数学物理方法习题指导

2.2. (1) 设函数在 $f(z), g(z)$ 在 $z=z_0$ 的邻域内解析，且 $f(z_0)=0$，$g(z_0)=0$，则有

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{g(z)-g(z_0)}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\Big/\frac{g(z)-g(z_0)}{z-z_0}$$

$$=\lim_{z\to z_0}\frac{f'(z)}{g'(z)}.$$

特别是，若 $g'(z_0)$ 与 $f'(z_0)\neq 0$ 均存在，且 $g'(z_0)\neq 0$，就有

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}$$

(2) 因 $f(z)=u(x,y)+iv(x,y)$ 在 G 内解析，应满足 C-R 条件

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

若在 G 内有 $f'(z)=0$，则有

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0,\quad \frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0.$$

因此，$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$，$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=0$

这就是说，函数 $u(x,y)$ 与 $v(x,y)$ 在 G 内均为实常数。

因而，函数 $f(z)$ 在 G 内为一复常数。

(3) 因 $f(z)$ 解析，故 C-R 条件

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

又因函数 $f^*(z)=u(x,y)-iv(x,y)$ 也在 G 内解析，故又应该有

$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$$

因此，$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0$，$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$

于是 $u(x,y)$，$v(x,y)$ 在 G 内是常数，函数 $f(z)$ 在 G 内也是常数。

(4) 注意到 $x=\frac{z+z^*}{2}$，$y=\frac{z-z^*}{2i}$，直接求偏导数

$$\frac{\partial f(z,z^*)}{\partial z^*}=\frac{\partial(u+iv)}{\partial x}\cdot\frac{\partial x}{\partial z^*}+\frac{\partial(u+iv)}{\partial y}\cdot\frac{\partial y}{\partial z^*}$$

$$=\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]$$

由于函数 $f(z,z^*)=u(x,y)+iv(x,y)$ 在 G 内解析，它必满足 C-R 条件，所以

$$\frac{\partial f(z,z^*)}{\partial z^*}=0,$$

即解析函数一定不显含 z^*，此结论是直接由 C-R 条件推导出的，也是函数解析的必要条件。

2.3 (1) $u(x,y)=e^y\sin x$

WAY 1. 因 $f(z)=u(x,y)+iv(x,y)$ 解析，故虚部 $v(x,y)$ 可微，由 C-R 条件可得

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$

$$=-e^y\sin x\,dx+e^y\cos x\,dy=d(e^y\cos x).$$

所以 $v(x,y)=e^y\cos x+C$，这里 C 是任意实常数，得解

$$f(z)=u(x,y)+iv(x,y)=e^y\sin x+ie^y\cos x+iC$$

$$=ie^y(\cos x-i\sin x)+iC$$

$$=ie^{-iz}+iC$$

WAY 2. 用线积分的方法。

$$v(x,y)=\int_{(x_0,y_0)}^{(x,y)}\left(-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy\right)+C\qquad 取 (0,0)\to(x,0)\to(x,y).$$

$$=\int_0^x(-e^y\sin x)_{y=0}\,dx+\int_0^y e^y\cos x\,dy+C$$

$$=\cos x-1+(e^y-1)\cos x+C$$

$$=e^y\cos x+C'$$

WAY 3. 设 $v(x,y)=\int\frac{\partial v}{\partial x}dx+\phi(y)=\int\left(-\frac{\partial u}{\partial y}\right)dx+\phi(y)$

$$\frac{\partial v}{\partial y}=-\frac{\partial}{\partial y}\int\frac{\partial u}{\partial y}dx+\phi'(y)=\frac{\partial u}{\partial x}$$

WAY 4. $\begin{cases}x=\dfrac{z+z^*}{2}\\ y=\dfrac{z-z^*}{2i}\end{cases}$

同样，这会有

$$\begin{cases}f(z)=u+iv\\ f^*(z)=u-iv\end{cases}\quad 即\quad \begin{cases}u=\dfrac{f(z)+f^*(z)}{2}\\ v=\dfrac{f(z)-f^*(z)}{2i}\end{cases}$$

我们对 $u(x,y)=e^y\sin x$ 作代换，得到

$$u(x,y)=e^{(z-z^*)/2i}\sin\frac{z+z^*}{2}$$

$$=e^{-i(z-z^*)/2}\cdot\frac{1}{2i}\left[e^{i(z+z^*)/2}-e^{-i(z+z^*)/2}\right]$$

$$=\frac{1}{2i}(e^{iz^*}-e^{-iz})=\frac{i}{2}(e^{-iz}-e^{iz^*})=\frac{1}{2}[ie^{-iz}+(ie^{-iz})^*]$$

这样就把 $u(x,y)$ 化成 $(f+f^*)/2$ 的形式，因而就能定出 $f(z)=ie^{-iz}+iC$

若 $u(x,y)+iv(x,y)$ 在 G 内解析，它一定不显含 z^*，即

$$\omega=u\left(\frac{z+z^*}{2},\frac{z+z^*}{2i}\right)+iv\left(\frac{z+z^*}{2},\frac{z+z^*}{2i}\right)=f(z)$$

(2) 已知 $v(x,y)=e^{-x}(x\sin y-y\cos y)$，因 $u(x,y)$ 可微，所以

$$u(x,y)=\int\frac{\partial u}{\partial x}dx+\phi(y)=\int\frac{\partial v}{\partial y}dx+\phi(y)$$

$$=\int e^{-x}(x\cos y-\cos y+y\sin y)dx+\phi(y)$$

$$=-e^{-x}(x\cos y+y\sin y)dx+\phi(y)$$

由 C-R 方程 $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$ 得

$$-e^{-x}(x\sin y-y\cos y-\sin y)=e^{-x}(-x\sin y+\sin y+y\cos y)+\phi'(y)$$

解得 $\phi'(y)=0$，即 $\phi(y)=C$，C 为实数，于是得到

$$u(x,y)=-e^{-x}(x\cos y+y\sin y)+C$$

$$f(z)=-e^{-x}(x\cos y+y\sin y)+C+ie^{-x}(x\sin y-y\cos y)$$

$$=[u(x,y)+iv(x,y)]_{x=z,y=0}=-ze^{-z}+C$$

(3). 已知 $f(z)=u(x,y)+iv(x,y)$ 解析，且

$$u(x,y)+v(x,y)=x^2-y^2-2xy$$

假设

$$F(z)=U(x,y)+iV(x,y)=f(z)-if(z)=(1-i)f(z)$$

$$=(u+iv)-i(u+iv)$$

$$=(u+v)+i(v-u)$$

因 $f(z)$ 解析，所以 $F(z)=(1-i)f(z)$ 也解析。因此可通过 $F(z)$ 的实部 $U(x,y)=u(x,y)+v(x,y)=x^2-y^2-2xy$ 求其虚部 $V(x,y)$：

$$V(x,y)=\int\frac{\partial V}{\partial x}dx+\phi(y)=-\int\frac{\partial U}{\partial y}dx+\phi(y)$$

$$=\int(2y+2x)dx+\phi(y)$$

$$=2xy+x^2+\phi(y)$$

由 $\frac{\partial V}{\partial y}=\frac{\partial U}{\partial x}$ 得

$$2x+\phi'(y)=2x-2y.$$

于是解得 $\phi(y)=-y^2+2C$，C 为实常数，所以

$$F(z)=x^2-y^2-2xy+i(2xy+x^2-y^2+2C)$$

$$=(U+iV)_{x=z,y=0}$$

$$=z^2+i(z^2+2C)$$

$$=i(1-i)z^2+2iC$$

从而得到 $f(z)=\frac{1}{1-i}F(z)=iz^2+C'$，常数 $C'=\frac{2iC}{1-i}=(-1+i)C$

2.4 因为 $e^{i(z_1+z_2)}=e^{iz_1}e^{iz_2}$，根据 Euler 公式，就有

$$\cos(z_1+z_2)+i\sin(z_1+z_2)$$

$$=(\cos z_1+i\sin z_1)(\cos z_2+i\sin z_2)$$

$$=(\cos z_1\cos z_2-\sin z_1\sin z_2)+i(\sin z_1\cos z_2+\cos z_1\sin z_2)\qquad\triangle$$

同理，因 $e^{-i(z_1+z_2)}=e^{-iz_1}\cdot e^{-iz_2}$，又有

$$\cos(z_1+z_2)-i\sin(z_1+z_2)$$

$$=(\cos z_1\cos z_2-\sin z_1\sin z_2)-i(\sin z_1\cos z_2+\cos z_1\sin z_2)\qquad\square$$

$\triangle+\square\Rightarrow\cos(z_1+z_2)=\cos z_1\cos z_2-\sin z_1\sin z_2$

$\triangle-\square\Rightarrow\sin(z_1+z_2)=\sin z_1\cos z_2+\cos z_1\sin z_2$

类似可以证明 $\cos(z_1-z_2)$ 和 $\sin(z_1-z_2)$ 的公式

$$\tan(z_1\pm z_2)=\frac{\sin(z_1\pm z_2)}{\cos(z_1\pm z_2)}$$

2.5. 根据 de Moivre 公式，我们可以得到

$$\frac{\sin(2n+1)\theta}{\sin\theta}=\frac{e^{i(2n+1)\theta}-e^{-i(2n+1)\theta}}{e^{i\theta}-e^{-i\theta}}=~~\frac{e^{i2n\theta}(1-e^{-i4n\theta})}{1-e^{-i2\theta}}~~$$

$$=e^{i2n\theta}+e^{i(2n-2)\theta}+\cdots+e^{-i(2n-2)\theta}+e^{-i2n\theta}$$

$$=A_0+A_1\sin^2\theta+\cdots+A_{n-1}\sin^{2n-2}\theta+A_n\sin^{2n}\theta$$

$$\frac{e^{i(2n+1)\theta}-e^{-i(2n+1)\theta}}{e^{i\theta}-e^{-i\theta}}=\frac{e^{i(2n+1)\theta}(1-e^{-i(4n+2)\theta})}{e^{i\theta}(1-e^{-i2\theta})}$$

$$=\frac{e^{i2n\theta}(1-e^{-i(4n+2)\theta})}{1-e^{-i2\theta}},\quad a_1=e^{-i2n\theta},\ q=e^{-i2\theta}$$

$$=\frac{a_1(1-q^{2n+1})}{1-q}$$ 所以是 $2n+1$ 项的等比数列。

当 $(2n+1)\theta=\pm\pi,\pm2\pi,\cdots,\pm n\pi$ 时，$\sin(2n+1)\theta/\sin\theta=0$，因此，

$$\frac{\sin(2n+1)\theta}{\sin\theta}=A\left(1-\frac{\sin^2\theta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\theta}{\sin^2 2\alpha}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2 n\alpha}\right)$$

其 $\alpha=\frac{\pi}{2n+1}$。再令 $\theta\to0$，即可定出常数 $A=2n+1$，于是有

$$\frac{\sin(2n+1)\theta}{\sin\theta}=(2n+1)\left(1-\frac{\sin^2\theta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\theta}{\sin^2 2\alpha}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2 n\alpha}\right)$$

类似的等式还可以列出，

$$\frac{\sin 2n\theta}{\sin\theta\cos\theta}=2n\left(1-\frac{\sin^2\theta}{\sin^2\alpha'}\right)\left(1-\frac{\sin^2\theta}{\sin^2 2\alpha'}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2(n-1)\alpha'}\right)$$

$$\frac{\cos(2n+1)\theta}{\cos\theta}=\left(1-\frac{\sin^2\theta}{\sin^2\beta}\right)\left(1-\frac{\sin^2\theta}{\sin^2 3\beta}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2(2n-1)\beta}\right)$$

$$\cos 2n\theta=\left(1-\frac{\sin^2\theta}{\sin^2\beta'}\right)\left(1-\frac{\sin^2\theta}{\sin^2 3\beta'}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2(2n-1)\beta'}\right)$$

其 $\alpha'=\frac{\pi}{2n}$，$\beta=\frac{\pi}{2(2n+1)}$，$\beta'=\frac{\pi}{4n}$

从这些关系式中可以得到如下的结果，例如，

$$\csc^2\frac{\pi}{2n+1}+\csc^2\frac{2\pi}{2n+1}+\cdots+\csc^2\frac{n\pi}{2n+1}=\frac{2}{3}n(n+1)$$

$$\csc\frac{\pi}{2n}+\csc^2\frac{2\pi}{2n}+\cdots+\csc^2\frac{(n-1)\pi}{2n}=\frac{2}{3}(n^2-1)$$

$$\csc^2\frac{\pi}{2(2n+1)}+\csc^2\frac{3\pi}{2(2n+1)}+\cdots+\csc^2\frac{(2n-1)\pi}{2(2n+1)}=2n(n+1)$$

$$\csc^2\frac{\pi}{4n}+\csc^2\frac{3\pi}{4n}+\cdots+\csc^2\frac{(2n-1)\pi}{4n}=2n^2$$

2.6. 证明 Jacobi 变换公式

$$令\ f_n=\frac{d^{n-1}}{d\mu^{n-1}}(1-\mu^2)^{n-\frac{1}{2}}$$

则当 $n=1$ 时显然

$$f_1=\sin\theta$$

变换成立。再设 $n=k-1$ 时

$$f_{k-1}\equiv\frac{d^{k-2}}{d\mu^{k-2}}(1-\mu^2)^{k-\frac{3}{2}}=(-)^{k-2}\frac{(2k-3)!!}{k-1}\sin(k-1)\theta$$

成立，则根据 f_n 的递推关系

$$f_{n+1}=\frac{d^n}{d\mu^n}(1-\mu^2)^{n+\frac{1}{2}}=\frac{d^n}{d\mu^n}\left[(1-\mu^2)(1-\mu^2)^{n-\frac{1}{2}}\right]$$

$$=(1-\mu^2)\frac{d^{n-1}}{d\mu^{n-1}}(1-\mu^2)^{n-\frac{1}{2}}-2n\mu\frac{d^{n-1}}{d\mu^{n-1}}(1-\mu)^{n-\frac{1}{2}}-n(n-1)\frac{d^{n-2}}{d\mu^{n-2}}(1-\mu^2)^{n-\frac{1}{2}}$$

$$=(1-\mu^2)\frac{df_n}{d\mu}-2n\mu f_n-n(n-1)\int_1^\mu f_n\,d\mu$$

以及 $\frac{d\mu}{d\theta}=-\sin\theta$，就立即得到

$$f_k=(1-\mu)^2\frac{df_{k-1}}{d\mu}-2(k-1)\mu f_{k-1}-(k-1)(k-2)\int_1^\mu f_{k-1}\,d\mu$$

$$=-\sin\theta\frac{df_{k-1}}{d\theta}-2(k-1)\cos\theta f_{k-1}+(k-1)(k-2)\int_0^\theta f_{k-1}\sin\theta\,d\theta$$

$$=(-)^{k-1}(2k-3)!!\left[\sin\theta\cos(k-1)\theta+2\cos\theta\sin(k-1)\theta-(k-2)\int_0^\theta\sin(k-1)\theta\sin\theta\,d\theta\right]$$

$$=(-)^{k-1}(2k-3)!!\left\{\frac{\sin k\theta-\sin(k-2)\theta}{2}+[\sin k\theta+\sin(k-2)\theta]-\frac{k-2}{2}\left[\frac{\sin(k-2)\theta}{k-2}-\frac{\sin k\theta}{k}\right]\right\}$$

$$=(-)^{k-1}(2k-3)!!\left(\frac{1}{2}+1+\frac{k-2}{2k}\right)\sin k\theta$$

$$=(-)^{k-1}\frac{(2k-1)!!}{k}\sin k\theta$$

因此 Jacobi 变换成立。

$$即\quad\frac{d^{n-1}}{d\mu^{n-1}}\sin^{2n-1}\theta=(-)^{n-1}\frac{1\cdot3\cdot5\cdots(2n-1)}{n}\sin n\theta\qquad(\mu=\cos\theta)$$