Dynamic Oligopoly and Innovation

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Motivation & Research Question

- Product market competition shapes incentives for R&D
- Two key externalities of innovation:
  - – Business stealing effect
  - + Technology spillover effect

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- Product market competition shapes incentives for R&D
- Two key externalities of innovation:
  – Business stealing effect
  + Technology spillover effect
- How do inter-firm networks of product market rivalry and technology spillover shape equilibrium and optimal R&D allocation, growth, and welfare?

Framework

- Existing Schumpeterian growth models:
  - Monopolistic competition (no strategic interaction)
  - Very few firms in Markov perfect equilibrium

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- Existing Schumpeterian growth models:
  - Monopolistic competition (no strategic interaction)
  - Very few firms in Markov perfect equilibrium
- This paper:
  - Many oligopolists engage in a dynamic R&D game
  - Two inter-firm networks: (i) product market rivalry and (ii) technology spillovers
- LQ differential game avoids the curse of dimensionality
- Characterize the existence of BGP, firm distribution, and endogenous growth rate

Quantitative Analysis

- Identify networks using data for $\approx$ 1,000 public & patenting U.S. firms
- Product market rivalry: business descriptions (Hoberg and Phillips, 2016; Pellegrino, 2025)
- Technology spillovers: patent classification (Jaffe, 1986; Bloom et al., 2013)

Proximity: Example

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Proximity: Example

	Equilibrium	**Optimal R&D Allocation**	**Optimal Uniform R&D Subsidy (32%)**
R&D Share	3.6%	8.0%	8.6%
Growth Rate	1.61%	2.03%	2.07%
CE Welfare Gain	—	+1.29%	+1.17%

Literature

- Competition & innovation:
  d'Aspremont and Jacquemin (1988); Aghion et al. (2001, 2005); Acemoglu and Akcigit (2012); Aghion et al. (2013); Bloom et al. (2013); Lopez and Vives (2019); Peters (2020); Akcigit and Ates (2021, 2023); Liu et al. (2022); Cavenaile et al. (2023)
  Quantitative Schumpeterian growth model matched to firm-(pair-)level data
- Hedonic demand / empirical IO:
  Lancaster (1966); Rosen (1974); Epple (1987); Berry et al. (1995); Nevo (2001); Pellegrino (2025)
  Dynamic GE / endogenous growth
- Oligopoly / market power:
  Neary (2003); Atkeson and Burstein (2008); Gutierrez and Philippon (2017); Autor et al. (2020); Baqaee and Farhi (2020); De Loecker et al. (2020); Azar and Vives (2021); Edmond et al. (2023)

Demand and Production Technology

- Firm $i \in \{1, \ldots, n\}$

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$$Y_t = \boldsymbol{q}_t^T \boldsymbol{b}_t - \frac{1}{2} \boldsymbol{q}_t^T \boldsymbol{\Sigma} \boldsymbol{q}_t$$

  - $\boldsymbol{\Sigma} = \left[\sigma_{ij}\right]$: product market rivalry matrix (network)
  - Linear inverse demand: $p_{i,t} = b_{i,t} - \sum_{j \neq i} \sigma_{ij} q_{j,t} - q_{i,t}$

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$$q_{i,t} = a_{i,t} l_{i,t}$$

- Each firm has knowledge capital (state variable): $z_{i,t}$
- Each firm allocates knowledge capital to improve labor productivity and product quality:

$$\zeta a_{i,t} + b_{i,t} = z_{i,t}$$

Law of Motion of Knowledge Capital

$$\dot{z}_t = \underbrace{\boldsymbol{\Omega} z_t}_{\text{Tech Spillover}} + \underbrace{\mu x_t}_{\text{R\&D}}$$

- $\boldsymbol{\Omega} = \left[\omega_{ij}\right]$: technology spillover matrix (network)
- $x_{i,t} = \sqrt{d_{i,t}}$
  - $d_{i,t}$: R&D input in terms of final good
  - Innovation elasticity is 0.5
- μ: positive scalar
- Can incorporate shocks (not today)

Market Clearing and Preference

- Inelastic labor supply:

$$\sum_i l_{i,t} = L$$

- Final good market clearing:

$$C_t + \underbrace{\sum_i d_{i,t}}_{\text{R\&D input}} = Y_t$$

- Risk neutral representative household:

$$U_t = \int_t^{\infty} \exp\left(-\rho s\right) C_s ds$$

Static Cournot-Nash Equilibrium

- Firm i's gross profit before subtracting dynamic R&D cost:

$$\pi_{i,t}\left(a_{i,t}, b_{i,t}, \boldsymbol{q}_t\right) = p_{i,t}q_{i,t} - w_t l_{i,t} = \left(b_{i,t} - \sum_{j\neq i}\sigma_{ij}q_{j,t} - q_{i,t} - \frac{w_t}{a_{i,t}}\right)q_{i,t} \quad \text{where} \quad \zeta a_{i,t} + b_{i,t} = z_{i,t}$$

- Given w_t, $z_{i,t}$, and $\{q_{j,t}\}_{j\neq i}$, firm i simultaneously chooses $a_{i,t}$, $b_{i,t}$, and $q_{i,t}$ to maximize $\pi_{i,t}$

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- Given w_t, $z_{i,t}$, and $\{q_{j,t}\}_{j \neq i}$, firm i simultaneously chooses $a_{i,t}$, $b_{i,t}$, and $q_{i,t}$ to maximize $\pi_{i,t}$
- Quantity is a linear function of knowledge capital:

$$\boldsymbol{q}(\boldsymbol{z}_t) = \left\{ \underbrace{2\frac{\zeta}{L}\mathbf{1}_{n\times n}}_{\text{labor cost}} + \underbrace{\boldsymbol{I}}_{\text{diminishing demand}} + \underbrace{\boldsymbol{\Sigma}}_{\text{substitutability}} \right\}^{-1} \boldsymbol{z}_t$$

- Gross profits are expressed in quadratic form: $\pi_{i,t} = q_{i,t}^2 = \boldsymbol{z}_t^T \boldsymbol{Q}^i \boldsymbol{z}_t$ Q^i

Linear-Quadratic Differential Game

- Given other firms' R&D $\{x_{j,t}\}_{j\neq i, t\geq 0}$, firm i chooses R&D $\{x_{i,t}\}_{t\geq 0}$ to maximize

$$\max_{\{x_{i,t}\}_{t\geq 0}} \quad V^i(\boldsymbol{z}_0) \equiv \int_0^\infty \exp(-\rho t)\left\{\pi_{i,t} - x_{i,t}^2\right\} dt$$

subject to $\dot{\boldsymbol{z}}_t = \boldsymbol{\Omega}\boldsymbol{z}_t + \mu \boldsymbol{x}_t$

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- Firm i's HJB equation:

$$\rho V^i(\boldsymbol{z}) = \max_{x_i}\left\{\boldsymbol{z}^T\boldsymbol{Q}^i\boldsymbol{z} - x_i^2 + V_{\boldsymbol{z}}^i(\boldsymbol{z})\left[\boldsymbol{\Omega}\boldsymbol{z} + \mu\boldsymbol{x}\right]\right\}$$

HJB Equations $\Longrightarrow$ Riccati Equations

- Guess and verify $V^i(z) = z^T X^i z$ (for any z)
- X^i is the solution of stacked algebraic Riccati equations

Riccati Equations

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- Guess and verify $V^i(z) = z^T X^i z$ (for any z)
- X^i is the solution of stacked algebraic Riccati equations
  Riccati Equations
- All public & patenting firms in the U.S. in our dataset ≈1000 firms $\Longrightarrow$ $1000^3 = 1$ billion undetermined coefficients (< 1 min on my laptop)

Oligopolistic Schumpeterian	Computation time	# of firms	Productivity space
Cavenaile et al. (2023)	$O(2^n)$	4	6 grid
Our model	$O(n^4)$	≈1000	Continuous

BGP

- Linear R&D strategy: $x_t = \mu \widetilde{\boldsymbol{X}} z_t$ where $\widetilde{\boldsymbol{X}} = \left[\boldsymbol{X}_1^1 \ \cdots \ \boldsymbol{X}_n^n \right]^T$ and $\boldsymbol{X}_i^i$ is the i th column of $\boldsymbol{X}^i$
- The law of motion is rewritten as $\dot{z}_t = \boldsymbol{\Phi} z_t$ where

$$\boldsymbol{\Phi} \equiv \underbrace{\boldsymbol{\Omega}}_{\text{Tech Spillover}} + \underbrace{\mu^2 \widetilde{\boldsymbol{X}}}_{\text{R\&D}}$$

BGP

- Linear R&D strategy: $x_t = \mu\widetilde{X}z_t$ where $\widetilde{X} = \left[X_1^1 \cdots X_n^n\right]^T$ and X_i^i is the i th column of X^i
- The law of motion is rewritten as $\dot{z}_t = \Phi z_t$ where

$$\Phi \equiv \underbrace{\Omega}_{\text{Tech Spillover}} + \underbrace{\mu^2\widetilde{X}}_{\text{R\&D}}$$

Theorem

If Φ is irreducible, then:
(i) There exists largest positive eigenvalue of Φ, g, and associated positive eigenvector, z^*.
(ii) There exists a globally stable BGP such that the knowledge capital growth rate of all firms is g, and the knowledge capital distribution is a scalar multiple of z^*.

- Proof: Perron–Frobenius Theorem
- "Φ is irreducible" $\Longleftrightarrow$ "All firms are directly or indirectly connected technologically"

Symmetric Example

Intuition Behind BGP

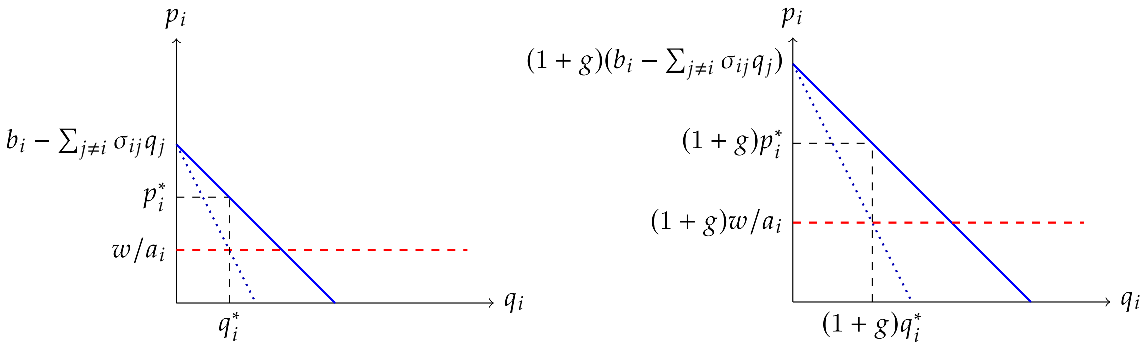
- On the BGP, a_t, b_t, z_t, and q_t grow at the same rate

Technological Choice:	$\zeta a_{i,t} + b_{i,t} = z_{i,t}$
Linear Production Technology:	$q_{i,t} = a_{i,t} l_{i,t}$
Inelastic Labor Supply:	$L = \sum_i l_{i,t}$

- Linear and quadratic terms in q_t of output grow at the same rate:

$$Y_t = \boldsymbol{q}_t^T \boldsymbol{b}_t - \frac{1}{2} \boldsymbol{q}_t^T \boldsymbol{\Sigma} \boldsymbol{q}_t$$

Partial Equilibrium Diagram: CES on BGP despite non-CES



- a_i, b_i, $q_i (= a_i l_i)$, p_i, and w/a_i grow at the same rate g
- Price elasticity stays the same
- (i) (consumer surplus / producer surplus) and (ii) (cost / revenue) stay the same

Lifetime Utility

- Lifetime utility is expressed in quadratic form:

$$\int_t^{\infty} \exp\left(-\rho s\right) C_s ds = \boldsymbol{z}_t^T X \boldsymbol{z}_t$$

X

- Solve the equilibrium once $\Longrightarrow$ Can compute lifetime utility for any initial z_t

Growth Decomposition

- $dz_t/dt = \mathbf{\Phi} z_t$ where $\mathbf{\Phi} = \mathbf{\Omega} + \mu^2 \widetilde{\mathbf{X}}$

$$\frac{d\log Y_t}{dt} = \underbrace{\frac{z_t^T\left(\mathbf{Q\Omega} + \mathbf{\Omega Q}\right)z_t}{Y_t}}_{\text{Tech Spillover}} + \underbrace{\frac{\mu^2 z_t^T\left(\mathbf{Q}\widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}^T\mathbf{Q}\right)z_t}{Y_t}}_{\text{R\&D}}$$

Product Market Rivalry Σ

- Hoberg and Phillips (2016) estimates product proximity using business descriptions in 10-K
- Pellegrino (2025) estimates α to align with the cross-price elasticity of demand

$$\underbrace{\sigma_{ij}}_{\text{substitutability}} = \alpha \times \text{product proximity b/w } i \text{ and } j \quad (i \neq j)$$

micro estimates

Technological Proximity $\widetilde{\mathbf{\Omega}}$

- Technological profile of firm i
  - The vector of the share of patents held by firm i in each patent class
  - Baseline: group-level patent classifications (≈ 4000)

- Jaffe (1986) technological proximity measure $\tilde{\omega}_{ij}$:
  - Cosine similarity of the technological profiles b/w firm i and j

Distribution of Knowledge Capital z_t

Variables	Identification
$\pi_{i,t}$	Gross profit (before R&D cost) = Revenue − Cost of goods sold
$\boldsymbol{q}_t$	$\pi_{i,t} = q_{i,t}^2$
ζ/L	Matches sample firms' cost share (average markup)
$\boldsymbol{z}_t$	$\boldsymbol{z}_t = \left\{ 2\frac{\zeta}{L}\mathbf{1}_{n\times n} + \boldsymbol{I} + \boldsymbol{\Sigma} \right\} \boldsymbol{q}_t$

Technology Spillover $\mathbf{\Omega} = \beta \times$Technological Proximity $\widetilde{\mathbf{\Omega}}$

$$z_{i,t+1} - z_{i,t} = \beta \sum_{j \neq i} \tilde{\omega}_{ij,t} z_{j,t} + \text{Year FE}_t + \epsilon_{i,t}$$

	(1)	(2)	(3)
$\sum_{j \neq i} \tilde{\omega}_{ij,t} z_{j,t}$	0.000178***	0.000120***	0.000148***
	(0.000032)	(0.000033)	(0.000035)
$\sqrt{\text{R\&D Expenditure}}$		0.051598***	
		(0.018485)	
Year Fixed Effects	✓	✓	✓
IV			✓
IV 1st Stage F-statistics			4176
No. observations	21,825	21,825	21,825

SEs clustered by years and 4-digit NAICS industries are reported in parentheses. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

- IV: Tax price of R&D from state-specific rules (Bloom et al., 2013)

First Stage

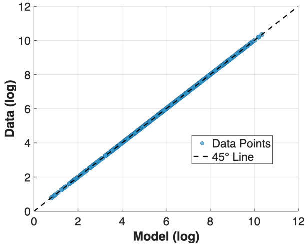
Identification Summary

- Publicly available data + Compustat

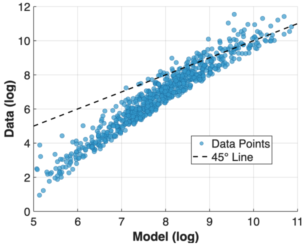
Notation	Description	Value	Source
$\mathbf{\Sigma}$	Product proximity		Form 10-K, Hoberg and Phillips (2016)
$\widetilde{\mathbf{\Omega}}$	Technological proximity		USPTO, Patent classification
α	Product proximity $\to$ Substitutability	0.12	Pellegrino (2025)
β	Technological proximity $\to$ Spillover	0.00015	Estimate the law of motion
ζ/L	Labor augmentation efficiency	0.0038	Compustat, Cost of goods sold
ρ	Discount rate	0.10	$>$ risk-free rates, $<$ private R&D returns
μ	R&D efficiency	0.06	1.6% economic growth rate

Fit b/w Model and Data: Profit, Sales, and R&D

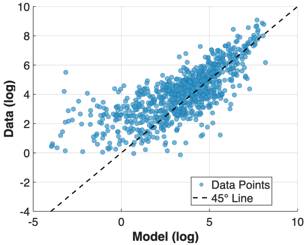
- Comparison of firm-level model-generated values (x-axis) with observed data (y-axis)



(a) Log of Profit (Targeted)



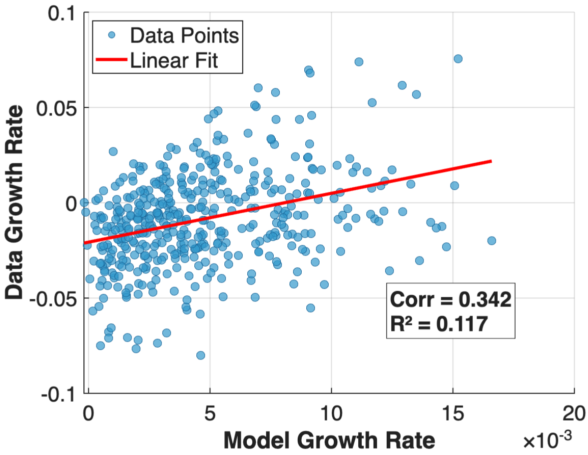
(b) Log of Sales



(c) Log of R&D Expenditure

Fit b/w Model and Data: Firm-level Growth Rates

- Data: Average growth rate of z_{it} b/w 2010 and 2019



Variable Definitions

Variable	Definition
Total Output	(competitive equilibrium = 100)
R&D Share	Total R&D / Total output (%)
Growth Rate	(%)
R&D Contribution	Growth due to R&D / Growth rate (%)
Social Welfare	(competitive equilibrium = 100)
Firm Value Share	Total firm value / Social welfare (%)

Counterfactuals: R&D (Given Static Cournot Competition)

	Total Output	R&D Share	Growth Rate	R&D Contribution	Social Welfare	Firm Value Share
Competitive Equilibrium	100.00	3.61%	1.61%	52.85%	100.00	31.07%

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Competitive Equilibrium	100.00	3.61%	1.61%	52.85%	100.00	31.07%
Social Optimum	100.00	8.02%	2.03%	62.55%	101.29	28.69%

- Social optimum: Internalize the impact of innovation on consumer surplus $\Longrightarrow$ R&D $\uparrow$

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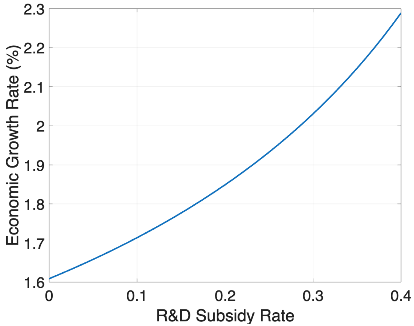
Social Optimum	100.00	8.02%	2.03%	62.55%	101.29	28.69%
Monopoly	100.00	2.18%	1.40%	45.74%	98.92	32.13%

- Social optimum: Internalize the impact of innovation on consumer surplus $\Longrightarrow$ R&D $\uparrow$
- Monopoly: Internalize business stealing $>$ technological spillover $\Longrightarrow$ R&D $\downarrow$

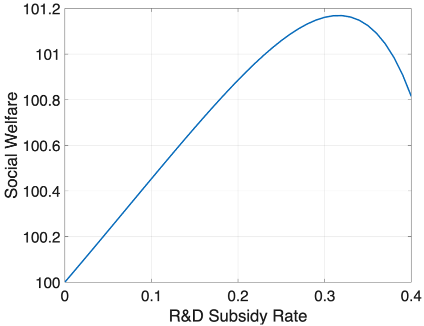
Counterfactuals: Both R&D and Product Market Competition

	Total Output	R&D Share	Growth Rate	R&D Contribution	Social Welfare	Firm Value Share
Competitive Equilibrium	100.00	3.61%	1.61%	52.85%	100.00	31.07%
Social Optimum	109.68	13.57%	2.73%	71.71%	117.79	–25.59%
Monopoly	96.17	2.54%	1.45%	48.50%	95.46	46.86%

R&D Subsidy



(a) Growth Rates



(b) Social Welfare

- Optimal subsidy rate: 32%
    - Growth rate increases by 0.46 p.p
    - Welfare increases by 1.17%

Conclusion

- Quantitative Schumpeterian growth model with inter-firm networks
  - Product market rivalry network ($\boldsymbol{\Sigma}$)
  - Technology spillover network ($\boldsymbol{\Omega}$)
- LQ differential game avoids curse of dimensionality
  - Can solve with thousands of oligopolistic firms
  - Utilize recently available micro data and computational power
- Potential applications:
  - Trade liberalization (war) and international technology spillovers

Technology & Product Proximity: Example

Tesla vs. Ford	
Technology Proximity	0.11
Product Proximity	0.15

Apple vs. Intel	
Technology Proximity	0.57
Product Proximity	0.00

Back

Generalized Hedonic-Linear Demand (Pellegrino, 2025) (1/2)

- $i \in \{1, 2, \ldots, n\}$: firms / products
- 1 unit of product i provides
  - 1 unit of idiosyncratic characteristic $k \in \{1, 2, \ldots, n\}$
  - $\psi_{k,i}$ unit of shared characteristic $k \in \{n+1, n+2, \ldots, n+n_k\}$ where $\sum_k \psi_{k,i}^2 = 1$
- Aggregate each characteristic:

$$y_{k,t} = \begin{cases} q_{k,t} & k = 1, 2, \ldots, n \\ \sum_i \psi_{k,i} q_{i,t} & k = n+1, n+2, \ldots, n+n_k \end{cases}$$

- Linear quadratic aggregator over characteristics:

$$Y_t = (1-\alpha) \sum_{k=1}^{n} \left(\underbrace{\hat{b}_{k,t} y_{k,t} - \frac{1}{2} y_{k,t}^2}_{\text{idiosyncratic characteristic}} \right) + \alpha \sum_{k=n+1}^{n+n_k} \left(\underbrace{\hat{b}_{k,t} y_{k,t} - \frac{1}{2} y_{k,t}^2}_{\text{shared characteristic}} \right)$$

Generalized Hedonic-Linear Demand (Pellegrino, 2025) (2/2)

- Quality:

$$b_i = (1 - \alpha)\,\hat{b}_i + \alpha \sum_{k=n+1}^{n+n_k} \psi_k \hat{b}_k$$

- Inverse demand:

$$\boldsymbol{p} = \boldsymbol{b} - \boldsymbol{\Sigma} \boldsymbol{q}$$

- Inverse cross price elasticity of demand:

$$\frac{\partial \log p_i}{\partial \log q_j} = -\frac{q_j}{p_i} \cdot \sigma_{ij}$$

- Cross price elasticity of demand:

$$\frac{\partial \log q_i}{\partial \log p_j} = -\frac{p_j}{q_i} (\boldsymbol{\Sigma}^{-1})_{ij}$$

Static Profits

- Gross profit: $\pi_{i,t} = q_{i,t}^2$
- Firms choose labor productivity and product quality: $\zeta a_{i,t} = \sqrt{\zeta w_t}$, $b_{i,t} = z_{i,t} - \sqrt{\zeta w_t}$
- Labor market clearing: $L = \sum_i \frac{q_{i,t}}{a_{i,t}} \Longrightarrow \sqrt{\zeta w_t} = \frac{\zeta}{L} \sum_i q_{i,t}$
- $\boldsymbol{q}_t = \boldsymbol{N} \boldsymbol{z}_t$ where $\boldsymbol{N} \equiv \left\{ 2 \frac{\zeta}{L} \mathbf{1}_{n \times n} + \boldsymbol{I} + \boldsymbol{\Sigma} \right\}^{-1}$
- $\boldsymbol{N}_i$: the i th row of $\boldsymbol{N}$
- Profit:

$$\pi_{i,t} = q_{i,t}^2 = \boldsymbol{z}_t^T \boldsymbol{Q}^{\boldsymbol{i}} \boldsymbol{z}_t$$

where $\boldsymbol{Q}^{\boldsymbol{i}} \equiv \boldsymbol{N}_i^T \boldsymbol{N}_i$

Back

Riccati Equations

- Stochastic law of motion:

$$dz_t = (\boldsymbol{\Omega} z_t + \mathrm{diag}(\mu)\, x_t)\, dt + \mathrm{diag}(\gamma)\, z_t\, d\boldsymbol{W}_t$$

- $V^i(z) = z^T \boldsymbol{X}^i z$ where $\boldsymbol{X}^i$ is the solution of the stacked Riccati equation

$$0 = \boldsymbol{Q}^i - \mu^2 \boldsymbol{X}_i^i \left(\boldsymbol{X}_i^i\right)^T + \left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)^T \boldsymbol{X}^i + \boldsymbol{X}^i \left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)$$

  - $\boldsymbol{X}_i^i \equiv$ the i th column of $\boldsymbol{X}^i$
  - $\boldsymbol{\Phi} \equiv \boldsymbol{\Omega} + \mu^2 \begin{bmatrix} \boldsymbol{X}_1^1 & \cdots & \boldsymbol{X}_n^n \end{bmatrix}^T$

- Algorithm: Given $\begin{bmatrix} \boldsymbol{X}_\tau^1 & \cdots & \boldsymbol{X}_\tau^n \end{bmatrix}$, update $\begin{bmatrix} \boldsymbol{X}_{\tau-\Delta}^1 & \cdots & \boldsymbol{X}_{\tau-\Delta}^n \end{bmatrix}$ by

$$-\frac{\boldsymbol{X}_\tau^i - \boldsymbol{X}_{\tau-\Delta}^i}{\Delta} = \boldsymbol{Q}^i - \mu^2 \boldsymbol{X}_{i,\tau}^i \left(\boldsymbol{X}_{i,\tau}^i\right)^T + \left(\boldsymbol{\Phi}_\tau - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)^T \boldsymbol{X}_\tau^i + \boldsymbol{X}_\tau^i \left(\boldsymbol{\Phi}_\tau - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)$$

Output and Lifetime Utility

- Stochastic law of motion:

$$d\boldsymbol{z}_t = \left(\boldsymbol{\Omega}\boldsymbol{z}_t + \operatorname{diag}\left(\mu\right)\boldsymbol{x}_t\right)dt + \operatorname{diag}\left(\gamma\right)\boldsymbol{z}_t d\boldsymbol{W}_t$$

- Output: $Y_t = \boldsymbol{q}_t^T \boldsymbol{Q} \boldsymbol{q}_t$ where

$$\boldsymbol{Q} = \frac{\zeta}{L}\boldsymbol{1}_{n\times n} + \boldsymbol{I} + \frac{1}{2}\boldsymbol{\Sigma}$$

- Expected utility:

$$V\left(\boldsymbol{z}_t\right) \equiv \boldsymbol{E}_t\left[\int_t^\infty \exp\left(-\rho s\right) C_s ds \middle| \boldsymbol{z}_t\right] = \boldsymbol{z}_t^T \boldsymbol{X} \boldsymbol{z}_t$$

where X is the solution of the Lyapunov equation (obtained from households' HJB equation):

$$0 = \boldsymbol{Q} - \mu^2 \tilde{\boldsymbol{X}}^T \tilde{\boldsymbol{X}} + \boldsymbol{X}\left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right) + \left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)^T \boldsymbol{X}$$

Equilibrium Summary

Description	Expression
Production strategy	$\boldsymbol{q}_t = \boldsymbol{N}\boldsymbol{z}_t$
R&D strategy	$\boldsymbol{x}_t = \mu\tilde{\boldsymbol{X}}\boldsymbol{z}_t$
Profit of final producers	$\Pi_t^F = \boldsymbol{q}_t^T\left(\frac{1}{2}\boldsymbol{\Sigma}\right)\boldsymbol{q}_t$
Total operating profit of firms	$\Pi_t = \boldsymbol{q}_t^T\boldsymbol{q}_t$
Labor income	$w_tL = \boldsymbol{q}_t^T\left(\frac{\zeta}{L}\mathbf{1}_{n\times n}\right)\boldsymbol{q}_t$
Output	$Y_t = \boldsymbol{q}_t^T\left(\frac{\zeta}{L}\mathbf{1}_{n\times n} + \boldsymbol{I} + \frac{1}{2}\boldsymbol{\Sigma}\right)\boldsymbol{q}_t$
Consumption	$C_t = Y_t - \boldsymbol{x}_t^T\boldsymbol{x}_t$

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Example: Symmetric Equilibrium

Assumption

- Symmetric product substitutability and technology spillover: $\sigma_{ij} = \sigma, \omega_{ij} = \omega \quad \forall i \neq j$

- R&D strategy: $x^*_{i,t} = \mu \left(\tilde{x}_1 z_{i,t} + \tilde{x}_2 \sum_{j \neq i} z_j \right)$
  - $\tilde{x}_1$: market size effect (> 0)
  - $\tilde{x}_2$: strategic substitutability (< 0) / complementarity (> 0)

- Growth rate: $g = \underbrace{(n-1)\omega}_{\text{Tech Spillover}} + \underbrace{\mu^2 (\tilde{x}_1 + (n-1)\tilde{x}_2)}_{\text{R\&D}}$

- Stability (irreducibility) requires $\omega + \mu^2 \tilde{x}_2 > 0$
  - Tech spillover (ω) must be strong relative to strategic substitutability ($\tilde{x}_2 < 0$)

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Social Optimum

- Static optimal allocation: $\boldsymbol{q}_t^* = \boldsymbol{N}^* \boldsymbol{z}_t$ where $\boldsymbol{N}^* \equiv \left\{2\frac{\zeta}{L}\mathbf{1}_{n\times n} + \boldsymbol{\Sigma}\right\}^{-1}$
- Optimal output: $Y_t^* = \boldsymbol{z}_t^T \boldsymbol{Q}^* \boldsymbol{z}_t$ where $\boldsymbol{Q}^* = \frac{1}{2}\boldsymbol{N}^*$
- Optimal expected utility:

$$V^*(\boldsymbol{z}_t) \equiv \boldsymbol{E}_t\left[\int_t^\infty \exp(-\rho s)\, C_s\, ds \middle| \boldsymbol{z}_t\right] = \boldsymbol{z}_t^T \boldsymbol{X}^* \boldsymbol{z}_t,$$

where $\boldsymbol{X}^*$ is the solution of the Riccati equation (obtained from planner's HJB equation):

$$0 = \boldsymbol{Q}^* - \mu^2(\boldsymbol{X}^*)^T\boldsymbol{X}^* + \boldsymbol{X}^*\left(\boldsymbol{\Phi}^* - \frac{1}{2}(\rho-\gamma^2)\,\boldsymbol{I}\right) + \left(\boldsymbol{\Phi}^* - \frac{1}{2}(\rho-\gamma^2)\,\boldsymbol{I}\right)\boldsymbol{X}^*$$

- Optimal R&D: $\boldsymbol{x}_t^* = \mu \boldsymbol{X}^* \boldsymbol{z}_t$
- Optimal technology transition matrix: $\boldsymbol{\Phi}^* = \boldsymbol{\Omega} + \mu^2 \boldsymbol{X}^*$

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Stochastic Growth

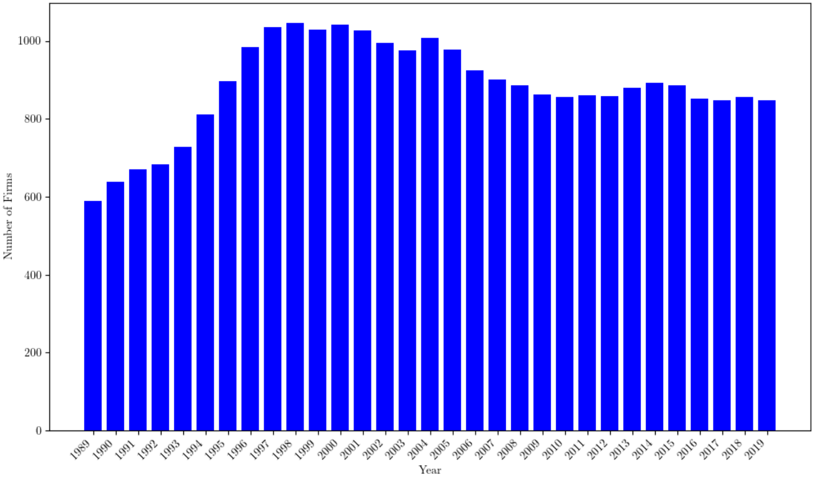
- Applying Itô's lemma,

$$d\log Y_t = \left[\frac{z_t^T\left(Q\Phi + \Phi^T Q\right)z_t}{Y_t} + \gamma^2\left\{\frac{\sum_i z_{i,t}^2 Q_{ii}}{Y_t} - \frac{2z_t^T Q\,\mathrm{diag}\left(z_t^2\right)Qz_t}{Y_t^2}\right\}\right]dt + \frac{2\gamma z_t^T Q\,\mathrm{diag}\left(z_t\right)}{Y_t}dW_t$$

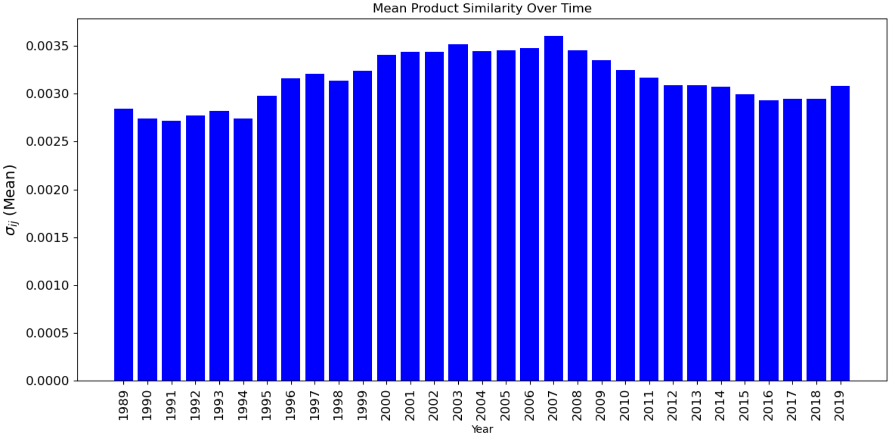
where $Y_t = z_t^T Q z_t$ and $\Phi = \Omega + \mu^2\widetilde{X}$

Tech Spillover	$\frac{z_t^T\left(Q\Omega + \Omega^T Q\right)z_t}{Y_t}$
R&D Contribution	$\frac{\mu^2 z_t^T\left(Q\widetilde{X} + \widetilde{X}^T Q\right)z_t}{Y_t}$
Itô Correction	$\gamma^2\left\{\frac{\sum_i z_{i,t}^2 Q_{ii}}{Y_t} - \frac{2z_t^T Q\,\mathrm{diag}\left(z_t^2\right)Qz_t}{Y_t^2}\right\}$
Total	$E\left[d\log Y_t\right]$

Trend of Number of Sample Firms

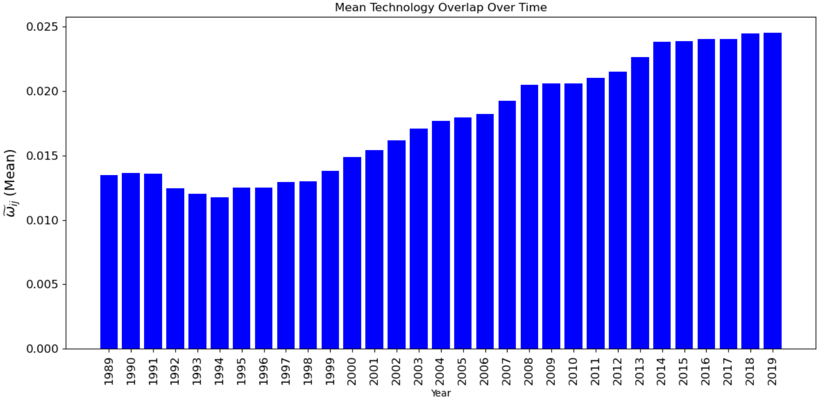


Trend of Product Substitutability



Trend of Technological Proximity

- Merge USPTO data with Compustat firms using DISCERN 2 dataset (Arora et al., 2024)
- Jaffe measure, Group-level patent classification, Stack for 5 years



Microeconometric Estimates vs. GHL (Pellegrino, 2025) (1/2)

Market	Firm i	Firm j	Micro Estimate	GHL
Auto	Ford	Ford	-4.320	-5.197
Auto	Ford	General Motors	0.034	0.056
Auto	Ford	Toyota	0.007	0.017
Auto	General Motors	Ford	0.065	0.052
Auto	General Motors	General Motors	-6.433	-4.685
Auto	General Motors	Toyota	0.008	0.005
Auto	Toyota	Ford	0.018	0.025
Auto	Toyota	General Motors	0.008	0.008
Auto	Toyota	Toyota	-3.085	-4.851

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Microeconometric Estimates vs. GHL (Pellegrino, 2025) (2/2)

Market	Firm i	Firm j	Micro Estimate	GHL
Cereals	Kellogg’s	Kellogg’s	-3.231	-1.770
Cereals	Kellogg’s	Quaker Oats	0.033	0.023
Cereals	Quaker Oats	Kellogg’s	0.046	0.031
Cereals	Quaker Oats	Quaker Oats	-3.031	-1.941
Computers	Apple	Apple	-11.979	-8.945
Computers	Apple	Dell	0.018	0.025
Computers	Dell	Apple	0.027	0.047
Computers	Dell	Dell	-5.570	-5.110

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First Stage

	(1)
Dependent Variable:	$z_{i,t}$
User cost of R&D	-48.15^{***}
	(4.25)
Year Fixed Effects	✓
No. observations	$20,665$

SEs clustered by years and 4-digit NAICS industries are reported in parentheses.

- IV: User cost of R&D, driven by state-level tax variations (Wilson, 2009; Bloom et al., 2013)

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