Evolution of cooperation in asymmetric interaction/influence networks

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Abstract. Studying the emergence and evolution of cooperation in society is a hard to model phenomenon. Better understanding it has obvious applications in fostering all kind of desirable social outcomes.
Population structure is a well known factor in cooperation. Recent studies show how uni-directional interactions actually play a critical role in facilitating cooperation. Using these mixed directed/undirected networks to model interactions, it has been proposed to model the propagation of cooperation throughout the population with a second influence network.
Our contributions are to test more sophisticated influence networks. Given an interaction network that is not scale free (Erdos Renyi and Random Regular graphs) generate an influence network that retains some of the interaction network structure. This could be understood as having global information but local interaction.

Keywords: cooperation · evolutionary game theory · asymetric relationships · interaction/influence asymetry

1 Introduction

Understanding how and why cooperation emerges in human societies is still not well understood. That is the reason why modeling cooperation and seeing how it evolves is not only interesting in itself as a theoretical topic, but it also has many practical applications (in sociology, anthropology and computer science for example).

In the article [2] they present a way of modeling cooperation in structured populations. The model used is based on a variant of the "prisoner's dilemma" from game theory, called the donation game. Each of these "games" is played unidirectionally, so they model the structure of the society using a directed graph. This network is called the interaction network, and must always be connected. They account for both bi-directional (reciprocal) and uni-directional interactions (not reciprocal [3]).

The authors showed the importance of having both kinds of links to foster cooperation. This was linked to the presence of hierarchical structures in advanced societies, but "hierarchical" in a very broad meaning. They later explore different motives (subgraphs) that also correlate with easier emergence of cooperation, and find that one example of this "hierarchical" structures was a size 3 directed loop.

They observed that these formations of closed chains of cooperation was an improvement from totally bi-directional (reciprocal) interactions. They also observed that depending on the kinds of structure motifs, even if cooperation emerged, simulations ended with more equality (loop hierarchies) or more inequality (concentrating/star hierarchies). A real life example by the authors of these concentrating hierarchies is families, where the 2 parents "donate" to the offspring with no reciprocity.

Studying these motifs could potentially help modify lightly real life networks (under this model) in an attempt to foster cooperation.

Unfortunately these models remain very theoretical, and in order to better model the cooperation phenomena a deeper refinement of the model is needed. Specifically in the modeling of influences on the network. The original paper hints at possible scenarios to further develop. We plan on using the idea they lay out on the article: making the cooperation influence depend on other structures and not only the neighbours.

1.1 Original scenario

Our first objective with this project is to recreate the donation game shown in the article. This objective is a baseline to ensure that our implementation of the experiments is at least comparable to the original paper. We will use the same idea, thus, we will have the same two networks, the interaction network and the dispersal network.

For the interaction network, individuals (nodes) can have either of two states, cooperating or defecting. A cooperative individual pays a cost c to generate a benefit b to a recipient at a target node. Cooperation is indiscriminate, a cooperative node donates for each outgoing edge. A defector pays no cost and generates no benefit to none.

The model uses accumulated payoffs to score the fitness of each node. Weighted by this fitness, a random individual is chosen to update it's strategy according to the dispersal network. All neighbours pointing to that individual in the dispersion network will spread/influence their strategies to it (the probability of a node spreading its strategy is proportional to the reproductive rate).

The original scenario was to use 4 well known families of graphs as interaction networks: Random Regular, Erdös-Renyi, Barabasi-Albert, and Watts-Strogatz. These undirected graphs are then transformed to directed using a parameter p. This parameter is the proportion of edges that will result in a non-reciprocal interaction. So the undirected edges are assigned one and only one random direction. The rest of undirected edges will be reciprocal and are transformed into the two directed edges.

Finally, the original scenario uses a copy of the interaction network as its dispersal network. This is called the downstream case as a node influences in the same direction as the donations flow. The second original scenario uses a copy of the interaction network but with all its edges reversed, appropriately named the upstream scenario.

1.2 Our proposals and hypothesis

We have two proposals, they both involve changing the influence network to one that is different from the interaction network. Those two scenarios are:

- **The influence of moral hubs.** To do this simulation, we will use a regular network as the interaction network and a scale free as the dispersal network. This way, individuals play a donation game with their neighbours but we distinguish between two types of individuals. The first type (extrovert) has a lot interactions, that means a lot of in-degree **and** out degree. They are hyper-social, tend setters and chasers at the same time. The second type (introverts) have lower interactions and their interactions or more likely to be with extroverts.

  Our hypothesis for this model is that there should not be cooperation at all, since the two graphs have no structural properties in common. Introverts only interact with extroverts, however, the influence direction is random. Extroverts are not exactly celebrities, they would be better described as moral/sociability hubs. Moreover, there is no proximity relation relating which introverts follow a certain extrovert, as the two graphs are generated independently.

- **The influence of communities.** In this case, we will also use a regular network as the interaction network. For the dispersal network we will augment the neighbourhoods by introducing some cliques, where each clique corresponds to a community. The communities are computed with the greedy algorithm.
  Our hypothesis for this model is that the will be some cooperation. Cooperation will be polarized, some cliques (communities) will collectively decide to cooperate, some to defect. But communities that cooperate will collectively have a positive balance when cooperating, even if some individuals have a negative balance. The same is true for communities that do not cooperate.
  To make more differentiable the cliques form the regular network, we have chosen a lower range of values for the average degree k. And instead of 7 values we only test 4 values due to computational time limits.

2 Results

We first will show the results of the original experiment (the one done in the paper). What we want is to check that our results are cohesive with the ones obtained in the article. Then follows our two scenarios, first the “moral hubs”, last “communities”

The color coding of the tables allows for a gross comparison. The blue cells have positive values for the benefit cost ratio, so the network has some tendency to cooperation. If the value is high, we have to give big incentives (a high benefit to cost ratio) in order to expect cooperation, so the lower the number, the more cooperation is fostered by structural properties of the networks. The converse is true for the negative cooperation, that is referred to as spite. This produces an asymptotic singularity, also observed in the results of the original paper.

Downstream | Upstream

RR

A (Downstream)

Proportion, p \ Average degree $\bar{k}$	20	40	48	50	52	54	56
1.0	33.3	198.1	1185.9	$+/-$	-129.3	-66.8	-46.2
0.9	29.5	144.1	414.6	665.5	1512.3	$+/-$	-1213.1
0.67	27.3	121.5	288.0	396.8	609.4	1209.9	14903.7
0.5	27.8	126.8	314.6	447.0	731.5	1782.1	-5541.0
0.1	31.4	174.6	729.1	2001.3	-3331.7	-955.6	-575.3
0.0	32.7	196.0	1176.0	∞	-1274.0	-661.5	-457.3

E (Upstream)

Proportion, p \ Average degree $\bar{k}$	20	40	48	50	52	54	56
1.0	31.8	193.9	1184.4	$+/-$	-1259.1	-653.4	-452.9
0.9	28.3	140.9	405.4	650.2	1464.4	$+/-$	-1218.9
0.67	26.7	120.0	284.3	391.4	599.7	1181.2	$+/-$
0.5	27.4	125.9	312.2	443.2	724.1	1752.0	-5786.4
0.1	31.4	174.5	728.6	1999.3	-3332.7	-955.2	-575.1
0.0	32.7	196.0	1176.0	∞	-1274.0	-661.5	-457.3

ER

B (Downstream)

Proportion, p \ Average degree $\bar{k}$	20	40	48	50	52	54	56
1.0	36.2	224.0	2245.4	-3196.6	-927.4	-564.6	-414.2
0.9	31.6	156.5	487.1	851.8	$+/-$	-2895.5	-941.8
0.67	28.8	129.2	317.8	449.7	728.6	1722.6	$+/-$
0.5	29.2	134.8	348.5	511.1	899.2	3083.6	-2598.1
0.1	33.0	188.3	915.3	4186.6	-1897.2	-797.9	-518.9
0.0	34.4	212.9	1735.1	-4132.3	-992.0	-583.1	-421.6

F (Upstream)

Proportion, p \ Average degree $\bar{k}$	20	40	48	50	52	54	56
1.0	32.5	203.1	1477.9	$+/-$	-1067.1	-602.2	-429.1
0.9	29.1	147.0	442.1	739.4	1978.9	$+/-$	-1038.3
0.67	27.7	125.5	305.7	429.2	683.6	1521.6	$+/-$
0.5	28.5	132.3	340.0	495.7	859.1	2702.4	-2848.5
0.1	32.9	187.8	909.5	4075.2	-1913.7	-800.2	-519.7
0.0	34.4	212.9	1735.1	-4132.3	-992.0	-583.1	-421.6

SW

C (Downstream)

Proportion, p \ Average degree $\bar{k}$	20	40	48	50	52	54	56
1.0	33.0	198.5	1233.0	$+/-$	-1242.3	-654.2	-454.5
0.9	29.3	144.5	420.4	681.8	1603.4	$+/-$	-1162.3
0.67	27.3	122.1	291.6	403.3	624.6	1271.5	$+/-$
0.5	27.8	127.6	319.0	455.8	754.4	1920.3	-4581.1
0.1	31.6	176.4	755.1	2199.5	-2923.9	-920.0	-562.8
0.0	32.8	198.3	1245.4	-22887.5	-1209.0	-644.4	-449.4

G (Upstream)

Proportion, p \ Average degree $\bar{k}$	20	40	48	50	52	54	56
1.0	32.4	198.7	1307.4	$+/-$	-1187.3	-635.1	-444.0
0.9	28.8	143.2	416.9	675.9	1597.4	$+/-$	-1161.4
0.67	26.9	121.2	289.1	399.5	617.5	1250.7	$+/-$
0.5	27.6	127.1	317.4	453.4	749.3	1896.9	-4708.7
0.1	31.5	176.4	755.1	2200.9	-2918.8	-919.4	-562.6
0.0	32.8	198.3	1245.4	-22887.5	-1209.0	-644.4	-449.4

SF

D (Downstream)

Proportion, p \ Average degree $\bar{k}$	20	40	48	50	52	54	56
1.0	$+/-$	$+/-$	-441.4	-334.0	-274.4	-236.8	-207.6
0.9	500.7	$+/-$	$+/-$	$+/-$	-701.7	-463.4	-347.1
0.67	223.9	310.5	474.7	1009.6	$+/-$	$+/-$	-867.1
0.5	210.0	303.1	460.4	883.1	$+/-$	$+/-$	-789.0
0.1	330.0	603.0	1836.7	$+/-$	-844.2	-510.8	-364.7
0.0	410.7	903.2	$+/-$	-1195.3	-594.0	-410.5	-312.0

H (Upstream)

Proportion, p \ Average degree $\bar{k}$	20	40	48	50	52	54	56
1.0	182.8	265.5	405.9	771.4	$+/-$	-1548.3	-634.9
0.9	146.7	197.9	269.1	397.0	691.9	$+/-$	$+/-$
0.67	146.5	186.8	247.3	347.3	546.3	1142.6	$+/-$
0.5	161.4	219.3	301.7	454.6	842.8	$+/-$	$+/-$
0.1	311.7	550.3	1397.7	$+/-$	-932.0	-538.4	-377.3
0.0	410.7	903.2	$+/-$	-1195.3	-594.0	-410.5	-312.0

Fig. S1. Intermediate frequency of uni-directional edges is optimal for cooperation on random networks. We consider four classes of networks: random regular networks (RR), Erdös-Rényi networks (ER) (8), Watts-Strogatz small-world networks (SW) with rewiring probability 0.1 (9), and Barabási-Albert scale-free networks (BA-SF) (10). For each class, we generated $10,000$ strictly bi-directional networks each with $N = 100$ nodes, for several values of the average node degree $\bar{k}$. For each such network we randomly select a proportion p of edges and convert them to be uni-directional, with randomly chosen orientation. For each resulting network we compute the critical benefit-to-cost ratio $(b/c)^*$ required to favor cooperation, and we plot the mean value, given p and $\bar{k}$, in the tables. In the regions displayed in blue, all ratios are positive and cooperation can evolve for some choice of benefits and costs. In red regions, all ratios are negative and spite is favored instead of cooperation. The symbol '$+/-$' means that a fraction of critical ratios are positive and a fraction negative; and the symbol '∞' means that cooperation is never favored, regardless of how large the benefit.

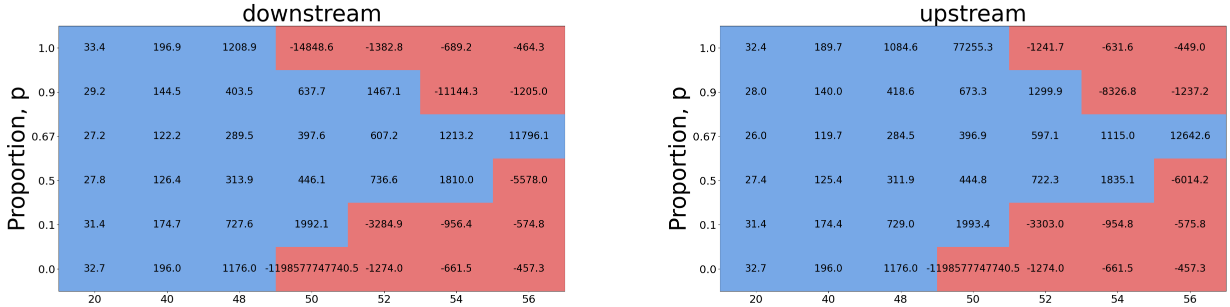

Fig. 1. Random regular

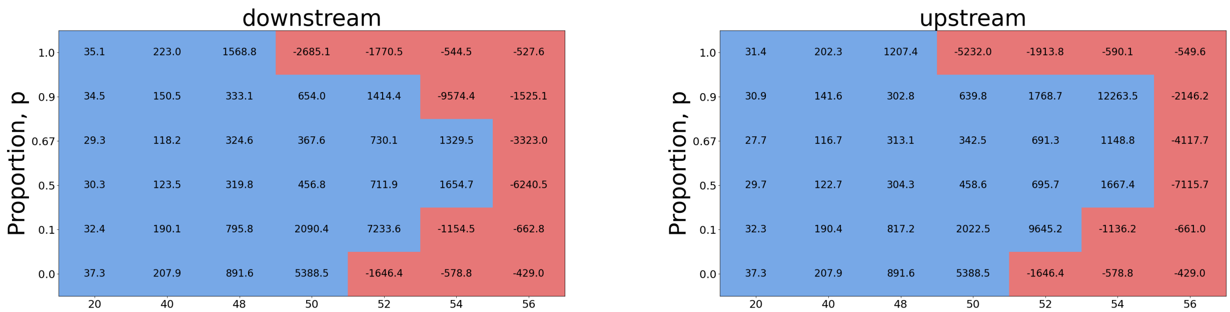

Fig. 2. Erdös-Renyi

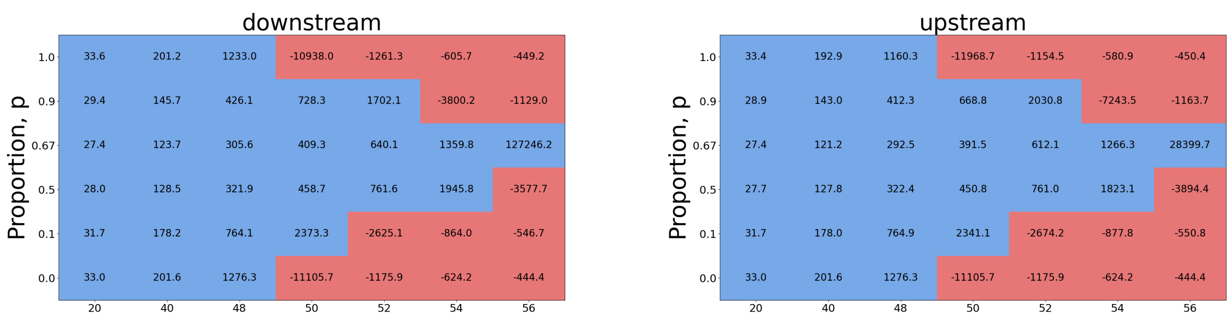

Fig. 3. Watts-Strogatz

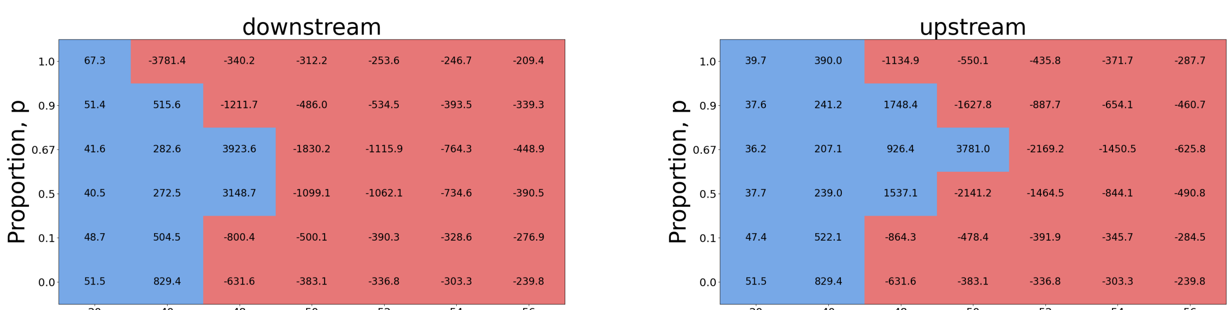

Fig. 4. Barabasi-Albert Scale Free

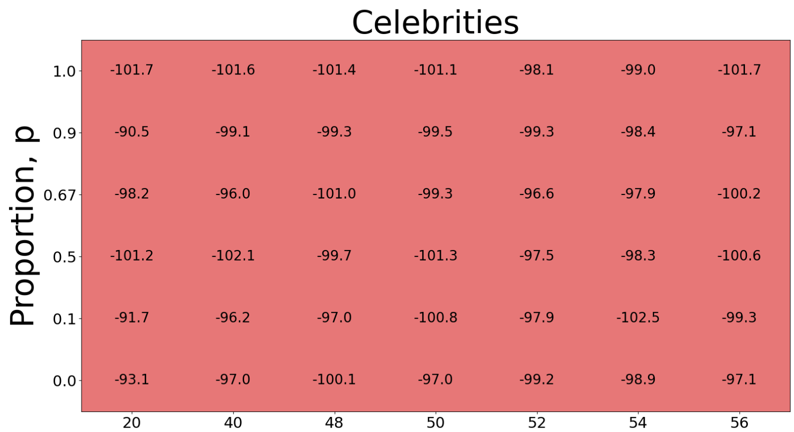

Fig. 5. Results on moral hubs (BA influence network) over a neighbourhood (RR interaction network)

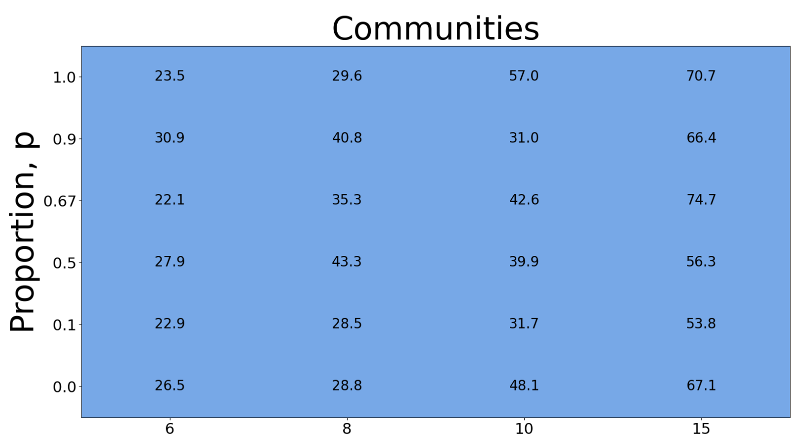

Fig. 6. Results on community influence (every community forms a clique) over a neighbourhood (RR interaction network)

3 Discussion

The color coding (for RR 1, ER 2 and WS 3) is very similar to the baseline (2), except for the BA network (4), that in both downstream and upstream has resulted in less cooperation than expected. On closer inspection of the values, some significant differences remain, but we also find that the singular/special values match. Specifically the case of $\hat{k} = 50$ and $p = 0.0$ for the RR network. The original results comment as how the expectancy of this cell tends to infinity. And indeed our computations ended up with an absurdly high value.

We suspect that given more iterations we would end up with even closer values to the baseline with high confidence. We meet the baseline and consider the experiment successfully replicated.

3.1 Moral hubs

The results for the moral hubs experiment (5) has been that cooperation is highly discouraged across all values of the parameters in a homogeneous manner.

This coincides with our hypothesis, and we are convinced that the lack of similarity from the interaction network with the dispersal network must be the cause. The relevance of this negative result is that for the ranges tested, the two variables (average degree and probability of reciprocity) seem to be non correlated to the cooperation.

Future work could investigate this in a methodical way by generating graphs of more and more similarity and study if there is a correlation with cooperation.

Our interpretation of this result for practical purposes, is that cooperation is obviously dependent on some kind of local socialization among the individuals. Remove it and spite is an expected outcome. If the individuals interact materially with some local neighbours, but are influenced by random individuals[3], and furthermore, these random network of influences has “hubs”, it results in spite.

3.2 Herd and Communities

The results for the communities experiment (*fig.* 6) is a high degree of cooperation, showing similar tendencies to the baseline RR experiment. As the average degree increases, so does the critical benefit cost ratio.

Our thesis is that this relation must be inherited from the original RR network, and adding the cliques, although seems positive for the range tested, might have some interesting consequences for higher values of $\hat{k}$. This contradicted our intuition on what parameters of$\hat{k}$ to explore.

If we closely compare the last column of the communities experiment with the first of the RR baseline, we already see that the values are quite different. This is our main motivation to suggest exploring more this higher range values.

[3] Like say on the internet, although influences in the internet aren’t random.

For this experiment we have no deeper interpretation. It seems that the polarization we could have introduced with the cliques, the herd behaviour ... did not impede cooperation. The tendency is similar for the parameter k, but not for p.

Our hypothesis is correct, but with low confidence and need further work to uncover why we observe differing patterns.

3.3 Limitations of the model

The models used, although having a strong theoretical foundation in evolutionary game theory, are just estimations of how societies behave. More specifically, how they behave in a simple donation game. So none of the results can be taken as a serious estimation on how societies behave.
We would also like to point to the underlying complexity of the model used, that has its roots on simulation the spread of mutations in biological populations (bacteria). They depend strongly on statistic tools of high complexity and lots of approximations and bounds. The critical cost benefit ratio we computed is an approximation.
With no intent to discredit the usefulness of these tools, we approach them not for recommendations on political policy or as philosophical guides for ethics of any kind. But still they retain great potential and are worthy of improvement. Having better models and understanding for cooperation, that match with empirical data (another setting tested in [2]) could shine light on the critical diagnosis of networks that encourage spite, so that we may reform them.

4 Method

4.1 Software used

For this project we needed a programming language with good linear algebra and graph libraries. We also needed the language to be easy to work and fast to program in, so we decided to use Python.

We have used Numpy which has linear algebra functions useful to solve some linear systems of equations. We have used igraph for Python which has all the functions needed to create and operate with graphs.

4.2 Graph generation

Since we wanted to simulate the results of the paper, we needed a way to recreate the different graphs they used. These are the following: Erdös-Renyi, Barabasi-Albert, Watts–Strogatz and random regular.

As they do in the paper, we have considered the following parameters:

- The number of nodes of the graph (n). The article does the study for n = 100.
- Proportion (p) of nodes that are converted from bi-directional to uni-directional.
- Average degree ($\bar{k}$) of the graph. We needed to compute the parameters needed for each of the graphs as a function of $\bar{k}$:

- The Erdös-Renyi model we used has probability $\frac{\bar{k}}{n}$. This way, the average degree of the graph yields approximately $\bar{k}$.

- The random regular model has $\bar{k}$ regularity as an input, so it trivially has average degree $\bar{k}$.

- The Watts-Strogatz model has dimension 1, rewiring probability 0.1 and a nei of $\frac{\bar{k}}{2}$ (this value indicates the level of the lattice up to which a node is connected). Given that for dimension 1 the model has has a ring topology then $\bar{k} = 2 * nei$.

- The Barabasi-Albert model has a diferent m (number of edges added at each iteration) for each of the $\bar{k}$ considered. The m values are taken from a linear regresion that estimated the m needed to have a $\bar{k}$ average degree. Using the hand-shake lemma or other graph properties to compute the average degree $\bar{k}$ as a function of m was not worth the effort. We experimentally fond the appropriate values for our desired values of $\bar{k}$.

4.3 Solving linear systems of equations

We use a linear algebra library. This is because in the article we are trying to replicate, they solved complex and large systems of equations. These systems are crucial for the computation of the critical cost-benefit values of a network.

The first system:

$$\pi_i = \sum_j p_{ij}\pi_j$$
$$\sum_i \pi_i = 1$$

The second one:

$$\tau_{ij} = \begin{cases} 1 + \frac{1}{2}\sum_{k\in\mathcal{N}} p_{ik}\tau_{kj} + \frac{1}{2}\sum_{k\in\mathcal{N}} p_{jk}\tau_{ik} & i \neq j \\ 0 & i = j \end{cases}$$

Solving these systems of equations allowed us to calculate the critical cost-benefit value:

$$\left(\frac{b}{c}\right)^* = \frac{v_2}{u_2 - u_0}$$

$$u_0 = \sum_{i,j,\ell\in\mathcal{N}} \pi_i p_{ij} \tau_{j\ell} w_{\ell j}^{[1]}, \qquad u_2 = \sum_{i,j,k,\ell\in\mathcal{N}} \pi_i p_{ij} p_{ik} \tau_{j\ell} w_{\ell k}^{[1]},$$
$$v_2 = \sum_{i,j,k,\ell\in\mathcal{N}} \pi_i p_{ij} p_{ik} \tau_{jk} w_{k\ell}^{[1]}.$$

As it is mentioned in the original paper, the computational cost of the first system is relatively cheap in comparison to the huge size of the second system. We have to build a matrix with N^4 elements, that has a prohibitive cost in RAM. We tested all manner of optimizations, i.e space matrices of all kinds. The problem is that as we have to solve the inverse of this matrix, we are trading space for time. And as it is abundantly clear we have neither of both resources. But if it is of any interest we tested *SciPy* implementations of *linalg.spsolve*.

All the relevant statistical parameters used to solve the linear system of equations and calculate the critical cost-benefit are explained in [1]. They broadly are the reproductive value, the coalescence time between any two nodes and other statistical estimators.

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