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SEQUENT SYSTEMS FOR MODAL LOGICS

INTRODUCTION

> [T]he framework of ordinary sequents is not capable of handling all interesting logics. There are logics with nice, simple semantics and obvious interest for which no decent, cut-free formulation seems to exist .... Larger, but still satisfactory frameworks should, therefore, be sought. A. Avron [1996, p. 3]

This chapter surveys the application of various kinds of sequent systems to modal and temporal logic, also called tense logic. The starting point are ordinary Gentzen sequents and their limitations both technically and philosophically. The rest of the chapter is devoted to generalizations of the ordinary notion of sequent. These considerations are restricted to formalisms that do not make explicit use of semantic parameters like possible worlds or truth values, thereby excluding, for instance, Gabbay's labelled deductive systems, indexed tableau calculi, and Kanger-style proof systems from being dealt with. Readers interested in these types of proof systems are referred to [Gabbay, 1996], [Goré, 1999] and [Pliuškevienė, 1998]. Also Orlowska's [1988; 1996] Rasiowa-Sikorski-style relational proof systems for normal modal logics will not be considered in the present chapter. In relational proof systems the logical object language is associated with a language of relational terms. These terms may contain subterms representing the accessibility relation in possible-worlds models, so that semantic information is available at the same level as syntactic information. The derivation rules in relational proof systems manipulate finite sequences of relational formulas constructed from relational terms and relational operations. An overview of *ordinary* sequent systems for non-classical logics is given in [Ono, 1998], and for a general background on proof theory the reader may consult [Troelstra and Schwichtenberg, 2000]. In this chapter we shall pay special attention to *display logic*, a general proof-theoretic approach developed by Belnap [1982]. Two applications of the modal display calculus are included as case studies: the formulas-as-types notion of construction for temporal logic and a display calculus for propositional bi-intuitionistic logic (also called Heyting-Brouwer logic). This logic comprises both constructive implication and coimplication (see, for example, [Goré, 2000], [Rauszer, 1980], [Wolter, 1998]), and its sequent-calculus presentation to be given is based on a modal translation into the temporal propositional logic **S4t**.[1]

[1]The chapter consists of revised and expanded material from [Wansing, 1998] and includes the contents of the unpublished report [Wansing, 2000] on formulas-as-types for temporal logics. Moreover, the sequent calculus for bi-intuitionistic logic and subsystems of bi-intuitionistic logics in Section 3.8 and the translation of multiple-sequent systems into higher-arity sequent systems in Section 4.1 are new.

A note on notation. In the present chapter, both classical and constructive logics will be considered. Therefore it makes sense to reflect this distinction in the notation for the logical operations. In particular, the following symbols will be used: $\rhd$ (constructive, intuitionistic implication), $\blacktriangleleft$ (coimplication), $\supset$ (Boolean implication), $\frown$ (intuitionistic negation), $\smile$ (conegation), $\neg$ (Boolean negation).

1 ORDINARY SEQUENT SYSTEMS

The presentation of normal modal logics as ordinary (standard) sequent systems has turned out to be problematic for both technical and philosophical reasons. The technical problems chiefly result from a lack of flexibility of the ordinary notion of sequent for dealing with the multitude of interesting and important modal logics in a uniform and perspicuous way. In this section a number of standard Gentzen systems for normal modal propositional logics is reviewed in order to give an impression of what has been and what can be done to present normal modal logics as ordinary Gentzen calculi. An ordinary Gentzen system is a collection of rule schemata for manipulating *Gentzen sequents*; these are derivability statements of the form $\Delta \rightarrow \Gamma$, where Δ and Γ are finite, possibly empty sets of formulas. The set terms 'Δ' and 'Γ' are called the antecedent and the succedent of $\Delta \rightarrow \Gamma$, respectively. Often, a sequent

$$\{A_1, \ldots, A_m\} \rightarrow \{B_1, \ldots, B_n\}$$

is written as $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$. This notation supports viewing the ',' (the comma) as a *structure connective* in the language of sequents. Indeed, the sequent arrow in Gentzen's [1934] denotes a derivability relation between finite *sequences* of formulas separated by the comma. Gentzen, however, postulated structural rules that justify thinking of antecedents and succedents as denoting sets:

$$\text{(permutation)} \quad \frac{\Delta, A, B, \Gamma \rightarrow \Sigma}{\Delta, B, A, \Gamma \rightarrow \Sigma} \quad \frac{\Delta \rightarrow \Sigma, A, B, \Gamma}{\Delta \rightarrow \Sigma, B, A, \Gamma}$$

$$\text{(contraction)} \quad \frac{\Delta, A, A, \Gamma \rightarrow \Sigma}{\Delta, A, \Gamma \rightarrow \Sigma} \quad \frac{\Delta \rightarrow \Sigma, A, A, \Gamma}{\Delta \rightarrow \Sigma, A, \Gamma}$$

Gentzen also postulated

$$\text{(monotonicity)} \quad \frac{\Delta, \Gamma \rightarrow \Sigma}{\Delta, A, \Gamma \rightarrow \Sigma} \quad \frac{\Delta \rightarrow \Gamma, \Sigma}{\Delta \rightarrow \Gamma, A, \Sigma}$$

These three rules are structural in the sense of exhibiting no operation from an underlying logical object language. If the polymorphic comma is interpreted as a binary structure connective that may or may not be associative, the antecedent and the succedent of a sequent are *Gentzen terms*, and in generalized sequent calculi, the sequents display Gentzen terms or other, much more complex data structures. We shall use '$\vdash$' to denote the derivability relation in a given axiomatic system or a consequence relation between finite sets of sequents and single sequents satisfying identity, cut, and monotonicity. In other words, if Δ and Γ are finite sets of sequents and s, s' are sequents, then we assume that $\{s\} \vdash s$,

$$\frac{\Delta \vdash s}{\Delta \cup \{s'\} \vdash s} \quad \text{and} \quad \frac{\Delta \vdash s \quad \Gamma \cup \{s\} \vdash s'}{\Delta \cup \Gamma \vdash s'} .$$

1.1 Ordinary Gentzen systems for normal modal logics

The syntax of modal propositional logic (in Backus-Naur form, see for example [Goldblatt, 1992, p. 3]) is given by:

$$A ::= p \mid \boldsymbol{t} \mid \boldsymbol{f} \mid \neg A \mid A \wedge B \mid A \vee B \mid A \supset B \mid A \equiv B \mid \Diamond A \mid \Box A.$$

The smallest normal modal propositional logic **K** admits a simple presentation as an ordinary Gentzen system (see, for instance, [Leivant, 1981], [Mints, 1990], [Sambin and Valentini, 1982]). In the language with $\Box$ ("necessarily") as the only primitive modal operator and $\Diamond A$ ("possibly A") being defined as $\neg\Box\neg A$, one may just add the rule

$$(\rightarrow \Box)_1 \quad \Delta \rightarrow A \vdash \Box\Delta \rightarrow \Box A$$

to the standard sequent system **LCPL** for classical propositional logic **CPL**, where $\Box\Delta = \{\Box A \mid A \in \Delta\}$. A sequent calculus **LK4** for **K4** can be obtained by supplementing **LCPL** with the rule

$$(\rightarrow \Box)_2 \quad \Delta, \Box\Delta \rightarrow A \vdash \Box\Delta \rightarrow \Box A$$

(see [Sambin and Valentini, 1982]). In [Goble, 1974] it is shown that the pair of modal sequent rules $(\rightarrow \Box)_1$ and

$$(\Box \rightarrow)_1 \quad \Delta, A \rightarrow \emptyset \vdash \Box\Delta, \Box A \rightarrow \emptyset$$

yields a sequent system for **KD** (where '$\emptyset$' denotes the empty set) and that a sequent calculus for **KD4** is obtained, if $(\rightarrow \Box)_1$ is replaced by the rule

$$(\rightarrow \Box)_3 \quad \Delta' \rightarrow A \vdash \Box\Delta \rightarrow \Box A,$$

where Δ' results from Δ by prefixing zero or more formulas in Δ by $\Box$. Shvarts [1989] gives a sequent calculus formulation of **KD45** by adjoining to **LCPL** the following rule for $\Box$:

$$[\Box] \quad \Box\Delta_1, \Delta_2 \rightarrow \Box\Gamma_1, \Gamma_2 \vdash \Box\Delta_1, \Box\Delta_2 \rightarrow \Box\Gamma_1, \Box\Gamma_2,$$

where Γ_2 contains at most one formula. If in addition Γ_1 and Γ_2 are required to be non-empty, this results in a sequent system for **K45**.

Among the most important modal logics are the almost ubiquitous systems **S4** and **S5**. Standard sequent systems for the axiomatic calculi **S4** (= **KT4**) and **S5** (= **KT5** = **KT4B**) were studied by Ohnishi and Matsumoto [1957]. They considered the following schematic sequent rules for $\Box$ and $\Diamond$:

$$
\begin{array}{ll}
(\rightarrow \Box)_0 & \Box\Delta \rightarrow \Box\Gamma, A \vdash \Box\Delta \rightarrow \Box\Gamma, \Box A; \\
(\Box \rightarrow)_0 & \Delta, A \rightarrow \Gamma \vdash \Delta, \Box A \rightarrow \Gamma; \\
(\rightarrow \Diamond)_0 & \Delta \rightarrow \Gamma, A \vdash \Delta \rightarrow \Gamma, \Diamond A; \\
(\Diamond \rightarrow)_0 & \Diamond\Gamma, A \rightarrow \Diamond\Delta \vdash \Diamond\Gamma, \Diamond A \rightarrow \Diamond\Delta;
\end{array}
$$

where $\Diamond\Delta = \{\Diamond A \mid A \in \Delta\}$. If either the rules $(\rightarrow \Box)_0$ and $(\Box \rightarrow)_0$ or the rules $(\rightarrow \Diamond)_0$ and $(\Diamond \rightarrow)_0$ are adjoined to **LCPL**, then the result is a sequent calculus **LS5*** for **S5**. If Γ is empty in $(\rightarrow \Box)_0$ or $(\Diamond \rightarrow)_0$, this yields a sequent calculus **LS4** for **S4**. Several other modal logics can be obtained by imposing suitable constraints on the structures exhibited in $(\rightarrow \Box)_0$ and $(\Diamond \rightarrow)_0$, respectively. Ohnishi and Matsumoto show that if $(\rightarrow \Box)_0$ and $(\Diamond \rightarrow)_0$ are replaced by $(\rightarrow \Box)_1$ and

$$(\Diamond \rightarrow)_1 \quad A \rightarrow \Gamma \vdash \Diamond A \rightarrow \Diamond\Gamma,$$

one obtains a Gentzen-system **LKT** for **KT** (= **T**). Kripke [1963] noted that the equivalences between $\Box A$ and $\neg\Diamond\neg A$ and between $\Diamond A$ and $\neg\Box\neg A$ cannot be proved by means of Ohnishi's and Matsumoto's rules. In the case of **S4**, Kripke suggested remedying this by using sequent rules which exhibit both $\Box$ and $\Diamond$, namely in addition to $(\Box \rightarrow)_0$ and $(\rightarrow \Diamond)_0$ the rules

$$
\begin{array}{lll}
 & (\rightarrow \Box)' & \Box\Gamma \rightarrow A, \Diamond\Delta \vdash \Box\Gamma \rightarrow \Box A, \Diamond\Delta \\
\text{and} & (\Diamond \rightarrow)' & A, \Box\Gamma \rightarrow \Diamond\Delta \vdash \Diamond A, \Box\Gamma \rightarrow \Diamond\Delta.
\end{array}
$$

Such rules fail to give a separate account of the inferential behaviour of $\Box$ and $\Diamond$, since only the combined use of these operations is specified. Another problem with Ohnishi's and Matsumoto's sequent rules for **S5** is that the cut-rule

$$\Delta \rightarrow \Sigma, A; \quad \Gamma, A \rightarrow \Theta \vdash \Gamma, \Delta \rightarrow \Sigma, \Theta$$

cannot be eliminated: the system without cut allows proving less formulas than the full system containing cut. Ohnishi and Matsumoto [1959] give the following counter-example to cut-elimination:

$$
\dfrac{\dfrac{\dfrac{\Box p \rightarrow \Box p}{\emptyset \rightarrow \neg\Box p, \Box p}}{\emptyset \rightarrow \Box\neg\Box p, \Box p} \qquad \dfrac{p \rightarrow p}{\Box p \rightarrow p}}{\emptyset \rightarrow \Box\neg\Box p, p}
$$

A solution to the problem of defining a cut-free ordinary Gentzen system for **S5** has been given in [Braüner, 2000].[2] The logic **S5** can be faithfully

[2]Another, perhaps less convincing solution has been presented by Ohnishi [1982]. Define the degree $\mathsf{deg}(A)$ of a modal formula in the language with $\Box$ primitive as follows:

embedded into monadic predicate logic, the first-order logic of unary predicates, under a translation $\mathtt{t}$ employing a single individual variable x, see for instance [Mints, 1992]. The translation $\mathtt{t}$ assigns to every propositional variable p an atomic formula $P(x)$, and for compound formulas it is defined as follows:

$$\begin{array}{rcl} \mathtt{t}(\boldsymbol{t}) & = & \boldsymbol{t}, \\ \mathtt{t}(\neg A) & = & \neg \mathtt{t}(A), \\ \mathtt{t}(A \sharp B) & = & \mathtt{t}(A) \sharp \mathtt{t}(B), \text{ for } \sharp \in \{\supset, \wedge, \vee\}, \\ \mathtt{t}(\Box A) & = & \forall x \mathtt{t}(A), \\ \mathtt{t}(\Diamond A) & = & \exists x \mathtt{t}(A). \end{array}$$

THEOREM 1. *A modal formula A is provable in* $\boldsymbol{S5}$ *if and only if* $\mathtt{t}(A)$ *is provable in monadic predicate logic.*

The familiar cut-free sequent calculus for monadic predicate logic can serve as a starting point for defining a cut-free ordinary sequent system for **S5** with side-conditions on the introduction rules for $\Box$ on the right and $\Diamond$ on the left of the sequent arrow. The side conditions are simple, though their precise formulation requires some terminology that will be useful also in other contexts. An inference *inf* is a pair (Δ, s), where Δ is a set of sequents (the premises of *inf*) and s is a single sequent (the conclusion of *inf*). A rule of inference R is a set of inferences. If *inf* $\in R$, then *inf* is said to be an instantiation of R. The rule R is an axiomatic rule, if $\Delta = \emptyset$ for every $(\Delta, s) \in R$. We assume that inference rules are stated by using variables for structures (in the present case finite sets of formulas) and formulas. Every structure occurrence in an inference *inf* (a sequent s) is called a constituent of *inf* (s). The *parameters* of *inf* $\in R$ are those constituents which occur as substructures of structures assigned to structure variables in the statement of R. Constituents of *inf* are defined as *congruent* in *inf* if and only if (iff) they are occupying similar positions in occurrences of structures assigned to the same structure variable, in the present case iff they belong to a set assigned to the same set variable.

DEFINITION 2. Two formula occurrences are immediately connected in a proof Π iff Π contains an inference *inf* such that one of the following

1. $\mathsf{deg}(p) = 0$, for every propositional variable p;
2. $\mathsf{deg}(\neg A) = \mathsf{deg}(A)$;
3. $\mathsf{deg}(A \wedge B) = \max(\mathsf{deg}(A), \mathsf{deg}(B))$;
4. $\mathsf{deg}(\Box A) = \mathsf{deg}(A) + 1$.

Ohnishi adds to $(\Box \rightarrow)_0$ and $(\rightarrow \Box)_0$ two further rules that deviate considerably from familiar introduction schemata:

$$\Gamma, A^*, \Delta \rightarrow \Sigma \vdash \Gamma, A, \Delta \rightarrow \Sigma \quad \text{and} \quad \Gamma \rightarrow \Delta, A^*, \Sigma \vdash \Gamma \rightarrow \Delta, A, \Sigma,$$

where the formula A^* is defined in such a way that (i) A and A^* are equivalent in **S5** and (ii) $\mathsf{deg}(A^*) \leq 1$.

conditions holds:

1. both occurrences are non-parametric, one in the conclusion and the other in a premise of *inf*;
2. *inf* belongs to an axiomatic sequent rule and both occurrences are non-parametric in *inf*;
3. *inf* $\in$ cut and both occurrences are non-parametric in *inf*;
4. the occurrences are parametric and congruent in *inf*.

A list of formula occurrences $A_1, \ldots, A_n$ in a proof Π is called a connection between A_1 and A_n in Π iff for every $i \in \{1, \ldots, n–1\}$, the occurrences A_i and A_{i+1} are immediately connected in Π. A formula is said to be modally closed if every propositional variable in the formula occurs in the scope of an occurrence $\Diamond$ or $\Box$.

DEFINITION 3. Two formula occurrences in a proof Π are said to be dependent on each other in Π iff there exists a connection between these occurrences that does not contain any modally closed formula.

The sequent system **LS5** extends **LCPL** by $(\Box \to)_0$, $(\to \Diamond)_0$ and the rules:

$$\begin{array}{lll} & (\to \Box)'' & \Gamma \to \Delta, A \vdash \Gamma \to \Delta, \Box A \\ \text{and} & (\Diamond \to)'' & \Gamma, A \to \Delta \vdash \Gamma, \Diamond A \to \Delta, \end{array}$$

where applications of $(\to \Box)''$ and $(\Diamond \to)''$ in a proof Π must be such that in Π none of the formula occurrences in Γ and Δ depends on the displayed occurrence of A. A cut-free proof of the notorious sequent $\emptyset \to \Box\neg\Box p, p$ is then easily available (as it is also in Ohnishi's [1982] calculus):

$$\dfrac{\dfrac{\dfrac{p \to p}{\Box p \to p}}{\emptyset \to \neg\Box p, p}}{\emptyset \to \Box\neg\Box p, p}$$

THEOREM 4. ([Braüner, 2000]) *A sequent* $\Delta \to \Gamma$ *is provable in* ***LS*5** *iff* $\bigwedge \Delta \supset \bigvee \Gamma$ *is provable in* ***S*5**.

Avron [1984] (see also [Shimura, 1991]) presents a sequent calculus **LS4Grz** for **S4Grz** (= **KGrz**). He replaces the rule $(\to \Box)_0$ in Ohnishi and Matsumoto's sequent calculus for **S4** by the rule

$$(\to \Box)_4 \quad \Box(A \supset \Box A), \Box\Delta \to A \vdash \Box\Delta \to \Box A$$

exhibiting both $\Box$ and $\supset$. In [Takano, 1992], Takano defines sequent calculi **LKB**, **LKTB**, **LKDB**, and **LK4B** for **KB**, **KTB** (= **B**), **KDB**, and **K4B**.

The systems **LKB** and **LK4B** are obtained from **LCPL** by including the rules

$$
\begin{array}{lll}
 & (\rightarrow \Box)_B & \Gamma \rightarrow \Box\Theta, A \vdash \Box\Gamma \rightarrow \Theta, \Box A \\
\text{and} & (\rightarrow \Box)_{4BE} & \Gamma, \Box\Gamma \rightarrow \Box\Theta, \Box\Delta, A \vdash \Box\Gamma \rightarrow \Box\Theta, \Delta, \Box A
\end{array}
$$

respectively. **LKTB** and **LKDB** result from **LKB** by adjoining $(\Box \rightarrow)_0$ and

$$(\Box \rightarrow)_D \quad \Gamma \rightarrow \Box\Delta \vdash \Box\Gamma \rightarrow \Delta$$

respectively. Standard sequent systems for several other modal logics can be found in [Goré, 1992] and [Zeman, 1973]. The sequent calculus for **S4.3** $(= \mathbf{S4} + \Box(\Box A \supset B) \vee \Box(\Box B \supset A))$ in [Zeman, 1973] results from **LS4** by the addition of the *axiomatic sequent*

$$\Box(A \vee \Box B), \Box(\Box A \vee B) \rightarrow \Box A, \Box B.$$

Shimura [1991] obtains a cut-free sequent system **LS4.3** by adding to **LCPL** the rules $(\Box \rightarrow)_0$ and

$$(\rightarrow \Box)_5 \quad \Box\Gamma \rightarrow (\Box\Delta) \setminus \{\Box A_1\} \ldots \Box\Gamma \rightarrow (\Box\Delta) \setminus \{\Box A_n\} \vdash \Box\Gamma \rightarrow \Box\Delta,$$

where $\Delta = \{A_1, \ldots, A_n\}$ and $\setminus$ is set-theoretic difference.

1.2 Ordinary Gentzen systems for normal temporal logics

The syntax of temporal propositional logic is given by:

$$
\begin{array}{l}
A ::= p \mid \boldsymbol{t} \mid \boldsymbol{f} \mid \neg A \mid A \wedge B \mid A \vee B \mid A \supset B \mid A \equiv \\
\quad B \mid \langle P \rangle A \mid [P]A \mid \langle F \rangle A \mid [F]A.
\end{array}
$$

Also a number of normal temporal propositional logics have been presented as ordinary sequent calculi. Nishimura [1980], for example, defines sequent systems **LKt** and **LK4t** for the minimal normal temporal logic **Kt** and the tense-logical counterpart **K4t** of **K4**. The sequent calculus **LKt** comprises the following introduction rules for forward-looking necessity $[F]$ (“always in the future”) and backward-looking necessity $[P]$ (“always in the past”):[3]

$$
\begin{array}{ll}
(\rightarrow [F]) & \Gamma \rightarrow A, [P]\Delta \vdash [F]\Gamma \rightarrow [F]A, \Delta; \\
(\rightarrow [P]) & \Gamma \rightarrow A, [F]\Delta \vdash [P]\Gamma \rightarrow [P]A, \Delta,
\end{array}
$$

where $[F]\Delta = \{[F]A \mid A \in \Delta\}$ and $[P]\Delta = \{[P]A \mid A \in \Delta\}$. In **K4t**, these rules are replaced by the following pair of rules:

$$
\begin{array}{ll}
(\rightarrow [F])_4 & [F]\Gamma, \Gamma \rightarrow A, [P]\Delta, [P]\Sigma \vdash [F]\Gamma \rightarrow [F]A, \Delta, [P]\Sigma; \\
(\rightarrow [P])_4 & [P]\Gamma, \Gamma \rightarrow A, [F]\Delta, [F]\Sigma \vdash [P]\Gamma \rightarrow [P]A, \Delta, [F]\Sigma.
\end{array}
$$

[3] Nishimura allows infinite sets in antecedent and succedent position. It is proved, however, that if a sequent $\Gamma \rightarrow \Delta$ is provable, then there are finite sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that the sequent $\Gamma' \rightarrow \Delta'$ is provable.

In both systems, $\langle P \rangle$ ("sometimes in the past") and $\langle F \rangle$ ("sometimes in the future") are treaded not as primitive but as defined by $\langle P \rangle A := \neg[P]\neg A$ and $\langle F \rangle A := \neg[F]\neg A$. Note also that this approach gives completely parallel rules for $[F]$ and $[P]$ and that these rules do not exploit the interrelation between the backward and the forward-looking modalities, that shows up, for instance, in the provability of $A \supset [F]\langle P \rangle A$ and $A \supset [P]\langle F \rangle A$.

In summary, it may be said that many normal modal and temporal logics are presentable as ordinary Gentzen calculi, and that in some cases suitable constraints on the structures exhibited in the statement of the sequent rules for the modal operators allow for a number of variations. However, no uniform way of presenting only the most important normal modal and temporal propositional logics as ordinary Gentzen calculi is known. Further, the standard approach fails to be *modular*: in general it is not the case that a single axiom schema is captured by a single sequent rule (or a finite set of such rules). In the following section a more philosophical critique of ordinary Gentzen systems is advanced.

1.3 Introduction schemata and the meaning of the logical operations

The philosophical (and methodological) problems with applying the notion of a Gentzen sequent to modal logics have to do with the idea of *defining* the logical operations by means of introduction schemata (together with structural assumptions about derivability formulated in terms of structural rules). This 'anti-realistic' conception of the meaning of the logical operations is often traced back to a certain passage on natural deduction from Gentzen's *Investigations into Logical Deduction* [Gentzen, 1934, p. 80]:

> [I]ntroductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.

To qualify as a definition of a logical operation, an introduction schema must satisfy certain adequacy criteria. Such conditions are discussed, for instance, by Hacking [1994]. Following Hacking, if introduction rules are to be regarded as defining logical operations, these rules must be such that the structural rules monotonicity (also called weakening, thinning, or dilution), reflexivity, and cut can be eliminated. Hacking claims that

> [i]t is not provability of cut-elimination that excludes modal logic, but dilution-elimination .... The serious modal logics such as **T**, **S4** and **S5** have cut-free sequent-calculus formalizations, but the rules place restrictions on side formulas. Gentzen's rules for sentential connections are all 'local' in that they concern

> only the components from which the principal formula is built up, and place no restrictions on the side formulas. Gentzen's own first-order rules, though not strictly local, are equivalent to local ones. That is why dilution-elimination goes through for first-order logic but not for modal logics ([Hacking, 1994, p. 24]).

By dilution-elimination Hacking means that the monotonicity rules

$$\Delta \to \Gamma \vdash \Delta, A \to \Gamma, \qquad \Delta \to \Gamma \vdash \Delta \to \Gamma, A$$

may be replaced by atomic thinning rules

$$\Delta \to \Gamma \vdash \Delta, p \to \Gamma, \qquad \Delta \to \Gamma \vdash \Delta \to \Gamma, p.$$

without changing the set of provable sequents. Similarly, reflexivity-elimination amounts to replaceability of $\vdash A \to A$ by $\vdash p \to p$. The term "cut-elimination" is reserved for something stronger than replaceability of cut by the atomic cut-rule

$$\Delta \to \Sigma, p; \quad \Gamma, p \to \Theta \vdash \Gamma, \Delta \to \Sigma, \Theta.$$

A cut-elimination proof shows the admissibility of cut: the rule has no effect on the set of provable sequents.

The introduction rules for $\Box$ in **LS4** prevent dilution-elimination. Obviously, the sequent $\Box B, \Box A \to \Box A$, for example, cannot be proved using only these rules and atomic thinning. A problem with the requirement of dilution-elimination is the weak status monotonicity has acquired as a defining characteristic of logical deduction. In view of the substantial work on relevance logic, many other substructural logics, and a plethora of non-monotonic reasoning formalisms extending a monotonic base system, monotonicity of inference is not generally viewed as a touchstone of logicality anymore. Moreover, also reflexivity and cut have been questioned. Unrestricted transitivity of deduction as expressed by the cut-rule does not hold, for instance, in Tennant's intuitionistic relevant logic [1994], and both reflexivity and cut fail to be validated by Update-to-Test semantic consequence as defined in Dynamic Logic, see [van Benthem, 1996]. Reflexivity-elimination and cut-elimination are, however, important. According to Belnap [1982, p. 383], the provability of $A \to A$ constitutes

> half of what is required to show that the "meaning" of formulas ... is not context-sensitive, but that instead formulas "mean the same" in both antecedent and consequent position. (The [Cut] Elimination Theorem ... is the other half of what is required for this purpose).

A similar remark can be found in [Girard, 1989, p. 31]. Cut-elimination is indispensable, because it amounts to the familiar non-creativity requirement for definitions (see, for instance, [Hacking, 1994], [von Kutschera, 1968]). If one adds introduction rules for a (finitary) operation f to a sequent calculus, this addition ought to be conservative, so that in the extended formalism, every proof of an f-free formula A is convertible into a proof of A without any application of an introduction rule for f.

There are other reasons why the eliminability of cut is a desirable property. Usually, cut-elimination implies the subformula property: every cut-free proof of a sequent s contains only subformulas of formulas in s. In sequent calculi for decidable logics, the subformula property can often be used to give a syntactic proof of decidability. According to Sambin and Valentini [1982, p. 316], it

> is usually not difficult to choose suitable [sequent] rules for each modal logic if one is content with completeness of rules. The real problem however is to find a set of rules also satisfying the subformula-property.

The sequent calculi for **S5** in [Mints, 1970], [Sato, 1977], and [Sato, 1980], although admitting cut-elimination, do not have the subformula property. In a sequent calculus with an enriched structural language, the subformula property need not be accompanied by a *substructure property*. In such systems the subformula property for the logical vocabulary need neither imply nor be of direct use for syntactic decidability proofs. Avron [1996, p. 2] requires of a decent sequent calculus simplicity of the structures employed and a 'real' subformula property. But even without the substructure property, the subformula property may be useful, for instance in proving conservative extension results, see also Section 3.8.

It is well-known that cut-elimination itself does not guarantee efficient proof search (see [D'Agostino and Mondadori, 1994], [Boolos, 1984]), so that it may be attractive to work with an analytic, subformula property preserving cut-rule, if possible. An application of cut

$$\Delta \to \Sigma, A \quad \Gamma, A \to \Theta \vdash \Gamma, \Delta \to \Sigma, \Theta$$

is *analytic* (see [Smullyan, 1968]), if the cut-formula A is a subformula of some formula in the conclusion sequent $\Gamma, \Delta \to \Sigma, \Theta$. Let $\mathrm{Sub}(\Delta)$ denote the set of all subformulas of formulas in Δ. Applications of the sequent rules

$$\begin{array}{lll} & (\to \Box)_B & \Gamma \to \Box\Theta, A \vdash \Box\Gamma \to \Theta, \Box A \\ & (\to \Box)_{4BE} & \Gamma, \Box\Gamma \to \Box\Theta, \Box\Delta, A \vdash \Box\Gamma \to \Box\Theta, \Delta, \Box A \\ \text{and} & (\Box \to)_D & \Gamma \to \Box\Delta \vdash \Box\Gamma \to \Delta \end{array}$$

may be said to be analytic if $\Box\Theta \subseteq \mathrm{Sub}(\Gamma \cup \{A\})$, $\Box\Delta \subseteq \mathrm{Sub}(\Box\Gamma \cup \Box\Theta \cup \{A\})$, and $\Box\Delta \subseteq \mathrm{Sub}(\Gamma)$, respectively. Takano [1992] shows that the cut-rule in

LS5*, **LKB**, **LKTB**, **LKDB**, **LK4B**, **LKt** and **LK4t** can be replaced by the analytic cut-rule: every proof in these sequent calculi can be transformed into a proof of the same sequent such that every application of cut (and, moreover, every application of the rules $(\rightarrow \Box)_B$, $(\rightarrow \Box)_{4BE}$, and $(\rightarrow \Box)_D$) in this proof is analytic.

Although admissibility of analytic cut is a welcome property, in general, unrestricted cut-elimination is to be preferred over elimination of analytic cut. Admissibility of cut has great conceptual significance. The cut-rule justifies certain substitutions of data; in particular it justifies the use of previously proved formulas. Moreover, if the cut-rule is assumed, the non-creativity requirement for definitions implies that cut must be eliminable.

There are other nice properties of introduction schemata as definitions in addition to enabling cut-elimination and reflexivity-elimination. The assignment of meaning to the logical operations should, for instance, be non-holistic, and hence sequent rules like the above $(\rightarrow \Box)'$ and $(\Diamond \rightarrow)'$ are unsuitable. If (the statement of) an introduction rule for a logical operation f exhibits no connective other than f, the rule is called *separated*, see [Zucker and Tragesser, 1978]. An even stronger condition is *segregation*, requiring that the antecedent (succedent) of the conclusion sequent in a left (right) introduction rule must not exhibit any structure operation. Segregation has been suggested (although not under this name) by Belnap [1996] who explains that

> [t]he nub is this. If a rule for $\supset$ only shows how $A \supset B$ behaves *in context*, then that rule is not *merely* explaining the meaning of $\supset$. It is also and inextricably explaining the meaning of the context. Suppose we give sufficient conditions for
>
> $$A \supset B, \Delta \rightarrow \Gamma$$
>
> in part by the rule
>
> $$\frac{\Delta \rightarrow A \quad B \rightarrow \Gamma}{A \supset B, \Delta \rightarrow \Gamma}$$
>
> Then we are not explaining $A \supset B$ alone. We are simultaneously involving the comma not just in our explicans (that would surely be all right), but in our explicandum. We are explaining two things at once. There is no way around this. You do not have to take it as a defect, but it is a fact. ... If you are a ‘holist’, probably you will not care; but then there is not much about which holists much care. [Belnap, 1996, p. 81 f.] (notation adjusted)

Moreover, the rules for f may be required to be *weakly symmetrical* in the sense that every rule should either belong to a set of rules $(f \rightarrow)$ which introduce f on the left side of $\rightarrow$ in the conclusion sequent or to a set of rules $(\rightarrow f)$ which introduce f on the right side of $\rightarrow$ in the conclusion sequent. The introduction rules for f are called *symmetrical*, if they are weakly symmetrical and both $(\rightarrow f)$ and $(f \rightarrow)$ are non-empty. The sequent rules for f are called *weakly explicit*, if the rules $(\rightarrow f)$ and $(f \rightarrow)$ exhibit f in their conclusion sequents only, and they are called *explicit*, if in addition to being weakly explicit, the rules in $(\rightarrow f)$ and $(f \rightarrow)$ exhibit only one occurrence of f on the right, respectively the left side of $\rightarrow$. Separation, symmetry, and explicitness of the rules imply that in a sequent calculus for a given logic Λ, every connective that is explicitly definable in Λ also has separate, symmetrical, and explicit introduction rules. These rules can be found by decomposition of the defined connective, if it is assumed that the deductive role of $f(A_1, \ldots, A_n)$ only depends on the deductive relationships between $A_1, \ldots, A_n$. It is therefore desirable to have introduction rules for $\Box$, $\Diamond$, $\langle P \rangle$, $[P]$, $\langle F \rangle$ and $[F]$ as primitive operations, so that the familiar mutual definitions are derivable.

A further desirable property, reminiscent of implicit definability in predicate logic, is the unique characterization of f by its introduction rules. Suppose that Λ is a logical system with a syntactic presentation S in which f occurs. Let S^* be the result of rewriting f everywhere in S as f^*, and let $\Lambda\Lambda^*$ be the system presented by the union SS^* of S and S^* in the combined language with both f and f^*. Let A_f denote a formula (in this language) that contains a certain occurrence of f, and let A_{f^*} denote the result of replacing this occurrence of f in A by f^*. The connectives f and f^* are said to be *uniquely characterized* in $\Lambda\Lambda^*$ iff for every formula A_f in the language of $\Lambda\Lambda^*$, A_f is provable in SS^* iff A_{f^*} is provable in SS^*. Došen [1985] has proved that unique characterization is a non-trivial property and that the connectives in his higher-level systems S4p/D and S5p/D for **S4** and **S5**, respectively, are uniquely characterized.

As we have seen, the standard sequent-style proof-theory for normal modal and temporal logic fails to be modular. The idea that modularity can be achieved by systematically varying structural features of the derivability relation while keeping the introduction rules for the logical operations untouched can be traced back to Gentzen [1934] and has been referred to as Došen's Principle in [Wansing, 1994]. In [Došen, 1988, p. 352], Došen suggests that "the rules for the logical operations are never changed: all changes are made in the structural rules." This methodology is adopted, for example in Došen's [1985] higher-level sequent systems for **S4** and **S5**, Blamey and Humberstone's [1991] higher-arity sequent calculi for certain extensions of **K**, Nishimura's [1980] higher-arity sequent systems for **Kt** and **K4t**, and the presentation of normal modal and temporal logics as cut-free display sequent calculi.

Another methodological aspect is generality. Is there a type of sequent system that allows not only a uniform treatment of the most important modal and temporal logics but also a treatment of substructural logics, other non-classical logics and systems combining operations from different families of logics and that, moreover, is rich enough to suggest important, hitherto unexplored logics? The framework of display logic to be presented in the next section has been devised explicitly as an instrument for combining logics (see [Belnap, 1982]), and has been suggested, for example, as a tool for defining subsystems of classical predicate logic (see [Wansing, 1999]). In addition to generality, a 'real' subformula property, and Došen's principle, Avron [1996] requires of a good sequent calculus framework also *semantics independence*. The framework should not be so closely tied to a particular semantics that one can more or less read off the semantic structures in question. Moreover, the proof systems instantiating the framework should lead to a better understanding of the respective logics and the differences between them.

Note that each of the ordinary sequent systems presented in the present section fails to satisfy some of the more philosophical requirements mentioned. The same holds true for the ordinary sequent systems for various non-normal, classical modal logics investigated in [Lavendhomme and Lucas, 2000]. There are thus not only technical but also methodological and philosophical reasons for investigating generalizations of the notion of a Gentzen sequent.

2 GENERALIZED SEQUENT SYSTEMS

In this section the application of a number of generalizations of the ordinary notion of sequent to normal modal propositional and temporal logics is surveyed.

2.1 Higher-level sequent systems

Došen [1985] has developed certain non-standard sequent systems for **S4** and **S5**. In these Gentzen-style systems one is dealing with sequents of arbitrary finite level. Sequents of level 1 are like ordinary sequents, whereas sequents of level $n+1$ $(0 < n < \omega)$ have finite sets of sequents of level n on both sides of the sequent arrow. The main sequent arrow in a sequent of level n carries the superscript ^{n}, and $\emptyset$ is regarded as a set of any finite level. The rules for logical operations are presented as *double-line* rules. A double-line rule

$$\frac{s_1, \ldots, s_n}{\overline{\quad s_0 \quad}}$$

involving sequents $s_0, \ldots, s_n$, denotes the rules

$$\frac{s_1, \ldots, s_n}{s_0}, \ \frac{s_0}{s_1}, \ldots, \frac{s_0}{s_n}.$$

Došen gives the following double-line sequent rules for $\Box$ and $\Diamond$:

$$\frac{X + \{\emptyset \rightarrow^1 \{A\}\} \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_4\}}{X_1 \rightarrow^2 X_2 + \{X_3 + \{\Box A\} \rightarrow^1 X_4\}}$$

$$\frac{X_1 + \{\{A\} \rightarrow^1 \emptyset\} \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_4\}}{X_1 \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_4 + \{\Diamond A\}\}},$$

where $+$ refers to the union of disjoint sets. If these rules are added to Došen's higher-level sequent calculus Cp/D for **CPL**, this results in the sequent system S5p/D for **S5**. The sequent calculus S4p/D for **S4** is then obtained by imposing a structural restriction on the monotonicity rule of level 2:

$$X \rightarrow^2 Y \vdash X \cup Z_1 \rightarrow^2 Y \cup Z_2.$$

The restriction is this: if $Y = \emptyset$, then Z_2 must be a singleton or empty; if $Y \neq \emptyset$, then Z_2 must be empty. If the same restriction is applied to monotonicity of level 1 in Cp/D, then this gives a higher-level sequent system for intuitionistic propositional logic **IPL**.

Note that $\Diamond$ and $\Box$ are interdefinable in S4p/D and S5p/D. The double-line rules for $\Box$ and $\Diamond$, however, do not satisfy weak symmetry and weak explicitness, but the upward directions of these rules can be replaced by:

$$\emptyset \rightarrow^1 \{A\} \vdash \emptyset \rightarrow^1 \{\Box A\} \quad \text{and} \quad \{A\} \rightarrow^1 \emptyset \vdash \{\Diamond A\} \rightarrow^1 \emptyset.$$

Whereas cut can be eliminated at levels 1 and 2, cut of all levels fails to be eliminable [Došen, 1985, Lemma 1]. Moreover, in Došen's higher-level framework it is not clear how restrictions similar to the one used to obtain S4p/D from S5p/D would allow to capture further axiomatic systems of normal modal propositional logic.

2.2 Higher-dimensional sequent systems

A 'higher-dimensional' proof theory for modal logics has been developed by Masini [1992; 1996]. This approach is based on the notion of a 2-*sequent*. In order to define this notion, various preparatory definitions are useful. Any finite sequence of modal formulas is called a 1-sequence. The empty 1-sequence is denoted by ϵ. A 2-sequence is an infinite 'vertical' succession of 1 sequences, $\Gamma = \{\alpha_i\}_{0<i<\omega}$ such that $\exists j \geq 1$, $\forall k \geq j : \alpha_k = \epsilon$. For each i, α_i is said to be at level i. The depth of Γ ($\natural\Gamma$) is defined as $min\{i \mid i \geq 0,\ \forall k > i : \alpha_k = \epsilon\}$. A *2-sequent* is an expression $\Gamma \rightarrow \Delta$, where Γ and Δ

are 2-sequences. The depth of $\Gamma \to \Delta$ ($\natural(\Gamma \to \Delta)$) is defined as $max(\natural\Gamma, \natural\Delta)$. If $\Gamma \to \Delta$ is a 2-sequent and A an occurrence of a modal formula in $\Gamma \to \Delta$, then A is said to be maximal in $\Gamma \to \Delta$, if A is at level k in Γ or in Δ and $k = \natural(\Gamma \to \Delta)$. A is the maximum in $\Gamma \to \Delta$, if A is the unique maximal formula in $\Gamma \to \Delta$. The sequent rules for $\Box$ are based on the idea of "internalizing the level structure of 2-sequents" [Masini, 1992, p. 231]:

$$(\Box \to) \quad \frac{\begin{array}{l}\Gamma \\ \alpha \\ \beta, A \\ \Gamma'\end{array} \to \Delta}{\begin{array}{l}\Gamma \\ \alpha, \Box A \\ \beta \\ \Gamma'\end{array} \to \Delta} \qquad (\to \Box) \quad \frac{\Gamma \to \begin{array}{l}\Delta \\ \mu \\ A\end{array}}{\Gamma \to \begin{array}{l}\Delta \\ \mu, \Box A\end{array}}$$

$$(\Diamond \to) \quad \frac{\begin{array}{l}\Gamma \\ \alpha \\ A\end{array} \to \Delta}{\begin{array}{l}\Gamma \\ \alpha, \Diamond A\end{array} \to \Delta} \qquad (\to \Diamond) \quad \frac{\Gamma \to \begin{array}{l}\Delta \\ \mu \\ \pi, A \\ \Delta'\end{array}}{\Gamma \to \begin{array}{l}\Delta \\ \mu, \Diamond A \\ \pi \\ \Delta'\end{array}}$$

where α, β, π, and μ denote arbitrary 1-sequences, and A must be the maximum of the premise 2-sequent in $(\to \Box)$ and $(\Diamond \to)$. According to Masini, these introduction rules give rise to a "general basic proof theory of modalities" [Masini, 1992, p. 232]. If added to a 2-sequent calculus for **CPL**, the above rules result, however, in a sequent calculus for **KD** instead of the basic system **K**. This sequent system for **KD** admits cut-elimination, $\Box$ and $\Diamond$ are interdefinable, and the introduction rules are separate, symmetrical, and explicit, but no indication is given of how to present axiomatic extensions of **KD** as higher-dimensional sequent systems. Moreover, it is not clear how Masini's framework may be modified in order to obtain a 2-sequent calculus for **K**.

2.3 Higher-arity sequent systems

In search of generalizations of the standard Gentzen-style sequent format, it is a natural move to consider consequence relations with an arity greater than 2. It seems that the first higher-arity sequent calculus was formulated by Schröter [1955], see also [Gottwald, 1989]. This formalism is a natural generalization of Gentzen's sequent calculus for **CPL** to truth-functional

n-valued logic. The intended truth-functional reading of a Gentzen sequent $s = \Delta \to \Gamma$ is given by a translation σ of s into a formula:

$$\sigma(\Delta \to \Gamma) \;=\; \bigwedge \Delta \supset \bigvee \Gamma,$$

The sequent s thus is true under a given interpretation if either some formula in Δ is false, or some formula in Γ is true, and the two places of the sequent arrow correspond to the two truth-values of classical logic. In general, in n-valued logic (with $2 \leq n$) one obtains n-place sequents $s = \Delta_1; \Delta_2; \ldots; \Delta_n$, with the understanding that s is true under an interpretation if for every $i \leq n$, some formula in Δ_i has truth-value i; for a comprehensive treatment of sequent calculi for truth-functional many-valued logics see [Zach, 1993]. We shall here briefly review some relevant parts of the work of Blamey and Humberstone [1991], who investigate an application of three-place and, ultimately, four-place sequent arrows to normal modal logic. This approach is congenial to display logic with respect to a realization of the Došen-Principle insofar as Blamey and Humberstone emphasize that distinctions between various well-known normal modal logics can "be reflected at the purely structural level, if an appropriate notion of *sequent*" is adopted [Blamey and Humberstone, 1991, p. 763]. Let Γ, Δ, Θ, and Σ range over finite sets of formulas in the modal propositional language with $\Box$ as primitive. The four-place sequent

$$\Gamma \to^{\Theta}_{\Sigma} \Delta$$

has the following heuristic reading:

$$(\bigwedge \Gamma \wedge \bigwedge \Box\Sigma) \supset (\bigvee \Delta \vee \bigvee \Box\Theta).$$

This kind of sequent had independently been used by Sato [1977], where a cut-free sequent calculus for **S5** is presented containing a left introduction rule for $\Box$ that fails to be weakly explicit. Blamey's and Humberstone's introduction rules for $\Box$ are:

$$(\Box\downarrow)_0 \quad \vdash \emptyset \to^{\emptyset}_{A} \Box A \qquad\qquad (\Box\uparrow)_0 \quad \vdash \Box A \to^{A}_{\emptyset} \emptyset.$$

In order to obtain a sequent calculus for **K** the following structural rules are assumed:

$$\begin{array}{rl}
(R) & \vdash A \to^{\emptyset}_{\emptyset} A \qquad (\text{vertical } R) \quad \vdash \emptyset \to^{A}_{A} \emptyset \\
(M) & \Gamma \to^{\Theta}_{\Sigma} \Delta \vdash \Gamma, \Gamma' \to^{\Theta,\Theta'}_{\Sigma,\Sigma'} \Delta, \Delta' \\
(\text{undercut}) & \Sigma \to^{\emptyset}_{\emptyset} A \quad \Gamma \to^{\Theta}_{\Sigma',A} \Delta \vdash \Gamma \to^{\Theta}_{\Sigma,\Sigma'} \Delta \\
(T) & \Gamma, A \to^{\Theta}_{\Sigma} \Delta \quad \Gamma \to^{\Theta}_{\Sigma} A, \Delta \vdash \Gamma \to^{\Theta}_{\Sigma} \Delta \\
(\text{vertical } T) & \Gamma \to^{\Theta}_{\Sigma,A} \Delta \quad \Gamma \to^{\Theta,A}_{\Sigma} \Delta \vdash \Gamma \to^{\Theta}_{\Sigma} \Delta.
\end{array}$$

Against the background of these rules, the introduction rules $(\Box\downarrow)_0$ and $(\Box\uparrow)_0$ are interreplaceable with the following rules, respectively:

$$(\Box\downarrow) \qquad \Gamma, \Box A \rightarrow^{\Theta}_{\Sigma} \Delta \vdash \Gamma \rightarrow^{\Theta}_{\Sigma,A} \Delta$$
$$(\Box\uparrow) \qquad \Gamma \rightarrow^{\Theta}_{\Sigma,A} \Delta \vdash \Gamma, \Box A \rightarrow^{\Theta}_{\Sigma} \Delta.$$

The introduction rules for the Boolean operations are adaptations of the familiar rules to the higher-arity case. Here is a simple example of a derivation in this formalism (using some obvious notational simplifications):

$$\dfrac{\dfrac{\dfrac{A\wedge B \rightarrow A}{A \wedge B \rightarrow A, B} \quad \dfrac{\dfrac{\Box A, \Box B \rightarrow \Box A \wedge \Box B}{\Box A \rightarrow_{B} \Box A \wedge \Box B}(\Box\downarrow)}{\emptyset \rightarrow_{A,B} \Box A \wedge \Box B}(\Box\downarrow)}{\emptyset \rightarrow_{A\wedge B} \Box A \wedge \Box B}\text{(undercut) twice}}{\Box(A\wedge B) \rightarrow \Box A \wedge \Box B}(\Box\uparrow)$$

The axiom schemata D, T, 4, and B are captured by purely structural rules not exhibiting any logical operations:

$$\begin{array}{ll} D & \Sigma \rightarrow^{\emptyset}_{\emptyset} \emptyset \vdash \emptyset \rightarrow^{\emptyset}_{\Sigma} \emptyset \\ T & \vdash \emptyset \rightarrow^{\emptyset}_{A} A \\ 4 & \Sigma \rightarrow^{\Theta}_{\Sigma} A \quad \Gamma \rightarrow^{\Theta}_{\Sigma',A} \Delta \vdash \Gamma \rightarrow^{\Theta}_{\Sigma,\Sigma'} \Delta \\ B & \Sigma \rightarrow^{\Delta}_{\emptyset} A \quad \Gamma \rightarrow^{\Theta}_{\Sigma,A} \Delta \vdash \Gamma \rightarrow^{\Theta}_{\Sigma} \Delta. \end{array}$$

Since Blamey and Humberstone are primarily interested in semantical aspects of their sequent systems, they do not consider cut-elimination. Although their calculi satisfy Došen's Principle, it remains unclear whether their approach is fully modular for the most important systems of normal modal propositional logic. They do not present a structural equivalent of the 5-axiom schema, but rather treat **S5** as **KTB4**.

In [Nishimura, 1980], Nishimura uses six-place sequents

$$\Theta_1; \Gamma; \Theta_2 \rightarrow \Sigma_1; \Delta; \Sigma_2.$$

These higher-arity sequents can intuitively be read as follows:

$$(\bigwedge[P]\Theta_1 \wedge \bigwedge\Gamma \wedge \bigwedge[F]\Theta_2) \supset (\bigvee[P]\Sigma_1 \vee \bigvee\Delta \vee \bigvee[F]\Sigma_2).$$

Nishimura defines introduction rules for the tense logical operations $[F]$ and $[P]$, which are explicit in the sense of Section 1.3:

$$\begin{array}{ll} (\rightarrow [F])' & \dfrac{\Theta_1; \Gamma; \Theta_2 \rightarrow \Sigma_1; \Delta; A, \Sigma_2}{\Theta_1; \Gamma; \Theta_2 \rightarrow \Sigma_1; \Delta, [F]A; \Sigma_2} \\ ([F] \rightarrow)' & \dfrac{\Theta_1; \Gamma; A, \Theta_2 \rightarrow \Sigma_1; \Delta; \Sigma_2}{\Theta_1; \Gamma, [F]A; \Theta_2 \rightarrow \Sigma_1; \Delta; \Sigma_2} \\ (\rightarrow [P])' & \dfrac{\Theta_1; \Gamma; \Theta_2 \rightarrow \Sigma_1, A; \Delta; \Sigma_2}{\Theta_1; \Gamma; \Theta_2 \rightarrow \Sigma_1; [P]A, \Delta\Sigma_2} \\ ([P] \rightarrow)' & \dfrac{\Theta_1, A; \Gamma; \Theta_2 \rightarrow \Sigma_1; \Delta; \Sigma_2}{\Theta_1; [P]A, \Gamma; \Theta_2 \rightarrow \Sigma_1; \Delta; \Sigma_2} \end{array}$$

In accordance with the Došen Principle, these rules are held constant in sequent systems for **Kt** and **K4t**. The difference between these logics is accounted for by different structural rules, namely

$$\text{(r-trans)} \quad \frac{\emptyset; \Gamma; \emptyset \to \Delta; A; \emptyset}{\emptyset; \emptyset; \Gamma \to \emptyset; \Delta; A} \qquad \text{(l-trans)} \quad \frac{\emptyset; \Gamma; \emptyset \to \emptyset; A; \Delta}{\Gamma; \emptyset; \emptyset \to A; \Delta; \emptyset}$$

in the case of **Kt** and

$$\text{(r-trans)}_4 \quad \frac{\emptyset; \Gamma; \Gamma \to \Delta, \Sigma; A; \emptyset}{\emptyset; \emptyset; \Gamma \to \Sigma; \Delta; A} \qquad \text{(l-trans)}_4 \quad \frac{\Gamma; \Gamma; \emptyset \to \emptyset; A; \Delta, \Sigma}{\Gamma; \emptyset; \emptyset \to A; \Delta; \Sigma}$$

in the case of **K4t**. Nishimura observes that although in the introduction rules for $\langle F\rangle$ and $\langle P\rangle$ subformulas are preserved from premise sequent to conclusion sequent, cut-elimination fails to hold in the six-place sequent systems for **Kt** and **K4t**. There is, for instance, no cut-free proof of $; p; \to ; [F]\neg[P]\neg p;$.[4]

2.4 Multiple-sequent systems

Indrzejczak, in [1997; 1998], suggested non-standard sequent systems for certain extensions of the minimal regular modal logic **C** using three sequent arrows $\to$, $\Box\!\!\to$, and $\Diamond\!\!\to$. These sequent arrows denote binary relations between finite sets of S-formulas, where the set of S-formulas is defined as the union of the set of modal formulas and $\{-A \mid A \text{ is a modal formula}\}$. As before, we shall use A, B, C, ... to denote modal formulas. The symbol '$-$' is a unary structure connective that may not be nested, and the sequent arrows $\Box\!\!\to$ and $\Diamond\!\!\to$ are auxiliary in the sense that they fail to represent consequence relations, because (in general) neither $\vdash A \Box\!\!\to A$ nor $\vdash A \Diamond\!\!\to A$. The logics presented by such multiple-sequent systems are given by the set of provable sequents $\Delta \to \Gamma$. The intended meaning of a sequent is captured by a translation σ from sequents into ordinary sequents using a translation δ from S-formulas to modal formulas. For every modal formula A, $\delta(-A) := \neg A$ and $\delta(A) := A$. The translation σ is defined as follows:

$$\begin{aligned} \sigma(\Gamma \to \Delta) &= \bigwedge \delta(\Gamma) \to \bigvee \delta(\Delta) \\ \sigma(\Gamma \Box\!\!\to \Delta) &= \bigwedge \delta(\Gamma) \to \Box \bigvee \delta(\Delta) \\ \sigma(\Gamma \Diamond\!\!\to \Delta) &= \Diamond \bigwedge \delta(\Gamma) \to \bigvee \delta(\Delta) \end{aligned}$$

Here $\delta(\Gamma) := \{A \mid A \in \Gamma\} \cup \{\neg A \mid -A \in \Gamma\}$. For every modal formula A, A^* is defined as $-A$ and $-A^*$ as A. If Δ is a set of S-formulas, $\Delta^* :=$

[4]Note that Nishimura allows infinite sets in antecedent and succedent position. It is, however, shown that if a sequent $\Theta_1; \Gamma; \Theta_2 \to \Sigma_1; \Delta; \Sigma_2$ is provable, then there are finite sets $\Theta_i' \subseteq \Theta_i$, $\Sigma_i' \subseteq \Sigma_i$, $(i = 1, 2)$, $\Gamma' \subseteq \Gamma$, and $\Delta' \subseteq \Delta$ such that the sequent $\Theta_1'; \Gamma'; \Theta_2' \to \Sigma_1'; \Delta'; \Sigma_2'$ is provable.

$\{A \mid -A \in \Delta\} \cup \{-A \mid A \in \Delta\}$. Let $(\rightarrow)$ be any of $\rightarrow$, $\Box\!\!\rightarrow$, $\Diamond\!\!\rightarrow$. The following reflexivity and monotonicity rules are assumed:

$$\vdash A \rightarrow A; \quad \Delta\,(\rightarrow)\,\Gamma \vdash \Delta\,(\rightarrow)\,\Gamma, A; \quad \Delta\,(\rightarrow)\,\Gamma \vdash \Delta, A\,(\rightarrow)\,\Gamma.$$

Next, there are further structural rules called shifting rules:

$[\rightarrow^*]$	$A, \Delta \rightarrow \Gamma \vdash \Delta \rightarrow \Gamma, A^*$	$[^*\!\rightarrow]$	$\Delta \rightarrow \Gamma, A \vdash \Delta, A^* \rightarrow \Gamma$
[TR]	$\Delta \Box\!\!\rightarrow \Gamma \vdash \Gamma^* \Diamond\!\!\rightarrow \Delta^*$		$\Delta \Diamond\!\!\rightarrow \Gamma \vdash \Gamma^* \Box\!\!\rightarrow \Delta^*$

The introduction rules for $\wedge$, $\vee$, $\supset$ and $\neg$ are formulated for arbitrary sequent arrows. Whereas the rules for $\wedge$ and $\vee$ are versions of the familiar introduction rules, the rules for $\neg$ and $\supset$ can be formulated such that they make use of the structure connective $-$:

$$\Delta, -A\,(\rightarrow)\,\Gamma \vdash \Delta, \neg A\,(\rightarrow)\,\Gamma$$
$$\Delta\,(\rightarrow)\,\Gamma, -A \vdash \Delta\,(\rightarrow)\,\Gamma, \neg A$$
$$\Delta, -A\,(\rightarrow)\,\Gamma \quad \Sigma, B\,(\rightarrow)\,\Theta \vdash \Delta, \Sigma, A \supset B\,(\rightarrow)\,\Gamma, \Theta$$
$$\Delta\,(\rightarrow)\,\Gamma, -A, B \vdash \Delta\,(\rightarrow)\,\Gamma, A \supset B$$

The introduction rules for the modal operators are not formulated for arbitrary sequent arrows:

$[\Box\Box\!\!\rightarrow]$	$A \rightarrow \Delta \vdash \Box A \Box\!\!\rightarrow \Delta$	$[\rightarrow \Box]$	$\Delta \Box\!\!\rightarrow A \vdash \Delta \rightarrow \Box A$
$[\Diamond\Diamond\!\!\rightarrow]$	$A \Diamond\!\!\rightarrow \Delta \vdash \Diamond A \rightarrow \Delta$	$[\Diamond\!\!\rightarrow \Diamond]$	$\Delta \rightarrow A \vdash \Delta \Diamond\!\!\rightarrow \Diamond A$
$[\Diamond\!\!\rightarrow \Diamond]$	$-A, \Delta \Diamond\!\!\rightarrow \Gamma \vdash \Delta \Diamond\!\!\rightarrow \Gamma, \Diamond A$	$[\Box\Box\!\!\rightarrow]$	$\Delta \Box\!\!\rightarrow \Gamma, -A \vdash \Delta \Box A \Box\!\!\rightarrow \Gamma$

The above collection of sequent rules forms a multiple-sequent calculus **MC** for the system **C**. An axiomatization of **C** can be obtained by replacing the necessitation rule in the familiar axiomatization of **K** by the weaker rule

(RR) if $(A \wedge B) \supset C$ is provable, then so is $(\Box A \wedge \Box B) \supset \Box C$,

see [Chellas, 1980]. The necessitation rule and the modal axiom schemata D, T, and 4 can be captured in a modular fashion by pairs of sequent rules:

[nec]	$\Delta \rightarrow \emptyset \vdash \Delta \Diamond\!\!\rightarrow \emptyset$	$\emptyset \rightarrow \Delta \vdash \emptyset \Box\!\!\rightarrow \Delta$
$[D]$	$\Delta \Box\!\!\rightarrow \emptyset \vdash \Delta \rightarrow \emptyset$	$\emptyset \Diamond\!\!\rightarrow \Delta \vdash \emptyset \rightarrow \Delta$
$[T]$	$\Delta \Box\!\!\rightarrow \Gamma \vdash \Delta \rightarrow \Gamma$	$\Delta \Diamond\!\!\rightarrow \Gamma \vdash \Delta \rightarrow \Gamma$
$[4]$	$\Delta \rightarrow \Sigma \vdash \Delta \Box\!\!\rightarrow \Sigma$	$\Theta \rightarrow \Gamma \vdash \Theta \Diamond\!\!\rightarrow \Gamma$,

where in rule $[4]$, every S-formula in Δ has the shape $\Box A$ or $-\Diamond A$ and every S-formula in Γ has the shape $\Diamond A$ or $-\Box A$. All sequent systems obtained in this way satisfy a generalized subformula property: for every modal formula A, it holds that if A or $-A$ is used in a proof of $\Delta \rightarrow \Gamma$, then A is a subformula of $\Delta \cup \Gamma$ (where the notion of a subformula of an S-formula is defined in the obvious way). Indrzejczak does not investigate the admissibility of cut for $\to$ or the admissibility of cut for $\Box\!\!\to$ and $\Diamond\!\!\to$ in extensions of **CT** (where $\vdash A \Box\!\!\to A$ and $\vdash A \Diamond\!\!\to A$). Note that the introduction rules for the modal operators fail to be symmetrical, since there are no introduction rule for $\Box$ on the left and $\Diamond$ on the right of $\to$. Moreover, the side conditions on [*4*] are such that the status of this rule as a purely structural rule is doubtful. The multiple-sequent systems for extensions of **KB** make use of denumerably many sequent arrows $\xrightarrow{n}$ $(n \geq 0)$, where logics are defined by the provable sequents $\Delta \xrightarrow{0} \Gamma$. The introduction rules

$$\begin{array}{ll} A \xrightarrow{n} \Delta \vdash \Box A \xrightarrow{n+1} \Delta & \Delta \xrightarrow{n+1} A \vdash \Delta \xrightarrow{n} \Box A \\ A \xrightarrow{n+1} \Delta \vdash \Diamond A \xrightarrow{n} \Delta & \Delta \xrightarrow{n} A \vdash \Delta \xrightarrow{n+1} \Diamond A \end{array}$$

fail to introduce $\Box$ on the left and $\Diamond$ on the right of $\xrightarrow{0}$, so that also these rules are not symmetrical.

In Section 4.1, we shall point to a simple relation between Indrzejczak's multiple-sequent systems and higher-arity sequent systems for modal logics.

2.5 Hypersequents

Hypersequents were introduced into the literature by Pottinger [1983], and have later systematically been studied by Avron [1991; 1991a; 1996]. A *hypersequent* is a sequence

$$\Gamma_1 \to \Delta_1 \mid \Gamma_2 \to \Delta_2 \mid \ldots \mid \Gamma_n \to \Delta_n$$

of ordinary sequents (or, more generally, sequents in which Δ_i and Γ_i are sequences of formula occurrences) as their *components*. The symbol '|' in the statement of a hypersequent enriches the language of sequents and is intuitively to be read as disjunction. This expressive enhancement "makes it possible to introduce *new* types of structural rules, and ... to allow greater versatility in developing interesting logical systems" [Avron, 1996, p. 6]. In particular, a distinction may be drawn between internal and external versions of structural rules. The internal rules deal with formulas within a certain component, whereas the external rules deal with components within a hypersequent. Let G, H, H_1, H_2 etc. be schematic letters for possibly empty hypersequents. External monotonicity, for instance, can be contrasted with internal monotonicity:

$$H_1 \mid H_2 \vdash H_1 \mid G \mid H_2 \quad \text{vs.} \quad H_1 \mid \Gamma \to \Delta \mid H_2 \vdash H_1 \mid A, \Gamma \to \Delta \mid H_2.$$

Cut only has an internal version:

$$\frac{G_1 \mid \Gamma_1 \to \Delta_1, A \mid H_1 \qquad G_2 \mid A, \Gamma_2 \to \Delta_2 \mid H_2}{G_1 \mid G_2 \mid \Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2 \mid H_1 \mid H_2}$$

The use of hypersequents allows a cut-free presentation **GS5** of **S5** satisfying the subformula property. The system **GS5** consists of hypersequential versions of the rules of **LS4**, in particular, external and internal versions of contraction and monotonicity, the above cut-rule, and a structural rule of a new kind, namely the *modalized splitting rule*:

$$(MS) \qquad G \mid \Box\Gamma_1, \Gamma_2 \to \Box\Delta_1, \Delta_2 \mid H \vdash G \mid \Box\Gamma_1 \to \Box\Delta_1 \mid \Gamma_2 \to \Delta_2 \mid H.$$

In the next section we shall define display sequents, and in Section 4.2 we shall define a translation of hypersequents into display sequents.

3 DISPLAY LOGIC

We shall develop display logic only to the extent needed to cover a variety of normal modal and temporal logics based on classical or intuitionistic logic. A more comprehensive presentation of display logic and its application to modal and non-classical logics can be found in [Belnap, 1982], [Belnap, 1990], [Belnap, 1996], [Goré, 1998], [Kracht, 1996], [Restall, 1998], [Wansing, 1998]. Note that except for the substructure property, all requirements examined in the previous sections are satisfied by the display sequent systems to be presented.

3.1 Introduction rules through residuation

Whereas the ordinary sequent systems for temporal logics presented in Section 1.2 fail to exploit the interaction between the backward and the forward looking modalities, the modal display calculus is based on observing that the operators $\langle P\rangle$ and $[F]$ form a residuated pair. The following definition is taken from Dunn [1990, p. 32]:

DEFINITION 5. Consider two partially ordered sets $\mathcal{A} = (\mathbf{A}, \leq)$ and $\mathcal{B} = (\mathbf{B}, \leq')$ with functions f: $\mathbf{A} \longrightarrow \mathbf{B}$ and g: $\mathbf{B} \longrightarrow \mathbf{A}$. The pair (f, g) is called

$$\begin{array}{rll} \textit{residuated} & \text{iff} & (fa \leq' b \ \text{ iff } \ a \leq gb); \\ \text{a } \textit{Galois connection} & \text{iff} & (b \leq' fa \ \text{ iff } \ a \leq gb); \\ \text{a } \textit{dual Galois connection} & \text{iff} & (fa \leq' b \ \text{ iff } \ gb \leq a); \\ \text{a } \textit{dual residuated pair} & \text{iff} & (b \leq' fa \ \text{ iff } \ gb \leq a). \end{array}$$

Obviously, $(\langle P\rangle, [F])$ forms a residuated pair with respect to the provability relation in normal extensions of **Kt**, and $(\neg\langle F\rangle, \neg\langle P\rangle)$ is a Galois connection.[5] These ideas of residuation and Galois connection can be generalized. In [Dunn, 1990], [Dunn, 1993], Dunn has defined an abstract law of

[5]The fact that $\langle P\rangle$ and $[F]$ form a residuated pair is also used in Kashima's [1994] sequent calculi for various normal temporal logics. The approach of Kashima is similar to the modal display calculus and the modal signs approach developed by Cerrato [1993; 1996] insofar as the structural language of sequents is extended by unary structure operations. Whereas nesting of these operations is not allowed in Cerrato's sequent systems for normal modal propositional logics, Kashima allows iteration. Kashima inductively defines a notion of sequent as follows:

residuation for n-place connectives f and g. The formulation of this principle refers to *traces* of operations and assumes the presence (or definability) of a truth constant $\boldsymbol{t}$ and a falsity constant $\boldsymbol{f}$. We shall use $A \dashv\vdash B$ to express that A and B are interderivable in a given axiom system.

DEFINITION 6. An n-place connective f ($n \geq 0$) has a trace $(\rho_1, \ldots, \rho_n) \mapsto +$ (in symbols $T(f) = (\rho_1, \ldots, \rho_n) \mapsto +$) iff

$$\begin{array}{l}
f(A_1, \ldots, \boldsymbol{t}, \ldots, A_n) \dashv\vdash \boldsymbol{t}, \text{ if } \rho_i = + \text{ (the indicated } \boldsymbol{t} \text{ is in position } i);\\
f(A_1, \ldots, \boldsymbol{f}, \ldots, A_n) \dashv\vdash \boldsymbol{t}, \text{ if } \rho_i = - \text{ (the indicated } \boldsymbol{f} \text{ is in position } i);\\
\text{if } A \vdash B \text{ and } \rho_i = +, \text{ then } f(A_1, \ldots, A, \ldots, A_n) \vdash f(A_1, \ldots, B, \ldots, A_n);\\
\text{if } A \vdash B \text{ and } \rho_i = -, \text{ then } f(A_1, \ldots, B, \ldots, A_n) \vdash f(A_1, \ldots, A, \ldots, A_n).
\end{array}$$

The operation f has a trace $(\rho_1, \ldots, \rho_n) \mapsto -$ ($T(f) = (\rho_1, \ldots, \rho_n) \mapsto -$) iff

$$\begin{array}{l}
f(A_1, \ldots, \boldsymbol{f}, \ldots, A_n) \dashv\vdash \boldsymbol{f}, \text{ if } \rho_i = + \text{ (the indicated } \boldsymbol{f} \text{ is in position } i);\\
f(A_1, \ldots, \boldsymbol{t}, \ldots, A_n) \dashv\vdash \boldsymbol{f}, \text{ if } \rho_i = - \text{ (the indicated } \boldsymbol{t} \text{ is in position } i);\\
\text{if } A \vdash B \text{ and } \rho_i = +, \text{ then } f(A_1, \ldots, B, \ldots, A_n) \vdash f(A_1, \ldots, A, \ldots, A_n);\\
\text{if } A \vdash B \text{ and } \rho_i = -, \text{ then } f(A_1, \ldots, A, \ldots, A_n) \vdash f(A_1, \ldots, B, \ldots, A_n).
\end{array}$$

In **Kt**, $\neg$ has traces $- \mapsto +$ and $+ \mapsto -$, whereas $[F]$ has trace $+ \mapsto +$ and $\langle P \rangle$ has trace $- \mapsto -$.

DEFINITION 7. Two n-place operations f and g are *contrapositives in place* j iff $T(f) = (\rho_1, \ldots, \rho_j, \ldots, \rho_n) \mapsto \rho$ implies $T(g) = (\rho_1, \ldots, -\rho, \ldots, \rho_n) \mapsto -\rho_j$, where $-+ = -$ and $-- = +$.

DEFINITION 8. Let

$$S(f, A_1, \ldots, A_n, B) \text{ iff } \begin{cases} B \vdash f(A_1, \ldots, A_n) & \text{if } T(f) = (\ldots) \mapsto + \\ f(A_1, \ldots, A_n) \vdash B & \text{if } T(f) = (\ldots) \mapsto - \end{cases}$$

---

1. every temporal formula is a sequent;
2. if Γ is a sequent, then so is ${}^{P}\{\Gamma\}$ and ${}^{F}\{\Gamma\}$;
3. if $n \geq 0$ and every Γ_i ($1 \leq i \leq n$) is a sequent, then so is $\Gamma_1, \ldots, \Gamma_i$.

The intuitive meaning of a sequent is given by the following inductively defined translation $(\cdot)^*$ from sequents into formulas:

1. $(\Gamma)^* = A$, if Γ is the formula A;
2. $({}^{P}\{\Gamma\})^* = [P]^{P}(\Gamma)^*$; $({}^{F}\{\Gamma\})^* = [F]^{F}(\Gamma)^*$;
3. if $n > 0$, then $(\Gamma_1, \ldots, \Gamma n)^* = \bigvee\{(\Gamma_1)^*, \ldots, (\Gamma_n)^*\}$;
4. $(\,)^* = (p \wedge \neg p)$, for some atom p.

Residuation then shows up in Kashima's "turn rules":

$$\Gamma, {}^{F}\{\Delta\} \vdash {}^{P}\Gamma, \Delta; \qquad \Gamma, {}^{P}\{\Delta\} \vdash {}^{F}\Gamma, \Delta.$$

Most of Kashima's sequent rules used to capture various structural properties of accessibility either fail to be explicit or separated in the sense of Section 1.3. Cut-elimination for these systems is shown semantically, i.e., in a non-constructive way.

A pair of n-place connectives f and g satisfies the abstract law of residuation just in case for some j $(1 \leq j \leq n)$, f and g are contrapositives in place j, and

$$S(f, A_1, \ldots, A_j, \ldots, A_n, B) \text{ iff } S(f, A_1, \ldots, B, \ldots, A_n, A_j).$$

OBSERVATION 9. The abstract law of residuation holds for the pairs $(\boldsymbol{t}, \boldsymbol{f})$, $(\neg, \neg)$, $(\langle P\rangle, [F])$, $(\wedge, \rhd)$, $(\blacktriangleleft, \vee)$, $(\wedge, \neg\ldots\vee\ldots)$, and $(\ldots\wedge\neg\ldots, \vee)$, where $\rhd$ is intuitionistic implication and $\blacktriangleleft$ is coimplication.

Coimplication $\blacktriangleleft$ is characterized by

$$A \vdash B \vee C, \Delta \quad \text{iff} \quad A \blacktriangleleft B \vdash C, \Delta.$$

In classical logic, the residual of disjunction is definable, since

$$A \vdash B \vee C, \Delta \quad \text{iff} \quad A \wedge \neg B \vdash C, \Delta \quad \text{iff} \quad \neg(A \supset B) \vdash C,$$

but in bi-intuitionistic logic it is not, see Section 3.8. For each of the pairs $(\boldsymbol{t}, \boldsymbol{f})$, $(\neg, \neg)$, $(\langle P\rangle, [F])$, $(\wedge, \rhd)$, $(\blacktriangleleft, \vee)$, the structural language of display sequents contains one structure connective. Since in classical logic $\wedge$ and $\vee$ are interdefinable using $\neg$, the pairs $(\wedge, \neg\ldots\vee\ldots)$ and $(\ldots\wedge\neg\ldots, \vee)$ require only a single structure connective in addition to the unary structure operation associated with $(\neg, \neg)$. We shall use X, Y, Z (possibly with subscripts) as variables for structures. A display sequent is an expression $X \rightarrow Y$; X is called the antecedent and Y is called the succedent of $X \rightarrow Y$. The structures are defined by:

$$X ::= A \mid \mathbf{I} \mid *X \mid \bullet X \mid X \circ Y \mid X \rtimes Y \mid X \ltimes Y.$$

The association of structure connectives with pairs of operations satisfying the abstract law of residuation is accomplished by the following translations τ_1 of antecedents and τ_2 of succedents into formulas:

$$\begin{array}{rcl@{\qquad}rcl}
\tau_1(A) & = & A & \tau_2(A) & = & A \\
\tau_1(\mathbf{I}) & = & \boldsymbol{t} & \tau_2(\mathbf{I}) & = & \boldsymbol{f} \\
\tau_1(*X) & = & \neg\tau_2(X) & \tau_2(*X) & = & \neg\tau_1(X) \\
\tau_1(\bullet X) & = & \langle P\rangle\tau_1(X) & \tau_2(\bullet X) & = & [F]\tau_2(X) \\
\tau_1(X \rtimes Y) & = & \tau_1(X) \wedge \tau_1(Y) & \tau_2(X \rtimes Y) & = & \tau_2(X) \rhd \tau_2(Y) \\
\tau_1(X \ltimes Y) & = & \tau_1(X) \blacktriangleleft \tau_1(Y) & \tau_2(X \ltimes Y) & = & \tau_2(X) \vee \tau_2(Y) \\
\tau_1(X \circ Y) & = & \tau_1(X) \wedge \tau_1(Y) & \tau_2(X \circ Y) & = & \tau_2(X) \vee \tau_2(Y)
\end{array}$$

Under these translations, the following basic structural rules are valid ((1)–(4) in normal temporal logic; (5) and (6) in bi-intuitionistic logic) if $\rightarrow$ is understood as provability:

Basic structural rules

(1) $X \circ Y \to Z \dashv\vdash X \to Z \circ *Y \dashv\vdash Y \to *X \circ Z$
(2) $X \to Y \circ Z \dashv\vdash X \circ *Z \to Y \dashv\vdash *Y \circ X \to Z$
(3) $X \to Y \dashv\vdash *Y \to *X \dashv\vdash X \to **Y$
(4) $X \to \bullet Y \dashv\vdash \bullet X \to Y$
(5) $X \rtimes Y \to Z \dashv\vdash Y \to X \rtimes Z \dashv\vdash X \to Y \rtimes Z$
(6) $X \to Y \ltimes Z \dashv\vdash X \ltimes Y \to Z \dashv\vdash X \ltimes Z \to Y$,

where $X_1 \to Y_1 \dashv\vdash X_2 \to Y_2$ abbreviates $X_1 \to Y_1 \vdash X_2 \to Y_2$ and $X_2 \to Y_2 \vdash X_1 \to Y_1$. If two sequents are interderivable by means of (1)–(6), then these sequents are said to be *structurally* or *display equivalent*. The following pairs of sequents, for example, are display equivalent on the strength of (1)–(3):

$X \circ Y \to Z$	$*Z \to *Y \circ *X$;	$X \to Y \circ Z$	$*Z \circ *Y \to *X$;
$X \to Y$	$*Y \to X$;	$X \to *Y$	$Y \to *X$;
$X \to Y$	$**X \to Y$.		

The name 'display logic' derives from the fact that any substructure of a given display sequent s may be *displayed* as the entire antecedent or succedent of a structurally equivalent sequent s'. In order to state this fact precisely, we define the notion of a polarity vector and antecedent and succedent part of a sequent (cf. [Goré, 1998]).

DEFINITION 10. To each n-place structure connective c we assign two *polarity vectors* $ap(c, \pm_1, \ldots, \pm_n)$ and $sp(c, \pm_1, \ldots, \pm_n)$, where $\pm_i \in \{+, -\}$ and $1 \leq i \leq n$:

$$\begin{array}{lllll} ap(*,-) & ap(\bullet,+) & ap(\circ,+,+) & ap(\rtimes,+,+) & ap(\ltimes,+,-) \\ sp(*,-) & sp(\bullet,+) & sp(\circ,+,+) & sp(\rtimes,-,+) & sp(\ltimes,+,+) \end{array}$$

We write $ap(c, j, \pm)$ and $sp(c, j, \pm)$ to express that c has antecedent, respectively succedent polarity $\pm$ at place j.

DEFINITION 11. Let $s = X \to Y$. The exhibited occurrence of X is an antecedent part of s, and the exhibited occurrence of Y is a succedent part of s. If $c(X_1, \ldots, X_n)$ is an antecedent [succedent] part of s, then the substructure occurrence X_j ($1 \leq j \leq n$) is

1. an antecedent [succedent] part of s if $ap(c, j, +)$ [$sp(c, j, +)$];
2. a succedent [antecedent] part of s if $ap(c, j, -)$ [$sp(c, j, -)$].

THEOREM 12. (Display Theorem, Belnap) *For each display sequent s and each antecedent [succedent] part X of s there exists a display sequent s' structurally equivalent to s such that X is the entire antecedent [succedent] of s'.*

Proof. The theorem was first proved in [Belnap, 1982]; we shall follow the proof in [Restall, 1998]. A *context* results from a structure by replacing one occurrence of a substructure by the 'Void' (in symbols '$-$'). If f is a context and X is a structure, then $f(X)$ is the result of substituting X for the Void in f. A context f is called *antecedent positive* (*negative*) if the indicated X is an antecedent part (a succedent part) of $f(X) \to Y$; f is said to be *succedent positive* (*negative*) if the indicated X is a succedent part (an antecedent part) of $Y \to f(X)$. A contextual sequent has the shape $f \to Z$ or $Z \to f$, and a pair of contextual sequents is said to be structurally equivalent if the sequents are interderivable by means of rules (1)–(6). The Display Theorem then follows from the following lemma.

LEMMA 13. *(i) Suppose f is a context in antecedent position. If f is antecedent positive, then $f(X) \to Y$ is structurally equivalent to $X \to f^a(Y)$, where f^a is a context obtained by unraveling the Void in f. If f is antecedent negative, then $f(X) \to Y$ is structurally equivalent to $f^a(Y) \to X$. (ii) Suppose f is a context in succedent position. If f is succedent positive, then $Y \to f(X)$ is structurally equivalent to $f^c(Y) \to X$, where f^c is a context obtained by unraveling the Void in f. If f is succedent negative, then $Y \to f(X)$ is structurally equivalent to $X \to f^c(Y)$.*

The proof is by induction on the complexity of contexts.
Case 1: $f = -$. Then f is antecedent and succedent positive, and $f^a(Y) = f^c(Y) = Y$.
Case 2: $f = \bullet g$. Then $f(X) \to Y$ is structurally equivalent to $g(X) \to \bullet Y$, and $Y \to f(X)$ is equivalent to $\bullet Y \to g(X)$. By the induction hypothesis, these sequents are equivalent to $X \to f^a(\bullet Y)$, $f^a(\bullet Y) \to X$, $f^c(\bullet Y) \to X$, or $X \to f^c(\bullet Y)$. Hence $f^a = g^a(\bullet -)$ and $f^c = g^c(\bullet -)$.
Case 3: $f = *g$. Then $f(X) \to Y$ is equivalent to $*Y \to g(X)$. Depending on whether g is succedent positive or negative, $f(X) \to Y$ is structurally equivalent to $g^c(*Y) \to X$ or to $X \to g^c(*Y)$. Therefore, by the induction hypothesis, $f^a = g^c(*-)$. Similarly, $f^c = g^a(*-)$.
Case 4: $f = Z \circ g$. Then $f(X) \to Y$ is equivalent to $g(X) \to *Z \circ Y$. By the induction hypothesis, this sequent is equivalent to $X \to g^a(*Z \circ Y)$ or $g^a(*Z \circ Y) \to X$, and hence $f^a = g^a(*Z \circ -)$. Similarly, $f^c = g^a(- \circ *Z)$.
Case 5: $f = g \circ Z$. Similar to Case 4.
Case 6: $f = g \rtimes Z$. Then $Y \to f(X)$ is equivalent to $g(X) \to Y \rtimes Z$, and by the induction hypothesis, the latter is equivalent to $X \to g^a(Y \rtimes Z)$ or to $g^a(Y \rtimes Z) \to X$. Thus $f^c = g^a(- \rtimes Z)$. Similarly, $f^a = g^c(Z \rtimes -)$.
Case 7: $f = Z \rtimes g$. Analogous to the previous case.
Cases 8 and 9: $f = g \ltimes Z$ and $f = Z \ltimes g$. Analogous to Cases 6 and 7. ■

If (for suitable notions of structural equivalence, antecedent part, and succedent part) a sequent calculus satisfies the Display Theorem, it is said to enjoy the *display property.* Note that the set of rules (1)–(6) is not the only

	truth and falsity rules
$(\to \boldsymbol{f})$	$X \to \mathbf{I} \vdash X \to \boldsymbol{f}$
$(\boldsymbol{f} \to)$	$\vdash \boldsymbol{f} \to \mathbf{I}$
$(\to \boldsymbol{t})$	$\vdash \mathbf{I} \to \boldsymbol{t}$
$(\boldsymbol{t} \to)$	$\mathbf{I} \to X \vdash \boldsymbol{t} \to X$
	Boolean introduction rules
$(\to \neg)$	$X \to *A \vdash X \to \neg A$
$(\neg \to)$	$*A \to X \vdash \neg A \to X$
$(\to \wedge)$	$X \to A \quad Y \to B \vdash X \circ Y \to A \wedge B$
$(\wedge \to)$	$A \circ B \to X \vdash A \wedge B \to X$
$(\to \vee)$	$X \to A \circ B \vdash X \to A \vee B$
$(\vee \to)$	$A \to X \quad B \to Y \vdash A \vee B \to X \circ Y$
$(\to \supset)$	$X \circ A \to B \vdash X \to A \supset B$
$(\supset \to)$	$X \to A \quad B \to Y \vdash A \supset B \to *X \circ Y$
$(\to \equiv)$	$X \circ A \to B \quad X \circ B \to A \vdash X \to A \equiv B$
$(\equiv \to)$	$X \to A \quad B \to Y \quad X \to B \quad A \to Y \vdash A \equiv B \to *X \circ Y$
	tense logical introduction rules
$(\to [F])$	$\bullet X \to A \vdash X \to [F]A$
$([F] \to)$	$A \to X \vdash [F]A \to \bullet X$
$(\to \langle F \rangle)$	$X \to A \vdash * \bullet * X \to \langle F \rangle A$
$(\langle F \rangle \to)$	$* \bullet * A \to Y \vdash \langle F \rangle A \to Y$
$(\to [P])$	$X \to * \bullet * A \vdash X \to [P]A$
$([P] \to)$	$A \to X \vdash [P]A \to * \bullet * X$
$(\to \langle P \rangle)$	$X \to A \vdash \bullet X \to \langle P \rangle A$
$(\langle P \rangle \to)$	$A \to \bullet X \vdash \langle P \rangle A \to X$
	nonclassical introduction rules
$(\to \wedge)'$	$X \to A \quad Y \to B \vdash X \rtimes Y \to A \wedge B$
$(\wedge \to)'$	$A \rtimes B \to X \vdash A \wedge B \to X$
$(\to \triangleright)$	$X \to A \rtimes B \vdash X \to A \triangleright B$
$(\triangleright \to)$	$X \to A \quad B \to Y \vdash A \triangleright B \to X \rtimes Y$
$(\to \vee)'$	$X \to A \ltimes B \vdash X \to A \vee B$
$(\vee \to)'$	$A \to X \quad B \to Y \vdash A \vee B \to X \ltimes Y$
$(\to \blacktriangleleft)$	$X \to A \quad B \to Y \vdash X \ltimes Y \to A \blacktriangleleft B$
$(\blacktriangleleft \to)$	$A \ltimes B \to X \vdash A \blacktriangleleft B \to X$

Table 1. Introduction rules.

$(\mathbf{I}^\circ_+)$	$X \to Z \vdash \mathbf{I} \circ X \to Z$	$X \to Z \vdash X \circ \mathbf{I} \to Z$
	$X \to Z \vdash X \to Z \circ \mathbf{I}$	$X \to Z \vdash X \to \mathbf{I} \circ Z$
$(\mathbf{I}^\circ_-)$	$\mathbf{I} \circ X \to Z \vdash X \to Z$	$X \circ \mathbf{I} \to Z \vdash X \to Z$
	$X \to Z \circ \mathbf{I} \vdash X \to Z$	$X \to \mathbf{I} \circ Z \vdash X \to Z$
$(\mathbf{I})$	$\mathbf{I} \to X \vdash Z \to X$	$X \to \mathbf{I} \vdash X \to Z$
$(\mathbf{I}^*)$	$\mathbf{I} \to X \dashv\vdash *\mathbf{I} \to X$	$X \to \mathbf{I} \dashv\vdash X \to *\mathbf{I}$
$(\mathbf{P}\circ)$	$X_1 \circ X_2 \to Z \vdash X_2 \circ X_1 \to Z$	$Z \to X_1 \circ X_2 \vdash Z \to X_2 \circ X_1$
$(\mathbf{C}\circ)$	$X \circ X \to Z \vdash X \to Z$	$Z \to X \circ X \vdash Z \to X$
$(\mathbf{E}\circ)$	$X \to Z \vdash X \circ X \to Z$	$Z \to X \vdash Z \to X \circ X$
$(\mathbf{M}\circ)$	$X_1 \to Z \vdash X_1 \circ X_2 \to Z$	$X_1 \to Z \vdash X_2 \circ X_1 \to Z$
	$Z \to X_1 \vdash Z \to X_1 \circ X_2$	$Z \to X_1 \vdash Z \to X_2 \circ X_1$
$(\mathbf{A}\circ)$	$X_1 \circ (X_2 \circ X_3) \to Z \dashv\vdash (X_1 \circ X_2) \circ X_3 \to Z$	
	$Z \to X_1 \circ (X_2 \circ X_3) \dashv\vdash (X_1 \circ X_2) \circ X_3 \to Z$	
$(\mathbf{MN})$	$\mathbf{I} \to X \vdash \mathbf{I} \to \bullet X$	$X \to \mathbf{I} \vdash X \to \bullet\mathbf{I}$
	$\mathbf{I} \to X \vdash \mathbf{I} \to * \bullet * X$	$X \to \mathbf{I} \vdash X \to * \bullet * \mathbf{I}$

Table 2. Additional structural rules.

possible choice of display rules warranting the display property, see [Belnap, 1996] and [Goré, 1998].[6] The display property allows an "'essentials-only' proof of cut elimination relying on easily established and maximally general properties of structural and connective rules" [Belnap, 1996, p. 80]. Further, the display property enables a statement of the introduction rules that satisfies the segregation requirement. Belnap emphasizes that the display property may be used to keep certain proof-theoretic components as separate as possible. In a sequent calculus enjoying the display property, the behaviour of the structural elements can be described by the structural rules, and the right (left) introductions rules for an n-place logical operation f can be formulated with $f(A_1, \ldots, A_n)$ standing alone as the entire succedent (antecedent) of the conclusion sequent. Since $f(A_1, \ldots, A_n)$ plays no inferential roles beyond being derived and allowing to derive, these left and right rules provide a complete explanation of the inferential meaning of f. The constant $\mathbf{I}$ induces introduction rules for $\boldsymbol{t}$ and $\boldsymbol{f}$. The operations $*$ and $\circ$ give rise to introduction rules for the Boolean connectives. The structure operation $\bullet$ permits formulating introduction rules for the modal-

[6] Goré [Goré, 1998] introduces binary structure connectives $<$ and $>$ to be interpreted as directional versions of implication in succedent position and coimplication in antecedent position. The display property is guaranteed by the following structural rules (notation adjusted):

$$X \to Z < Y \dashv\vdash X \circ Y \to Z \dashv\vdash Y \to X > Z$$
$$Z < Y \to X \dashv\vdash Z \to X \circ Y \dashv\vdash X > Z \to Y.$$

ities, whereas $\rtimes$ and $\ltimes$ give rise to introduction schemata for conjunction, disjunction, implication, and coimplication in bi-intuitionistic logic. These introduction rules are assembled in Table 1. The further structural rules in Table 2 contain many redundancies when they are assumed as a set. Such a rich inventory of structural inference rules is, however, an advantage in a treatment of substructural subsystems of normal modal and temporal logics, see [Goré, 1998]. In addition to a set of structural rules and a set of introduction rules, every display sequent system contains two *logical* rules exhibiting neither structural nor logical operations, namely reflexivity for atoms (alias identity) and cut:

$$\text{(id)} \quad \vdash p \to p \quad \text{and} \quad \text{(cut)} \quad X \to A \quad A \to Y \vdash X \to Y.$$

The identity rule (id) can be generalized to arbitrary formulas from temporal or bi-intuitionistic logic.

OBSERVATION 14. For every formula A, $\vdash A \to A$.

Proof. The proof is by induction on the complexity of A. For example,

$$\frac{\dfrac{A \to A}{[P]A \to * \bullet * A}}{[P]A \to [P]A} \qquad \frac{\dfrac{\dfrac{A \to A}{\bullet A \to \langle P\rangle A}}{A \to \bullet\langle P\rangle A}}{\langle P\rangle A \to \langle P\rangle A} \qquad \frac{\dfrac{A \to A \quad B \to B}{A \ltimes B \to A \blacktriangleleft B}}{A \blacktriangleleft B \to A \blacktriangleleft B}.$$

■

DEFINITION 15. The display sequent system $\boldsymbol{DCPL}$ is given by (id), (cut), the Boolean rules, and the structural rules exhibiting $\mathbf{I}$, $*$, and $\circ$. The system $\boldsymbol{DKt}$ consists of $\boldsymbol{DCPL}$ plus the tense logical rules and the structural rules exhibiting $\bullet$. The system $\boldsymbol{DK}$ results from $\boldsymbol{DKt}$ by removing the introduction rules for $[P]$ and $\langle P\rangle$.

A sequent rule is invertible if every premise sequent can be derived from the conclusion sequent.

OBSERVATION 16. The following holds in every purely structural extension of $\boldsymbol{DKt}$ and $\boldsymbol{DK}$. (i) The logical operations are uniquely characterized. (ii) The introduction rules for $\neg$, $\wedge$, and $\vee$, the left introduction rules for $\boldsymbol{t}$, $\langle P\rangle$, and $\langle F\rangle$, and the right introduction rules for $\boldsymbol{f}$, $\supset$, $\equiv$, $[P]$, and $[F]$ are invertible. (iii) The modalities $[F]$ and $\langle F\rangle$ ($[P]$ and $\langle P\rangle$) are interdefinable using $\neg$.

Note that there exist various duality and symmetry transformations on proofs in display logic, see [Goré, 1998], [Kracht, 1996].

3.2 Completeness

We shall first consider weak completeness of **DKt** and **DK**, that is, the coincidence of **Kt** (**K**) and **DKt** (**DK**) with respect to provable formulas. We shall then strengthen this result and in Section 3.4 turn to axiomatic extensions of **K** and **Kt**.

THEOREM 17. *(i) If* $\vdash A$ *in* ***Kt****, then* $\vdash \mathbf{I} \to A$ *in* ***DKt****. (ii) If* $\vdash X \to Y$ *in* ***DKt****, then* $\tau_1(X) \vdash \tau_2(Y)$ *in* ***Kt****.*

Proof. (i) We may take any axiomatization of **Kt** and show that the axiom schemata are provable in **DKt**, and the proof rules preserve provability in **DKt**. The following is a cut-free proof of the K axiom schema for $[F]$; the proof for $[P]$ is analogous:

$$\begin{array}{l}
A \to A \\ \hline
[F]A \to \bullet A \\ \hline
[F](A \supset B) \circ [F]A \to \bullet A \\ \hline
\bullet([F](A \supset B) \circ [F]A) \to A \quad\; B \to B \\ \hline
A \supset B \to * \bullet ([F](A \supset B) \circ [F]A) \circ B \\ \hline
[F](A \supset B) \to \bullet(* \bullet ([F](A \supset B) \circ [F]A) \circ B) \\ \hline
[F](A \supset B) \circ [F]A \to \bullet(* \bullet ([F](A \supset B) \circ [F]A) \circ B) \\ \hline
\bullet([F](A \supset B) \circ [F]A) \to * \bullet ([F](A \supset B) \circ [F]A) \circ B \\ \hline
\bullet([F](A \supset B) \circ [F]A) \circ \bullet([F](A \supset B) \circ [F]A) \to B \\ \hline
\bullet([F](A \supset B) \circ [F]A) \to B \\ \hline
[F](A \supset B) \circ [F]A \to [F]B \\ \hline
[F](A \supset B) \to [F]A \supset [F]B \\ \hline
\mathbf{I} \circ [F](A \supset B) \to [F]A \supset [F]B \\ \hline
\mathbf{I} \to [F](A \supset B) \supset [F]A \supset [F]B
\end{array}$$

Necessitation for $[F]$ and $[P]$ is taken care of by the (**MN**) rules. It remains to derive the tense logical interaction schemata $A \supset [F]\langle P \rangle A$ and $A \supset [P]\langle F \rangle A$:

$$\begin{array}{c}
A \to A \\ \hline
\bullet A \to \langle P \rangle A \\ \hline
A \to [F]\langle P \rangle A
\end{array}
\qquad\qquad
\begin{array}{c}
A \to A \\ \hline
* \bullet * A \to \langle F \rangle A \\ \hline
*\langle F \rangle A \to \bullet * A \\ \hline
\bullet * \langle F \rangle A \to * A \\ \hline
A \to * \bullet * \langle F \rangle A \\ \hline
A \to [P]\langle F \rangle A
\end{array}$$

(ii) By induction on the complexity of proofs in **DKt**. ■

COROLLARY 18. *(i) In* ***Kt****,* $\vdash A$ *iff* $\vdash \mathbf{I} \to A$ *in* ***DKt****. (ii) In* ***K****,* $\vdash A$ *iff* $\vdash \mathbf{I} \to A$ *in* ***DK****.*

Proof. (i) By the previous theorem. (ii) This follows from the fact that every frame complete normal propositional tense logic is a conservative extension of its modal fragment. ∎

LEMMA 19. *In every extension of* $\boldsymbol{DKt}$ *by structural inference rules, it holds that* $\vdash X \rightarrow \tau_1(X)$ *and* $\vdash \tau_2(X) \rightarrow X$.

Proof. By induction on the complexity of X. ∎

This lemma allows one to prove strong completeness.

THEOREM 20. *In* $\boldsymbol{DKt}$, $\vdash X \rightarrow Y$ *iff* $\tau_1(X) \vdash \tau_2(Y)$ *in* $\boldsymbol{Kt}$.

Proof. ($\Rightarrow$): This is Theorem 17, (ii). ($\Leftarrow$): Suppose that in **Kt**, $\tau_1(X) \vdash \tau_2(Y)$. Hence $\vdash_{\mathbf{Kt}} \tau_1(X) \supset \tau_2(Y)$. By Corollary 18, $\vdash_{\mathbf{DKt}} \mathbf{I} \rightarrow \tau_1(X) \supset \tau_2(Y)$ and thus $\vdash_{\mathbf{DKt}} \tau_1(X) \rightarrow \tau_2(Y)$. Since by Lemma 19, $\vdash X \rightarrow \tau_1(X)$ and $\vdash \tau_2(Y) \rightarrow Y$ in **DKt**, an application of cut gives $\vdash X \rightarrow Y$. ∎

COROLLARY 21. $\boldsymbol{DK}$ *is strongly sound and complete with respect to* $\boldsymbol{K}$.

COROLLARY 22. $\boldsymbol{DCPL}$ *is strongly sound and complete with respect to* $\boldsymbol{CPL}$.

3.3 Strong cut-elimination

A remarkable quality of display logic is that a strong cut-elimination theorem holds for every properly displayable and every displayable logic. Proper displayability and displayability are easily checkable properties. A *proper display calculus* is a calculus of sequents whose rules of inference satisfy the following eight conditions (recall the terminology from Section 1.1):

C1 *Preservation of formulas.* Each formula which is a constituent of some premise of *inf* is a subformula of some formula in the conclusion of *inf*.

C2 *Shape-alikeness of parameters.* Congruent parameters are occurrences of the same structure.

C3 *Non-proliferation of parameters.* Each parameter of *inf* is congruent to at most one constituent in the conclusion of *inf*.

C4 *Position-alikeness of parameters.* Congruent parameters are either all antecedent or all succedent parts of their respective sequents.

C5 *Display of principal constituents.* A principal formula of *inf* is either the entire antecedent or the entire succedent of the conclusion of *inf*.

C6 *Closure under substitution for consequent parts.* Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas which are consequent parts.

C7 *Closure under substitution for antecedent parts.* Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas which are antecedent parts.

C8 *Eliminability of matching principal formulas.* If there are inferences inf_1 and inf_2 with respective conclusions (1) $X \to A$ and (2) $A \to Y$ with A principal in both inferences, and if cut is applied to obtain (3) $X \to Y$, then either (3) is identical to one of (1) or (2), or there is a proof of (3) from the premises of inf_1 and inf_2 in which every cut-formula of any application of cut is a proper subformula of A.

Obviously, every display calculus satisfying C1 enjoys the subformula property, that is, every cut-free proof of any sequent s contains no formulas which are not subformulas of constituents of s. If a logical system can be presented as a proper display calculus, it is said to be *properly displayable.* Belnap [1982] showed that in every properly displayable logic, a proof of a sequent s can be converted into a proof of s not containing any application of cut

$$\frac{(1)\ \ X \to A \qquad (2)\ \ A \to Y}{(3)\ \ X \to Y}.$$

The proof of strong cut-elimination reveals that every sufficiently long sequence of steps in a certain process of cut-elimination terminates with a cut-free proof. The elimination process consists of various kinds of actions, *principal moves*, *parametric moves*, and a combination of parametric and principal moves. If the cut-formula A is principal in the final inference in the proofs of both (1) and (2), a principal move is performed. Otherwise, if there is no previous application of cut, a parametric move or a combination of parametric and principal moves is executed. According to this distinction we define primitive reductions of proofs Π ending in an application of cut. Recently, Rajeev Goré and Jeremy Dawson discovered a gap in the proof of strong normalization presented in [Wansing, 1998]. To avoid the problem, the primitive reduction steps have to be redefined. Let Π_i be the proof of (i) we are dealing with, $(i = 1, 2)$.

Principal moves. By C8, there are two subcases:

Case 1. (3) is the same as (i): $\dfrac{\Pi_1 \quad \Pi_2}{(3)} \rightsquigarrow \Pi_i$

Case 2. There is a proof Π of (3) from the premises $s_1, \ldots, s_n$ of (1) and $s'_1, \ldots, s'_m$ of (2) in which every cut-formula of any application of cut is a proper subformula of A:

$$\dfrac{\begin{array}{c}\Pi^1\\ s_1,\ldots,s_n\\ (1)\end{array} \qquad \begin{array}{c}\Pi^2\\ s'_1,\ldots,s'_m\\ (2)\end{array}}{(3)} \quad \rightsquigarrow \quad \begin{array}{c}\Pi^1 \quad \Pi^2\\ \Pi\\ (3)\end{array}$$

Parametric moves. The parametric moves modify proofs on a larger scale than the principal moves. The parametric moves show that applications of structural rules need never immediately precede applications of cut. Suppose that A is parametric in the inference ending in (1). The case for (2) is completely symmetrical. In order to define the parametric moves, we inductively define a set Q of occurrences of A, called the set of 'parametric ancestors' of A (in Π_1), cf. [Belnap, 1982, p. 394]. We start with putting the displayed occurrence of A in (1) into Q. Then, by working up Π_1, we add for every inference *inf* in Π_1 each constituent of a premise of *inf* which is congruent (with respect to *inf*) to a constituent of the conclusion of *inf* already in Q. What we obtain is a finite tree of parametric ancestors of A rooted in the displayed occurrence of A in (1). This tree and the tree of parametric ancestors of the displayed occurrence of A in (2) either contain an application of cut or not. If so, we do *not* perform a reduction, but instead consider one of these applications of cut above (1) or (2) for reduction. If not, that is, if there is no application of cut in the trees of parametric ancestors, then for each path of parametric ancestors of A in Π_1, we distinguish two subcases. Let A_u be the uppermost element of the path and let *inf* be the inference ending in the sequent s which contains A_u.

Case 1. A_u is not parametric in *inf.* By C4 and C5, it is the entire consequent of s. We cut with Π_2 and replace every parametric ancestor of A below A_u in the path by Y.

Case 2. A_u is parametric in *inf.* Then, with respect to *inf*, A_u is congruent only to itself, and we just replace every parametric ancestor of A below A_u in the path by Y. Moreover, we delete Π_2, which is now superfluous.

Call the result of simultaneously carrying out these operations for every path of parametric ancestors of A in Π_1 and removing the initial occurrence of (3) (since now (2) = (3)) Π^l. If the tree of parametric ancestors of the displayed occurrence of A in (1) contains at most one element A_u that is not parametric in *inf*, Π reduces to Π^l: $\Pi \rightsquigarrow \Pi^l$. Typically we have the situation of Figure 1.

By C3 and the bottom-up definition of Q, for every inference *inf* in Π_1, Q must contain the whole congruence class of *inf*, if Q is inhabited at all. By C4, Q only consists of consequent parts. Hence, by C2 and C6, the result of such a reduction is in fact a proof of (3), since on the path from (1) to $Z \rightarrow A$ we have the same sequence of inference rules being applied as on the path from (3) to $Z \rightarrow Y$. If the cut-formula A is parametric in the inference ending in (2), we rely on C7 instead of C6 and obtain a proof Π^r.

If the tree of parametric ancestors of the displayed occurrence of A in (1) contains more than one element A_u that is not parametric in *inf*, parametric and principal moves have to be combined. If A is parametric in the final inference of Π_2, we apply to Π^l a principal move on every cut with Π_2.

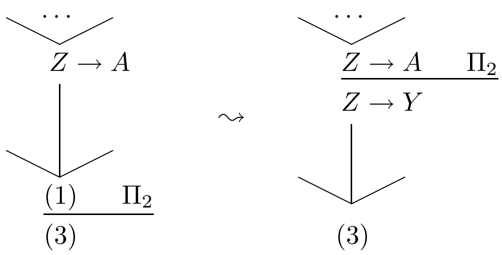

Figure 1.

Call the resulting proof Π^{l*}: $\Pi \rightsquigarrow \Pi^{l*}$. If A is not parametric in the final inference of Π_2, consider Π^{lr}. We apply to Π^{lr} a principal move on every cut with any subproof of Π_2 ending in a sequent containing a parametric ancestor A_u. Call the resulting proof Π^{lr*}: $\Pi \rightsquigarrow \Pi^{lr*}$. Thus, if the tree of parametric ancestors of the displayed occurrence of A in (1) contains more than one element A_u that is not parametric in *inf*, the primitive reduction of Π gives a proof that is calculated via some intermediate steps. Moreover, instead of a cut with cut-formula A, we obtain several cuts with subformulas of A as the cut-formula. Here is a worked out example:

$$\Pi = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi^1\\ *(A\circ B)\circ X\to(A\circ B)\end{array}}{*(A\circ B)\circ X\to(A\vee B)}}{*(A\vee B)\circ X\to(A\circ B)}}{*(A\vee B)\circ X\to(A\vee B)}}{X\to(A\vee B)\circ(A\vee B)}}{X\to(A\vee B)} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi^{2_1}\\ A\to Y\end{array}\qquad\begin{array}{c}\Pi^{2_2}\\ B\to Z\end{array}}{(A\vee B)\to(Y\circ Z)}}{(A\vee B)\to(Y\circ Z)\circ W}}{X\to(Y\circ Z)\circ W}$$

$$\Pi^l = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi^1\\ *(A\circ B)\circ X\to(A\circ B)\end{array}}{*(A\circ B)\circ X\to(A\vee B)}\qquad\Pi_2}{*(A\circ B)\circ X\to(Y\circ Z)\circ W}}{*((Y\circ Z)\circ W)\circ X\to(A\circ B)}}{*((Y\circ Z)\circ W)\circ X\to(A\vee B)}\qquad\Pi_2}{*((Y\circ Z)\circ W)\circ X\to(Y\circ Z)\circ W}}{X\to((Y\circ Z)\circ W)\circ((Y\circ Z)\circ W)}}{X\to((Y\circ Z)\circ W)}$$

$$\Pi^{lr} = \quad
\begin{prooftree}
\AxiomC{Π^1}
\noLine
\UnaryInfC{$*(A \circ B) \circ X \rightarrow (A \circ B)$}
\UnaryInfC{$*(A \circ B) \circ X \rightarrow (A \vee B)$}
\AxiomC{Π^{2_1}}
\noLine
\UnaryInfC{$A \rightarrow Y$}
\AxiomC{Π^{2_2}}
\noLine
\UnaryInfC{$B \rightarrow Z$}
\BinaryInfC{$(A \vee B) \rightarrow (Y \circ Z)$}
\BinaryInfC{$*(A \circ B) \circ X \rightarrow (Y \circ Z)$}
\UnaryInfC{$*(A \circ B) \circ X \rightarrow (Y \circ Z) \circ W$}
\UnaryInfC{$*((Y \circ Z) \circ W) \circ X \rightarrow (A \circ B)$}
\UnaryInfC{$*((Y \circ Z) \circ W) \circ X \rightarrow (A \vee B)$}
\AxiomC{Π^{2_1}}
\noLine
\UnaryInfC{$A \rightarrow Y$}
\AxiomC{Π^{2_2}}
\noLine
\UnaryInfC{$B \rightarrow Z$}
\BinaryInfC{$(A \vee B) \rightarrow (Y \circ Z)$}
\BinaryInfC{$*((Y \circ Z) \circ W) \circ X \rightarrow (Y \circ Z)$}
\UnaryInfC{$*((Y \circ Z) \circ W) \circ X \rightarrow (Y \circ Z) \circ W$}
\UnaryInfC{$X \rightarrow ((Y \circ Z) \circ W) \circ ((Y \circ Z) \circ W)$}
\UnaryInfC{$X \rightarrow ((Y \circ Z) \circ W)$}
\end{prooftree}$$

$$\Pi \rightsquigarrow \Pi^{lr*} = \quad
\begin{prooftree}
\AxiomC{Π^1}
\noLine
\UnaryInfC{$*(A \circ B) \circ X \rightarrow (A \circ B)$}
\UnaryInfC{$(*(A \circ B) \circ X) \circ *B \rightarrow A$}
\AxiomC{Π^{2_1}}
\noLine
\UnaryInfC{$A \rightarrow Y$}
\BinaryInfC{$(*(A \circ B) \circ X) \circ *B \rightarrow Y$}
\UnaryInfC{$*Y \circ (*(A \circ B) \circ X) \rightarrow B$}
\AxiomC{Π^{2_2}}
\noLine
\UnaryInfC{$B \rightarrow Z$}
\BinaryInfC{$*Y \circ (*(A \circ B) \circ X) \rightarrow Z$}
\UnaryInfC{$*(A \circ B) \circ X \rightarrow (Y \circ Z)$}
\UnaryInfC{$*(A \circ B) \circ X \rightarrow (Y \circ Z) \circ W$}
\UnaryInfC{$*((Y \circ Z) \circ W) \circ X \rightarrow A \circ B$}
\UnaryInfC{$(*((Y \circ Z) \circ W) \circ X) \circ *B \rightarrow A$}
\AxiomC{Π^{2_1}}
\noLine
\UnaryInfC{$A \rightarrow Y$}
\BinaryInfC{$(*((Y \circ Z) \circ W) \circ X) \circ *B \rightarrow Y$}
\UnaryInfC{$*Y \circ (*((Y \circ Z) \circ W) \circ X) \rightarrow B$}
\AxiomC{Π^{2_2}}
\noLine
\UnaryInfC{$B \rightarrow Z$}
\BinaryInfC{$*Y \circ (*((Y \circ Z) \circ W) \circ X) \rightarrow Z$}
\UnaryInfC{$*((Y \circ Z) \circ W) \circ X \rightarrow (Y \circ Z)$}
\UnaryInfC{$*((Y \circ Z) \circ W) \circ X \rightarrow ((Y \circ Z) \circ W)$}
\UnaryInfC{$X \rightarrow ((Y \circ Z) \circ W) \circ ((Y \circ Z) \circ W)$}
\UnaryInfC{$X \rightarrow ((Y \circ Z) \circ W)$}
\end{prooftree}$$

THEOREM 23. *Every proper display calculus enjoys strong cut-elimination.*

Proof. See Appendix A. ■

COROLLARY 24. *Cut is an admissible rule of every proper display calculus.*

Theorem 23 can straightforwardly be applied to **DK** and **DKt**. It can easily be checked that in these systems conditions C1 – C7 are satisfied.

Verification of C8 is also a simple exercise. We have for instance:

$$\dfrac{\dfrac{\bullet X \to A}{X \to [F]A} \quad \dfrac{A \to Y}{[F]A \to \bullet Y}}{X \to \bullet Y} \quad \rightsquigarrow \quad \dfrac{\dfrac{\bullet X \to A \quad A \to Y}{\bullet X \to Y}}{X \to \bullet Y}$$

$$\dfrac{\dfrac{X \to A}{* \bullet * X \to \langle A \rangle} \quad \dfrac{* \bullet * A \to Y}{\langle A \rangle \to Y}}{* \bullet * X \to Y} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{X \to A \quad \dfrac{\dfrac{\dfrac{* \bullet * A \to Y}{* Y \to \bullet * A}}{\bullet * Y \to * A}}{A \to * \bullet * Y}}{X \to * \bullet * Y}}{\bullet * Y \to * X}}{* Y \to \bullet * X}}{* \bullet * X \to Y.}$$

THEOREM 25. *Strong cut-elimination holds for* $\boldsymbol{DK}$ *and* $\boldsymbol{DKt}$.

COROLLARY 26. $\boldsymbol{DKt}$ *is a conservative extension of* $\boldsymbol{DK}$.

We shall now briefly consider generalizations of Theorem 23. By conditions C6 and C7, the inference rules of a proper display calculus are closed under simultaneous substitution of arbitrary structures for congruent formulas. The proof of strong normalization can be generalized to logics which for formulas of a certain shape satisfy closure under substitution either only for congruent formulas (of this shape) which are consequent parts or only for congruent formulas (of this shape) which are antecedent parts. In order to extend the proof of strong cut-elimination to such systems, C6 and C7 have to be replaced by the more general condition of *regularity*, see [Belnap, 1990]. A formula A is defined as *cons-regular* if the following holds: (i) if A occurs as a consequent parameter of an inference *inf* in a certain rule R, then R contains also the inference resulting by replacing every member of the congruence class of A in *inf* with an arbitrary structure X, and (ii) if A occurs as an antecedent parameter of an inference *inf* in a certain rule R, then R contains also the inference resulting by replacing every member of the congruence class of A in *inf* with any structure X such that $X \to A$ is the conclusion of an inference in which A is *not* parametric. The notion of *ant-regularity* is defined in exactly the dual way. The new condition on rules then is

C6/C7 *Regularity.* Every formula is regular.

A display calculus *simpliciter* is a calculus of sequents satisfying C1 - C5, C6/7, and C8. If a logic can be presented as a display calculus, then it is said to be displayable. Obviously, every properly displayable logic is displayable. Also the parametric moves must be redefined. Suppose in what follows that the cut-formula A is parametric in both the final inference of Π_1 and the final inference of Π_2. Moreover, suppose that the trees of parametric ancestors of A in Π_1 and in Π_2 do not contain any application of cut. If A_u is the tip of a path of parametric ancestors of A in Π_i, let *inf* be the inference ending in the sequent which contains A_u. Let us call A_u significant, if it is not parametric in *inf*. Then, in a *proper* display calculus we may choose whether we cut every significant tip A_u in the tree of parametric ancestors of A in Π_1 with Π_2 or whether we cut every significant tip A_u in the tree of parametric ancestors of A in Π_2 with Π_1 to obtain Π^l or Π^r. Both operations form an essential part in the definition of certain primitive reductions. In a display calculus simpliciter this indeterministic choice has to be abandoned. If the cut-formula is cons-regular, we cut with Π_2, and if the cut-formula is ant-regular, we cut with Π_1. This further restriction on parametric moves does not affect the proof of strong cut-elimination.

THEOREM 27. *Every displayable logic enjoys strong cut-elimination.*

A further strengthening of the strong cut-elimination theorem has recently been proved in [Demri and Goré, 1999], where it is shown that condition C8 may be relaxed. A proof Π ending in a principal application of cut may also be replaced by a proof Π' of the same sequent if the degree of any application of cut in Π' is the same as the degree of the cut-formula in Π, and in Π', every inference except possibly one falls under a structural rule with a single premise. Moreover, in [Demri and Goré, 1999] a display sequent calculus for the minimal nominal tense logic is defined, and it is shown that every extension of this calculus by structural rules satisfying conditions C1 – C7 enjoys strong cut-elimination.

3.4 Kracht's algorithm

The class of all properly displayable normal propositional tense logics has been characterized by Kracht [1996]. The idea is to obtain a canonical way of capturing axiomatic extensions of **Kt** by purely structural inference rules over **DKt**.

DEFINITION 28. Let $\boldsymbol{Kt} + \alpha$ be an extension of $\boldsymbol{Kt}$ by a tense logical axiom schema α, and let $\boldsymbol{DKt} + \alpha'$ be an extension of $\boldsymbol{DKt}$ by a set α' of purely structural inference rules. $\boldsymbol{Kt} + \alpha$ is said to be *properly displayed* by $\boldsymbol{DKt} + \alpha'$ if (i) $\boldsymbol{DKt} + \alpha'$ is a proper display calculus and (ii) every derived rule of $\boldsymbol{Kt} + \alpha$ is the τ-translation of a sequent rule derivable in $\boldsymbol{DKt} + \alpha'$.

Now, every axiom schema is equivalent to a schema of the form $A \supset B$, where A and B are implication-free. The schema $A \supset B$ has the same deductive strength as the rule

$$B \to X \vdash A \to X.$$

Moreover, if A and B are only built up from propositional variables, $\boldsymbol{t}$, $\wedge$,

$\vee$, $\langle F\rangle$, and $\langle P\rangle$, then by classical logic and distribution of $\langle F\rangle$ and $\langle P\rangle$ over disjunction, we have

$$A \equiv \bigvee_{i\leq m} C_i \quad \text{and} \quad B \equiv \bigvee_{j\leq n} D_j,$$

where every C_i and D_j is only built up from $\boldsymbol{t}$, $\wedge$, $\langle F\rangle$, and $\langle P\rangle$. Therefore $A \supset B$ may as well be replaced by the rule schemata

$$\frac{D_1 \to Y \quad \ldots \quad D_n \to Y}{C_i \to Y.}$$

These rule schemata can now be translated into purely structural display sequent rules, using the following translation η from formulas of the fragment under consideration into structures:

$$\begin{array}{rclrcl} \eta(p) & = & p & \eta(\boldsymbol{t}) & = & \mathbf{I} \\ \eta(\langle F\rangle A) & = & * \bullet * \eta(A) & \eta(\langle P\rangle A) & = & \bullet\eta(A) \\ \eta(A \wedge B) & = & \eta(A) \wedge \eta(B) & & & \end{array}$$

The resulting structural rules

$$\frac{\eta(D_1) \to Y \quad \ldots \quad \eta(D_n) \to Y}{\eta(C_i) \to Y}$$

may still violate condition C3. In order to avoid this obstruction of proper display, it must be required that in the inducing schema $A \supset B$, the schematic formula A contains each formula variable *only once*. A tense logical formula schema is then said to be *primitive* if it has the form $A \supset B$, A contains each formula variable only once, and A, B are built up from $\boldsymbol{t}$, $\wedge$, $\vee$, $\langle F\rangle$, and $\langle P\rangle$.

LEMMA 29. *Every extension of $\boldsymbol{Kt}$ by primitive axiom schemata can be properly displayed.*

Next, if $\mathbf{DKt} + \alpha'$ properly displays $\mathbf{Kt} + \alpha$, by condition (ii) of Definition 28, the structural rules in α' may all have the form

$$\frac{X_1 \to Y \quad \ldots \quad X_n \to Y}{Z \to Y.}$$

This rule has the same deductive strength as the axiom schema

$$\tau_1(Z) \supset \bigvee_i \tau_1(X_i),$$

which is a primitive formula schema.

THEOREM 30. (Kracht) *An axiomatic extension of $\boldsymbol{Kt}$ can be properly displayed in precisely the case that it is axiomatizable by a set of primitive axiom schemata.*

The question whether an axiomatically presented normal temporal logic Λ is properly displayable thus reduces to the question whether Λ can be axiomatized by primitive axioms over **Kt**. The implicit use of tense logic in the structural language of sequents may help to find simple structural sequent rules expressing less simple modal axiom schemata. The following example is taken from [Kracht, 1996]. The .3 axiom schema $\Box(\Box A \supset \Box B) \vee \Box(\Box B \supset \Box A)$ has the primitive modal equivalent

$$(\Diamond A \wedge \Diamond B) \supset ((\Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A)) \vee \Diamond(A \wedge B)),$$

which in tense logic is equivalent to the simpler primitive schema

$$\langle P\rangle\langle F\rangle A \supset (((\langle F\rangle A \vee A) \vee \langle P\rangle A).$$

Application of Kracht's algorithm results in the following structural rule:

$$X \to Y \quad \bullet X \to Y \quad * \bullet * X \to Y \vdash \bullet * \bullet * X \to Y.$$

Kracht also proves a semantic characterization of the properly displayable tense logics. Let $\mathcal{F}$ be a class of Kripke frames $\langle W, \mathcal{R}, \mathcal{R}^{-1}\rangle$ for temporal logics, where $\mathcal{R}^{-1}$ is the inverse of $\mathcal{R}$ (i.e., $\mathcal{R} = \{(x,y) \mid (y,x) \in \mathcal{R}\}$). A first-order sentence (open formula) over two binary relation symbols R and R^{-1} is said to be *primitive* if it has the form $(\forall)(\exists)A$, where every quantifier is restricted with respect to R or R^{-1}, and A is built up from $\wedge$, $\vee$, and atomic formulas $x = y$, xRy, $xR^{-1}y$, where at least one of x, y is not in the scope of an existential quantifier.

THEOREM 31. (Kracht) *A class $\mathcal{F}$ of Kripke frames for temporal logics is describable by a set of primitive first-order sentences iff the tense logic of $\mathcal{F}$ can be properly displayed.*

The characteristic axiom schemata of quite a few fundamental systems of modal and tense logic are equivalent to primitive schemata, and therefore these systems can be presented as proper display calculi, cf. Table 3.[7] A set of structural sequent rules α' is said to *correspond* to a property of an accessibility relation $\mathcal{R}$ (with a modal or tense logical axiom schema α) iff under the τ-translation the rules in α' are admissible just in the event that

[7] Goré recently observed that Theorem 20 in [Kracht, 1996] is incorrect. This theorem states that an axiomatic extension of **K** can be properly displayed iff it is axiomatizable by a set of primitive *modal* axiom schemata. There are, however, first-order frame properties that correspond to a primitive tense logical schema but fail to correspond to a primitive *modal* axiom schema. An example of such a frame property is weak directedness:

$$\forall s \forall t \forall u(sRt \wedge sRu \supset \exists v(tRv \wedge uRv)).$$

Weak directedness corresponds to the .2 schema $\Diamond\Box A \supset \Box\Diamond A$ (alias $\langle F\rangle[F]A \supset [F]\langle F\rangle A$). Although .2 has no primitive modal equivalent, it has a primitive tense logical equivalent, namely $\langle P\rangle\langle F\rangle A \supset \langle F\rangle\langle P\rangle A$. The latter schema induces a structural rule that may be added to display calculi for (extensions of) **K**. Therefore, **K.2** is properly displayable, although .2 is not primitive.

$\mathcal{R}$ enjoys the property (the rules in α' have the same deductive strength as α). Every axiom schema α in Table 3 corresponds to a purely structural sequent rule α' which can directly be determined from α, see Table 4.

	schema	*primitive equivalent*
D	$[F]A \supset \langle F\rangle A$	$\mathbf{t} \supset \langle F\rangle \mathbf{t}$
T	$[F]A \supset A$	$A \supset \langle F\rangle A$
4	$[F]A \supset [F][F]A$	$\langle F\rangle\langle F\rangle A \supset \langle F\rangle A$
5	$\langle F\rangle A \supset [F]\langle F\rangle A$	$\langle P\rangle\langle F\rangle A \supset \langle F\rangle A$
B	$A \supset [F]\langle F\rangle A$	$(A \wedge \langle F\rangle B) \supset \langle F\rangle(B \wedge \langle F\rangle A)$
Alt1	$\langle F\rangle A \supset [F]A$	$((\langle F\rangle A \wedge \langle F\rangle B) \supset \langle F\rangle(A \wedge B)$
T^c	$A \supset [F]A$	$\langle F\rangle A \supset A$
4^c	$[F][F]A \supset [F]A$	$\langle F\rangle A \supset \langle F\rangle\langle F\rangle A$
.2	$\langle F\rangle[F]A \supset [F]\langle F\rangle A$	$\langle P\rangle\langle F\rangle A \supset \langle F\rangle\langle P\rangle A$
.3	$[F]([F]A \supset [F]B) \vee [F]([F]B \supset [F]A)$	$\langle P\rangle\langle F\rangle A \supset (((\langle F\rangle A \vee A) \vee \langle P\rangle A)$
linf	$\langle F\rangle A \supset [F](((\langle F\rangle A \vee A) \vee \langle P\rangle A)$	$\langle P\rangle\langle F\rangle A \supset (((\langle F\rangle A \vee A) \vee \langle P\rangle A)$
linp	$\langle P\rangle A \supset [P](((\langle P\rangle A \vee A) \vee \langle F\rangle A)$	$\langle F\rangle\langle P\rangle A \supset (((\langle P\rangle A \vee A) \vee \langle F\rangle A)$
V	$[F]A$	$\langle P\rangle \mathbf{t} \supset A$
D_p	$[P]A \supset \langle P\rangle A$	$\mathbf{t} \supset \langle P\rangle \mathbf{t}$
T_p	$[P]A \supset A$	$A \supset \langle P\rangle A$
4_p	$[P]A \supset [P][P]A$	$\langle P\rangle\langle P\rangle A \supset \langle P\rangle A$
5_p	$\langle P\rangle A \supset [P]\langle P\rangle A$	$\langle F\rangle\langle P\rangle A \supset \langle P\rangle A$
B_p	$A \supset [P]\langle P\rangle A$	$(A \wedge \langle P\rangle B) \supset \langle P\rangle(B \wedge \langle P\rangle A)$
Alt1$_p$	$\langle P\rangle A \supset [P]A$	$((\langle P\rangle A \wedge \langle P\rangle B) \supset \langle P\rangle(A \wedge B)$
T^c_p	$A \supset [P]A$	$\langle P\rangle A \supset A$
4^c_p	$[P][P]A \supset [P]A$	$\langle P\rangle A \supset \langle P\rangle\langle P\rangle A$
V_p	$[P]A$	$\langle F\rangle \mathbf{t} \supset A$

Table 3. Axioms and primitive axioms.

Let Γ (Θ) be the set of all (all purely modal) axiom schemata from Table 3, $\overline{\Gamma} \subseteq \Gamma$, $\overline{\Theta} \subseteq \Theta$, $\Gamma' = \{\alpha' \mid \alpha \in \overline{\Gamma}\}$, and $\Theta' = \{\alpha' \mid \alpha \in \overline{\Theta}\}$.

THEOREM 32. *In* $\boldsymbol{DKt}\cup\Gamma'$, $\vdash X \to Y$ *iff* $\vdash \tau_1(X) \supset \tau_2(Y)$ *in* $\boldsymbol{Kt}\cup\Gamma$. *In* $\boldsymbol{DK}\cup\Theta'$, $\vdash X \to Y$ *iff* $\vdash \tau_1(X) \supset \tau_2(Y)$ *in* $\boldsymbol{K}\cup\Theta$.

Proof. This follows from axiomatizability by primitive schemata. ■

THEOREM 33. *Strong cut-elimination holds for* $\boldsymbol{DKt} \cup\Gamma'$ *and* $\boldsymbol{DK} \cup\Theta'$.

Proof. The rules in Γ' and Θ' satisfy conditions C2 – C7. ■

COROLLARY 34. $\boldsymbol{DKt} \cup\Gamma'$ *is a conservative extension of* $\boldsymbol{DK} \cup\Gamma'$.

Kracht's algorithm can be dualized. Every schema $A \supset B$ is interreplaceable with the rule

$$X \to A \vdash X \to B.$$

D'	$* \bullet * \mathbf{I} \rightarrow Y \vdash \mathbf{I} \rightarrow Y$
T'	$* \bullet * X \rightarrow Y \vdash X \rightarrow Y$
$4'$	$* \bullet * X \rightarrow Y \vdash * \bullet \bullet * X \rightarrow Y$
$5'$	$* \bullet * X \rightarrow Y \vdash \bullet * \bullet * X \rightarrow Y$
B'	$* \bullet * (X \circ * \bullet * Y) \rightarrow Z \vdash Y \circ * \bullet * X \rightarrow Z$
$Alt1'$	$* \bullet * (X \circ Y) \rightarrow Z \vdash * \bullet * X \circ * \bullet * Y \rightarrow Z$
$T^{c\prime}$	$X \rightarrow Y \vdash * \bullet * X \rightarrow Y$
$4^{c\prime}$	$* \bullet \bullet * X \rightarrow Y \vdash * \bullet * X \rightarrow Y$
$.2'$	$* \bullet * \bullet X \rightarrow Y \vdash \bullet * \bullet * X \rightarrow Y$
$.3'$	$X \rightarrow Y \quad \bullet X \rightarrow Y \quad * \bullet * X \rightarrow Y \vdash \bullet * \bullet * X \rightarrow Y$
$linf'$	$= .3'$
$linp'$	$X \rightarrow Y \quad \bullet X \rightarrow Y \quad * \bullet * X \rightarrow Y \vdash * \bullet * \bullet X \rightarrow Y$
V'	$X \rightarrow Y \vdash \bullet \mathbf{I} \rightarrow Y$
$D_p{}'$	$\bullet \mathbf{I} \rightarrow Y \vdash \mathbf{I} \rightarrow Y$
$T_p{}'$	$\bullet X \rightarrow Y \vdash X \rightarrow Y$
$4_p{}'$	$\bullet X \rightarrow Y \vdash \bullet \bullet X \rightarrow Y$
$5_p{}'$	$\bullet X \rightarrow Y \vdash * \bullet * \bullet X \rightarrow Y$
$B_p{}'$	$\bullet (X \circ \bullet Y) \rightarrow Z \vdash Y \circ \bullet X \rightarrow Z$
$Alt1_p{}'$	$\bullet (X \circ Y) \rightarrow Z \vdash \bullet X \circ \bullet Y \rightarrow Z$
$T_p^{c\prime}$	$X \rightarrow Y \vdash \bullet X \rightarrow Y$
$4_p^{c\prime}$	$\bullet \bullet X \rightarrow Y \vdash \bullet X \rightarrow Y$
$V_p{}'$	$X \rightarrow Y \vdash * \bullet * \mathbf{I} \rightarrow Y$

Table 4. Structural rules corresponding to axiom schemata.

If A and B are only built up from propositional variables, $\boldsymbol{f}$, $\wedge$, $\vee$, $[F]$, and $[P]$, then by classical logic and distribution of $[F]$ and $[P]$ over conjunction, we have

$$A \equiv \bigwedge\nolimits_{i \leq m} C_i \quad \text{and} \quad B \equiv \bigwedge\nolimits_{j \leq n} D_j,$$

where every C_i and D_j is only built up from $\boldsymbol{f}$, $\vee$, $[F]$, and $[P]$. Therefore $A \supset B$ may be replaced by the rule schemata

$$\frac{X \rightarrow C_1 \quad \ldots \quad X \rightarrow C_m}{X \rightarrow D_j.}$$

These schemata are translatable into purely structural sequent rules using the following translation η' from formulas of the fragment under consideration into structures:

$$\begin{array}{rclrcl} \eta'(p) & = & p & \eta'(\mathbf{f}) & = & \mathbf{I} \\ \eta'([F]A) & = & \bullet \eta'(A) & \eta'([P]A) & = & * \bullet * \eta'(A) \\ \eta'(A \vee B) & = & \eta'(A) \vee \eta'(B) & & & \end{array}$$

The resulting structural rules

$$\frac{X \to \eta'(C_1) \ \ldots \ X \to \eta'(C_m)}{X \to \eta'(D_j)}$$

again may still violate condition C3. In order to avoid the obstruction of proper display, it must be required that in the inducing schema $A \supset B$, the schematic formula B contains each formula variable only once. A tense logical formula schema is then said to be *dually primitive* if it has the form $A \supset B$, B contains each formula variable only once, and A, B are built up from $\boldsymbol{f}$, $\wedge$, $\vee$, $[F]$, and $[P]$.

THEOREM 35. *An axiomatic extension of* $\boldsymbol{Kt}$ *can be properly displayed iff it is axiomatizable by a set of dually primitive axiom schemata.*

For instance, rule T' is equivalent to $X \to \bullet Y \vdash X \to Y$ and $4'$ with $X \to \bullet Y \vdash X \to \bullet \bullet Y$. Moreover, D' is equivalent to $\bullet X \circ \bullet Y \to *\mathbf{I} \vdash X \to *Y$, *Alt1'* with $X \to Y \vdash X \to * \bullet * \bullet Y$, and V' with $\vdash \bullet \mathbf{I} \to X$, see [Wansing, 1994].

The properly displayable modal and tense logics satisfy Došen's Principle. They are all based on the same set of left and right introduction rules, so that the logical operations indeed have the same proof-theoretic, operational meaning in each of these systems. Kracht's characterization results show that many interesting and important intensional logics admit a cut-free display sequent calculus presentation. In Sections 3.8 and 4 other applications of the display calculus are pointed out. Display sequent systems for various non-normal modal logics may be found in [Belnap, 1982].

3.5 Formulas-as-types for temporal logics

It is well-known that every derivation in Gentzen's natural deduction calculus for intuitionistic implicational logic can be encoded by a typed λ-term, and vice versa [Howard, 1980]. In particular, every natural deduction proof can be encoded by a closed term, and every closed term encodes a proof. It is also well-known that every pair of non-convertible typed λ-terms defines different functionals of finite type [Friedman, 1975]. Every type A is associated with an infinite set D^A, every term variable x^A of type A denotes an element from D^A, and every term $M^{(A \rhd B)}$ of type $A \rhd B$ denotes an element from the set $(D^B)^{D^A}$ of all functions from D^A to D^B. Together with the encoding, this interpretation results in a set-theoretic semantics of proofs in intuitionistic implicational logic. In this section, we shall develop a set-theoretic interpretation of sequent proofs in the $\{\boldsymbol{t}, [F], \langle P\rangle, \rhd, \wedge\}$–fragment of the smallest normal temporal intuitionistic (or, for that purpose, minimal) logic **IntKt**. The interpretation is based on the observation that the modalities $\langle P\rangle$ and $[F]$ form a residuated pair with respect to derivability. The encoding of proofs by typed terms should be such that proof-simplification (or normalization) corresponds with a suitable reduction relation on terms, and therefore the set-theoretic semantics of terms has to validate the equalities underlying the reduction rules. The principal cut-elimination steps for $\langle P\rangle$ and $[F]$ reveal that two pairs of term forming operations o_1 and o_2 are needed such that $o_1(o_2(M)) = M$. We shall use the following identities:

$$\bigcup \mathcal{P}a = a \ \text{ and } \ \bigcap \mathcal{S}a = a,$$

where $\mathcal{P}$ is the familiar powerset operation and $\mathcal{S}a =_{\mathrm{def}} \{b \mid a \subseteq b\}$). Since in general $\mathcal{S}a$ is a proper class, we shall restrict the denotations of terms to the universe V_{ω_1}. This is enough to accommodate the sets used as domains of the intended models in Section 3.7.

We shall first define a display sequent system **DIntKt** for the fragment of **IntKt** under consideration, and then present an extension λ_t of the typed λ-calculus. The set of types in λ_t is the set of all formulas in the language $\mathcal{L} = \{\boldsymbol{t}, [F], \langle P\rangle, \rhd, \wedge\}$ based on a denumerable set *Atom* of propositional variables. In Section 3.6 it is proved that term reduction is a homomorphic image of proof-simplification. Next, an encoding of terms by proofs is presented. A set-theoretic semantics of proofs in **DIntKt** is obtained in Section 3.7 by showing that every pair of non-convertible λ_t-terms defines different sets in the set-theoretic universe under consideration. In particular, every term $M^{[F]A}$ denotes an element from $\{\mathcal{P}a \mid a \in D^A\}$, and every term $M^{\langle P\rangle A}$ denotes an element from $\{\mathcal{S}a \mid a \in D^A\}$. Also the formulas-as-types notion of construction for various extensions of **DIntKt** is dealt with and remarks on some related work about formulas-as-types for modal logics are made.

First, we shall define the sequent system **DIntKt**. We assume the following language of structures:

$$X ::= A \mid \mathbf{I} \mid \bullet X \mid X \rtimes Y.$$

A sequent now is an expression $X \to Y$, provided $Y \neq \mathbf{I}$. The declarative meaning of the structure connectives can be made explicit by a translation τ from the set of sequents into the set of $\mathcal{L}$-formulas:

$$\tau(X \to Y) := \tau_1(X) \rhd \tau_2(Y),$$

where τ_i $(i = 1, 2)$ is defined as follows:

$$\begin{array}{rclrcl}
\tau_i(A) & = & A & \tau_1(\mathbf{I}) & = & \boldsymbol{t} \\
\tau_1(X \rtimes Y) & = & \tau_1(X) \wedge \tau_1(Y) & \tau_2(X \rtimes Y) & = & \tau_1(X) \rhd \tau_2(Y) \\
\tau_1(\bullet X) & = & \langle P\rangle\tau_1(X) & \tau_2(\bullet X) & = & [F]\tau_2(X)
\end{array}$$

Given this understanding of the structure connectives, the basic structural rules (4) and (5) from Section 3.1 are assumed. Clearly, the Display Theorem holds for this structural language and calculus.

DEFINITION 36. The display sequent calculus ***DIntKt*** is given by the logical rules (id) and (cut), the basic structural rules (4) and (5), the introduction rules for $\boldsymbol{t}$, $\rhd$, $\langle P\rangle$, $[F]$, and the rules $(\to\wedge)'$ and $(\wedge\to)'$, together with the following structural rules:

$$
\begin{array}{rl}
\text{(empty structure)} & X \to Y \vdash \mathbf{I} \rtimes X \to Y, \quad X \to Y \vdash X \rtimes \mathbf{I} \to Y \\
 & \mathbf{I} \rtimes X \to Y \vdash X \to Y, \quad X \rtimes \mathbf{I} \to Y \vdash X \to Y \\
\text{(associativity)} & (X_1 \rtimes X_2) \rtimes X_3 \to Y \dashv\vdash X_1 \rtimes (X_2 \rtimes X_3) \to Y \\
\text{(permutation)} & X \rtimes Y \to Z \vdash Y \rtimes X \to Z \\
\text{(contraction)} & X \rtimes X \to Y \vdash X \to Y \\
\text{(expansion)} & X \to Y \vdash X \rtimes X \to Y \\
\text{(monotonicity)} & X \to Z \vdash X \rtimes Y \to Z, \quad X \to Z \vdash Y \rtimes X \to Z \\
\text{(necessitation)} & \mathbf{I} \to X \vdash \bullet\mathbf{I} \to X.
\end{array}
$$

To show that **DIntKt** is a display calculus for **IntKt**, we define an axiomatic calculus **HIntKt**.

DEFINITION 37. The system ***HIntKt*** consists of the axiom schemata and rules of the $\{\boldsymbol{t}, \wedge, \rhd\}$–fragment of positive intuitionistic logic, together with

1. $([F]A \wedge [F]B) \rhd [F](A \wedge B)$
2. $[F]\boldsymbol{t}$
3. $A \rhd [F]\langle P\rangle A$
4. $\dfrac{\vdash A \rhd B}{\vdash [F]A \rhd [F]B}$
5. $\dfrac{\vdash A \rhd B}{\vdash \langle P\rangle A \rhd \langle P\rangle B}$

The relational semantics to be presented is a straightforward adaptation of the semantics developed by Bošić and Došen [1984]. A comprehensive survey of intuitionistic modal logics and their algebraic and relational semantics is [Wolter and Zakharyaschev, 1999]. A temporal frame is defined as a structure $\langle W, R_I, R_T\rangle$, where W is a non-empty set (of states), R_I and R_T are binary relations on W, R_I is both reflexive and transitive, and, moreover, (i) $R_I R_T \subseteq R_T R_I$ (i.e. the composition of R_T and R_I is a subset of the composition of R_I and R_T) and (ii) $R_I^{-1} R_T^{-1} \subseteq R_T^{-1} R_I^{-1}$. If $\mathcal{F} = \langle W, R_I, R_T\rangle$ is a temporal frame, the temporal model based on $\mathcal{F}$ is the structure $\langle \mathcal{F}, v\rangle$, where v is a function from $Atom \times W$ into $\{0, 1\}$ satisfying:

(Heredity) $\quad (v(p, u) = 1 \text{ and } uR_I t)$ implies $v(p, t) = 1$.

Let $\mathcal{M} = \langle W, R_I, R_T, v\rangle$ be a temporal model. Verification of a formula A at a state $u \in W$ ($\mathcal{M}, u \models A$) is inductively defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, u \models p & \text{iff} & v(p, u) = 1 \\
\mathcal{M}, u \models \boldsymbol{t} & & \\
\mathcal{M}, u \models A \wedge B & \text{iff} & \mathcal{M}, u \models A \text{ and } \mathcal{M}, u \models B \\
\mathcal{M}, u \models A \rhd B & \text{iff} & (\forall t \in W)\, uR_I t \text{ implies } [\mathcal{M}, t \not\models A \text{ or } \mathcal{M}, t \models B] \\
\mathcal{M}, u \models [F]A & \text{iff} & (\forall t \in W)\, uR_T t \text{ implies } \mathcal{M}, t \models A \\
\mathcal{M}, u \models \langle P\rangle A & \text{iff} & (\exists t \in W)\, tR_T u \text{ and } \mathcal{M}, t \models A
\end{array}
$$

For every formula A, if A is verified at state u and $uR_I t$, then A is also verified at t. Condition (i) ensures this general heredity property for formulas $[F]A$, and condition (ii) ensures it for formulas $\langle P\rangle A$. A formula A is true in a model $\langle W, R_T, R_I, v\rangle$ if A is verified at every $u \in W$, and A is said to be true on a frame $\mathcal{F}$, if A is valid in every model based on $\mathcal{F}$. If $\mathcal{K}$ is a class of models (frames), A is said to be valid in $\mathcal{K}$ iff A is valid in every model (valid on every frame) in $\mathcal{K}$.

THEOREM 38. ***HIntKt*** *is sound and complete with respect to the class of all temporal frames, i.e. for every $\mathcal{L}$–formula A, A is provable in* ***HIntKt*** *iff A is valid in the class of all temporal frames.*

Proof. Soundness is shown by induction on proofs in **HIntKt**; for completeness see Appendix B. ■

LEMMA 39. *(1) If $\vdash A$ in* ***HIntKt****, then $\vdash \mathbf{I} \to A$ in* ***DIntKt****, and (2) If $\vdash X \to Y$ in* ***DIntKt****, then $\vdash \tau(X \to Y)$ in* ***HIntKt****.*

Proof. (1) By induction on proofs in **HIntKt**. We shall consider only two example cases:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\bullet A \to \langle P\rangle A}}{A \to [F]\langle P\rangle A}}{A \rtimes \mathbf{I} \to [F]\langle P\rangle A}}{\mathbf{I} \to A \rtimes [F]\langle P\rangle A}}{\mathbf{I} \to A \rhd [F]\langle P\rangle A}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{[F]A \to \bullet A}}{[F]A \rtimes [F]B \to \bullet A}}{\bullet([F]A \rtimes [F]B) \to A} \qquad \dfrac{\dfrac{\dfrac{B \to B}{[F]B \to \bullet B}}{[F]A \rtimes [F]B \to \bullet B}}{\bullet([F]A \rtimes [F]B) \to B}}{\bullet([F]A \rtimes [F]B) \rtimes \bullet([F]A \rtimes [F]B) \to A \wedge B}}{\bullet([F]A \rtimes [F]B) \to A \wedge B}}{([F]A \rtimes [F]B) \to [F](A \wedge B)}}{([F]A \wedge [F]B) \to [F](A \wedge B)}}{([F]A \wedge [F]B) \rtimes \mathbf{I} \to [F](A \wedge B)}}{\mathbf{I} \to ([F]A \wedge [F]B) \rtimes [F](A \wedge B)}}{\mathbf{I} \to ([F]A \wedge [F]B) \rhd [F](A \wedge B)}
$$

(2) By induction on proofs in **DIntKt**. ■

COROLLARY 40. *In* $\boldsymbol{HIntKt}$, $\vdash A$ *iff* $\vdash \mathbf{I} \to A$ *in* $\boldsymbol{DIntKt}$.

By induction on the complexity of X, one can prove the following

LEMMA 41. *In every extension of* $\boldsymbol{DIntKt}$ *by structural rules, it holds that* $\vdash X \to \tau_1(X)$ *and* $\vdash \tau_2(X) \to X$.

THEOREM 42. *In* $\boldsymbol{DIntKt}$, $\vdash X \to Y$ *iff* $\vdash \tau(X \to Y)$ *in* $\boldsymbol{HIntKt}$.

Proof. Analogous to the proof of Theorem 20. ■

Since **DIntKt** is a proper display calculus, we have the following

THEOREM 43. $\boldsymbol{DIntKt}$ *enjoys strong cut-elimination.*

Take any terminating cut-elimination algorithm $elim_c$ for **DIntKt**. We may also define a binary relation $\rightsquigarrow_s$ on the set of proofs in **DIntKt** by the following stipulations:

$$\dfrac{\dfrac{A \to A \qquad B \to B}{A \rtimes B \to A \wedge B}}{A \wedge B \to A \wedge B} \quad \rightsquigarrow_s \quad A \wedge B \to A \wedge B$$

$$\dfrac{\dfrac{A \to A \qquad B \to B}{A \rhd B \to A \rtimes B}}{A \rhd B \to A \rhd B} \quad \rightsquigarrow_s \quad A \rhd B \to A \rhd B$$

If $\Pi \rightsquigarrow_s \Pi'$, we say that in Π' a redundant part of Π has been removed. Let $elim_r$ denote the terminating algorithm that removes redundant parts of a proof in top-down left to right order, so that a redundant part is removed only if it has no redundant part above it. Obviously, in any extension of **DIntKt**, every proof of a sequent s can be converted into a proof of s containing no redundant part. Let *elim* denote $elim_r elim_c$, i.e. the composition of $elim_r$ and $elim_c$. The algorithm *elim* is the process of proof simplification to be considered. We assume that $elim(\Pi) = \Pi$ if Π contains no application of (cut) and no redundant part.

3.6 The typed λ-calculus λ_t

The set T of type symbols (or just types) is the set of all $\mathcal{L}$-formulas. The set V of term variables is defined as $\{v_i^A \mid 0 < i \in \omega, A \in T\}$.

DEFINITION 44. The set Term of typed terms is defined as the smallest set Δ such that

1. $V \subseteq \Delta$;
2. if M^A, $N^B \in \Delta$, then $\langle M^A, N^B\rangle^{(A\wedge B)} \in \Delta$;
3. $M^{(A\wedge B)} \in \Delta$, then $(M^{(A\wedge B)})_0^A$, $(M^{(A\wedge B)})_1^B \in \Delta$;

4. if $x^A \in V$ and $M^B \in \Delta$, then $(\lambda x^A M^B)^{(A \rhd B)} \in \Delta$;
5. if $M^{(A \rhd B)}$, $N^A \in \Delta$, then $(M^{(A \rhd B)}, N^A)^B \in \Delta$;
6. if $M^A \in \Delta$, then $(\mathcal{P}M)^{[F]A}$, $(\mathcal{S}M)^{\langle P \rangle A} \in \Delta$;
7. if $M^{[F]A} \in \Delta$, then $(\cup M^{[F]A})^A \in \Delta$;
8. if $M^{\langle P \rangle A} \in \Delta$, then $(\cap M^{\langle P \rangle A})^A \in \Delta$.

A term M^A is said to be a term of type A; obviously, every term has a unique type. If confusion is unlikely to arise, we shall often write M instead of M^A and omit parentheses not needed for disambiguation. The set $fv(M)$ of free variables of M, the set of subterms of M, and $M[x^A := N^A]$, the result of substituting term N of type A for every occurrence of x^A in M are inductively defined in the obvious way. If a variable x in M is not an element of $fv(M)$, x is said to be a bound variable of M. The set of bound variables of M is denoted as $bv(M)$. We shall also write $M(x_1^{A_1}, \ldots, x_n^{A_n})$ to express that $x_1, \ldots, x_n \in fv(M)$. If $M(x_1^{A_1}, \ldots, x_n^{A_n})$ and $N_1, \ldots, N_n$ are terms of types $A_1, \ldots, A_n$, then $M(N_1, \ldots, N_n)$ is the result of substituting in M the variables x_i by N_i. We shall use '$\equiv$' to denote syntactic identity between term.

DEFINITION 45. The typed λ-calculus λ_t consists of the following rules and axiom schemata:

1. $\lambda x^A M = (\lambda y^A M[x := y])$, if $y \notin (fv(M) \cup bv(M))$;
2. $\lambda x(M, x) = M$, if $x \notin fv(M)$;
3. $(\lambda x M)N = M[x := N]$, if $bv(M) \cup fv(N) = \emptyset$;
4. $(\langle M_0, M_1 \rangle)_i = M_i$;
5. $\langle (M)_0, (M)_1 \rangle = M$;
6. $\cup \mathcal{P} M = M$;
7. $\cap \mathcal{S} M = M$;
8. $M^A = M^A$;
9. $M = N \vdash N = M$; $M = N,\ N = G \vdash M = G$;
10. $M = N \vdash (G, M) = (G, N)$; $M = N \vdash (M, G) = (N, G)$;
11. $M = N \vdash \lambda x M = \lambda x N$;
12. $M = N \vdash \mathcal{P} M = \mathcal{P} N$; $M = N \vdash \cup M = \cup N$.

DEFINITION 46. The binary relations on Term, $\rightarrow_r$ (one-step reduction), $\twoheadrightarrow_r$ (reduction), and $=_r$ (equality) are defined as follows:

1.
   - $\lambda x(Mx) \rightarrow_r M$, if $x \notin fv(M)$;
   - $(\lambda x M)N \rightarrow_r M[x := N]$, if $bv(M) \cup fv(N) = \emptyset$;
   - $(\langle M, N\rangle)_0 \rightarrow_r M$; $(\langle M, N\rangle)_1 \rightarrow_r N$;
   - $\langle (M)_0, (M)_1\rangle \rightarrow_r M$;
   - $\cup \mathcal{P} M \rightarrow_r M$; $\cap \mathcal{S} M \rightarrow_r M$;
   - if $M^{A \rhd B} \rightarrow_r N^{A \rhd B}$, then $(M, G^A) \rightarrow_r (N, G)$;
   - if $M^{A \wedge B} \rightarrow_r N^{A \wedge B}$, then $(M)_i \rightarrow_r (N)_i$;
   - if $M^A \rightarrow_r N^A$, then $\lambda x M \rightarrow_r \lambda x N$, $(G^{A \rhd B} M) \rightarrow_r (GN)$, $\langle M, G\rangle \rightarrow_r \langle N, G\rangle$, $\langle G, M\rangle \rightarrow_r \langle G, N\rangle$, $\mathcal{P}M \rightarrow_r \mathcal{P}N$, $\mathcal{S}M \rightarrow_r \mathcal{S}N$, $\cap M \rightarrow_r \cap N$, $\cup M \rightarrow_r \cup N$.
2. $\twoheadrightarrow_r$ is the reflexive transitive closure of $\rightarrow_r$;
3. $=_r$ is the equivalence relation generated by $\twoheadrightarrow_r$.

DEFINITION 47. λ_t-terms $\lambda x(Mx)$ (where $x \notin fv(M)$), $(\lambda x M)N$ (where $bv(M) \cup fv(N) = \emptyset$), $(\langle M, N\rangle)_0$, $(\langle M, N\rangle)_1$, $\langle (M)_0, (M)_1\rangle$, $\cup \mathcal{P} M$, and $\cap \mathcal{S} M$ are called redexes. A term M is a normal form (*nf*) if it has no redex as a subterm, and M has a *nf* if there is a *nf* N such that $M =_r N$. M is said to be strongly normalizable with respect to $\twoheadrightarrow_r$ ($sn(M)$) if every sequence of reduction steps starting at M is finite.

THEOREM 48. *Every $M \in$ Term is strongly normalizable with respect to $\twoheadrightarrow_r$.*

Proof. See Appendix C. ∎

Let *norm*(M) refer to the iterated contraction of the leftmost redex in M. Since by the previous theorem, every reduction starting at M is finite, *norm* is a terminating normalization algorithm with respect to $\twoheadrightarrow_r$.

We shall now encode proofs by giving recipes for building up constructions of sequents. Every formula occurring in an antecedent part of a sequent s is said to be an antecedent formula component of s.

DEFINITION 49. A construction of a sequent s is a term M^A such that an occurrence of A is the succedent part of s, and every type of a free variable of M is an antecedent formula component of s.

This notion of construction is a straightforward adaptation of the notion of construction for ordinary natural deduction and sequent calculi. The *set* of types of the free variables occurring in the term encoding a derivation Π is a subset of the set of assumptions on which Π depends. Therefore applications of structural inference rules are not reflected by term modifications, and variations of structural rules are captured by imposing conditions on variable binding and occurrences of free variables in the encoding terms (see, for instance, [van Benthem, 1986, Chapter 7], [van Benthem, 1991], [Helman, 1977], [Wansing, 1992]).

OBSERVATION 50. Given a proof in ***DIntKt*** of a sequent s, one can find a construction M of s.

Proof. We define a function f from the set Π**DIntKt** of proofs in **DIntKt** to $Term$ such that $f(\Pi)$ is a construction of the conclusion sequent of Π. The pairs of sequent rules and terms or term construction rules in Table 5 amount to an inductive definition of f. The variables newly introduced into the conclusion of a term construction rule are the numerically first variables of the types indicated not occurring in the premise term. ■

Clearly, *norm* is a function on *Term*. Let Π^+**DIntKt** denote the set of all proofs in **DIntKt** containing an application of (cut) or a redundant part, and let Π^-**DIntKt** denote the set of all cut-free proofs in **DIntKt** containing no redundant part. Let $+Term$ denote the set of all terms that are not normal forms, and let $-Term$ denote the set of all terms that are normal forms.

THEOREM 51. *Let* $\mathcal{A} = \langle \Pi\mathbf{DIntKt}, elim \rangle$ *and* $\mathcal{B} = \langle Term, norm \rangle$. *The function* f *defined in the proof of Observation 50 is a homomorphism from* $\mathcal{A}$ *to* $\mathcal{B}$.

Proof. See Appendix D. ■

Under the encoding of proofs by terms, surjective pairing ($\langle (M)_0, (M)_1 \rangle \rightarrow_r M$) and η–reduction ($\lambda x(Mx) \rightarrow_r M$, if $x \notin fv(M)$) correspond with replacing proofs

$$\dfrac{\dfrac{A \rightarrow A \quad B \rightarrow B}{A \rtimes B \rightarrow A \wedge B}}{A \wedge B \rightarrow A \wedge B} \qquad \text{and} \qquad \dfrac{\dfrac{A \rightarrow A \quad B \rightarrow B}{A \rhd B \rightarrow A \rtimes B}}{A \rhd B \rightarrow A \rhd B}$$

by the axiomatic sequents $A \wedge B \rightarrow A \wedge B$ and $A \rhd B \rightarrow A \rhd B$, respectively. Note that there are no analogues of surjective pairing and η–reduction that correspond with a replacement of proofs of $[F]A \rightarrow [F]A$ and $\langle P \rangle A \rightarrow \langle P \rangle A$ from $A \rightarrow A$ by the axiomatic sequents $[F]A \rightarrow [F]A$ and $\langle P \rangle A \rightarrow \langle P \rangle A$. Moreover, since in the encoding applications of structural rules are not reflected by term formation steps, it is in general *not* the case that if $M = f(\Pi)$, Π can be uniquely reconstructed from M.

Logical rules	
$A \to A$	v_1^A
$\dfrac{X \to A \quad A \to Y}{X \to Y}$	$\dfrac{M^A \quad N(x^A)}{N[x := M]}$
Structural rules	
$\dfrac{s}{s'}$	$\dfrac{M}{M}$
Intuitionistic connective rules	
$\mathbf{I} \to \boldsymbol{t}$	$v_1^{\boldsymbol{t}}$
$\dfrac{\mathbf{I} \to X}{\boldsymbol{t} \to X}$	$\dfrac{M}{M}$
$\dfrac{X \to A \quad Y \to B}{X \rtimes Y \to A \wedge B}$	$\dfrac{M^A \quad N^B}{\langle M, N \rangle}$
$\dfrac{A \rtimes B \to X}{A \wedge B \to X}$	$\dfrac{M(x^A, y^B)}{M((z^{A \wedge B})_0, (z^{A \wedge B})_1)}$
$\dfrac{X \to A \rtimes B}{X \to A \rhd B}$	$\dfrac{M(x^A)}{\lambda x^A M}$
$\dfrac{X \to A \quad B \to Y}{A \rhd B \to X \rtimes Y}$	$\dfrac{M^A \quad N(x^B)}{N[x := (y^{(A \rhd B)}, M)]}$
Modal connective rules	
$\dfrac{\bullet X \to A}{X \to [F]A}$	$\dfrac{M}{\mathcal{P}M}$
$\dfrac{A \to X}{[F]A \to \bullet X}$	$\dfrac{M(x^A)}{M(\cup y^{[F]A})}$
$\dfrac{X \to A}{\bullet X \to \langle P \rangle A}$	$\dfrac{M}{\mathcal{S}M}$
$\dfrac{A \to \bullet X}{\langle P \rangle A \to X}$	$\dfrac{M(x^A)}{M(\cap y^{\langle P \rangle A})}$

Table 5. Sequent rules and term construction rules.

3.7 A denotational semantics of proofs

We shall now define models for λ_t. The completeness proof to be given straightforwardly extends H. Friedman's [1975] completeness proof for typed λ–calculus. The plan of the proof is as follows: first it is shown that λ_t is sound and complete with respect to the class of all models. This is achieved by defining a canonical model that itself characterizes λ_t. Then a notion of intended model is defined. In such models the typed terms have their intended set-theoretic interpretation. In order to characterize provable equality of terms in λ_t by validity in all intended models, it is shown that for every intended model $\mathcal{M}$, there exists a 'partial homomorphism' from $\mathcal{M}$ onto the canonical model. Since such partial homomorphisms turn out to preserve validity, λ_t is sound and complete with respect to the class of all intended models.

DEFINITION 52. A structure $\mathcal{F} = \langle \{D^A\}, \{\mathsf{AP}_{A,B}\}, \{\mathsf{PRO}^0_{A,B}\}, \{\mathsf{PRO}^1_{A,B}\}, \{\mathsf{PAIR}_{A,B}\}, \{\mathsf{P}_A\}, \{\mathsf{S}_A\}, \{\mathsf{P}{\downarrow}_A\}, \{\mathsf{S}{\downarrow}_A\} \rangle$ is called a type structure frame (or just a frame) iff for all types A, B:

1. D^A (the domain of type A) is a non-empty set;
2. $\mathsf{AP}_{A,B} : D^{(A \rhd B)} \times D^A \longrightarrow D^B$,
   $\mathsf{PRO}^0_{A,B} : D^{(A \wedge B)} \longrightarrow D^A$,
   $\mathsf{PRO}^1_{A,B} : D^{(A \wedge B)} \longrightarrow D^B$,
   $\mathsf{PAIR}_{A,B} : D^A \times D^B \longrightarrow D^{(A \wedge B)}$,
   $\mathsf{P}_A : D^A \longrightarrow D^{[F]A}$,
   $\mathsf{S}_A : D^A \longrightarrow D^{\langle P \rangle A}$,
   $\mathsf{P}{\downarrow}_A$: $D^{[F]A} \longrightarrow D^A$,
   $\mathsf{S}{\downarrow}_A$: $D^{\langle P \rangle A} \longrightarrow D^A$;
3. *(extensionality)* if $a, b \in D^{(A \rhd B)}$ and $(\forall c \in D^A)$ we have $(\mathsf{AP}_{A,B}(a,c) = \mathsf{AP}_{A,B}(b,c))$, then $a = b$;
4. *(pro)* for all $a \in D^A$, $b \in D^B$:
   $\mathsf{PRO}^0_{A,B}(\mathsf{PAIR}_{A,B}(a,b)) = a$, $\mathsf{PRO}^1_{A,B}(\mathsf{PAIR}_{A,B}(a,b)) = b$;
5. *(pair)* for all $a \in D^{A \wedge B}$: $\mathsf{PAIR}_{A,B}(\mathsf{PRO}^0_{A,B}(a), \mathsf{Pro}^1_{A,B}(a)) = a$;
6. *(future)* for all $a \in D^A$: $\mathsf{P}{\downarrow}\ (\mathsf{P}a) = a$;
7. *(past)* for all $a \in D^A$: $\mathsf{S}{\downarrow}\ (\mathsf{S}a) = a$.

An assignment in a frame $\langle \{D^A\}, \{\mathsf{AP}_{A,B}\}, \{\mathsf{PRO}^0_{A,B}\}, \{\mathsf{PRO}^1_{A,B}\}, \{\mathsf{PAIR}_{A,B}\}, \{\mathsf{P}_A\}, \{\mathsf{S}_A\}, \{\mathsf{P}{\downarrow}_A\}, \{\mathsf{S}{\downarrow}_A\} \rangle$ is a function f defined on the set V of term variables such that $f(x^A) \in D^A$. The set of all assignments in a given frame is denoted by Asg. If $y \in V$, then f^y_a is defined by $f^y_a(x) = f(x)$, if $x \neq y$, $f^y_a(y) = a$.

DEFINITION 53. Suppose that $\mathcal{F} = \langle \{D^A\}, \{\mathsf{AP}_{A,B}\}, \{\mathsf{PRO}^0_{A,B}\}, \{\mathsf{PRO}^1_{A,B}\}, \{\mathsf{PAIR}_{A,B}\}, \{\mathsf{P}_A\}, \{\mathsf{S}_A\}, \{\mathsf{P}\downarrow_A\}, \{\mathsf{S}\downarrow_A\} \rangle$ is a frame. Then $\langle \mathcal{F}, val \rangle$ is said to be a type structure model (or just a model) based on $\mathcal{F}$ iff val is the valuation function from $Term \times Asg$ to $\bigcup_{A \in T} D^A$ such that:

1. $val(x, f) = f(x)$;
2. $\mathsf{AP}_{A,B}(val((\lambda x M), f), a) = val(M, f^x_a)$, $\forall a \in D^A$;
3. $val((M^{(A \triangleright B)}, N^B), f) = \mathsf{AP}_{A,B}(val(M, f), val(N, f))$;
4. $val(\langle M^A, N^B \rangle, f) = \mathsf{PAIR}_{A,B}(val(M, f), val(N, f))$;
5. $val((M^{(A \wedge B)})_i, f) = \mathsf{PRO}^i_{A,B}(val(M, f))$, $i = 0, 1$;
6. $val((\mathcal{P}M^A)^{[F]A}, f) = \mathsf{P}_A(val(M, f))$;
7. $val((\mathcal{S}M^A)^{\langle P \rangle A}, f) = \mathsf{S}_A(val(M, f))$;
8. $val((\cup M^{[F]A})^A, f) = \mathsf{P}\!\downarrow_A (val(M, f))$;
9. $val((\cap M^{\langle P \rangle A})^A, f) = \mathsf{S}\!\downarrow_A (val(M, f))$.

Let $\mathcal{M} = \langle \mathcal{F}, val \rangle$ be a model.

LEMMA 54. *(1) $val(M[x := N], f) = val(M, f^x_{val(N,f)})$, if $bv(M) \cap fv(N) = \emptyset$. (2) $val(M[x := y], f^y_a) = val(M, f^x_a)$, if $y \notin bv(M) \cup fv(M)$.*

Proof. (1) By induction on M, for fixed N; (2) by (1). ■

The equality $M = N$ is said to hold in $\mathcal{M}$ under assignment f ($\mathcal{M}, f \models M = N$) iff $val(M, f) = val(N, f)$. $M = N$ is called valid in $\mathcal{M}$ ($\mathcal{M} \models M = N$) iff $\mathcal{M}, f \models M = N$, for all $f \in Asg$. $M = N$ is said to be valid in a class $\mathcal{K}$ of models, if $\mathcal{M} \models M = N$, for each $\mathcal{M} \in \mathcal{K}$.

OBSERVATION 55. (Soundness) If $M = N$ is provable in λ_t, then $M = N$ is valid in the class of all models.

Proof. By induction on proofs in λ_t. We must show that every axiom is valid in every model, and that the rules of inference preserve validity. We shall consider two cases not already dealt with in [Friedman, 1975].

$$\begin{array}{rl}
& \langle (M)_0, (M)_1 \rangle = M: \\
& val((\langle M,N\rangle)_0, (\langle M,N\rangle)_1\rangle, f) \\
= & \mathsf{PAIR}(val((\langle M,N\rangle)_0, f), val((\langle M,N\rangle)_1, f)) \\
= & \mathsf{PAIR}(\mathsf{PRO}^0(\mathsf{PAIR}(val(M,f), val(N,f))), \\
& \mathsf{PRO}^1(\mathsf{PAIR}(val(M,f), val(N,f)))) \\
= & \mathsf{PAIR}(val(M,f), val(N,f)) = val(\langle M,N\rangle, f).
\end{array}$$

$$\begin{array}{rl}
& \cap \mathcal{S} M = M: \\
& val(\cap \mathcal{S} M, f) \\
= & \mathsf{S}{\downarrow}\ (val(\mathcal{S} M, f) \\
= & \mathsf{S}{\downarrow}\ (\mathsf{S}(val(M,f)) \\
= & val(M,f)
\end{array}$$ ■

Next, we define the frame $\mathcal{F}_0$ on which the canonical model is based. Let $\mid M \mid = \{N \mid \vdash_{\lambda_t} M = N\}$; $\mid M \mid$ is the equivalence class of M with respect to provable equality in λ_t.

DEFINITION 56. $\mathcal{F}_0 = \langle \{D^A\}, \{\mathsf{AP}_{A,B}\}, \{\mathsf{PRO}^0_{A,B}\}, \{\mathsf{PRO}^1_{A,B}\}, \{\mathsf{PAIR}_{A,B}\}, \{\mathsf{P}_A\}, \{\mathsf{S}_A\}, \{\mathsf{P}{\downarrow}_A\}, \{\mathsf{S}{\downarrow}_A\} \rangle$ is defined as follows:

- $D^A = \{\mid M \mid \mid M$ is of type $A\}$;
- $\mathsf{AP}_{A,B}(\mid M^{A \triangleright B} \mid, \mid N^A \mid) = \mid (M,N) \mid$;
- $\mathsf{PRO}^0_{A,B}(\mid M^{A \wedge B} \mid) = \mid (M)_0 \mid$;
- $\mathsf{PRO}^1_{A,B}(\mid M^{A \wedge B} \mid) = \mid (M)_1 \mid$;
- $\mathsf{PAIR}_{A,B}(\mid M^A \mid, \mid N^B \mid) = \mid \langle M,N\rangle \mid$;
- $\mathsf{P}_A(\mid M^A \mid) = \mid \mathcal{P} M \mid$;
- $\mathsf{S}_A(\mid M^A \mid) = \mid \mathcal{S} M \mid$;
- $\mathsf{P}{\downarrow}_A\ (\mid M^A \mid) = \mid \cup M \mid$;
- $\mathsf{S}{\downarrow}_A\ (\mid M^A \mid) = \mid \cap M \mid$.

LEMMA 57. *$\mathcal{F}_0$ is a frame.*

Proof. Clearly, D^A is a non-empty set, and $\mathsf{AP}_{A,B}$, $\mathsf{PRO}^0_{A,B}$, $\mathsf{PRO}^1_{A,B}$, $\mathsf{PAIR}_{A,B}$, P_A, S_A, $\mathsf{P}{\downarrow}_A$, and $\mathsf{S}{\downarrow}_A$ are functions with appropriate domain and range, for all types A and B. For (extensionality) see [Friedman, 1975]. For (pro), (pair), (future), and (past), use the obvious equalities. ■

A function $g : V \longrightarrow Term$ is called a substitution, if $g(x)$ and x are of the same type. A substitution is called regular, if for pairwise distinct variables x, y, $fv(g(x)) \cap fv(g(y)) = \emptyset$. Let $M(g)$ denote the result of simultaneously replacing in M every free occurrence of each variable x by $g(x)$. It can easily be shown that if $M \in \mathit{Term}$ and Γ is a finite set of variables, then there is an N such that $\vdash_{\lambda_t} M = N$, $fv(M) = fv(N)$, and $bv(N) \cap \Gamma = \emptyset$.

DEFINITION 58. Suppose f is an assignment in $\mathcal{F}_0$ and g is a regular substitution such that $f(x) = \mid g(x) \mid$, for every $x \in V$. For a given term M, choose a term N such that $\vdash_{\lambda_t} M = N$ and for every $x \in fv(N)$, $bv(N) \cap fv(g(x)) = \emptyset$. Then $val(M, f)$ is defined by $val(M, f) = \mid N(g) \mid$.

It can be shown that $val : \mathit{Term} \times \mathit{Asg} \longrightarrow \bigcup_A D^A$, and $\vdash_{\lambda_t} M = N$ implies $val(M, f) = val(N, f)$, cf. [Friedman, 1975].

LEMMA 59. $\mathcal{M}_0 = \langle \mathcal{F}_0, val \rangle$ *is a type structure model.*

Proof. We consider those conditions not already assumed in Friedman's paper. Let g be a regular substitution and $f(x) = \mid g(x) \mid$, for $f \in \mathit{Asg}$. Choose M_1, N_1 such that $\vdash_{\lambda_t} M = M_1$, $\vdash_{\lambda_t} N = N_1$, and $bv(M_1) \cap fv(g(x))$ $= bv(N_1) \cap fv(g(x)) = \emptyset$, for every $x \in fv(M_1) \cup fv(N_1)$.

$4 : val(\langle M, N \rangle, f) = \mid \langle M_1, N_1 \rangle(g) \mid =$
$\quad \mathsf{PAIR}(\mid M_1(g) \mid, \mid N_1(g) \mid) = \mathsf{PAIR}(val(M, f), val(N, f))$.

$5 : val((M)_i, f) = \mid (M_1)_i(g) \mid = \mathsf{PRO}^i(\mid M_1(g) \mid) = \mathsf{PRO}^i(val(M, f))$.

$6 : val((\mathcal{P}M^A)^{[F]A}, f) = \mid \mathcal{P}M_1(g) \mid = \mathsf{P}_A(\mid M_1(g) \mid) = \mathsf{P}_A(val(M, f))$.

$8 : val((\cup M^{[F]A})^A, f) = \mid \cup M_1(g) \mid = \mathsf{P}{\downarrow_A} (\mid M_1(g) \mid) = \mathsf{P}{\downarrow_A} (val(M, f))$.

7 and 9 : analogous to the previous two cases. ■

THEOREM 60. (Completeness) *If $M = N$ is valid in the class of all models, then $\vdash_{\lambda_t} M = N$.*

Proof. Suppose $\not\vdash_{\lambda_t} M = N$. Choose M_1, N_1 such that $\vdash_{\lambda_t} M = N_1$, $\vdash_{\lambda^c} N = N_1$, and $bv(M_1) \cap fv(M_1) = bv(N_1) \cap fv(N_1) = \emptyset$. Then $val(M, f)$ $= \mid M_1 \mid \neq \mid N_1 \mid = val(N, f)$, for $f(x) = \mid \mathsf{id}(x) \mid$, for all $x \in V$, where id is the identity function on V. Thus, $\mathcal{M}_0 \not\models M = N$. ■

We now define the intended models. Following the terminology of Friedman, we shall call the frames underlying an intended model 'full temporal type structures over infinite sets'.

DEFINITION 61. A type structure frame $\mathcal{F} = \langle \{\mathbf{D}^A\}, \{\mathbf{AP}_{A,B}\}, \{\mathbf{PRO}^0_{A,B}\}, \{\mathbf{PRO}^1_{A,B}\}, \{\mathbf{PAIR}_{A,B}\}, \{\mathbf{P}_A\}, \{\mathbf{S}_A\}, \{\mathbf{P}{\downarrow_A}\}, \{\mathbf{S}{\downarrow_A}\} \rangle$ is said to be a full temporal type structure over infinite sets, if

- $\mathbf{D}^{\boldsymbol{t}}$ is infinite, and for every $p \in \mathit{Atom}$, $\mathbf{D}^p$ is infinite;
- $\mathbf{D}^{A \wedge B} = \mathbf{D}^A \times \mathbf{D}^B$;
- $\mathbf{D}^{A \rhd B} = (\mathbf{D}^B)^{\mathbf{D}^A}$;

- $\mathbf{D}^{[F]A} = \{\mathcal{P}a \mid a \in \mathbf{D}^A\}$; $\mathbf{D}^{\langle P\rangle A} = \{\mathcal{S}a \mid a \in \mathbf{D}^A\}$;
- $\mathbf{AP}_{A,B}(a,b) = a(b)$;
- $\mathbf{PRO}^0_{A,B}(\langle a,b\rangle) = a$; $\mathbf{PRO}^1_{A,B}(\langle a,b\rangle) = b$;
- $\mathbf{PAIR}_{A,B}(a,b) = \langle a,b\rangle$;
- $\mathbf{P}_A(a) = \mathcal{P}a$; $\mathbf{S}_A(a) = \mathcal{S}a$;
- $\mathbf{P}{\downarrow_A}\ (a) = \cup a$; $\mathbf{S}{\downarrow_A}\ (a) = \cap a$.

DEFINITION 62. Let $\mathcal{F} = \langle\ \{D^A\},\ \{\mathsf{AP}_{A,B}\},\ \{\mathsf{PRO}^0_{A,B}\},\ \{\mathsf{PRO}^1_{A,B}\},\ \{\mathsf{PAIR}_{A,B}\},\ \{\mathsf{P}_A\},\ \{\mathsf{S}_A\},\ \{\mathsf{P}\downarrow_A\},\ \{\mathsf{S}\downarrow_A\}\ \rangle$, $\mathcal{F}^* = \langle\{D^{*A}\},\ \{\mathsf{AP}^*_{A,B}\},\ \{\mathsf{PRO}^{*0}_{A,B}\},\ \{\mathsf{PRO}^{*1}_{A,B}\},\ \{\mathsf{PAIR}^*_{A,B}\},\ \{\mathsf{P}^*_A\},\ \{\mathsf{S}^*_A\},\ \{\mathsf{P}{\downarrow}^*_A\},\ \{\mathsf{S}{\downarrow}^*_A\}\ \rangle$ be frames, and let $\mathcal{M} = \langle\mathcal{F}, val\rangle$ and $\mathcal{M}^* = \langle\mathcal{F}^*, val^*\rangle$ be models. A family of functions $\{f_A\}$ is called a partial homomorphism from $\mathcal{M}$ onto $\mathcal{M}^*$ iff

1. for each type A, f_A is a partial function from D^A onto D^{*A};
2. if $f_{A\rhd B}(a)$ exists, then $f_B(\mathsf{AP}_{A,B}(a,b)) = \mathsf{AP}^*_{A,B}(f_{A\rhd B}(a), f_A(b))$, for all b in the domain of f_A,
3. if $f_A(a)$, $f_B(b)$ exist, then $f_{A\wedge B}(\mathsf{PAIR}_{A,B}(a,b)) = \mathsf{PAIR}^*_{A,B}(f_A(a), f_B(b))$;
4. if $f_{A\wedge B}(a)$ exists, then $f_A(\mathsf{PRO}^0_{A,B}(a)) = \mathsf{PRO}^{*0}_{A,B}(f_{A\wedge B}(a))$;
5. if $f_{A\wedge B}(a)$ exists, then $f_B(\mathsf{PRO}^1_{A,B}(a)) = \mathsf{PRO}^{*1}_{A,B}(f_{A\wedge B}(a))$;
6. if $f_A(a)$ exists, then $f_{[F]A}(\mathsf{P}_A(a)) = \mathsf{P}^*_A(f_A(a))$; $f_{\langle P\rangle A}(\mathsf{S}_A(a)) = \mathsf{S}^*_A(f_A(a))$;
7. if $f_{[F]A}(a)$, $f_{\langle P\rangle A}(b)$ exist, then $f_A(\mathsf{P}{\downarrow_A}\ (a)) = \mathsf{P}{\downarrow}^*_A\ (f_{[F]A}(a))$; $f_A(\mathsf{S}{\downarrow_A}\ (b)) = \mathsf{S}{\downarrow}^*_A\ (f_{\langle P\rangle A}(b))$.

LEMMA 63. *Let $\mathcal{M}$, $\mathcal{M}^*$ be as in the previous definition, and let $\{f_A\}$ be a partial homomorphism from $\mathcal{M}$ onto $\mathcal{M}^*$. If g, $g*$ are assignments in $\mathcal{F}$ and $\mathcal{F}^*$ respectively, and $f_A(g(x^A)) = g^*(x)$, then $f_A(val(M^A, g)) = val^*(M, g^*)$.*

Proof. By induction on M. We consider the cases not already dealt with in [Friedman, 1975]. Note that we may assume $f_A(g(x^A)) = g^*(x)$, since f_A is onto.

- $M \equiv \langle N^A, G^B\rangle$: $f_{A\wedge B}(val(\langle N,G\rangle, g))$
$= f_{A\wedge B}(\mathsf{PAIR}(val(N,g), val(G,g)))$
$= \mathsf{PAIR}^*_{A,B}(f_A(val(N,g)), f_B(val(G,g)))$
$= \mathsf{PAIR}^*_{A,B}(val^*(N,g^*), val^*(G,g^*))$ by the induction hypothesis
$= val^*(\langle N,G\rangle, g^*)$.

- $M \equiv (N^{A\wedge B})_i$: $f(val((N)_i, g)) = f(\mathsf{PRO}^i(val(N,g)))$
  $= \mathsf{PRO}^{*i}(f_{A\wedge B}(val(N,g)))$
  $= \mathsf{PRO}^{*i}(val^*(N,g^*))$ by the induction hypothesis
  $= val^*((N)_i, g^*)$.
- $M \equiv \mathcal{P}N^A$: $f_{[F]A}(val(\mathcal{P}N,g)) = f_{[F]A}(\mathsf{P}_A(val(N,g)))$
  $= \mathsf{P}^*_A(f_A(val(N,g))) = \mathsf{P}^*_A(val^*(N,g^*)) = val^*(\mathcal{P}N, g^*)$.
- $M \equiv \cup N^{[F]A}$: $f_A(val(\cup N, g)) = f_A(\mathsf{P}{\downarrow}_A\ (val(N,g)))$
  $= \mathsf{P}{\downarrow}^*_A\ (f_{[F]A}(val(N,g))) = \mathsf{P}{\downarrow}^*_A\ (val^*(N,g^*)) = val^*(\cup N, g^*)$.
- $M \equiv \mathcal{S}N, \cap N$: analogous to the previous two cases. ■

COROLLARY 64. *Let $\mathcal{M} = \langle \mathcal{F}, val\rangle$, $\mathcal{M}^* = \langle \mathcal{F}^*, val^*\rangle$ be models. If there is a partial homomorphism from $\mathcal{M}$ onto $\mathcal{M}^*$, then $\mathcal{M} \models M = N$ implies $\mathcal{M}^* \models M = N$.*

Proof. Suppose $\mathcal{M} \models M^B = N^B$, $\{f_A\}$ is a partial homomorphism from $\mathcal{M}$ onto $\mathcal{M}^*$, and g^* is an assignment in $\mathcal{M}^*$. We choose an assignment g in $\mathcal{M}$ such that for every $A \in T$, $g^*(x) = f_A(g(x^A))$. By the previous lemma, $val^*(M, g^*) = f_B(val(M,g) = f_B(val(N,g) = val^*(N,g^*)$ ■

THEOREM 65. *Let $\mathcal{M}$ be a model based on a full temporal type structure over infinite sets. Then $\vdash_{\lambda_t}\ M = N$ iff $\mathcal{M} \models M = N$.*

Proof. It suffices to show that $\mathcal{M} \models M = N$ implies $\mathcal{M}_0 \models M = N$. To prove this, we define by induction on A a partial homomorphism $\{f_A\}$ from $\mathcal{M}$ onto $\mathcal{M}_0$ as follows:

- $A = p$, $A = \boldsymbol{t}$, $p \in Atom$:
  f_A is any function from $\mathbf{D}^A$ onto $\mathcal{M}_0$'s domain D^A.
  (Such a function exists, since $\mathbf{D}^A$ is infinite and D^A is denumerable.)
- $A = (B \wedge C)$:
  If $f_B(b)$, $f_C(c)$ exist, then $f_{B\wedge C}(\langle b, c\rangle) = f_{B\wedge C}(\mathbf{PAIR}(b,c))$ is defined as $\mathsf{PAIR}_{B,C}(f_B(b), f_C(c))$.
- $A = (B \rhd C)$:
  $f_{B\rhd C}(a)$ is defined as the unique member of $D^{(B\rhd C)}$ (if it exists) such that $f_C(a(b)) = \mathsf{AP}_{B,C}(f_{B\rhd C}(a), f_B(b))$, for all b in the domain of f_B.
- $A = [F]A$:
  $f_{[F]A}(a) = f_{[F]A}(\mathbf{P}_A(b))$ for some $b \in \mathbf{D}^A$ is defined as $\mathsf{P}_A(f_A(b))$ if $f_A(b)$ exists.
- $A = \langle P\rangle A$:
  $f_{\langle P\rangle A}(a) = f_{\langle P\rangle A}(\mathbf{S}_A(b))$ for some $b \in \mathbf{D}^A$ is defined as $\mathsf{S}_A(f_A(b))$ if $f_A(b)$ exists.

That $\{f_A\}$ is a partial homomorphism follows from the definition of $\{f_A\}$ and the following equations:

$$\begin{array}{ll} & f_A(\mathbf{PRO}^0_{A,B}(\langle a,b\rangle)) \\ = & f_A(a) \\ = & \mathsf{PRO}^0_{A,B}(\mathsf{PAIR}_{A,B}(f_A(a), f_B(b))) \\ = & \mathsf{PRO}^0_{A,B}(f_{A\wedge B}(\mathbf{PAIR}_{A,B}(a,b))) \\ = & \mathsf{PRO}^0_{A,B}(f_{A\wedge B}(\langle a,b\rangle)) \end{array} \qquad \begin{array}{ll} & f_B(\mathbf{PRO}^1_{A,B}(\langle a,b\rangle)) \\ = & f_B(b) \\ = & \mathsf{PRO}^1_{A,B}(\mathsf{PAIR}_{A,B}(f_A(a), f_B(b))) \\ = & \mathsf{PRO}^1_{A,B}(f_{A\wedge B}(\mathbf{PAIR}_{A,B}(a,b))) \\ = & \mathsf{PRO}^1_{A,B}(f_{A\wedge B}(\langle a,b\rangle)) \end{array}$$

$$\begin{array}{ll} & f_A(\mathbf{P}\downarrow_A(\mathcal{P}a)) \\ = & f_A(a) \\ = & \mathsf{P}\downarrow_A(f_{[F]A}(\mathcal{P}a)) \end{array} \qquad \begin{array}{ll} & f_A(\mathbf{S}\downarrow_A(\mathcal{S}a)) \\ = & f_A(a) \\ = & \mathsf{S}\downarrow_A(f_{\langle P\rangle A}(\mathcal{S}a)) \end{array}$$

It remains to be shown that f_A is onto, for every type A. For $A = \boldsymbol{t}$ and $A = p \in Atom$, this follows from the definitions of $f_{\boldsymbol{t}}$, f_p and $\mathcal{F}_0$. For the remaining cases we consider two examples. $A = [F]B$. Assume $d = \mid \mathcal{P}M \mid \in D^{[F]B}$. Choose $a \in \mathbf{D}^{[F]B}$ such that $a = \mathcal{P}b$ for $b \in \mathbf{D}^B$ and $b = f_B^{-1}(\mid M^B \mid)$. Since f_B is onto, such an element a from $\mathbf{D}^{[F]B}$ exists. Then $f_{[F]B}(a) = f_{[F]B}(\mathbf{P}_B(b)) = \mathsf{P}_B(f_B(b)) = \mid \mathcal{P}M \mid = d$. Consider now $A = (B \rhd C)$, and assume $d \in D^{(B \rhd C)}$. Choose $a \in \mathbf{D}^{(B\rhd C)}$ such that for every b in the domain of f_B, $a(b) \in f_C^{-1}(\mathsf{Ap}(d, f_B(b)))$. Then $f_{(B\rhd C)}(a) = d$. Since f_C and f_B may be assumed to be onto, the set of such $a \in \mathbf{D}^{(B\rhd C)}$ is non-empty. ■

Whereas the encoding of substructural subsystems of **DIntKt** obtained by giving up all or part of **DIntKt**'s structural rules will require modifications of the notion of construction, in order to encode structural extensions of **DIntKt**, the notion of construction need not be altered. Various extensions of **HIntKt** can be presented as structural extensions of **DIntKt**. The following axiom schemata are those schematic axioms from Table 3, which are in $\mathcal{L}$. Each axiom schema Ax in this table corresponds with the associated structural rule Ax' in the sense that an $\mathcal{L}$–formula A is provable in **HIntKt** + Ax iff $\mathbf{I} \rightarrow A$ is provable in **DIntKt** + Ax'.

In the literature, several proposals have been made to extend the formulas-as-types notion of construction from positive logic to modal logics based on it. We shall here briefly point to five such approaches.

1. Gabbay and de Queiroz [1992] interpret the necessity modality $\Box$ "as a sort of second-order universal quantification (quantification over structured collections of formulas)" [Gabbay and de Quieroz, 1992, p. 1359]. Using the framework of Labelled Natural Deduction [de Queiroz and Gabbay, 1999], proofs in various modal logics are encoded by imposing conditions on abstraction over possible-world variables [de Queiroz and Gabbay, 1997]. However, Gabbay and de Queiroz do not consider a Friedman-style completeness proof for the λ–calculi under consideration.

name	*axiom schema*	*name*	*structural rule*
T	$[F]A \rhd A$	T'	$X \to \bullet Y \vdash X \to Y$
4	$[F]A \rhd [F][F]A$	$4'$	$X \to \bullet Y \vdash X \to \bullet\bullet Y$
V	$[F]A$	V'	$X \to Y \vdash \bullet\mathbf{I} \to Y$
T^c	$A \rhd [F]A$	$T^{c\,\prime}$	$X \to Y \vdash X \to \bullet Y$
4^c	$[F][F]A \rhd [F]A$	$4^{c\,\prime}$	$X \to \bullet\bullet Y \vdash X \to \bullet Y$
D_p	$\boldsymbol{t} \rhd \langle P\rangle\boldsymbol{t}$	$D_p{}'$	$\bullet\mathbf{I} \to Y \vdash \mathbf{I} \to Y$
T_p	$A \rhd \langle P\rangle A$	$T_p{}'$	$\bullet X \to Y \vdash X \to Y$
4_p	$\langle P\rangle\langle P\rangle A \rhd \langle P\rangle A$	$4_p{}'$	$\bullet X \to Y \vdash \bullet\bullet X \to Y$
B_p	$(A \wedge \langle P\rangle B) \rhd \langle P\rangle(B \wedge \langle P\rangle A)$	$B_p{}'$	$\bullet(X \rtimes \bullet Y) \to Z \vdash Y \rtimes \bullet X \to Z$
$Alt1_p$	$(\langle P\rangle A \wedge \langle P\rangle B) \rhd \langle P\rangle(A \wedge B)$	$Alt1_p{}'$	$\bullet(X \rtimes Y) \to Z \vdash \bullet X \rtimes \bullet Y \to Z$
$T_p{}^c$	$\langle P\rangle A \rhd A$	$T_p{}^{c\,\prime}$	$X \to Y \vdash \bullet X \to Y$
4_p^c	$\langle P\rangle A \rhd \langle P\rangle\langle P\rangle A$	$4_p^{c\,\prime}$	$\bullet\bullet X \to Y \vdash \bullet X \to Y$

Table 6. Axioms in $\mathcal{L}$.

2. Borghuis [1993; 1994; 1998] investigates the formulas-as-types-notion of construction for several normal modal propositional logics based on **CPL**. Fitch-style natural deduction proofs in these modal logics are interpreted in a second-order λ–calculus. In this approach, unary type-forming operators are introduced to encode applications of import and export rules for $\Box$ in Fitch-style natural deduction. The operations $\hat{k}$ and $\check{k}$ encoding the export and import rules for $\Box$ in the smallest normal modal logic **K**, for example, satisfy the following reduction rule: $\hat{k}(\check{k}M) \to_r M$. Borghuis proves strong normalization results for the modal typed λ–calculi under consideration. However, the term-forming operations used to encode applications of import and export rules for $\Box$ are not provided with a set-theoretic interpretation.

3. Martini and Masini [1996] consider formulas-as-types for 2-sequent calculi, cf. Section 2.2. They introduce two unary term-forming operations gen and ungen to encode applications of $\Box$–introduction and $\Box$–elimination rules. A strong normalization theorem is proved for the typed λ–calculus encoding proofs in the 2-sequent calculus for the modal logic **S4**. However, the typed terms do not receive a set-theoretic interpretation.

4. Recently, Sasaki [1999] suggested understanding a λ–term of type $\Box A$ as either denoting an element from the domain associated with A, or being undefined. A term $M^{A \rhd \Box B}$ would then denote a partial function from D^A to D^B. Sasaki defines an extended typed λ–calculus with various formation rules for obtaining terms of type $\Box A$. Moreover, natural deduction proofs in the extension of the intuitionistic modal logic **IntK** by the axiom schemata

$$T_c \;\; A \rhd \Box A \quad \text{and} \quad 4_c \;\; \Box\Box A \rhd \Box A$$

are encoded by terms in the extended typed λ–calculus. Unfortunately, no denotational semantics for this λ–calculus is developed.

5. The approach that comes closest to the one presented here is Restall's [1999, Chapter 7], who also applies Belnap's display calculus. Introductions of $[F]$ on the right (left) of the sequent arrow are encoded using a unary operator up (down), lifting (lowering) terms of type A ($[F]A$) to terms of type $[F]A$ (A), just like the operation $\mathcal{P}$ ($\cup$). Backward-looking possibility is treated quite differently. Introductions of $\langle P\rangle$ (in Restall's notation $\ominus$) on the right are encoded using a unary type-lifting operation $\bullet$ (not to be confused with the structure connective $\bullet$). Introductions on the left are encoded by a unary term-forming operation turning terms N^B, $M^{\langle P\rangle A}$ into the term $\mathsf{let}\ M\ \mathsf{be}\ \bullet x\ \mathsf{in}\ N$ of type B. Whereas the term $\mathsf{down}\ \mathsf{up}N$ reduces in one step to N, $\mathsf{let}\ \bullet G\ \mathsf{be}\ \bullet x\ \mathsf{in}\ N$ reduces in one step to $N[x := G]$. Restall proves normalization for the extended typed λ–calculus under consideration, however, no set-theoretic interpretation of up, down, $\bullet$, and $\mathsf{let}\ M\ \mathsf{be}\ \bullet x\ \mathsf{in}$ is suggested.

In the literature on functional programming there are various proposals for providing an *operational* semantics of proofs in modal logics, notably in intuitionistic **S4**. Natural deduction in the framework of Martin-Löf's type theory is considered in [Davis and Pfenning, 2000] and [Pfenning, 2000]. Also, further references can be found in these papers.

3.8 Bi-intuitionistic logic

Suppose a connective f_1 is introduced in a finite-set-to-formula sequent calculus, whereas another connective f_2 is introduced in a formula-to-finite-set sequent system. Then the right introduction rules for f_1 and the left introduction rules for f_2 satisfy the segregation condition. However, if we just combine the sets of rules of both sequent calculi, neither Af_1B nor Af_2B is introduced in the most general context, namely in an arbitrary finite set of formulas, because there are no structure operations like in display logic that allow keeping track of succedent (antecedent) formulas on the left (right) of $\rightarrow$. This leads to a problem encountered in formulating an ordinary sequent calculus for bi-intuitionistic logic **BiInt**, the combination of intuitionistic logic and dual-intuitionistic logic. It can be shown that in the ordinary finite-set-to-formula sequent calculus no binary operation $\sharp$ is definable such that $\sharp$ satisfies (in the finite-set-to-formula setting) the dual Deduction Theorem characteristic of coimplication: $A \rightarrow B$ iff $A\sharp B \rightarrow \emptyset$, see [Goré, 2000]. Bi-intuitionistic logic extends the language of intuitionistic logic by *coimplication*, the residual of disjunction, and *conegation*. The syntax of **BiInt** is given by:

$$A ::= p \mid \frown A \mid \smile A \mid A \wedge B \mid A \vee B \mid A \rhd B \mid A \blacktriangleleft B.$$

In the presence of a falsity constant $\boldsymbol{f}$, intuitionistic negation $\frown$ can be defined by $\frown A := (A \rhd \boldsymbol{f})$, and in the presence of a truth constant $\boldsymbol{t}$, conegation $\smile$ can be defined by $\smile A := (\boldsymbol{t} \blacktriangleleft A)$.

Bi-intuitionistic logic has a natural algebraic and possible-worlds semantics, see [Rauszer, 1980]. The possible-worlds semantics adds to Kripke models for intuitionistic logic evaluation clauses for conegation and coimplication. A frame is a pair $\langle I, \sqsubseteq \rangle$, where I a is non-empty set (of states), and $\sqsubseteq$ is a reflexive and transitive binary relation on I. A structure $\langle I, \sqsubseteq, v \rangle$ is a bi-intuitionistic model if v is a function assigning to every propositional variable p a subset $v(p)$ of I and, moreover, for every $t, u \in I$, if $t \sqsubseteq u$ and $t \in v(p)$, then $u \in v(p)$. Verification of a formula A in the model $\mathcal{M} = \langle I, \sqsubseteq, v \rangle$ at state t (in symbols $\mathcal{M}, t \models A$) is inductively defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, t \models p & \text{iff} & t \in v(p), \text{ for every propositional variable } p; \\
\mathcal{M}, t \models \frown A & \text{iff} & \text{for all } u \in I,\ t \sqsubseteq u \text{ implies } \mathcal{M}, u \not\models A \\
\mathcal{M}, t \models \smile A & \text{iff} & \text{there exists } u \in I,\ u \sqsubseteq t, \text{ and } \mathcal{M}, u \not\models A \\
\mathcal{M}, t \models A \wedge B & \text{iff} & \mathcal{M}, t \models A \text{ and } \mathcal{M}, t \models B; \\
\mathcal{M}, t \models A \vee B & \text{iff} & \mathcal{M}, t \models A \text{ or } \mathcal{M}, t \models B; \\
\mathcal{M}, t \models A \rhd B & \text{iff} & \text{for all } u \in I, \text{ if } t \sqsubseteq u \text{ then } \mathcal{M}, u \not\models A \text{ or} \mathcal{M}, u \models B; \\
\mathcal{M}, t \models A \blacktriangleleft B & \text{iff} & \text{there is a } u \in I,\ u \sqsubseteq t\ \mathcal{M}, u \models A \text{ and } \mathcal{M}, u \not\models B;
\end{array}
$$

where $\mathcal{M}, t \not\models A$ is the (classical) negation of $\mathcal{M}, t \models A$. A formula A is valid in $\mathcal{M} = \langle I, \sqsubseteq, v \rangle$ if for every $t \in I$, $\mathcal{M}, t \models A$; and A is valid on a frame $\mathcal{F} = \langle I, \sqsubseteq \rangle$ if A is valid in every model $\langle \mathcal{F}, v \rangle$ based on $\mathcal{F}$. A formula A is said to be valid in a class $\mathcal{K}$ of models (frames) if A is valid in every model (frame) from $\mathcal{K}$.

The axiomatic system **HBiInt** consists of axiom schemata for intuitionistic logic **Int**, modus ponens, the rule

from A infer $\frown\smile A$

and the following axiom schemata:

1. $A \rhd (B \vee (A \blacktriangleleft B))$
2. $(A \blacktriangleleft B) \rhd \smile(A \rhd B)$
3. $((A \blacktriangleleft B) \blacktriangleleft C) \rhd (A \blacktriangleleft (B \vee C))$
4. $\frown(A \blacktriangleleft B) \rhd (A \rhd B)$
5. $(A \rhd (B \blacktriangleleft B)) \rhd \frown A$
6. $\frown A \rhd (A \rhd (B \blacktriangleleft B))$
7. $((B \rhd B) \blacktriangleleft A) \rhd \smile A$
8. $\smile A \rhd ((B \rhd B) \blacktriangleleft A)$

THEOREM 66. *A formula A in the language of **BiInt** is valid in the class of all models iff A is provable in **HBiInt**.*

In the present section, we shall apply the modal display calculus and use a modal translation of **BiInt** into **S4t** to give a display sequent calculus for **BiInt** based on the structure connectives **I**, $*$, $\circ$, and $\bullet$, cf. [Goré, 1995], [Wansing, 1998, Chapter 10]. A direct display sequent system for **BiInt** not relying on a modal translation has been presented in [Goré, 2000]. Sometimes making a detour via a modal translation may be useful. In [Wansing, 1999], a modal translation into **S4** has been used to give a cut-free display sequent calculus for a certain constructive modal logic of consistency, for which no other proof system is known. In view of the possible-worlds semantics for **BiInt** and the familiar modal translation of **Int** into **S4** (see [Gödel, 1933]), a faithful modal translation m of **BiInt** into **S4t** can be straightforwardly defined as follows:

1. $\mathsf{m}(p) = [F]p$, for every propositional variable p;
2. $\mathsf{m}(\boldsymbol{t}) = \boldsymbol{t}$;
3. $\mathsf{m}(\boldsymbol{f}) = \boldsymbol{f}$;
4. $\mathsf{m}(A\sharp B) = \mathsf{m}(A)\sharp\mathsf{m}(B)$, $\sharp \in \{\wedge, \vee\}$;
5. $\mathsf{m}(A \rhd B) = [\mathrm{F}](\mathsf{m}(A) \supset \mathsf{m}(B))$;
6. $\mathsf{m}(A \blacktriangleleft B) = \langle P\rangle\neg(\mathsf{m}(A) \supset \mathsf{m}(B))$.

THEOREM 67. ([Łukowski, 1996]) *A formula A in the language of **BiInt** is provable in **HBiInt** iff $\mathsf{m}(A)$ is provable in **S4t**.*

DEFINITION 68. The display sequent system ***DBiInt*** consists of (id), (cut), the basic structural rules (1) – (4) of Section 1.3, rules $(\rightarrow \boldsymbol{t})$, $(\boldsymbol{t} \rightarrow)$, $(\rightarrow \boldsymbol{f})$, $(\boldsymbol{f} \rightarrow)$, $(\rightarrow \wedge)$, $(\wedge \rightarrow)$, $(\rightarrow \vee)$, $(\vee \rightarrow)$, the structural rules from Table 2 and:

$$
\begin{array}{rl}
(\rightarrow \frown) & \bullet X \rightarrow *A \vdash X \rightarrow \frown A \\
(\frown \rightarrow) & *A \rightarrow X \vdash \frown A \rightarrow \bullet X \\
(\rightarrow \smile) & X \rightarrow *A \vdash \bullet X \rightarrow \smile A \\
(\smile \rightarrow) & *A \rightarrow \bullet X \vdash \smile A \rightarrow X \\
(\rightarrow\rhd)^{\mathsf{m}} & \bullet X \circ A \rightarrow B \vdash X \rightarrow A \rhd B \\
(\rhd\rightarrow)^{\mathsf{m}} & X \rightarrow A \quad B \rightarrow Y \vdash A \rhd B \rightarrow \bullet(*X \circ Y) \\
(\rightarrow\blacktriangleleft)^{\mathsf{m}} & X \rightarrow A \quad B \rightarrow *X \vdash \bullet X \rightarrow A \blacktriangleleft B \\
(\blacktriangleleft\rightarrow)^{\mathsf{m}} & * \bullet X \circ A \rightarrow B \vdash A \blacktriangleleft B \rightarrow X \\
(persistence) & p \rightarrow X \vdash \bullet p \rightarrow X \\
(reflexivity) & X \rightarrow \bullet Y \vdash X \rightarrow Y \\
(transitivity) & X \rightarrow \bullet Y \vdash X \rightarrow \bullet \bullet Y
\end{array}
$$

It can be shown that the persistence rule for arbitrary formulas is an admissible rule of **DBiInt**. This can be used to prove weak completeness of **DBiInt** with respect to **HBiInt**.

LEMMA 69. *In* ***DBiInt***, $A \to X \vdash \bullet A \to X$.

Proof. By induction on A; for example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{*A \to *A}}{\frown A \to \bullet * A}}{\frown A \to \bullet \bullet * A}}{\bullet \frown A \to \bullet * A}}{\bullet \bullet \frown A \to * A}}{\bullet \frown A \to \frown A} \qquad \frown A \to X}{\bullet \frown A \to X}\ (\text{cut})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{*A \to *A}}{*A \to *A \circ B}}{*(*A \circ B) \to A} \quad \dfrac{\dfrac{B \to B}{B \to *A \circ B}}{B \to * * (*A \circ B)}}{\bullet * (*A \circ B) \to A \blacktriangleleft B}}{*(*A \circ B) \to \bullet(A \blacktriangleleft B)}}{*(*A \circ B) \to \bullet \bullet (A \blacktriangleleft B)}}{* \bullet \bullet(A \blacktriangleleft B) \to *A \circ B}}{A \circ * \bullet \bullet(A \blacktriangleleft B) \to B}}{* \bullet \bullet(A \blacktriangleleft B) \circ A \to B}}{A \blacktriangleleft B \to \bullet(A \blacktriangleleft B)}}{\bullet(A \blacktriangleleft B) \to A \blacktriangleleft B} \qquad A \blacktriangleleft B \to X}{\bullet(A \blacktriangleleft B) \to X}$$

■

THEOREM 70. *In* ***DBiInt*** $\vdash \mathbf{I} \to A$ *iff in* ***HBiInt*** $\vdash A$.

Proof. $\Leftarrow$: By induction on proofs in **HBiInt**. As an example, we here consider only the proof of one axiom schema of **HBiInt**:

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\quad B \to B \\ \hline
\qquad\qquad\qquad\qquad\quad \ast\bullet\ast\bullet\mathbf{I}\circ B \to B \\ \hline
A \to A \qquad\qquad B \blacktriangleleft B \to \ast\bullet\mathbf{I} \\ \hline
A \triangleright (B \blacktriangleleft B) \to \bullet(\ast A \circ \ast\bullet\mathbf{I}) \\ \hline
A \triangleright (B \blacktriangleleft B) \to \ast A \circ \ast\bullet\mathbf{I} \ (reflexivity) \\ \hline
A \triangleright (B \blacktriangleleft B) \to \ast\bullet\mathbf{I} \circ \ast A \\ \hline
\bullet\mathbf{I} \circ (A \triangleright (B \blacktriangleleft B)) \to \ast A \\ \hline
\bullet\mathbf{I} \circ (A \triangleright (B \blacktriangleleft B)) \to \bullet\ast A \ (persistence) \\ \hline
\bullet(\bullet\mathbf{I} \circ (A \triangleright (B \blacktriangleleft B))) \to \bullet\ast A \\ \hline
\bullet\mathbf{I} \circ (A \triangleright (B \blacktriangleleft B)) \to \frown A \\ \hline
\mathbf{I} \to (A \triangleright (B \blacktriangleleft B)) \triangleright \frown A
\end{array}
$$

⇒: We define the translations τ_1 and τ_2 from structures into tense logical formulas as in Section 1.3, except that now $\tau_1(A) = \tau_2(A) = \mathrm{m}(A)$. By induction on proofs in **DBiInt**, it can be shown that $\vdash X \to Y$ in **DBiInt** implies $\vdash \tau_1(X) \supset \tau_2(Y)$ in **S4t**. Therefore, $\vdash \mathbf{I} \to A$ in **DBiInt** implies $\vdash \mathrm{m}(A)$ in **S4t**. By the previous theorem we have $\vdash A$ in **HBiInt**. ■

THEOREM 71. *Strong cut-elimination holds for* ***DBiInt****.*

Proof. **DBiInt** is a proper display calculus. As to the fulfillment of condition C8, the derivation on the left, for example, reduces to the derivation on the right, using contraction:

$$
\dfrac{\dfrac{X \to A \quad B \to \ast X}{\bullet X \to A \blacktriangleleft B} \qquad \dfrac{\ast\bullet Y \circ A \to B}{A \blacktriangleleft B \to Y}}{\bullet X \to Y}
\qquad\qquad
\begin{array}{l}
\qquad\qquad \ast\bullet Y \circ A \to B \\ \hline
X \to A \quad A \to \bullet Y \circ B \\ \hline
X \to \bullet Y \circ B \\ \hline
\ast\bullet Y \circ X \to B \quad B \to \ast X \\ \hline
\ast\bullet Y \circ X \to \ast X \\ \hline
X \to \bullet Y \circ \ast X \\ \hline
X \circ X \to \bullet Y \\ \hline
X \to \bullet Y \\ \hline
\bullet X \to Y
\end{array}
$$

■

COROLLARY 72. ***DBiInt*** $\cup$ ***DS4t*** *is a conservative extension of both* ***DBiInt*** *and* ***DS4t****.*

As in Section 3.1, let for modal formulas A the translations τ_i $(i = 1, 2)$ be defined by $\tau_i(A) = A$.

LEMMA 73. *In* ***DBiInt*** $\cup$ ***DS4t****, (i)* $\vdash X \to \tau_1(X)$ *and (ii)* $\vdash \tau_2(X) \to X$.

Proof. Both (i) and (ii) are proved simultaneously by induction on X. In particular we have to verify that for every formula of the language of **BiInt**, $\vdash A \to \mathsf{m}(A)$ and $\vdash \mathsf{m}(A) \to A$. But this is the case, see for example:

$$\begin{array}{l}
A \to \mathsf{m}(A) \quad \mathsf{m}(B) \to B \\ \hline
\mathsf{m}(A) \supset \mathsf{m}(B) \to *A \circ B \\ \hline
*(*A \circ B) \to *(\mathsf{m}(A) \supset \mathsf{m}(B)) \\ \hline
*(*A \circ B) \to \neg(\mathsf{m}(A) \supset \mathsf{m}(B)) \\ \hline
\bullet *(*A \circ B) \to \langle P\rangle\neg(\mathsf{m}(A) \supset \mathsf{m}(B)) \\ \hline
*(*A \circ B) \to \bullet\langle P\rangle\neg(\mathsf{m}(A) \supset \mathsf{m}(B)) \\ \hline
* \bullet \langle P\rangle\neg(\mathsf{m}(A) \supset \mathsf{m}(B)) \to *A \circ B \\ \hline
* \bullet \langle P\rangle\neg(\mathsf{m}(A) \supset \mathsf{m}(B)) \to B \circ *A \\ \hline
* \bullet \langle P\rangle\neg(\mathsf{m}(A) \supset \mathsf{m}(B)) \circ A \to B \\ \hline
A \blacktriangleleft B \to \langle P\rangle\neg(\mathsf{m}(A) \supset \mathsf{m}(B))
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\mathsf{m}(A) \to A \\ \hline
\mathsf{m}(A) \to A \circ \mathsf{m}(B) \\ \hline
*A \circ \mathsf{m}(A) \to \mathsf{m}(B) \\ \hline
*A \to \mathsf{m}(A) \supset \mathsf{m}(B) \\ \hline
*(\mathsf{m}(A) \supset \mathsf{m}(B)) \to A \\ \hline
\neg(\mathsf{m}(A) \supset \mathsf{m}(B)) \to A
\end{array}
\quad
\begin{array}{c}
B \to \mathsf{m}(B) \\ \hline
B \circ \mathsf{m}(A) \to \mathsf{m}(B) \\ \hline
B \to \mathsf{m}(A) \supset \mathsf{m}(B) \\ \hline
*(\mathsf{m}(A) \supset \mathsf{m}(B)) \to *B \\ \hline
\neg(\mathsf{m}(A) \supset \mathsf{m}(B)) \to *B \\ \hline
B \to *\neg(\mathsf{m}(A) \supset \mathsf{m}(B))
\end{array} \\ \hline
\bullet\neg(\mathsf{M}(A) \supset \mathsf{m}(B)) \to A \blacktriangleleft B \\ \hline
\neg(\mathsf{M}(A) \supset \mathsf{m}(B)) \to \bullet(A \blacktriangleleft B) \\ \hline
\langle P\rangle\neg(\mathsf{M}(A) \supset \mathsf{m}(B)) \to A \blacktriangleleft B
\end{array}$$

■

THEOREM 74. *In* $\boldsymbol{DBiInt} \vdash X \to Y$ *iff* $\tau_1(X) \supset \tau_2(Y)$ *is valid on every frame (understood as a frame for* $\boldsymbol{S4t}$*).*

Proof. ($\Rightarrow$): This follows by induction on proofs in **DBiInt**. ($\Leftarrow$): Suppose that $\tau_1(X) \supset \tau_2(Y)$ is valid on every frame. Hence $\tau_1(X) \supset \tau_2(Y)$ is a theorem of **S4t** and hence $\vdash \tau_1(X) \to \tau_2(Y)$ in $\boldsymbol{DBiInt} \cup \boldsymbol{DS4t}$. By the previous lemma, $\vdash X \to Y$ in $\boldsymbol{DBiInt} \cup \boldsymbol{DS4t}$ and by Corollary 72, $\vdash X \to Y$ in $\boldsymbol{DBiInt}$. ■

One advantage of the translation-based sequent system **DBiInt** is that by abandoning combinations of the structural rules ($persistence$), ($reflexivity$), and ($transitivity$), one obtains cut-free sequent calculus presentations of the subsystems of **BiInt** that arise from giving up the corresponding semantic requirements: persistence of atomic information, reflexivity, and transitivity of the relation $\sqsubseteq$. Also seriality of $\sqsubseteq$, a weakening of reflexivity, is expressible by a purely structural sequent rule, see condition D' in Table 4.

4 INTERRELATIONS AND EXTENSIONS

While the existence of a rich inventory of types of proof systems for modal and other logics may be welcomed, for instance, from the point of view of designing and combining logics, there also exists the need of comparing different approaches and investigating their interrelations and their relative advantages and disadvantages. Mints [1997], for example, presents cut-free systems of indexed sequents for certain extensions of **K** and defines a translation of these sequent systems into equivalent display calculi. In this final section a translation of multiple-sequent systems into higher-arity sequent systems and a translation of hypersequents into display sequents are defined, showing that multiple-sequent systems can be simulated within higher-arity proof systems and that the method of hypersequents can be simulated within display logic. Moreover, one interesting aspect of extending the sequent-style proof systems for modal and temporal *propositional* logics to sequent calculi for modal and temporal *predicate* logics is considered, namely avoiding the provability of the Barcan formula and its converse. We also briefly refer to recent work on display calculi for extended modal languages. Finally, the relation between display logic and Dunn's Gaggle Theory is pointed out.

4.1 Translation of multiple-sequent systems

The translation σ in Section 2.4 reveals a straightforward relation between Indrzejczak's multiple-sequent systems and higher-arity sequent systems for modal logics. The intended meaning of the multiple-sequents can be expressed by four-place sequents using a translation μ:

$$\begin{array}{rcl}\mu(\Gamma \rightarrow \Delta) & = & \delta(\Gamma) \rightarrow^{\emptyset}_{\emptyset} \delta(\Delta) \\ \mu(\Gamma \Box\!\!\rightarrow \Delta) & = & \delta(\Gamma) \rightarrow^{\bigvee \delta(\Delta)}_{\emptyset} \emptyset \\ \mu(\Gamma \Diamond\!\!\rightarrow \Delta) & = & \emptyset \rightarrow^{\neg \bigwedge \delta(\Gamma)}_{\emptyset} \delta(\Delta).\end{array}$$

If **S** is a multiple-sequent system, then let $\mu(\mathbf{S})$ be the result of the μ-translation of the rules of **S**. Let μ^* denote the translation of four-place sequents into modal formulas stated in Section 2.3. If $s_1, \ldots, s_n/s$ is a rule of **MC**, then $\mu^*(\mu(s_1)), \ldots, \mu^*(\mu(s_n))/\mu^*(\mu(s))$ is validity preserving in **C**. For the rule $[TR]$, for instance, we have $\mu^*(\mu([TR])) =$

$$\frac{\bigwedge \delta(\Delta) \supset \Box \bigvee \delta(\Gamma)}{\Diamond \neg \bigvee \delta(\Gamma) \supset \neg \bigwedge \delta(\Delta)} \quad = \quad \frac{\bigwedge \delta(\Delta) \supset \Box \bigvee \delta(\Gamma)}{\Diamond \bigwedge \delta(\Gamma^*) \supset \bigvee \delta(\Delta^*)}$$

Moreover, (RR) is derivable and **CPL** is contained in $\mu(\mathbf{MC})$. Hence,

OBSERVATION 75. The system $\mu(\boldsymbol{MC})$ is sound and complete with respect to $\boldsymbol{C}$: $\vdash \Gamma \rightarrow \Delta$ in $\mu(\boldsymbol{MC})$ iff $\mu^*(\mu(\Gamma \rightarrow \Delta))$ is valid in $\boldsymbol{C}$.

The translation μ is also faithful for the extension of **MC** by the rules [nec], [D], [T], and [*4*] and extensions of **C** by the necessitation rule and the axiom schemata D, T and 4.

4.2 Translation of hypersequents

In order to characterize various non-classical logics by means of hypersequential calculi, Avron [1996] uses different semantical readings of hypersequents. Basically a distinction can be drawn between interpreting the sequent arrow of a component in a hypersequent as material implication or as a constructive implication not definable in terms of Boolean negation and disjunction. This difference in interpretation requires different translations of hypersequents into display sequents. If the sequent arrow is interpreted constructively, a suitable translation may, for example, exploit a faithful embedding of the logic under consideration into a normal modal or temporal logic. In such a case, the sequent arrow is interpreted as strict material implication. In [Wansing, 1998, Chapter 11] translations of hypersequents into display sequents are defined that simulate hypersequents in Avron's hypersequential calculi **GL3**, **GS5**, and **GLC** for Łukasiewicz 3-valued logic **L3**, **S5**, and Dummett's superintuitionistic logic **LC**, also called Gödel-Dummett logic. We shall here consider only the translations suitable for **S5** and **LC**. The treatment of **GL3** is slightly more involved, because **L3** comprises connectives from different 'families' of logical operations. To deal with this composite character of **L3** in display logic, the structure connective $\circ$ is replaced by two binary structure operations $\circ_c$ and $\circ_i$, see [Wansing, 1998]. If $\Delta = \{A_1, \ldots, A_n\}$, let $*\Delta = \{*A_1, \ldots, *A_n\}$. Since $\circ$ is assumed to be associative and commutative, we may put $(\circ\Delta) = A_1 \circ \ldots \circ A_n$. If $\Delta = \emptyset$, let $*\Delta = (\circ\Delta) = \mathbf{I}$. Recall the notion of hypersequent from Section 2.5.

DEFINITION 76. The translation η_0 of ordinary sequents into display structures is defined by

$$\eta_0(\Delta \rightarrow \Gamma) = \bullet((\circ * \Delta) \circ (\circ\Gamma)),$$

and the translation η of non-empty hypersequents into display sequents is defined by

$$\eta(s_1 \mid \ldots \mid s_n) = \mathbf{I} \rightarrow \eta_0(s_1) \circ \ldots \circ \eta_0(s_n).$$

THEOREM 77. *For every hypersequent H, $\vdash \eta(H)$ in **DS5** iff $\vdash H$ in **GS5**.*

In the hypersequential system **GLC** the components of a hypersequent are restricted to be ordinary Gentzen sequents with at most a single conclusion. Dummett's **LC** is the logic of linearly ordered intuitionistic Kripke models. An axiomatization of **LC** is obtained from an axiomatization **HInt** of **Int** by adding the axiom schema $(A \rhd B) \vee (B \rhd A)$. It is well-known that the modal translation m defined in Section 3.8 (restricted to the language of intuitionistic logic, i.e. the language of **LC**) is a faithful embedding of **LC** into **S4.3**, the logic of linearly ordered modal Kripke models.

THEOREM 78. *For every formula A in the language of* ***LC***, $\vdash A$ *in* ***LC*** *iff* $\vdash \mathsf{m}(A)$ *in* ***S4.3***.

DEFINITION 79. The translation ζ_0 of a single-conclusion ordinary sequent $s = A_1, \ldots, A_n \rightarrow B$ is defined by

$$\zeta_0(s) = \bullet(*A_1 \circ \bullet(*A_2 \circ \ldots \bullet (*A_n \circ B) \ldots)).$$

If $s = A_1, \ldots, A_n \rightarrow \emptyset$, then $\zeta(s) = \bullet(*A_1 \circ \bullet(*A_2 \circ \ldots \bullet (*A_n \circ \mathbf{I}) \ldots))$. If $s = \emptyset \rightarrow B$, then $\zeta(s) = \bullet(*\mathbf{I} \circ B)$, and if $s = \emptyset \rightarrow \emptyset$, $\zeta(s) = \bullet(*\mathbf{I} \circ \mathbf{I})$. The translation ζ of hypersequents with at most single-conclusion components into display sequents is defined by

$$\zeta(s_1 \mid \ldots \mid s_n) = \mathbf{I} \rightarrow \zeta_0(s_1) \circ \ldots \circ \zeta_0(s_n).$$

THEOREM 80. *For every hypersequent H with at most single-conclusion components,* $\vdash \zeta(H)$ *in* ***DLC*** *iff* $\vdash H$ *in* ***GLC***.

4.3 Predicate logics and other logics

Modal predicate logic is still a largely unexplored area. As to sequent systems for modal predicate logics, one notorious problem is providing introduction rules for the modal operators and the quantifiers such that neither the Barcan formula (BF) $\forall x \Box A \supset \Box \forall x A$ nor its converse (BFc) $\Box \forall x A \supset \forall x \Box A$ are provable on the strength of only these rules. It is well-known that (BF) corresponds to the assumption of constant domains and (BFc) to the persistence of individuals along the accessibility relation; cf. for example [Fitting, 1993]. One way of avoiding the provability of the Barcan formula and its converse is described in [Wansing, 1998, Chapter 12]. The idea is to exploit the well-known similarity between $\Box$ $[\Diamond]$ and $\forall x$ $[\exists x]$ to develop display introduction rules for $\forall x$ $[\exists x]$; i.e., instead of thinking of the modal operators as quantifiers, one thinks of the quantifiers as modal operators, see also [Andreka *et al.*, 1998]. The addition of quantifiers to display logic is briefly discussed in [Belnap, 1982]:

> Quantifiers may be added with the obvious rules:
>
> $$(UQ) \quad \frac{Aa \vdash X}{(x)Ax \vdash X} \quad \frac{X \vdash Aa}{X \vdash (x)Ax}$$
>
> provided, for the right rule, that a does not occur free in the conclusion. ... The rule for the existential quantifier would be

> dual. ... [A]s yet this addition provides no extra illumination. I think that is because these rules for quantifiers are "structure free" (no structure connectives are involved; ...). One upshot is that adding these quantifiers to modal logic brings along Barcan and its converse ... willy-nilly, which is an indication of an unrefined account; alternatives therefore need investigating. [Belnap, 1982, p. 408 f.]

Using the structure-independent rules (UQ), we would have the following proofs of (BF) and (BFc):

$$
\begin{array}{ll}
\underline{A \rightarrow A} & \qquad \underline{A \rightarrow A}\ {\scriptstyle (UQ)} \\
\underline{\Box A \rightarrow \bullet A}\ {\scriptstyle (UQ)} & \qquad \underline{\forall x A \rightarrow A} \\
\underline{\forall x \Box A \rightarrow \bullet A} & \qquad \underline{\Box \forall x A \rightarrow \bullet A} \\
\underline{\bullet \forall x \Box A \rightarrow A}\ {\scriptstyle (UQ)} & \qquad \underline{\bullet \Box \forall x A \rightarrow A} \\
\underline{\bullet \forall x \Box A \rightarrow \forall x A} & \qquad \underline{\Box \forall x A \rightarrow \Box A}\ {\scriptstyle (UQ)} \\
\underline{\forall x \Box A \rightarrow \Box \forall x A} & \qquad \underline{\Box \forall x A \rightarrow \forall x \Box A} \\
\underline{\mathbf{I} \circ \forall x \Box A \rightarrow \Box \forall x A} & \qquad \underline{\mathbf{I} \circ \Box \forall x A \rightarrow \forall x \Box A} \\
\mathbf{I} \rightarrow \forall x \Box A \supset \Box \forall x A & \qquad \mathbf{I} \rightarrow \Box \forall x A \supset \forall x \Box A
\end{array}
$$

Structure-dependent introduction rules for $\forall x$ and $\exists x$ are, however, available. For every binary relation $\mathcal{R}_x$ on a non-empty set S of states, we may define the following functions on the powerset of S:

$$
\begin{array}{ll}
\forall x\mathsf{A} := \{a \mid \forall b\,(a\mathcal{R}_x b \ \Rightarrow\ b \in \mathsf{A})\}, & \exists x^{\smallsmile}\mathsf{A} := \{a \mid \exists b\,(b\mathcal{R}_x a \ \&\ b \in \mathsf{A})\}, \\
\forall x^{\smallsmile}\mathsf{A} := \{a \mid \forall b\,(b\mathcal{R}_x a \ \Rightarrow\ b \in \mathsf{A})\}, & \exists x\mathsf{A} := \{a \mid \exists b\,(a\mathcal{R}_x b \ \&\ b \in \mathsf{A})\}.
\end{array}
$$

We then have

$$\exists x^{\smallsmile}\mathsf{A} \subseteq \mathsf{B} \text{ iff } \mathsf{A} \subseteq \forall x\mathsf{B}, \qquad \exists x\mathsf{A} \subseteq \mathsf{B} \text{ iff } \mathsf{A} \subseteq \forall x^{\smallsmile}\,\mathsf{B},$$

and for every individual variable x, we may introduce a structure connective $\bullet_x$, which in succedent position is to be understood as $\forall x$ and in antecedent position as a backward-looking existential quantifier $\exists x^{\smallsmile}$. Semantically, what is required to account for these quantifiers is a generalization of the Tarskian semantics for first-order logic, see [Andreka *et al.*, 1998]. Let $\mathcal{M}$ be any first-order model and let α, β, ... range over variable assignments in $\mathcal{M}$. Tarski's truth definition for the existential quantifier is:

$$
\begin{array}{lll}
\mathcal{M} \models \exists x A[\alpha] & \text{iff} & \text{for some assignment } \beta \text{ on } |\mathcal{M}|: \\
 & & \quad \alpha =_x \beta \text{ and } \mathcal{M} \models A[\beta],
\end{array}
$$

where $\alpha =_x \beta$ means that α and β differ at most with respect to the object assigned to x. In the more general semantics the concrete relations $=_x$ between variable assignments are replaced by abstract binary relations $\mathcal{R}_x$ of 'variable update' between 'states' α, β, γ, ... from a set of states S.

Assuming an interpretation of atoms containing free variables, the truth definition for the existential quantifier becomes:

$$\mathcal{M}, \alpha \models \exists x A \quad \text{iff} \quad \text{for some } \beta \in S: \ \alpha \mathcal{R}_x \beta \text{ and } \mathcal{M}, \beta \models A$$

Thus, to every individual variable x there is associated a transition relation $\mathcal{R}_x$ on states. The resulting minimal predicate logic, **KFOL**, is nothing but the ω-modal version of the minimal normal modal logic **K**. In order to obtain an axiomatization of **KFOL**, one may just take any axiomatic presentation of **K** and replace every occurrence of $\Diamond$ and $\Box$ by one of $\exists x$ and $\forall x$, respectively. The basic structural rules for the structure connective $\bullet_x$ are:

$$X \to \bullet_x Y \dashv\vdash \bullet_x X \to Y.$$

In analogy to the case for $\Box$ and $\Diamond$, we obtain the following structure-dependent introduction rules for $\forall x$ and $\exists x$:

$$\begin{array}{llll} (\to \forall x) & \bullet_x X \to A \vdash X \to \forall x A & (\to \exists x) & X \to A \vdash * \bullet_x * X \to \exists x A \\ (\forall x \to) & A \to X \vdash \forall x A \to \bullet_x X & (\exists x \to) & * \bullet_x * A \to X \vdash \exists x A \to X \end{array}$$

In addition to these introduction rules we need further structural assumption in order to take care of the necessitation rules in axiomatic presentations of normal modal and tense logics:

$$(MN\bullet_x) \quad \mathbf{I} \to X \vdash \mathbf{I} \to \bullet_x X \qquad X \to \mathbf{I} \vdash X \to \bullet_x \mathbf{I}$$

The structural account of the quantifiers as modal operators blocks the above proofs of (BF) and (BFc). In the presence of additional structural sequent rules, however, these schemata become derivable:

OBSERVATION 81. BF and BFc correspond to the structural rules

$$\text{rBF} \quad X \to \bullet_x \bullet Y \vdash X \to \bullet \bullet_x Y; \qquad \text{rBFc} \quad X \to \bullet \bullet_x \vdash X \to \bullet_x \bullet Y.$$

The apparatus of display logic has also been applied to other extensions of normal modal propositional logic. A result of Kracht concerns the undecidability of decidability of display calculi. Consider the fusion or 'independent sum' of **Kf** and **Kf**, i.e. the bimodal logic **Kf** $\otimes$ **Kf** of two functional accessibility relations $\mathcal{R}_1$, $\mathcal{R}_2$. In this system there are two pairs of modal operators, say, [*1*], $\langle 1 \rangle$ and [*2*], $\langle 2 \rangle$ each satisfying the D and the *Alt1* axiom schemata. The structural language of sequents for this logic comes with two unary operations $\bullet_1$ and $\bullet_2$ satisfying the display equivalence

$$\bullet_i X \to Y \dashv\vdash X \to \bullet_i Y,$$

$i = 1, 2$. Clearly, **Kf** $\otimes$ **Kf** has many properly displayable extensions. Using an encoding of Thue-processes into frames of **Kf** $\otimes$ **Kf**, Grefe and Kracht [1996] have proved a theorem about the undecidability of decidability.

THEOREM 82. (Grefe and Kracht) *It is undecidable whether or not a display calculus is decidable.*

According to Kracht, Theorem 82 indicates a serious weakness of display logic. In any case, the theorem provides insight into the expressive power of display logic; it shows that the subformula property and the strong cut-elimination theorem for displayable logics fail to guarantee decidability. Undecidability of the decidability of properly displayable extensions of $\mathbf{Kf} \otimes \mathbf{Kf}$ is a remarkable property of this particular family of bimodal logics, but is *not* a defect of the modal display calculus, at least insofar as the proof of the theorem also shows that it is undecidable whether or not a finite axiomatic calculus is decidable. Would it be desirable to have a proof-theoretic framework in which only decidable logics can be presented? A weakness of display logic is that it does not lend itself easily to obtain decidability proofs. Restall [1998] uses a display presentation to prove, among other things, decidability of certain relevance logics which are not known to have the finite model property. In [Wansing, 1998, Chapter 6] display logic is used to prove decidability of **Kf** and deterministic dynamic propositional logic without Kleene star.

Display calculi for logics with relative accessibility relations can be found in [Demri and Goré, 2000] and for nominal tense logics in [Demri and Goré, 1999]. In both cases the calculi are obtained using modal translations.

4.4 Gaggle Theory

The generality of display logic has been highlighted by Restall [1995], who observes a close relation between display logic and J. Michael Dunn's *Gaggle Theory* [1990; 1993; 1995]. The relation between gaggle theory and display logic has also been investigated and worked out by Goré [1998]. A *gaggle* is an algebra $\mathcal{G} = \langle \mathbf{G}, \leq, OP \rangle$, where $\leq$ is a distributive lattice ordering on $\mathbf{G}$, and OP is a founded family of operations. The latter means that there is an $f \in OP$ such that for every $g \in OP$, f and g satisfy the abstract law of residuation, see Section 3. If one only requires that $\leq$ is a partial order, and every $f \in OP$ has a trace, then $\mathcal{G}$ is said to be a *tonoid*. Restall defines the notion of *mimicing structure*. An n-place logical operation f mimics antecedent structure if there is a possibly complex n-place structure connective $\sharp$ such that the following rules are admissible:

$$s = \sharp(A_1, \ldots, A_n) \rightarrow X \vdash f(A_1, \ldots, A_n) \rightarrow X$$
$$\mathcal{C}(X_1, A_1) \ldots \mathcal{C}(X_n, A_n) \vdash \sharp(A_1, \ldots, A_n) \rightarrow f(A_1, \ldots, A_n)$$

where $\sharp(A_1, \ldots, A_n)$ is an antecedent part of s, $\mathcal{C}(X_i, A_i) = X_i \rightarrow A_i$, if A_i is an antecedent part of $\sharp(A_1, \ldots, A_n)$, and $\mathcal{C}(X_i, A_i) = A_i \rightarrow X_i$, if A_i is a succedent part of $\sharp(A_1, \ldots, A_n)$. Dually, f mimics succedent structure if there is a possibly complex n-place structure connective $\sharp$ such that the following rules are admissible:

$$s = X \rightarrow \sharp(A_1, \ldots, A_n) \vdash X \rightarrow f(A_1, \ldots, A_n)$$
$$\mathcal{C}(X_1, A_1) \ldots \mathcal{C}(X_n, A_n) \vdash f(A_1, \ldots, A_n) \rightarrow \sharp(A_1, \ldots, A_n)$$

where $\sharp(A_1, \ldots, A_n)$ is a succedent part of s, $\mathcal{C}(X_i, A_i) = X_i \rightarrow A_i$, if A_i is an antecedent part of $\sharp(A_1, \ldots, A_n)$, and $\mathcal{C}(X_i, A_i) = A_i \rightarrow X_i$, if A_i is a succedent part of $\sharp(A_1, \ldots, A_n)$.

THEOREM 83. (Restall [1995]) *If a logical operation f in a display calculus presentation $\mathbf{D}\Lambda$ of a logic Λ mimics structure, then f is a tonoid operator on the Lindenbaum algebra of Λ.*

If every logical operation of $\mathbf{D}\Lambda$ mimics structure, mutual provability is a congruence relation and Λ has an algebraic semantics. Dunn's representation theorem for tonoids supplies also a Kripke-style relational semantics.

5 APPENDICES

5.1 Appendix A

The proof of Theorem 23 takes its pattern from the proof of strong normalization for typed λ-calculus (see for instance [Hindley and Seldin, 1986, Appendix 2]) and follows the argument given in [Roorda, 1991, Chapter 2, reprinted in [Troelstra, 1992]]. This proof has been extracted from the proof of strong cut-elimination for classical predicate logic in [Dragalin, 1988, Appendix B]. Suppose that Π is a proof containing an application of cut. A (one-step) reduction of Π is the proof Σ resulting by applying a primitive reduction to a subproof of Π. If Π reduces to Σ, this is denoted by $\Pi > \Sigma$ (or $\Sigma < \Pi$). Π is said to be reducible iff there is a Σ such that $\Pi > \Sigma$.

LEMMA 84. *If a proof cannot be reduced, then it is cut-free.*

Proof. Since the case distinction in the definition of primitive reductions is exhaustive, every proof that contains an application of cut is reducible. ■

DEFINITION 85. We inductively define the set of *inductive* proofs.

a Every instantiation of an axiomatic rule is an inductive proof.

b If Π ends in an inference *inf* different from cut, and every premise s_i of *inf* has an inductive proof Π_i in Π, then Π is inductive.

c $\Pi = \dfrac{\Pi_1 \quad \Pi_2}{(3)}$ cut is inductive, if every Σ such that $\Pi > \Sigma$ is inductive.

LEMMA 86. *If* Π *is inductive, and* $\Pi > \Sigma$*, then* Σ *is inductive.*

Proof. By induction on the construction of Π. If Π is inductive by **a**, then no reduction can be performed. If Π is inductive by **b**, then every reduction on Π takes place in the Π_i's, which are inductive. Hence, by the induction hypothesis, Σ is inductive due to **b**. If Π is inductive by **c**, then Σ is inductive by definition. ■

DEFINITION 87. Let Π be an inductive proof. The size $ind(\Pi)$ of Π is inductively defined as follows (the clauses correspond to those in the previous definition):

a $ind(\Pi) = 1$;

b $ind(\Pi) = \sum_i ind(\Pi_i) + 1$;

c $ind(\Pi) = \sum_{\Sigma<\Pi} ind(\Sigma) + 1$.

A proof Π is said to be *strongly normalizable* iff every sequence of reductions starting at Π terminates.

LEMMA 88. *Every inductive proof is strongly normalizable.*

Proof. By induction on $ind(\Pi)$. If $ind(\Pi) = 1$, no reduction is feasible. If Π is inductive by **b**, then every reduction is in the premises Π_i, and we can apply the induction hypothesis. If Π is inductive by **c**, then every proof to which Π reduces is inductive and therefore every such proof is strongly normalizable, by the induction hypothesis. But then Π is also strongly normalizable. ■

LEMMA 89. *Let* Π *be an inductive proof and let* inf *be the final inference of* Π*. If* $\Pi > \Pi'$ *by reducing a proof* Π_j *of a premise sequent of* inf*, then* $ind(\Pi) > ind(\Pi')$*.*

Proof. By induction on $ind(\Pi)$. If $ind(\Pi) = 1$, then Π cannot be reduced. Whence Π is inductive by **b** or **c**. If Π is inductive by **c**, then by definition, $ind(\Pi) > ind(\Pi')$. If Π is inductive by **b**, then Π_j is inductive by definition. If Π_j is inductive by **a**, it cannot be reduced. If Π_j is inductive by **b**, then the reduction of Π_j to Π'_j takes place in the proof of some premise sequent of the final inference of Π_j. By the induction hypothesis, $ind(\Pi_j) > ind(\Pi'_j)$. Hence $ind(\Pi) > ind(\Pi')$. If Π_j is inductive by **c**, then by definition, $ind(\Pi_j) > ind(\Pi'_j)$ and thus $ind(\Pi) > ind(\Pi')$. ■

LEMMA 90. *Suppose* Π *ends in an application* inf *of cut, and* Π_1 *and* Π_2 *are the proofs of the premises of* inf*. If* Π_1 *and* Π_2 *are inductive, then so is* Π*.*

Proof. We must show that every $\Sigma < \Pi$ is inductive. For this purpose, we define two complexity measures for Π: $r(\Pi)$, the rank of Π, and $h(\Pi)$, the height of Π. $r(\Pi)$ is the number of symbols in the cut-formula. $h(\Pi)$ is defined by:

$$h(\Pi) = ind(\Pi_1) + ind(\Pi_2).$$

We use induction on $r(\Pi)$ and, for fixed rank, induction on $h(\Pi)$.

Case 1. Σ is obtained by reduction in Π_1 or Π_2, say $\Pi_1 > \Pi_1'$. It follows from Lemma 89 that $ind(\Pi_1') < ind(\Pi_1)$. Then $h(\Sigma) < h(\Pi)$. Since Π_1 and Π_2 are inductive, by Lemma 86, Σ has inductive premises, and by the induction hypothesis for $h(\Pi)$, Σ is inductive.

Case 2. Σ is obtained by reducing *inf*. Then this reduction was either a principal or a parametric move.

Principal move.
Case 1. Since Σ proves one of (1) or (2), Σ is inductive by assumption.

Case 2. Since for every new proof Π' ending in an application of cut, $r(\Pi) > r(\Pi')$, Σ is inductive by the induction hypothesis for $r(\Pi)$.

Parametric move. Suppose A is parametric in the inference ending in (1) (the case for (2) is analogous). If the tree of parametric ancestors of the displayed occurrence of A in (1) contains at most one element A_u that is not parametric in *inf*, we have Figure 1, and we may assume that there is no application of cut on the path from (1) to $Z \to A$.

$$\text{Let } \Pi' = \dfrac{\begin{matrix}\Pi^1\\ Z \to A\end{matrix} \quad \Pi_2}{Z \to Y} \quad \text{and } \Pi'' = \begin{matrix}\Pi^1\\ Z \to A\end{matrix}\ .$$

Consider Π and Π'. Clearly, $r(\Pi) = r(\Pi')$, hence we use induction on the height. Since both Π_1 and Π'' are inductive by **b**, $ind(\Pi'') < ind(\Pi_1)$. Hence $h(\Pi') < h(\Pi)$. By the induction hypothesis for $h(\Pi)$, Π' is inductive, and thus Σ is inductive by definition. If the primitive reduction of Π to Σ requires cutting with Π_2 more than once, analogously every new Π' and hence Σ can be shown to be inductive.

If the tree of parametric ancestors of the displayed occurrence of A in (1) contains *more than one* element A_u that is not parametric in *inf*, $\Sigma = \Pi^{l*}$ or $\Sigma = \Pi^{lr*}$. Since for every new proof Π' ending in an application of cut, $r(\Pi) > r(\Pi')$, Σ is inductive by the induction hypothesis for $r(\Pi)$. ∎

COROLLARY 91. *Every proof is inductive.*

Now Theorem 23 follows by Lemma 88 and Corollary 91, and cut is an admissible rule by Lemma 84.

5.2 Appendix B

To prove completeness of **HIntKt** with respect to the class of all temporal models we shall adopt completely standard methods as applied, for example, in [Schütte 1969, pp. 48–51]. Suppose Δ and Γ are finite sets of formulas, where Γ is empty or a singleton, and let p be a new propositional variable not already in *Atom*. The formula $\Delta \rhd \Gamma$ is defined as follows:

$$\Delta \rhd \Gamma = \begin{cases} \bigwedge \Delta \rhd B & \text{if } \Delta \neq \emptyset,\ \Gamma = \{B\} \\ \boldsymbol{t} \rhd B & \text{if } \Delta = \emptyset,\ \Gamma = \{B\} \\ \bigwedge \Delta \rhd p & \text{if } \Delta \neq \emptyset,\ \Gamma = \emptyset \\ \boldsymbol{t} \rhd p & \text{if } \Delta = \Gamma = \emptyset \end{cases}$$

The pair (Δ, Γ) is said to be consistent if $\Delta \rhd \Gamma$ is unprovable in **HIntKt** based on $\mathcal{L}^+ = \mathcal{L} \cup \{p\}$. In what follows, let $A \in \mathcal{L}$. Let $sub(A)$ denote the finite set of all subformulas of A. If $C = (A_1 \rhd \ldots (A_{n-1} \rhd A_n) \ldots)$, then $sub^*(\{C\}) = (\bigcup_{1 \leq i \leq n} sub(A_i)) \setminus \{p\}$; $sub^*(\emptyset) = \emptyset$. The pair (Δ, Γ) is called A-complete, if $\Delta \cup sub^*(\Gamma) = sub(A)$. A pair (Δ^*, Γ^*) is called an expansion of (Δ, Γ), if Δ^* is a finite superset of Δ, and either $\Gamma^* = \Gamma$ or Γ^* has the shape $(A_1 \rhd \ldots (A_{n-1} \rhd A_n) \ldots)$ and $n > 1$.

LEMMA 92. *If (Δ, Γ) is consistent, then so is $(\Delta \cup \{A\}, \Gamma)$ or $(\Delta, \{A \rhd B\})$, where $B = p$ if $\Gamma = \emptyset$, and $\Gamma = \{B\}$ otherwise.*

Proof. Suppose neither $(\Delta \cup \{A\}, \Gamma)$ nor $(\Delta, \{A \rhd B\})$ are consistent. Then both $(\bigwedge \Delta \wedge A) \rhd B$ and $\bigwedge \Delta \rhd (A \rhd B)$ are derivable in **HIntKt** based on $\mathcal{L}^+$. But then also $\bigwedge \Delta \rhd B$ is derivable, and hence (Δ, Γ) is not consistent; a contradiction. ■

COROLLARY 93. *Every consistent pair (Δ, Γ) such that $\Delta, sub^*(\Gamma) \subseteq sub(A)$ can be expanded to an A–complete consistent pair.*

Let $\Delta \subseteq sub(A)$. Then Δ is said to be A–designated, if some A–complete pair (Δ, Γ), where $sub^*(\Gamma) = sub(A) \setminus \Delta$ is consistent. By soundness of **HIntKt** based on $\mathcal{L}^+$, the formula $\boldsymbol{t} \rhd p$ fails to be provable. Therefore $(\emptyset, \emptyset)$ is consistent. By the previous corollary, for every formula A, $(\emptyset, \emptyset)$ can be expanded to an A–complete consistent pair. Hence, for every A, the set $\mathcal{D}(A)$ of all A–designated subsets of $sub(A)$ is non-empty.

LEMMA 94. *If $C \in sub(A)$, then C belongs to an A–designated set Δ iff $\Delta \rhd \{C\}$ is provable in* **HIntKt**.

Proof. If $C \in \Delta$, then clearly $\Delta \rhd \{C\}$ is provable in **HIntKt**. If $C \notin \Delta$, then $C \in sub(A) \setminus \Delta$, and since Δ is A–designated, $(\Delta, \{C\})$ is consistent. In other words, $\Delta \rhd \{C\}$ is not provable in **HIntKt**. ■

DEFINITION 95. For every formula A, the structure $\mathcal{M}^A = \langle W^A, R_I^A, R_T^A, v^A\rangle$ is called the canonical model for A if

$$\begin{array}{rcl} W^A & = & \mathcal{D}(A) \\ R_I^A & = & \subseteq \\ uR_T^A t & \text{iff} & [F]B \in u \text{ implies } B \in t \\ v^A(p,u) = 1 & \text{iff} & p \in u. \end{array}$$

As we have seen, the set W^A is non-empty, and it can easily be shown that $\mathcal{M}^A$ is indeed a temporal model.

LEMMA 96. *Let $u, t \in \mathcal{D}(A)$. For every formula B, ($[F]B \in u$ implies $B \in t$) iff for every formula C, ($C \in u$ implies $\langle P\rangle C \in t$).*

Proof. First, suppose (i) for all B, $[F]B \in u$ implies $B \in t$ but (ii) there is a formula $C \in u$ such that $\langle P\rangle C \notin t$. By (i), $[F]\langle P\rangle C \notin u$. By the previous lemma, $u \rhd [F]\langle P\rangle C$ is not provable in **HIntKt**. Since $C \rhd [F]\langle P\rangle C$ is provable, also $u \rhd C$ fails to be provable. But then, by the previous lemma, $C \notin u$, which contradicts (ii). Suppose now (iii) for all C, $C \in u$ implies $\langle P\rangle C \in t$ but (iv) there is a formula $[F]B \in u$ such that $B \notin t$. By (iii), $\langle P\rangle[F]B \in t$, and by the previous lemma, $t \rhd \langle P\rangle[F]B$ is provable in **HIntKt**. Since $\langle P\rangle[F]B \rhd B$ is provable, also $t \rhd B$ is provable. Hence $B \in t$, a contradiction with (iv). ■

LEMMA 97. (Verification Lemma) *Consider $\mathcal{M}^A = \langle W^A, R_I^A, R_T^A, v^A\rangle$. For every $C \in sub(A)$ and every $u \in \mathcal{D}(A)$, $\mathcal{M}^A, u \models C$ iff $C \in u$.*

Proof. By induction on C. We shall consider only two cases. Let $\bigwedge u$ denote $\boldsymbol{t}$, if $u = \emptyset$, and note that for all $B \in u$, $\vdash \langle P\rangle \bigwedge u \rhd \langle P\rangle B$. Hence for every $u, t \in W^A$ we have: (*) if $\langle P\rangle \bigwedge u \in t$, then for every $B \in u$, $\langle P\rangle B \in t$.
1. $C = [F]B$.
$\Rightarrow$: Suppose $[F]B \notin u$. This is the case iff

	$\bigwedge u \rhd [F]B$ cannot be proved	
iff	$\langle P\rangle \bigwedge u \rhd B$ cannot be proved	
iff	$(\langle P\rangle \bigwedge u, \{B\})$ is consistent	
iff	$(\exists t \in \mathcal{D}(A))\ u \subseteq t,\ \langle P\rangle \bigwedge u \in t,\ B \notin t$	by Corollary 93
only if	$(\exists t \in \mathcal{D}(A))\ uR_T^A t,\ B \notin t$	by Lemma 96 and (*)
iff	$\mathcal{M}, u \not\models [F]B$	by the ind. hyp.

$\Leftarrow$: Suppose $[F]B \in u$. Then for all $t \in W^A$, $uR_T^A t$ implies $B \in t$. By the induction hypothesis, $\mathcal{M}^c, u \models [F]B$.

2. $C = \langle P \rangle B$.
$\Rightarrow$: Suppose $\mathcal{M}^A, u \models \langle P \rangle B$. This is the case iff

	$(\exists t \in W^A)\, tR^A_T u$ and $\mathcal{M}^A, t \models B$	
only if	$(\exists t \in W^A)\, (B \in t$ implies $\langle P \rangle B \in u),\ B \in t$	by Lem. 96, ind. hyp.
only if	$\langle P \rangle B \in u$.	

$\Leftarrow$: Suppose $\langle P \rangle B \in u$. Put $t' := \{C \mid \langle P \rangle C \in u\}$. Clearly, the pair $(t', \emptyset)$ is consistent. Hence

	$(\exists t \in W^A)\, t' \subseteq t,\ \bigwedge t' \in t$	by Corollary 85
only if	$(\exists t \in W^A)\, tR^A_T u$ and $\mathcal{M}^A, t \models B$	by Lemma 88 and the ind. hyp.
iff	$\mathcal{M}^A, u \models \langle P \rangle B$	

■

COROLLARY 98. *If A is valid in every temporal model, then A is provable in* **HIntKt**.

Proof. Suppose A is not provable in **HIntKt**. Then the pair $(\emptyset, \{A\})$ is consistent, and, by the previous corollary, there exists a $u \in \mathcal{D}(A)$ such that $A \notin u$. By the Verification Lemma, $\mathcal{M}^A, u \not\models A$. ■

COROLLARY 99. **HIntKt** *is decidable.*

Proof. This follows easily by the fact that $sub(A)$ is finite. ■

5.3 Appendix C

In order to prove strong normalization for λ_t, we shall follow R. de Vrijer's [1987] proof of strong normalization for typed λ–calculus with pairing and projections satisfying surjective pairing. Let $h(M)$ (the height of the reduction tree of M) be the length of a reduction sequence of M that has maximal length.

DEFINITION 100. $M^A \in \mathit{Term}$ is said to be computable iff

1. $sn(M)$;
2. if $A = B \rhd C$, $M \twoheadrightarrow_r N_1$, and N_2^B is computable, then $(N_1, N_2)^C$ is computable;
3. if $A = B \wedge C$ and $M \twoheadrightarrow_r \langle N_1, N_2 \rangle$, then N_1^B, N_2^C are computable;
4. if $A = [F]B$ and $M \twoheadrightarrow_r \mathcal{P}N$, then N^B is computable;
5. if $A = \langle P \rangle B$ and $M \twoheadrightarrow_r \mathcal{S}N$, then N^B is computable.

The set of all computable terms is denoted by C.

By this definition, every computable term is strongly normalizable. The aim is to show that every term is computable.

LEMMA 101.

(a) *If* $M \in \mathsf{C}$ *and* $M \twoheadrightarrow_r N$*, then* $N \in \mathsf{C}$.

(b) C *is closed under repeated formation of application terms* (M, N).

(c) *If* $x \in V$*, then* $x \in \mathsf{C}$.

(d) *If for every* $N^A \in \mathsf{C}$, $(M^{(A \rhd B)}, N) \in \mathsf{C}$, *then* $M \in \mathsf{C}$.

(e) *If* $(M^{A \wedge B})_0 \in \mathsf{C}$ *and* $(M^{A \wedge B})_1 \in \mathsf{C}$, *then* $M \in \mathsf{C}$.

(f) *If* N_1, $N_2 \in \mathsf{C}$, *and* $G \in \mathsf{C}$, *for every* G *such that* $\langle N_1, N_2 \rangle \rightarrow_r G$, *then* $\langle N_1, N_2 \rangle \in \mathsf{C}$.

(g) *If* $N \in \mathsf{C}$ *and* $G \in \mathsf{C}$, *for all* G *such that* $(N)_i \rightarrow_r G$, *then* $(N)_i \in \mathsf{C}$.

(h) *If* $N \in \mathsf{C}$, *and* $G \in \mathsf{C}$, *for all* G *such that* $\mathcal{P}N \rightarrow_r G$, *then* $\mathcal{P}N \in \mathsf{C}$.

(i) *If* $N \in \mathsf{C}$, *and* $G \in \mathsf{C}$, *for all* G *such that* $\mathcal{S}N \rightarrow_r G$, *then* $\mathcal{S}N \in \mathsf{C}$.

(j) *If* $N \in \mathsf{C}$, *and* $G \in \mathsf{C}$, *for all* G *such that* $\cup N \rightarrow_r G$, *then* $\cup N \in \mathsf{C}$.

(k) *If* $N \in \mathsf{C}$, *and* $G \in \mathsf{C}$, *for all* G *such that* $\cap N \rightarrow_r G$, *then* $\cap N \in \mathsf{C}$.

Proof. (a): By induction on $h(M)$. (b) By reflexivity of $\twoheadrightarrow_r$ and Clause 2 in the definition of C. (c): By induction on $A \in T$. If $A = B \rhd C$, the claim follows by (b). (d): If for every $N^A \in \mathsf{C}$, $(M, N) \in \mathsf{C}$, then $sn(M)$, since by (c) and the assumption $(M, x^B) \in \mathsf{C}$. Now suppose $M \twoheadrightarrow_r N_1$, $N_2 \in \mathsf{C}$, and for every N, $(M, N) \in \mathsf{C}$. Then $(M, N_2) \twoheadrightarrow (N_1, N_2)$ and, by (a), $(N_1, N_2) \in \mathsf{C}$. Thus $M \in \mathsf{C}$. (e): Since $sn((M)_i)$, also $sn(M)$. Suppose $M \twoheadrightarrow_r \langle N_0, N_1 \rangle$. Then $(M)_i \twoheadrightarrow_r (\langle N_0, N_1 \rangle)_i \rightarrow_r N_i$. Since $(M)_i \in \mathsf{C}$ and C is closed under $\twoheadrightarrow_r$, also $N_i \in \mathsf{C}$. (f): Obviously, for every M, $sn(M)$ iff $sn(N)$, for each N such that $M \rightarrow_r N$. Moreover, suppose that $\langle N_1, N_2 \rangle \twoheadrightarrow_r \langle G_1, G_2 \rangle$. This is the case iff $\langle N_1, N_2 \rangle \equiv \langle G_1, G_2 \rangle$ or there is a term M^* such that $\langle N_1, N_2 \rangle \rightarrow_r M^*$, and $M^* \twoheadrightarrow_r \langle G_1, G_2 \rangle$. In both cases $G_1, G_2 \in \mathsf{C}$. (g): By induction on the type A of $(N)_i$. If A is atomic, Clauses 2–5 in the definition of C hold trivially. $A = \langle P \rangle B$: Suppose $(N)_i \twoheadrightarrow_r \mathcal{S}M$. If $N \equiv \langle M_1, M_2 \rangle$, then $(N)_i \rightarrow_r M_i$, and $M_i \in \mathsf{C}$. If $\mathcal{S}M \not\equiv M_i$, then $M_i \twoheadrightarrow_r \mathcal{S}M$, and $\mathcal{S}M \in \mathsf{C}$, by closure of C under $\twoheadrightarrow_r$. If $N \not\equiv \langle M_1, M_2 \rangle$, then there is a term $M^* \in \mathsf{C}$ such that $(N)_i \rightarrow_r M^*$ and $M^* \twoheadrightarrow_r \mathcal{S}M$. In each subcase, $M \in \mathsf{C}$. The cases $A = [F]B$ and $A = B \wedge C$ are analogous. If $A = B \rhd C$, we may use closure of C under application. (h): Suppose $\mathcal{P}N \twoheadrightarrow_r \mathcal{P}G$. This holds iff $N \equiv G$ or there is a term M^* such that $\mathcal{P}N \rightarrow_r M^*$ and $M^* \twoheadrightarrow_r \mathcal{P}G$. In both cases $G \in C$. (i): Analogous to (h). (j): By induction on the type A of $\cup N$. The only interesting case is $A = [F]B$. Suppose $\cup N \twoheadrightarrow_r \mathcal{P}M$. If $N \equiv \mathcal{P}N_1$, then $\cup N \rightarrow_r N_1$ and $N_1 \in \mathsf{C}$. If $\mathcal{P}M \not\equiv \mathcal{P}N_1$, then $N_1 \twoheadrightarrow_r \mathcal{P}M$, and $\mathcal{P}M \in \mathsf{C}$. In each case $M \in \mathsf{C}$. (k): Analogous to (j). ■

THEOREM 102. *If $M \in$ Term is λ–free, then $M \in \mathsf{C}$.*

Proof. By induction on M. (1): M is a variable: Lemma 101 (c). (2) $M \equiv (N_1, N_2)$: Lemma 101 (b) and the induction hypothesis. (3) $M = \langle N_1^A, N_2^B \rangle$: In view of Lemma 101 (f), it is enough to show that $G \in \mathsf{C}$, for every G such that $\langle N_1, N_2 \rangle \rightarrow_r G$. There are tow subcases. (i): $N_1 \equiv (G)_0$ and $N_2 \equiv (G)_1$. Then the claim follows by (e). (ii): $G \equiv \langle N_1, N^* \rangle$ and $N_2 \rightarrow_r N^*$ or $G \equiv \langle M^*, N_2 \rangle$ and $N_1 \rightarrow_r M^*$. We may use induction on $h(N_1) + h(N_2)$. (4) $M \equiv (N)_i$. In view of Lemma 101 (g), it is enough to show that $G \in \mathsf{C}$, for every G such that $M \rightarrow_r G$. There are tow cases. (i) $N \equiv \langle N_0, N_1 \rangle$ and $G \equiv N_i$. Then we may use the induction hypothesis. (ii) $G \equiv (N^*)_i$, $N \rightarrow_r N^*$, and we may use induction on $h(N)$. (5) $M \equiv \mathcal{P}N$: In view of Lemma 101 (h), it is enough to show that $G \in \mathsf{C}$, for every G such that $M \rightarrow_r G$. If $M \rightarrow_r G$, then $G \equiv \mathcal{P}N^*$, $N \rightarrow_r N^*$, and we may use induction on $h(N)$. (6) $M \equiv \mathcal{S}N$: Analogous to (5), using Lemma 101 (i). (7) $M \equiv \cap N$: Given Lemma 101 (k), it suffices to show that $G \in \mathsf{C}$, for every G such that $M \rightarrow_r G$. There are two cases. (i) $N \equiv \mathcal{S}G_1$ and $G \equiv G_1$. Then we may use the induction hypothesis. (ii) $G \equiv \cap N^*$, $N \rightarrow_r N^*$, and we may use induction on $h(N)$. (8) $M \equiv \cup N$: Analogous to (7), using Lemma 101 (j). ■

Strong normalizability of all terms is derived from computability of all terms under substitution.

DEFINITION 103. $M^A \in$ *Term* is said to be computable under substitution iff any substitution of free variables in M by computable terms of suitable type results in a computable term.

Let C^s denote the set of all terms computable under substitution.

THEOREM 104. *Every λ_t–term M is computable under substitution.*

Proof. By induction on M. For term variables the claim is obvious. Moreover, since C is closed under application, C^s is also closed under application. If $M \equiv \langle N_1, N_2 \rangle$, $M \equiv (N)_i$, $M \equiv \mathcal{P}N$, or $M \equiv \mathcal{S}N$, the claim follows by the induction hypothesis. If $M \equiv \lambda x^A N$, it must be show that $\lambda x N \in \mathsf{C}^s$ if $N \in \mathsf{C}^s$. Suppose that $\lambda x N^*$ is the result of substituting a computable term for a free variable in $\lambda x N$, and suppose that G^A is a computable term such that (M, G) does not have a type $B \rhd C$. Then, by Lemma 101 (f) – (k), $((\lambda x N^*)G) \in \mathsf{C}$, if for every term H, $((\lambda x N^*)G) \rightarrow_r H$ implies $H \in \mathsf{C}$. Since by assumption $N \in \mathsf{C}^s$, we have $N^* \in \mathsf{C}$. Therefore we may use induction on $h(N^*) + h(G)$ to show that $((\lambda x N^*)G) \in \mathsf{C}$. There are three

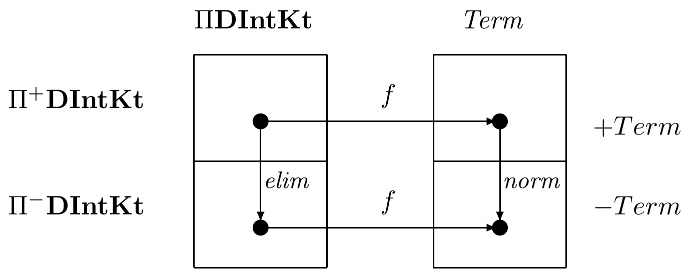

Figure 2. Normalization as a homomorphic image of proof-simplification.

subcases. (i) $H \equiv N^*[x := G]$ and $x \in fv(N^*)$. Then $N^* \in \mathsf{C}^*$ implies $H \in \mathsf{C}$. (ii) $H \equiv N^*[x := G]$ and $x \notin fv(N^*)$. Then $H \equiv N^* \in \mathsf{C}$. (iii) H is obtained from $((\lambda x N^*)G)$ by executing one reduction step either in N^* or G. In this case we may use the induction hypothesis. ■

COROLLARY 105. *If M is a λ_t–term, then M is strongly normalizable.*

5.4 Appendix D

It has to be shown that f is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$, i.e., for every $\Pi \in \Pi^+\mathbf{DIntKt}$, we have $f(\mathit{elim}(\Pi)) = \mathit{norm}(f(\Pi))$, see Figure 2. The proof is by induction on Π. If the rule applied to obtain the conclusion sequent s_c of Π is an axiomatic sequent $A \to A$, then $f(\mathit{elim}(\Pi)) = f(\Pi)$, and $f(\Pi)$ is a *nf*. If the rule applied to obtain s_c is such that the term construction step associated with it cannot generate a redex, we may apply the induction hypothesis. We shall consider the remaining cases.

Case 1. $\Pi = \dfrac{\begin{array}{c}\Pi' \\ A \rtimes B \to X\end{array}}{A \wedge B \to X}$

A redex could be generated if the free variables x^A, y^B in the construction of $A \rtimes B \to X$ occur in the context $\langle x, y \rangle$. But then $X = A \wedge B$, $A \rtimes B \to X$ has been derived from $\{A \to A, B \to B\}$, and $\mathit{elim}(\Pi) = A \wedge B \to A \wedge B$. The claim holds, since $\langle (v_1^{A \wedge B})_0, (v_1^{A \wedge B})_1 \rangle \to_r v_1^{A \wedge B}$.

Case 2. $\Pi = \dfrac{\begin{array}{c}\Pi' \\ X \to A \rtimes B\end{array}}{X \to A \rhd B}$

A redex could be generated if the free variable x^A in the construction of $X \to A \rtimes B$ occurs in the context $(N^{A \rhd B}, x^A)$. But then $X = A \rhd B$,

$X \to A \rtimes B$ has been derived from $\{A \to A, B \to B\}$, and $elim(\Pi) = A \rhd B \to A \rhd B$. The claim holds, since $\lambda v_1^A(v_1^{A\rhd B}, v_1^A) \to_r v_1^{A\rhd B}$.

Case 3. $\Pi = \dfrac{\overset{\Pi_1}{X \to A} \quad \overset{\Pi_2}{A \to Y}}{X \to Y}$

Suppose the exhibited application of cut in Π is not principal. If this application is reduced in one step, either the f–images of the resulting proof and Π are the same, or some principal cuts have been performed on subformulas of A. Thus, there are five remaining cases to be considered.

Case 3.1 ($\boldsymbol{t}$):

$$\dfrac{\mathbf{I} \to \boldsymbol{t} \quad \dfrac{\overset{\Pi}{\mathbf{I} \to X}}{\boldsymbol{t} \to X}}{\mathbf{I} \to X} \quad \text{is converted into} \quad \dfrac{\Pi}{\mathbf{I} \to X}$$

$$\downarrow f \qquad\qquad\qquad \downarrow f$$

$$\dfrac{v_1^{\boldsymbol{t}} \quad \dfrac{M}{M}}{M} \qquad \to_r \qquad M$$

Case 3.2 ($\wedge$):

$$\dfrac{\dfrac{\overset{\Pi_1}{X \to A} \quad \overset{\Pi_2}{Y \to B}}{X \rtimes Y \to A \wedge B} \quad \dfrac{\overset{\Pi_3}{A \rtimes B \to Z}}{A \wedge B \to Z}}{X \rtimes Y \to Z} \quad \text{is conv. into} \quad \dfrac{\overset{\Pi_2}{Y \to B} \quad \dfrac{\dfrac{\dfrac{\overset{\Pi_1}{X \to A} \quad \dfrac{\overset{\Pi_3}{A \rtimes B \to Z}}{A \to B \rtimes Z}}{X \to B \rtimes Z}}{X \rtimes B \to Z}}{B \to X \rtimes Z}}{\dfrac{Y \to X \rtimes Z}{X \rtimes Y \to Z}}$$

$$\downarrow f \qquad\qquad\qquad \downarrow f$$

$$\dfrac{\dfrac{M_1^A \quad M_2^B}{\langle M_1, M_2\rangle} \quad \dfrac{N(x^A, y^B)}{N((z^{A\wedge B})_0, (z^{A\wedge B})_1)}}{N((\langle M_1, M_2\rangle)_0, (\langle M_1, M_2\rangle)_1)} \quad \twoheadrightarrow_r \quad \dfrac{M_2^B \quad \dfrac{\dfrac{\dfrac{M_1^A \quad \dfrac{N(x^A, y^B)}{N}}{N(M_1)}}{N(M_1)}}{N(M_1)}}{\dfrac{N(M_1, M_2)}{N(M_1, M_2)}}$$

Case 3.3 ($\rhd$):

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ X \to A \rtimes B\end{matrix}}{X \to A \rhd B} \quad \dfrac{\begin{matrix}\Pi_2\\ Y \to A\end{matrix} \quad \begin{matrix}\Pi_3\\ B \to Z\end{matrix}}{A \rhd B \to Y \rtimes Z}}{X \to Y \rtimes Z} \quad \text{is conv. into} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_2\\ Y \to A\end{matrix} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ X \to A \rtimes B\end{matrix}}{X \rtimes A \to B} \quad \begin{matrix}\Pi_3\\ B \to Z\end{matrix}}{X \rtimes A \to Z}}{A \to X \rtimes Z}}{Y \to X \rtimes Z}}{X \rtimes Y \to Z}}{X \to Y \rtimes Z}$$

$$\downarrow f \qquad\qquad\qquad \downarrow f$$

$$\dfrac{\dfrac{M^B(x^A)}{\lambda x^A M} \quad \dfrac{N_1^A \quad N_2(y^B)}{N_2(z^{A \rhd B}), N_1)}}{N_2(\lambda x^A M, N_1)} \quad \to_r \quad \dfrac{\dfrac{\dfrac{N_1^A \quad \dfrac{\dfrac{\dfrac{\dfrac{M^B(x^A)}{M^B} \quad N_2(y^B)}{N_2(M)}}{N_2(M)}}{N_2(M(x^A))}}{N_2(M(N_1))}}{N_2(M(N_1))}}{N_2(M(N_1))}$$

Case 3.4 ($[F]$):

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \bullet X \to A\end{matrix}}{X \to [F]A} \quad \dfrac{\begin{matrix}\Pi_2\\ A \to Y\end{matrix}}{[F]A \to \bullet Y}}{X \to \bullet Y} \quad \text{is converted into} \quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \bullet X \to A\end{matrix} \quad \begin{matrix}\Pi_2\\ A \to Y\end{matrix}}{\bullet X \to Y}}{X \to \bullet Y}$$

$$\downarrow f \qquad\qquad\qquad \downarrow f$$

$$\dfrac{\dfrac{M^A}{\mathcal{P}M} \quad \dfrac{N(x^A)}{N(\cup y^{[F]A})}}{N(\cup \mathcal{P}M)} \quad \to_r \quad \dfrac{\dfrac{M^A \quad N(x^A)}{N(M)}}{N(M)}$$

Case 3.5 ($\langle P \rangle$): analogous to the previous case. ■

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