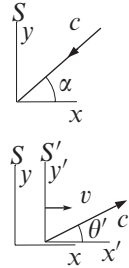


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^aFor frames S and S' coincident at $t=0$ in relative motion along x .^bCovariant components, using the $(1, -1, -1, -1)$ signature.

Propagation of light^a

Doppler effect	$\frac{v'}{v} = \gamma \left(1 + \frac{v}{c} \cos \alpha \right)$	v	frequency received in S
		v'	frequency emitted in S'
		α	arrival angle in S
Aberration ^b	$\cos \theta = \frac{\cos \theta' + v/c}{1 + (v/c) \cos \theta'}$	γ	Lorentz factor = $[1 - (v/c)^2]^{-1/2}$
	$\cos \theta' = \frac{\cos \theta - v/c}{1 - (v/c) \cos \theta}$	v	velocity of S' in S
		c	speed of light
		θ, θ'	emission angle of light in S and S'
Relativistic beaming ^c	$P(\theta) = \frac{\sin \theta}{2\gamma^2 [1 - (v/c) \cos \theta]^2}$	$P(\theta)$	angular distribution of photons in S

^aFor frames S and S' coincident at $t=0$ in relative motion along x .^bLight travelling in the opposite sense has a propagation angle of $\pi + \theta$ radians.^cAngular distribution of photons from a source, isotropic and stationary in S' . $\int_0^\pi P(\theta) d\theta = 1$.

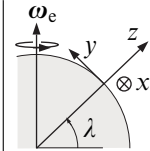
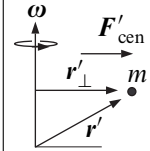
Four-vectors^a

Covariant and contravariant components	$x_0 = x^0$ $x_1 = -x^1$ $x_2 = -x^2$ $x_3 = -x^3$	x_i	covariant vector components
		x^i	contravariant components
Scalar product	$x^i y_i = x^0 y_0 + x^1 y_1 + x^2 y_2 + x^3 y_3$		
Lorentz transformations		x^i, x'^i	four-vector components in frames S and S'
$x^0 = \gamma[x'^0 + (v/c)x'^1];$	$x'^0 = \gamma[x^0 - (v/c)x^1]$	γ	Lorentz factor = $[1 - (v/c)^2]^{-1/2}$
$x^1 = \gamma[x'^1 + (v/c)x'^0];$	$x'^1 = \gamma[x^1 - (v/c)x^0]$	v	velocity of S' in S
$x^2 = x'^2;$	$x'^3 = x^3$	c	speed of light

^aFor frames S and S' , coincident at $t=0$ in relative motion along the (1) direction. Note that the $(1, -1, -1, -1)$ signature used here is common in special relativity, whereas $(-1, 1, 1, 1)$ is often used in connection with general relativity (page 67).

Rotating frames

Vector transformation	$\left[\frac{d\mathbf{A}}{dt} \right]_S = \left[\frac{d\mathbf{A}}{dt} \right]_{S'} + \boldsymbol{\omega} \times \mathbf{A} \quad (3.31)$	\mathbf{A} any vector S stationary frame S' rotating frame $\boldsymbol{\omega}$ angular velocity of S' in S
Acceleration	$\dot{\mathbf{v}} = \dot{\mathbf{v}}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \quad (3.32)$	$\dot{\mathbf{v}}, \dot{\mathbf{v}}'$ accelerations in S and S' \mathbf{v}' velocity in S' \mathbf{r}' position in S'
Coriolis force	$\mathbf{F}'_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}' \quad (3.33)$	\mathbf{F}'_{cor} coriolis force m particle mass
Centrifugal force	$\mathbf{F}'_{\text{cen}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \quad (3.34)$	\mathbf{F}'_{cen} centrifugal force
	$= +m\omega^2 \mathbf{r}'_{\perp} \quad (3.35)$	\mathbf{r}'_{\perp} perpendicular to particle from rotation axis
Motion relative to Earth	$m\ddot{\mathbf{x}} = F_x + 2m\omega_e(\dot{y} \sin \lambda - \dot{z} \cos \lambda) \quad (3.36)$	F_i nongravitational force
	$m\ddot{y} = F_y - 2m\omega_e \dot{x} \sin \lambda \quad (3.37)$	λ latitude
	$m\ddot{z} = F_z - mg + 2m\omega_e \dot{x} \cos \lambda \quad (3.38)$	z local vertical axis
		y northerly axis x easterly axis
Foucault's pendulum ^a	$\Omega_f = -\omega_e \sin \lambda \quad (3.39)$	Ω_f pendulum's rate of turn ω_e Earth's spin rate

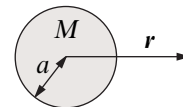


^aThe sign is such as to make the rotation clockwise in the northern hemisphere.

3.3 Gravitation

Newtonian gravitation

Newton's law of gravitation	$\mathbf{F}_1 = \frac{Gm_1m_2}{r_{12}^2} \hat{\mathbf{r}}_{12} \quad (3.40)$	$m_{1,2}$ masses \mathbf{F}_1 force on m_1 ($= -\mathbf{F}_2$) \mathbf{r}_{12} vector from m_1 to m_2 $\hat{\mathbf{r}}$ unit vector
Newtonian field equations ^a	$\mathbf{g} = -\nabla \phi \quad (3.41)$	G constant of gravitation
	$\nabla^2 \phi = -\nabla \cdot \mathbf{g} = 4\pi G\rho \quad (3.42)$	\mathbf{g} gravitational field strength ϕ gravitational potential ρ mass density
Fields from an isolated uniform sphere, mass M , \mathbf{r} from the centre	$\mathbf{g}(\mathbf{r}) = \begin{cases} -\frac{GM}{r^2} \hat{\mathbf{r}} & (r > a) \\ -\frac{GM}{a^3} \mathbf{r} & (r < a) \end{cases} \quad (3.43)$	\mathbf{r} vector from sphere centre M mass of sphere a radius of sphere
	$\phi(\mathbf{r}) = \begin{cases} -\frac{GM}{r} & (r > a) \\ \frac{GM}{2a^3} (r^2 - 3a^2) & (r < a) \end{cases} \quad (3.44)$	



^aThe gravitational force on a mass m is $m\mathbf{g}$.

General relativity^a

Line element	$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -d\tau^2$	(3.45)	ds	invariant interval
			dτ	proper time interval
			$g_{\mu\nu}$	metric tensor
			dx^μ	differential of x^μ
Christoffel symbols and covariant differentiation	$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$	(3.46)	$\Gamma^\alpha_{\beta\gamma}$	Christoffel symbols
	$\phi_{;\gamma} = \phi_{,\gamma} \equiv \partial\phi/\partial x^\gamma$	(3.47)	$_{,\alpha}$	partial diff. w.r.t. x^α
	$A^\alpha_{;\gamma} = A^\alpha_{,\gamma} + \Gamma^\alpha_{\beta\gamma} A^\beta$	(3.48)	$_{;\alpha}$	covariant diff. w.r.t. x^α
	$B_{\alpha;\gamma} = B_{\alpha,\gamma} - \Gamma^\beta_{\alpha\gamma} B_\beta$	(3.49)	ϕ	scalar
			A^α	contravariant vector
			B_α	covariant vector
	$R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\mu\gamma} \Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta} \Gamma^\mu_{\beta\gamma} + \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta}$	(3.50)		
Riemann tensor	$B_{\mu;\alpha;\beta} - B_{\mu;\beta;\alpha} = R^\gamma_{\mu\alpha\beta} B_\gamma$	(3.51)	$R^\alpha_{\beta\gamma\delta}$	Riemann tensor
	$R_{\alpha\beta\gamma\delta} = -R_{\alpha\delta\beta\gamma} ; \quad R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}$	(3.52)		
	$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$	(3.53)		
Geodesic equation	$\frac{Dv^\mu}{D\lambda} = 0$	(3.54)	v^μ	tangent vector (= $dx^\mu/d\lambda$)
	where $\frac{DA^\mu}{D\lambda} \equiv \frac{dA^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} A^\alpha v^\beta$	(3.55)	λ	affine parameter (e.g., τ for material particles)
Geodesic deviation	$\frac{D^2 \xi^\mu}{D\lambda^2} = -R^\mu_{\alpha\beta\gamma} v^\alpha \xi^\beta v^\gamma$	(3.56)	ξ^μ	geodesic deviation
Ricci tensor	$R_{\alpha\beta} \equiv R^\sigma_{\alpha\sigma\beta} = g^{\sigma\delta} R_{\delta\alpha\sigma\beta} = R_{\beta\alpha}$	(3.57)	$R_{\alpha\beta}$	Ricci tensor
Einstein tensor	$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$	(3.58)	$G^{\mu\nu}$	Einstein tensor
			R	Ricci scalar (= $g^{\mu\nu} R_{\mu\nu}$)
Einstein's field equations	$G^{\mu\nu} = 8\pi T^{\mu\nu}$	(3.59)	$T^{\mu\nu}$	stress-energy tensor
			p	pressure (in rest frame)
Perfect fluid	$T^{\mu\nu} = (p + \rho) u^\mu u^\nu + p g^{\mu\nu}$	(3.60)	ρ	density (in rest frame)
			u^ν	fluid four-velocity
Schwarzschild solution (exterior)	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$	(3.61)	M	spherically symmetric mass (see page 183)
			(r, θ, ϕ)	spherical polar coords.
			t	time
Kerr solution (outside a spinning black hole)	$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\varrho^2} dt^2 - 2a \frac{2Mr \sin^2\theta}{\varrho^2} dt d\phi + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}{\varrho^2} \sin^2\theta d\phi^2 + \frac{\varrho^2}{\Delta} dr^2 + \varrho^2 d\theta^2$	(3.62)	J	angular momentum (along z)
			a	$\equiv J/M$
			Δ	$\equiv r^2 - 2Mr + a^2$
			ϱ^2	$\equiv r^2 + a^2 \cos^2\theta$

^aGeneral relativity conventionally uses the $(-1, 1, 1, 1)$ metric signature and “geometrized units” in which $G = 1$ and $c = 1$. Thus, $1 \text{ kg} = 7.425 \times 10^{-28} \text{ m}$ etc. Contravariant indices are written as superscripts and covariant indices as subscripts. Note also that ds^2 means $(ds)^2$ etc.

3.4 Particle motion

Dynamics definitions^a

Newtonian force	$\mathbf{F} = m\ddot{\mathbf{r}} = \dot{\mathbf{p}}$	(3.63)	\mathbf{F} force m mass of particle \mathbf{r} particle position vector
Momentum	$\mathbf{p} = m\dot{\mathbf{r}}$	(3.64)	\mathbf{p} momentum
Kinetic energy	$T = \frac{1}{2}mv^2$	(3.65)	T kinetic energy v particle velocity
Angular momentum	$\mathbf{J} = \mathbf{r} \times \mathbf{p}$	(3.66)	\mathbf{J} angular momentum
Couple (or torque)	$\mathbf{G} = \mathbf{r} \times \mathbf{F}$	(3.67)	\mathbf{G} couple
Centre of mass (ensemble of N particles)	$\mathbf{R}_0 = \frac{\sum_{i=1}^N m_i \mathbf{r}_i}{\sum_{i=1}^N m_i}$	(3.68)	\mathbf{R}_0 position vector of centre of mass m_i mass of i th particle \mathbf{r}_i position vector of i th particle

^aIn the Newtonian limit, $v \ll c$, assuming m is constant.

Relativistic dynamics^a

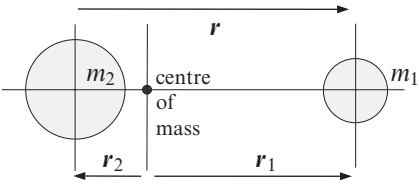
Lorentz factor	$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$	(3.69)	γ Lorentz factor v particle velocity c speed of light
Momentum	$\mathbf{p} = \gamma m_0 \mathbf{v}$	(3.70)	\mathbf{p} relativistic momentum m_0 particle (rest) mass
Force	$\mathbf{F} = \frac{d\mathbf{p}}{dt}$	(3.71)	\mathbf{F} force on particle t time
Rest energy	$E_r = m_0 c^2$	(3.72)	E_r particle rest energy
Kinetic energy	$T = m_0 c^2 (\gamma - 1)$	(3.73)	T relativistic kinetic energy
Total energy	$E = \gamma m_0 c^2$	(3.74)	E total energy ($= E_r + T$)
	$= (p^2 c^2 + m_0^2 c^4)^{1/2}$	(3.75)	

^aIt is now common to regard mass as a Lorentz invariant property and to drop the term “rest mass.” The symbol m_0 is used here to avoid confusion with the idea of “relativistic mass” ($= \gamma m_0$) used by some authors.

Constant acceleration

$v = u + at$	(3.76)	u initial velocity
$v^2 = u^2 + 2as$	(3.77)	v final velocity
$s = ut + \frac{1}{2}at^2$	(3.78)	t time
$s = \frac{u+v}{2}t$	(3.79)	s distance travelled
		a acceleration

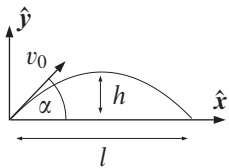
Reduced mass (of two interacting bodies)

				
Reduced mass	$\mu = \frac{m_1 m_2}{m_1 + m_2}$	(3.80)	μ	reduced mass
Distances from centre of mass	$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2} \mathbf{r}$	(3.81)	m_i	interacting masses
	$\mathbf{r}_2 = \frac{-m_1}{m_1 + m_2} \mathbf{r}$	(3.82)	\mathbf{r}_i	position vectors from centre of mass
Moment of inertia	$I = \mu \mathbf{r} ^2$	(3.83)	\mathbf{r}	$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$
Total angular momentum	$\mathbf{J} = \mu \mathbf{r} \times \dot{\mathbf{r}}$	(3.84)	$ \mathbf{r} $	distance between masses
Lagrangian	$L = \frac{1}{2} \mu \dot{\mathbf{r}} ^2 - U(\mathbf{r})$	(3.85)	I	moment of inertia
			\mathbf{J}	angular momentum
			L	Lagrangian
			U	potential energy of interaction

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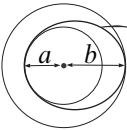
Ballistics^a

Velocity	$\mathbf{v} = v_0 \cos \alpha \hat{\mathbf{x}} + (v_0 \sin \alpha - gt) \hat{\mathbf{y}}$	(3.86)	v_0	initial velocity
	$v^2 = v_0^2 - 2gy$	(3.87)	\mathbf{v}	velocity at t
Trajectory	$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$	(3.88)	α	elevation angle
			g	gravitational acceleration
Maximum height	$h = \frac{v_0^2}{2g} \sin^2 \alpha$	(3.89)	$\hat{}$	unit vector
Horizontal range	$l = \frac{v_0^2}{g} \sin 2\alpha$	(3.90)	t	time
			h	maximum height
			l	range



^aIgnoring the curvature and rotation of the Earth and frictional losses, g is assumed constant.

Rocketry

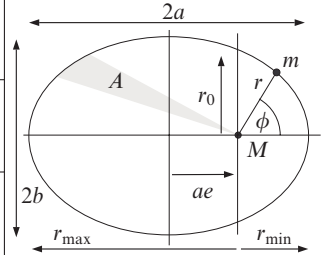
Escape velocity ^a	$v_{\text{esc}} = \left(\frac{2GM}{r} \right)^{1/2} \quad (3.91)$	v_{esc} escape velocity G constant of gravitation M mass of central body r central body radius
Specific impulse	$I_{\text{sp}} = \frac{u}{g} \quad (3.92)$	I_{sp} specific impulse u effective exhaust velocity g acceleration due to gravity
Exhaust velocity (into a vacuum)	$u = \left[\frac{2\gamma RT_c}{(\gamma - 1)\mu} \right]^{1/2} \quad (3.93)$	R molar gas constant γ ratio of heat capacities T_c combustion temperature μ effective molecular mass of exhaust gas
Rocket equation ($g = 0$)	$\Delta v = u \ln \left(\frac{M_i}{M_f} \right) \equiv u \ln \mathcal{M} \quad (3.94)$	Δv rocket velocity increment M_i pre-burn rocket mass M_f post-burn rocket mass \mathcal{M} mass ratio
Multistage rocket	$\Delta v = \sum_{i=1}^N u_i \ln \mathcal{M}_i \quad (3.95)$	N number of stages \mathcal{M}_i mass ratio for i th burn u_i exhaust velocity of i th burn
In a constant gravitational field	$\Delta v = u \ln \mathcal{M} - gt \cos \theta \quad (3.96)$	t burn time θ rocket zenith angle
Hohmann cotangential transfer ^b	$\Delta v_{ah} = \left(\frac{GM}{r_a} \right)^{1/2} \left[\left(\frac{2r_b}{r_a + r_b} \right)^{1/2} - 1 \right] \quad (3.97)$ $\Delta v_{hb} = \left(\frac{GM}{r_b} \right)^{1/2} \left[1 - \left(\frac{2r_a}{r_a + r_b} \right)^{1/2} \right] \quad (3.98)$	Δv_{ah} velocity increment, a to h Δv_{hb} velocity increment, h to b r_a radius of inner orbit r_b radius of outer orbit 

^aFrom the surface of a spherically symmetric, nonrotating body, mass M .

^bTransfer between coplanar, circular orbits a and b , via ellipse h with a minimal expenditure of energy.

Gravitationally bound orbital motion^a

Potential energy of interaction	$U(r) = -\frac{GMm}{r} \equiv -\frac{\alpha}{r}$	(3.99)	$U(r)$ potential energy G constant of gravitation M central mass m orbiting mass ($\ll M$) α GMm (for gravitation)
Total energy	$E = -\frac{\alpha}{r} + \frac{J^2}{2mr^2} = -\frac{\alpha}{2a}$	(3.100)	E total energy (constant) J total angular momentum (constant)
Virial theorem	$E = \langle U \rangle / 2 = -\langle T \rangle$	(3.101)	T kinetic energy
($1/r$ potential)	$\langle U \rangle = -2\langle T \rangle$	(3.102)	$\langle \cdot \rangle$ mean value
Orbital equation	$\frac{r_0}{r} = 1 + e \cos \phi$, or	(3.103)	r_0 semi-latus-rectum
(Kepler's 1st law)	$r = \frac{a(1-e^2)}{1+e \cos \phi}$	(3.104)	r distance of m from M e eccentricity ϕ phase (true anomaly)
Rate of sweeping area	$\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$	(3.105)	A area swept out by radius vector (total area = πab)
(Kepler's 2nd law)			
Semi-major axis	$a = \frac{r_0}{1-e^2} = \frac{\alpha}{2 E }$	(3.106)	a semi-major axis b semi-minor axis
Semi-minor axis	$b = \frac{r_0}{(1-e^2)^{1/2}} = \frac{J}{(2m E)^{1/2}}$	(3.107)	
Eccentricity ^b	$e = \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2} = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$	(3.108)	
Semi-latus-rectum	$r_0 = \frac{J^2}{m\alpha} = \frac{b^2}{a} = a(1-e^2)$	(3.109)	
Pericentre	$r_{\min} = \frac{r_0}{1+e} = a(1-e)$	(3.110)	r_{\min} pericentre distance
Apocentre	$r_{\max} = \frac{r_0}{1-e} = a(1+e)$	(3.111)	r_{\max} apocentre distance
Speed	$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right)$	(3.112)	v orbital speed
Period (Kepler's 3rd law)	$P = \pi\alpha \left(\frac{m}{2 E ^3} \right)^{1/2} = 2\pi a^{3/2} \left(\frac{m}{\alpha} \right)^{1/2}$	(3.113)	P orbital period



^aFor an inverse-square law of attraction between two isolated bodies in the nonrelativistic limit. If m is not $\ll M$, then the equations are valid with the substitutions $m \rightarrow \mu = Mm/(M+m)$ and $M \rightarrow (M+m)$ and with r taken as the body separation. The distance of mass m from the centre of mass is then $r\mu/m$ (see earlier table on *Reduced mass*). Other orbital dimensions scale similarly, and the two orbits have the same eccentricity.

^bNote that if the total energy, E , is < 0 then $e < 1$ and the orbit is an ellipse (a circle if $e = 0$). If $E = 0$, then $e = 1$ and the orbit is a parabola. If $E > 0$ then $e > 1$ and the orbit becomes a hyperbola (see *Rutherford scattering* on next page).

Rutherford scattering^a

Scattering potential energy	$U(r) = -\frac{\alpha}{r} \quad (3.114)$ $\alpha \begin{cases} < 0 & \text{repulsive} \\ > 0 & \text{attractive} \end{cases} \quad (3.115)$	$U(r)$ potential energy r particle separation α constant
Scattering angle	$\tan \frac{\chi}{2} = \frac{ \alpha }{2Eb} \quad (3.116)$	χ scattering angle E total energy (> 0) b impact parameter
Closest approach	$r_{\min} = \frac{ \alpha }{2E} \left(\csc \frac{\chi}{2} - \frac{\alpha}{ \alpha } \right) \quad (3.117)$ $= a(e \pm 1) \quad (3.118)$	r_{\min} closest approach a hyperbola semi-axis e eccentricity
Semi-axis	$a = \frac{ \alpha }{2E} \quad (3.119)$	
Eccentricity	$e = \left(\frac{4E^2 b^2}{\alpha^2} + 1 \right)^{1/2} = \csc \frac{\chi}{2} \quad (3.120)$	
Motion trajectory ^b	$\frac{4E^2}{\alpha^2} x^2 - \frac{y^2}{b^2} = 1 \quad (3.121)$	x, y position with respect to hyperbola centre
Scattering centre ^c	$x = \pm \left(\frac{\alpha^2}{4E^2} + b^2 \right)^{1/2} \quad (3.122)$	
Rutherford scattering formula ^d	$\frac{d\sigma}{d\Omega} = \frac{1}{n} \frac{dN}{d\Omega} \quad (3.123)$ $= \left(\frac{\alpha}{4E} \right)^2 \csc^4 \frac{\chi}{2} \quad (3.124)$	$\frac{d\sigma}{d\Omega}$ differential scattering cross section n beam flux density dN number of particles scattered into $d\Omega$ Ω solid angle

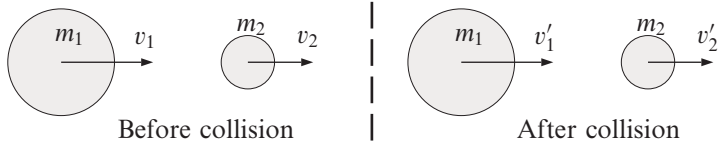
^aNonrelativistic treatment for an inverse-square force law and a fixed scattering centre. Similar scattering results from either an attractive or repulsive force. See also *Conic sections* on page 38.

^bThe correct branch can be chosen by inspection.

^cAlso the focal points of the hyperbola.

^d n is the number of particles per second passing through unit area perpendicular to the beam.

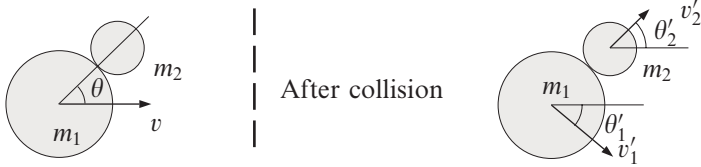
Inelastic collisions^a

			
Coefficient of restitution	$v'_2 - v'_1 = \epsilon(v_1 - v_2)$	(3.125)	ϵ coefficient of restitution
	$\epsilon = 1$ if perfectly elastic	(3.126)	v_i pre-collision velocities
	$\epsilon = 0$ if perfectly inelastic	(3.127)	v'_i post-collision velocities
Loss of kinetic energy ^b	$\frac{T - T'}{T} = 1 - \epsilon^2$	(3.128)	T, T' total KE in zero momentum frame before and after collision
Final velocities	$v'_1 = \frac{m_1 - \epsilon m_2}{m_1 + m_2} v_1 + \frac{(1 + \epsilon)m_2}{m_1 + m_2} v_2$	(3.129)	m_i particle masses
	$v'_2 = \frac{m_2 - \epsilon m_1}{m_1 + m_2} v_2 + \frac{(1 + \epsilon)m_1}{m_1 + m_2} v_1$	(3.130)	

^aAlong the line of centres, $v_1, v_2 \ll c$.

^bIn zero momentum frame.

Oblique elastic collisions^a

			
Directions of motion	$\tan \theta'_1 = \frac{m_2 \sin 2\theta}{m_1 - m_2 \cos 2\theta}$	(3.131)	θ angle between centre line and incident velocity
	$\theta'_2 = \theta$	(3.132)	θ'_i final trajectories
Relative separation angle	$\theta'_1 + \theta'_2 \begin{cases} > \pi/2 & \text{if } m_1 < m_2 \\ = \pi/2 & \text{if } m_1 = m_2 \\ < \pi/2 & \text{if } m_1 > m_2 \end{cases}$	(3.133)	m_i sphere masses
Final velocities	$v'_1 = \frac{(m_1^2 + m_2^2 - 2m_1 m_2 \cos 2\theta)^{1/2}}{m_1 + m_2} v$	(3.134)	v incident velocity of m_1
	$v'_2 = \frac{2m_1 v}{m_1 + m_2} \cos \theta$	(3.135)	v'_i final velocities

^aCollision between two perfectly elastic spheres: m_2 initially at rest, velocities $\ll c$.

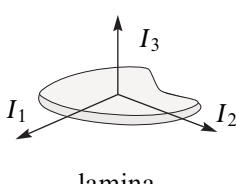
3.5 Rigid body dynamics

Moment of inertia tensor

Moment of inertia tensor ^a	$I_{ij} = \int (r^2 \delta_{ij} - x_i x_j) dm \quad (3.136)$	r $r^2 = x^2 + y^2 + z^2$ δ_{ij} Kronecker delta \mathbf{I} moment of inertia tensor dm mass element x_i position vector of dm I_{ij} components of \mathbf{I}
	$\mathbf{I} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{pmatrix} \quad (3.137)$	
Parallel axis theorem	$I_{12} = I_{12}^* - m a_1 a_2 \quad (3.138)$	I_{ij}^* tensor with respect to centre of mass
	$I_{11} = I_{11}^* + m(a_2^2 + a_3^2) \quad (3.139)$	a_i, \mathbf{a} position vector of centre of mass
	$I_{ij} = I_{ij}^* + m(\mathbf{a} ^2 \delta_{ij} - a_i a_j) \quad (3.140)$	m mass of body
Angular momentum	$\mathbf{J} = \mathbf{I} \boldsymbol{\omega} \quad (3.141)$	\mathbf{J} angular momentum $\boldsymbol{\omega}$ angular velocity
Rotational kinetic energy	$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} = \frac{1}{2} I_{ij} \omega_i \omega_j \quad (3.142)$	T kinetic energy

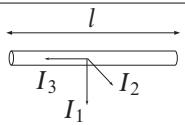
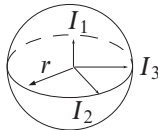
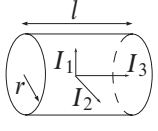
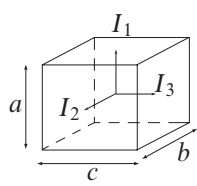
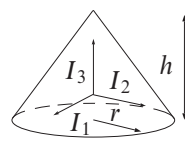
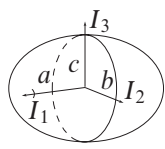
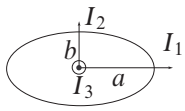
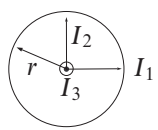
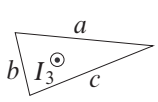
^a I_{ii} are the moments of inertia of the body. I_{ij} ($i \neq j$) are its products of inertia. The integrals are over the body volume.

Principal axes

Principal moment of inertia tensor	$\mathbf{I}' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad (3.143)$	\mathbf{I}' principal moment of inertia tensor I_i principal moments of inertia
Angular momentum	$\mathbf{J} = (I_1 \omega_1, I_2 \omega_2, I_3 \omega_3) \quad (3.144)$	\mathbf{J} angular momentum ω_i components of $\boldsymbol{\omega}$ along principal axes
Rotational kinetic energy	$T = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \quad (3.145)$	T kinetic energy
Moment of inertia ellipsoid ^a	$T = T(\omega_1, \omega_2, \omega_3) \quad (3.146)$ $J_i = \frac{\partial T}{\partial \omega_i} \quad (\mathbf{J} \text{ is } \perp \text{ ellipsoid surface}) \quad (3.147)$	 <p>lamina</p>
Perpendicular axis theorem	$I_1 + I_2 \begin{cases} \geq I_3 & \text{generally} \\ = I_3 & \text{flat lamina } \perp \text{ to 3-axis} \end{cases} \quad (3.148)$	
Symmetries	$I_1 \neq I_2 \neq I_3 \quad \text{asymmetric top} \\ I_1 = I_2 \neq I_3 \quad \text{symmetric top} \\ I_1 = I_2 = I_3 \quad \text{spherical top} $ (3.149)	

^aThe ellipsoid is defined by the surface of constant T .

Moments of inertia^a

Thin rod, length l	$I_1 = I_2 = \frac{ml^2}{12}$ $I_3 \simeq 0$	(3.150) (3.151)	
Solid sphere, radius r	$I_1 = I_2 = I_3 = \frac{2}{5}mr^2$	(3.152)	
Spherical shell, radius r	$I_1 = I_2 = I_3 = \frac{2}{3}mr^2$	(3.153)	
Solid cylinder, radius r , length l	$I_1 = I_2 = \frac{m}{4} \left(r^2 + \frac{l^2}{3} \right)$ $I_3 = \frac{1}{2}mr^2$	(3.154) (3.155)	
Solid cuboid, sides a, b, c	$I_1 = m(b^2 + c^2)/12$ $I_2 = m(c^2 + a^2)/12$ $I_3 = m(a^2 + b^2)/12$	(3.156) (3.157) (3.158)	
Solid circular cone, base radius r , height h ^b	$I_1 = I_2 = \frac{3}{20}m \left(r^2 + \frac{h^2}{4} \right)$ $I_3 = \frac{3}{10}mr^2$	(3.159) (3.160)	
Solid ellipsoid, semi-axes a, b, c	$I_1 = m(b^2 + c^2)/5$ $I_2 = m(c^2 + a^2)/5$ $I_3 = m(a^2 + b^2)/5$	(3.161) (3.162) (3.163)	
Elliptical lamina, semi-axes a, b	$I_1 = mb^2/4$ $I_2 = ma^2/4$ $I_3 = m(a^2 + b^2)/4$	(3.164) (3.165) (3.166)	
Disk, radius r	$I_1 = I_2 = mr^2/4$ $I_3 = mr^2/2$	(3.167) (3.168)	
Triangular plate ^c	$I_3 = \frac{m}{36}(a^2 + b^2 + c^2)$	(3.169)	

^aWith respect to principal axes for bodies of mass m and uniform density. The radius of gyration is defined as $k = (I/m)^{1/2}$.^bOrigin of axes is at the centre of mass ($h/4$ above the base).^cAround an axis through the centre of mass and perpendicular to the plane of the plate.

Centres of mass

Solid hemisphere, radius r	$d = 3r/8$ from sphere centre	(3.170)
Hemispherical shell, radius r	$d = r/2$ from sphere centre	(3.171)
Sector of disk, radius r , angle 2θ	$d = \frac{2}{3}r \frac{\sin\theta}{\theta}$ from disk centre	(3.172)
Arc of circle, radius r , angle 2θ	$d = r \frac{\sin\theta}{\theta}$ from circle centre	(3.173)
Arbitrary triangular lamina, height h^a	$d = h/3$ perpendicular from base	(3.174)
Solid cone or pyramid, height h	$d = h/4$ perpendicular from base	(3.175)
Spherical cap, height h , sphere radius r	solid: $d = \frac{3}{4} \frac{(2r-h)^2}{3r-h}$ from sphere centre	(3.176)
	shell: $d = r - h/2$ from sphere centre	(3.177)
Semi-elliptical lamina, height h	$d = \frac{4h}{3\pi}$ from base	(3.178)

^a h is the perpendicular distance between the base and apex of the triangle.

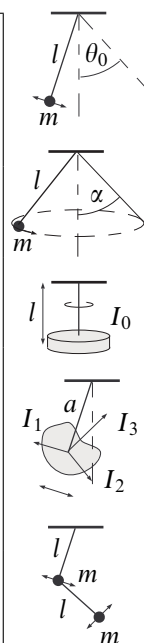
Pendulums

Simple pendulum	$P = 2\pi \sqrt{\frac{l}{g} \left(1 + \frac{\theta_0^2}{16} + \dots \right)}$	(3.179)	P period g gravitational acceleration l length θ_0 maximum angular displacement
Conical pendulum	$P = 2\pi \left(\frac{l \cos \alpha}{g} \right)^{1/2}$	(3.180)	α cone half-angle
Torsional pendulum ^a	$P = 2\pi \left(\frac{I I_0}{C} \right)^{1/2}$	(3.181)	I_0 moment of inertia of bob C torsional rigidity of wire (see page 81)
Compound pendulum ^b	$P \simeq 2\pi \left[\frac{1}{mga} (ma^2 + I_1 \cos^2 \gamma_1 + I_2 \cos^2 \gamma_2 + I_3 \cos^2 \gamma_3) \right]^{1/2}$	(3.182)	a distance of rotation axis from centre of mass m mass of body I_i principal moments of inertia γ_i angles between rotation axis and principal axes
Equal double pendulum ^c	$P \simeq 2\pi \left[\frac{l}{(2 \pm \sqrt{2})g} \right]^{1/2}$	(3.183)	

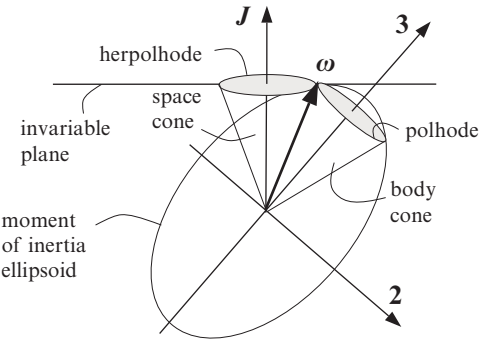
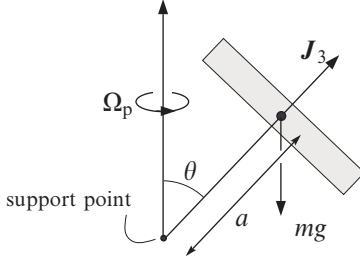
^aAssuming the bob is supported parallel to a principal rotation axis.

^bI.e., an arbitrary triaxial rigid body.

^cFor very small oscillations (two eigenmodes).



Tops and gyroscopes

 <p style="text-align: center;">prolate symmetric top</p>			 <p style="text-align: center;">gyroscope</p>		
Euler's equations ^a	$G_1 = I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3$	(3.184)	G_i	external couple (=0 for free rotation)	
	$G_2 = I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1$	(3.185)	I_i	principal moments of inertia	
	$G_3 = I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2$	(3.186)	ω_i	angular velocity of rotation	
Free symmetric top ^b ($I_3 < I_2 = I_1$)	$\Omega_b = \frac{I_1 - I_3}{I_1} \omega_3$	(3.187)	Ω_b	body frequency	
	$\Omega_s = \frac{J}{I_1}$	(3.188)	Ω_s	space frequency	
Free asymmetric top ^c	$\Omega_b^2 = \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2} \omega_3^2$	(3.189)	J	total angular momentum	
Steady gyroscopic precession	$\Omega_p^2 I_1' \cos \theta - \Omega_p J_3 + m g a = 0$	(3.190)	Ω_p	precession angular velocity	
	$\Omega_p \simeq \begin{cases} M g a / J_3 & (\text{slow}) \\ J_3 / (I_1' \cos \theta) & (\text{fast}) \end{cases}$	(3.191)	θ	angle from vertical	
Gyroscopic stability	$J_3^2 \geq 4 I_1' m g a \cos \theta$	(3.192)	J_3	angular momentum around symmetry axis	
Gyroscopic limit ("sleeping top")	$J_3^2 \gg I_1' m g a$	(3.193)	m	mass	
Nutation rate	$\Omega_n = J_3 / I_1'$	(3.194)	g	gravitational acceleration	
Gyroscope released from rest	$\Omega_p = \frac{m g a}{J_3} (1 - \cos \Omega_n t)$	(3.195)	a	distance of centre of mass from support point	
			I_1'	moment of inertia about support point	
			Ω_n	nutation angular velocity	
			t	time	

^aComponents are with respect to the principal axes, rotating with the body.^bThe body frequency is the angular velocity (with respect to principal axes) of ω around the 3-axis. The space frequency is the angular velocity of the 3-axis around J , i.e., the angular velocity at which the body cone moves around the space cone.^c J close to 3-axis. If $\Omega_b^2 < 0$, the body tumbles.

3.6 Oscillating systems

Free oscillations

Differential equation	$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0$	(3.196)	x oscillating variable t time γ damping factor (per unit mass) ω_0 undamped angular frequency
Underdamped solution ($\gamma < \omega_0$)	$x = Ae^{-\gamma t} \cos(\omega t + \phi)$	(3.197)	A amplitude constant ϕ phase constant ω angular eigenfrequency
	where $\omega = (\omega_0^2 - \gamma^2)^{1/2}$	(3.198)	
Critically damped solution ($\gamma = \omega_0$)	$x = e^{-\gamma t} (A_1 + A_2 t)$	(3.199)	A_i amplitude constants
Overdamped solution ($\gamma > \omega_0$)	$x = e^{-\gamma t} (A_1 e^{qt} + A_2 e^{-qt})$	(3.200)	
	where $q = (\gamma^2 - \omega_0^2)^{1/2}$	(3.201)	
Logarithmic decrement ^a	$\Delta = \ln \frac{a_n}{a_{n+1}} = \frac{2\pi\gamma}{\omega}$	(3.202)	Δ logarithmic decrement a_n n th displacement maximum
Quality factor	$Q = \frac{\omega_0}{2\gamma} \left[\simeq \frac{\pi}{\Delta} \text{ if } Q \gg 1 \right]$	(3.203)	Q quality factor

^aThe *decrement* is usually the ratio of successive displacement *maxima* but is sometimes taken as the ratio of successive displacement *extrema*, reducing Δ by a factor of 2. Logarithms are sometimes taken to base 10, introducing a further factor of $\log_{10} e$.

Forced oscillations

Differential equation	$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = F_0 e^{i\omega_f t}$	(3.204)	x oscillating variable t time γ damping factor (per unit mass) ω_0 undamped angular frequency F_0 force amplitude (per unit mass) ω_f forcing angular frequency A amplitude ϕ phase lag of response behind driving force
Steady-state solution ^a	$x = Ae^{i(\omega_f t - \phi)}$, where	(3.205)	
	$A = F_0 [(\omega_0^2 - \omega_f^2)^2 + (2\gamma\omega_f)^2]^{-1/2}$	(3.206)	
	$\simeq \frac{F_0/(2\omega_0)}{[(\omega_0 - \omega_f)^2 + \gamma^2]^{1/2}} \quad (\gamma \ll \omega_f)$	(3.207)	
	$\tan \phi = \frac{2\gamma\omega_f}{\omega_0^2 - \omega_f^2}$	(3.208)	
Amplitude resonance ^b	$\omega_{ar}^2 = \omega_0^2 - 2\gamma^2$	(3.209)	ω_{ar} amplitude resonant forcing angular frequency
Velocity resonance ^c	$\omega_{vr} = \omega_0$	(3.210)	ω_{vr} velocity resonant forcing angular frequency
Quality factor	$Q = \frac{\omega_0}{2\gamma}$	(3.211)	Q quality factor
Impedance	$Z = 2\gamma + i \frac{\omega_f^2 - \omega_0^2}{\omega_f}$	(3.212)	Z impedance (per unit mass)

^aExcluding the free oscillation terms.

^bForcing frequency for maximum displacement.

^cForcing frequency for maximum velocity. Note $\phi = \pi/2$ at this frequency.

3.7 Generalised dynamics

Lagrangian dynamics

Action	$S = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt \quad (3.213)$	S action ($\delta S = 0$ for the motion)
Euler–Lagrange equation	$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad (3.214)$	\mathbf{q} generalised coordinates $\dot{\mathbf{q}}$ generalised velocities L Lagrangian t time m mass
Lagrangian of particle in external field	$L = \frac{1}{2}mv^2 - U(\mathbf{r}, t) \quad (3.215)$ $= T - U \quad (3.216)$	\mathbf{v} velocity \mathbf{r} position vector U potential energy T kinetic energy
Relativistic Lagrangian of a charged particle	$L = -\frac{m_0 c^2}{\gamma} - e(\phi - \mathbf{A} \cdot \mathbf{v}) \quad (3.217)$	m_0 (rest) mass γ Lorentz factor $+e$ positive charge ϕ electric potential \mathbf{A} magnetic vector potential
Generalised momenta	$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (3.218)$	p_i generalised momenta

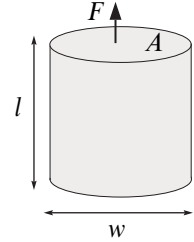
Hamiltonian dynamics

Hamiltonian	$H = \sum_i p_i \dot{q}_i - L \quad (3.219)$	L Lagrangian p_i generalised momenta \dot{q}_i generalised velocities
Hamilton's equations	$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (3.220)$	H Hamiltonian q_i generalised coordinates
Hamiltonian of particle in external field	$H = \frac{1}{2}mv^2 + U(\mathbf{r}, t) \quad (3.221)$ $= T + U \quad (3.222)$	v particle speed \mathbf{r} position vector U potential energy T kinetic energy
Relativistic Hamiltonian of a charged particle	$H = (m_0^2 c^4 + \mathbf{p} - e\mathbf{A} ^2 c^2)^{1/2} + e\phi \quad (3.223)$	m_0 (rest) mass c speed of light $+e$ positive charge ϕ electric potential \mathbf{A} vector potential
Poisson brackets	$[f, g] = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \quad (3.224)$ $[q_i, g] = \frac{\partial g}{\partial p_i}, \quad [p_i, g] = -\frac{\partial g}{\partial q_i} \quad (3.225)$ $[H, g] = 0 \quad \text{if} \quad \frac{\partial g}{\partial t} = 0, \quad \frac{dg}{dt} = 0 \quad (3.226)$	\mathbf{p} particle momentum t time f, g arbitrary functions $[\cdot, \cdot]$ Poisson bracket (also see <i>Commutators</i> on page 26)
Hamilton–Jacobi equation	$\frac{\partial S}{\partial t} + H \left(q_i, \frac{\partial S}{\partial q_i}, t \right) = 0 \quad (3.227)$	S action

3.8 Elasticity

Elasticity definitions (simple)^a

Stress	$\tau = F/A$	(3.228)	τ stress F applied force A cross-sectional area
Strain	$e = \delta l/l$	(3.229)	e strain δl change in length l length
Young modulus (Hooke's law)	$E = \tau/e = \text{constant}$	(3.230)	E Young modulus
Poisson ratio ^b	$\sigma = -\frac{\delta w/w}{\delta l/l}$	(3.231)	σ Poisson ratio δw change in width w width



^aThese apply to a thin wire under longitudinal stress.

^bSolids obeying Hooke's law are restricted by thermodynamics to $-1 \leq \sigma \leq 1/2$, but none are known with $\sigma < 0$. Non-Hookean materials can show $\sigma > 1/2$.

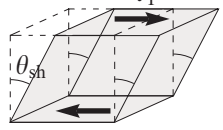
Elasticity definitions (general)

Stress tensor ^a	$\tau_{ij} = \frac{\text{force } \parallel i \text{ direction}}{\text{area } \perp j \text{ direction}}$	(3.232)	τ_{ij} stress tensor ($\tau_{ij} = \tau_{ji}$)
Strain tensor	$e_{kl} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$	(3.233)	e_{kl} strain tensor ($e_{kl} = e_{lk}$) u_k displacement \parallel to x_k x_k coordinate system
Elastic modulus	$\tau_{ij} = \lambda_{ijkl} e_{kl}$	(3.234)	λ_{ijkl} elastic modulus
Elastic energy ^b	$U = \frac{1}{2} \lambda_{ijkl} e_{ij} e_{kl}$	(3.235)	U potential energy
Volume strain (dilatation)	$e_v = \frac{\delta V}{V} = e_{11} + e_{22} + e_{33}$	(3.236)	e_v volume strain δV change in volume V volume
Shear strain	$e_{kl} = \underbrace{\left(e_{kl} - \frac{1}{3} e_v \delta_{kl} \right)}_{\text{pure shear}} + \underbrace{\frac{1}{3} e_v \delta_{kl}}_{\text{dilatation}}$	(3.237)	δ_{kl} Kronecker delta
Hydrostatic compression	$\tau_{ij} = -p \delta_{ij}$	(3.238)	p hydrostatic pressure

^a τ_{ii} are normal stresses, τ_{ij} ($i \neq j$) are torsional stresses.

^bAs usual, products are implicitly summed over repeated indices.

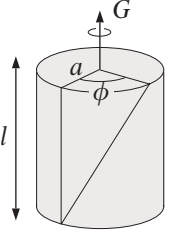
Isotropic elastic solids

Lamé coefficients	$\mu = \frac{E}{2(1+\sigma)} \quad (3.239)$	μ, λ Lamé coefficients
	$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \quad (3.240)$	E Young modulus σ Poisson ratio
Longitudinal modulus ^a	$M_1 = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} = \lambda + 2\mu \quad (3.241)$	M_1 longitudinal elastic modulus
Diagonalised equations ^b	$e_{ii} = \frac{1}{E} [\tau_{ii} - \sigma(\tau_{jj} + \tau_{kk})] \quad (3.242)$	e_{ii} strain in i direction τ_{ii} stress in i direction
	$\tau_{ii} = M_1 \left[e_{ii} + \frac{\sigma}{1-\sigma} (e_{jj} + e_{kk}) \right] \quad (3.243)$	\mathbf{e} strain tensor \mathbf{t} stress tensor
	$\mathbf{t} = 2\mu\mathbf{e} + \lambda\mathbf{1}\text{tr}(\mathbf{e}) \quad (3.244)$	$\mathbf{1}$ unit matrix $\text{tr}(\cdot)$ trace
Bulk modulus (compression modulus)	$K = \frac{E}{3(1-2\sigma)} = \lambda + \frac{2}{3}\mu \quad (3.245)$	K bulk modulus K_T isothermal bulk modulus
	$\frac{1}{K_T} = -\frac{1}{V} \frac{\partial V}{\partial p} \bigg _T \quad (3.246)$	V volume
	$-p = K e_v \quad (3.247)$	p pressure T temperature
Shear modulus (rigidity modulus)	$\mu = \frac{E}{2(1+\sigma)} \quad (3.248)$	e_v volume strain
	$\tau_T = \mu\theta_{\text{sh}} \quad (3.249)$	μ shear modulus τ_T transverse stress θ_{sh} shear strain
Young modulus	$E = \frac{9\mu K}{\mu + 3K} \quad (3.250)$	
Poisson ratio	$\sigma = \frac{3K - 2\mu}{2(3K + \mu)} \quad (3.251)$	

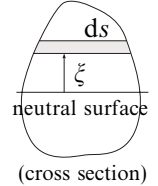
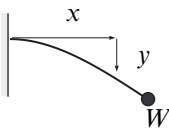
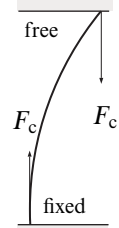
^aIn an extended medium.

^bAxes aligned along eigenvectors of the stress and strain tensors.

Torsion

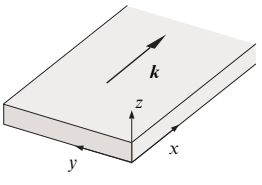
Torsional rigidity (for a homogeneous rod)	$G = C \frac{\phi}{l} \quad (3.252)$	G twisting couple C torsional rigidity l rod length ϕ twist angle in length l	
Thin circular cylinder	$C = 2\pi a^3 \mu t \quad (3.253)$	a radius t wall thickness μ shear modulus	
Thick circular cylinder	$C = \frac{1}{2} \mu \pi (a_2^4 - a_1^4) \quad (3.254)$	a_1 inner radius a_2 outer radius	
Arbitrary thin-walled tube	$C = \frac{4A^2 \mu t}{P} \quad (3.255)$	A cross-sectional area P perimeter	
Long flat ribbon	$C = \frac{1}{3} \mu w t^3 \quad (3.256)$	w cross-sectional width	

Bending beams^a

Bending moment	$G_b = \frac{E}{R_c} \int \xi^2 ds$	(3.257)	G_b bending moment	 <p>(cross section)</p>
	$= \frac{EI}{R_c}$	(3.258)	E Young modulus R_c radius of curvature ds area element ξ distance to neutral surface from ds	
Light beam, horizontal at $x=0$, weight at $x=l$	$y = \frac{W}{2EI} \left(l - \frac{x}{3}\right) x^2$	(3.259)	I moment of area y displacement from horizontal W end-weight l beam length x distance along beam	
Heavy beam	$EI \frac{d^4 y}{dx^4} = w(x)$	(3.260)	w beam weight per unit length	
Euler strut failure	$F_c = \begin{cases} \pi^2 EI / l^2 & \text{(free ends)} \\ 4\pi^2 EI / l^2 & \text{(fixed ends)} \\ \pi^2 EI / (4l^2) & \text{(1 free end)} \end{cases}$	(3.261)	F_c critical compression force l strut length	

^aThe radius of curvature is approximated by $1/R_c \approx d^2 y / dx^2$.

Elastic wave velocities^a

In an infinite isotropic solid ^b	$v_t = (\mu / \rho)^{1/2}$	(3.262)	v_t speed of transverse wave
	$v_l = (M_l / \rho)^{1/2}$	(3.263)	v_l speed of longitudinal wave
	$\frac{v_l}{v_t} = \left(\frac{2-2\sigma}{1-2\sigma} \right)^{1/2}$	(3.264)	μ shear modulus ρ density M_l longitudinal modulus ($= \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}$)
In a fluid	$v_l = (K / \rho)^{1/2}$	(3.265)	K bulk modulus
 On a thin plate (wave travelling along x , plate thin in z)	$v_l^{(x)} = \left[\frac{E}{\rho(1-\sigma^2)} \right]^{1/2}$	(3.266)	$v_l^{(i)}$ speed of longitudinal wave (displacement $\parallel i$)
	$v_t^{(y)} = (\mu / \rho)^{1/2}$	(3.267)	$v_t^{(i)}$ speed of transverse wave (displacement $\parallel i$)
	$v_t^{(z)} = k \left[\frac{Et^2}{12\rho(1-\sigma^2)} \right]^{1/2}$	(3.268)	E Young modulus σ Poisson ratio k wavenumber ($= 2\pi / \lambda$) t plate thickness (in z , $t \ll \lambda$)
In a thin circular rod	$v_l = (E / \rho)^{1/2}$	(3.269)	
	$v_\phi = (\mu / \rho)^{1/2}$	(3.270)	v_ϕ torsional wave velocity
	$v_t = \frac{ka}{2} \left(\frac{E}{\rho} \right)^{1/2}$	(3.271)	a rod radius ($\ll \lambda$)

^aWaves that produce “bending” are generally dispersive. Wave (phase) speeds are quoted throughout.

^bTransverse waves are also known as shear waves, or S-waves. Longitudinal waves are also known as pressure waves, or P-waves.

Waves in strings and springs^a

In a spring	$v_l = (\kappa l / \rho_l)^{1/2}$	(3.272)	v_l speed of longitudinal wave κ spring constant ^b l spring length ρ_l mass per unit length ^c
On a stretched string	$v_t = (T / \rho_l)^{1/2}$	(3.273)	v_t speed of transverse wave T tension
On a stretched sheet	$v_t = (\tau / \rho_A)^{1/2}$	(3.274)	τ tension per unit width ρ_A mass per unit area

^aWave amplitude assumed \ll wavelength.

^bIn the sense κ = force/extension.

^cMeasured along the axis of the spring.

Propagation of elastic waves

Acoustic impedance	$Z = \frac{\text{force}}{\text{response velocity}} = \frac{F}{\dot{u}}$	(3.275)	Z impedance F stress force u strain displacement
	$= (E' \rho)^{1/2}$	(3.276)	
Wave velocity/impedance relation	if $v = \left(\frac{E'}{\rho} \right)^{1/2}$	(3.277)	E' elastic modulus ρ density
	then $Z = (E' \rho)^{1/2} = \rho v$	(3.278)	v wave phase velocity
Mean energy density (nondispersive waves)	$\mathcal{U} = \frac{1}{2} E' k^2 u_0^2$	(3.279)	\mathcal{U} energy density k wavenumber
	$= \frac{1}{2} \rho \omega^2 u_0^2$	(3.280)	ω angular frequency u_0 maximum displacement
	$P = \mathcal{U} v$	(3.281)	P mean energy flux
Normal coefficients ^a	$r = \frac{u_r}{u_i} = -\frac{\tau_r}{\tau_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$	(3.282)	r reflection coefficient t transmission coefficient
	$t = \frac{2Z_1}{Z_1 + Z_2}$	(3.283)	τ stress
Snell's law ^b	$\frac{\sin \theta_i}{v_i} = \frac{\sin \theta_r}{v_r} = \frac{\sin \theta_t}{v_t}$	(3.284)	θ_i angle of incidence θ_r angle of reflection θ_t angle of refraction

^aFor stress and strain amplitudes. Because these reflection and transmission coefficients are usually defined in terms of displacement, u , rather than stress, there are differences between these coefficients and their equivalents defined in electromagnetism [see Equation (7.179) and page 154].

^bAngles defined from the normal to the interface. An incident plane pressure wave will generally excite both shear and pressure waves in reflection and transmission. Use the velocity appropriate for the wave type.

3.9 Fluid dynamics

Ideal fluids^a

Continuity ^b	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$	(3.285)	ρ density \mathbf{v} fluid velocity field t time
Kelvin circulation	$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = \text{constant}$	(3.286)	Γ circulation $d\mathbf{l}$ loop element
	$= \int_S \boldsymbol{\omega} \cdot d\mathbf{s}$	(3.287)	$d\mathbf{s}$ element of surface bounded by loop $\boldsymbol{\omega}$ vorticity ($= \nabla \times \mathbf{v}$)
Euler's equation ^c	$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}$	(3.288)	p pressure \mathbf{g} gravitational field strength
	or $\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) = \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})]$	(3.289)	$(\mathbf{v} \cdot \nabla)$ advective operator
Bernoulli's equation (incompressible flow)	$\frac{1}{2} \rho v^2 + p + \rho g z = \text{constant}$	(3.290)	z altitude
Bernoulli's equation (compressible adiabatic flow) ^d	$\frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + g z = \text{constant}$	(3.291)	γ ratio of specific heat capacities (c_p/c_v)
	$= \frac{1}{2} v^2 + c_p T + g z$	(3.292)	c_p specific heat capacity at constant pressure T temperature
Hydrostatics	$\nabla p = \rho \mathbf{g}$	(3.293)	
Adiabatic lapse rate (ideal gas)	$\frac{dT}{dz} = -\frac{g}{c_p}$	(3.294)	

^aNo thermal conductivity or viscosity.

^bTrue generally.

^cThe second form of Euler's equation applies to incompressible flow only.

^dEquation (3.292) is true only for an ideal gas.

Potential flow^a

Velocity potential	$\mathbf{v} = \nabla \phi$	(3.295)	\mathbf{v} velocity
	$\nabla^2 \phi = 0$	(3.296)	ϕ velocity potential
Vorticity condition	$\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0$	(3.297)	$\boldsymbol{\omega}$ vorticity \mathbf{F} drag force on moving sphere
Drag force on a sphere ^b	$\mathbf{F} = -\frac{2}{3} \pi \rho a^3 \dot{\mathbf{u}} = -\frac{1}{2} M_d \dot{\mathbf{u}}$	(3.298)	a sphere radius
			$\dot{\mathbf{u}}$ sphere acceleration
			ρ fluid density M_d displaced fluid mass

^aFor incompressible fluids.

^bThe effect of this drag force is to give the sphere an additional effective mass equal to half the mass of fluid displaced.

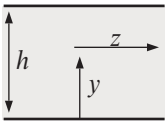
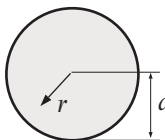
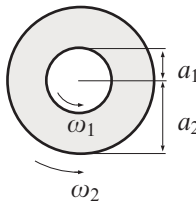
Viscous flow (incompressible)^a

Fluid stress	$\tau_{ij} = -p\delta_{ij} + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$	(3.299)	τ_{ij} fluid stress tensor p hydrostatic pressure η shear viscosity v_i velocity along i axis δ_{ij} Kronecker delta
Navier–Stokes equation ^b	$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} - \frac{\eta}{\rho} \nabla \times \boldsymbol{\omega} + \mathbf{g}$	(3.300)	\mathbf{v} fluid velocity field $\boldsymbol{\omega}$ vorticity \mathbf{g} gravitational acceleration ρ density
	$= -\frac{\nabla p}{\rho} + \frac{\eta}{\rho} \nabla^2 \mathbf{v} + \mathbf{g}$	(3.301)	
Kinematic viscosity	$\nu = \eta / \rho$	(3.302)	ν kinematic viscosity

^aI.e., $\nabla \cdot \mathbf{v} = 0$, $\eta \neq 0$.

^bNeglecting bulk (second) viscosity.

Laminar viscous flow

Between parallel plates	$v_z(y) = \frac{1}{2\eta} y(h-y) \frac{\partial p}{\partial z}$	(3.303)	v_z flow velocity z direction of flow y distance from plate η shear viscosity p pressure	
Along a circular pipe ^a	$v_z(r) = \frac{1}{4\eta} (a^2 - r^2) \frac{\partial p}{\partial z}$	(3.304)	r distance from pipe axis a pipe radius V volume	
	$Q = \frac{dV}{dt} = \frac{\pi a^4}{8\eta} \frac{\partial p}{\partial z}$	(3.305)		
Circulating between concentric rotating cylinders ^b	$G_z = \frac{4\pi\eta a_1^2 a_2^2}{a_2^2 - a_1^2} (\omega_2 - \omega_1)$	(3.306)	G_z axial couple between cylinders per unit length ω_i angular velocity of i th cylinder	
Along an annular pipe	$Q = \frac{\pi}{8\eta} \frac{\partial p}{\partial z} \left[a_2^4 - a_1^4 - \frac{(a_2^2 - a_1^2)^2}{\ln(a_2/a_1)} \right]$	(3.307)	a_1 inner radius a_2 outer radius Q volume discharge rate	

^aPoiseuille flow.

^bCouette flow.

Drag^a

On a sphere (Stokes's law)	$F = 6\pi a \eta v$	(3.308)	F drag force a radius
On a disk, broadside to flow	$F = 16a \eta v$	(3.309)	v velocity η shear viscosity
On a disk, edge on to flow	$F = 32a \eta v / 3$	(3.310)	

^aFor Reynolds numbers $\ll 1$.

Characteristic numbers

Reynolds number	$\text{Re} = \frac{\rho UL}{\eta} = \frac{\text{inertial force}}{\text{viscous force}} \quad (3.311)$	Re Reynolds number ρ density U characteristic velocity L characteristic scale-length η shear viscosity
Froude number ^a	$\text{F} = \frac{U^2}{Lg} = \frac{\text{inertial force}}{\text{gravitational force}} \quad (3.312)$	F Froude number g gravitational acceleration
Strouhal number ^b	$\text{S} = \frac{U\tau}{L} = \frac{\text{evolution scale}}{\text{physical scale}} \quad (3.313)$	S Strouhal number τ characteristic timescale
Prandtl number	$\text{P} = \frac{\eta c_p}{\lambda} = \frac{\text{momentum transport}}{\text{heat transport}} \quad (3.314)$	P Prandtl number c_p Specific heat capacity at constant pressure λ thermal conductivity
Mach number	$\text{M} = \frac{U}{c} = \frac{\text{speed}}{\text{sound speed}} \quad (3.315)$	M Mach number c sound speed
Rossby number	$\text{Ro} = \frac{U}{\Omega L} = \frac{\text{inertial force}}{\text{Coriolis force}} \quad (3.316)$	Ro Rossby number Ω angular velocity

^aSometimes the square root of this expression. L is usually the fluid depth.

^bSometimes the reciprocal of this expression.

Fluid waves

Sound waves	$v_p = \left(\frac{K}{\rho} \right)^{1/2} = \left(\frac{dp}{d\rho} \right)^{1/2} \quad (3.317)$	v_p wave (phase) speed K bulk modulus p pressure ρ density
In an ideal gas (adiabatic conditions) ^a	$v_p = \left(\frac{\gamma RT}{\mu} \right)^{1/2} = \left(\frac{\gamma p}{\rho} \right)^{1/2} \quad (3.318)$	γ ratio of heat capacities R molar gas constant T (absolute) temperature μ mean molecular mass
Gravity waves on a liquid surface ^b	$\omega^2 = gk \tanh kh \quad (3.319)$ $v_g \simeq \begin{cases} \frac{1}{2} \left(\frac{g}{k} \right)^{1/2} & (h \gg \lambda) \\ (gh)^{1/2} & (h \ll \lambda) \end{cases} \quad (3.320)$	v_g group speed of wave h liquid depth λ wavelength k wavenumber g gravitational acceleration ω angular frequency
Capillary waves (ripples) ^c	$\omega^2 = \frac{\sigma k^3}{\rho} \quad (3.321)$	σ surface tension
Capillary-gravity waves ($h \gg \lambda$)	$\omega^2 = gk + \frac{\sigma k^3}{\rho} \quad (3.322)$	

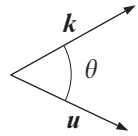
^aIf the waves are isothermal rather than adiabatic then $v_p = (p/\rho)^{1/2}$.

^bAmplitude \ll wavelength.

^cIn the limit $k^2 \gg g\rho/\sigma$.

Doppler effect^a

Source at rest, observer moving at u	$\frac{v'}{v} = 1 - \frac{ u }{v_p} \cos \theta$	(3.323)	v', v'' observed frequency v emitted frequency v_p wave (phase) speed in fluid
Observer at rest, source moving at u	$\frac{v''}{v} = \frac{1}{1 - \frac{ u }{v_p} \cos \theta}$	(3.324)	u velocity θ angle between wavevector, k , and u



^aFor plane waves in a stationary fluid.

Wave speeds

Phase speed	$v_p = \frac{\omega}{k} = v \lambda$	(3.325)	v_p phase speed v frequency ω angular frequency ($= 2\pi v$) λ wavelength k wavenumber ($= 2\pi/\lambda$)
Group speed	$v_g = \frac{d\omega}{dk}$	(3.326)	v_g group speed
	$= v_p - \lambda \frac{dv_p}{d\lambda}$	(3.327)	

Shocks

Mach wedge ^a	$\sin \theta_w = \frac{v_p}{v_b}$	(3.328)	θ_w wedge semi-angle v_p wave (phase) speed v_b body speed
Kelvin wedge ^b	$\lambda_K = \frac{4\pi v_b^2}{3g}$	(3.329)	λ_K characteristic wavelength
	$\theta_w = \arcsin(1/3) = 19^\circ.5$	(3.330)	g gravitational acceleration
Spherical adiabatic shock ^c	$r \simeq \left(\frac{Et^2}{\rho_0} \right)^{1/5}$	(3.331)	r shock radius E energy release t time ρ_0 density of undisturbed medium
Rankine– Hugoniot shock relations ^d	$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$	(3.332)	1 upstream values 2 downstream values
	$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{(\gamma - 1) + 2/M_1^2}$	(3.333)	p pressure v velocity
	$\frac{T_2}{T_1} = \frac{[2\gamma M_1^2 - (\gamma - 1)][2 + (\gamma - 1)M_1^2]}{(\gamma + 1)^2 M_1^2}$	(3.334)	T temperature ρ density γ ratio of specific heats M Mach number

^aApproximating the wake generated by supersonic motion of a body in a nondispersive medium.

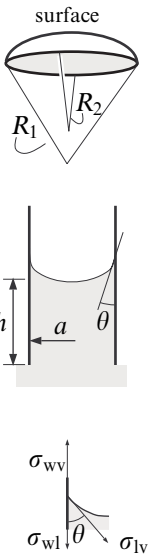
^bFor gravity waves, e.g., in the wake of a boat. Note that the wedge semi-angle is independent of v_b .

^cSedov–Taylor relation.

^dSolutions for a steady, normal shock, in the frame moving with the shock front. If $\gamma = 5/3$ then $v_1/v_2 \leq 4$.

Surface tension

Definition	$\sigma_{lv} = \frac{\text{surface energy}}{\text{area}} \quad (3.335)$ $= \frac{\text{surface tension}}{\text{length}} \quad (3.336)$	σ_{lv} surface tension (liquid/vapour interface)
Laplace's formula ^a	$\Delta p = \sigma_{lv} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad (3.337)$	Δp pressure difference over surface R_i principal radii of curvature
Capillary constant	$c_c = \left(\frac{2\sigma_{lv}}{g\rho} \right)^{1/2} \quad (3.338)$	c_c capillary constant ρ liquid density g gravitational acceleration
Capillary rise (circular tube)	$h = \frac{2\sigma_{lv} \cos \theta}{\rho g a} \quad (3.339)$	h rise height θ contact angle a tube radius
Contact angle	$\cos \theta = \frac{\sigma_{wv} - \sigma_{wl}}{\sigma_{lv}} \quad (3.340)$	σ_{wv} wall/vapour surface tension σ_{wl} wall/liquid surface tension



^aFor a spherical bubble in a liquid $\Delta p = 2\sigma_{lv}/R$. For a soap bubble (two surfaces) $\Delta p = 4\sigma_{lv}/R$.

Chapter 4 Quantum physics

4.1 Introduction

Quantum ideas occupy such a pivotal position in physics that different notations and algebras appropriate to each field have been developed. In the spirit of this book, only those formulas that are commonly present in undergraduate courses and that can be simply presented in tabular form are included here. For example, much of the detail of atomic spectroscopy and of specific perturbation analyses has been omitted, as have ideas from the somewhat specialised field of quantum electrodynamics. Traditionally, quantum physics is understood through standard “toy” problems, such as the potential step and the one-dimensional harmonic oscillator, and these are reproduced here. Operators are distinguished from observables using the “hat” notation, so that the momentum observable, p_x , has the operator $\hat{p}_x = -i\hbar\partial/\partial x$.

For clarity, many relations that can be generalised to three dimensions in an obvious way have been stated in their one-dimensional form, and wavefunctions are implicitly taken as normalised functions of space and time unless otherwise stated. With the exception of the last panel, all equations should be taken as nonrelativistic, so that “total energy” is the sum of potential and kinetic energies, excluding the rest mass energy.

4.2 Quantum definitions

Quantum uncertainty relations

De Broglie relation	$p = \frac{h}{\lambda}$	(4.1)	p, \mathbf{p}	particle momentum
	$\mathbf{p} = \hbar \mathbf{k}$	(4.2)	h	Planck constant
Planck–Einstein relation	$E = h\nu = \hbar\omega$	(4.3)	\hbar	$h/(2\pi)$
			λ	de Broglie wavelength
Dispersion ^a	$(\Delta a)^2 = \langle (a - \langle a \rangle)^2 \rangle$ $= \langle a^2 \rangle - \langle a \rangle^2$	(4.4)	\mathbf{k}	de Broglie wavevector
		(4.5)	E	energy
General uncertainty relation	$(\Delta a)^2 (\Delta b)^2 \geq \frac{1}{4} \langle [\hat{a}, \hat{b}] \rangle^2$	(4.6)	ν	frequency
			ω	angular frequency ($= 2\pi\nu$)
Momentum–position uncertainty relation ^c	$\Delta p \Delta x \geq \frac{\hbar}{2}$	(4.7)	a, b	observables ^b
Energy–time uncertainty relation	$\Delta E \Delta t \geq \frac{\hbar}{2}$	(4.8)	$\langle \cdot \rangle$	expectation value
Number–phase uncertainty relation	$\Delta n \Delta \phi \geq \frac{1}{2}$	(4.9)	$(\Delta a)^2$	dispersion of a
			\hat{a}	operator for observable a
			$[\cdot, \cdot]$	commutator (see page 26)
			x	particle position
			t	time
			n	number of photons
			ϕ	wave phase

^aDispersion in quantum physics corresponds to variance in statistics.

^bAn observable is a directly measurable parameter of a system.

^cAlso known as the “Heisenberg uncertainty relation.”

Wavefunctions

Probability density	$\text{pr}(x, t) \, dx = \psi(x, t) ^2 \, dx$	(4.10)	pr	probability density
Probability density current ^a	$j(x) = \frac{\hbar}{2im} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right)$	(4.11)	ψ	wavefunction
	$\mathbf{j} = \frac{\hbar}{2im} [\psi^*(\mathbf{r}) \nabla \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla \psi^*(\mathbf{r})]$	(4.12)	\mathbf{j}, j	probability density current
	$= \frac{1}{m} \Re(\psi^* \hat{\mathbf{p}} \psi)$	(4.13)	\hbar	(Planck constant)/(2π)
			x	position coordinate
Continuity equation	$\nabla \cdot \mathbf{j} = -\frac{\partial}{\partial t}(\psi \psi^*)$	(4.14)	$\hat{\mathbf{p}}$	momentum operator
Schrödinger equation	$\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$	(4.15)	m	particle mass
Particle stationary states ^b	$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$	(4.16)	\Re	real part of
			t	time
			H	Hamiltonian
			V	potential energy
			E	total energy

^aFor particles. In three dimensions, suitable units would be particles $\text{m}^{-2}\text{s}^{-1}$.

^bTime-independent Schrödinger equation for a particle, in one dimension.

Operators

Hermitian conjugate operator	$\int (\hat{a}\phi)^* \psi \, dx = \int \phi^* \hat{a}\psi \, dx \quad (4.17)$	\hat{a} Hermitian conjugate operator ψ, ϕ normalisable functions
Position operator	$\hat{x}^n = x^n \quad (4.18)$	$*$ complex conjugate x, y position coordinates
Momentum operator	$\hat{p}_x^n = \frac{\hbar^n}{i^n} \frac{\partial^n}{\partial x^n} \quad (4.19)$	n arbitrary integer ≥ 1 p_x momentum coordinate
Kinetic energy operator	$\hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \quad (4.20)$	T kinetic energy \hbar (Planck constant)/(2 π) m particle mass
Hamiltonian operator	$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \quad (4.21)$	H Hamiltonian V potential energy
Angular momentum operators	$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \quad (4.22)$ $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad (4.23)$	L_z angular momentum along z axis (sim. x and y) L total angular momentum
Parity operator	$\hat{P}\psi(\mathbf{r}) = \psi(-\mathbf{r}) \quad (4.24)$	\hat{P} parity operator \mathbf{r} position vector

Expectation value

Expectation value ^a	$\langle a \rangle = \langle \hat{a} \rangle = \int \Psi^* \hat{a} \Psi \, dx \quad (4.25)$ $= \langle \Psi \hat{a} \Psi \rangle \quad (4.26)$	$\langle a \rangle$ expectation value of a \hat{a} operator for a Ψ (spatial) wavefunction x (spatial) coordinate
Time dependence	$\frac{d}{dt} \langle \hat{a} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{a}] \rangle + \left\langle \frac{\partial \hat{a}}{\partial t} \right\rangle \quad (4.27)$	t time \hbar (Planck constant)/(2 π)
Relation to eigenfunctions	if $\hat{a}\psi_n = a_n\psi_n$ and $\Psi = \sum c_n\psi_n$ then $\langle a \rangle = \sum c_n ^2 a_n \quad (4.28)$	ψ_n eigenfunctions of \hat{a} a_n eigenvalues n dummy index c_n probability amplitudes
Ehrenfest's theorem	$m \frac{d}{dt} \langle \mathbf{r} \rangle = \langle \mathbf{p} \rangle \quad (4.29)$ $\frac{d}{dt} \langle \mathbf{p} \rangle = -\langle \nabla V \rangle \quad (4.30)$	m particle mass \mathbf{r} position vector \mathbf{p} momentum V potential energy

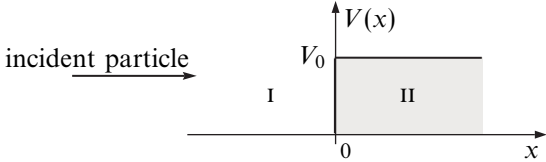
^aEquation (4.26) uses the Dirac “bra-ket” notation for integrals involving operators. The presence of vertical bars distinguishes this use of angled brackets from that on the left-hand side of the equations. Note that $\langle a \rangle$ and $\langle \hat{a} \rangle$ are taken as equivalent.

Dirac notation

Matrix element ^a	$a_{nm} = \int \psi_n^* \hat{a} \psi_m dx \quad (4.31)$ $= \langle n \hat{a} m \rangle \quad (4.32)$	n, m eigenvector indices a_{nm} matrix element ψ_n basis states \hat{a} operator x spatial coordinate
Bra vector	bra state vector $= \langle n $ (4.33)	$\langle \cdot $ bra
Ket vector	ket state vector $= m \rangle$ (4.34)	$ \cdot \rangle$ ket
Scalar product	$\langle n m \rangle = \int \psi_n^* \psi_m dx$ (4.35)	
Expectation	if $\Psi = \sum_n c_n \psi_n$ (4.36) then $\langle a \rangle = \sum_m \sum_n c_n^* c_m a_{nm}$ (4.37)	Ψ wavefunction c_n probability amplitudes

^aThe Dirac bracket, $\langle n | \hat{a} | m \rangle$, can also be written $\langle \psi_n | \hat{a} | \psi_m \rangle$.

4.3 Wave mechanics**Potential step^a**

		
Potential function	$V(x) = \begin{cases} 0 & (x < 0) \\ V_0 & (x \geq 0) \end{cases} \quad (4.38)$	V particle potential energy V_0 step height \hbar (Planck constant)/(2π)
Wavenumbers	$\hbar^2 k^2 = 2mE \quad (x < 0) \quad (4.39)$ $\hbar^2 q^2 = 2m(E - V_0) \quad (x > 0) \quad (4.40)$	k, q particle wavenumbers m particle mass E total particle energy
Amplitude reflection coefficient	$r = \frac{k - q}{k + q} \quad (4.41)$	r amplitude reflection coefficient
Amplitude transmission coefficient	$t = \frac{2k}{k + q} \quad (4.42)$	t amplitude transmission coefficient
Probability currents ^b	$j_I = \frac{\hbar k}{m} (1 - r ^2) \quad (4.43)$ $j_{II} = \frac{\hbar q}{m} t ^2 \quad (4.44)$	j_I particle flux in zone I j_{II} particle flux in zone II

^aOne-dimensional interaction with an incident particle of total energy $E = KE + V$. If $E < V_0$ then q is imaginary and $|r|^2 = 1$. $1/|q|$ is then a measure of the tunnelling depth.

^bParticle flux with the sign of increasing x .

Potential well^a

Potential function	$V(x) = \begin{cases} 0 & (x > a) \\ -V_0 & (x \leq a) \end{cases} \quad (4.45)$	V particle potential energy V_0 well depth \hbar (Planck constant)/(2π) $2a$ well width
Wavenumbers	$\hbar^2 k^2 = 2mE \quad (x > a) \quad (4.46)$	k, q particle wavenumbers
	$\hbar^2 q^2 = 2m(E + V_0) \quad (x < a) \quad (4.47)$	m particle mass E total particle energy
Amplitude reflection coefficient	$r = \frac{\mathbf{i}e^{-2ika}(q^2 - k^2)\sin 2qa}{2kq \cos 2qa - \mathbf{i}(q^2 + k^2)\sin 2qa} \quad (4.48)$	r amplitude reflection coefficient
Amplitude transmission coefficient	$t = \frac{2kqe^{-2ika}}{2kq \cos 2qa - \mathbf{i}(q^2 + k^2)\sin 2qa} \quad (4.49)$	t amplitude transmission coefficient
Probability currents ^b	$j_{\text{I}} = \frac{\hbar k}{m}(1 - r ^2) \quad (4.50)$	j_{I} particle flux in zone I
	$j_{\text{III}} = \frac{\hbar k}{m} t ^2 \quad (4.51)$	j_{III} particle flux in zone III
Ramsauer effect ^c	$E_n = -V_0 + \frac{n^2 \hbar^2 \pi^2}{8ma^2} \quad (4.52)$	n integer > 0 E_n Ramsauer energy
Bound states ($V_0 < E < 0$) ^d	$\tan qa = \begin{cases} k /q & \text{even parity} \\ -q/ k & \text{odd parity} \end{cases} \quad (4.53)$	
	$q^2 - k ^2 = 2mV_0/\hbar^2 \quad (4.54)$	

^aOne-dimensional interaction with an incident particle of total energy $E = KE + V > 0$.^bParticle flux in the sense of increasing x .^cIncident energy for which $2qa = n\pi$, $|r| = 0$, and $|t| = 1$.^dWhen $E < 0$, k is purely imaginary. $|k|$ and q are obtained by solving these implicit equations.

Barrier tunnelling^a

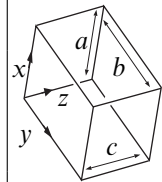
Potential function	$V(x) = \begin{cases} 0 & (x > a) \\ V_0 & (x \leq a) \end{cases} \quad (4.55)$	V particle potential energy V_0 well depth \hbar (Planck constant)/(2π) $2a$ barrier width
Wavenumber and tunnelling constant	$\hbar^2 k^2 = 2mE \quad (x > a) \quad (4.56)$	k incident wavenumber
	$\hbar^2 \kappa^2 = 2m(V_0 - E) \quad (x < a) \quad (4.57)$	κ tunnelling constant m particle mass E total energy ($< V_0$)
Amplitude reflection coefficient	$r = \frac{-i e^{-2ika} (k^2 + \kappa^2) \sinh 2\kappa a}{2k\kappa \cosh 2\kappa a - i(k^2 - \kappa^2) \sinh 2\kappa a} \quad (4.58)$	r amplitude reflection coefficient
Amplitude transmission coefficient	$t = \frac{2k\kappa e^{-2ika}}{2k\kappa \cosh 2\kappa a - i(k^2 - \kappa^2) \sinh 2\kappa a} \quad (4.59)$	t amplitude transmission coefficient
Tunnelling probability	$ t ^2 = \frac{4k^2 \kappa^2}{(k^2 + \kappa^2)^2 \sinh^2 2\kappa a + 4k^2 \kappa^2} \quad (4.60)$	$ t ^2$ tunnelling probability
	$\simeq \frac{16k^2 \kappa^2}{(k^2 + \kappa^2)^2} \exp(-4\kappa a) \quad (t ^2 \ll 1) \quad (4.61)$	
Probability currents ^b	$j_I = \frac{\hbar k}{m} (1 - r ^2) \quad (4.62)$	j_I particle flux in zone I
	$j_{III} = \frac{\hbar k}{m} t ^2 \quad (4.63)$	j_{III} particle flux in zone III

^aBy a particle of total energy $E = KE + V$, through a one-dimensional rectangular potential barrier height $V_0 > E$.

^bParticle flux in the sense of increasing x .

Particle in a rectangular box^a

Eigenfunctions	$\Psi_{lmn} = \left(\frac{8}{abc} \right)^{1/2} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \quad (4.64)$	Ψ_{lmn} eigenfunctions a, b, c box dimensions l, m, n integers ≥ 1
Energy levels	$E_{lmn} = \frac{h^2}{8M} \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) \quad (4.65)$	E_{lmn} energy h Planck constant M particle mass
Density of states	$\rho(E) dE = \frac{4\pi}{h^3} (2M^3 E)^{1/2} dE \quad (4.66)$	$\rho(E)$ density of states (per unit volume)



^aSpinless particle in a rectangular box bounded by the planes $x=0$, $y=0$, $z=0$, $x=a$, $y=b$, and $z=c$. The potential is zero inside and infinite outside the box.

Harmonic oscillator

Schrödinger equation	$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_n}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi_n = E_n \psi_n \quad (4.67)$	\hbar (Planck constant)/(2 π) m mass ψ_n n th eigenfunction x displacement
Energy levels ^a	$E_n = \left(n + \frac{1}{2}\right) \hbar \omega \quad (4.68)$	n integer ≥ 0 ω angular frequency E_n total energy in n th state
Eigen-functions	$\psi_n = \frac{H_n(x/a) \exp[-x^2/(2a^2)]}{(n! 2^n a \pi^{1/2})^{1/2}} \quad (4.69)$ where $a = \left(\frac{\hbar}{m\omega}\right)^{1/2}$	H_n Hermite polynomials
Hermite polynomials	$H_0(y) = 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2 - 2$ $H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y) \quad (4.70)$	y dummy variable

^a E_0 is the zero-point energy of the oscillator.

4

4.4 Hydrogenic atoms**Bohr model^a**

Quantisation condition	$\mu r_n^2 \Omega = n \hbar \quad (4.71)$	r_n n th orbit radius Ω orbital angular speed n principal quantum number (> 0)
Bohr radius	$a_0 = \frac{\epsilon_0 \hbar^2}{\pi m_e e^2} = \frac{\alpha}{4\pi R_\infty} \simeq 52.9 \text{ pm} \quad (4.72)$	a_0 Bohr radius μ reduced mass ($\simeq m_e$) $-e$ electronic charge
Orbit radius	$r_n = \frac{n^2}{Z} a_0 \frac{m_e}{\mu} \quad (4.73)$	Z atomic number h Planck constant \hbar $h/(2\pi)$
Total energy	$E_n = -\frac{\mu e^4 Z^2}{8\epsilon_0^2 \hbar^2 n^2} = -R_\infty h c \frac{\mu}{m_e} \frac{Z^2}{n^2} \quad (4.74)$	E_n total energy of n th orbit ϵ_0 permittivity of free space m_e electron mass
Fine structure constant	$\alpha = \frac{\mu_0 c e^2}{2\hbar} = \frac{e^2}{4\pi\epsilon_0 \hbar c} \simeq \frac{1}{137} \quad (4.75)$	α fine structure constant μ_0 permeability of free space
Hartree energy	$E_H = \frac{\hbar^2}{m_e a_0^2} \simeq 4.36 \times 10^{-18} \text{ J} \quad (4.76)$	E_H Hartree energy
Rydberg constant	$R_\infty = \frac{m_e c \alpha^2}{2\hbar} = \frac{m_e e^4}{8\hbar^3 \epsilon_0^2 c} = \frac{E_H}{2hc} \quad (4.77)$	R_∞ Rydberg constant c speed of light
Rydberg's formula ^b	$\frac{1}{\lambda_{mn}} = R_\infty \frac{\mu}{m_e} Z^2 \left(\frac{1}{n^2} - \frac{1}{m^2} \right) \quad (4.78)$	λ_{mn} photon wavelength m integer $> n$

^aBecause the Bohr model is strictly a two-body problem, the equations use reduced mass, $\mu = m_e m_{\text{nuc}}/(m_e + m_{\text{nuc}}) \simeq m_e$, where m_{nuc} is the nuclear mass, throughout. The orbit radius is therefore the electron–nucleus distance.

^bWavelength of the spectral line corresponding to electron transitions between orbits m and n .

Hydrogenlike atoms – Schrödinger solution^a

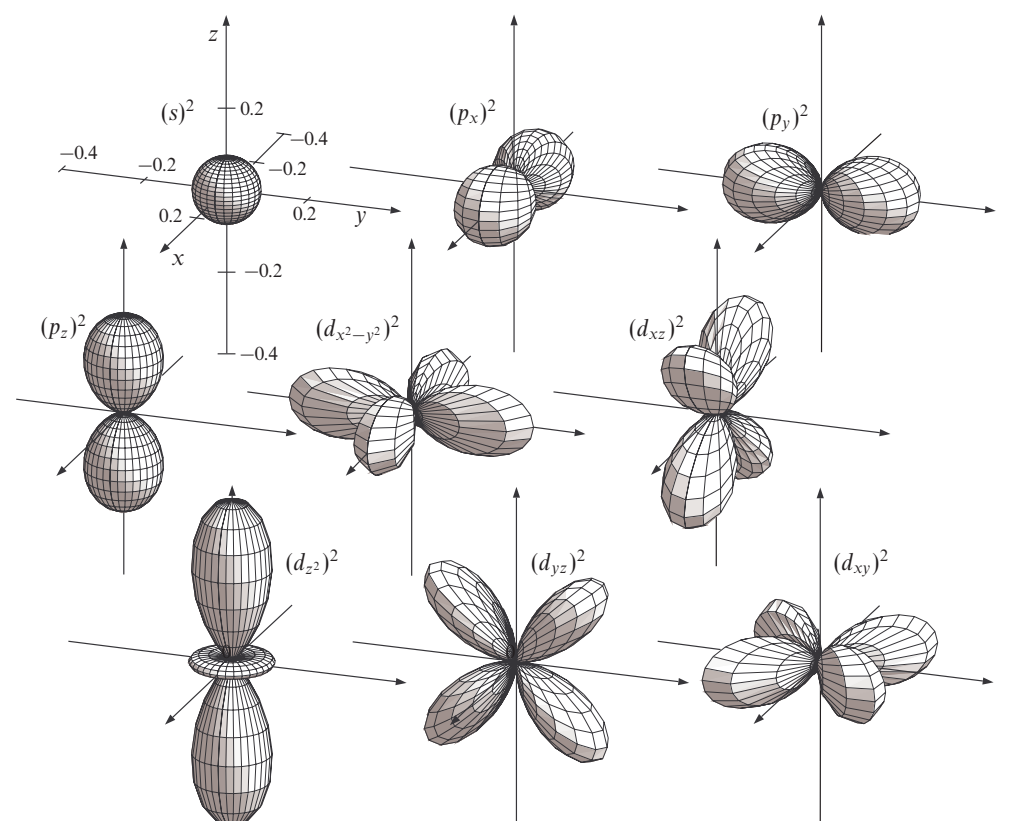
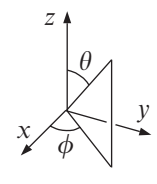
Schrödinger equation		
$-\frac{\hbar^2}{2\mu}\nabla^2\Psi_{nlm}-\frac{Ze^2}{4\pi\epsilon_0 r}\Psi_{nlm}=E_n\Psi_{nlm} \quad \text{with} \quad \mu=\frac{m_em_{\text{nuc}}}{m_e+m_{\text{nuc}}} \quad (4.79)$		
Eigenfunctions		
$\Psi_{nlm}(r,\theta,\phi)=\left[\frac{(n-l-1)!}{2n(n+l)!}\right]^{1/2}\left(\frac{2}{an}\right)^{3/2}x^l e^{-x/2}L_{n-l-1}^{2l+1}(x)Y_l^m(\theta,\phi) \quad (4.80)$		
with $a=\frac{m_e}{\mu}\frac{a_0}{Z}$, $x=\frac{2r}{an}$, and $L_{n-l-1}^{2l+1}(x)=\sum_{k=0}^{n-l-1}\frac{(l+n)!(-x)^k}{(2l+1+k)!(n-l-1-k)!k!}$		
Total energy	$E_n=-\frac{\mu e^4 Z^2}{8\epsilon_0^2 \hbar^2 n^2} \quad (4.81)$	E_n total energy ϵ_0 permittivity of free space
Radial expectation values	$\langle r \rangle = \frac{a}{2}[3n^2 - l(l+1)] \quad (4.82)$	h Planck constant m_e mass of electron
	$\langle r^2 \rangle = \frac{a^2 n^2}{2}[5n^2 + 1 - 3l(l+1)] \quad (4.83)$	\hbar $h/2\pi$ μ reduced mass ($\simeq m_e$)
	$\langle 1/r \rangle = \frac{1}{an^2} \quad (4.84)$	m_{nuc} mass of nucleus Ψ_{nlm} eigenfunctions
	$\langle 1/r^2 \rangle = \frac{2}{(2l+1)n^3 a^2} \quad (4.85)$	Ze charge of nucleus $-e$ electronic charge
Allowed quantum numbers and selection rules ^b	$n=1,2,3,\dots \quad (4.86)$	L_p^q associated Laguerre polynomials ^c
	$l=0,1,2,\dots,(n-1) \quad (4.87)$	a classical orbit radius, $n=1$
	$m=0,\pm 1,\pm 2,\dots,\pm l \quad (4.88)$	r electron–nucleus separation
	$\Delta n \neq 0 \quad (4.89)$	Y_l^m spherical harmonics
	$\Delta l = \pm 1 \quad (4.90)$	a_0 Bohr radius $= \frac{\epsilon_0 \hbar^2}{\pi m_e e^2}$
$\Delta m = 0 \quad \text{or} \quad \pm 1 \quad (4.91)$		
$\Psi_{100} = \frac{a^{-3/2}}{\pi^{1/2}} e^{-r/a} \quad \Psi_{200} = \frac{a^{-3/2}}{4(2\pi)^{1/2}} \left(2 - \frac{r}{a}\right) e^{-r/2a}$		
$\Psi_{210} = \frac{a^{-3/2}}{4(2\pi)^{1/2}} \frac{r}{a} e^{-r/2a} \cos\theta \quad \Psi_{21\pm 1} = \mp \frac{a^{-3/2}}{8\pi^{1/2}} \frac{r}{a} e^{-r/2a} \sin\theta e^{\pm i\phi}$		
$\Psi_{300} = \frac{a^{-3/2}}{81(3\pi)^{1/2}} \left(27 - 18\frac{r}{a} + 2\frac{r^2}{a^2}\right) e^{-r/3a} \quad \Psi_{310} = \frac{2^{1/2} a^{-3/2}}{81\pi^{1/2}} \left(6 - \frac{r}{a}\right) \frac{r}{a} e^{-r/3a} \cos\theta$		
$\Psi_{31\pm 1} = \mp \frac{a^{-3/2}}{81\pi^{1/2}} \left(6 - \frac{r}{a}\right) \frac{r}{a} e^{-r/3a} \sin\theta e^{\pm i\phi} \quad \Psi_{320} = \frac{a^{-3/2}}{81(6\pi)^{1/2}} \frac{r^2}{a^2} e^{-r/3a} (3\cos^2\theta - 1)$		
$\Psi_{32\pm 1} = \mp \frac{a^{-3/2}}{81\pi^{1/2}} \frac{r^2}{a^2} e^{-r/3a} \sin\theta \cos\theta e^{\pm i\phi} \quad \Psi_{32\pm 2} = \frac{a^{-3/2}}{162\pi^{1/2}} \frac{r^2}{a^2} e^{-r/3a} \sin^2\theta e^{\pm 2i\phi}$		

^aFor a single bound electron in a perfect nuclear Coulomb potential (nonrelativistic and spin-free).

^bFor dipole transitions between orbitals.

^cThe sign and indexing definitions for this function vary. This form is appropriate to Equation (4.80).

Orbital angular dependence

			
<i>s</i> orbital (<i>l</i> = 0)	$s = Y_0^0 = \text{constant}$	(4.92)	Y_l^m spherical harmonics ^a θ, ϕ spherical polar coordinates 
<i>p</i> orbitals (<i>l</i> = 1)	$p_x = \frac{-1}{2^{1/2}}(Y_1^1 - Y_1^{-1}) \propto \cos \phi \sin \theta$	(4.93)	
	$p_y = \frac{i}{2^{1/2}}(Y_1^1 + Y_1^{-1}) \propto \sin \phi \sin \theta$	(4.94)	
	$p_z = Y_1^0 \propto \cos \theta$	(4.95)	
<i>d</i> orbitals (<i>l</i> = 2)	$d_{x^2-y^2} = \frac{1}{2^{1/2}}(Y_2^2 + Y_2^{-2}) \propto \sin^2 \theta \cos 2\phi$	(4.96)	
	$d_{xz} = \frac{-1}{2^{1/2}}(Y_2^1 - Y_2^{-1}) \propto \sin \theta \cos \theta \cos \phi$	(4.97)	
	$d_{z^2} = Y_2^0 \propto (3 \cos^2 \theta - 1)$	(4.98)	
	$d_{yz} = \frac{i}{2^{1/2}}(Y_2^1 + Y_2^{-1}) \propto \sin \theta \cos \theta \sin \phi$	(4.99)	
	$d_{xy} = \frac{-i}{2^{1/2}}(Y_2^2 - Y_2^{-2}) \propto \sin^2 \theta \sin 2\phi$	(4.100)	

^aSee page 49 for the definition of spherical harmonics.

4.5 Angular momentum

Orbital angular momentum

Angular momentum operators	$\hat{\mathbf{L}} = \mathbf{r} \times \hat{\mathbf{p}}$	(4.101)	\mathbf{L} angular momentum
	$\hat{L}_z = \frac{\hbar}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$	(4.102)	\mathbf{p} linear momentum
	$= \frac{\hbar}{i} \frac{\partial}{\partial \phi}$	(4.103)	\mathbf{r} position vector
	$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$	(4.104)	xyz Cartesian coordinates
	$= -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$	(4.105)	$r\theta\phi$ spherical polar coordinates
Ladder operators	$\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y$	(4.106)	\hbar (Planck constant)/(2 π)
	$= \hbar e^{\pm i\phi} \left(i \cot \theta \frac{\partial}{\partial \phi} \pm \frac{\partial}{\partial \theta} \right)$	(4.107)	\hat{L}_{\pm} ladder operators
	$\hat{L}_{\pm} Y_l^{m_l} = \hbar [l(l+1) - m_l(m_l \pm 1)]^{1/2} Y_l^{m_l \pm 1}$	(4.108)	$Y_l^{m_l}$ spherical harmonics
Eigen-functions and eigenvalues	$\hat{L}^2 Y_l^{m_l} = l(l+1)\hbar^2 Y_l^{m_l} \quad (l \geq 0)$	(4.109)	l, m_l integers
	$\hat{L}_z Y_l^{m_l} = m_l \hbar Y_l^{m_l} \quad (m_l \leq l)$	(4.110)	
	$\hat{L}_z [\hat{L}_{\pm} Y_l^{m_l}(\theta, \phi)] = (m_l \pm 1) \hbar \hat{L}_{\pm} Y_l^{m_l}(\theta, \phi)$	(4.111)	
	$l\text{-multiplicity} = (2l+1)$	(4.112)	

Angular momentum commutation relations^a

Conservation of angular momentum ^b	$[\hat{H}, \hat{L}_z] = 0$	(4.113)	\mathbf{L} angular momentum
			p momentum
			H Hamiltonian
			\hat{L}_{\pm} ladder operators
$[\hat{L}_z, x] = i\hbar y$	(4.114)	$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$	(4.120)
$[\hat{L}_z, y] = -i\hbar x$	(4.115)	$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$	(4.121)
$[\hat{L}_z, z] = 0$	(4.116)	$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$	(4.122)
$[\hat{L}_x, \hat{p}_x] = i\hbar \hat{p}_y$	(4.117)	$[\hat{L}_+, \hat{L}_z] = -\hbar \hat{L}_+$	(4.123)
$[\hat{L}_z, \hat{p}_y] = -i\hbar \hat{p}_x$	(4.118)	$[\hat{L}_-, \hat{L}_z] = \hbar \hat{L}_-$	(4.124)
$[\hat{L}_z, \hat{p}_z] = 0$	(4.119)	$[\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z$	(4.125)
		$[\hat{L}^2, \hat{L}_{\pm}] = 0$	(4.126)
	$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0$	(4.127)	

^aThe commutation of a and b is defined as $[a, b] = ab - ba$ (see page 26). Similar expressions hold for S and J .

^bFor motion under a central force.

Clebsch–Gordan coefficients^a

$\langle j, -m_j | l_1, -m_1; l_2, -m_2 \rangle = (-1)^{l_1+l_2-j} \langle j, m_j | l_1, m_1; l_2, m_2 \rangle$

$l_1 \times l_2$		m_j	
m_1	m_2	j	j ...
m_1	m_2	coefficients $\langle j, m_j l_1, m_1; l_2, m_2 \rangle$	
\vdots	\vdots	\vdots	\vdots
1/2 × 1/2			
+1/2	+1/2	1	0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2
3/2 × 1/2			
+3/2	+1/2	2	1
+3/2	-1/2	1/4	3/4
+1/2	+1/2	3/4	-1/4
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2
1 × 1			
+1	+1	2	1
+1	0	1/2	1/2
0	+1	1/2	-1/2
+1	-1	1/6	1/2
0	0	2/3	0
-1	+1	1/6	-1/2
2 × 1			
+2	+1	3	2
+2	0	1/3	2/3
+1	+1	2/3	-1/3
+2	-1	1/15	1/3
+1	0	8/15	1/6
0	+1	6/15	-1/2
+1	-1	1/5	1/2
0	0	3/5	0
-1	+1	1/5	-1/2
3/2 × 3/2			
+3/2	+3/2	3	2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	-1/2
+3/2	-1/2	1/5	1/2
+1/2	+1/2	3/5	0
-1/2	+3/2	1/5	-1/2
+3/2	-3/2	1/20	1/4
+1/2	-1/2	9/20	1/4
-1/2	+1/2	9/20	-1/4
-3/2	+3/2	1/20	-1/4
2 × 3/2			
+2	+3/2	7/2	5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	-3/7
+2	-1/2	1/7	16/35
+1	+1/2	4/7	1/35
0	+3/2	2/7	-18/35
+2	-3/2	1/35	6/35
+1	-1/2	12/35	5/14
0	+1/2	18/35	-3/35
-1	+3/2	4/35	-27/70
+2	-2	1/70	1/10
+1	-1	8/35	2/5
0	0	18/35	0
-1	+1	8/35	-2/5
-2	+2	1/70	-1/10

^aOr “Wigner coefficients,” using the Condon–Shortley sign convention. Note that a square root is assumed over all coefficient digits, so that “-3/10” corresponds to $-\sqrt{3/10}$. Also for clarity, only values of $m_j \geq 0$ are listed here. The coefficients for $m_j < 0$ can be obtained from the symmetry relation $\langle j, -m_j | l_1, -m_1; l_2, -m_2 \rangle = (-1)^{l_1+l_2-j} \langle j, m_j | l_1, m_1; l_2, m_2 \rangle$.

Angular momentum addition^a

Total angular momentum	$\mathbf{J} = \mathbf{L} + \mathbf{S}$	(4.128)	\mathbf{J}, J total angular momentum
	$\hat{J}_z = \hat{L}_z + \hat{S}_z$	(4.129)	\mathbf{L}, L orbital angular momentum
	$\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\widehat{\mathbf{L} \cdot \mathbf{S}}$	(4.130)	\mathbf{S}, S spin angular momentum
	$\hat{J}_z \psi_{j,m_j} = m_j \hbar \psi_{j,m_j}$	(4.131)	ψ eigenfunctions
	$\hat{J}^2 \psi_{j,m_j} = j(j+1) \hbar^2 \psi_{j,m_j}$	(4.132)	m_j magnetic quantum number $ m_j \leq j$
	j -multiplicity $= (2l+1)(2s+1)$	(4.133)	j $(l+s) \geq j \geq l-s $
Mutually commuting sets	$\{L^2, S^2, J^2, J_z, \mathbf{L} \cdot \mathbf{S}\}$	(4.134)	{ } set of mutually commuting observables
	$\{L^2, S^2, L_z, S_z, J_z\}$	(4.135)	
Clebsch–Gordan coefficients ^b	$ j, m_j\rangle = \sum_{\substack{m_l, m_s \\ m_l + m_s = m_j}} \langle j, m_j l, m_l; s, m_s \rangle l, m_l\rangle s, m_s\rangle$	(4.136)	$ \cdot\rangle$ eigenstates $\langle \cdot \cdot \rangle$ Clebsch–Gordan coefficients

^aSumming spin and orbital angular momenta as examples, eigenstates $|s, m_s\rangle$ and $|l, m_l\rangle$.

^bOr “Wigner coefficients.” Assuming no L – S interaction.

Magnetic moments

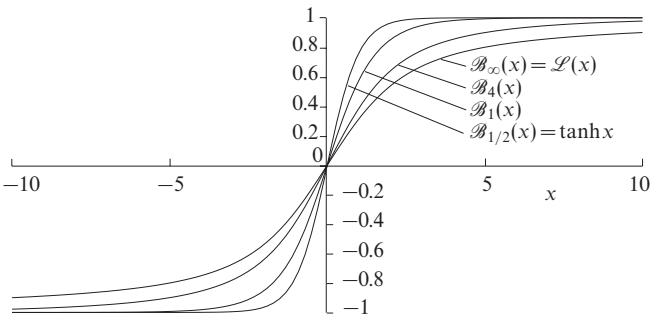
Bohr magneton	$\mu_B = \frac{e\hbar}{2m_e}$	(4.137)	μ_B Bohr magneton $-e$ electronic charge \hbar (Planck constant)/(2 π) m_e electron mass
Gyromagnetic ratio ^a	$\gamma = \frac{\text{orbital magnetic moment}}{\text{orbital angular momentum}}$	(4.138)	γ gyromagnetic ratio
Electron orbital gyromagnetic ratio	$\gamma_e = \frac{-\mu_B}{\hbar}$	(4.139)	γ_e electron gyromagnetic ratio
	$= \frac{-e}{2m_e}$	(4.140)	
Spin magnetic moment of an electron ^b	$\mu_{e,z} = -g_e \mu_B m_s$	(4.141)	$\mu_{e,z}$ z component of spin magnetic moment g_e electron g -factor ($\simeq 2.002$) m_s spin quantum number ($\pm 1/2$)
	$= \pm g_e \gamma_e \frac{\hbar}{2}$	(4.142)	
	$= \pm \frac{g_e e \hbar}{4m_e}$	(4.143)	
Landé g -factor ^c	$\mu_J = g_J \sqrt{J(J+1)} \mu_B$	(4.144)	μ_J total magnetic moment
	$\mu_{J,z} = -g_J \mu_B m_J$	(4.145)	$\mu_{J,z}$ z component of μ_J m_J magnetic quantum number
	$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$	(4.146)	J, L, S total, orbital, and spin quantum numbers g_J Landé g -factor

^aOr “magnetogyric ratio.”

^bThe electron g -factor equals exactly 2 in Dirac theory. The modification $g_e = 2 + \alpha/\pi + \dots$, where α is the fine structure constant, comes from quantum electrodynamics.

^cRelating the spin + orbital angular momenta of an electron to its total magnetic moment, assuming $g_e = 2$.

Quantum paramagnetism



$$\mathcal{B}_J(x) = \frac{2J+1}{2J} \coth \left[\frac{(2J+1)x}{2J} \right] - \frac{1}{2J} \coth \frac{x}{2J}$$

(4.147)

Brillouin function	$\mathcal{B}_J(x) \simeq \begin{cases} \frac{J+1}{3J} x & (x \ll 1) \\ \mathcal{L}(x) & (J \gg 1) \end{cases}$	(4.148)	$\mathcal{B}_J(x)$	Brillouin function
	$\mathcal{B}_{1/2}(x) = \tanh x$	(4.149)	J	total angular momentum quantum number
Mean magnetisation ^a	$\langle M \rangle = n \mu_B J g_J \mathcal{B}_J \left(J g_J \frac{\mu_B B}{k T} \right)$	(4.150)	$\mathcal{L}(x)$	Langevin function = $\coth x - 1/x$ (see page 144)
	$\langle M \rangle_{1/2} = n \mu_B \tanh \left(\frac{\mu_B B}{k T} \right)$	(4.151)	$\langle M \rangle$	mean magnetisation
$\langle M \rangle$ for isolated spins ($J = 1/2$)			n	number density of atoms
			g_J	Landé g-factor
			μ_B	Bohr magneton
			B	magnetic flux density
			k	Boltzmann constant
			T	temperature
			$\langle M \rangle_{1/2}$	mean magnetisation for $J = 1/2$ (and $g_J = 2$)

^aOf an ensemble of atoms in thermal equilibrium at temperature T , each with total angular momentum quantum number J .

4.6 Perturbation theory

Time-independent perturbation theory

Unperturbed states	$\hat{H}_0\psi_n = E_n\psi_n$ (ψ_n nondegenerate)	(4.152)	\hat{H}_0 unperturbed Hamiltonian ψ_n eigenfunctions of \hat{H}_0 E_n eigenvalues of \hat{H}_0 n integer ≥ 0
Perturbed Hamiltonian	$\hat{H} = \hat{H}_0 + \hat{H}'$	(4.153)	\hat{H} perturbed Hamiltonian \hat{H}' perturbation ($\ll \hat{H}_0$)
Perturbed eigenvalues ^a	$E'_k = E_k + \langle \psi_k \hat{H}' \psi_k \rangle$ $+ \sum_{n \neq k} \frac{ \langle \psi_k \hat{H}' \psi_n \rangle ^2}{E_k - E_n} + \dots$	(4.154)	E'_k perturbed eigenvalue ($\simeq E_k$) $\langle \rangle$ Dirac bracket
Perturbed eigenfunctions ^b	$\psi'_k = \psi_k + \sum_{n \neq k} \frac{\langle \psi_k \hat{H}' \psi_n \rangle}{E_k - E_n} \psi_n + \dots$	(4.155)	ψ'_k perturbed eigenfunction ($\simeq \psi_k$)

^aTo second order.

^bTo first order.

Time-dependent perturbation theory

Unperturbed stationary states	$\hat{H}_0\psi_n = E_n\psi_n$	(4.156)	\hat{H}_0 unperturbed Hamiltonian ψ_n eigenfunctions of \hat{H}_0 E_n eigenvalues of \hat{H}_0 n integer ≥ 0
Perturbed Hamiltonian	$\hat{H}(t) = \hat{H}_0 + \hat{H}'(t)$	(4.157)	\hat{H} perturbed Hamiltonian $\hat{H}'(t)$ perturbation ($\ll \hat{H}_0$) t time
Schrödinger equation	$[\hat{H}_0 + \hat{H}'(t)]\Psi(t) = i\hbar \frac{\partial \Psi(t)}{\partial t}$	(4.158)	Ψ wavefunction
	$\Psi(t=0) = \psi_0$	(4.159)	ψ_0 initial state \hbar (Planck constant)/(2π)
Perturbed wave-function ^a	$\Psi(t) = \sum_n c_n(t)\psi_n \exp(-iE_nt/\hbar)$	(4.160)	c_n probability amplitudes
	where		
	$c_n = \frac{-i}{\hbar} \int_0^t \langle \psi_n \hat{H}'(t') \psi_0 \rangle \exp[i(E_n - E_0)t'/\hbar] dt'$	(4.161)	
Fermi's golden rule	$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} \langle \psi_f \hat{H}' \psi_i \rangle ^2 \rho(E_f)$	(4.162)	$\Gamma_{i \rightarrow f}$ transition probability per unit time from state i to state f $\rho(E_f)$ density of final states

^aTo first order.

4.7 High energy and nuclear physics

Nuclear decay

Nuclear decay law	$N(t) = N(0)e^{-\lambda t}$	(4.163)	$N(t)$ number of nuclei remaining after time t t time
Half-life and mean life	$T_{1/2} = \frac{\ln 2}{\lambda}$	(4.164)	λ decay constant
	$\langle T \rangle = 1/\lambda$	(4.165)	$T_{1/2}$ half-life $\langle T \rangle$ mean lifetime
Successive decays $1 \rightarrow 2 \rightarrow 3$ (species 3 stable)	$N_1(t) = N_1(0)e^{-\lambda_1 t}$	(4.166)	N_1 population of species 1
	$N_2(t) = N_2(0)e^{-\lambda_2 t} + \frac{N_1(0)\lambda_1(e^{-\lambda_1 t} - e^{-\lambda_2 t})}{\lambda_2 - \lambda_1}$	(4.167)	N_2 population of species 2
	$N_3(t) = N_3(0) + N_2(0)(1 - e^{-\lambda_2 t}) + N_1(0)\left(1 + \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1}\right)$	(4.168)	N_3 population of species 3 λ_1 decay constant $1 \rightarrow 2$ λ_2 decay constant $2 \rightarrow 3$
Geiger's law ^a	$v^3 = a(R - x)$	(4.169)	v velocity of α particle x distance from source a constant
Geiger–Nuttall rule	$\log \lambda = b + c \log R$	(4.170)	R range b, c constants for each series α , β , and γ

^aFor α particles in air (empirical).

Nuclear binding energy

Liquid drop model ^a	$B = a_v A - a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_a \frac{(N - Z)^2}{A} + \delta(A)$	(4.171)	N number of neutrons A mass number ($= N + Z$) B semi-empirical binding energy Z number of protons a_v volume term ($\sim 15.8 \text{ MeV}$) a_s surface term ($\sim 18.0 \text{ MeV}$) a_c Coulomb term ($\sim 0.72 \text{ MeV}$) a_a asymmetry term ($\sim 23.5 \text{ MeV}$) a_p pairing term ($\sim 33.5 \text{ MeV}$)
	$\delta(A) \simeq \begin{cases} +a_p A^{-3/4} & Z, N \text{ both even} \\ -a_p A^{-3/4} & Z, N \text{ both odd} \\ 0 & \text{otherwise} \end{cases}$	(4.172)	
Semi-empirical mass formula	$M(Z, A) = Z M_H + N m_n - B$	(4.173)	$M(Z, A)$ atomic mass M_H mass of hydrogen atom m_n neutron mass

^aCoefficient values are empirical and approximate.

Nuclear collisions

Breit–Wigner formula ^a	$\sigma(E) = \frac{\pi}{k^2} g \frac{\Gamma_{ab}\Gamma_c}{(E - E_0)^2 + \Gamma^2/4} \quad (4.174)$	$\sigma(E)$ cross-section for $a + b \rightarrow c$
	$g = \frac{2J + 1}{(2s_a + 1)(2s_b + 1)} \quad (4.175)$	k incoming wavenumber g spin factor E total energy (PE + KE) E_0 resonant energy Γ width of resonant state R Γ_{ab} partial width into $a + b$ Γ_c partial width into c τ resonance lifetime J total angular momentum quantum number of R $s_{a,b}$ spins of a and b
Total width	$\Gamma = \Gamma_{ab} + \Gamma_c \quad (4.176)$	$\frac{d\sigma}{d\Omega}$ differential collision cross-section μ reduced mass $K = \mathbf{k}_{\text{in}} - \mathbf{k}_{\text{out}} $ (see footnote) r radial distance $V(r)$ potential energy of interaction
Resonance lifetime	$\tau = \frac{\hbar}{\Gamma} \quad (4.177)$	
Born scattering formula ^b	$\frac{d\sigma}{d\Omega} = \left \frac{2\mu}{\hbar^2} \int_0^\infty \frac{\sin Kr}{Kr} V(r) r^2 dr \right ^2 \quad (4.178)$	
Mott scattering formula ^c	$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E} \right)^2 \left[\csc^4 \frac{\chi}{2} + \sec^4 \frac{\chi}{2} + \frac{A \cos \left(\frac{\alpha}{\hbar v} \ln \tan^2 \frac{\chi}{2} \right)}{\sin^2 \frac{\chi}{2} \cos \frac{\chi}{2}} \right] \quad (4.179)$	\hbar (Planck constant)/ 2π α/r scattering potential energy χ scattering angle v closing velocity A = 2 for spin-zero particles, = -1 for spin-half particles
	$\frac{d\sigma}{d\Omega} \simeq \left(\frac{\alpha}{2E} \right)^2 \frac{4 - 3 \sin^2 \chi}{\sin^4 \chi} \quad (A = -1, \alpha \ll v\hbar) \quad (4.180)$	

^aFor the reaction $a + b \leftrightarrow R \rightarrow c$ in the centre of mass frame.

^bFor a central field. The Born approximation holds when the potential energy of scattering, V , is much less than the total kinetic energy. K is the magnitude of the change in the particle's wavevector due to scattering.

^cFor identical particles undergoing Coulomb scattering in the centre of mass frame. Nonidentical particles obey the Rutherford scattering formula (page 72).

Relativistic wave equations^a

Klein–Gordon equation (massive, spin zero particles)	$(\nabla^2 - m^2)\psi = \frac{\partial^2 \psi}{\partial t^2} \quad (4.181)$	ψ wavefunction m particle mass t time
Weyl equations (massless, spin 1/2 particles)	$\frac{\partial \psi}{\partial t} = \pm \left(\sigma_x \frac{\partial \psi}{\partial x} + \sigma_y \frac{\partial \psi}{\partial y} + \sigma_z \frac{\partial \psi}{\partial z} \right) \quad (4.182)$	ψ spinor wavefunction σ_i Pauli spin matrices (see page 26)
Dirac equation (massive, spin 1/2 particles)	$(\mathbf{i}\gamma^\mu \partial_\mu - m)\psi = 0 \quad (4.183)$	\mathbf{i} $\mathbf{i}^2 = -1$ γ^μ Dirac matrices:
	where $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad (4.184)$	$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}$
	$(\gamma^0)^2 = \mathbf{1}_4; \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbf{1}_4 \quad (4.185)$	$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$ $\mathbf{1}_n$ $n \times n$ unit matrix

^aWritten in natural units, with $c = \hbar = 1$.

Chapter 5 Thermodynamics

5.1 Introduction

The term *thermodynamics* is used here loosely and includes classical thermodynamics, statistical thermodynamics, thermal physics, and radiation processes. Notation in these subjects can be confusing and the conventions used here are those found in the majority of modern treatments. In particular:

- The internal energy of a system is defined in terms of the heat supplied *to* the system plus the work done *on* the system, that is, $dU = đQ + đW$.
- The lowercase symbol p is used for pressure. Probability density functions are denoted by $\text{pr}(x)$ and microstate probabilities by p_i .
- With the exception of *specific intensity*, quantities are taken as specific if they refer to unit mass and are distinguished from the extensive equivalent by using lowercase. Hence *specific volume*, v , equals V/m , where V is the volume of gas and m its mass. Also, the *specific heat capacity* of a gas at constant pressure is $c_p = C_p/m$, where C_p is the heat capacity of mass m of gas. Molar values take a subscript “m” (e.g., V_m for molar volume) and remain in upper case.
- The component held constant during a partial differentiation is shown after a vertical bar; hence $\left. \frac{\partial V}{\partial p} \right|_T$ is the partial differential of volume with respect to pressure, holding temperature constant.

The thermal properties of solids are dealt with more explicitly in the section on solid state physics (page 123). Note that in solid state literature *specific heat capacity* is often taken to mean heat capacity per unit volume.

5.2 Classical thermodynamics

Thermodynamic laws

Thermodynamic temperature ^a	$T \propto \lim_{p \rightarrow 0} (pV)$	(5.1)	T thermodynamic temperature V volume of a fixed mass of gas p gas pressure
Kelvin temperature scale	$T / \text{K} = 273.16 \frac{\lim_{p \rightarrow 0} (pV)_T}{\lim_{p \rightarrow 0} (pV)_{\text{tr}}}$	(5.2)	K kelvin unit tr temperature of the triple point of water
First law ^b	$dU = \delta Q + \delta W$	(5.3)	dU change in internal energy δW work done on system δQ heat supplied to system
Entropy ^c	$dS = \frac{\delta Q_{\text{rev}}}{T} \geq \frac{\delta Q}{T}$	(5.4)	S experimental entropy T temperature rev reversible change

^aAs determined with a gas thermometer. The idea of temperature is associated with the zeroth law of thermodynamics: *If two systems are in thermal equilibrium with a third, they are also in thermal equilibrium with each other.*

^bThe δ notation represents a differential change in a quantity that is not a function of state of the system.

^cAssociated with the second law of thermodynamics: *No process is possible with the sole effect of completely converting heat into work* (Kelvin statement).

Thermodynamic work^a

Hydrostatic pressure	$\delta W = -p dV$	(5.5)	p (hydrostatic) pressure dV volume change
Surface tension	$\delta W = \gamma dA$	(5.6)	δW work done on the system γ surface tension dA change in area
Electric field	$\delta W = \mathbf{E} \cdot d\mathbf{p}$	(5.7)	\mathbf{E} electric field $d\mathbf{p}$ induced electric dipole moment
Magnetic field	$\delta W = \mathbf{B} \cdot d\mathbf{m}$	(5.8)	\mathbf{B} magnetic flux density $d\mathbf{m}$ induced magnetic dipole moment
Electric current	$\delta W = \Delta\phi dq$	(5.9)	$\Delta\phi$ potential difference dq charge moved

^aThe sources of electric and magnetic fields are taken as being outside the thermodynamic system on which they are working.

Cycle efficiencies (thermodynamic)^a

Heat engine	$\eta = \frac{\text{work extracted}}{\text{heat input}} \leq \frac{T_h - T_l}{T_h}$	(5.10)	η efficiency T_h higher temperature T_l lower temperature
Refrigerator	$\eta = \frac{\text{heat extracted}}{\text{work done}} \leq \frac{T_l}{T_h - T_l}$	(5.11)	
Heat pump	$\eta = \frac{\text{heat supplied}}{\text{work done}} \leq \frac{T_h}{T_h - T_l}$	(5.12)	
Otto cycle ^b	$\eta = \frac{\text{work extracted}}{\text{heat input}} = 1 - \left(\frac{V_2}{V_1} \right)^{\gamma-1}$	(5.13)	$\frac{V_1}{V_2}$ compression ratio γ ratio of heat capacities (assumed constant)

^aThe equalities are for reversible cycles, such as Carnot cycles, operating between temperatures T_h and T_l .

^bIdealised reversible “petrol” (heat) engine.

Heat capacities

Constant volume	$C_V = \left. \frac{\delta Q}{dT} \right _V = \left. \frac{\partial U}{\partial T} \right _V = T \left. \frac{\partial S}{\partial T} \right _V$	(5.14)	C_V heat capacity, V constant Q heat T temperature V volume U internal energy S entropy
Constant pressure	$C_p = \left. \frac{\delta Q}{dT} \right _p = \left. \frac{\partial H}{\partial T} \right _p = T \left. \frac{\partial S}{\partial T} \right _p$	(5.15)	C_p heat capacity, p constant p pressure H enthalpy
Difference in heat capacities	$C_p - C_V = \left(\left. \frac{\partial U}{\partial V} \right _T + p \right) \left. \frac{\partial V}{\partial T} \right _p$	(5.16)	β_p isobaric expansivity
	$= \frac{VT\beta_p^2}{\kappa_T}$	(5.17)	κ_T isothermal compressibility
Ratio of heat capacities	$\gamma = \frac{C_p}{C_V} = \frac{\kappa_T}{\kappa_S}$	(5.18)	γ ratio of heat capacities κ_S adiabatic compressibility

Thermodynamic coefficients

Isobaric expansivity ^a	$\beta_p = \frac{1}{V} \left. \frac{\partial V}{\partial T} \right _p$	(5.19)	β_p isobaric expansivity V volume T temperature
Isothermal compressibility	$\kappa_T = -\frac{1}{V} \left. \frac{\partial V}{\partial p} \right _T$	(5.20)	κ_T isothermal compressibility p pressure
Adiabatic compressibility	$\kappa_S = -\frac{1}{V} \left. \frac{\partial V}{\partial p} \right _S$	(5.21)	κ_S adiabatic compressibility
Isothermal bulk modulus	$K_T = \frac{1}{\kappa_T} = -V \left. \frac{\partial p}{\partial V} \right _T$	(5.22)	K_T isothermal bulk modulus
Adiabatic bulk modulus	$K_S = \frac{1}{\kappa_S} = -V \left. \frac{\partial p}{\partial V} \right _S$	(5.23)	K_S adiabatic bulk modulus

^aAlso called “cubic expansivity” or “volume expansivity.” The linear expansivity is $\alpha_p = \beta_p/3$.

Expansion processes

Joule expansion ^a	$\eta = \frac{\partial T}{\partial V} \Big _U = -\frac{T^2}{C_V} \frac{\partial(p/T)}{\partial T} \Big _V$	(5.24)	η Joule coefficient
	$= -\frac{1}{C_V} \left(T \frac{\partial p}{\partial T} \Big _V - p \right)$	(5.25)	T temperature p pressure U internal energy C_V heat capacity, V constant
Joule–Kelvin expansion ^b	$\mu = \frac{\partial T}{\partial p} \Big _H = \frac{T^2}{C_p} \frac{\partial(V/T)}{\partial T} \Big _p$	(5.26)	μ Joule–Kelvin coefficient
	$= \frac{1}{C_p} \left(T \frac{\partial V}{\partial T} \Big _p - V \right)$	(5.27)	V volume H enthalpy C_p heat capacity, p constant

^aExpansion with no change in internal energy.

^bExpansion with no change in enthalpy. Also known as a “Joule–Thomson expansion” or “throttling” process.

Thermodynamic potentials^a

Internal energy	$dU = T dS - p dV + \mu dN$	(5.28)	U internal energy T temperature S entropy μ chemical potential N number of particles
Enthalpy	$H = U + pV$	(5.29)	H enthalpy
	$dH = T dS + V dp + \mu dN$	(5.30)	p pressure V volume
Helmholtz free energy ^b	$F = U - TS$	(5.31)	F Helmholtz free energy
	$dF = -S dT - p dV + \mu dN$	(5.32)	
Gibbs free energy ^c	$G = U - TS + pV$	(5.33)	G Gibbs free energy
	$= F + pV = H - TS$	(5.34)	
	$dG = -S dT + V dp + \mu dN$	(5.35)	
Grand potential	$\Phi = F - \mu N$	(5.36)	Φ grand potential
	$d\Phi = -S dT - p dV - N d\mu$	(5.37)	
Gibbs–Duhem relation	$-S dT + V dp - N d\mu = 0$	(5.38)	
Availability	$A = U - T_0 S + p_0 V$	(5.39)	A availability
	$dA = (T - T_0) dS - (p - p_0) dV$	(5.40)	T_0 temperature of surroundings p_0 pressure of surroundings

^a $dN=0$ for a closed system.

^bSometimes called the “work function.”

^cSometimes called the “thermodynamic potential.”

Maxwell's relations

Maxwell 1	$\left. \frac{\partial T}{\partial V} \right _S = - \left. \frac{\partial p}{\partial S} \right _V \quad \left(= \frac{\partial^2 U}{\partial S \partial V} \right)$	(5.41)	U internal energy T temperature V volume
Maxwell 2	$\left. \frac{\partial T}{\partial p} \right _S = \left. \frac{\partial V}{\partial S} \right _p \quad \left(= \frac{\partial^2 H}{\partial p \partial S} \right)$	(5.42)	H enthalpy S entropy p pressure
Maxwell 3	$\left. \frac{\partial p}{\partial T} \right _V = \left. \frac{\partial S}{\partial V} \right _T \quad \left(= \frac{\partial^2 F}{\partial T \partial V} \right)$	(5.43)	F Helmholtz free energy
Maxwell 4	$\left. \frac{\partial V}{\partial T} \right _p = - \left. \frac{\partial S}{\partial p} \right _T \quad \left(= \frac{\partial^2 G}{\partial p \partial T} \right)$	(5.44)	G Gibbs free energy

Gibbs–Helmholtz equations

$U = -T^2 \left. \frac{\partial(F/T)}{\partial T} \right _V$	(5.45)	F Helmholtz free energy U internal energy G Gibbs free energy
$G = -V^2 \left. \frac{\partial(F/V)}{\partial V} \right _T$	(5.46)	H enthalpy T temperature
$H = -T^2 \left. \frac{\partial(G/T)}{\partial T} \right _p$	(5.47)	p pressure V volume

Phase transitions

Heat absorbed	$L = T(S_2 - S_1)$	(5.48)	L (latent) heat absorbed ($1 \rightarrow 2$) T temperature of phase change S entropy
Clausius–Clapeyron equation ^a	$\frac{dp}{dT} = \frac{S_2 - S_1}{V_2 - V_1}$	(5.49)	p pressure V volume
	$= \frac{L}{T(V_2 - V_1)}$	(5.50)	1,2 phase states
Coexistence curve ^b	$p(T) \propto \exp\left(\frac{-L}{RT}\right)$	(5.51)	R molar gas constant
Ehrenfest's equation ^c	$\frac{dp}{dT} = \frac{\beta_{p2} - \beta_{p1}}{\kappa_{T2} - \kappa_{T1}}$	(5.52)	β_p isobaric expansivity κ_T isothermal compressibility
	$= \frac{1}{VT} \frac{C_{p2} - C_{p1}}{\beta_{p2} - \beta_{p1}}$	(5.53)	C_p heat capacity (p constant)
Gibbs's phase rule	$P + F = C + 2$	(5.54)	P number of phases in equilibrium F number of degrees of freedom C number of components

^aPhase boundary gradient for a first-order transition. Equation (5.50) is sometimes called the "Clapeyron equation."

^bFor $V_2 \gg V_1$, e.g., if phase 1 is a liquid and phase 2 a vapour.

^cFor a second-order phase transition.

5.3 Gas laws

Ideal gas

Joule's law	$U = U(T)$	(5.55)	U internal energy T temperature
Boyle's law	$pV _T = \text{constant}$	(5.56)	p pressure V volume
Equation of state (Ideal gas law)	$pV = nRT$	(5.57)	n number of moles R molar gas constant
Adiabatic equations	$pV^\gamma = \text{constant}$	(5.58)	γ ratio of heat capacities (C_p/C_V) ΔW work done on system
	$TV^{(\gamma-1)} = \text{constant}$	(5.59)	
	$T^\gamma p^{(1-\gamma)} = \text{constant}$	(5.60)	
	$\Delta W = \frac{1}{\gamma-1} (p_2 V_2 - p_1 V_1)$	(5.61)	
Internal energy	$U = \frac{nRT}{\gamma-1}$	(5.62)	ΔQ heat supplied to system 1,2 initial and final states
Reversible isothermal expansion	$\Delta Q = nRT \ln(V_2/V_1)$	(5.63)	
Joule expansion ^a	$\Delta S = nR \ln(V_2/V_1)$	(5.64)	ΔS change in entropy of the system

^aSince $\Delta Q = 0$ for a Joule expansion, ΔS is due entirely to irreversibility. Because entropy is a function of state it has the same value as for the reversible isothermal expansion, where $\Delta S = \Delta Q/T$.

Virial expansion

Virial expansion	$pV = RT \left(1 + \frac{B_2(T)}{V} + \frac{B_3(T)}{V^2} + \dots \right)$	(5.65)	p pressure V volume R molar gas constant T temperature B_i virial coefficients
Boyle temperature	$B_2(T_B) = 0$	(5.66)	T_B Boyle temperature

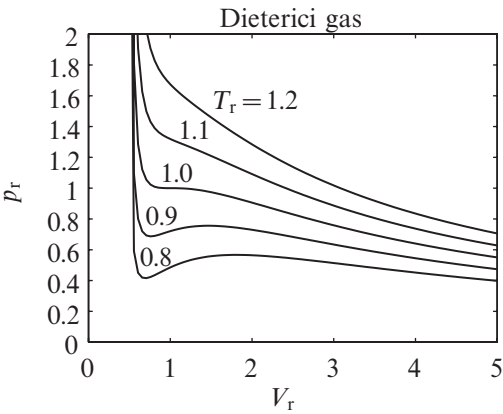
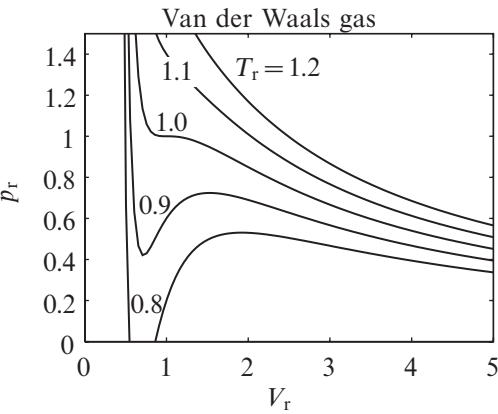
Van der Waals gas

Equation of state	$\left(p + \frac{a}{V_m^2}\right)(V_m - b) = RT \quad (5.67)$	p pressure V_m molar volume R molar gas constant T temperature a, b van der Waals' constants
Critical point	$T_c = 8a/(27Rb) \quad (5.68)$	T_c critical temperature
	$p_c = a/(27b^2) \quad (5.69)$	p_c critical pressure
	$V_{mc} = 3b \quad (5.70)$	V_{mc} critical molar volume
Reduced equation of state	$\left(p_r + \frac{3}{V_r^2}\right)(3V_r - 1) = 8T_r \quad (5.71)$	$p_r = p/p_c$ $V_r = V_m/V_{mc}$ $T_r = T/T_c$

Dieterici gas

Equation of state	$p = \frac{RT}{V_m - b'} \exp\left(\frac{-a'}{RTV_m}\right) \quad (5.72)$	p pressure V_m molar volume R molar gas constant T temperature a', b' Dieterici's constants
Critical point	$T_c = a'/(4Rb') \quad (5.73)$	T_c critical temperature
	$p_c = a'/(4b'^2e^2) \quad (5.74)$	p_c critical pressure
	$V_{mc} = 2b' \quad (5.75)$	V_{mc} critical molar volume $e = 2.71828\dots$
Reduced equation of state	$p_r = \frac{T_r}{2V_r - 1} \exp\left(2 - \frac{2}{V_r T_r}\right) \quad (5.76)$	$p_r = p/p_c$ $V_r = V_m/V_{mc}$ $T_r = T/T_c$

5



5.4 Kinetic theory

Monatomic gas

Pressure	$p = \frac{1}{3}nm\langle c^2 \rangle$	(5.77)	p pressure n number density = N/V m particle mass $\langle c^2 \rangle$ mean squared particle velocity
Equation of state of an ideal gas	$pV = NkT$	(5.78)	V volume k Boltzmann constant N number of particles T temperature
Internal energy	$U = \frac{3}{2}NkT = \frac{N}{2}m\langle c^2 \rangle$	(5.79)	U internal energy
Heat capacities	$C_V = \frac{3}{2}Nk$	(5.80)	C_V heat capacity, constant V
	$C_p = C_V + Nk = \frac{5}{2}Nk$	(5.81)	C_p heat capacity, constant p
	$\gamma = \frac{C_p}{C_V} = \frac{5}{3}$	(5.82)	γ ratio of heat capacities
Entropy (Sackur–Tetrode equation) ^a	$S = Nk \ln \left[\left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} e^{5/2} \frac{V}{N} \right]$	(5.83)	S entropy \hbar = (Planck constant)/(2π) e = 2.71828...

^aFor the uncondensed gas. The factor $\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}$ is the quantum concentration of the particles, n_Q . Their thermal de Broglie wavelength, λ_T , approximately equals $n_Q^{-1/3}$.

Maxwell–Boltzmann distribution^a

Particle speed distribution	$\text{pr}(c) \, dc = \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(\frac{-mc^2}{2kT} \right) 4\pi c^2 \, dc$	(5.84)	pr probability density m particle mass k Boltzmann constant T temperature c particle speed
Particle energy distribution	$\text{pr}(E) \, dE = \frac{2E^{1/2}}{\pi^{1/2}(kT)^{3/2}} \exp\left(\frac{-E}{kT} \right) dE$	(5.85)	E particle kinetic energy ($= mc^2/2$)
Mean speed	$\langle c \rangle = \left(\frac{8kT}{\pi m} \right)^{1/2}$	(5.86)	$\langle c \rangle$ mean speed
rms speed	$c_{\text{rms}} = \left(\frac{3kT}{m} \right)^{1/2} = \left(\frac{3\pi}{8} \right)^{1/2} \langle c \rangle$	(5.87)	c_{rms} root mean squared speed
Most probable speed	$\hat{c} = \left(\frac{2kT}{m} \right)^{1/2} = \left(\frac{\pi}{4} \right)^{1/2} \langle c \rangle$	(5.88)	\hat{c} most probable speed

^aProbability density functions normalised so that $\int_0^\infty \text{pr}(x) \, dx = 1$.

Transport properties

Mean free path ^a	$l = \frac{1}{\sqrt{2}\pi d^2 n}$	(5.89)	l mean free path d molecular diameter n particle number density
Survival equation ^b	$\text{pr}(x) = \exp(-x/l)$	(5.90)	pr probability x linear distance
Flux through a plane ^c	$J = \frac{1}{4}n\langle c \rangle$	(5.91)	J molecular flux $\langle c \rangle$ mean molecular speed
Self-diffusion (Fick's law of diffusion) ^d	$\mathbf{J} = -D\nabla n$	(5.92)	D diffusion coefficient
	where $D \simeq \frac{2}{3}l\langle c \rangle$	(5.93)	
Thermal conductivity ^d	$\mathbf{H} = -\lambda\nabla T$	(5.94)	\mathbf{H} heat flux per unit area
	$\nabla^2 T = \frac{1}{D} \frac{\partial T}{\partial t}$	(5.95)	λ thermal conductivity T temperature
	for monatomic gas $\lambda \simeq \frac{5}{4}\rho l\langle c \rangle c_V$	(5.96)	ρ density c_V specific heat capacity, V constant
Viscosity ^d	$\eta \simeq \frac{1}{2}\rho l\langle c \rangle$	(5.97)	η dynamic viscosity x displacement of sphere in x direction after time t
Brownian motion (of a sphere)	$\langle x^2 \rangle = \frac{kTt}{3\pi\eta a}$	(5.98)	k Boltzmann constant t time interval a sphere radius
Free molecular flow (Knudsen flow) ^e	$\frac{dM}{dt} = \frac{4R_p^3}{3L} \left(\frac{2\pi m}{k} \right)^{1/2} \left(\frac{p_1}{T_1^{1/2}} - \frac{p_2}{T_2^{1/2}} \right)$	(5.99)	$\frac{dM}{dt}$ mass flow rate R_p pipe radius L pipe length m particle mass p pressure

^aFor a perfect gas of hard, spherical particles with a Maxwell-Boltzmann speed distribution.

^bProbability of travelling distance x without a collision.

^cFrom the side where the number density is n , assuming an isotropic velocity distribution. Also known as “collision number.”

^dSimplistic kinetic theory yields numerical coefficients of 1/3 for D , λ and η .

^eThrough a pipe from end 1 to end 2, assuming $R_p \ll l$ (i.e., at very low pressure).

Gas equipartition

Classical equipartition ^a	$E_q = \frac{1}{2}kT$	(5.100)	E_q energy per quadratic degree of freedom k Boltzmann constant T temperature
Ideal gas heat capacities	$C_V = \frac{1}{2}fNk = \frac{1}{2}fnR$	(5.101)	C_V heat capacity, V constant
	$C_p = Nk \left(1 + \frac{f}{2} \right)$	(5.102)	C_p heat capacity, p constant N number of molecules
	$\gamma = \frac{C_p}{C_V} = 1 + \frac{2}{f}$	(5.103)	f number of degrees of freedom n number of moles
			R molar gas constant γ ratio of heat capacities

^aSystem in thermal equilibrium at temperature T .

5.5 Statistical thermodynamics

Statistical entropy

Boltzmann formula ^a	$S = k \ln W \quad (5.104)$ $\simeq k \ln g(E) \quad (5.105)$	S entropy k Boltzmann constant W number of accessible microstates $g(E)$ density of microstates with energy E
Gibbs entropy ^b	$S = -k \sum_i p_i \ln p_i \quad (5.106)$	\sum_i sum over microstates p_i probability that the system is in microstate i
N two-level systems	$W = \frac{N!}{(N-n)!n!} \quad (5.107)$	N number of systems n number in upper state
N harmonic oscillators	$W = \frac{(Q+N-1)!}{Q!(N-1)!} \quad (5.108)$	Q total number of energy quanta available

^aSometimes called “configurational entropy.” Equation (5.105) is true only for large systems.

^bSometimes called “canonical entropy.”

Ensemble probabilities

Microcanonical ensemble ^a	$p_i = \frac{1}{W} \quad (5.109)$	p_i probability that the system is in microstate i W number of accessible microstates
Partition function ^b	$Z = \sum_i e^{-\beta E_i} \quad (5.110)$	Z partition function \sum_i sum over microstates $\beta = 1/(kT)$ E_i energy of microstate i
Canonical ensemble (Boltzmann distribution) ^c	$p_i = \frac{1}{Z} e^{-\beta E_i} \quad (5.111)$	k Boltzmann constant T temperature
Grand partition function	$\Xi = \sum_i e^{-\beta(E_i - \mu N_i)} \quad (5.112)$	Ξ grand partition function μ chemical potential N_i number of particles in microstate i
Grand canonical ensemble (Gibbs distribution) ^d	$p_i = \frac{1}{\Xi} e^{-\beta(E_i - \mu N_i)} \quad (5.113)$	

^aEnergy fixed.

^bAlso called “sum over states.”

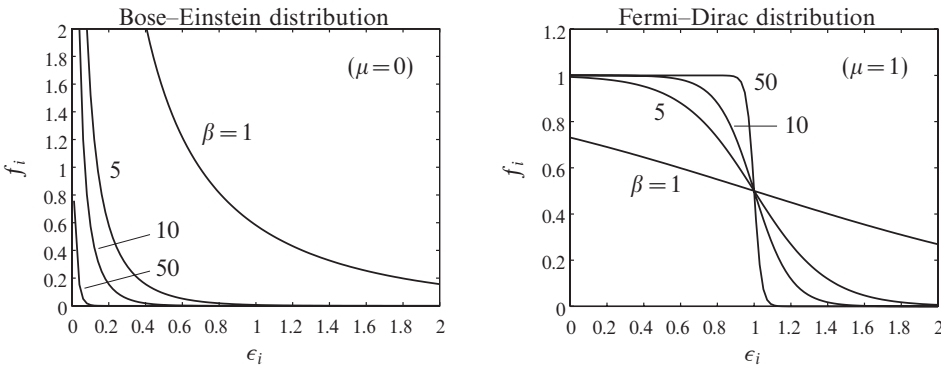
^cTemperature fixed.

^dTemperature fixed. Exchange of both heat and particles with a reservoir.

Macroscopic thermodynamic variables

Helmholtz free energy	$F = -kT \ln Z$	(5.114)	F Helmholtz free energy k Boltzmann constant T temperature Z partition function
Grand potential	$\Phi = -kT \ln \Xi$	(5.115)	Φ grand potential Ξ grand partition function
Internal energy	$U = F + TS = -\frac{\partial \ln Z}{\partial \beta} \Big _{V,N}$	(5.116)	U internal energy $\beta = 1/(kT)$
Entropy	$S = -\frac{\partial F}{\partial T} \Big _{V,N} = \frac{\partial (kT \ln Z)}{\partial T} \Big _{V,N}$	(5.117)	S entropy N number of particles
Pressure	$p = -\frac{\partial F}{\partial V} \Big _{T,N} = \frac{\partial (kT \ln Z)}{\partial V} \Big _{T,N}$	(5.118)	p pressure
Chemical potential	$\mu = \frac{\partial F}{\partial N} \Big _{V,T} = -\frac{\partial (kT \ln Z)}{\partial N} \Big _{V,T}$	(5.119)	μ chemical potential

Identical particles

			
Bose-Einstein distribution ^a	$f_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$	(5.120)	f_i mean occupation number of i th state $\beta = 1/(kT)$
Fermi-Dirac distribution ^b	$f_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}$	(5.121)	ϵ_i energy quantum for i th state μ chemical potential
Fermi energy ^c	$\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{g} \right)^{2/3}$	(5.122)	ϵ_F Fermi energy \hbar (Planck constant)/(2π) n particle number density m particle mass
Bose condensation temperature	$T_c = \frac{2\pi\hbar^2}{mk} \left[\frac{n}{g\zeta(3/2)} \right]^{2/3}$	(5.123)	g spin degeneracy ($=2s+1$) ζ Riemann zeta function $\zeta(3/2) \simeq 2.612$ T_c Bose condensation temperature

^aFor bosons. $f_i \geq 0$.

^bFor fermions. $0 \leq f_i \leq 1$.

^cFor noninteracting particles. At low temperatures, $\mu \simeq \epsilon_F$.

Population densities^a

Boltzmann excitation equation	$\frac{n_{mj}}{n_{lj}} = \frac{g_{mj}}{g_{lj}} \exp \left[\frac{-(\chi_{mj} - \chi_{lj})}{kT} \right] \quad (5.124)$	n_{ij} number density of atoms in excitation level i of ionisation state j ($j=0$ if not ionised)
	$= \frac{g_{mj}}{g_{lj}} \exp \left(\frac{-h\nu_{lm}}{kT} \right) \quad (5.125)$	g_{ij} level degeneracy χ_{ij} excitation energy relative to the ground state
Partition function	$Z_j(T) = \sum_i g_{ij} \exp \left(\frac{-\chi_{ij}}{kT} \right) \quad (5.126)$	ν_{ij} photon transition frequency h Planck constant
	$\frac{n_{ij}}{N_j} = \frac{g_{ij}}{Z_j(T)} \exp \left(\frac{-\chi_{ij}}{kT} \right) \quad (5.127)$	k Boltzmann constant T temperature
Saha equation (general)		Z_j partition function for ionisation state j
$n_{ij} = n_{0,j+1} n_e \frac{g_{ij}}{g_{0,j+1}} \frac{h^3}{2} (2\pi m_e kT)^{-3/2} \exp \left(\frac{\chi_{Ij} - \chi_{ij}}{kT} \right) \quad (5.128)$		N_j total number density in ionisation state j
Saha equation (ion populations)		n_e electron number density
$\frac{N_j}{N_{j+1}} = n_e \frac{Z_j(T)}{Z_{j+1}(T)} \frac{h^3}{2} (2\pi m_e kT)^{-3/2} \exp \left(\frac{\chi_{Ij}}{kT} \right) \quad (5.129)$		m_e electron mass χ_{Ij} ionisation energy of atom in ionisation state j

^aAll equations apply only under conditions of local thermodynamic equilibrium (LTE). In atoms with no magnetic splitting, the degeneracy of a level with total angular momentum quantum number J is $g_{ij} = 2J + 1$.

5.6 Fluctuations and noise

Thermodynamic fluctuations^a

Fluctuation probability	$\text{pr}(x) \propto \exp[S(x)/k] \quad (5.130)$	pr probability density
	$\propto \exp \left[\frac{-A(x)}{kT} \right] \quad (5.131)$	x unconstrained variable S entropy A availability
General variance	$\text{var}[x] = kT \left[\frac{\partial^2 A(x)}{\partial x^2} \right]^{-1} \quad (5.132)$	$\text{var}[\cdot]$ mean square deviation k Boltzmann constant T temperature
Temperature fluctuations	$\text{var}[T] = kT \frac{\partial T}{\partial S} \Big _V = \frac{kT^2}{C_V} \quad (5.133)$	V volume C_V heat capacity, V constant
Volume fluctuations	$\text{var}[V] = -kT \frac{\partial V}{\partial p} \Big _T = \kappa_T V kT \quad (5.134)$	p pressure κ_T isothermal compressibility
Entropy fluctuations	$\text{var}[S] = kT \frac{\partial S}{\partial T} \Big _p = kC_p \quad (5.135)$	C_p heat capacity, p constant
Pressure fluctuations	$\text{var}[p] = -kT \frac{\partial p}{\partial V} \Big _S = \frac{K_S kT}{V} \quad (5.136)$	K_S adiabatic bulk modulus
Density fluctuations	$\text{var}[n] = \frac{n^2}{V^2} \text{var}[V] = \frac{n^2}{V} \kappa_T kT \quad (5.137)$	n number density

^aIn part of a large system, whose mean temperature is fixed. Quantum effects are assumed negligible.

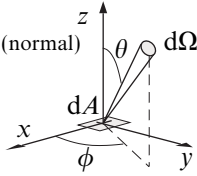
Noise

Nyquist's noise theorem	$dw = kT \cdot \beta \epsilon (e^{\beta \epsilon} - 1)^{-1} dv$	(5.138)	w	exchangeable noise power
	$= kT_N dv$	(5.139)	k	Boltzmann constant
	$\simeq kT dv \quad (hv \ll kT)$	(5.140)	T	temperature
			T_N	noise temperature
Johnson (thermal) noise voltage ^a			$\beta \epsilon$	$= hv / (kT)$
			v	frequency
			h	Planck constant
			v_{rms}	rms noise voltage
Shot noise (electrical)	$v_{\text{rms}} = (4kT_N R \Delta v)^{1/2}$	(5.141)	R	resistance
			Δv	bandwidth
			I_{rms}	rms noise current
			$-e$	electronic charge
Noise figure ^b	$I_{\text{rms}} = (2eI_0 \Delta v)^{1/2}$	(5.142)	I_0	mean current
			f_{dB}	noise figure (decibels)
			T_0	ambient temperature (usually taken as 290 K)
Relative power	$f_{\text{dB}} = 10 \log_{10} \left(1 + \frac{T_N}{T_0} \right)$	(5.143)		
	$G = 10 \log_{10} \left(\frac{P_2}{P_1} \right)$	(5.144)	G	decibel gain of P_2 over P_1
			P_1, P_2	power levels

^aThermal voltage over an open-circuit resistance.
^bNoise figure can also be defined as $f = 1 + T_N/T_0$, when it is also called “noise factor.”

5.7 Radiation processes

Radiometry^a

Radiant energy ^b	$Q_e = \iiint L_e \cos \theta \, dA \, d\Omega \, dt \quad \text{J} \quad (5.145)$	Q_e radiant energy L_e radiance (generally a function of position and direction) θ angle between dir. of $d\Omega$ and normal to dA Ω solid angle
Radiant flux ("radiant power")	$\Phi_e = \frac{\partial Q_e}{\partial t} \quad \text{W} \quad (5.146)$ $= \iint L_e \cos \theta \, dA \, d\Omega \quad (5.147)$	A area t time Φ_e radiant flux
Radiant energy density ^c	$W_e = \frac{\partial Q_e}{\partial V} \quad \text{J m}^{-3} \quad (5.148)$	W_e radiant energy density dV differential volume of propagation medium
Radiant exitance ^d	$M_e = \frac{\partial \Phi_e}{\partial A} \quad \text{W m}^{-2} \quad (5.149)$ $= \int L_e \cos \theta \, d\Omega \quad (5.150)$	M_e radiant exitance
Irradiance ^e	$E_e = \frac{\partial \Phi_e}{\partial A} \quad \text{W m}^{-2} \quad (5.151)$ $= \int L_e \cos \theta \, d\Omega \quad (5.152)$	
Radiant intensity	$I_e = \frac{\partial \Phi_e}{\partial \Omega} \quad \text{W sr}^{-1} \quad (5.153)$ $= \int L_e \cos \theta \, dA \quad (5.154)$	
Radiance	$L_e = \frac{1}{\cos \theta} \frac{\partial^2 \Phi_e}{\partial A \, d\Omega} \quad \text{W m}^{-2} \text{sr}^{-1} \quad (5.155)$ $= \frac{1}{\cos \theta} \frac{\partial I_e}{\partial A} \quad (5.156)$	

^aRadiometry is concerned with the treatment of light as energy.

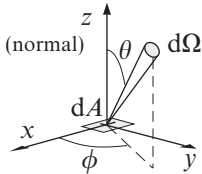
^bSometimes called "total energy." Note that we assume opaque radiant surfaces, so that $0 \leq \theta \leq \pi/2$.

^cThe instantaneous amount of radiant energy contained in a unit volume of propagation medium.

^dPower per unit area leaving a surface. For a perfectly diffusing surface, $M_e = \pi L_e$.

^ePower per unit area incident on a surface.

Photometry^a

Luminous energy ("total light")	$Q_v = \iiint L_v \cos \theta \, dA \, d\Omega \, dt \quad \text{lm s} \quad (5.157)$	Q_v luminous energy L_v luminance (generally a function of position and direction) θ angle between dir. of $d\Omega$ and normal to dA Ω solid angle
Luminous flux	$\Phi_v = \frac{\partial Q_v}{\partial t} \quad \text{lumen (lm)} \quad (5.158)$ $= \iint L_v \cos \theta \, dA \, d\Omega \quad (5.159)$	A area t time Φ_v luminous flux
Luminous density ^b	$W_v = \frac{\partial Q_v}{\partial V} \quad \text{lm s m}^{-3} \quad (5.160)$	W_v luminous density V volume
Luminous exitance ^c	$M_v = \frac{\partial \Phi_v}{\partial A} \quad \text{lx} \quad (\text{lm m}^{-2}) \quad (5.161)$ $= \int L_v \cos \theta \, d\Omega \quad (5.162)$	M_v luminous exitance
Illuminance ("illumination") ^d	$E_v = \frac{\partial \Phi_v}{\partial A} \quad \text{lm m}^{-2} \quad (5.163)$ $= \int L_v \cos \theta \, d\Omega \quad (5.164)$	
Luminous intensity ^e	$I_v = \frac{\partial \Phi_v}{\partial \Omega} \quad \text{cd} \quad (5.165)$ $= \int L_v \cos \theta \, dA \quad (5.166)$	E_v illuminance I_v luminous intensity
Luminance ("photometric brightness")	$L_v = \frac{1}{\cos \theta} \frac{\partial^2 \Phi_v}{\partial A \, d\Omega} \quad \text{cd m}^{-2} \quad (5.167)$ $= \frac{1}{\cos \theta} \frac{\partial I_v}{\partial A} \quad (5.168)$	
Luminous efficacy	$K = \frac{\Phi_v}{\Phi_e} = \frac{L_v}{L_e} = \frac{I_v}{I_e} \quad \text{lm W}^{-1} \quad (5.169)$	K luminous efficacy L_e radiance Φ_e radiant flux I_e radiant intensity
Luminous efficiency	$V(\lambda) = \frac{K(\lambda)}{K_{\max}} \quad (5.170)$	V luminous efficiency λ wavelength K_{\max} spectral maximum of $K(\lambda)$

^aPhotometry is concerned with the treatment of light as seen by the human eye.

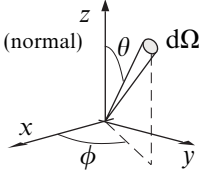
^bThe instantaneous amount of luminous energy contained in a unit volume of propagating medium.

^cLuminous emitted flux per unit area.

^dLuminous incident flux per unit area. The derived SI unit is the lux (lx). 1 lx = 1 lm m⁻².

^eThe SI unit of luminous intensity is the candela (cd). 1 cd = 1 lm sr⁻¹.

Radiative transfer^a

Flux density (through a plane)	$F_v = \int I_v(\theta, \phi) \cos \theta \, d\Omega \quad \text{W m}^{-2} \text{Hz}^{-1}$	(5.171)	 <p> F_v flux density I_v specific intensity ($\text{W m}^{-2} \text{Hz}^{-1} \text{sr}^{-1}$) J_v mean intensity u_v spectral energy density Ω solid angle θ angle between normal and direction of Ω j_v specific emission coefficient ϵ_v emission coefficient ($\text{W m}^{-3} \text{Hz}^{-1} \text{sr}^{-1}$) ρ density α_v linear absorption coefficient n particle number density σ_v particle cross section l_v mean free path κ_v opacity τ_v optical depth, or optical thickness ds line element S_v source function </p>
Mean intensity ^b	$J_v = \frac{1}{4\pi} \int I_v(\theta, \phi) \, d\Omega \quad \text{W m}^{-2} \text{Hz}^{-1}$	(5.172)	
Spectral energy density ^c	$u_v = \frac{1}{c} \int I_v(\theta, \phi) \, d\Omega \quad \text{J m}^{-3} \text{Hz}^{-1}$	(5.173)	
Specific emission coefficient	$j_v = \frac{\epsilon_v}{\rho} \quad \text{W kg}^{-1} \text{Hz}^{-1} \text{sr}^{-1}$	(5.174)	
Gas linear absorption coefficient ($\alpha_v \ll 1$)	$\alpha_v = n\sigma_v = \frac{1}{l_v} \quad \text{m}^{-1}$	(5.175)	
Opacity ^d	$\kappa_v = \frac{\alpha_v}{\rho} \quad \text{kg}^{-1} \text{m}^2$	(5.176)	
Optical depth	$\tau_v = \int \kappa_v \rho \, ds$	(5.177)	
Transfer equation ^e	$\frac{1}{\rho} \frac{dI_v}{ds} = -\kappa_v I_v + j_v \quad (5.178)$ or $\frac{dI_v}{ds} = -\alpha_v I_v + \epsilon_v \quad (5.179)$		
Kirchhoff's law ^f	$S_v \equiv \frac{j_v}{\kappa_v} = \frac{\epsilon_v}{\alpha_v}$	(5.180)	
Emission from a homogeneous medium	$I_v = S_v(1 - e^{-\tau_v})$	(5.181)	

^aThe definitions of these quantities vary in the literature. Those presented here are common in meteorology and astrophysics. Note particularly that the ambiguous term *specific* is taken to mean “per unit frequency interval” in the case of specific intensity and “per unit mass per unit frequency interval” in the case of specific emission coefficient.

^bIn radio astronomy, flux density is usually taken as $S = 4\pi J_v$.

^cAssuming a refractive index of 1.

^dOr “mass absorption coefficient.”

^eOr “Schwarzschild's equation.”

^fUnder conditions of local thermal equilibrium (LTE), the source function, S_v , equals the Planck function, $B_v(T)$ [see Equation (5.182)].

Blackbody radiation

Planck function ^a	$B_v(T) = \frac{2hv^3}{c^2} \left[\exp\left(\frac{hv}{kT}\right) - 1 \right]^{-1} \quad (5.182)$ $B_\lambda(T) = B_v(T) \frac{dv}{d\lambda} \quad (5.183)$ $= \frac{2hc^2}{\lambda^5} \left[\exp\left(\frac{hc}{\lambda kT}\right) - 1 \right]^{-1} \quad (5.184)$
Spectral energy density	$u_v(T) = \frac{4\pi}{c} B_v(T) \quad \text{J m}^{-3} \text{ Hz}^{-1} \quad (5.185)$ $u_\lambda(T) = \frac{4\pi}{c} B_\lambda(T) \quad \text{J m}^{-3} \text{ m}^{-1} \quad (5.186)$
Rayleigh–Jeans law ($hv \ll kT$)	$B_v(T) = \frac{2kT}{c^2} v^2 = \frac{2kT}{\lambda^2} \quad (5.187)$
Wien's law ($hv \gg kT$)	$B_v(T) = \frac{2hv^3}{c^2} \exp\left(\frac{-hv}{kT}\right) \quad (5.188)$
Wien's displacement law	$\lambda_m T = \begin{cases} 5.1 \times 10^{-3} \text{ m K} & \text{for } B_v \\ 2.9 \times 10^{-3} \text{ m K} & \text{for } B_\lambda \end{cases} \quad (5.189)$
Stefan–Boltzmann law ^b	$M = \pi \int_0^\infty B_v(T) dv \quad (5.190)$ $= \frac{2\pi^5 k^4}{15c^2 h^3} T^4 = \sigma T^4 \quad \text{W m}^{-2} \quad (5.191)$
Energy density	$u(T) = \frac{4}{c} \sigma T^4 \quad \text{J m}^{-3} \quad (5.192)$
Greybody	$M = \epsilon \sigma T^4 = (1 - A) \sigma T^4 \quad (5.193)$

^aWith respect to the projected area of the surface. Surface brightness is also known simply as “brightness.” “Specific intensity” is used for reception.

^bSometimes “Stefan’s law.” Exitance is the total radiated energy from unit area of the body per unit time.

Chapter 6 Solid state physics

6.1 Introduction

This section covers a few selected topics in solid state physics. There is no attempt to do more than scratch the surface of this vast field, although the basics of many undergraduate texts on the subject are covered. In addition a period table of elements, together with some of their physical properties, is displayed on the next two pages.

Periodic table (overleaf) Data for the periodic table of elements are taken from *Pure Appl. Chem.*, **71**, 1593–1607 (1999), from the 16th edition of Kaye and Laby *Tables of Physical and Chemical Constants* (Longman, 1995) and from the 74th edition of the CRC *Handbook of Chemistry and Physics* (CRC Press, 1993). Note that melting and boiling points have been converted to kelvins by adding 273.15 to the Celsius values listed in Kaye and Laby. The standard atomic masses reflect the relative isotopic abundances in samples found naturally on Earth, and the number of significant figures reflect the variations between samples. Elements with atomic masses shown in square brackets have no stable nuclides, and the values reflect the mass numbers of the longest-lived isotopes. Crystallographic data are based on the most common forms of the elements (the α -form, unless stated otherwise) stable under standard conditions. Densities are for the solid state. For full details and footnotes for each element, the reader is advised to consult the original texts.

Elements 110, 111, 112 and 114 are known to exist but their names are not yet permanent.

1	<div>Hydrogen</div> <div>1.007 94</div> <div>1 H</div> <div>$1s^1$</div> <div>89 (β) 378</div> <div>HEX 1.632</div> <div>13.80 20.28</div>	2	<div>Lithium</div> <div>6.941</div> <div>3 Li</div> <div>$[He]2s^1$</div> <div>533 (β) 351</div> <div>BCC 1.568</div> <div>453.65 1 613</div>	<div>Beryllium</div> <div>9.012 182</div> <div>4 Be</div> <div>$[He]2s^2$</div> <div>1 846 229</div> <div>HEX 1.568</div> <div>1 560 2 745</div>	3	<div>Sodium</div> <div>22.989 770</div> <div>11 Na</div> <div>$[Ne]3s^1$</div> <div>966 429</div> <div>BCC 1.624</div> <div>370.8 1 153</div>	<div>Magnesium</div> <div>24.305 0</div> <div>12 Mg</div> <div>$[Ne]3s^2$</div> <div>1 738 321</div> <div>HEX 1.624</div> <div>923 1 560 2 745</div>	4	<div>Potassium</div> <div>39.098 3</div> <div>19 K</div> <div>$[Ar]4s^1$</div> <div>862 532</div> <div>BCC 336.5</div> <div>1 033</div>	<div>Calcium</div> <div>40.078</div> <div>20 Ca</div> <div>$[Ar]4s^2$</div> <div>1 530 559</div> <div>FCC 1 113</div> <div>1 757</div>	<div>Scandium</div> <div>44.955 910</div> <div>21 Sc</div> <div>$[Ca]3d^1$</div> <div>2 992 331</div> <div>HEX 1.592</div> <div>1 813 3 103</div>	<div>Titanium</div> <div>47.867</div> <div>22 Ti</div> <div>$[Ca]3d^2$</div> <div>4 508 295</div> <div>HEX 1.587</div> <div>1 943 3 563</div>	<div>Vanadium</div> <div>50.941 5</div> <div>23 V</div> <div>$[Ca]3d^3$</div> <div>6 090 302</div> <div>BCC 2 193</div> <div>3 673</div>	<div>Chromium</div> <div>51.996 1</div> <div>24 Cr</div> <div>$[Ar]3d^5 4s^1$</div> <div>7 194 388</div> <div>BCC 2 180</div> <div>2 943</div>	<div>Manganese</div> <div>54.938 049</div> <div>25 Mn</div> <div>$[Ca]3d^5$</div> <div>7 473 891</div> <div>FCC 1 523</div> <div>2 333</div>	<div>Iron</div> <div>55.845</div> <div>26 Fe</div> <div>$[Ca]3d^6$</div> <div>7 873 287</div> <div>BCC 1 813</div> <div>3 133</div>	<div>Cobalt</div> <div>58.933 200</div> <div>27 Co</div> <div>$[Ca]3d^7$</div> <div>8 800 (ε) 251</div> <div>HEX 1.623</div> <div>1 768 3 203</div>	5	<div>Rubidium</div> <div>85.467 8</div> <div>37 Rb</div> <div>$[Kr]5s^1$</div> <div>1 533 571</div> <div>BCC 312.4</div> <div>963.1</div>	<div>Strontium</div> <div>87.62</div> <div>38 Sr</div> <div>$[Kr]5s^2$</div> <div>2 583 608</div> <div>FCC 1 050</div> <div>1 653</div>	<div>Yttrium</div> <div>88.905 85</div> <div>39 Y</div> <div>$[Sr]4d^1$</div> <div>4 475 365</div> <div>HEX 1.571</div> <div>1 798 3 613</div>	<div>Zirconium</div> <div>91.224</div> <div>40 Zr</div> <div>$[Sr]4d^2$</div> <div>6 507 323</div> <div>HEX 1.593</div> <div>2 123 4 673</div>	<div>Niobium</div> <div>92.906 38</div> <div>41 Nb</div> <div>$[Kr]4d^4 5s^1$</div> <div>8 578 330</div> <div>BCC 2 750</div> <div>4 973</div>	<div>Molybdenum</div> <div>95.94</div> <div>42 Mo</div> <div>$[Kr]4d^5 5s^1$</div> <div>10 222 315</div> <div>BCC 2 896</div> <div>4 913</div>	<div>Technetium</div> <div>[98]</div> <div>43 Tc</div> <div>$[Sr]4d^5$</div> <div>11 496 274</div> <div>HEX 1.604</div> <div>2 433 4 533</div>	<div>Ruthenium</div> <div>101.07</div> <div>44 Ru</div> <div>$[Kr]4d^7 5s^1$</div> <div>12 360 270</div> <div>HEX 1.582</div> <div>2 603 4 423</div>	<div>Rhodium</div> <div>102.905 50</div> <div>45 Rh</div> <div>$[Kr]4d^8 5s^1$</div> <div>12 420 380</div> <div>FCC 2 236</div> <div>3 973</div>	6	<div>Caesium</div> <div>132.905 45</div> <div>55 Cs</div> <div>$[Xe]6s^1$</div> <div>1 900 614</div> <div>BCC 301.6</div> <div>943.2</div>	<div>Barium</div> <div>137.327</div> <div>56 Ba</div> <div>$[Xe]6s^2$</div> <div>3 594 502</div> <div>BCC 1 001</div> <div>2 173</div>	<div>Lanthanides</div> <div>57 – 71</div>	<div>Hafnium</div> <div>178.49</div> <div>72 Hf</div> <div>$[Yb]5d^2$</div> <div>13 276 319</div> <div>HEX 1.581</div> <div>2 503 4 873</div>	<div>Tantalum</div> <div>180.947 9</div> <div>73 Ta</div> <div>$[Yb]5d^3$</div> <div>16 670 330</div> <div>BCC 3 293</div> <div>5 833</div>	<div>Tungsten</div> <div>183.84</div> <div>74 W</div> <div>$[Yb]5d^4$</div> <div>19 254 316</div> <div>BCC 3 695</div> <div>5 823</div>	<div>Rhenium</div> <div>186.207</div> <div>75 Re</div> <div>$[Yb]5d^5$</div> <div>21 023 276</div> <div>HEX 1.615</div> <div>3 459 5 873</div>	<div>Osmium</div> <div>190.23</div> <div>76 Os</div> <div>$[Yb]5d^6$</div> <div>22 580 273</div> <div>HEX 1.606</div> <div>3 303 5 273</div>	<div>Iridium</div> <div>192.217</div> <div>77 Ir</div> <div>$[Yb]5d^7$</div> <div>22 550 384</div> <div>FCC 2 270</div> <div>4 703</div>	7	<div>Francium</div> <div>[223]</div> <div>87 Fr</div> <div>$[Rn]7s^1$</div> <div>300 923</div>	<div>Radium</div> <div>[226]</div> <div>88 Ra</div> <div>$[Rn]7s^2$</div> <div>5 000 515</div> <div>BCC 973 1 773</div>	<div>Actinides</div> <div>89 – 103</div>	<div>Rutherfordium</div> <div>[261]</div> <div>104 Rf</div> <div>$[Ra]5f^{14} 6d^2$</div>	<div>Dubnium</div> <div>[262]</div> <div>105 Db</div> <div>$[Ra]5f^{14} 6d^3 ?$</div>	<div>Seaborgium</div> <div>[263]</div> <div>106 Sg</div> <div>$[Ra]5f^{14} 6d^4 ?$</div>	<div>Bohrium</div> <div>[264]</div> <div>107 Bh</div> <div>$[Ra]5f^{14} 6d^5 ?$</div>	<div>Hassium</div> <div>[265]</div> <div>108 Hs</div> <div>$[Ra]5f^{14} 6d^6 ?$</div>	<div>Meitnerium</div> <div>[268]</div> <div>109 Mt</div> <div>$[Ra]5f^{14} 6d^7 ?$</div>
Lanthanides			<div>Lanthanum</div> <div>138.905 5</div> <div>57 La</div> <div>$[Ba]5d^1$</div> <div>6 174 377</div> <div>HEX 3.23</div> <div>1 193 3 733</div>	<div>Cerium</div> <div>140.116</div> <div>58 Ce</div> <div>$[Ba]4f^1 5d^1$</div> <div>6 711 (γ) 516</div> <div>FCC 1 073</div> <div>3 693</div>	<div>Praseodymium</div> <div>140.907 65</div> <div>59 Pr</div> <div>$[Ba]$</div>																																										

										18							
										Helium 4.002 602 2 He 1s ² 120 356 HEX 1.631 3-5 4.22							
BCC CUB DIA FCC HEX MCL ORC RHL TET (t-pt)	body-centred cubic simple cubic diamond face-centred cubic hexagonal monoclinic orthorhombic rhombohedral tetragonal triple point		13		14		15		16		17		18				
			Boron 10.811 5 B [Be]2p ¹ 2 466 1017 RHL 65° 7' 2 348 4 273		Carbon 12.0107 6 C [Be]2p ² 2 266 357 DIA 4 763 (t-pt)		Nitrogen 14.006 74 7 N [Be]2p ³ 1 035 (β) 405 HEX 1.631 63 77.35		Oxygen 15.999 4 8 O [Be]2p ⁴ 1 460 (γ) 683 CUB 54.36 90.19		Fluorine 18.998 403 2 9 F [Be]2p ⁵ 1 140 550 MCL 1.32 0.61 53.55 85.05		Neon 20.179 7 10 Ne [Be]2p ⁶ 1 442 446 FCC 24.56 27.07				
			Aluminium 26.981 538 13 Al [Mg]3p ¹ 2 698 405 FCC 933.47 2 793		Silicon 28.085 5 14 Si [Mg]3p ² 2 329 543 DIA 1 683 3 533		Phosphorus 30.973 761 15 P [Mg]3p ³ 1 820 331 ORC 1.320 3.162 317.3 550		Sulfur 32.066 16 S [Mg]3p ⁴ 2 086 1 046 ORC 1.340 1.229 388.47 717.82		Chlorine 35.452 7 17 Cl [Mg]3p ⁵ 2 030 624 ORC 1.324 0.719 172 239.1		Argon 39.948 18 Ar [Mg]3p ⁶ 1 656 532 FCC 83.81 87.30				
10		11		12		13		14		15		16		17		18	
Nickel 58.693 4 28 Ni [Ca]3d ⁸ 8 907 352 FCC 1 728 3 263		Copper 63.546 29 Cu [Ar]3d ¹⁰ 4s ¹ 8 933 361 FCC 1 357.8 2 833		Zinc 65.39 30 Zn [Ca]3d ¹⁰ 7 135 266 HEX 1.856 692.68 1 183		Gallium 69.723 31 Ga [Zn]4p ¹ 5 905 452 ORC 1.001 1.695 302.9 2 473		Germanium 72.61 32 Ge [Zn]4p ² 5 323 566 DIA 1 211 3 103		Arsenic 74.921 60 33 As [Zn]4p ³ 5 776 413 RHL 54° 7' 883 (t-pt)		Selenium 78.96 34 Se [Zn]4p ⁴ 4 808 (γ) 436 HEX 1.135 493 958		Bromine 79.904 35 Br [Zn]4p ⁵ 3 120 668 ORC 1.308 0.672 265.90 332.0		Krypton 83.80 36 Kr [Zn]4p ⁶ 3 000 581 FCC 115.8 119.9	
Palladium 106.42 46 Pd [Kr]4d ¹⁰ 11 995 389 FCC 1 828 3 233		Silver 107.868 2 47 Ag [Pd]5s ¹ 10 500 409 FCC 1 235 2 433		Cadmium 112.411 48 Cd [Pd]5s ² 8 647 298 HEX 1.886 594.2 1 043		Indium 114.818 49 In [Cd]5p ¹ 7 290 325 TET 1.521 429.75 2 343		Tin 118.710 50 Sn [Cd]5p ² 7 285 (β) 583 TET 0.546 505.08 2 893		Antimony 121.760 51 Sb [Cd]5p ³ 6 692 451 RHL 57° 7' 903.8 1 860		Tellurium 127.60 52 Te [Cd]5p ⁴ 6 247 446 HEX 1.33 723 1 263		Iodine 126.904 47 53 I [Cd]5p ⁵ 4 953 727 ORC 1.347 0.659 386.7 457		Xenon 131.29 54 Xe [Cd]5p ⁶ 3 560 635 FCC 161.3 165.0	
Platinum 195.078 78 Pt [Xe]4f ¹⁴ 5d ⁹ 6s ¹ 21 450 392 FCC 2 041 4 093		Gold 196.966 55 79 Au [Xe]4f ¹⁴ 5d ¹⁰ 6s ¹ 19 281 408 FCC 1 337.3 3 123		Mercury 200.59 80 Hg [Yb]5d ¹⁰ 13 546 300 RHL 70° 32' 234.32 629.9		Thallium 204.383 3 81 Tl [Hg]6p ¹ 11 871 346 HEX 1.598 577 1 743		Lead 207.2 82 Pb [Hg]6p ² 11 343 495 FCC 600.7 2 023		Bismuth 208.980 38 83 Bi [Hg]6p ³ 9 803 475 RHL 57° 14' 544.59 1 833		Polonium [209] 84 Po [Hg]6p ⁴ 9 400 337 CUB 527 1 233		Astatine [210] 85 At [Hg]6p ⁵ 573 623		Radon [222] 86 Rn [Hg]6p ⁶ 440 202 211	
Ununnilium [271] 110 Uun		Unununium [272] 111 Uuu		Ununbium [285] 112 Uub		Ununquadium [289] 114 Uuq											
Europium 151.964 63 Eu [Ba]4f ⁷ 5 248 458 BCC 1 095 1 873		Gadolinium 157.25 64 Gd [Ba]4f ⁷ 5d ¹ 7 870 363 HEX 1.591 1 587 3 533		Terbium 158.925 34 65 Tb [Ba]4f ⁹ 8 267 361 HEX 1.580 1 633 3 493		Dysprosium 162.50 66 Dy [Ba]4f ¹⁰ 8 531 359 HEX 1.573 1 683 2 833		Holmium 164.930 32 67 Ho [Ba]4f ¹¹ 8 797 358 HEX 1.570 1 743 2 973		Erbium 167.26 68 Er [Ba]4f ¹² 9 044 356 HEX 1.570 1 803 3 133		Thulium 168.934 21 69 Tm [Ba]4f ¹³ 9 325 354 HEX 1.570 1 823 2 223		Ytterbium 173.04 70 Yb [Ba]4f ¹⁴ 6 966 (β) 549 FCC 1 097 1 473		Lutetium 174.967 71 Lu [Yb]5d ¹ 9 842 351 HEX 1.583 1 933 3 663	
Americium [243] 95 Am [Ra]5f ⁷ 13 670 347 HEX 3.24 1 449 2 873		Curium [247] 96 Cm [Rn]5f ⁷ 6d ¹ 7s ² [Ra]5f ⁹ 13 510 350 HEX 3.24 1 618 3 383		Berkelium [247] 97 Bk [Ra]5f ⁹ 14 780 342 HEX 3.24 1 323		Californium [251] 98 Cf [Ra]5f ¹⁰ 15 100 338 HEX 3.24 1 173		Einsteinium [252] 99 Es [Ra]5f ¹¹ HEX 1 133		Fermium [257] 100 Fm [Ra]5f ¹² 1 803		Mendelevium [258] 101 Md [Ra]5f ¹³ 1 103		Nobelium [259] 102 No [Ra]5f ¹⁴ 1 103		Lawrencium [262] 103 Lr [Ra]5f ¹⁴ 7p ¹ 1 903	

6.3 Crystalline structure

Bravais lattices

Volume of primitive cell	$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$	(6.1)	$\mathbf{a}, \mathbf{b}, \mathbf{c}$ V	primitive base vectors volume of primitive cell
Reciprocal primitive base vectors ^a	$\mathbf{a}^* = 2\pi \mathbf{b} \times \mathbf{c} / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(6.2)	$\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$	reciprocal primitive base vectors
	$\mathbf{b}^* = 2\pi \mathbf{c} \times \mathbf{a} / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(6.3)		
	$\mathbf{c}^* = 2\pi \mathbf{a} \times \mathbf{b} / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(6.4)		
	$\mathbf{a} \cdot \mathbf{a}^* = \mathbf{b} \cdot \mathbf{b}^* = \mathbf{c} \cdot \mathbf{c}^* = 2\pi$	(6.5)		
	$\mathbf{a} \cdot \mathbf{b}^* = \mathbf{a} \cdot \mathbf{c}^* = 0$ (etc.)	(6.6)		
Lattice vector	$\mathbf{R}_{uvw} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c}$	(6.7)	\mathbf{R}_{uvw} u, v, w	lattice vector $[uvw]$ integers
Reciprocal lattice vector	$\mathbf{G}_{hkl} = h\mathbf{a}^* + k\mathbf{b}^* + l\mathbf{c}^*$	(6.8)	\mathbf{G}_{hkl}	reciprocal lattice vector $[hkl]$
	$\exp(i\mathbf{G}_{hkl} \cdot \mathbf{R}_{uvw}) = 1$	(6.9)	\mathbf{i}	$\mathbf{i}^2 = -1$
Weiss zone equation ^b	$hu + kv + lw = 0$	(6.10)	(hkl)	Miller indices of plane ^c
Interplanar spacing (general)	$d_{hkl} = \frac{2\pi}{G_{hkl}}$	(6.11)	d_{hkl}	distance between (hkl) planes
Interplanar spacing (orthogonal basis)	$\frac{1}{d_{hkl}^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}$	(6.12)		

^aNote that this is 2π times the usual definition of a “reciprocal vector” (see page 20).

^bCondition for lattice vector $[uvw]$ to be parallel to lattice plane (hkl) in an arbitrary Bravais lattice.

^cMiller indices are defined so that \mathbf{G}_{hkl} is the shortest reciprocal lattice vector normal to the (hkl) planes.

Weber symbols

Converting $[uvw]$ to $[UVTW]$	$U = \frac{1}{3}(2u - v)$	(6.13)	U, V, T, W u, v, w $[UVTW]$ $[uvw]$	Weber indices zone axis indices Weber symbol zone axis symbol
	$V = \frac{1}{3}(2v - u)$	(6.14)		
	$T = -\frac{1}{3}(u + v)$	(6.15)		
	$W = w$	(6.16)		
Converting $[UVTW]$ to $[uvw]$	$u = (U - T)$	(6.17)		
	$v = (V - T)$	(6.18)		
	$w = W$	(6.19)		
Zone law ^a	$hU + kV + iT + lW = 0$	(6.20)	$(hkil)$	Miller–Bravais indices

^aFor trigonal and hexagonal systems.

Cubic lattices

lattice	primitive (P)	body-centred (I)	face-centred (F)
lattice parameter	a	a	a
volume of conventional cell	a^3	a^3	a^3
lattice points per cell	1	2	4
1st nearest neighbours ^a	6	8	12
1st n.n. distance	a	$a\sqrt{3}/2$	$a/\sqrt{2}$
2nd nearest neighbours	12	6	6
2nd n.n. distance	$a\sqrt{2}$	a	a
packing fraction ^b	$\pi/6$	$\sqrt{3}\pi/8$	$\sqrt{2}\pi/6$
reciprocal lattice ^c	P	F	I
primitive base vectors ^d	$\mathbf{a}_1 = a\hat{\mathbf{x}}$	$\mathbf{a}_1 = \frac{a}{2}(\hat{\mathbf{y}} + \hat{\mathbf{z}} - \hat{\mathbf{x}})$	$\mathbf{a}_1 = \frac{a}{2}(\hat{\mathbf{y}} + \hat{\mathbf{z}})$
	$\mathbf{a}_2 = a\hat{\mathbf{y}}$	$\mathbf{a}_2 = \frac{a}{2}(\hat{\mathbf{z}} + \hat{\mathbf{x}} - \hat{\mathbf{y}})$	$\mathbf{a}_2 = \frac{a}{2}(\hat{\mathbf{z}} + \hat{\mathbf{x}})$
	$\mathbf{a}_3 = a\hat{\mathbf{z}}$	$\mathbf{a}_3 = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}})$	$\mathbf{a}_3 = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$

^aOr “coordination number.”

^bFor close-packed spheres. The maximum possible packing fraction for spheres is $\sqrt{2}\pi/6$.

^cThe lattice parameters for the reciprocal lattices of P, I, and F are $2\pi/a$, $4\pi/a$, and $4\pi/a$ respectively.

^d $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors.

Crystal systems^a

system	symmetry	unit cell ^b	lattices ^c
triclinic	none	$a \neq b \neq c$; $\alpha \neq \beta \neq \gamma \neq 90^\circ$	P
monoclinic	one diad $\parallel [010]$	$a \neq b \neq c$; $\alpha = \gamma = 90^\circ$, $\beta \neq 90^\circ$	P, C
orthorhombic	three orthogonal diads	$a \neq b \neq c$; $\alpha = \beta = \gamma = 90^\circ$	P, C, I, F
tetragonal	one tetrad $\parallel [001]$	$a = b \neq c$; $\alpha = \beta = \gamma = 90^\circ$	P, I
trigonal ^d	one triad $\parallel [111]$	$a = b = c$; $\alpha = \beta = \gamma < 120^\circ \neq 90^\circ$	P, R
hexagonal	one hexad $\parallel [001]$	$a = b \neq c$; $\alpha = \beta = 90^\circ$, $\gamma = 120^\circ$	P
cubic	four triads $\parallel \langle 111 \rangle$	$a = b = c$; $\alpha = \beta = \gamma = 90^\circ$	P, F, I

^aThe symbol “ \neq ” implies that equality is not required by the symmetry, but neither is it forbidden.

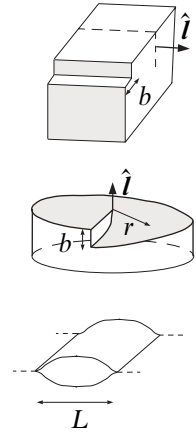
^bThe cell axes are a , b , and c with α , β , and γ the angles between $b:c$, $c:a$, and $a:b$ respectively.

^cThe lattice types are primitive (P), body-centred (I), all face-centred (F), side-centred (C), and rhombohedral primitive (R).

^dA primitive hexagonal unit cell, with a triad $\parallel [001]$, is generally preferred over this rhombohedral unit cell.

Dislocations and cracks

Edge dislocation	$\hat{l} \cdot \mathbf{b} = 0$	(6.21)	\hat{l} unit vector \parallel line of dislocation
Screw dislocation	$\hat{l} \cdot \mathbf{b} = b$	(6.22)	\mathbf{b}, b Burgers vector ^a
Screw dislocation energy per unit length ^b	$U = \frac{\mu b^2}{4\pi} \ln \frac{R}{r_0}$	(6.23)	U dislocation energy per unit length
	$\sim \mu b^2$	(6.24)	μ shear modulus
Critical crack length ^c	$L = \frac{4\alpha E}{\pi(1-\sigma^2)p_0^2}$	(6.25)	R outer cutoff for r
			r_0 inner cutoff for r
			L critical crack length
			α surface energy per unit area
			E Young modulus
			σ Poisson ratio
			p_0 applied widening stress



^aThe Burgers vector is a Bravais lattice vector characterising the total relative slip were the dislocation to travel throughout the crystal.

^bOr “tension.” The energy per unit length of an edge dislocation is also $\sim \mu b^2$.

^cFor a crack cavity (long $\perp L$) within an isotropic medium. Under uniform stress p_0 , cracks $\geq L$ will grow and smaller cracks will shrink.

Crystal diffraction

Laue equations	$a(\cos \alpha_1 - \cos \alpha_2) = h\lambda$	(6.26)	a, b, c lattice parameters
	$b(\cos \beta_1 - \cos \beta_2) = k\lambda$	(6.27)	$\alpha_1, \beta_1, \gamma_1$ angles between lattice base vectors and input wavevector
	$c(\cos \gamma_1 - \cos \gamma_2) = l\lambda$	(6.28)	$\alpha_2, \beta_2, \gamma_2$ angles between lattice base vectors and output wavevector
			h, k, l integers (Laue indices)
Bragg's law ^a	$2\mathbf{k}_{\text{in}} \cdot \mathbf{G} + \mathbf{G} ^2 = 0$	(6.29)	λ wavelength
			\mathbf{k}_{in} input wavevector
			\mathbf{G} reciprocal lattice vector
Atomic form factor	$f(\mathbf{G}) = \int_{\text{vol}} e^{-i\mathbf{G} \cdot \mathbf{r}} \rho(\mathbf{r}) d^3r$	(6.30)	$f(\mathbf{G})$ atomic form factor
			\mathbf{r} position vector
			$\rho(\mathbf{r})$ atomic electron density
Structure factor ^b	$S(\mathbf{G}) = \sum_{j=1}^n f_j(\mathbf{G}) e^{-i\mathbf{G} \cdot \mathbf{d}_j}$	(6.31)	$S(\mathbf{G})$ structure factor
			n number of atoms in basis
			\mathbf{d}_j position of j th atom within basis
Scattered intensity ^c	$I(\mathbf{K}) \propto N^2 S(\mathbf{K}) ^2$	(6.32)	\mathbf{K} change in wavevector ($= \mathbf{k}_{\text{out}} - \mathbf{k}_{\text{in}}$)
			$I(\mathbf{K})$ scattered intensity
			N number of lattice points illuminated
Debye–Waller factor ^d	$I_T = I_0 \exp \left[-\frac{1}{3} \langle u^2 \rangle \mathbf{G} ^2 \right]$	(6.33)	I_T intensity at temperature T
			I_0 intensity from a lattice with no motion
			$\langle u^2 \rangle$ mean-squared thermal displacement of atoms

^aAlternatively, see Equation (8.32).

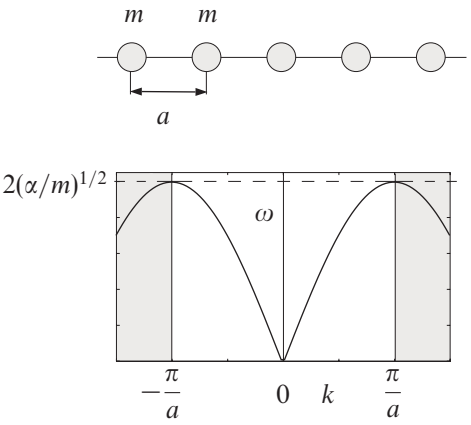
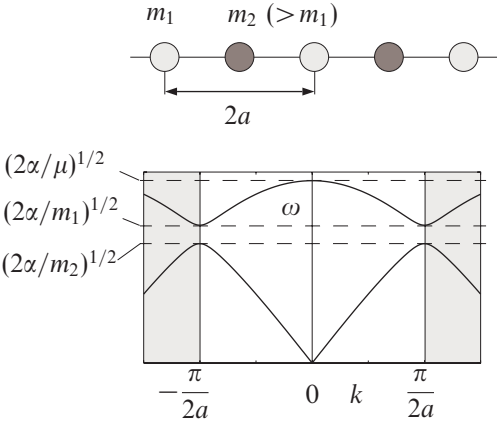
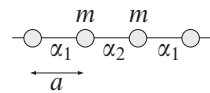
^bThe summation is over the atoms in the basis, i.e., the atomic motif repeating with the Bravais lattice.

^cThe Bragg condition makes \mathbf{K} a reciprocal lattice vector, with $|\mathbf{k}_{\text{in}}| = |\mathbf{k}_{\text{out}}|$.

^dEffect of thermal vibrations.

6.4 Lattice dynamics

Phonon dispersion relations^a

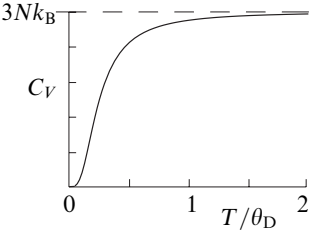
 <p style="text-align: center;">monatomic chain</p>		 <p style="text-align: center;">diatomic chain</p>	
Monatomic linear chain	$\omega^2 = 4 \frac{\alpha}{m} \sin^2 \left(\frac{ka}{2} \right) \quad (6.34)$	ω phonon angular frequency α spring constant ^b m atomic mass v_p phase speed ($\text{sinc } x \equiv \frac{\sin \pi x}{\pi x}$) v_g group speed λ phonon wavelength	k wavenumber ($= 2\pi/\lambda$) a atomic separation m_i atomic masses ($m_2 > m_1$) μ reduced mass $[= m_1 m_2 / (m_1 + m_2)]$
	$v_p = \frac{\omega}{k} = a \left(\frac{\alpha}{m} \right)^{1/2} \text{sinc} \left(\frac{a}{\lambda} \right) \quad (6.35)$		
	$v_g = \frac{\partial \omega}{\partial k} = a \left(\frac{\alpha}{m} \right)^{1/2} \cos \left(\frac{ka}{2} \right) \quad (6.36)$		
Diatomic linear chain ^c	$\omega^2 = \frac{\alpha}{\mu} \pm \alpha \left[\frac{1}{\mu^2} - \frac{4}{m_1 m_2} \sin^2(ka) \right]^{1/2} \quad (6.37)$		
Identical masses, alternating spring constants	$\omega^2 = \frac{\alpha_1 + \alpha_2}{m} \pm \frac{1}{m} (\alpha_1^2 + \alpha_2^2 + 2\alpha_1 \alpha_2 \cos ka)^{1/2} \quad (6.38)$	α_i alternating spring constants 	
	$= \begin{cases} 0, & 2(\alpha_1 + \alpha_2)/m & \text{if } k=0 \\ 2\alpha_1/m, & 2\alpha_2/m & \text{if } k=\pi/a \end{cases} \quad (6.39)$		

^aAlong infinite linear atomic chains, considering simple harmonic nearest-neighbour interactions only. The shaded region of the dispersion relation is outside the first Brillouin zone of the reciprocal lattice.

^bIn the sense α = restoring force/relative displacement.

^cNote that the repeat distance for this chain is $2a$, so that the first Brillouin zone extends to $|k| < \pi/(2a)$. The optic and acoustic branches are the + and − solutions respectively.

Debye theory

Mean energy per phonon mode ^a	$\langle E \rangle = \frac{1}{2} \hbar \omega + \frac{\hbar \omega}{\exp[\hbar \omega / (k_B T)] - 1}$ (6.40)	$\langle E \rangle$ mean energy in a mode at ω \hbar (Planck constant)/(2 π) ω phonon angular frequency k_B Boltzmann constant T temperature
Debye frequency	$\omega_D = v_s (6\pi^2 N/V)^{1/3}$ (6.41) where $\frac{3}{v_s^3} = \frac{1}{v_l^3} + \frac{2}{v_t^3}$ (6.42)	ω_D Debye (angular) frequency v_s effective sound speed v_l longitudinal phase speed v_t transverse phase speed
Debye temperature	$\theta_D = \hbar \omega_D / k_B$ (6.43)	N number of atoms in crystal V crystal volume θ_D Debye temperature
Phonon density of states	$g(\omega) d\omega = \frac{3V\omega^2}{2\pi^2 v_s^3} d\omega$ (6.44) (for $0 < \omega < \omega_D$, $g = 0$ otherwise)	$g(\omega)$ density of states at ω C_V heat capacity, V constant U thermal phonon energy within crystal $D(x)$ Debye function
Debye heat capacity	$C_V = 9Nk_B \frac{T^3}{\theta_D^3} \int_0^{\theta_D/T} \frac{x^4 e^x}{(e^x - 1)^2} dx$ (6.45)	
Dulong and Petit's law	$\simeq 3Nk_B \quad (T \gg \theta_D)$ (6.46)	
Debye T^3 law	$\simeq \frac{12\pi^4}{5} Nk_B \frac{T^3}{\theta_D^3} \quad (T \ll \theta_D)$ (6.47)	
Internal thermal energy ^b	$U(T) = \frac{9N}{\omega_D^3} \int_0^{\omega_D} \frac{\hbar \omega^3}{\exp[\hbar \omega / (k_B T)] - 1} d\omega \equiv 3Nk_B T D(\theta_D/T)$ (6.48) where $D(x) = \frac{3}{x^3} \int_0^x \frac{y^3}{e^y - 1} dy$ (6.49)	

^aOr any simple harmonic oscillator in thermal equilibrium at temperature T .^bNeglecting zero-point energy.

Lattice forces (simple)

Van der Waals interaction ^a	$\phi(r) = -\frac{3}{4} \frac{\alpha_p^2 \hbar \omega}{(4\pi\epsilon_0)^2 r^6} \quad (6.50)$	$\phi(r)$ two-particle potential energy r particle separation α_p particle polarisability
Lennard–Jones 6-12 potential (molecular crystals)	$\phi(r) = -\frac{A}{r^6} + \frac{B}{r^{12}} \quad (6.51)$	\hbar (Planck constant)/(2 π)
	$= 4\epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^6 \right] \quad (6.52)$	ϵ_0 permittivity of free space ω angular frequency of polarised orbital
	$\sigma = (B/A)^{1/6}; \quad \epsilon = A^2/(4B)$ $\phi_{\min} \quad \text{at} \quad r = \frac{2^{1/6}}{\sigma} \quad (6.53)$	A, B constants ϵ, σ Lennard–Jones parameters
De Boer parameter	$\Lambda = \frac{h}{\sigma(m\epsilon)^{1/2}} \quad (6.54)$	Λ de Boer parameter h Planck constant m particle mass
Coulomb interaction (ionic crystals)	$U_C = -\alpha_M \frac{e^2}{4\pi\epsilon_0 r_0} \quad (6.55)$	U_C lattice Coulomb energy per ion pair α_M Madelung constant $-e$ electronic charge r_0 nearest neighbour separation

^aLondon’s formula for fluctuating dipole interactions, neglecting the propagation time between particles.

Lattice thermal expansion and conduction

Grüneisen parameter ^a	$\gamma = -\frac{\partial \ln \omega}{\partial \ln V} \quad (6.56)$	γ Grüneisen parameter ω normal mode frequency V volume
Linear expansivity ^b	$\alpha = \frac{1}{3K_T} \left. \frac{\partial p}{\partial T} \right _V = \frac{\gamma C_V}{3K_T V} \quad (6.57)$	α linear expansivity K_T isothermal bulk modulus p pressure T temperature C_V lattice heat capacity, constant V
Thermal conductivity of a phonon gas	$\lambda = \frac{1}{3} \frac{C_V}{V} v_s l \quad (6.58)$	λ thermal conductivity v_s effective sound speed l phonon mean free path
Umklapp mean free path ^c	$l_u \propto \exp(\theta_u/T) \quad (6.59)$	l_u umklapp mean free path θ_u umklapp temperature ($\sim \theta_D/2$)

^aStrictly, the Grüneisen parameter is the mean of γ over all normal modes, weighted by the mode’s contribution to C_V .

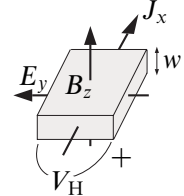
^bOr “coefficient of thermal expansion,” for an isotropically expanding crystal.

^cMean free path determined solely by “umklapp processes” – the scattering of phonons outside the first Brillouin zone.

6.5 Electrons in solids

Free electron transport properties

Current density	$\mathbf{J} = -nev_d$	(6.60)	\mathbf{J} current density n free electron number density $-e$ electronic charge v_d mean electron drift velocity τ mean time between collisions (relaxation time) m_e electronic mass
Mean electron drift velocity	$v_d = -\frac{e\tau}{m_e}\mathbf{E}$	(6.61)	\mathbf{E} applied electric field σ_0 d.c. conductivity ($\mathbf{J} = \sigma\mathbf{E}$)
d.c. electrical conductivity	$\sigma_0 = \frac{ne^2\tau}{m_e}$	(6.62)	ω a.c. angular frequency $\sigma(\omega)$ a.c. conductivity
a.c. electrical conductivity ^a	$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$	(6.63)	C_V total electron heat capacity, V constant V volume $\langle c^2 \rangle$ mean square electron speed k_B Boltzmann constant T temperature T_F Fermi temperature
Thermal conductivity	$\lambda = \frac{1}{3} \frac{C_V}{V} \langle c^2 \rangle \tau$ $= \frac{\pi^2 nk_B^2 \tau T}{3m_e} \quad (T \ll T_F)$	(6.64) (6.65)	L Lorenz constant ($\simeq 2.45 \times 10^{-8} \text{ W } \Omega \text{ K}^{-2}$) λ thermal conductivity
Wiedemann–Franz law ^b	$\frac{\lambda}{\sigma T} = L = \frac{\pi^2 k_B^2}{3e^2}$	(6.66)	R_H Hall coefficient E_y Hall electric field J_x applied current density B_z magnetic flux density V_H Hall voltage I_x applied current ($= J_x \times \text{cross-sectional area}$) w strip thickness in z
Hall coefficient ^c	$R_H = -\frac{1}{ne} = \frac{E_y}{J_x B_z}$	(6.67)	
Hall voltage (rectangular strip)	$V_H = R_H \frac{B_z I_x}{w}$	(6.68)	



^aFor an electric field varying as $e^{-i\omega t}$.

^bHolds for an arbitrary band structure.

^cThe charge on an electron is $-e$, where e is the elementary charge (approximately $+1.6 \times 10^{-19} \text{ C}$). The Hall coefficient is therefore a negative number when the dominant charge carriers are electrons.

Fermi gas

Electron density of states ^a	$g(E) = \frac{V}{2\pi^2} \left(\frac{2m_e}{\hbar^2} \right)^{3/2} E^{1/2} \quad (6.69)$ $g(E_F) = \frac{3nV}{2E_F} \quad (6.70)$	E electron energy (>0) $g(E)$ density of states V “gas” volume m_e electronic mass \hbar (Planck constant)/(2π)
Fermi wavenumber	$k_F = (3\pi^2 n)^{1/3} \quad (6.71)$	k_F Fermi wavenumber n number of electrons per unit volume
Fermi velocity	$v_F = \hbar k_F / m_e \quad (6.72)$	v_F Fermi velocity
Fermi energy ($T=0$)	$E_F = \frac{\hbar^2 k_F^2}{2m_e} = \frac{\hbar^2}{2m_e} (3\pi^2 n)^{2/3} \quad (6.73)$	E_F Fermi energy
Fermi temperature	$T_F = \frac{E_F}{k_B} \quad (6.74)$	T_F Fermi temperature k_B Boltzmann constant
Electron heat capacity ^b ($T \ll T_F$)	$C_{Ve} = \frac{\pi^2}{3} g(E_F) k_B^2 T \quad (6.75)$ $= \frac{\pi^2 k_B^2}{2E_F} T \quad (6.76)$	C_{Ve} heat capacity per electron T temperature
Total kinetic energy ($T=0$)	$U_0 = \frac{3}{5} n V E_F \quad (6.77)$	U_0 total kinetic energy
Pauli paramagnetism	$\mathbf{M} = \chi_{HP} \mathbf{H} \quad (6.78)$ $= \frac{3n}{2E_F} \mu_0 \mu_B^2 \mathbf{H} \quad (6.79)$	χ_{HP} Pauli magnetic susceptibility \mathbf{H} magnetic field strength \mathbf{M} magnetisation μ_0 permeability of free space μ_B Bohr magneton
Landau diamagnetism	$\chi_{HL} = -\frac{1}{3} \chi_{HP} \quad (6.80)$	χ_{HL} Landau magnetic susceptibility

^aThe density of states is often quoted per unit volume in real space (i.e., $g(E)/V$ here).

^bEquation (6.75) holds for any density of states.

Thermoelectricity

Thermopower ^a	$\mathcal{E} = \frac{J}{\sigma} + S_T \nabla T \quad (6.81)$	\mathcal{E} electrochemical field ^b \mathbf{J} current density σ electrical conductivity S_T thermopower T temperature \mathbf{H} heat flux per unit area
Peltier effect	$\mathbf{H} = \Pi \mathbf{J} - \lambda \nabla T \quad (6.82)$	Π Peltier coefficient λ thermal conductivity
Kelvin relation	$\Pi = T S_T \quad (6.83)$	

^aOr “absolute thermoelectric power.”

^bThe electrochemical field is the gradient of $(\mu/e) - \phi$, where μ is the chemical potential, $-e$ the electronic charge, and ϕ the electrical potential.

Band theory and semiconductors

Bloch's theorem	$\Psi(\mathbf{r} + \mathbf{R}) = \exp(i\mathbf{k} \cdot \mathbf{R})\Psi(\mathbf{r})$	(6.84)	Ψ electron eigenstate \mathbf{k} Bloch wavevector \mathbf{R} lattice vector \mathbf{r} position vector
Electron velocity	$\mathbf{v}_b(\mathbf{k}) = \frac{1}{\hbar} \nabla_{\mathbf{k}} E_b(\mathbf{k})$	(6.85)	\mathbf{v}_b electron velocity (for wavevector \mathbf{k}) \hbar (Planck constant)/ 2π b band index $E_b(\mathbf{k})$ energy band
Effective mass tensor	$m_{ij} = \hbar^2 \left[\frac{\partial^2 E_b(\mathbf{k})}{\partial k_i \partial k_j} \right]^{-1}$	(6.86)	m_{ij} effective mass tensor k_i components of \mathbf{k}
Scalar effective mass ^a	$m^* = \hbar^2 \left[\frac{\partial^2 E_b(k)}{\partial k^2} \right]^{-1}$	(6.87)	m^* scalar effective mass $k = \mathbf{k} $
Mobility	$\mu = \frac{ \mathbf{v}_d }{ \mathbf{E} } = \frac{eD}{k_B T}$	(6.88)	μ particle mobility \mathbf{v}_d mean drift velocity \mathbf{E} applied electric field $-e$ electronic charge D diffusion coefficient T temperature
Net current density	$\mathbf{J} = (n_e \mu_e + n_h \mu_h) e \mathbf{E}$	(6.89)	\mathbf{J} current density $n_{e,h}$ electron, hole, number densities $\mu_{e,h}$ electron, hole, mobilities
Semiconductor equation	$n_e n_h = \frac{(k_B T)^3}{2(\pi \hbar^2)^3} (m_e^* m_h^*)^{3/2} e^{-E_g/(k_B T)}$	(6.90)	k_B Boltzmann constant E_g band gap $m_{e,h}^*$ electron, hole, effective masses
p-n junction	$I = I_0 \left[\exp\left(\frac{eV}{k_B T}\right) - 1 \right]$	(6.91)	I current I_0 saturation current V bias voltage (+ for forward)
	$I_0 = e n_i^2 A \left(\frac{D_e}{L_e N_a} + \frac{D_h}{L_h N_d} \right)$	(6.92)	n_i intrinsic carrier concentration A area of junction $D_{e,h}$ electron, hole, diffusion coefficients
	$L_e = (D_e \tau_e)^{1/2}$	(6.93)	$L_{e,h}$ electron, hole, diffusion lengths
	$L_h = (D_h \tau_h)^{1/2}$	(6.94)	$\tau_{e,h}$ electron, hole, recombination times
			$N_{a,d}$ acceptor, donor, concentrations

^aValid for regions of k -space in which $E_b(\mathbf{k})$ can be taken as independent of the direction of \mathbf{k} .

Chapter 7 Electromagnetism

7.1 Introduction

The electromagnetic force is central to nearly every physical process around us and is a major component of classical physics. In fact, the development of electromagnetic theory in the nineteenth century gave us much mathematical machinery that we now apply quite generally in other fields, including potential theory, vector calculus, and the ideas of divergence and curl.

It is therefore not surprising that this section deals with a large array of physical quantities and their relationships. As usual, SI units are assumed throughout. In the past electromagnetism has suffered from the use of a variety of systems of units, including the cgs system in both its electrostatic (esu) and electromagnetic (emu) forms. The fog has now all but cleared, but some specialised areas of research still cling to these historical measures. Readers are advised to consult the section on unit conversion if they come across such exotica in the literature.

Equations cast in the rationalised units of SI can be readily converted to the once common Gaussian (unrationalised) units by using the following symbol transformations:

Equation conversion: SI to Gaussian units

$\epsilon_0 \mapsto 1/(4\pi)$	$\mu_0 \mapsto 4\pi/c^2$	$\mathbf{B} \mapsto \mathbf{B}/c$
$\chi_E \mapsto 4\pi\chi_E$	$\chi_H \mapsto 4\pi\chi_H$	$\mathbf{H} \mapsto c\mathbf{H}/(4\pi)$
$\mathbf{A} \mapsto \mathbf{A}/c$	$\mathbf{M} \mapsto c\mathbf{M}$	$\mathbf{D} \mapsto \mathbf{D}/(4\pi)$

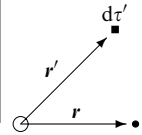
The quantities ρ , \mathbf{J} , \mathbf{E} , ϕ , σ , \mathbf{P} , ϵ_r , and μ_r are all unchanged.

7.2 Static fields

Electrostatics

Electrostatic potential	$\mathbf{E} = -\nabla\phi$	(7.1)	\mathbf{E} electric field ϕ electrostatic potential
Potential difference ^a	$\phi_a - \phi_b = \int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \mathbf{E} \cdot d\mathbf{l}$	(7.2)	ϕ_a potential at a ϕ_b potential at b $d\mathbf{l}$ line element
Poisson's Equation (free space)	$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$	(7.3)	ρ charge density ϵ_0 permittivity of free space
Point charge at \mathbf{r}'	$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 \mathbf{r}-\mathbf{r}' }$	(7.4)	q point charge
	$\mathbf{E}(\mathbf{r}) = \frac{q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0 \mathbf{r}-\mathbf{r}' ^3}$	(7.5)	
Field from a charge distribution (free space)	$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{ \mathbf{r}-\mathbf{r}' ^3} d\tau'$	(7.6)	$d\tau'$ volume element \mathbf{r}' position vector of $d\tau'$

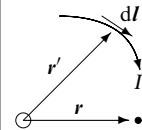
^aBetween points a and b along a path \mathbf{l} .



Magnetostatics^a

Magnetic scalar potential	$\mathbf{B} = -\mu_0\nabla\phi_m$	(7.7)	ϕ_m magnetic scalar potential \mathbf{B} magnetic flux density
ϕ_m in terms of the solid angle of a generating current loop	$\phi_m = \frac{I\Omega}{4\pi}$	(7.8)	Ω loop solid angle I current
Biot-Savart law (the field from a line current)	$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\text{line}} \frac{d\mathbf{l} \times (\mathbf{r}-\mathbf{r}')}{ \mathbf{r}-\mathbf{r}' ^3}$	(7.9)	$d\mathbf{l}$ line element in the direction of the current \mathbf{r}' position vector of $d\mathbf{l}$
Ampère's law (differential form)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	(7.10)	\mathbf{J} current density μ_0 permeability of free space
Ampère's law (integral form)	$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{tot}}$	(7.11)	I_{tot} total current through loop

^aIn free space.



Capacitance^a

Of sphere, radius a	$C = 4\pi\epsilon_0\epsilon_r a$	(7.12)
Of circular disk, radius a	$C = 8\epsilon_0\epsilon_r a$	(7.13)
Of two spheres, radius a , in contact	$C = 8\pi\epsilon_0\epsilon_r a \ln 2$	(7.14)
Of circular solid cylinder, radius a , length l	$C \simeq [8 + 4.1(l/a)^{0.76}] \epsilon_0\epsilon_r a$	(7.15)
Of nearly spherical surface, area S	$C \simeq 3.139 \times 10^{-11} \epsilon_r S^{1/2}$	(7.16)
Of cube, side a	$C \simeq 7.283 \times 10^{-11} \epsilon_r a$	(7.17)
Between concentric spheres, radii $a < b$	$C = 4\pi\epsilon_0\epsilon_r ab(b-a)^{-1}$	(7.18)
Between coaxial cylinders, radii $a < b$	$C = \frac{2\pi\epsilon_0\epsilon_r}{\ln(b/a)}$ per unit length	(7.19)
Between parallel cylinders, separation $2d$, radii a	$C = \frac{\pi\epsilon_0\epsilon_r}{\operatorname{arcosh}(d/a)}$ per unit length	(7.20)
	$\simeq \frac{\pi\epsilon_0\epsilon_r}{\ln(2d/a)} \quad (d \gg a)$	(7.21)
Between parallel, coaxial circular disks, separation d , radii a	$C \simeq \frac{\epsilon_0\epsilon_r\pi a^2}{d} + \epsilon_0\epsilon_r a [\ln(16\pi a/d) - 1]$	(7.22)

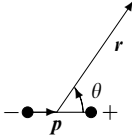
^aFor conductors, in an embedding medium of relative permittivity ϵ_r .

Inductance^a

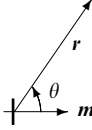
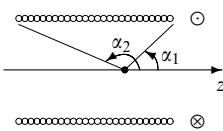
Of N -turn solenoid (straight or toroidal), length l , area A ($\ll l^2$)	$L = \mu_0 N^2 A / l$	(7.23)
Of coaxial cylindrical tubes, radii a, b ($a < b$)	$L = \frac{\mu_0}{2\pi} \ln \frac{b}{a}$ per unit length	(7.24)
Of parallel wires, radii a , separation $2d$	$L \simeq \frac{\mu_0}{\pi} \ln \frac{2d}{a}$ per unit length, ($2d \gg a$)	(7.25)
Of wire of radius a bent in a loop of radius $b \gg a$	$L \simeq \mu_0 b \left(\ln \frac{8b}{a} - 2 \right)$	(7.26)

^aFor currents confined to the surfaces of perfect conductors in free space.

Electric fields^a

Uniformly charged sphere, radius a , charge q	$E(\mathbf{r}) = \begin{cases} \frac{q}{4\pi\epsilon_0 a^3} \mathbf{r} & (r < a) \\ \frac{q}{4\pi\epsilon_0 r^3} \mathbf{r} & (r \geq a) \end{cases} \quad (7.27)$	
Uniformly charged disk, radius a , charge q (on axis, z)	$E(z) = \frac{q}{2\pi\epsilon_0 a^2} z \left(\frac{1}{ z } - \frac{1}{\sqrt{z^2 + a^2}} \right) \quad (7.28)$	
Line charge, charge density λ per unit length	$E(\mathbf{r}) = \frac{\lambda}{2\pi\epsilon_0 r^2} \mathbf{r} \quad (7.29)$	
Electric dipole, moment \mathbf{p} (spherical polar coordinates, θ angle between \mathbf{p} and \mathbf{r})	$E_r = \frac{p \cos \theta}{2\pi\epsilon_0 r^3} \quad (7.30)$	
	$E_\theta = \frac{p \sin \theta}{4\pi\epsilon_0 r^3} \quad (7.31)$	
Charge sheet, surface density σ	$E = \frac{\sigma}{2\epsilon_0} \quad (7.32)$	

^aFor $\epsilon_r = 1$ in the surrounding medium.**Magnetic fields^a**

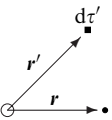
Uniform infinite solenoid, current I , n turns per unit length	$B = \begin{cases} \mu_0 n I & \text{inside (axial)} \\ 0 & \text{outside} \end{cases} \quad (7.33)$	
Uniform cylinder of current I , radius a	$B(r) = \begin{cases} \mu_0 I r / (2\pi a^2) & r < a \\ \mu_0 I / (2\pi r) & r \geq a \end{cases} \quad (7.34)$	
Magnetic dipole, moment \mathbf{m} (θ angle between \mathbf{m} and \mathbf{r})	$B_r = \mu_0 \frac{m \cos \theta}{2\pi r^3} \quad (7.35)$	
	$B_\theta = \frac{\mu_0 m \sin \theta}{4\pi r^3} \quad (7.36)$	
Circular current loop of N turns, radius a , along axis, z	$B(z) = \frac{\mu_0 N I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \quad (7.37)$	
The axis, z , of a straight solenoid, n turns per unit length, current I	$B_{\text{axis}} = \frac{\mu_0 n I}{2} (\cos \alpha_1 - \cos \alpha_2) \quad (7.38)$	

^aFor $\mu_r = 1$ in the surrounding medium.**Image charges**

Real charge, $+q$, at a distance:	image point	image charge
b from a conducting plane	$-b$	$-q$
b from a conducting sphere, radius a	a^2/b	$-qa/b$
b from a plane dielectric boundary:		
seen from free space	$-b$	$-q(\epsilon_r - 1)/(\epsilon_r + 1)$
seen from the dielectric	b	$+2q/(\epsilon_r + 1)$

7.3 Electromagnetic fields (general)


Field relationships

Conservation of charge	$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$	(7.39)	\mathbf{J} current density ρ charge density t time
Magnetic vector potential	$\mathbf{B} = \nabla \times \mathbf{A}$	(7.40)	\mathbf{A} vector potential
Electric field from potentials	$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$	(7.41)	ϕ electrical potential
Coulomb gauge condition	$\nabla \cdot \mathbf{A} = 0$	(7.42)	
Lorenz gauge condition	$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$	(7.43)	c speed of light
Potential field equations ^a	$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0}$	(7.44)	
	$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$	(7.45)	
Expression for ϕ in terms of ρ^a	$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{\rho(\mathbf{r}', t - \mathbf{r} - \mathbf{r}' /c)}{ \mathbf{r} - \mathbf{r}' } d\tau'$	(7.46)	$d\tau'$ volume element \mathbf{r}' position vector of $d\tau'$
Expression for \mathbf{A} in terms of \mathbf{J}^a	$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\text{volume}} \frac{\mathbf{J}(\mathbf{r}', t - \mathbf{r} - \mathbf{r}' /c)}{ \mathbf{r} - \mathbf{r}' } d\tau'$	(7.47)	μ_0 permeability of free space

^aAssumes the Lorenz gauge.

7

Liénard–Wiechert potentials^a

Electrical potential of a moving point charge	$\phi = \frac{q}{4\pi\epsilon_0(\mathbf{r} - \mathbf{v} \cdot \mathbf{r}/c)}$	(7.48)	q charge \mathbf{r} vector from charge to point of observation \mathbf{v} particle velocity
Magnetic vector potential of a moving point charge	$\mathbf{A} = \frac{\mu_0 q \mathbf{v}}{4\pi(\mathbf{r} - \mathbf{v} \cdot \mathbf{r}/c)}$	(7.49)	

^aIn free space. The right-hand sides of these equations are evaluated at retarded times, i.e., at $t' = t - |\mathbf{r}'|/c$, where \mathbf{r}' is the vector from the charge to the observation point at time t' .

Maxwell's equations

Differential form:	Integral form:
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (7.50)$	$\oint_{\text{closed surface}} \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\epsilon_0} \int_{\text{volume}} \rho \, d\tau \quad (7.51)$
$\nabla \cdot \mathbf{B} = 0 \quad (7.52)$	$\oint_{\text{closed surface}} \mathbf{B} \cdot d\mathbf{s} = 0 \quad (7.53)$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.54)$	$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \quad (7.55)$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (7.56)$	$\oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I + \mu_0 \epsilon_0 \int_{\text{surface}} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{s} \quad (7.57)$
Equation (7.51) is "Gauss's law" Equation (7.55) is "Faraday's law" \mathbf{E} electric field \mathbf{B} magnetic flux density \mathbf{J} current density ρ charge density	$d\mathbf{s}$ surface element $d\tau$ volume element $d\mathbf{l}$ line element Φ linked magnetic flux ($= \int \mathbf{B} \cdot d\mathbf{s}$) I linked current ($= \int \mathbf{J} \cdot d\mathbf{s}$) t time

Maxwell's equations (using \mathbf{D} and \mathbf{H})

Differential form:	Integral form:
$\nabla \cdot \mathbf{D} = \rho_{\text{free}} \quad (7.58)$	$\oint_{\text{closed surface}} \mathbf{D} \cdot d\mathbf{s} = \int_{\text{volume}} \rho_{\text{free}} \, d\tau \quad (7.59)$
$\nabla \cdot \mathbf{B} = 0 \quad (7.60)$	$\oint_{\text{closed surface}} \mathbf{B} \cdot d\mathbf{s} = 0 \quad (7.61)$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.62)$	$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \quad (7.63)$
$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t} \quad (7.64)$	$\oint_{\text{loop}} \mathbf{H} \cdot d\mathbf{l} = I_{\text{free}} + \int_{\text{surface}} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad (7.65)$
\mathbf{D} displacement field ρ_{free} free charge density (in the sense of $\rho = \rho_{\text{induced}} + \rho_{\text{free}}$) \mathbf{B} magnetic flux density \mathbf{H} magnetic field strength \mathbf{J}_{free} free current density (in the sense of $\mathbf{J} = \mathbf{J}_{\text{induced}} + \mathbf{J}_{\text{free}}$)	\mathbf{E} electric field $d\mathbf{s}$ surface element $d\tau$ volume element $d\mathbf{l}$ line element Φ linked magnetic flux ($= \int \mathbf{B} \cdot d\mathbf{s}$) I_{free} linked free current ($= \int \mathbf{J}_{\text{free}} \cdot d\mathbf{s}$) t time


Relativistic electrodynamics

Lorentz transformation of electric and magnetic fields	$E'_{\parallel} = E_{\parallel}$	(7.66)	E	electric field
	$E'_{\perp} = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\perp}$	(7.67)	\mathbf{B}	magnetic flux density
	$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$	(7.68)	$'$	measured in frame moving at relative velocity \mathbf{v}
	$\mathbf{B}'_{\perp} = \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2)_{\perp}$	(7.69)	γ	Lorentz factor $= [1 - (v/c)^2]^{-1/2}$
Lorentz transformation of current and charge densities	$\rho' = \gamma(\rho - vJ_{\parallel}/c^2)$	(7.70)	\parallel	parallel to \mathbf{v}
	$J'_{\perp} = J_{\perp}$	(7.71)	\perp	perpendicular to \mathbf{v}
	$J'_{\parallel} = \gamma(J_{\parallel} - v\rho)$	(7.72)	\mathbf{J}	current density
Lorentz transformation of potential fields	$\phi' = \gamma(\phi - vA_{\parallel})$	(7.73)	ρ	charge density
	$A'_{\perp} = A_{\perp}$	(7.74)	ϕ	electric potential
	$A'_{\parallel} = \gamma(A_{\parallel} - v\phi/c^2)$	(7.75)	\mathbf{A}	magnetic vector potential
Four-vector fields ^a	$\tilde{\mathbf{J}} = (\rho c, \mathbf{J})$	(7.76)		
	$\tilde{\mathbf{A}} = \left(\frac{\phi}{c}, \mathbf{A} \right)$	(7.77)	$\tilde{\mathbf{J}}$	current density four-vector
	$\square^2 = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2}, -\nabla^2 \right)$	(7.78)	$\tilde{\mathbf{A}}$	potential four-vector
	$\square^2 \tilde{\mathbf{A}} = \mu_0 \tilde{\mathbf{J}}$	(7.79)	\square^2	D'Alembertian operator

^aOther sign conventions are common here. See page 65 for a general definition of four-vectors.

7.4 Fields associated with media

Polarisation

Definition of electric dipole moment	$\mathbf{p} = q\mathbf{a}$	(7.80)	$\pm q$ end charges \mathbf{a} charge separation vector (from $-$ to $+$)	
Generalised electric dipole moment	$\mathbf{p} = \int_{\text{volume}} \mathbf{r}' \rho \, d\tau'$	(7.81)	\mathbf{p} dipole moment ρ charge density $d\tau'$ volume element \mathbf{r}' vector to $d\tau'$	
Electric dipole potential	$\phi(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3}$	(7.82)	ϕ dipole potential \mathbf{r} vector from dipole ϵ_0 permittivity of free space	
Dipole moment per unit volume (polarisation) ^a	$\mathbf{P} = n\mathbf{p}$	(7.83)	\mathbf{P} polarisation n number of dipoles per unit volume	
Induced volume charge density	$\nabla \cdot \mathbf{P} = -\rho_{\text{ind}}$	(7.84)	ρ_{ind} volume charge density	
Induced surface charge density	$\sigma_{\text{ind}} = \mathbf{P} \cdot \hat{\mathbf{s}}$	(7.85)	σ_{ind} surface charge density $\hat{\mathbf{s}}$ unit normal to surface	
Definition of electric displacement	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(7.86)	\mathbf{D} electric displacement \mathbf{E} electric field	
Definition of electric susceptibility	$\mathbf{P} = \epsilon_0 \chi_E \mathbf{E}$	(7.87)	χ_E electrical susceptibility (may be a tensor)	
Definition of relative permittivity ^b	$\epsilon_r = 1 + \chi_E$ $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$ $= \epsilon \mathbf{E}$	(7.88) (7.89) (7.90)	ϵ_r relative permittivity ϵ permittivity	
Atomic polarisability ^c	$\mathbf{p} = \alpha \mathbf{E}_{\text{loc}}$	(7.91)	α polarisability \mathbf{E}_{loc} local electric field	
Depolarising fields	$\mathbf{E}_d = -\frac{N_d \mathbf{P}}{\epsilon_0}$	(7.92)	\mathbf{E}_d depolarising field N_d depolarising factor =1/3 (sphere) =1 (thin slab \perp to \mathbf{P}) =0 (thin slab \parallel to \mathbf{P}) =1/2 (long circular cylinder, axis \perp to \mathbf{P})	
Clausius–Mossotti equation ^d	$\frac{n\alpha}{3\epsilon_0} = \frac{\epsilon_r - 1}{\epsilon_r + 2}$	(7.93)		

^aAssuming dipoles are parallel. The equivalent of Equation (7.112) holds for a hot gas of electric dipoles.

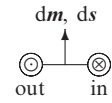
^bRelative permittivity as defined here is for a linear isotropic medium.

^cThe polarisability of a conducting sphere radius a is $\alpha = 4\pi\epsilon_0 a^3$. The definition $\mathbf{p} = \alpha\epsilon_0 \mathbf{E}_{\text{loc}}$ is also used.

^dWith the substitution $\eta^2 = \epsilon_r$ [cf. Equation (7.195) with $\mu_r = 1$] this is also known as the “Lorentz–Lorenz formula.”

Magnetisation

Definition of magnetic dipole moment	$d\mathbf{m} = I d\mathbf{s}$	(7.94)	$d\mathbf{m}$ dipole moment I loop current $d\mathbf{s}$ loop area (right-hand sense with respect to loop current)
Generalised magnetic dipole moment	$\mathbf{m} = \frac{1}{2} \int_{\text{volume}} \mathbf{r}' \times \mathbf{J} d\tau'$	(7.95)	\mathbf{m} dipole moment \mathbf{J} current density $d\tau'$ volume element \mathbf{r}' vector to $d\tau'$
Magnetic dipole (scalar) potential	$\phi_m(\mathbf{r}) = \frac{\mu_0 \mathbf{m} \cdot \mathbf{r}}{4\pi r^3}$	(7.96)	ϕ_m magnetic scalar potential \mathbf{r} vector from dipole μ_0 permeability of free space
Dipole moment per unit volume (magnetisation) ^a	$\mathbf{M} = n\mathbf{m}$	(7.97)	\mathbf{M} magnetisation n number of dipoles per unit volume
Induced volume current density	$\mathbf{J}_{\text{ind}} = \nabla \times \mathbf{M}$	(7.98)	\mathbf{J}_{ind} volume current density (i.e., A m ⁻²)
Induced surface current density	$\mathbf{j}_{\text{ind}} = \mathbf{M} \times \hat{\mathbf{s}}$	(7.99)	\mathbf{j}_{ind} surface current density (i.e., A m ⁻¹) $\hat{\mathbf{s}}$ unit normal to surface
Definition of magnetic field strength, \mathbf{H}	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	(7.100)	\mathbf{B} magnetic flux density \mathbf{H} magnetic field strength
Definition of magnetic susceptibility	$\mathbf{M} = \chi_H \mathbf{H}$	(7.101)	χ_H magnetic susceptibility. χ_B is also used (both may be tensors)
	$= \frac{\chi_B \mathbf{B}}{\mu_0}$	(7.102)	
	$\chi_B = \frac{\chi_H}{1 + \chi_H}$	(7.103)	
Definition of relative permeability ^b	$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$	(7.104)	μ_r relative permeability μ permeability
	$= \mu \mathbf{H}$	(7.105)	
	$\mu_r = 1 + \chi_H$	(7.106)	
	$= \frac{1}{1 - \chi_B}$	(7.107)	



^aAssuming all the dipoles are parallel. See Equation (7.112) for a classical paramagnetic gas and page 101 for the quantum generalisation.

^bRelative permeability as defined here is for a linear isotropic medium.

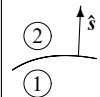
Paramagnetism and diamagnetism

Diamagnetic moment of an atom	$\mathbf{m} = -\frac{e^2}{6m_e} Z \langle r^2 \rangle \mathbf{B} \quad (7.108)$	\mathbf{m} magnetic moment $\langle r^2 \rangle$ mean squared orbital radius (of all electrons) Z atomic number \mathbf{B} magnetic flux density m_e electron mass $-e$ electronic charge
Intrinsic electron magnetic moment ^a	$\mathbf{m} \simeq -\frac{e}{2m_e} g \mathbf{J} \quad (7.109)$	\mathbf{J} total angular momentum g Landé g-factor (=2 for spin, =1 for orbital momentum)
Langevin function	$\mathcal{L}(x) = \coth x - \frac{1}{x} \quad (7.110)$ $\simeq x/3 \quad (x \lesssim 1) \quad (7.111)$	$\mathcal{L}(x)$ Langevin function
Classical gas paramagnetism ($ \mathbf{J} \gg \hbar$)	$\langle M \rangle = nm_0 \mathcal{L} \left(\frac{m_0 B}{kT} \right) \quad (7.112)$	$\langle M \rangle$ apparent magnetisation m_0 magnitude of magnetic dipole moment n dipole number density T temperature k Boltzmann constant
Curie's law	$\chi_H = \frac{\mu_0 n m_0^2}{3kT} \quad (7.113)$	χ_H magnetic susceptibility
Curie–Weiss law	$\chi_H = \frac{\mu_0 n m_0^2}{3k(T - T_c)} \quad (7.114)$	μ_0 permeability of free space T_c Curie temperature

^aSee also page 100.

Boundary conditions for E , D , B , and H ^a

Parallel component of the electric field	E_{\parallel} continuous	(7.115)	\parallel component parallel to interface
Perpendicular component of the magnetic flux density	B_{\perp} continuous	(7.116)	\perp component perpendicular to interface
Electric displacement ^b	$\hat{s} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma$	(7.117)	$\mathbf{D}_{1,2}$ electrical displacements in media 1 & 2 \hat{s} unit normal to surface, directed 1 \rightarrow 2 σ surface density of free charge
Magnetic field strength ^c	$\hat{s} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{j}_s$	(7.118)	$\mathbf{H}_{1,2}$ magnetic field strengths in media 1 & 2 \mathbf{j}_s surface current per unit width



^aAt the plane surface between two uniform media.

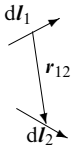
^bIf $\sigma = 0$, then D_{\perp} is continuous.

^cIf $\mathbf{j}_s = \mathbf{0}$ then H_{\parallel} is continuous.

7.5 Force, torque, and energy

Electromagnetic force and torque

Force between two static charges: Coulomb's law	$\mathbf{F}_2 = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}^2} \hat{\mathbf{r}}_{12} \quad (7.119)$	\mathbf{F}_2 force on q_2 $q_{1,2}$ charges \mathbf{r}_{12} vector from 1 to 2 $\hat{\mathbf{r}}$ unit vector ϵ_0 permittivity of free space
Force between two current-carrying elements	$d\mathbf{F}_2 = \frac{\mu_0 I_1 I_2}{4\pi r_{12}^2} [d\mathbf{I}_2 \times (d\mathbf{I}_1 \times \hat{\mathbf{r}}_{12})] \quad (7.120)$	$d\mathbf{I}_{1,2}$ line elements $I_{1,2}$ currents flowing along $d\mathbf{I}_1$ and $d\mathbf{I}_2$ $d\mathbf{F}_2$ force on $d\mathbf{I}_2$ μ_0 permeability of free space
Force on a current-carrying element in a magnetic field	$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B} \quad (7.121)$	$d\mathbf{l}$ line element \mathbf{F} force I current flowing along $d\mathbf{l}$ \mathbf{B} magnetic flux density
Force on a charge (Lorentz force)	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (7.122)$	\mathbf{E} electric field \mathbf{v} charge velocity
Force on an electric dipole ^a	$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E} \quad (7.123)$	\mathbf{p} electric dipole moment
Force on a magnetic dipole ^b	$\mathbf{F} = (\mathbf{m} \cdot \nabla) \mathbf{B} \quad (7.124)$	\mathbf{m} magnetic dipole moment
Torque on an electric dipole	$\mathbf{G} = \mathbf{p} \times \mathbf{E} \quad (7.125)$	\mathbf{G} torque
Torque on a magnetic dipole	$\mathbf{G} = \mathbf{m} \times \mathbf{B} \quad (7.126)$	
Torque on a current loop	$\mathbf{G} = I_L \oint_{\text{loop}} \mathbf{r} \times (d\mathbf{I}_L \times \mathbf{B}) \quad (7.127)$	$d\mathbf{I}_L$ line-element (of loop) \mathbf{r} position vector of $d\mathbf{I}_L$ I_L current around loop



^a \mathbf{F} simplifies to $\nabla(\mathbf{p} \cdot \mathbf{E})$ if \mathbf{p} is intrinsic, $\nabla(pE/2)$ if \mathbf{p} is induced by \mathbf{E} and the medium is isotropic.
^b \mathbf{F} simplifies to $\nabla(\mathbf{m} \cdot \mathbf{B})$ if \mathbf{m} is intrinsic, $\nabla(mB/2)$ if \mathbf{m} is induced by \mathbf{B} and the medium is isotropic.

Electromagnetic energy

Electromagnetic field energy density (in free space)	$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}$	(7.128)	u energy density E electric field B magnetic flux density
Energy density in media	$u = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})$	(7.129)	ϵ_0 permittivity of free space μ_0 permeability of free space \mathbf{D} electric displacement \mathbf{H} magnetic field strength
Energy flow (Poynting) vector	$\mathbf{N} = \mathbf{E} \times \mathbf{H}$	(7.130)	c speed of light N energy flow rate per unit area \perp to the flow direction
Mean flux density at a distance r from a short oscillating dipole	$\langle N \rangle = \frac{\omega^4 p_0^2 \sin^2 \theta}{32\pi^2 \epsilon_0 c^3 r^3} \mathbf{r}$	(7.131)	p_0 amplitude of dipole moment \mathbf{r} vector from dipole (\gg wavelength) θ angle between \mathbf{p} and \mathbf{r} ω oscillation frequency
Total mean power from oscillating dipole ^a	$W = \frac{\omega^4 p_0^2 / 2}{6\pi \epsilon_0 c^3}$	(7.132)	W total mean radiated power
Self-energy of a charge distribution	$U_{\text{tot}} = \frac{1}{2} \int_{\text{volume}} \phi(\mathbf{r}) \rho(\mathbf{r}) d\tau$	(7.133)	U_{tot} total energy $d\tau$ volume element \mathbf{r} position vector of $d\tau$ ϕ electrical potential ρ charge density
Energy of an assembly of capacitors ^b	$U_{\text{tot}} = \frac{1}{2} \sum_i \sum_j C_{ij} V_i V_j$	(7.134)	V_i potential of i th capacitor C_{ij} mutual capacitance between capacitors i and j
Energy of an assembly of inductors ^c	$U_{\text{tot}} = \frac{1}{2} \sum_i \sum_j L_{ij} I_i I_j$	(7.135)	L_{ij} mutual inductance between inductors i and j
Intrinsic dipole in an electric field	$U_{\text{dip}} = -\mathbf{p} \cdot \mathbf{E}$	(7.136)	U_{dip} energy of dipole \mathbf{p} electric dipole moment
Intrinsic dipole in a magnetic field	$U_{\text{dip}} = -\mathbf{m} \cdot \mathbf{B}$	(7.137)	\mathbf{m} magnetic dipole moment
Hamiltonian of a charged particle in an EM field ^d	$H = \frac{ \mathbf{p}_m - q\mathbf{A} ^2}{2m} + q\phi$	(7.138)	H Hamiltonian \mathbf{p}_m particle momentum q particle charge m particle mass \mathbf{A} magnetic vector potential

^aSometimes called “Larmor’s formula.”

^b C_{ii} is the self-capacitance of the i th body. Note that $C_{ij} = C_{ji}$.

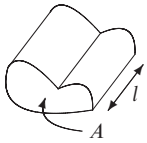
^c L_{ii} is the self-inductance of the i th body. Note that $L_{ij} = L_{ji}$.

^dNewtonian limit, i.e., velocity $\ll c$.

7.6 LCR circuits

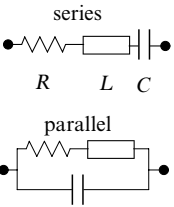
LCR definitions

Current	$I = \frac{dQ}{dt}$	(7.139)	I current Q charge
Ohm's law	$V = IR$	(7.140)	R resistance V potential difference over R I current through R
Ohm's law (field form)	$\mathbf{J} = \sigma \mathbf{E}$	(7.141)	\mathbf{J} current density \mathbf{E} electric field σ conductivity ρ resistivity
Resistivity	$\rho = \frac{1}{\sigma} = \frac{RA}{l}$	(7.142)	A area of face (I is normal to face) l length
Capacitance	$C = \frac{Q}{V}$	(7.143)	C capacitance V potential difference across C
Current through capacitor	$I = C \frac{dV}{dt}$	(7.144)	I current through C t time
Self-inductance	$L = \frac{\Phi}{I}$	(7.145)	Φ total linked flux I current through inductor
Voltage across inductor	$V = -L \frac{dI}{dt}$	(7.146)	V potential difference over L
Mutual inductance	$L_{12} = \frac{\Phi_1}{I_2} = L_{21}$	(7.147)	Φ_1 total flux from loop 2 linked by loop 1 L_{12} mutual inductance I_2 current through loop 2
Coefficient of coupling	$ L_{12} = k \sqrt{L_1 L_2}$	(7.148)	k coupling coefficient between L_1 and L_2 (≤ 1)
Linked magnetic flux through a coil	$\Phi = N\phi$	(7.149)	Φ linked flux N number of turns around ϕ ϕ flux through area of turns



Resonant LCR circuits

Phase resonant frequency ^a	$\omega_0^2 = \begin{cases} 1/LC & \text{(series)} \\ 1/LC - R^2/L^2 & \text{(parallel)} \end{cases}$ (7.150)	ω_0 resonant angular frequency L inductance C capacitance R resistance
Tuning ^b	$\frac{\delta\omega}{\omega_0} = \frac{1}{Q} = \frac{R}{\omega_0 L}$ (7.151)	$\delta\omega$ half-power bandwidth Q quality factor
Quality factor	$Q = 2\pi \frac{\text{stored energy}}{\text{energy lost per cycle}}$ (7.152)	



^aAt which the impedance is purely real.
^bAssuming the capacitor is purely reactive. If L and R are parallel, then $1/Q = \omega_0 L/R$.

Energy in capacitors, inductors, and resistors

Energy stored in a capacitor	$U = \frac{1}{2} C V^2 = \frac{1}{2} Q V = \frac{1}{2} \frac{Q^2}{C}$ (7.153)	U stored energy C capacitance Q charge V potential difference
Energy stored in an inductor	$U = \frac{1}{2} L I^2 = \frac{1}{2} \Phi I = \frac{1}{2} \frac{\Phi^2}{L}$ (7.154)	L inductance Φ linked magnetic flux I current
Power dissipated in a resistor ^a (Joule's law)	$W = I V = I^2 R = \frac{V^2}{R}$ (7.155)	W power dissipated R resistance
Relaxation time	$\tau = \frac{\epsilon_0 \epsilon_r}{\sigma}$ (7.156)	τ relaxation time ϵ_r relative permittivity σ conductivity

^aThis is d.c., or instantaneous a.c., power.

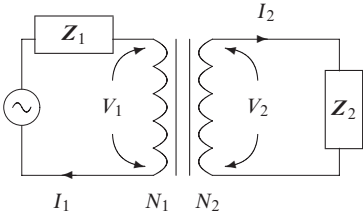
Electrical impedance

Impedances in series	$Z_{\text{tot}} = \sum_n Z_n$ (7.157)
Impedances in parallel	$Z_{\text{tot}} = \left(\sum_n Z_n^{-1} \right)^{-1}$ (7.158)
Impedance of capacitance	$Z_C = -\frac{i}{\omega C}$ (7.159)
Impedance of inductance	$Z_L = i\omega L$ (7.160)
Impedance: Z Inductance: L Conductance: $G = 1/R$ Admittance: $Y = 1/Z$	Capacitance: C Resistance: $R = \text{Re}[Z]$ Reactance: $X = \text{Im}[Z]$ Susceptance: $S = 1/X$

Kirchhoff’s laws

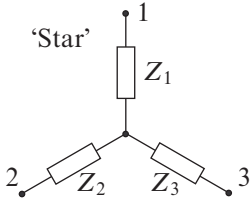
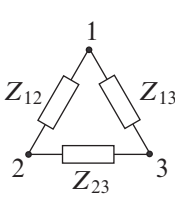
Current law	$\sum_{\text{node}} I_i = 0$	(7.161)	I_i currents impinging on node V_i potential differences around loop
Voltage law	$\sum_{\text{loop}} V_i = 0$	(7.162)	

Transformers^a

	n turns ratio N_1 number of primary turns N_2 number of secondary turns V_1 primary voltage V_2 secondary voltage I_1 primary current I_2 secondary current Z_{out} output impedance Z_{in} input impedance Z_1 source impedance Z_2 load impedance	
Turns ratio	$n = N_2/N_1$	(7.163)
Transformation of voltage and current	$V_2 = nV_1$	(7.164)
	$I_2 = I_1/n$	(7.165)
Output impedance (seen by Z_2)	$Z_{\text{out}} = n^2 Z_1$	(7.166)
Input impedance (seen by Z_1)	$Z_{\text{in}} = Z_2/n^2$	(7.167)

^aIdeal, with a coupling constant of 1 between loss-free windings.

Star–delta transformation

<div><div><p>‘Star’</p></div><div><p>‘Delta’</p></div></div>	<div>i, j, k node indices (1,2, or 3)</div> <div>Z_i impedance on node i</div> <div>Z_{ij} impedance connecting nodes i and j</div>	
Star impedances	$Z_i = \frac{Z_{ij}Z_{ik}}{Z_{ij} + Z_{ik} + Z_{jk}}$	(7.168)
Delta impedances	$Z_{ij} = Z_i Z_j \left(\frac{1}{Z_i} + \frac{1}{Z_j} + \frac{1}{Z_k} \right)$	(7.169)

7.7 Transmission lines and waveguides

Transmission line relations

Loss-free transmission line equations	$\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t} \quad (7.170)$ $\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t} \quad (7.171)$	V potential difference across line I current in line L inductance per unit length C capacitance per unit length
Wave equation for a lossless transmission line	$\frac{1}{LC} \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2} \quad (7.172)$ $\frac{1}{LC} \frac{\partial^2 I}{\partial x^2} = \frac{\partial^2 I}{\partial t^2} \quad (7.173)$	x distance along line t time
Characteristic impedance of lossless line	$Z_c = \sqrt{\frac{L}{C}} \quad (7.174)$	Z_c characteristic impedance
Characteristic impedance of lossy line	$Z_c = \sqrt{\frac{R + i\omega L}{G + i\omega C}} \quad (7.175)$	R resistance per unit length of conductor G conductance per unit length of insulator ω angular frequency
Wave speed along a lossless line	$v_p = v_g = \frac{1}{\sqrt{LC}} \quad (7.176)$	v_p phase speed v_g group speed
Input impedance of a terminated lossless line	$Z_{in} = Z_c \frac{Z_t \cos kl - iZ_c \sin kl}{Z_c \cos kl - iZ_t \sin kl} \quad (7.177)$ $= Z_c^2 / Z_t \quad \text{if } l = \lambda/4 \quad (7.178)$	Z_{in} (complex) input impedance Z_t (complex) terminating impedance k wavenumber ($= 2\pi/\lambda$)
Reflection coefficient from a terminated line	$r = \frac{Z_t - Z_c}{Z_t + Z_c} \quad (7.179)$	l distance from termination r (complex) voltage reflection coefficient
Line voltage standing wave ratio	$\text{VSWR} = \frac{1 + r }{1 - r } \quad (7.180)$	

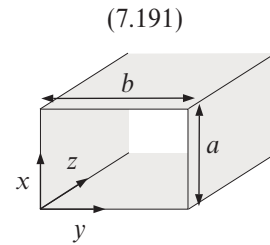
Transmission line impedances^a

Coaxial line	$Z_c = \sqrt{\frac{\mu}{4\pi^2\epsilon}} \ln \frac{b}{a} \simeq \frac{60}{\sqrt{\epsilon_r}} \ln \frac{b}{a} \quad (7.181)$	Z_c characteristic impedance (Ω) a radius of inner conductor b radius of outer conductor ϵ permittivity ($= \epsilon_0 \epsilon_r$)
Open wire feeder	$Z_c = \sqrt{\frac{\mu}{\pi^2\epsilon}} \ln \frac{l}{r} \simeq \frac{120}{\sqrt{\epsilon_r}} \ln \frac{l}{r} \quad (7.182)$	μ permeability ($= \mu_0 \mu_r$) r radius of wires l distance between wires ($\gg r$)
Paired strip	$Z_c = \sqrt{\frac{\mu}{\epsilon}} \frac{d}{w} \simeq \frac{377}{\sqrt{\epsilon_r}} \frac{d}{w} \quad (7.183)$	d strip separation w strip width ($\gg d$)
Microstrip line	$Z_c \simeq \frac{377}{\sqrt{\epsilon_r} [(w/h) + 2]} \quad (7.184)$	h height above earth plane ($\ll w$)

^aFor lossless lines.

Waveguides^a

Waveguide equation	$k_g^2 = \frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}$	(7.185)	k_g wavenumber in guide ω angular frequency a guide height b guide width m, n mode indices with respect to a and b (integers) c speed of light
Guide cutoff frequency	$v_c = c\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}$	(7.186)	v_c cutoff frequency $\omega_c = 2\pi v_c$
Phase velocity above cutoff	$v_p = \frac{c}{\sqrt{1 - (v_c/v)^2}}$	(7.187)	v_p phase velocity v frequency
Group velocity above cutoff	$v_g = c^2/v_p = c\sqrt{1 - (v_c/v)^2}$	(7.188)	v_g group velocity
Wave impedances ^b	$Z_{TM} = Z_0\sqrt{1 - (v_c/v)^2}$	(7.189)	Z_{TM} wave impedance for transverse magnetic modes
	$Z_{TE} = Z_0/\sqrt{1 - (v_c/v)^2}$	(7.190)	Z_{TE} wave impedance for transverse electric modes
			Z_0 impedance of free space ($= \sqrt{\mu_0/\epsilon_0}$)
<p>Field solutions for TE_{<i>mn</i>} modes^c</p> $B_x = \frac{\mathbf{i}k_g c^2}{\omega_c^2} \frac{\partial B_z}{\partial x} \quad E_x = \frac{\mathbf{i}\omega c^2}{\omega_c^2} \frac{\partial B_z}{\partial y}$ $B_y = \frac{\mathbf{i}k_g c^2}{\omega_c^2} \frac{\partial B_z}{\partial y} \quad E_y = \frac{-\mathbf{i}\omega c^2}{\omega_c^2} \frac{\partial B_z}{\partial x}$ $B_z = B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \quad E_z = 0$ <p>Field solutions for TM_{<i>mn</i>} modes^c</p> $E_x = \frac{\mathbf{i}k_g c^2}{\omega_c^2} \frac{\partial E_z}{\partial x} \quad B_x = \frac{-\mathbf{i}\omega}{\omega_c^2} \frac{\partial E_z}{\partial y}$ $E_y = \frac{\mathbf{i}k_g c^2}{\omega_c^2} \frac{\partial E_z}{\partial y} \quad B_y = \frac{\mathbf{i}\omega}{\omega_c^2} \frac{\partial E_z}{\partial x}$ $E_z = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad B_z = 0$			

^aEquations are for lossless waveguides with rectangular cross sections and no dielectric.^bThe ratio of the electric field to the magnetic field strength in the xy plane.^cBoth TE and TM modes propagate in the z direction with a further factor of $\exp[\mathbf{i}(k_g z - \omega t)]$ on all components. B_0 and E_0 are the amplitudes of the z components of magnetic flux density and electric field respectively.

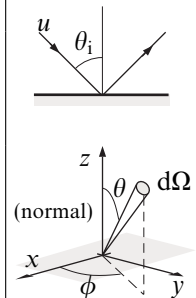
7.8 Waves in and out of media

Waves in lossless media

Electric field	$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$	(7.193)	\mathbf{E} electric field μ permeability ($=\mu_0\mu_r$) ϵ permittivity ($=\epsilon_0\epsilon_r$)
Magnetic field	$\nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2}$	(7.194)	\mathbf{B} magnetic flux density t time
Refractive index	$\eta = \sqrt{\epsilon_r\mu_r}$	(7.195)	
Wave speed	$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\eta}$	(7.196)	v wave phase speed η refractive index c speed of light
Impedance of free space	$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 376.7 \Omega$	(7.197)	Z_0 impedance of free space
Wave impedance	$Z = \frac{E}{H} = Z_0 \sqrt{\frac{\mu_r}{\epsilon_r}}$	(7.198)	Z wave impedance H magnetic field strength

Radiation pressure^a

Radiation momentum density	$\mathbf{G} = \frac{\mathbf{N}}{c^2}$	(7.199)	\mathbf{G} momentum density \mathbf{N} Poynting vector c speed of light
Isotropic radiation	$p_n = \frac{1}{3}u(1+R)$	(7.200)	p_n normal pressure u incident radiation energy density R (power) reflectance coefficient
Specular reflection	$p_n = u(1+R)\cos^2\theta_i$	(7.201)	p_t tangential pressure
	$p_t = u(1-R)\sin\theta_i\cos\theta_i$	(7.202)	θ_i angle of incidence
From an extended source ^b	$p_n = \frac{1+R}{c} \iint I_v(\theta, \phi) \cos^2\theta \, d\Omega \, dv$	(7.203)	I_v specific intensity v frequency Ω solid angle θ angle between $d\Omega$ and normal to plane
From a point source, ^c luminosity L	$p_n = \frac{L(1+R)}{4\pi r^2 c}$	(7.204)	L source luminosity (i.e., radiant power) r distance from source

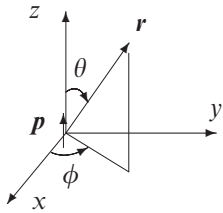


^aOn an opaque surface.

^bIn spherical polar coordinates. See page 120 for the meaning of specific intensity.

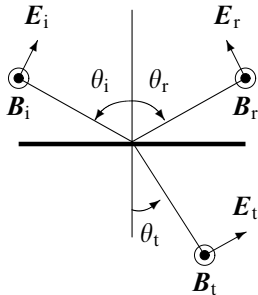
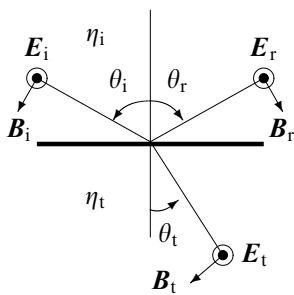
^cNormal to the plane.

Antennas

Spherical polar geometry:			
Field from a short ($l \ll \lambda$) electric dipole in free space ^a	$E_r = \frac{1}{2\pi\epsilon_0} \left(\frac{[\dot{p}]}{r^2 c} + \frac{[p]}{r^3} \right) \cos \theta \quad (7.205)$ $E_\theta = \frac{1}{4\pi\epsilon_0} \left(\frac{[\dot{p}]}{r c^2} + \frac{[\ddot{p}]}{r^2 c} + \frac{[p]}{r^3} \right) \sin \theta \quad (7.206)$ $B_\phi = \frac{\mu_0}{4\pi} \left(\frac{[\dot{p}]}{r c} + \frac{[p]}{r^2} \right) \sin \theta \quad (7.207)$	r distance from dipole θ angle between r and p $[p]$ retarded dipole moment $[p] = p(t - r/c)$ c speed of light	
Radiation resistance of a short electric dipole in free space	$R = \frac{\omega^2 l^2}{6\pi\epsilon_0 c^3} = \frac{2\pi Z_0}{3} \left(\frac{l}{\lambda} \right)^2 \quad (7.208)$ $\simeq 789 \left(\frac{l}{\lambda} \right)^2 \text{ ohm} \quad (7.209)$	l dipole length ($\ll \lambda$) ω angular frequency λ wavelength Z_0 impedance of free space	
Beam solid angle	$\Omega_A = \int_{4\pi} P_n(\theta, \phi) d\Omega \quad (7.210)$	Ω_A beam solid angle P_n normalised antenna power pattern $P_n(0,0) = 1$ $d\Omega$ differential solid angle	
Forward power gain	$G(0) = \frac{4\pi}{\Omega_A} \quad (7.211)$	G antenna gain	
Antenna effective area	$A_e = \frac{\lambda^2}{\Omega_A} \quad (7.212)$	A_e effective area	
Power gain of a short dipole	$G(\theta) = \frac{3}{2} \sin^2 \theta \quad (7.213)$		
Beam efficiency	$\text{efficiency} = \frac{\Omega_M}{\Omega_A} \quad (7.214)$	Ω_M main lobe solid angle	
Antenna temperature ^b	$T_A = \frac{1}{\Omega_A} \int_{4\pi} T_b(\theta, \phi) P_n(\theta, \phi) d\Omega \quad (7.215)$	T_A antenna temperature T_b sky brightness temperature	

^aAll field components propagate with a further phase factor equal to $\exp i(kr - \omega t)$, where $k = 2\pi/\lambda$.^bThe brightness temperature of a source of specific intensity I_ν is $T_b = \lambda^2 I_\nu / (2k_B)$.

Reflection, refraction, and transmission^a

<div style="display: flex; justify-content: space-around; align-items: flex-start;"> <div style="text-align: center;"> <p>parallel incidence</p>  </div> <div style="text-align: center;"> <p>perpendicular incidence</p>  </div> </div>		<p>E electric field</p> <p>B magnetic flux density</p> <p>η_i refractive index on incident side</p> <p>η_t refractive index on transmitted side</p> <p>θ_i angle of incidence</p> <p>θ_r angle of reflection</p> <p>θ_t angle of refraction</p>	
Law of reflection	$\theta_i = \theta_r$	(7.216)	
Snell's law ^b	$\eta_i \sin \theta_i = \eta_t \sin \theta_t$	(7.217)	
Brewster's law	$\tan \theta_B = \eta_t / \eta_i$	(7.218)	θ_B Brewster's angle of incidence for plane-polarised reflection ($r_{\parallel} = 0$)
Fresnel equations of reflection and refraction			
$r_{\parallel} = \frac{\sin 2\theta_i - \sin 2\theta_t}{\sin 2\theta_i + \sin 2\theta_t}$	(7.219)	$r_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$	(7.223)
$t_{\parallel} = \frac{4\cos \theta_i \sin \theta_t}{\sin 2\theta_i + \sin 2\theta_t}$	(7.220)	$t_{\perp} = \frac{2\cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t)}$	(7.224)
$R_{\parallel} = r_{\parallel}^2$	(7.221)	$R_{\perp} = r_{\perp}^2$	(7.225)
$T_{\parallel} = \frac{\eta_t \cos \theta_t}{\eta_i \cos \theta_i} t_{\parallel}^2$	(7.222)	$T_{\perp} = \frac{\eta_t \cos \theta_t}{\eta_i \cos \theta_i} t_{\perp}^2$	(7.226)
Coefficients for normal incidence ^c			
$R = \frac{(\eta_i - \eta_t)^2}{(\eta_i + \eta_t)^2}$	(7.227)	$r = \frac{\eta_i - \eta_t}{\eta_i + \eta_t}$	(7.230)
$T = \frac{4\eta_i \eta_t}{(\eta_i + \eta_t)^2}$	(7.228)	$t = \frac{2\eta_i}{\eta_i + \eta_t}$	(7.231)
$R + T = 1$	(7.229)	$t - r = 1$	(7.232)
\parallel	electric field parallel to the plane of incidence	\perp	electric field perpendicular to the plane of incidence
R	(power) reflectance coefficient	r	amplitude reflection coefficient
T	(power) transmittance coefficient	t	amplitude transmission coefficient

^aFor the plane boundary between lossless dielectric media. All coefficients refer to the electric field component and whether it is parallel or perpendicular to the plane of incidence. Perpendicular components are out of the paper.

^bThe incident wave suffers total internal reflection if $\frac{n_t}{n_i} \sin \theta_i > 1$.

^cI.e., $\theta_i = 0$. Use the diagram labelled "perpendicular incidence" for correct phases.

Propagation in conducting media^a

Electrical conductivity ($B = 0$)	$\sigma = n_e e \mu = \frac{n_e e^2}{m_e} \tau_c$	(7.233)	σ electrical conductivity n_e electron number density τ_c electron relaxation time μ electron mobility B magnetic flux density
Refractive index of an ohmic conductor ^b	$\eta = (1 + i) \left(\frac{\sigma}{4\pi\nu\epsilon_0} \right)^{1/2}$	(7.234)	m_e electron mass $-e$ electronic charge η refractive index ϵ_0 permittivity of free space
Skin depth in an ohmic conductor	$\delta = (\mu_0 \sigma \pi \nu)^{-1/2}$	(7.235)	ν frequency δ skin depth μ_0 permeability of free space

^aAssuming a relative permeability, μ_r , of 1.^bTaking the wave to have an $e^{-i\omega t}$ time dependence, and the low-frequency limit ($\sigma \gg 2\pi\nu\epsilon_0$).**Electron scattering processes^a**

Rayleigh scattering cross section ^b	$\sigma_R = \frac{\omega^4 \alpha^2}{6\pi\epsilon_0 c^4}$	(7.236)	σ_R Rayleigh cross section ω radiation angular frequency α particle polarisability ϵ_0 permittivity of free space
Thomson scattering cross section ^c	$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2} \right)^2$ $= \frac{8\pi}{3} r_e^2 \simeq 6.652 \times 10^{-29} \text{ m}^2$	(7.237) (7.238)	σ_T Thomson cross section m_e electron (rest) mass r_e classical electron radius c speed of light
Inverse Compton scattering ^d	$P_{\text{tot}} = \frac{4}{3} \sigma_T c u_{\text{rad}} \gamma^2 \left(\frac{v^2}{c^2} \right)$	(7.239)	P_{tot} electron energy loss rate u_{rad} radiation energy density γ Lorentz factor $= [1 - (v/c)^2]^{-1/2}$ v electron speed
Compton scattering ^e	$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)$ $h\nu' = \frac{m_e c^2}{1 - \cos \theta + (1/\epsilon)}$ $\cot \phi = (1 + \epsilon) \tan \frac{\theta}{2}$	(7.240) (7.241) (7.242)	λ, λ' incident & scattered wavelengths ν, ν' incident & scattered frequencies θ photon scattering angle $\frac{h}{m_e c}$ electron Compton wavelength $\epsilon = h\nu/(m_e c^2)$
Klein–Nishina cross section (for a free electron)	$\sigma_{\text{KN}} = \frac{\pi r_e^2}{\epsilon} \left\{ \left[1 - \frac{2(\epsilon + 1)}{\epsilon^2} \right] \ln(2\epsilon + 1) + \frac{1}{2} + \frac{4}{\epsilon} - \frac{1}{2(2\epsilon + 1)^2} \right\}$ $\simeq \sigma_T \quad (\epsilon \ll 1)$ $\simeq \frac{\pi r_e^2}{\epsilon} \left(\ln 2\epsilon + \frac{1}{2} \right) \quad (\epsilon \gg 1)$	(7.243) (7.244) (7.245)	σ_{KN} Klein–Nishina cross section

^aFor Rutherford scattering see page 72.^bScattering by bound electrons.^cScattering from free electrons, $\epsilon \ll 1$.^dElectron energy loss rate due to photon scattering in the Thomson limit ($\gamma h\nu \ll m_e c^2$).^eFrom an electron at rest.

Cherenkov radiation

Cherenkov cone angle	$\sin \theta = \frac{c}{\eta v}$	(7.246)	θ cone semi-angle c (vacuum) speed of light $\eta(\omega)$ refractive index v particle velocity
Radiated power ^a	$P_{\text{tot}} = \frac{e^2 \mu_0}{4\pi} v \int_0^{\omega_c} \left[1 - \frac{c^2}{v^2 \eta^2(\omega)} \right] \omega \, d\omega$ where $\eta(\omega) \geq \frac{c}{v}$ for $0 < \omega < \omega_c$	(7.247)	P_{tot} total radiated power $-e$ electronic charge μ_0 free space permeability ω angular frequency ω_c cutoff frequency

^aFrom a point charge, e , travelling at speed v through a medium of refractive index $\eta(\omega)$.

7.9 Plasma physics

Warm plasmas

Landau length	$l_L = \frac{e^2}{4\pi\epsilon_0 k_B T_e}$ $\simeq 1.67 \times 10^{-5} T_e^{-1} \text{ m}$	(7.248) (7.249)	l_L Landau length $-e$ electronic charge ϵ_0 permittivity of free space k_B Boltzmann constant T_e electron temperature (K)
Electron Debye length	$\lambda_{De} = \left(\frac{\epsilon_0 k_B T_e}{n_e e^2} \right)^{1/2}$ $\simeq 69 (T_e / n_e)^{1/2} \text{ m}$	(7.250) (7.251)	λ_{De} electron Debye length n_e electron number density (m^{-3})
Debye screening ^a	$\phi(r) = \frac{q \exp(-2^{1/2} r / \lambda_{De})}{4\pi\epsilon_0 r}$	(7.252)	ϕ effective potential q point charge r distance from q
Debye number	$N_{De} = \frac{4}{3} \pi n_e \lambda_{De}^3$	(7.253)	N_{De} electron Debye number
Relaxation times ($B=0$) ^b	$\tau_e = 3.44 \times 10^5 \frac{T_e^{3/2}}{n_e \ln \Lambda} \text{ s}$ $\tau_i = 2.09 \times 10^7 \frac{T_i^{3/2}}{n_e \ln \Lambda} \left(\frac{m_i}{m_p} \right)^{1/2} \text{ s}$	(7.254) (7.255)	τ_e electron relaxation time τ_i ion relaxation time T_i ion temperature (K) $\ln \Lambda$ Coulomb logarithm (typically 10 to 20) B magnetic flux density
Characteristic electron thermal speed ^c	$v_{te} = \left(\frac{2k_B T_e}{m_e} \right)^{1/2}$ $\simeq 5.51 \times 10^3 T_e^{1/2} \text{ ms}^{-1}$	(7.256) (7.257)	v_{te} electron thermal speed m_e electron mass

^aEffective (Yukawa) potential from a point charge q immersed in a plasma.

^bCollision times for electrons and singly ionised ions with Maxwellian speed distributions, $T_i \lesssim T_e$. The Spitzer conductivity can be calculated from Equation (7.233).

^cDefined so that the Maxwellian velocity distribution $\propto \exp(-v^2/v_{te}^2)$. There are other definitions (see *Maxwell–Boltzmann distribution* on page 112).

Electromagnetic propagation in cold plasmas^a

Plasma frequency	$(2\pi\nu_p)^2 = \frac{n_e e^2}{\epsilon_0 m_e} = \omega_p^2$ (7.258)	ν_p plasma frequency
	$\nu_p \simeq 8.98 n_e^{1/2} \text{ Hz}$ (7.259)	ω_p plasma angular frequency
		n_e electron number density (m^{-3})
		m_e electron mass
Plasma refractive index ($B=0$)	$\eta = [1 - (\nu_p/\nu)^2]^{1/2}$ (7.260)	$-e$ electronic charge
		ϵ_0 permittivity of free space
		η refractive index
		ν frequency
Plasma dispersion relation ($B=0$)	$c^2 k^2 = \omega^2 - \omega_p^2$ (7.261)	k wavenumber ($=2\pi/\lambda$)
		ω angular frequency ($=2\pi/\nu$)
		c speed of light
Plasma phase velocity ($B=0$)	$v_\phi = c/\eta$ (7.262)	v_ϕ phase velocity
Plasma group velocity ($B=0$)	$v_g = c\eta$ (7.263)	
	$v_\phi v_g = c^2$ (7.264)	v_g group velocity
Cyclotron (Larmor, or gyro-) frequency	$2\pi\nu_C = \frac{qB}{m} = \omega_C$ (7.265)	ν_C cyclotron frequency
	$\nu_{Ce} \simeq 28 \times 10^9 B \text{ Hz}$ (7.266)	ω_C cyclotron angular frequency
	$\nu_{Cp} \simeq 15.2 \times 10^6 B \text{ Hz}$ (7.267)	ν_{Ce} electron ν_C
		ν_{Cp} proton ν_C
		q particle charge
		B magnetic flux density (T)
Larmor (cyclotron, or gyro-) radius	$r_L = \frac{v_\perp}{\omega_C} = v_\perp \frac{m}{qB}$ (7.268)	m particle mass (γm if relativistic)
	$r_{Le} = 5.69 \times 10^{-12} \left(\frac{v_\perp}{B}\right) \text{ m}$ (7.269)	r_L Larmor radius
	$r_{Lp} = 10.4 \times 10^{-9} \left(\frac{v_\perp}{B}\right) \text{ m}$ (7.270)	r_{Le} electron r_L
		r_{Lp} proton r_L
		v_\perp speed \perp to \mathbf{B} (ms^{-1})
Mixed propagation modes ^b	$\eta^2 = 1 - \frac{X(1-X)}{(1-X) - \frac{1}{2}Y^2 \sin^2 \theta_B \pm S}$, (7.271)	θ_B angle between wavefront normal ($\hat{\mathbf{k}}$) and \mathbf{B}
	where $X = (\omega_p/\omega)^2$, $Y = \omega_{Ce}/\omega$,	
	and $S^2 = \frac{1}{4}Y^4 \sin^4 \theta_B + Y^2(1-X)^2 \cos^2 \theta_B$	
Faraday rotation ^c	$\Delta\psi = \underbrace{\frac{\mu_0 e^3}{8\pi^2 m_e^2 c}}_{2.63 \times 10^{-13}} \lambda^2 \int_{\text{line}} n_e \mathbf{B} \cdot d\mathbf{l}$ (7.272)	$\Delta\psi$ rotation angle
	$= R\lambda^2$ (7.273)	λ wavelength ($=2\pi/k$)
		$d\mathbf{l}$ line element in direction of wave propagation
		R rotation measure

^aI.e., plasmas in which electromagnetic force terms dominate over thermal pressure terms. Also taking $\mu_r = 1$.^bIn a collisionless electron plasma. The ordinary and extraordinary modes are the + and - roots of S^2 when $\theta_B = \pi/2$. When $\theta_B = 0$, these roots are the right and left circularly polarised modes respectively, using the optical convention for handedness.^cIn a tenuous plasma, SI units throughout. $\Delta\psi$ is taken positive if \mathbf{B} is directed towards the observer.

Magnetohydrodynamics^a

Sound speed	$v_s = \left(\frac{\gamma p}{\rho} \right)^{1/2} = \left(\frac{2\gamma k_B T}{m_p} \right)^{1/2} \quad (7.274)$ $\simeq 166 T^{1/2} \text{ ms}^{-1} \quad (7.275)$	v_s sound (wave) speed γ ratio of heat capacities p hydrostatic pressure ρ plasma mass density k_B Boltzmann constant T temperature (K)
Alfvén speed	$v_A = \frac{B}{(\mu_0 \rho)^{1/2}} \quad (7.276)$ $\simeq 2.18 \times 10^{16} B n_e^{-1/2} \text{ ms}^{-1} \quad (7.277)$	m_p proton mass v_A Alfvén speed B magnetic flux density (T) μ_0 permeability of free space n_e electron number density (m^{-3})
Plasma beta	$\beta = \frac{2\mu_0 p}{B^2} = \frac{4\mu_0 n_e k_B T}{B^2} = \frac{2v_s^2}{\gamma v_A^2} \quad (7.278)$	β plasma beta (ratio of hydrostatic to magnetic pressure)
Direct electrical conductivity	$\sigma_d = \frac{n_e^2 e^2 \sigma}{n_e^2 e^2 + \sigma^2 B^2} \quad (7.279)$	$-e$ electronic charge σ_d direct conductivity σ conductivity ($B=0$)
Hall electrical conductivity	$\sigma_H = \frac{\sigma B}{n_e e} \sigma_d \quad (7.280)$	σ_H Hall conductivity
Generalised Ohm's law	$\mathbf{J} = \sigma_d (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \sigma_H \hat{\mathbf{B}} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (7.281)$	\mathbf{J} current density \mathbf{E} electric field \mathbf{v} plasma velocity field $\hat{\mathbf{B}} = \mathbf{B}/ \mathbf{B} $
Resistive MHD equations (single-fluid model) ^b	$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \quad (7.282)$	μ_0 permeability of free space η magnetic diffusivity [$= 1/(\mu_0 \sigma)$] ν kinematic viscosity \mathbf{g} gravitational field strength
	$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{v} + \frac{1}{3} \nu \nabla (\nabla \cdot \mathbf{v}) + \mathbf{g} \quad (7.283)$	
Shear Alfvénic dispersion relation ^c	$\omega = k v_A \cos \theta_B \quad (7.284)$	ω angular frequency ($= 2\pi \nu$) \mathbf{k} wavevector ($k = 2\pi/\lambda$) θ_B angle between \mathbf{k} and \mathbf{B}
Magnetosonic dispersion relation ^d	$\omega^2 k^2 (v_s^2 + v_A^2) - \omega^4 = v_s^2 v_A^2 k^4 \cos^2 \theta_B \quad (7.285)$	

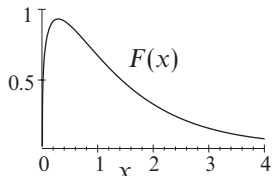
^aFor a warm, fully ionised, electrically neutral p^+/e^- plasma, $\mu_r = 1$. Relativistic and displacement current effects are assumed to be negligible and all oscillations are taken as being well below all resonance frequencies.

^bNeglecting bulk (second) viscosity.

^cNonresistive, inviscid flow.

^dNonresistive, inviscid flow. The greater and lesser solutions for ω^2 are the fast and slow magnetosonic waves respectively.

Synchrotron radiation

Power radiated by a single electron ^a	$P_{\text{tot}} = 2\sigma_{\text{T}}cu_{\text{mag}}\gamma^2\left(\frac{v}{c}\right)^2\sin^2\theta \quad (7.286)$ $\simeq 1.59 \times 10^{-14}B^2\gamma^2\left(\frac{v}{c}\right)^2\sin^2\theta \quad \text{W} \quad (7.287)$	P_{tot} total radiated power σ_{T} Thomson cross section u_{mag} magnetic energy density = $B^2/(2\mu_0)$ v electron velocity ($\sim c$) γ Lorentz factor = $[1 - (v/c)^2]^{-1/2}$ θ pitch angle (angle between v and B) B magnetic flux density c speed of light
... averaged over pitch angles	$P_{\text{tot}} = \frac{4}{3}\sigma_{\text{T}}cu_{\text{mag}}\gamma^2\left(\frac{v}{c}\right)^2 \quad (7.288)$ $\simeq 1.06 \times 10^{-14}B^2\gamma^2\left(\frac{v}{c}\right)^2 \quad \text{W} \quad (7.289)$	
Single electron emission spectrum ^b	$P(\nu) = \frac{3^{1/2}e^3B\sin\theta}{4\pi\epsilon_0cm_e}F(\nu/\nu_{\text{ch}}) \quad (7.290)$ $\simeq 2.34 \times 10^{-25}B\sin\theta F(\nu/\nu_{\text{ch}}) \quad \text{W Hz}^{-1} \quad (7.291)$	$P(\nu)$ emission spectrum ν frequency ν_{ch} characteristic frequency $-e$ electronic charge ϵ_0 free space permittivity m_e electronic (rest) mass
Characteristic frequency	$\nu_{\text{ch}} = \frac{3}{2}\gamma^2\frac{eB}{2\pi m_e}\sin\theta \quad (7.292)$ $\simeq 4.2 \times 10^{10}\gamma^2B\sin\theta \quad \text{Hz} \quad (7.293)$	
Spectral function	$F(x) = x \int_x^\infty K_{5/3}(y)dy \quad (7.294)$ $\simeq \begin{cases} 2.15x^{1/3} & (x \ll 1) \\ 1.25x^{1/2}e^{-x} & (x \gg 1) \end{cases} \quad (7.295)$	F spectral function $K_{5/3}$ modified Bessel fn. of the 2nd kind, order 5/3 

^aThis expression also holds for cyclotron radiation ($v \ll c$).

^bI.e., total radiated power per unit frequency interval.

Bremsstrahlung^aSingle electron and ion^b

$$\frac{dW}{d\omega} = \frac{Z^2 e^6}{24\pi^4 \epsilon_0^3 c^3 m_e^2} \frac{\omega^2}{\gamma^2 v^4} \left[\frac{1}{\gamma^2} K_0^2 \left(\frac{\omega b}{\gamma v} \right) + K_1^2 \left(\frac{\omega b}{\gamma v} \right) \right] \quad (7.296)$$

$$\simeq \frac{Z^2 e^6}{24\pi^4 \epsilon_0^3 c^3 m_e^2 b^2 v^2} \quad (\omega b \ll \gamma v) \quad (7.297)$$

Thermal bremsstrahlung radiation ($v \ll c$; Maxwellian distribution)

$$\frac{dP}{dV dv} = 6.8 \times 10^{-51} Z^2 T^{-1/2} n_i n_e g(v, T) \exp \left(\frac{-hv}{kT} \right) \quad \text{W m}^{-3} \text{Hz}^{-1} \quad (7.298)$$

$$\text{where } g(v, T) \simeq \begin{cases} 0.28 [\ln(4.4 \times 10^{16} T^3 v^{-2} Z^{-2}) - 0.76] & (hv \ll kT \lesssim 10^5 kZ^2) \\ 0.55 \ln(2.1 \times 10^{10} T v^{-1}) & (hv \ll 10^5 kZ^2 \lesssim kT) \\ (2.1 \times 10^{10} T v^{-1})^{-1/2} & (hv \gg kT) \end{cases} \quad (7.299)$$

$$\frac{dP}{dV} \simeq 1.7 \times 10^{-40} Z^2 T^{1/2} n_i n_e \quad \text{W m}^{-3} \quad (7.300)$$

ω angular frequency ($= 2\pi\nu$)
 Ze ionic charge
 $-e$ electronic charge
 ϵ_0 permittivity of free space
 c speed of light
 m_e electronic mass
 b collision parameter^c

v electron velocity
 K_i modified Bessel functions of order i (see page 47)
 γ Lorentz factor
 $= [1 - (v/c)^2]^{-1/2}$
 P power radiated
 V volume
 ν frequency (Hz)

W energy radiated
 T electron temperature (K)
 n_i ion number density (m^{-3})
 n_e electron number density (m^{-3})
 k Boltzmann constant
 h Planck constant
 g Gaunt factor

^aClassical treatment. The ions are at rest, and all frequencies are above the plasma frequency.^bThe spectrum is approximately flat at low frequencies and drops exponentially at frequencies $\gtrsim \gamma v/b$.^cDistance of closest approach.

Chapter 8 Optics

8.1 Introduction

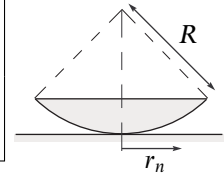
Any attempt to unify the notations and terminology of optics is doomed to failure. This is partly due to the long and illustrious history of the subject (a pedigree shared only with mechanics), which has allowed a variety of approaches to develop, and partly due to the disparate fields of physics to which its basic principles have been applied. Optical ideas find their way into most wave-based branches of physics, from quantum mechanics to radio propagation.

Nowhere is the lack of convention more apparent than in the study of polarisation, and so a cautionary note follows. The conventions used here can be taken largely from context, but the reader should be aware that alternative sign and handedness conventions do exist and are widely used. In particular we will take a circularly polarised wave as being right-handed if, for an observer looking *towards* the source, the electric field vector in a plane perpendicular to the line of sight rotates clockwise. This convention is often used in optics textbooks and has the conceptual advantage that the electric field orientation describes a right-hand corkscrew in space, with the direction of energy flow defining the screw direction. It is however opposite to the system widely used in radio engineering, where the handedness of a helical antenna generating or receiving the wave defines the handedness and is also in the opposite sense to the wave's own angular momentum vector.

8.2 Interference

Newton's rings^a

n th dark ring	$r_n^2 = nR\lambda_0$	(8.1)	r_n radius of n th ring
			n integer (≥ 0)
			R lens radius of curvature
n th bright ring	$r_n^2 = \left(n + \frac{1}{2}\right) R\lambda_0$	(8.2)	λ_0 wavelength in external medium



^aViewed in reflection.

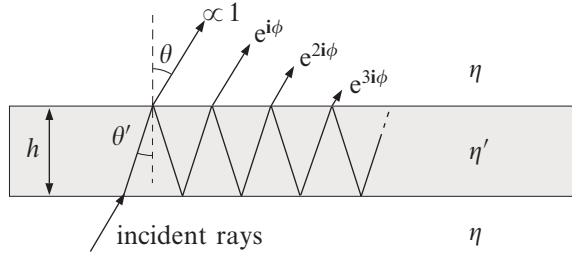
Dielectric layers^a

Quarter-wave condition	$a = \frac{m}{\eta_2} \frac{\lambda_0}{4}$	(8.3)	a film thickness m thickness integer ($m \geq 0$) η_2 film refractive index λ_0 free-space wavelength R power reflectance coefficient
Single-layer reflectance ^b	$R = \begin{cases} \left(\frac{\eta_1 \eta_3 - \eta_2^2}{\eta_1 \eta_3 + \eta_2^2} \right)^2 & (m \text{ odd}) \\ \left(\frac{\eta_1 - \eta_3}{\eta_1 + \eta_3} \right)^2 & (m \text{ even}) \end{cases}$	(8.4)	η_1 entry-side refractive index η_3 exit-side refractive index
Dependence of R on layer thickness, m	$\max \text{ if } (-1)^m (\eta_1 - \eta_2)(\eta_2 - \eta_3) > 0$ (8.5) $\min \text{ if } (-1)^m (\eta_1 - \eta_2)(\eta_2 - \eta_3) < 0$ (8.6) $R = 0 \text{ if } \eta_2 = (\eta_1 \eta_3)^{1/2} \text{ and } m \text{ odd}$ (8.7)		
Multilayer reflectance ^c	$R_N = \left[\frac{\eta_1 - \eta_3 (\eta_a / \eta_b)^{2N}}{\eta_1 + \eta_3 (\eta_a / \eta_b)^{2N}} \right]^2$	(8.8)	R_N multilayer reflectance N number of layer pairs η_a refractive index of top layer η_b refractive index of bottom layer

^aFor normal incidence, assuming the quarter-wave condition. The media are also assumed lossless, with $\mu_r = 1$.

^bSee page 154 for the definition of R .

^cFor a stack of N layer pairs, giving an overall refractive index sequence $\eta_1 \eta_a \eta_b \eta_a \dots \eta_a \eta_b \eta_3$ (see right-hand diagram). Each layer in the stack meets the quarter-wave condition with $m = 1$.

Fabry-Perot etalon^a

<p>Incremental phase difference^b</p> $\phi = 2k_0 h \eta' \cos \theta' \quad (8.9)$ $= 2k_0 h \eta' \left[1 - \left(\frac{\eta \sin \theta}{\eta'} \right)^2 \right]^{1/2} \quad (8.10)$ $= 2\pi n \quad \text{for a maximum} \quad (8.11)$	<p>ϕ incremental phase difference k_0 free-space wavenumber ($= 2\pi/\lambda_0$) h cavity width θ fringe inclination (usually $\ll 1$) θ' internal angle of refraction η' cavity refractive index η external refractive index n fringe order (integer)</p>
<p>Coefficient of finesse</p> $F = \frac{4R}{(1-R)^2} \quad (8.12)$	<p>F coefficient of finesse R interface power reflectance</p>
<p>Finesse</p> $\mathcal{F} = \frac{\pi}{2} F^{1/2} \quad (8.13)$ $= \frac{\lambda_0}{\eta' h} Q \quad (8.14)$	<p>\mathcal{F} finesse λ_0 free-space wavelength Q cavity quality factor</p>
<p>Transmitted intensity</p> $I(\theta) = \frac{I_0(1-R)^2}{1 + R^2 - 2R \cos \phi} \quad (8.15)$ $= \frac{I_0}{1 + F \sin^2(\phi/2)} \quad (8.16)$ $= I_0 A(\theta) \quad (8.17)$	<p>I transmitted intensity I_0 incident intensity A Airy function</p>
<p>Fringe intensity profile</p> $\Delta\phi = 2 \arcsin(F^{-1/2}) \quad (8.18)$ $\simeq 2F^{-1/2} \quad (8.19)$	<p>$\Delta\phi$ phase difference at half intensity point</p>
<p>Chromatic resolving power</p> $\frac{\lambda_0}{\delta\lambda} \simeq \frac{R^{1/2} \pi n}{1-R} = n\mathcal{F} \quad (8.20)$ $\simeq \frac{2\mathcal{F} h \eta'}{\lambda_0} \quad (\theta \ll 1) \quad (8.21)$	<p>$\delta\lambda$ minimum resolvable wavelength difference</p>
<p>Free spectral range^c</p> $\delta\lambda_f = \mathcal{F} \delta\lambda \quad (8.22)$ $\delta\nu_f = \frac{c}{2\eta' h} \quad (8.23)$	<p>$\delta\lambda_f$ wavelength free spectral range $\delta\nu_f$ frequency free spectral range</p>

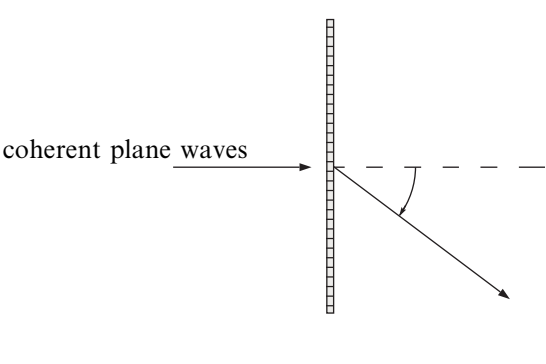
^aNeglecting any effects due to surface coatings on the etalon. See also *Lasers* on page 174.

^bBetween adjacent rays. Highest order fringes are near the centre of the pattern.

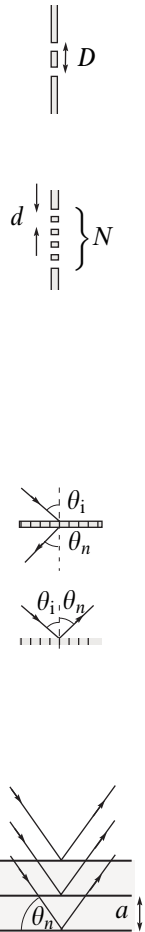
^cAt near-normal incidence ($\theta \simeq 0$), the orders of two spectral components separated by $< \delta\lambda_f$ will not overlap.

8.3 Fraunhofer diffraction

Gratings^a

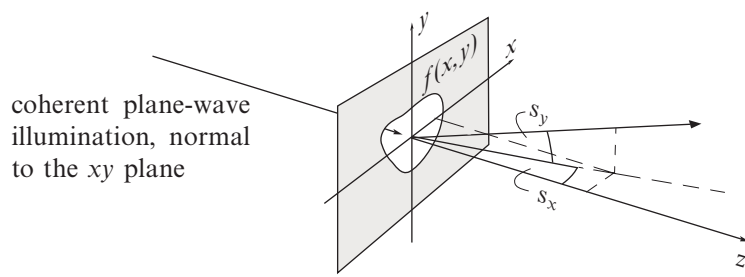
		
Young's double slits ^b	$I(s) = I_0 \cos^2 \frac{kDs}{2}$	(8.24)
N equally spaced narrow slits	$I(s) = I_0 \left[\frac{\sin(Nkds/2)}{N \sin(kds/2)} \right]^2$	(8.25)
Infinite grating	$I(s) = I_0 \sum_{n=-\infty}^{\infty} \delta \left(s - \frac{n\lambda}{d} \right)$	(8.26)
Normal incidence	$\sin \theta_n = \frac{n\lambda}{d}$	(8.27)
Oblique incidence	$\sin \theta_n + \sin \theta_i = \frac{n\lambda}{d}$	(8.28)
Reflection grating	$\sin \theta_n - \sin \theta_i = \frac{n\lambda}{d}$	(8.29)
Chromatic resolving power	$\frac{\lambda}{\delta \lambda} = Nn$	(8.30)
Grating dispersion	$\frac{\partial \theta}{\partial \lambda} = \frac{n}{d \cos \theta}$	(8.31)
Bragg's law ^c	$2a \sin \theta_n = n\lambda$	(8.32)

$I(s)$ diffracted intensity
 I_0 peak intensity
 θ diffraction angle
 $s = \sin \theta$
 D slit separation
 λ wavelength
 N number of slits
 k wavenumber ($= 2\pi/\lambda$)
 d slit spacing
 n diffraction order
 δ Dirac delta function
 θ_n angle of diffracted maximum
 θ_i angle of incident illumination
 $\delta \lambda$ diffraction peak width
 a atomic plane spacing



^aUnless stated otherwise, the illumination is normal to the grating.
^bTwo narrow slits separated by D .
^cThe condition is for Bragg reflection, with $\theta_n = \theta_i$.

Aperture diffraction

 <p>coherent plane-wave illumination, normal to the xy plane</p>		
General 1-D aperture ^a	$\psi(s) \propto \int_{-\infty}^{\infty} f(x) e^{-iksx} dx \quad (8.33)$ $I(s) \propto \psi \psi^*(s) \quad (8.34)$	ψ diffracted wavefunction I diffracted intensity θ diffraction angle $s = \sin \theta$
General 2-D aperture in (x, y) plane (small angles)	$\psi(s_x, s_y) \propto \iint_{\infty} f(x, y) e^{-ik(s_x x + s_y y)} dx dy \quad (8.35)$	f aperture amplitude transmission function x, y distance across aperture k wavenumber ($= 2\pi/\lambda$) s_x deflection \parallel xz plane s_y deflection \perp xz plane
Broad 1-D slit ^b	$I(s) = I_0 \frac{\sin^2(kas/2)}{(kas/2)^2} \quad (8.36)$ $\equiv I_0 \text{sinc}^2(as/\lambda) \quad (8.37)$	I_0 peak intensity a slit width (in x) λ wavelength
Sidelobe intensity	$\frac{I_n}{I_0} = \left(\frac{2}{\pi}\right)^2 \frac{1}{(2n+1)^2} \quad (n > 0) \quad (8.38)$	I_n n th sidelobe intensity
Rectangular aperture (small angles)	$I(s_x, s_y) = I_0 \text{sinc}^2 \frac{as_x}{\lambda} \text{sinc}^2 \frac{bs_y}{\lambda} \quad (8.39)$	a aperture width in x b aperture width in y
Circular aperture ^c	$I(s) = I_0 \left[\frac{2J_1(kDs/2)}{kDs/2} \right]^2 \quad (8.40)$	J_1 first-order Bessel function D aperture diameter
First minimum ^d	$s = 1.22 \frac{\lambda}{D} \quad (8.41)$	λ wavelength
First subside. maximum	$s = 1.64 \frac{\lambda}{D} \quad (8.42)$	
Weak 1-D phase object	$f(x) = \exp[i\phi(x)] \simeq 1 + i\phi(x) \quad (8.43)$	$\phi(x)$ phase distribution i $i^2 = -1$
Fraunhofer limit ^e	$L \gg \frac{(\Delta x)^2}{\lambda} \quad (8.44)$	L distance of aperture from observation point Δx aperture size

^aThe Fraunhofer integral.

^bNote that $\text{sinc } x = (\sin \pi x)/(\pi x)$.

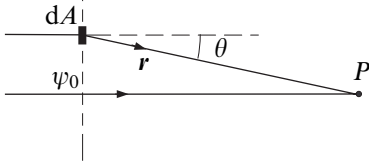
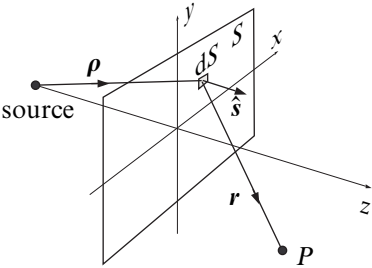
^cThe central maximum is known as the “Airy disk.”

^dThe “Rayleigh resolution criterion” states that two point sources of equal intensity can just be resolved with diffraction-limited optics if separated in angle by $1.22\lambda/D$.

^ePlane-wave illumination.

8.4 Fresnel diffraction

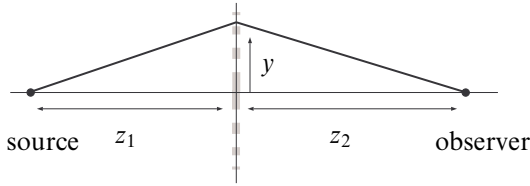
Kirchhoff's diffraction formula^a

 <p>(source at infinity)</p>		
Source at infinity where: Obliquity factor (cardioid)	$\psi_P = -\frac{i}{\lambda} \psi_0 \int_{\text{plane}} K(\theta) \frac{e^{ikr}}{r} dA \quad (8.45)$ $K(\theta) = \frac{1}{2}(1 + \cos \theta) \quad (8.46)$	ψ_P complex amplitude at P λ wavelength k wavenumber ($= 2\pi/\lambda$) ψ_0 incident amplitude θ obliquity angle r distance of dA from P ($\gg \lambda$) dA area element on incident wavefront K obliquity factor dS element of closed surface \hat{s} unit vector s vector normal to dS r vector from P to dS ρ vector from source to dS E_0 amplitude (see footnote)
Source at finite distance ^b	$\psi_P = -\frac{iE_0}{\lambda} \oint_{\text{closed surface}} \frac{e^{ik(\rho+r)}}{2\rho r} [\cos(\hat{s} \cdot \hat{r}) - \cos(\hat{s} \cdot \hat{\rho})] dS \quad (8.47)$	

^aAlso known as the “Fresnel–Kirchhoff formula.” Diffraction by an obstacle coincident with the integration surface can be approximated by omitting that part of the surface from the integral.

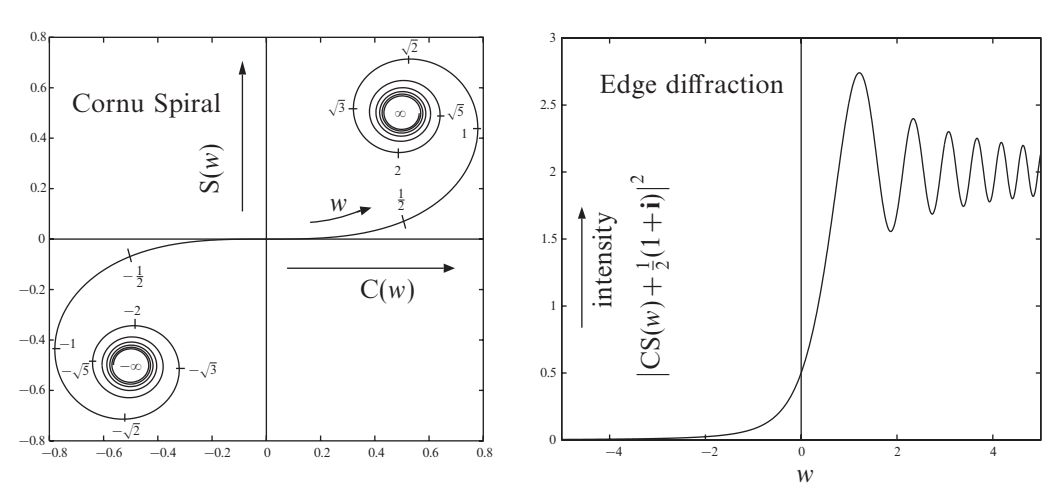
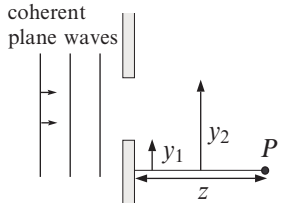
^bThe source amplitude at ρ is $\psi(\rho) = E_0 e^{ik\rho}/\rho$. The integral is taken over a surface enclosing the point P .

Fresnel zones

		
Effective aperture distance ^a	$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2} \quad (8.48)$	z effective distance z_1 source–aperture distance z_2 aperture–observer distance
Half-period zone radius	$y_n = (n\lambda z)^{1/2} \quad (8.49)$	n half-period zone number λ wavelength y_n n th half-period zone radius
Axial zeros (circular aperture)	$z_m = \frac{R^2}{2m\lambda} \quad (8.50)$	z_m distance of m th zero from aperture R aperture radius

^aI.e., the aperture–observer distance to be employed when the source is not at infinity.

Cornu spiral

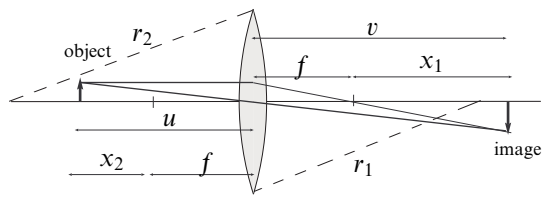
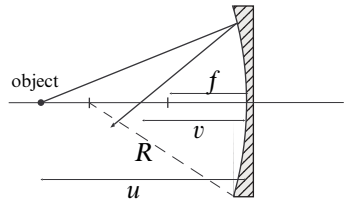
		
Fresnel integrals ^a	$C(w) = \int_0^w \cos \frac{\pi t^2}{2} dt \quad (8.51)$ $S(w) = \int_0^w \sin \frac{\pi t^2}{2} dt \quad (8.52)$	C Fresnel cosine integral S Fresnel sine integral
Cornu spiral	$CS(w) = C(w) + iS(w) \quad (8.53)$ $CS(\pm\infty) = \pm \frac{1}{2}(1 + i) \quad (8.54)$	CS Cornu spiral v, w length along spiral
Edge diffraction	$\psi_P = \frac{\psi_0}{2^{1/2}} [CS(w) + \frac{1}{2}(1 + i)] \quad (8.55)$ $\text{where } w = y \left(\frac{2}{\lambda z} \right)^{1/2} \quad (8.56)$	ψ_P complex amplitude at P ψ_0 unobstructed amplitude λ wavelength z distance of P from aperture plane [see (8.48)] y position of edge
Diffraction from a long slit ^b	$\psi_P = \frac{\psi_0}{2^{1/2}} [CS(w_2) - CS(w_1)] \quad (8.57)$ $\text{where } w_i = y_i \left(\frac{2}{\lambda z} \right)^{1/2} \quad (8.58)$	
Diffraction from a rectangular aperture	$\psi_P = \frac{\psi_0}{2} [CS(v_2) - CS(v_1)] \times [CS(w_2) - CS(w_1)] \quad (8.59)$ $\text{where } v_i = x_i \left(\frac{2}{\lambda z} \right)^{1/2} \quad (8.60)$ $\text{and } w_i = y_i \left(\frac{2}{\lambda z} \right)^{1/2} \quad (8.61)$ $\text{and } w_i = y_i \left(\frac{2}{\lambda z} \right)^{1/2} \quad (8.62)$	

^aSee also Equation (2.393) on page 45.

^bSlit long in x .

8.5 Geometrical optics

Lenses and mirrors^a

<div><div></div><div></div></div>																							
<div><div>lens</div><div>mirror</div><table><thead><tr><th colspan="3">sign convention</th></tr><tr><th></th><th>+</th><th>-</th></tr></thead><tbody><tr><td>r</td><td>centred to right</td><td>centred to left</td></tr><tr><td>u</td><td>real object</td><td>virtual object</td></tr><tr><td>v</td><td>real image</td><td>virtual image</td></tr><tr><td>f</td><td>converging lens/ concave mirror</td><td>diverging lens/ convex mirror</td></tr><tr><td>M_T</td><td>erect image</td><td>inverted image</td></tr></tbody></table></div>			sign convention				+	-	r	centred to right	centred to left	u	real object	virtual object	v	real image	virtual image	f	converging lens/ concave mirror	diverging lens/ convex mirror	M_T	erect image	inverted image
sign convention																							
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M_T	erect image	inverted image																					
Fermat's principle ^b	$L = \int \eta \, dl$ is stationary	(8.63)																					
Gauss's lens formula	$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$	(8.64)																					
Newton's lens formula	$x_1 x_2 = f^2$	(8.65)																					
Lensmaker's formula	$\frac{1}{u} + \frac{1}{v} = (\eta - 1) \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$	(8.66)																					
Mirror formula ^c	$\frac{1}{u} + \frac{1}{v} = -\frac{2}{R} = \frac{1}{f}$	(8.67)																					
Dioptre number	$D = \frac{1}{f}$ m ⁻¹	(8.68)																					
Focal ratio ^d	$n = \frac{f}{d}$	(8.69)																					
Transverse linear magnification	$M_T = -\frac{v}{u}$	(8.70)																					
Longitudinal linear magnification	$M_L = -M_T^2$	(8.71)																					

L optical path length

η refractive index

dl ray path element

u object distance

v image distance

f focal length

$x_1 = v - f$

$x_2 = u - f$

r_i radii of curvature of lens surfaces

R mirror radius of curvature

D dioptre number (f in metres)

n focal ratio

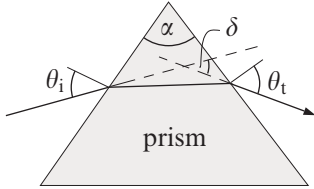
d lens or mirror diameter

M_T transverse magnification

M_L longitudinal magnification

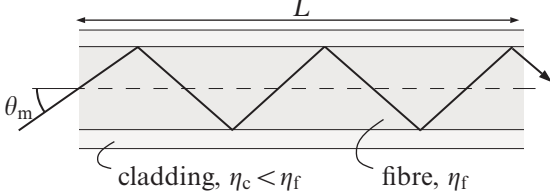
^aFormulas assume “Gaussian optics,” i.e., all lenses are thin and all angles small. Light enters from the left.
^bA stationary optical path length has, to first order, a length identical to that of adjacent paths.
^cThe mirror is concave if $R < 0$, convex if $R > 0$.
^dOr “ f -number,” written $f/2$ if $n = 2$ etc.

Prisms (dispersing)

		
Transmission angle	$\sin \theta_t = (\eta^2 - \sin^2 \theta_i)^{1/2} \sin \alpha - \sin \theta_i \cos \alpha \quad (8.72)$	θ_i angle of incidence θ_t angle of transmission α apex angle η refractive index
Deviation	$\delta = \theta_i + \theta_t - \alpha \quad (8.73)$	δ angle of deviation
Minimum deviation condition	$\sin \theta_i = \sin \theta_t = \eta \sin \frac{\alpha}{2} \quad (8.74)$	
Refractive index	$\eta = \frac{\sin[(\delta_m + \alpha)/2]}{\sin(\alpha/2)} \quad (8.75)$	δ_m minimum deviation
Angular dispersion ^a	$D = \frac{d\delta}{d\lambda} = \frac{2 \sin(\alpha/2)}{\cos[(\delta_m + \alpha)/2]} \frac{d\eta}{d\lambda} \quad (8.76)$	D dispersion λ wavelength

^aAt minimum deviation.

Optical fibres

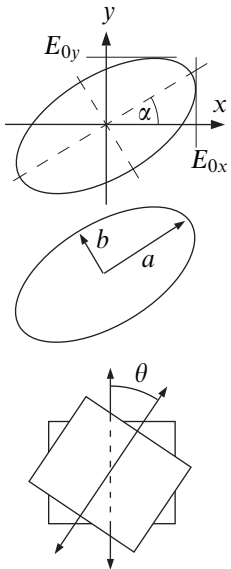
		
Acceptance angle	$\sin \theta_m = \frac{1}{\eta_0} (\eta_f^2 - \eta_c^2)^{1/2} \quad (8.77)$	θ_m maximum angle of incidence η_0 exterior refractive index η_f fibre refractive index η_c cladding refractive index
Numerical aperture	$N = \eta_0 \sin \theta_m \quad (8.78)$	N numerical aperture
Multimode dispersion ^a	$\frac{\Delta t}{L} = \frac{\eta_f}{c} \left(\frac{\eta_f}{\eta_c} - 1 \right) \quad (8.79)$	Δt temporal dispersion L fibre length c speed of light

^aOf a pulse with a given wavelength, caused by the range of incident angles up to θ_m . Sometimes called “intermodal dispersion” or “modal dispersion.”

8.6 Polarisation

Elliptical polarisation^a

Elliptical polarisation	$E = (E_{0x}, E_{0y}e^{i\delta})e^{i(kz - \omega t)}$ (8.80)	E electric field k wavevector z propagation axis ωt angular frequency \times time
Polarisation angle ^b	$\tan 2\alpha = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2} \cos \delta$ (8.81)	E_{0x} x amplitude of E E_{0y} y amplitude of E δ relative phase of E_y with respect to E_x α polarisation angle
Ellipticity ^c	$e = \frac{a-b}{a}$ (8.82)	e ellipticity a semi-major axis b semi-minor axis
Malus's law ^d	$I(\theta) = I_0 \cos^2 \theta$ (8.83)	$I(\theta)$ transmitted intensity I_0 incident intensity θ polariser–analyser angle



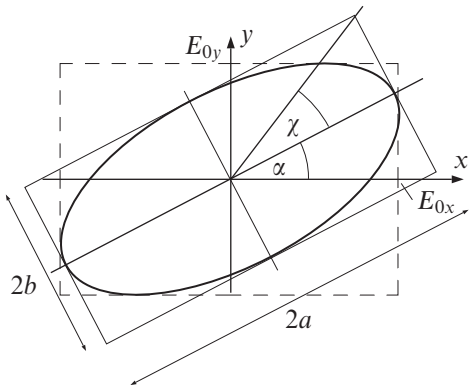
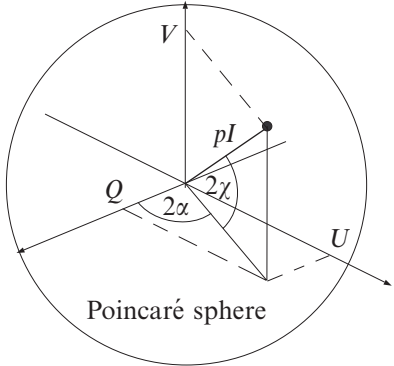
^aSee the introduction (page 161) for a discussion of sign and handedness conventions.
^bAngle between ellipse major axis and x axis. Sometimes the polarisation angle is defined as $\pi/2 - \alpha$.
^cThis is one of several definitions for ellipticity.
^dTransmission through skewed polarisers for unpolarised incident light.

Jones vectors and matrices

Normalised electric field ^a	$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}; \quad \mathbf{E} = 1$	(8.84)	\mathbf{E} electric field E_x x component of \mathbf{E} E_y y component of \mathbf{E}
Example vectors:	$E_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $E_r = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix}$	$E_{45} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $E_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix}$	E_{45} 45° to x axis E_r right-hand circular E_l left-hand circular
Jones matrix	$\mathbf{E}_t = \mathbf{A}\mathbf{E}_i$	(8.85)	\mathbf{E}_t transmitted vector \mathbf{E}_i incident vector \mathbf{A} Jones matrix
Example matrices:			
Linear polariser $\parallel x$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	Linear polariser $\parallel y$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Linear polariser at 45°	$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	Linear polariser at -45°	$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
Right circular polariser	$\frac{1}{2} \begin{pmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 1 \end{pmatrix}$	Left circular polariser	$\frac{1}{2} \begin{pmatrix} 1 & -\mathbf{i} \\ \mathbf{i} & 1 \end{pmatrix}$
$\lambda/4$ plate (fast $\parallel x$)	$\mathbf{e}^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{pmatrix}$	$\lambda/4$ plate (fast $\perp x$)	$\mathbf{e}^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{i} \end{pmatrix}$

^aKnown as the “normalised Jones vector.”

Stokes parameters^a

							
Electric fields	$E_x = E_{0x}e^{i(kz-\omega t)}$	(8.86)	k	wavevector			
	$E_y = E_{0y}e^{i(kz-\omega t+\delta)}$	(8.87)	ωt	angular frequency \times time			
Axial ratio ^b	$\tan\chi = \pm r = \pm \frac{b}{a}$	(8.88)	δ	relative phase of E_y with respect to E_x			
			χ	(see diagram)			
Stokes parameters			r	axial ratio			
	$I = \langle E_x^2 \rangle + \langle E_y^2 \rangle$	(8.89)	E_x	electric field component \parallel x			
	$Q = \langle E_x^2 \rangle - \langle E_y^2 \rangle$	(8.90)	E_y	electric field component \parallel y			
	$= pI \cos 2\chi \cos 2\alpha$	(8.91)	E_{0x}	field amplitude in x direction			
	$U = 2\langle E_x E_y \rangle \cos \delta$	(8.92)	E_{0y}	field amplitude in y direction			
	$= pI \cos 2\chi \sin 2\alpha$	(8.93)	α	polarisation angle			
	$V = 2\langle E_x E_y \rangle \sin \delta$	(8.94)	p	degree of polarisation			
	$= pI \sin 2\chi$	(8.95)	$\langle \cdot \rangle$	mean over time			
Degree of polarisation	$p = \frac{(Q^2 + U^2 + V^2)^{1/2}}{I} \leq 1$	(8.96)					
	Q/I	U/I	V/I	Q/I	U/I	V/I	
left circular	0	0	-1	right circular	0	0	1
linear \parallel x	1	0	0	linear \parallel y	-1	0	0
linear 45° to x	0	1	0	linear -45° to x	0	-1	0
unpolarised	0	0	0				

^aUsing the convention that right-handed circular polarisation corresponds to a clockwise rotation of the electric field in a given plane when looking towards the source. The propagation direction in the diagram is out of the plane. The parameters I , Q , U , and V are sometimes denoted s_0 , s_1 , s_2 , and s_3 , and other nomenclatures exist. There is no generally accepted definition – often the parameters are scaled to be dimensionless, with $s_0 = 1$, or to represent power flux through a plane \perp the beam, i.e., $I = (\langle E_x^2 \rangle + \langle E_y^2 \rangle)/Z_0$ etc., where Z_0 is the impedance of free space.

^bThe axial ratio is positive for right-handed polarisation and negative for left-handed polarisation using our definitions.

8.7 Coherence (scalar theory)

Mutual coherence function	$\Gamma_{12}(\tau) = \langle \psi_1(t) \psi_2^*(t + \tau) \rangle$	(8.97)	Γ_{ij} mutual coherence function τ temporal interval ψ_i (complex) wave disturbance at spatial point i
Complex degree of coherence	$\gamma_{12}(\tau) = \frac{\langle \psi_1(t) \psi_2^*(t + \tau) \rangle}{[\langle \psi_1 ^2 \rangle \langle \psi_2 ^2 \rangle]^{1/2}}$	(8.98)	t time $\langle \cdot \rangle$ mean over time
	$= \frac{\Gamma_{12}(\tau)}{[\Gamma_{11}(0) \Gamma_{22}(0)]^{1/2}}$	(8.99)	γ_{ij} complex degree of coherence $*$ complex conjugate
Combined intensity ^a	$I_{\text{tot}} = I_1 + I_2 + 2(I_1 I_2)^{1/2} \Re[\gamma_{12}(\tau)]$	(8.100)	I_{tot} combined intensity I_i intensity of disturbance at point i \Re real part of
Fringe visibility	$V(\tau) = \frac{2(I_1 I_2)^{1/2}}{I_1 + I_2} \gamma_{12}(\tau) $	(8.101)	
if $ \gamma_{12}(\tau) $ is a constant:	$V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}$	(8.102)	I_{max} max. combined intensity I_{min} min. combined intensity
if $I_1 = I_2$:	$V(\tau) = \gamma_{12}(\tau) $	(8.103)	
Complex degree of temporal coherence ^b	$\gamma(\tau) = \frac{\langle \psi_1(t) \psi_1^*(t + \tau) \rangle}{\langle \psi_1(t) ^2 \rangle}$	(8.104)	$\gamma(\tau)$ degree of temporal coherence
	$= \frac{\int I(\omega) e^{-i\omega\tau} d\omega}{\int I(\omega) d\omega}$	(8.105)	$I(\omega)$ specific intensity ω radiation angular frequency c speed of light
Coherence time and length	$\Delta\tau_c = \frac{\Delta l_c}{c} \sim \frac{1}{\Delta\nu}$	(8.106)	$\Delta\tau_c$ coherence time Δl_c coherence length $\Delta\nu$ spectral bandwidth
Complex degree of spatial coherence ^c	$\gamma(\mathbf{D}) = \frac{\langle \psi_1 \psi_2^* \rangle}{[\langle \psi_1 ^2 \rangle \langle \psi_2 ^2 \rangle]^{1/2}}$	(8.107)	$\gamma(\mathbf{D})$ degree of spatial coherence \mathbf{D} spatial separation of points 1 and 2
	$= \frac{\int I(\hat{s}) e^{ik\mathbf{D} \cdot \hat{s}} d\Omega}{\int I(\hat{s}) d\Omega}$	(8.108)	$I(\hat{s})$ specific intensity of distant extended source in direction \hat{s} $d\Omega$ differential solid angle \hat{s} unit vector in the direction of $d\Omega$ k wavenumber
Intensity correlation ^d	$\frac{\langle I_1 I_2 \rangle}{[\langle I_1 \rangle^2 \langle I_2 \rangle^2]^{1/2}} = 1 + \gamma^2(\mathbf{D})$	(8.109)	
Speckle intensity distribution ^e	$\text{pr}(I) = \frac{1}{\langle I \rangle} e^{-I/\langle I \rangle}$	(8.110)	pr probability density
Speckle size (coherence width)	$\Delta w_c \simeq \frac{\lambda}{\alpha}$	(8.111)	Δw_c characteristic speckle size λ wavelength α source angular size as seen from the screen

^aFrom interfering the disturbances at points 1 and 2 with a relative delay τ .

^bOr “autocorrelation function.”

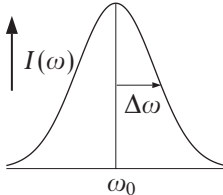
^cBetween two points on a wavefront, separated by \mathbf{D} . The integral is over the entire extended source.

^dFor wave disturbances that have a Gaussian probability distribution in amplitude. This is “Gaussian light” such as from a thermal source.

^eAlso for Gaussian light.

8.8 Line radiation

Spectral line broadening

Natural broadening ^a	$I(\omega) = \frac{(2\pi\tau)^{-1}}{(2\tau)^{-2} + (\omega - \omega_0)^2} \quad (8.112)$	$I(\omega)$ normalised intensity ^b τ lifetime of excited state ω angular frequency ($= 2\pi\nu$)
Natural half-width	$\Delta\omega = \frac{1}{2\tau} \quad (8.113)$	$\Delta\omega$ half-width at half-power ω_0 centre frequency
Collision broadening	$I(\omega) = \frac{(\pi\tau_c)^{-1}}{(\tau_c)^{-2} + (\omega - \omega_0)^2} \quad (8.114)$	τ_c mean time between collisions p pressure d effective atomic diameter m gas particle mass k Boltzmann constant T temperature c speed of light
Collision and pressure half-width ^c	$\Delta\omega = \frac{1}{\tau_c} = p\pi d^2 \left(\frac{\pi mkT}{16} \right)^{-1/2} \quad (8.115)$	
Doppler broadening	$I(\omega) = \left(\frac{mc^2}{2kT\omega_0^2\pi} \right)^{1/2} \exp \left[-\frac{mc^2}{2kT} \frac{(\omega - \omega_0)^2}{\omega_0^2} \right] \quad (8.116)$	
Doppler half-width	$\Delta\omega = \omega_0 \left(\frac{2kT \ln 2}{mc^2} \right)^{1/2} \quad (8.117)$	

^aThe transition probability per unit time for the state is $= 1/\tau$. In the classical limit of a damped oscillator, the e-folding time of the electric field is 2τ . Both the natural and collision profiles described here are Lorentzian.

^bThe intensity spectra are normalised so that $\int I(\omega) d\omega = 1$, assuming $\Delta\omega/\omega_0 \ll 1$.

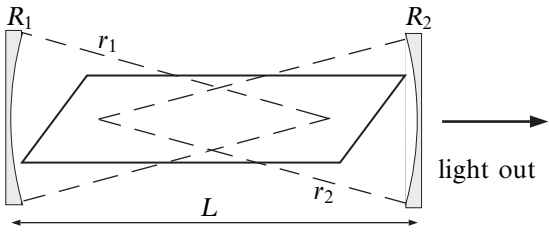
^cThe pressure-broadening relation combines Equations (5.78), (5.86) and (5.89) and assumes an otherwise perfect gas of finite-sized atoms. More accurate expressions are considerably more complicated.

Einstein coefficients^a

Absorption	$R_{12} = B_{12}I_\nu n_1 \quad (8.118)$	R_{ij} transition rate, level $i \rightarrow j$ ($\text{m}^{-3} \text{s}^{-1}$) B_{ij} Einstein B coefficients I_ν specific intensity of radiation field
Spontaneous emission	$R_{21} = A_{21}n_2 \quad (8.119)$	A_{21} Einstein A coefficient n_i number density of atoms in quantum level i (m^{-3})
Stimulated emission	$R'_{21} = B_{21}I_\nu n_2 \quad (8.120)$	
Coefficient ratios	$\frac{A_{21}}{B_{12}} = \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} \quad (8.121)$	h Planck constant ν frequency c speed of light
	$\frac{B_{21}}{B_{12}} = \frac{g_1}{g_2} \quad (8.122)$	g_i degeneracy of i th level

^aNote that the coefficients can also be defined in terms of spectral energy density, $u_\nu = 4\pi I_\nu/c$ rather than I_ν . In this case $\frac{A_{21}}{B_{12}} = \frac{8\pi h\nu^3}{c^3} \frac{g_1}{g_2}$. See also *Population densities* on page 116.

Lasers^a

		
Cavity stability condition	$0 \leq \left(1 - \frac{L}{r_1}\right) \left(1 - \frac{L}{r_2}\right) \leq 1 \quad (8.123)$	$r_{1,2}$ radii of curvature of end-mirrors L distance between mirror centres
Longitudinal cavity modes ^b	$v_n = \frac{c}{2L} n \quad (8.124)$	v_n mode frequency n integer c speed of light
Cavity Q	$Q = \frac{2\pi L (R_1 R_2)^{1/4}}{\lambda [1 - (R_1 R_2)^{1/2}]} \quad (8.125)$	Q quality factor $R_{1,2}$ mirror (power) reflectances λ wavelength
	$\simeq \frac{4\pi L}{\lambda (1 - R_1 R_2)} \quad (8.126)$	
Cavity line width	$\Delta v_c = \frac{v_n}{Q} = 1/(2\pi\tau_c) \quad (8.127)$	Δv_c cavity line width (FWHP) τ_c cavity photon lifetime
Schawlow–Townes line width	$\frac{\Delta v}{v_n} = \frac{2\pi h (\Delta v_c)^2}{P} \left(\frac{g_l N_u}{g_l N_u - g_u N_l} \right) \quad (8.128)$	Δv line width (FWHP) P laser power $g_{u,l}$ degeneracy of upper/lower levels $N_{u,l}$ number density of upper/lower levels
Threshold lasing condition	$R_1 R_2 \exp[2(\alpha - \beta)L] > 1 \quad (8.129)$	α gain per unit length of medium β loss per unit length of medium

^aAlso see the *Fabry-Perot etalon* on page 163. Note that “cavity” refers to the empty cavity, with no lasing medium present.

^bThe mode spacing equals the cavity free spectral range.

Chapter 9 Astrophysics

9.1 Introduction

Many of the formulas associated with astronomy and astrophysics are either too specialised for a general work such as this or are common to other fields and can therefore be found elsewhere in this book. The following section includes many of the relationships that fall into neither of these categories, including equations to convert between various astronomical coordinate systems and some basic formulas associated with cosmology.

Exceptionally, this section also includes data on the Sun, Earth, Moon, and planets. Observational astrophysics remains a largely inexact science, and parameters of these (and other) bodies are often used as approximate base units in measurements. For example, the masses of stars and galaxies are frequently quoted as multiples of the mass of the Sun ($1M_{\odot} = 1.989 \times 10^{30} \text{ kg}$), extra-solar system planets in terms of the mass of Jupiter, and so on. Astronomers seem to find it particularly difficult to drop arcane units and conventions, resulting in a profusion of measures and nomenclatures throughout the subject. However, the convention of using suitable astronomical objects in this way is both useful and widely accepted.

9.2 Solar system data

Solar data

equatorial radius	R_{\odot}	=	$6.960 \times 10^8 \text{ m}$	=	$109.1 R_{\oplus}$
mass	M_{\odot}	=	$1.9891 \times 10^{30} \text{ kg}$	=	$3.32946 \times 10^5 M_{\oplus}$
polar moment of inertia	I_{\odot}	=	$5.7 \times 10^{46} \text{ kg m}^2$	=	$7.09 \times 10^8 I_{\oplus}$
bolometric luminosity	L_{\odot}	=	$3.826 \times 10^{26} \text{ W}$		
effective surface temperature	T_{\odot}	=	5770 K		
solar constant ^a			$1.368 \times 10^3 \text{ W m}^{-2}$		
absolute magnitude	M_V	=	+4.83;	M_{bol}	= +4.75
apparent magnitude	m_V	=	-26.74;	m_{bol}	= -26.82

^aBolometric flux at a distance of 1 astronomical unit (AU).

Earth data

equatorial radius	R_{\oplus}	=	$6.37814 \times 10^6 \text{ m}$	=	$9.166 \times 10^{-3} R_{\odot}$
flattening ^a	f	=	0.00335364	=	1/298.183
mass	M_{\oplus}	=	$5.9742 \times 10^{24} \text{ kg}$	=	$3.0035 \times 10^{-6} M_{\odot}$
polar moment of inertia	I_{\oplus}	=	$8.037 \times 10^{37} \text{ kg m}^2$	=	$1.41 \times 10^{-9} I_{\odot}$
orbital semi-major axis ^b	1 AU	=	$1.495979 \times 10^{11} \text{ m}$	=	$214.9 R_{\odot}$
mean orbital velocity			$2.979 \times 10^4 \text{ m s}^{-1}$		
equatorial surface gravity	g_e	=	$9.780327 \text{ m s}^{-2}$	(includes rotation)	
polar surface gravity	g_p	=	$9.832186 \text{ m s}^{-2}$		
rotational angular velocity	ω_e	=	$7.292115 \times 10^{-5} \text{ rad s}^{-1}$		

^a f equals $(R_{\oplus} - R_{\text{polar}})/R_{\oplus}$. The mean radius of the Earth is $6.3710 \times 10^6 \text{ m}$.

^bAbout the Sun.

Moon data

equatorial radius	R_m	=	$1.7374 \times 10^6 \text{ m}$	=	$0.27240 R_{\oplus}$
mass	M_m	=	$7.3483 \times 10^{22} \text{ kg}$	=	$1.230 \times 10^{-2} M_{\oplus}$
mean orbital radius ^a	a_m	=	$3.84400 \times 10^8 \text{ m}$	=	$60.27 R_{\oplus}$
mean orbital velocity			$1.03 \times 10^3 \text{ m s}^{-1}$		
orbital period (sidereal)			27.32166 d		
equatorial surface gravity			1.62 m s^{-2}	=	$0.166 g_e$

^aAbout the Earth.

Planetary data^a

	M/M_{\oplus}	R/R_{\oplus}	$T(\text{d})$	$P(\text{yr})$	$a(\text{AU})$	M	mass
Mercury	0.055 274	0.382 51	58.646	0.240 85	0.387 10	R	equatorial radius
Venus ^b	0.815 00	0.948 83	243.018	0.615 228	0.723 35	T	rotational period
Earth	1	1	0.997 27	1.000 04	1.000 00	P	orbital period
Mars	0.107 45	0.532 60	1.025 96	1.880 93	1.523 71	a	mean distance
Jupiter	317.85	11.209	0.413 54	11.861 3	5.202 53	M_{\oplus}	$5.9742 \times 10^{24} \text{ kg}$
Saturn	95.159	9.449 1	0.444 01	29.628 2	9.575 60	R_{\oplus}	$6.37814 \times 10^6 \text{ m}$
Uranus ^b	14.500	4.007 3	0.718 33	84.746 6	19.293 4	1 d	86400 s
Neptune	17.204	3.882 6	0.671 25	166.344	30.245 9	1 yr	$3.15569 \times 10^7 \text{ s}$
Pluto ^b	0.002 51	0.187 36	6.387 2	248.348	39.509 0	1 AU	$1.495979 \times 10^{11} \text{ m}$

^aUsing the osculating orbital elements for 1998. Note that P is the instantaneous orbital period, calculated from the planet's daily motion. The radii of gas giants are taken at 1 atmosphere pressure.

^bRetrograde rotation.

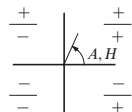
9.3 Coordinate transformations (astronomical)

Time in astronomy

Julian day number ^a $JD = D - 32075 + 1461 * (Y + 4800 + (M - 14)/12)/4$ $+ 367 * (M - 2 - (M - 14)/12 * 12)/12$ $- 3 * ((Y + 4900 + (M - 14)/12)/100)/4$ (9.1)	<i>JD</i> Julian day number <i>D</i> day of month number <i>Y</i> calendar year, e.g., 1963 <i>M</i> calendar month (Jan=1) <i>*</i> integer multiply <i>/</i> integer divide <i>MJD</i> modified Julian day number
Modified Julian day number $MJD = JD - 2400000.5$ (9.2)	
Day of week $W = (JD + 1) \mod 7$ (9.3)	<i>W</i> day of week (0=Sunday, 1=Monday, ...)
Local civil time $LCT = UTC + TZC + DSC$ (9.4)	<i>LCT</i> local civil time <i>UTC</i> coordinated universal time <i>TZC</i> time zone correction <i>DSC</i> daylight saving correction
Julian centuries $T = \frac{JD - 2451545.5}{36525}$ (9.5)	<i>T</i> Julian centuries between 12 ^h UTC 1 Jan 2000 and 0 ^h UTC <i>D/M/Y</i>
Greenwich sidereal time $GMST = 6^h 41^m 50^s.54841$ $+ 8640184^s.812866T$ $+ 0^s.093104T^2$ $- 0^s.0000062T^3$ (9.6)	<i>GMST</i> Greenwich mean sidereal time at 0 ^h UTC <i>D/M/Y</i> (for later times use 1s=1.002738 sidereal seconds)
Local sidereal time $LST = GMST + \frac{\lambda^\circ}{15^\circ}$ (9.7)	<i>LST</i> local sidereal time λ° geographic longitude, degrees east of Greenwich


^aFor the Julian day starting at noon on the calendar day in question. The routine is designed around integer arithmetic with “truncation towards zero” (so that $-5/3 = -1$) and is valid for dates from the onset of the Gregorian calendar, 15 October 1582. *JD* represents the number of days since Greenwich mean noon 1 Jan 4713 BC. For reference, noon, 1 Jan 2000 = *JD*2451545 and was a Saturday (*W* = 6).

Horizon coordinates^a

Hour angle $H = LST - \alpha$ (9.8)	<i>LST</i> local sidereal time <i>H</i> (local) hour angle
Equatorial to horizon $\sin a = \sin \delta \sin \phi + \cos \delta \cos \phi \cos H$ (9.9) $\tan A \equiv \frac{-\cos \delta \sin H}{\sin \delta \cos \phi - \sin \phi \cos \delta \cos H}$ (9.10)	<i>α</i> right ascension <i>δ</i> declination <i>a</i> altitude <i>A</i> azimuth (E from N) <i>φ</i> observer's latitude
Horizon to equatorial $\sin \delta = \sin a \sin \phi + \cos a \cos \phi \cos A$ (9.11) $\tan H \equiv \frac{-\cos a \sin A}{\sin a \cos \phi - \sin \phi \cos a \cos A}$ (9.12)	

^aConversions between horizon or alt-azimuth coordinates, (*a*, *A*), and celestial equatorial coordinates, (*δ*, *α*). There are a number of conventions for defining azimuth. For example, it is sometimes taken as the angle west from south rather than east from north. The quadrants for *A* and *H* can be obtained from the signs of the numerators and denominators in Equations (9.10) and (9.12) (see diagram).

Ecliptic coordinates^a

Obliquity of the ecliptic	$\varepsilon = 23^\circ 26' 21''.45 - 46''.815 T$ $- 0''.0006 T^2$ $+ 0''.00181 T^3$	(9.13)	ε mean ecliptic obliquity T Julian centuries since J2000.0 ^b
Equatorial to ecliptic	$\sin \beta = \sin \delta \cos \varepsilon - \cos \delta \sin \varepsilon \sin \alpha$ $\tan \lambda \equiv \frac{\sin \alpha \cos \varepsilon + \tan \delta \sin \varepsilon}{\cos \alpha}$	(9.14) (9.15)	α right ascension δ declination λ ecliptic longitude β ecliptic latitude
Ecliptic to equatorial	$\sin \delta = \sin \beta \cos \varepsilon + \cos \beta \sin \varepsilon \sin \lambda$ $\tan \alpha \equiv \frac{\sin \lambda \cos \varepsilon - \tan \beta \sin \varepsilon}{\cos \lambda}$	(9.16) (9.17)	

^aConversions between ecliptic, (β, λ) , and celestial equatorial, (δ, α) , coordinates. β is positive above the ecliptic and λ increases eastwards. The quadrants for λ and α can be obtained from the signs of the numerators and denominators in Equations (9.15) and (9.17) (see diagram).

^bSee Equation (9.5).

Galactic coordinates^a

Galactic frame	$\alpha_g = 192^\circ 15'$ $\delta_g = 27^\circ 24'$ $l_g = 33^\circ$	(9.18) (9.19) (9.20)	α_g right ascension of north galactic pole δ_g declination of north galactic pole
Equatorial to galactic	$\sin b = \cos \delta \cos \delta_g \cos(\alpha - \alpha_g) + \sin \delta \sin \delta_g$ $\tan(l - l_g) \equiv \frac{\tan \delta \cos \delta_g - \cos(\alpha - \alpha_g) \sin \delta_g}{\sin(\alpha - \alpha_g)}$	(9.21) (9.22)	l_g ascending node of galactic plane on equator
Galactic to equatorial	$\sin \delta = \cos b \cos \delta_g \sin(l - l_g) + \sin b \sin \delta_g$ $\tan(\alpha - \alpha_g) \equiv \frac{\cos(l - l_g)}{\tan b \cos \delta_g - \sin \delta_g \sin(l - l_g)}$	(9.23) (9.24)	δ declination α right ascension b galactic latitude l galactic longitude

^aConversions between galactic, (b, l) , and celestial equatorial, (δ, α) , coordinates. The galactic frame is defined at epoch B1950.0. The quadrants of l and α can be obtained from the signs of the numerators and denominators in Equations (9.22) and (9.24).

Precession of equinoxes^a

In right ascension	$\alpha \simeq \alpha_0 + (3^s.075 + 1^s.336 \sin \alpha_0 \tan \delta_0) N$	(9.25)	α right ascension of date α_0 right ascension at J2000.0 N number of years since J2000.0
In declination	$\delta \simeq \delta_0 + (20''.043 \cos \alpha_0) N$	(9.26)	δ declination of date δ_0 declination at J2000.0

^aRight ascension in hours, minutes, and seconds; declination in degrees, arcminutes, and arcseconds. These equations are valid for several hundred years each side of J2000.0.

9.4 Observational astrophysics

Astronomical magnitudes

Apparent magnitude	$m_1 - m_2 = -2.5 \log_{10} \frac{F_1}{F_2}$	(9.27)	m_i	apparent magnitude of object i
			F_i	energy flux from object i
Distance modulus ^a	$m - M = 5 \log_{10} D - 5$	(9.28)	M	absolute magnitude
	$= -5 \log_{10} p - 5$	(9.29)	$m - M$	distance modulus
			D	distance to object (parsec)
			p	annual parallax (arcsec)
Luminosity–magnitude relation	$M_{\text{bol}} = 4.75 - 2.5 \log_{10} \frac{L}{L_{\odot}}$	(9.30)	M_{bol}	bolometric absolute magnitude
	$L \simeq 3.04 \times 10^{(28 - 0.4 M_{\text{bol}})}$	(9.31)	L	luminosity (W)
			L_{\odot}	solar luminosity (3.826×10^{26} W)
Flux–magnitude relation	$F_{\text{bol}} \simeq 2.559 \times 10^{-(8 + 0.4 m_{\text{bol}})}$	(9.32)	F_{bol}	bolometric flux (W m^{-2})
			m_{bol}	bolometric apparent magnitude
Bolometric correction	$BC = m_{\text{bol}} - m_V$	(9.33)	BC	bolometric correction
	$= M_{\text{bol}} - M_V$	(9.34)	m_V	V -band apparent magnitude
			M_V	V -band absolute magnitude
Colour index ^b	$B - V = m_B - m_V$	(9.35)	$B - V$	observed $B - V$ colour index
	$U - B = m_U - m_B$	(9.36)	$U - B$	observed $U - B$ colour index
Colour excess ^c	$E = (B - V) - (B - V)_0$	(9.37)	E	$B - V$ colour excess
			$(B - V)_0$	intrinsic $B - V$ colour index

^aNeglecting extinction.

^bUsing the UBV magnitude system. The bands are centred around 365 nm (U), 440 nm (B), and 550 nm (V).

^cThe $U - B$ colour excess is defined similarly.

Photometric wavelengths

Mean wavelength	$\lambda_0 = \frac{\int \lambda R(\lambda) d\lambda}{\int R(\lambda) d\lambda}$	(9.38)	λ_0	mean wavelength
			λ	wavelength
			R	system spectral response
Isophotal wavelength	$F(\lambda_i) = \frac{\int F(\lambda) R(\lambda) d\lambda}{\int R(\lambda) d\lambda}$	(9.39)	$F(\lambda)$	flux density of source (in terms of wavelength)
			λ_i	isophotal wavelength
Effective wavelength	$\lambda_{\text{eff}} = \frac{\int \lambda F(\lambda) R(\lambda) d\lambda}{\int F(\lambda) R(\lambda) d\lambda}$	(9.40)	λ_{eff}	effective wavelength

Planetary bodies

Bode's law ^a	$D_{\text{AU}} = \frac{4 + 3 \times 2^n}{10}$	(9.41)	D_{AU}	planetary orbital radius (AU)
			n	index: Mercury = $-\infty$, Venus = 0, Earth = 1, Mars = 2, Ceres = 3, Jupiter = 4, ...
Roche limit	$R \gtrsim \left(\frac{100M}{9\pi\rho} \right)^{1/3}$	(9.42)	R	satellite orbital radius
	$\gtrsim 2.46R_0$ (if densities equal)	(9.43)	M	central mass
			ρ	satellite density
			R_0	central body radius
Synodic period ^b	$\frac{1}{S} = \left \frac{1}{P} - \frac{1}{P_{\oplus}} \right $	(9.44)	S	synodic period
			P	planetary orbital period
			P_{\oplus}	Earth's orbital period

^aAlso known as the "Titius–Bode rule." Note that the asteroid Ceres is counted as a planet in this scheme. The relationship breaks down for Neptune and Pluto.

^bOf a planet.

Distance indicators

Hubble law	$v = H_0 d$	(9.45)	v	cosmological recession velocity
			H_0	Hubble parameter (present epoch)
			d	(proper) distance
Annual parallax	$D_{\text{pc}} = p^{-1}$	(9.46)	D_{pc}	distance (parsec)
			p	annual parallax ($\pm p$ arcsec from mean)
Cepheid variables ^a	$\log_{10} \frac{\langle L \rangle}{L_{\odot}} \simeq 1.15 \log_{10} P_d + 2.47$	(9.47)	$\langle L \rangle$	mean cepheid luminosity
	$M_V \simeq -2.76 \log_{10} P_d - 1.40$	(9.48)	L_{\odot}	Solar luminosity
			P_d	pulsation period (days)
			M_V	absolute visual magnitude
Tully–Fisher relation ^b	$M_I \simeq -7.68 \log_{10} \left(\frac{2v_{\text{rot}}}{\sin i} \right) - 2.58$	(9.49)	M_I	I -band absolute magnitude
			v_{rot}	observed maximum rotation velocity (kms^{-1})
			i	galactic inclination (90° when edge-on)
Einstein rings	$\theta^2 = \frac{4GM}{c^2} \left(\frac{d_s - d_l}{d_s d_l} \right)$	(9.50)	θ	ring angular radius
			M	lens mass
			d_s	distance from observer to source
			d_l	distance from observer to lens
Sunyaev–Zel'dovich effect ^c	$\frac{\Delta T}{T} = -2 \int \frac{n_e k T_e \sigma_T}{m_e c^2} dl$	(9.51)	T	apparent CMBR temperature
			dl	path element through cloud
			R	cloud radius
			n_e	electron number density
			k	Boltzmann constant
			T_e	electron temperature
... for a homogeneous sphere	$\frac{\Delta T}{T} = -\frac{4Rn_e k T_e \sigma_T}{m_e c^2}$	(9.52)	σ_T	Thomson cross section
			m_e	electron mass
			c	speed of light

^aPeriod–luminosity relation for classical Cepheids. Uncertainty in M_V is ± 0.27 (Madore & Freedman, 1991, Publications of the Astronomical Society of the Pacific, **103**, 933).

^bGalaxy rotation velocity–magnitude relation in the infrared I waveband, centred at $0.90 \mu\text{m}$. The coefficients depend on waveband and galaxy type (see Giovanelli *et al.*, 1997, The Astronomical Journal, **113**, 1).

^cScattering of the cosmic microwave background radiation (CMBR) by a cloud of electrons, seen as a temperature decrement, ΔT , in the Rayleigh–Jeans limit ($\lambda \gg 1 \text{ mm}$).

9.5 Stellar evolution

Evolutionary timescales

Free-fall timescale ^a	$\tau_{\text{ff}} = \left(\frac{3\pi}{32G\rho_0} \right)^{1/2}$	(9.53)	τ_{ff} free-fall timescale G constant of gravitation ρ_0 initial mass density
Kelvin–Helmholtz timescale	$\tau_{\text{KH}} = \frac{-U_{\text{g}}}{L}$	(9.54)	τ_{KH} Kelvin–Helmholtz timescale U_{g} gravitational potential energy
	$\simeq \frac{GM^2}{R_0 L}$	(9.55)	M body's mass R_0 body's initial radius L body's luminosity

^aFor the gravitational collapse of a uniform sphere.

Star formation

Jeans length ^a	$\lambda_{\text{J}} = \left(\frac{\pi}{G\rho} \frac{dp}{d\rho} \right)^{1/2}$	(9.56)	λ_{J} Jeans length G constant of gravitation ρ cloud mass density p pressure
Jeans mass	$M_{\text{J}} = \frac{\pi}{6} \rho \lambda_{\text{J}}^3$	(9.57)	M_{J} (spherical) Jeans mass
Eddington limiting luminosity ^b	$L_{\text{E}} = \frac{4\pi G M m_{\text{p}} c}{\sigma_{\text{T}}}$	(9.58)	L_{E} Eddington luminosity M stellar mass M_{\odot} solar mass
	$\simeq 1.26 \times 10^{31} \frac{M}{M_{\odot}} \text{ W}$	(9.59)	m_{p} proton mass c speed of light σ_{T} Thomson cross section

^aNote that $(dp/d\rho)^{1/2}$ is the sound speed in the cloud.

^bAssuming the opacity is mostly from Thomson scattering.

Stellar theory^a

Conservation of mass	$\frac{dM_r}{dr} = 4\pi\rho r^2$	(9.60)	r radial distance M_r mass interior to r ρ mass density
Hydrostatic equilibrium	$\frac{dp}{dr} = \frac{-G\rho M_r}{r^2}$	(9.61)	p pressure G constant of gravitation
Energy release	$\frac{dL_r}{dr} = 4\pi\rho r^2 \epsilon$	(9.62)	L_r luminosity interior to r ϵ power generated per unit mass
Radiative transport	$\frac{dT}{dr} = \frac{-3}{16\sigma} \frac{\langle\kappa\rangle\rho}{T^3} \frac{L_r}{4\pi r^2}$	(9.63)	T temperature σ Stefan–Boltzmann constant $\langle\kappa\rangle$ mean opacity
Convective transport	$\frac{dT}{dr} = \frac{\gamma-1}{\gamma} \frac{T}{p} \frac{dp}{dr}$	(9.64)	γ ratio of heat capacities, c_p/c_V

^aFor stars in static equilibrium with adiabatic convection. Note that ρ is a function of r . κ and ϵ are functions of temperature and composition.

Stellar fusion processes^a

PP I chain	PP II chain	PP III chain
$p^+ + p^+ \rightarrow {}^2_1\text{H} + e^+ + \nu_e$ ${}^2_1\text{H} + p^+ \rightarrow {}^3_2\text{He} + \gamma$ ${}^3_2\text{He} + {}^3_2\text{He} \rightarrow {}^4_2\text{He} + 2p^+$	$p^+ + p^+ \rightarrow {}^2_1\text{H} + e^+ + \nu_e$ ${}^2_1\text{H} + p^+ \rightarrow {}^3_2\text{He} + \gamma$ ${}^3_2\text{He} + {}^4_2\text{He} \rightarrow {}^7_4\text{Be} + \gamma$ ${}^7_4\text{Be} + e^- \rightarrow {}^7_3\text{Li} + \nu_e$ ${}^7_3\text{Li} + p^+ \rightarrow 2{}^4_2\text{He}$	$p^+ + p^+ \rightarrow {}^2_1\text{H} + e^+ + \nu_e$ ${}^2_1\text{H} + p^+ \rightarrow {}^3_2\text{He} + \gamma$ ${}^3_2\text{He} + {}^4_2\text{He} \rightarrow {}^7_4\text{Be} + \gamma$ ${}^7_4\text{Be} + p^+ \rightarrow {}^8_5\text{B} + \gamma$ ${}^8_5\text{B} \rightarrow {}^8_4\text{Be} + e^+ + \nu_e$ ${}^8_4\text{Be} \rightarrow 2{}^4_2\text{He}$
CNO cycle	triple- α process	
${}^{12}_6\text{C} + p^+ \rightarrow {}^{13}_7\text{N} + \gamma$ ${}^{13}_7\text{N} \rightarrow {}^{13}_6\text{C} + e^+ + \nu_e$ ${}^{13}_6\text{C} + p^+ \rightarrow {}^{14}_7\text{N} + \gamma$ ${}^{14}_7\text{N} + p^+ \rightarrow {}^{15}_8\text{O} + \gamma$ ${}^{15}_8\text{O} \rightarrow {}^{15}_7\text{N} + e^+ + \nu_e$ ${}^{15}_7\text{N} + p^+ \rightarrow {}^{12}_6\text{C} + {}^4_2\text{He}$	${}^4_2\text{He} + {}^4_2\text{He} \rightleftharpoons {}^8_4\text{Be} + \gamma$ ${}^8_4\text{Be} + {}^4_2\text{He} \rightleftharpoons {}^{12}_6\text{C}^*$ ${}^{12}_6\text{C}^* \rightarrow {}^{12}_6\text{C} + \gamma$	γ photon p^+ proton e^+ positron e^- electron ν_e electron neutrino

^aAll species are taken as fully ionised.

Pulsars

Braking index	$\dot{\omega} \propto -\omega^n$ (9.65) $n = 2 - \frac{P\ddot{P}}{\dot{P}^2}$ (9.66)	ω rotational angular velocity P rotational period ($= 2\pi/\omega$) n braking index
Characteristic age ^a	$T = \frac{1}{n-1} \frac{P}{\dot{P}}$ (9.67)	T characteristic age L luminosity μ_0 permeability of free space c speed of light
Magnetic dipole radiation	$L = \frac{\mu_0 \ddot{m} ^2 \sin^2 \theta}{6\pi c^3}$ (9.68) $= \frac{2\pi R^6 B_p^2 \omega^4 \sin^2 \theta}{3c^3 \mu_0}$ (9.69)	m pulsar magnetic dipole moment R pulsar radius B_p magnetic flux density at magnetic pole θ angle between magnetic and rotational axes
Dispersion measure	$\text{DM} = \int_0^D n_e dl$ (9.70)	DM dispersion measure D path length to pulsar dl path element n_e electron number density
Dispersion ^b	$\frac{d\tau}{dv} = \frac{-e^2}{4\pi^2 \epsilon_0 m_e c v^3} \text{DM}$ (9.71) $\Delta\tau = \frac{e^2}{8\pi^2 \epsilon_0 m_e c} \left(\frac{1}{v_1^2} - \frac{1}{v_2^2} \right) \text{DM}$ (9.72)	τ pulse arrival time $\Delta\tau$ difference in pulse arrival time v_i observing frequencies m_e electron mass

^aAssuming $n \neq 1$ and that the pulsar has already slowed significantly. Usually n is assumed to be 3 (magnetic dipole radiation), giving $T = P/(2\dot{P})$.

^bThe pulse arrives first at the higher observing frequency.

Compact objects and black holes

Schwarzschild radius	$r_s = \frac{2GM}{c^2} \simeq 3 \frac{M}{M_\odot} \text{ km}$	(9.73)	r_s Schwarzschild radius G constant of gravitation M mass of body c speed of light M_\odot solar mass
Gravitational redshift	$\frac{v_\infty}{v_r} = \left(1 - \frac{2GM}{rc^2}\right)^{1/2}$	(9.74)	r distance from mass centre v_∞ frequency at infinity v_r frequency at r
Gravitational wave radiation ^a	$L_g = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5}$	(9.75)	m_i orbiting masses a mass separation L_g gravitational luminosity
Rate of change of orbital period	$\dot{P} = -\frac{96}{5} (4\pi^2)^{4/3} \frac{G^{5/3}}{c^5} \frac{m_1 m_2 P^{-5/3}}{(m_1 + m_2)^{1/3}}$	(9.76)	P orbital period
Neutron star degeneracy pressure (nonrelativistic)	$p = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m_n} \left(\frac{\rho}{m_n}\right)^{5/3} = \frac{2}{3} u$	(9.77)	p pressure \hbar (Planck constant)/(2 π) m_n neutron mass ρ density
Relativistic ^b	$p = \frac{\hbar c (3\pi^2)^{1/3}}{4} \left(\frac{\rho}{m_n}\right)^{4/3} = \frac{1}{3} u$	(9.78)	u energy density
Chandrasekhar mass ^c	$M_{\text{Ch}} \simeq 1.46 M_\odot$	(9.79)	M_{Ch} Chandrasekhar mass
Maximum black hole angular momentum	$J_m = \frac{GM^2}{c}$	(9.80)	J_m maximum angular momentum
Black hole evaporation time	$\tau_e \sim \frac{M^3}{M_\odot^3} \times 10^{66} \text{ yr}$	(9.81)	τ_e evaporation time
Black hole temperature	$T = \frac{\hbar c^3}{8\pi G M k} \simeq 10^{-7} \frac{M_\odot}{M} \text{ K}$	(9.82)	T temperature k Boltzmann constant

^aFrom two bodies, m_1 and m_2 , in circular orbits about their centre of mass. Note that the frequency of the radiation is twice the orbital frequency.

^bParticle velocities $\sim c$.

^cUpper limit to mass of a white dwarf.

9.6 Cosmology

Cosmological model parameters

Hubble law	$v_r = Hd$	(9.83)	v_r radial velocity H Hubble parameter d proper distance
Hubble parameter ^a	$H(t) = \frac{\dot{R}(t)}{R(t)}$ $H(z) = H_0 [\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0} + (1 - \Omega_{m0} - \Omega_{\Lambda 0})(1+z)^2]^{1/2}$	(9.84) (9.85)	0 present epoch R cosmic scale factor t cosmic time z redshift
Redshift	$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{R_0}{R(t_{\text{em}})} - 1$	(9.86)	λ_{obs} observed wavelength λ_{em} emitted wavelength t_{em} epoch of emission
Robertson–Walker metric ^b	$ds^2 = c^2 dt^2 - R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]$	(9.87)	ds interval c speed of light r, θ, ϕ comoving spherical polar coordinates
Friedmann equations ^c	$\ddot{R} = -\frac{4\pi}{3}GR\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda R}{3}$ $\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 - kc^2 + \frac{\Lambda R^2}{3}$	(9.88) (9.89)	k curvature parameter G constant of gravitation p pressure Λ cosmological constant
Critical density	$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$	(9.90)	ρ (mass) density ρ_{crit} critical density
Density parameters	$\Omega_m = \frac{\rho}{\rho_{\text{crit}}} = \frac{8\pi G\rho}{3H^2}$ $\Omega_\Lambda = \frac{\Lambda}{3H^2}$ $\Omega_k = -\frac{kc^2}{R^2H^2}$ $\Omega_m + \Omega_\Lambda + \Omega_k = 1$	(9.91) (9.92) (9.93) (9.94)	Ω_m matter density parameter Ω_Λ lambda density parameter Ω_k curvature density parameter
Deceleration parameter	$q_0 = -\frac{R_0\ddot{R}_0}{\dot{R}_0^2} = \frac{\Omega_{m0}}{2} - \Omega_{\Lambda 0}$	(9.95)	q_0 deceleration parameter

^aOften called the Hubble “constant.” At the present epoch, $60 \lesssim H_0 \lesssim 80 \text{ km s}^{-1} \text{ Mpc}^{-1} \equiv 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$, where h is a dimensionless scaling parameter. The Hubble time is $t_H = 1/H_0$. Equation (9.85) assumes a matter dominated universe and mass conservation.

^bFor a homogeneous, isotropic universe, using the $(-1, 1, 1, 1)$ metric signature. r is scaled so that $k = 0, \pm 1$. Note that $ds^2 \equiv (ds)^2$ etc.

^c $\Lambda = 0$ in a Friedmann universe. Note that the cosmological constant is sometimes defined as equalling the value used here divided by c^2 .

Cosmological distance measures

Look-back time	$t_{\text{lb}}(z) = t_0 - t(z)$	(9.96)	$t_{\text{lb}}(z)$ light travel time from an object at redshift z t_0 present cosmic time $t(z)$ cosmic time at z
Proper distance	$d_p = R_0 \int_0^r \frac{dr}{(1 - kr^2)^{1/2}} = cR_0 \int_t^{t_0} \frac{dt}{R(t)}$	(9.97)	d_p proper distance R cosmic scale factor c speed of light t_0 present epoch
Luminosity distance ^a	$d_L = d_p(1 + z) = c(1 + z) \int_0^z \frac{dz}{H(z)}$	(9.98)	d_L luminosity distance z redshift H Hubble parameter ^b
Flux density–redshift relation	$F(\nu) = \frac{L(\nu')}{4\pi d_L^2(z)}$ where $\nu' = (1 + z)\nu$	(9.99)	F spectral flux density ν frequency $L(\nu)$ spectral luminosity ^c
Angular diameter distance ^d	$d_a = d_L(1 + z)^{-2}$	(9.100)	d_a angular diameter distance k curvature parameter

^aAssuming a flat universe ($k=0$). The apparent flux density of a source varies as d_L^{-2} .

^bSee Equation (9.85).

^cDefined as the output power of the body per unit frequency interval.

^dTrue for all k . The angular diameter of a source varies as d_a^{-1} .

Cosmological models^a

	$d_p = \frac{2c}{H_0} [1 - (1 + z)^{-1/2}]$	(9.101)	d_p proper distance
Einstein – de Sitter model	$H(z) = H_0(1 + z)^{3/2}$	(9.102)	H Hubble parameter
($\Omega_k = 0$, $\Lambda = 0$, $p = 0$ and $\Omega_{m0} = 1$)	$q_0 = 1/2$	(9.103)	t_0 present epoch
	$t(z) = \frac{2}{3H(z)}$	(9.104)	z redshift
	$\rho = (6\pi G t^2)^{-1}$	(9.105)	c speed of light
	$R(t) = R_0(t/t_0)^{2/3}$	(9.106)	q deceleration parameter
Concordance model	$d_p = \frac{c}{H_0} \int_0^z \frac{\Omega_{m0}^{-1/2} dz'}{[(1 + z')^3 - 1 + \Omega_{m0}^{-1}]^{1/2}}$	(9.107)	$t(z)$ time at redshift z
($\Omega_k = 0$, $\Lambda = 3(1 - \Omega_{m0})H_0^2$, $p = 0$ and $\Omega_{m0} < 1$)	$H(z) = H_0[\Omega_{m0}(1 + z)^3 + (1 - \Omega_{m0})]$	(9.108)	R cosmic scale factor
	$q_0 = 3\Omega_{m0}/2 - 1$	(9.109)	Ω_{m0} present mass density parameter
	$t(z) = \frac{2}{3H_0} (1 - \Omega_{m0})^{-1/2} \text{arsinh} \left[\frac{(1 - \Omega_{m0})^{1/2}}{(1 + z)^{3/2}} \right]$	(9.110)	G constant of gravitation
			ρ mass density

^aCurrently popular.

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