Stanford | ENGINEERING
Aeronautics and Astronautics

Problem Set 4
Total of 125 points
Due on May 27, 2025, at 6 pm, Pacific Time

Aspects of two-body problem and *N*-body problems (20 points)
1.a) The position on an orbit in the perifocal coordinate system is given by $x_{orb} = r\cos\nu$ and $y_{orb} = r\sin\nu$, where r is the distance from the orbit's focus and ν is the true anomaly. Derive analytically the velocities $\dot{x}_{orb}$ and $\dot{y}_{orb}$ in the perifocal plane using the second Kepler's law $|h| = r^2\dot{\nu}$ and orbit equation $r = \frac{h^2}{\mu}\frac{1}{1+e\cos(\nu)}$, where e is the eccentricity, $|h| = \sqrt{\mu a(1-e^2)}$ is the specific angular momentum vector magnitude and $\mu = GM$ is the gravitational parameter of the planet. Velocities $\dot{x}_{orb}$ and $\dot{y}_{orb}$ need to be derived in terms of orbital elements e, ν, semimajor axis a and gravitational parameter μ.

1.b) The equations of motion for *N* bodies under their mutual gravity are:

$$m_i\,\ddot{\vec{r}}_i = \sum_{\substack{j=1\\ i\neq j}}^{N} \frac{Gm_im_j(\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|^3}$$

Derive a general expression for the motion (coordinates as a function of time) of the center of mass of the system of *N* bodies:

$$x_{COM} = f(t)$$
$$y_{COM} = f(t)$$
$$z_{COM} = f(t)$$

Remember that the center of mass of N bodies can be found as:

$$\vec{r}_{COM} = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{\sum_{i=1}^{N} m_i}$$

Hint: use 3rd Newton's law.

Bonus problem (10 points)
1.c) Derive the Taylor expansion of the mean anomaly in terms of the true anomaly in powers of eccentricity. Keep the terms up to the first power of eccentricity. You would need to use Kepler's equation $M = E - e\sin E$ and the relationship between the eccentric and mean anomalies: $\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$.
Another bonus problem (15 points)
1.d) Do the same as in 1.c, but up to the second power of eccentricity.

2) Orbital speed and orbital transfers (25 points)
2.a) Compute orbital speeds of all planets on their orbit around the Sun. You can data for semimajor axes of planets from https://ssd.jpl.nasa.gov/planets/approx_pos.html. You can neglect eccentricity here.

2.b) Using vis-viva law $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$, where a is the orbit's semimajor axis, and the orbit equation $r = \frac{a(1-e^2)}{1+e\,cos(v)}$, derive general expressions for the minimum and maximum velocity on a Keplerian elliptic orbit.

2.c) Now, we are going to the Moon. We start with LEO circular orbit with an altitude of 400 km. We are going to solve the problem in a simplified way, ignoring the gravity of the Moon. We assume the spacecraft performs an impulsive maneuver at LEO – it fires its engine, which leads to an instantaneous change in velocity ΔV in the direction of motion. What ΔV is needed to raise its apocenter to the Earth-Moon distance?

2.d) Now, you decide to stay at the Moon. What ΔV is needed be applied at the apocenter to circularize the orbit that has a pericenter altitude of 400 km and an apocenter at Earth-Moon distance?

Hint: you can neglect lunar gravity in 2.d.

3) Orbital Elements (65 points)

3.a) Write a Python function that converts orbital elements to Cartesian coordinates and velocities in the Earth-Centered Inertial frame. Assume that all orbital elements can change in time. Useful materials are available in slides for lecture 12 and Gill & Montenbrook textbook, pages 26 – 27. The syntaxis of this function would look like this:

```
def elem2coord(a, e, i, omega, Omega, M0, t0, t, mu):
  #% INPUTS
  #% a - semimajor axis
  #% e - eccentricity
  #% i - inclination
  #% omega - argument of pericenter (pericenter is the same as periapsis)
  #% Omega - longitude of the ascending node
  #% M0 - mean anomaly at time t0
  #% t0 - initial time
  #% t  - time
  #% mu  - gravitational constant (mu = GM)

  #% OUTPUT
  #% x,y,z     - position
  #% vx, vy, vz – velocity

  ##########################
  # IMPLEMENT YOUR FUNCTION HERE
  ##########################

  return x, y, z, vx, vy, vz
```

Here is a test case to validate your function:

Input

Parameter	**Value**
Semi-major axis, a	26600000.0 [m]
Eccentricity, e	0.74
Inclination, i	$63.4°$
Argument of periapsis, ω	$270°$
Right ascension of the ascending node, Ω	$45°$
Mean anomaly at epoch, M_0	$10°$
Initial time, t_0	100 [s]
Final time, t_f	$t_0 + 21600$ [s]
Gravitational parameter, μ	$3.986004418 \times 10^{14}$ [m^3 s^{-2}]

Expected Output (Cartesian state at the final time, t_f)

Component	**Value**
x	-15891749.923216064 [m]
y	13329971.701149576 [m]
z	41262812.92841874 [m]
v_x	-983.4914204373653 [m/s]
v_y	-1126.4374128032644 [m/s]
v_z	-201.84826266167386 [m/s]

3.b) You are tasked to design an orbit similar to the Molniya orbit we discussed in class. As a reminder the Molniya orbit has the following orbital elements:

a = 26600.0 km
e = 0.74
i = 63.4°
ω = 270°
Ω = 170.0°

You goal is to design an orbit the has the following properties:
- The orbit's shape is the same as the shape of the Molniya orbit.
- The orbit's apocenter is at the southernmost point of the orbit.
- Each subsequent spacecraft apocenter passage occurs 120 degrees of longitude apart.
- One of the apocenter passages occurs over Australia.

Hints:
- *Orbital inclination determines the range of latitudes for the orbital track.*
- *Changing Ω rotates the orbit around the Earth's rotation axis. In other words, changing Ω shifts the orbital tracking in longitudinal direction. To get the right Ω, you would need to*

compute the trajectory in the ECEF frame using the Greenwich Hour Angle Θ. Look at the Hint after question 4.b for this part.

- *Changing ω rotates the orbit in the orbital plane. If ω is zero, the pericenter of the orbit is at its node (i.e., at the Earth equator). If ω is 180 degrees, the pericenter is at the other node.*
- *Changing "a" can affect the orbit period: $T^2 = 4\pi^2 \cdot \frac{a^3}{\mu}$.*

Your goal here is to choose a new set of values for the orbital elements that define the desired orbit. Justify your choice for each orbital element. Confirm by plotting the ground tracks in the Earth-Centered Earth-Fixed coordinate system.

Bonus problem (15 points). *I highly recommend you solve this one as it will be extremely useful in your aerospace career.*

3.c) Following Gill & Montenbruck textbook, pages 28 – 29 (or scroll all the way down), write a Python function that converts Cartesian coordinates and velocities in the Earth-Centered Inertial frame to orbital elements ($a, i, e, \omega, \Omega, M$). The syntax of this function would look like this:

```
def coord2elem(x, y, z, vx, vy, vz, mu):
  ##########################
  # IMPLEMENT FUNCTION HERE
  ##########################
  return (a, e, i, omega, Omega, M)
```

Hint 1: a good way to check if coord2elem works correctly is 1) to choose orbital elements (e.g., from 3.b), 2) run elem2coord to get coordinates and velocities, 3) run coord2elem and see if you get back the original values.

Hint 2: You will need to you atan2 function to get the right quadrant in multiple places.

Example test case (same case as 3a)

Input

Component	**Value**
x	-15891749.923216064 [m]
y	13329971.701149576 [m]
z	41262812.92841874 [m]
v_x	-983.4914204373653 [m/s]
v_y	-1126.4374128032644 [m/s]
v_z	-201.84826266167386 [m/s]
Gravitational parameter, μ	$3.986004418 \times 10^{14}$ [m^3 s^{-2}]

Expected Output

Parameter	**Value**
Semi-major axis, a	26600000.0 [m]
Eccentricity, e	0.74
Inclination, i	1.106538745764405 [rad]
Argument of periapsis, ω	4.71238898038469 [rad]
Right ascension of the ascending node, Ω	0.7853981633974483 [rad]
Mean anomaly, M	3.31793679921364 [rad]

4) Orbital track of ISS (15 points)

You are given the following Two-Line Elements (TLE) for the International Space Station:

```
ISS (ZARYA)
1 25544U 98067A   25133.44462271  .00008689  00000-0  16281-3 0  9996
2 25544  51.6344 119.1760 0002307 106.2285 253.8958 15.49506546509779
```

You can use slides from Lecture 13 to identify individual orbital elements.

4.a) Compute the positions in the ECEF coordinate system using elem2coord.
4.b) Plot the ground track of the ISS in the ECEF coordinate system from May 28, 2025 noon Pacific time to May 29, 2025, noon Pacific Daylight Time.

Hint: to solve this problem you need to rotate the trajectory of ISS from the ECI frame to the ECEF frame. This is done by a rotation around the Earth-rotation axis (the Z-axis). Take a look at Gill & Montenbruck textbook, page 33 for reference. We need to find the Θ *angle, which is called the Greenwich Hour Angle. This angle is given by:*

$$\Theta = 280.4606^\circ + 360.9856473^\circ \cdot d$$

where d is the number of days since 12 h UTC on 1st of January 2000. So, you would need to find "d". This can be use using so-called Julian dates. Julian day is the number of days since… yes, you guessed it right, since 12:00 January 1, 4713 BC. You might ask what happened at 12:00 January 1, 4713 BC. As far as I know, nothing of interest. It is just a date someone named Joseph

Scaliger decided to make the starting point of the Julian date time scale. Julian days are simply a continuous time variable that counts the number of days since that starting point.

Take a look at this code to compute "d"

JD_J2000 = coord.gregorian2JD(1.0,1.5,2000.0) # Julian day at 12 h UTC on 1st of January 2000

JD_J2025 = coord.gregorian2JD(1.0,1.0,2025.0) # Julian day at 0 h UTC on 1st of January 2025

d = JD_J2025 - JD_J2000 + t / 86400.0 # where t is the number of seconds within the year of 2025.

Bonus problem (15 points)

4.c) Predict visibility over Stanford from May 20 to June 20, 2025

2.2.4 Orbital Elements from Position and Velocity

As has been shown, a total of six independent parameters are required to describe the motion of a satellite around the Earth. Two of these orbital elements (a and e) describe the form of the orbit, one element (M) defines the position along the orbit and the three others (Ω, i, and ω) finally define the orientation of the orbit in space. Given these six elements, it is always possible to uniquely calculate the position and velocity vector.

Vice versa there is exactly one set of orbital elements that corresponds to given initial values of $\boldsymbol{r}$ and $\boldsymbol{v}$, and one may ask how to find these elements. Part of the answer is already evident from the solution of the two-body problem presented above. First of all the areal velocity vector

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} = \begin{pmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{pmatrix} \tag{2.56}$$

and its modulus h can be obtained from the position and velocity. Then, from the representation of $\boldsymbol{h}$ or $\boldsymbol{W} = \boldsymbol{h}/h$ as a function of i and Ω in (2.54), it follows that

$$\begin{pmatrix} \sin i \sin \Omega \\ \sin i \cos \Omega \\ \cos i \end{pmatrix} = \begin{pmatrix} +h_x/h \\ -h_y/h \\ +h_z/h \end{pmatrix} = \begin{pmatrix} +W_x \\ -W_y \\ +W_z \end{pmatrix} \quad . \tag{2.57}$$

Hence the inclination and the right ascension of the ascending node are given by[2]

$$i = \arctan\left(\frac{\sqrt{W_x^2 + W_y^2}}{W_z}\right) \qquad \Omega = \arctan\left(\frac{W_x}{-W_y}\right) \quad . \tag{2.58}$$

The areal velocity can further be used to derive the semi-latus rectum

$$p = \frac{h^2}{GM_\oplus} \quad . \tag{2.59}$$

Next, the vis-viva law yields the semi-major axis

$$a = \left(\frac{2}{r} - \frac{v^2}{GM_\oplus}\right)^{-1} \tag{2.60}$$

and consequently the mean motion

$$n = \sqrt{\frac{GM_\oplus}{a^3}} \quad . \tag{2.61}$$

[2] In evaluating expressions of the form $\alpha = \arctan(y/x)$ the quadrant of α must be chosen in such a way that the sign of the denominator (x) is equal to the sign of $\cos\alpha$, i.e. $-90° < \alpha < +90°$ for $x > 0$ and $+90° < \alpha < +270°$ for $x < 0$.

For elliptic orbits a will always be positive. The eccentricity e follows from

$$e = \sqrt{1 - \frac{p}{a}} \quad . \tag{2.62}$$

Considering (2.31) and the identity

$$\begin{aligned} \boldsymbol{r} \cdot \dot{\boldsymbol{r}} &= -a\,(\cos(E) - e) \cdot a \sin(E)\dot{E} \\ &\qquad + a\sqrt{1-e^2}\sin(E) \cdot a\sqrt{1-e^2}\cos(E)\dot{E} \\ &= a^2 ne \sin(E) \end{aligned} \tag{2.63}$$

one may solve for $e \sin(E)$ and $e \cos(E)$ to find the eccentric anomaly from

$$E = \arctan\left(\frac{\boldsymbol{r} \cdot \dot{\boldsymbol{r}}/(a^2 n)}{1 - r/a}\right) \quad . \tag{2.64}$$

The eccentric anomaly may now be used to obtain the mean anomaly from Kepler's equation

$$M(t) = E(t) - e \sin E(t) \qquad \text{(in radians)} \tag{2.65}$$

with t being the epoch of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$.

In order to find the remaining orbital element ω, one has to determine the argument of latitude u first. Solving (2.51) for $\cos u$ and $\sin u$ yields

$$u = \arctan\left(\frac{z/\sin i}{x\cos\Omega + y\sin\Omega}\right) = \arctan\left(\frac{z}{-xW_y + yW_x}\right) \quad . \tag{2.66}$$

Furthermore, the true anomaly is given by

$$\nu = \arctan\left(\frac{\sqrt{1-e^2}\sin E}{\cos E - e}\right) \tag{2.67}$$

taking proper care of the correct quadrant (cf. (2.30)). The result may finally be used to obtain the argument of perigee from

$$\omega = u - \nu \quad . \tag{2.68}$$