Dynamic Oligopoly and Innovation

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Motivation & Research Question

- Product market competition shapes incentives for R&D
- Two key inter-firm externalities of innovation:
  – Business stealing effect
  + Technology spillover effect

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- Product market competition shapes incentives for R&D
- Two key inter-firm externalities of innovation:
  - – Business stealing effect
  - + Technology spillover effect
- How do inter-firm networks of product market rivalry and technology spillover shape equilibrium and optimal R&D allocation, growth, and welfare?

Framework

- Existing Schumpeterian growth models:
    - Monopolistic competition (no strategic interaction)
    - Very few firms in Markov perfect equilibrium

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- Existing Schumpeterian growth models:
  - Monopolistic competition (no strategic interaction)
  - Very few firms in Markov perfect equilibrium
- This paper:
  - Many oligopolists engage in a dynamic R&D game
  - Two inter-firm networks: (i) product market rivalry and (ii) technology spillovers
- LQ differential game avoids the curse of dimensionality
- Characterize the existence of BGP, firm distribution, and endogenous growth rate

Quantitative Analysis

- Identify networks using data for $\approx$ 1,000 public & patenting U.S. firms
- Product market rivalry: business descriptions (Hoberg and Phillips, 2016; Pellegrino, 2025)
- Technology spillovers: patent classification (Jaffe, 1986; Bloom et al., 2013)

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	Equilibrium	**Optimal R&D Allocation**	**Optimal Uniform R&D Subsidy (34%)**
R&D Share	2.60%	6.27%	6.17%
CE Welfare Gain	—	+4.36%	+0.53%

- R&D subsidies should take the network structure into account and target firms with
  - low product-market centrality
  - high technology-spillover centrality

Literature

- Competition & innovation:
  d'Aspremont and Jacquemin (1988); Aghion et al. (2001, 2005); Acemoglu and Akcigit (2012); Aghion et al. (2013); Bloom et al. (2013); Lopez and Vives (2019); Peters (2020); Akcigit and Ates (2021, 2023); Liu et al. (2022); Cavenaile et al. (2023)
  Quantitative Schumpeterian growth model matched to firm-(pair-)level data
- Hedonic demand / empirical IO:
  Lancaster (1966); Rosen (1974); Epple (1987); Berry et al. (1995); Nevo (2001); Pellegrino (2025)
  Dynamic GE / endogenous growth
- Oligopoly / market power:
  Neary (2003); Atkeson and Burstein (2008); Gutierrez and Philippon (2017); Autor et al. (2020); Baqaee and Farhi (2020); De Loecker et al. (2020); Azar and Vives (2021); Edmond et al. (2023)

Demand and Production Technology

- Firm $i \in \{1, \ldots, n\}$ has knowledge capital $z_{i,t}$ and produces a single differentiated product

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$$p_{i,t} = b_{i,t} - \sum_{j \neq i} \sigma_{ij} q_{j,t} - q_{i,t}$$

  - $\boldsymbol{\Sigma} = [\sigma_{ij}]$: product-market rivalry matrix (network) $(\sigma_{ii} = 1)$

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- CRS production technology:

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- Each firm allocates knowledge capital to improve labor productivity and product quality:

$$\zeta a_{i,t} + b_{i,t} = z_{i,t}$$

Law of Motion of Knowledge Capital

$$\dot{z}_t = \underbrace{\boldsymbol{\Omega} z_t}_{\text{Tech Spillover}} + \underbrace{\mu x_t}_{\text{R\&D}} - \underbrace{\delta z_t}_{\text{Depreciation}}$$

- $\boldsymbol{\Omega} = \left[\omega_{ij}\right]$: technology spillover matrix (network)
- $x_{i,t} = \sqrt{d_{i,t}}$
  - $d_{i,t}$: R&D input in terms of final good
  - Innovation elasticity is 0.5
- μ, δ: positive scalars
- Can incorporate idiosyncratic & aggregate shocks (not today)

Market Clearing and Preference

- Inelastic labor supply:

$$L = \sum_i l_{i,t}$$

- Linear-quadratic aggregator:

$$Y_t = \boldsymbol{q}_t^T \boldsymbol{b}_t - \frac{1}{2} \boldsymbol{q}_t^T \boldsymbol{\Sigma} \boldsymbol{q}_t$$

- Final good market clearing:

$$C_t + \underbrace{\sum_i d_{i,t}}_{\text{R\&D input}} = Y_t$$

- Risk-neutral representative household:

$$U_t = \int_t^\infty \exp\left(-\rho s\right) C_s ds$$

Static Cournot-Nash Equilibrium

- Firm i's gross profit before subtracting dynamic R&D cost:

$$\pi_{i,t}\left(a_{i,t}, b_{i,t}, \boldsymbol{q}_t\right) = p_{i,t} q_{i,t} - w_t l_{i,t} = \left(b_{i,t} - \sum_{j \neq i} \sigma_{ij} q_{j,t} - q_{i,t} - \frac{w_t}{a_{i,t}}\right) q_{i,t} \quad \text{where} \quad \zeta a_{i,t} + b_{i,t} = z_{i,t}$$

- Given w_t, $z_{i,t}$, and $\{q_{j,t}\}_{j \neq i}$, firm i simultaneously chooses $a_{i,t}$, $b_{i,t}$, and $q_{i,t}$ to maximize $\pi_{i,t}$

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- Given w_t, $z_{i,t}$, and $\{q_{j,t}\}_{j \neq i}$, firm i simultaneously chooses $a_{i,t}$, $b_{i,t}$, and $q_{i,t}$ to maximize $\pi_{i,t}$
- Quantity is a linear function of knowledge capital:

$$\boldsymbol{q}(\boldsymbol{z}_t) = \left\{ \underbrace{2\frac{\zeta}{L}\mathbf{1}_{n \times n}}_{\text{labor cost}} + \underbrace{\boldsymbol{I}}_{\text{diminishing demand}} + \underbrace{\boldsymbol{\Sigma}}_{\text{substitutability}} \right\}^{-1} \boldsymbol{z}_t$$

- Gross profits are expressed in quadratic form: $\pi_{i,t} = q_{i,t}^2 = \boldsymbol{z}_t^T \boldsymbol{Q}^i \boldsymbol{z}_t$ Q^i

Linear-Quadratic Differential Game

- Given other firms' R&D $\left\{x_{j,t}\right\}_{j\neq i,t\geq 0}$, firm i chooses R&D $\left\{x_{i,t}\right\}_{t\geq 0}$ to maximize

$$\max_{\left\{x_{i,t}\right\}_{t\geq 0}} \quad V^{i}\left(z_0\right) \equiv \int_0^{\infty} \exp\left(-\rho t\right)\left\{\pi_{i,t} - x_{i,t}^2\right\} dt$$

subject to $\dot{\boldsymbol{z}}_t = \boldsymbol{\Omega} \boldsymbol{z}_t + \mu \boldsymbol{x}_t$

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- Firm i's HJB equation:

$$\rho V^i(\boldsymbol{z}) = \max_{x_i} \left\{ \boldsymbol{z}^T \boldsymbol{Q}^i \boldsymbol{z} - x_i^2 + V_{\boldsymbol{z}}^i(\boldsymbol{z})\left[\boldsymbol{\Omega}\boldsymbol{z} + \mu \boldsymbol{x}\right]\right\}$$

HJB Equations $\Longrightarrow$ Riccati Equations

- Guess and verify $V^i(z) = z^T X^i z$ (for any z)
- X^i is the solution of stacked algebraic Riccati equations

Riccati Equations

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  Riccati Equations
- All public & patenting firms in the U.S. in our dataset ≈1000 firms $\Longrightarrow$ $1000^3 = 1$ billion undetermined coefficients (< 1 min on my laptop)

Oligopolistic Schumpeterian	Computation time	# of firms	Productivity space
Cavenaile et al. (2023)	$O(2^n)$	4	6 grids
Our model	$O(n^4)$	≈1000	Continuous

BGP

- Linear R&D strategy: $x_t = \mu \widetilde{\boldsymbol{X}} z_t$ where $\widetilde{\boldsymbol{X}} = \left[\boldsymbol{X}_1^1 \;\cdots\; \boldsymbol{X}_n^n\right]^T$ and $\boldsymbol{X}_i^i$ is the ith column of $\boldsymbol{X}^i$
- The law of motion is rewritten as $\dot{z}_t = \boldsymbol{\Phi} z_t$ where

$$\boldsymbol{\Phi} \equiv \underbrace{\boldsymbol{\Omega}}_{\text{Tech Spillover}} + \underbrace{\mu^2 \widetilde{\boldsymbol{X}}}_{\text{R\&D}} - \underbrace{\delta \boldsymbol{I}}_{\text{Depreciation}}$$

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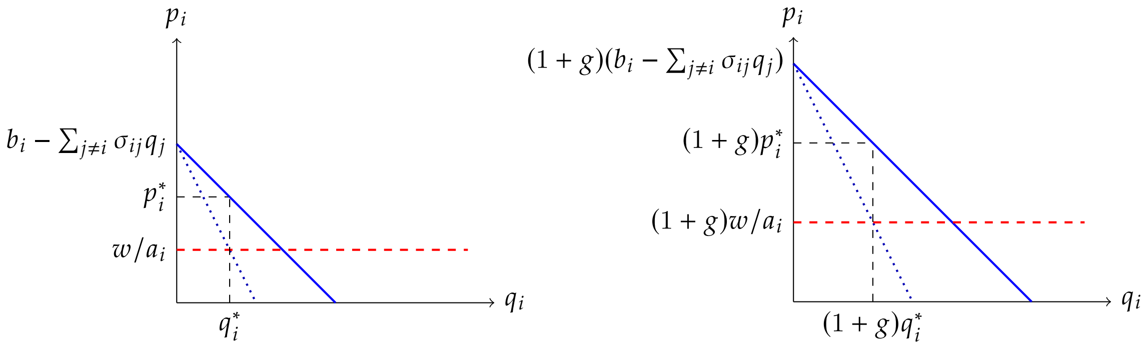
Theorem

If $\boldsymbol{\Phi}$ is irreducible, then:
(i) There exists a largest positive eigenvalue of $\boldsymbol{\Phi}$, g, and an associated positive eigenvector, z^*.
(ii) There exists a globally stable BGP such that the knowledge capital growth rate of all firms is g, and the knowledge capital distribution is a scalar multiple of z^*.

- Proof: Perron–Frobenius Theorem
- "$\boldsymbol{\Phi}$ is irreducible" $\Longleftrightarrow$ "All firms are directly or indirectly connected technologically"

Symmetric Example

Partial Equilibrium Diagram: CES on BGP despite non-CES



- a_i, b_i, $q_i (= a_i l_i)$, p_i, and w/a_i grow at the same rate g
- Price elasticity stays the same
- (i) (consumer surplus / producer surplus) and (ii) (cost / revenue) stay the same

Lifetime Utility

- Lifetime utility is expressed in quadratic form:

$$\int_t^\infty \exp\left(-\rho s\right) C_s ds = \boldsymbol{z}_t^T X \boldsymbol{z}_t$$

X

- Solve the equilibrium once $\Longrightarrow$ Can compute lifetime utility for any initial z_t

Product Market Rivalry Σ

- Hoberg and Phillips (2016) estimates product proximity using business descriptions in 10-K
- Pellegrino (2025) estimates α to align with the cross-price elasticity of demand

$$\underbrace{\sigma_{ij}}_{\text{substitutability}} = \alpha \times \text{product proximity b/w } i \text{ and } j \quad (i \neq j)$$

micro estimates

Technological Proximity $\widetilde{\Omega}$

- Technological profile of firm i
  - The vector of the share of patents held by firm i in each technology class
  - Baseline: group-level patent classifications (≈ 4000), five years window
- Jaffe (1986) technological proximity measure $\tilde{\omega}_{ij}$:
  - Cosine similarity of the technological profiles between firms i and j
  - Impose $\sum_{j \neq i} \tilde{\omega}_{ij} = 1$ for each i

Distribution of Knowledge Capital z_t

Variables	Identification
$\pi_{i,t}$	Gross profit (before R&D cost) = Revenue − Cost of goods sold
$\boldsymbol{q}_t$	$\pi_{i,t} = q_{i,t}^2$
ζ/L	Matches sample firms' cost share (average markup)
$\boldsymbol{z}_t$	$\boldsymbol{z}_t = \left\{2\frac{\zeta}{L}\mathbf{1}_{n\times n} + \boldsymbol{I} + \boldsymbol{\Sigma}\right\}\boldsymbol{q}_t$

Technology Spillover Ω = $\beta\times$Technological Proximity $\tilde{\Omega}$ First Stage

$$\log z_{i,t+1} - \log z_{i,t} = \beta \sum_{j \neq i} \tilde{\omega}_{ij,t} \frac{z_{j,t}}{z_{i,t}} + \text{Controls}_{i,t} + \epsilon_{i,t}$$

	(1)	(2)	(3)
	$\log z_{i,t+1} - \log z_{i,t}$	$\log z_{i,t+1} - \log z_{i,t}$	$\log z_{i,t+1} - \log z_{i,t}$
$\sum_{j \neq i} \tilde{\omega}_{ij,t} \frac{z_{j,t}}{z_{i,t}}$	0.026**	0.024**	0.073*
	(0.010)	(0.010)	(0.038)
$\frac{x_{i,t}}{z_{i,t}}$		0.514***	
		(0.063)	
Firm & Year FEs	✓	✓	✓
Controls	✓	✓	✓
IV			✓
Observations	14,576	14,576	14,576

SEs clustered by years and 4-digit NAICS industries are reported in parentheses. Control variables include $\log z_{i,t}$, firm fixed effects, and year fixed effects. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

- IV: User cost of R&D, driven by federal and state-specific rules variations (Bloom et al., 2013)

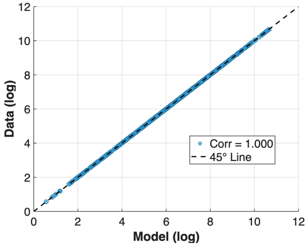
Identification Summary

- Publicly available data + Compustat

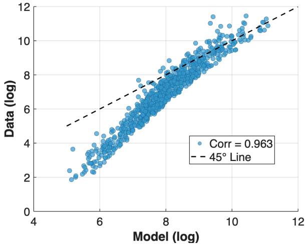
Notation	Description	Value	Source
$\boldsymbol{\Sigma}$	Product proximity		Form 10-K, Hoberg and Phillips (2016)
$\widetilde{\boldsymbol{\Omega}}$	Technological proximity		USPTO, Patent classification
α	Product proximity $\rightarrow$ Substitutability	0.12	Pellegrino (2025)
β	Technological proximity $\rightarrow$ Spillover	0.024	Estimate the law of motion
ζ/L	Labor-augmentation efficiency	0.004	Compustat, Cost of goods sold
ρ	Discount rate	0.100	$>$ risk-free rates, $<$ private R&D returns
μ	R&D efficiency	0.054	2.6% R&D share (moment match)
δ	Depreciation rate	0.016	1.5% economic growth rate (moment match)

Fit b/w Model and Data: Profit, Sales, and R&D

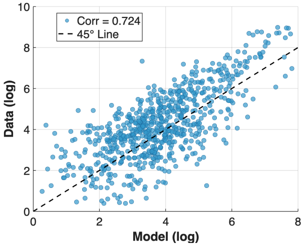
- Comparison of firm-level model-generated values (x-axis) with observed data (y-axis)



(a) Log of Profit (Targeted)



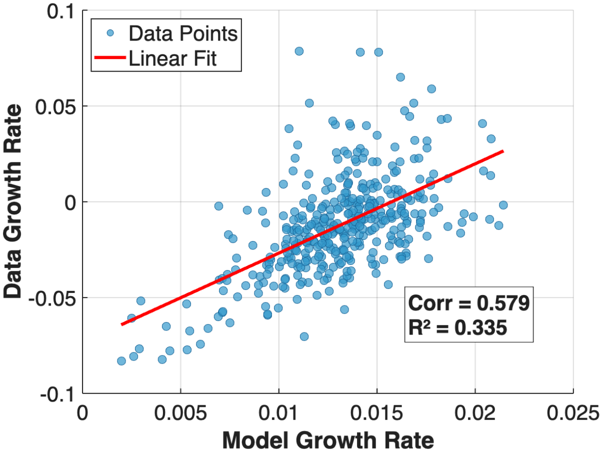
(b) Log of Sales



(c) Log of R&D Expenditure

Fit b/w Model and Data: Firm-level Growth Rates

- Data: Average growth rate of z_{it} b/w 2010 and 2017



Variable Definitions

Variable	Definition
Total Output	(competitive equilibrium = 100)
R&D Share	Total R&D / Total output (%)
Growth Rate	(%)
Social Welfare	(competitive equilibrium = 100)
Firm Value Share	Total firm value / Social welfare (%)

Counterfactuals: R&D (Given Static Cournot Competition)

	Total Output	R&D Share	Growth Rate	Social Welfare	Firm Value Share
Competitive Equilibrium	100.00	2.59%	1.51%	100.00	26.4%

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Monopoly (R&D)	100.00	1.34%	1.36%	98.89	27.39%

- Monopoly: Internalize business stealing > technological spillover $\Longrightarrow$ R&D $\downarrow$

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Competitive Equilibrium	100.00	2.59%	1.51%	100.00	26.4%
Monopoly (R&D)	100.00	1.34%	1.36%	98.89	27.39%
Planner (R&D)	100.00	6.27%	1.80%	104.36	24.09%

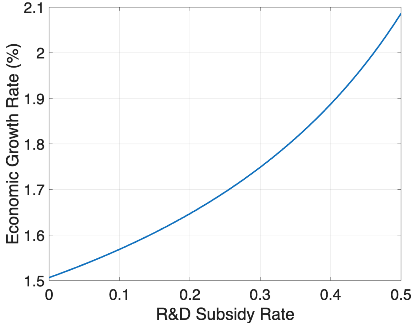
- Monopoly: Internalize business stealing > technological spillover $\Longrightarrow$ R&D $\downarrow$
- Social optimum: Internalize the impact of innovation on consumer surplus $\Longrightarrow$ R&D $\uparrow$

Counterfactuals: Both R&D and Product Market Competition

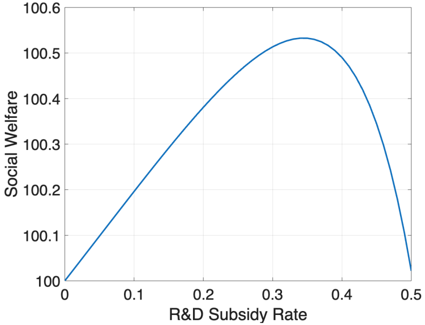
	Total Output	R&D Share	Growth Rate	Social Welfare	Firm Value Share
Competitive Equilibrium	100.00	2.59%	1.51%	100.00	26.54%
Monopoly (R&D)	100.00	1.34%	1.36%	98.89	27.39%
Planner (R&D)	100.00	6.27%	1.80%	104.36	24.09%
Monopoly (Both)	96.33	2.02%	1.40%	95.67	43.09%
Planner (Both)	109.29	4.76%	1.83%	114.54	−5.55%

- R&D in Monopoly (Both) > Monopoly (R&D): Less competition $\Longrightarrow$ private R&D return $\uparrow$
- R&D in Planner (Both) > Planner (R&D): Efficient production $\Longrightarrow$ social R&D return $\uparrow$

R&D Subsidy



(a) Growth Rates



(b) Social Welfare

- Optimal rate is $s = 34\%$, which increases g by 0.29 pp and CE welfare by 0.53%
- c.f. Welfare gains from optimal R&D allocation: 6.0%

Which Firms' R&D Should be Subsidized?

	Private R&D x	Social / Private value of R&D
Initial knowledge capital z	0.104***	0.000276
	(0.000670)	(0.000179)
Product market centrality	-15.5***	-0.523***
	(0.230)	(0.0615)
Tech spillover centrality	2.46***	1.70***
	(0.359)	(0.0959)
Intercept	-22.0***	0.726***
	(0.219)	(0.0585)
Observations	757	757
R^2	0.978	0.366

- R&D subsidies should target firms with
  - low product-market centrality
  - high technology-spillover centrality

Conclusion

- Quantitative Schumpeterian growth model with inter-firm networks
  - Product market rivalry network ($\boldsymbol{\Sigma}$)
  - Technology spillover network ($\boldsymbol{\Omega}$)
- LQ differential game avoids curse of dimensionality
  - Can solve with thousands of oligopolistic firms
  - Utilize recently available micro data and computational power

Technology & Product Proximity: Example

Tesla vs. Ford	
Technology Proximity	0.11
Product Proximity	0.15

Apple vs. Intel	
Technology Proximity	0.57
Product Proximity	0.00

Back

Generalized Hedonic-Linear Demand (Pellegrino, 2025) (1/2)

- $i \in \{1, 2, \ldots, n\}$: firms / products
- 1 unit of product i provides
  - 1 unit of idiosyncratic characteristic $k \in \{1, 2, \ldots, n\}$
  - $\psi_{k,i}$ unit of shared characteristic $k \in \{n+1, n+2, \ldots, n+n_k\}$ where $\sum_k \psi_{k,i}^2 = 1$
- Aggregate each characteristic:

$$y_{k,t} = \begin{cases} q_{k,t} & k = 1, 2, \ldots, n \\ \sum_i \psi_{k,i} q_{i,t} & k = n+1, n+2, \ldots, n+n_k \end{cases}$$

- Linear quadratic aggregator over characteristics:

$$Y_t = (1-\alpha) \sum_{k=1}^{n} \left(\underbrace{\hat{b}_{k,t} y_{k,t} - \frac{1}{2} y_{k,t}^2}_{\text{idiosyncratic characteristic}} \right) + \alpha \sum_{k=n+1}^{n+n_k} \left(\underbrace{\hat{b}_{k,t} y_{k,t} - \frac{1}{2} y_{k,t}^2}_{\text{shared characteristic}} \right)$$

Generalized Hedonic-Linear Demand (Pellegrino, 2025) (2/2)

- Quality:

$$b_i = (1-\alpha)\,\hat{b}_i + \alpha \sum_{k=n+1}^{n+n_k} \psi_k \hat{b}_k$$

- Inverse demand:

$$\boldsymbol{p} = \boldsymbol{b} - \boldsymbol{\Sigma q}$$

- Inverse cross price elasticity of demand:

$$\frac{\partial \log p_i}{\partial \log q_j} = -\frac{q_j}{p_i} \cdot \sigma_{ij}$$

- Cross price elasticity of demand:

$$\frac{\partial \log q_i}{\partial \log p_j} = -\frac{p_j}{q_i} (\boldsymbol{\Sigma}^{-1})_{ij}$$

Static Profits

- Gross profit: $\pi_{i,t} = q_{i,t}^2$
- Firms choose labor productivity and product quality: $\zeta a_{i,t} = \sqrt{\zeta w_t}$, $b_{i,t} = z_{i,t} - \sqrt{\zeta w_t}$
- Labor market clearing: $L = \sum_i \frac{q_{i,t}}{a_{i,t}} \Longrightarrow \sqrt{\zeta w_t} = \frac{\zeta}{L} \sum_i q_{i,t}$
- $\boldsymbol{q}_t = \boldsymbol{N} \boldsymbol{z}_t$ where $\boldsymbol{N} \equiv \left\{2\frac{\zeta}{L}\mathbf{1}_{n\times n} + \boldsymbol{I} + \boldsymbol{\Sigma}\right\}^{-1}$
- $\boldsymbol{N}_i$: the i th row of $\boldsymbol{N}$
- Profit:

$$\pi_{i,t} = q_{i,t}^2 = \boldsymbol{z}_t^T \boldsymbol{Q}^{\boldsymbol{i}} \boldsymbol{z}_t$$

where $\boldsymbol{Q}^{\boldsymbol{i}} \equiv \boldsymbol{N}_i^T \boldsymbol{N}_i$

Back

Riccati Equations

- Stochastic law of motion:

$$dz_t = \left(\boldsymbol{\Omega} z_t + \operatorname{diag}(\mu)\, x_t\right) dt + \operatorname{diag}(\gamma)\, z_t\, dW_t$$

- $V^i(z) = z^T X^i z$ where X^i is the solution of the stacked Riccati equation

$$0 = Q^i - \mu^2 X_i^i \left(X_i^i\right)^T + \left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right) I\right)^T X^i + X^i \left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right) I\right)$$

  - $X_i^i \equiv$ the i th column of X^i
  - $\boldsymbol{\Phi} \equiv \boldsymbol{\Omega} + \mu^2 \begin{bmatrix} X_1^1 & \cdots & X_n^n \end{bmatrix}^T$

- Algorithm: Given $\begin{bmatrix} X_\tau^1 & \cdots & X_\tau^n \end{bmatrix}$, update $\begin{bmatrix} X_{\tau-\Delta}^1 & \cdots & X_{\tau-\Delta}^n \end{bmatrix}$ by

$$-\frac{X_\tau^i - X_{\tau-\Delta}^i}{\Delta} = Q^i - \mu^2 X_{i,\tau}^i \left(X_{i,\tau}^i\right)^T + \left(\boldsymbol{\Phi}_\tau - \frac{1}{2}\left(\rho - \gamma^2\right) I\right)^T X_\tau^i + X_\tau^i \left(\boldsymbol{\Phi}_\tau - \frac{1}{2}\left(\rho - \gamma^2\right) I\right)$$

Output and Lifetime Utility

- Output: $Y_t = \boldsymbol{q}_t^T \boldsymbol{Q} \boldsymbol{q}_t$ where

$$\boldsymbol{Q} = \frac{\zeta}{L} \mathbf{1}_{n\times n} + \boldsymbol{I} + \frac{1}{2} \boldsymbol{\Sigma}$$

- Expected utility:

$$V(\boldsymbol{z}_t) \equiv \boldsymbol{E}_t \left[\int_t^{\infty} \exp(-\rho s)\, C_s ds \middle| \boldsymbol{z}_t \right] = \boldsymbol{z}_t^T \boldsymbol{X} \boldsymbol{z}_t$$

where X is the solution of the Lyapunov equation (obtained from households' HJB equation):

$$0 = \boldsymbol{Q} - \mu^2 \tilde{\boldsymbol{X}}^T \tilde{\boldsymbol{X}} + \boldsymbol{X} \left(\boldsymbol{\Phi} - \frac{1}{2} (\rho - \gamma^2) \boldsymbol{I} \right) + \left(\boldsymbol{\Phi} - \frac{1}{2} (\rho - \gamma^2) \boldsymbol{I} \right)^T \boldsymbol{X}$$

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Equilibrium Summary

Description	**Expression**
Production strategy	$\boldsymbol{q}_t = \boldsymbol{N}\boldsymbol{z}_t$
R&D strategy	$\boldsymbol{x}_t = \mu\tilde{\boldsymbol{X}}\boldsymbol{z}_t$
Profit of final producers	$\Pi_t^F = \boldsymbol{q}_t^T\left(\frac{1}{2}\boldsymbol{\Sigma}\right)\boldsymbol{q}_t$
Total operating profit of firms	$\Pi_t = \boldsymbol{q}_t^T\boldsymbol{q}_t$
Labor income	$w_t L = \boldsymbol{q}_t^T\left(\frac{\zeta}{L}\mathbf{1}_{n\times n}\right)\boldsymbol{q}_t$
Output	$Y_t = \boldsymbol{q}_t^T\left(\frac{\zeta}{L}\mathbf{1}_{n\times n} + \boldsymbol{I} + \frac{1}{2}\boldsymbol{\Sigma}\right)\boldsymbol{q}_t$
Consumption	$C_t = Y_t - \boldsymbol{x}_t^T\boldsymbol{x}_t$

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Example: Symmetric Equilibrium

Assumption

- Symmetric product substitutability and technology spillover: $\sigma_{ij} = \sigma, \omega_{ij} = \omega \quad \forall i \neq j$

- R&D strategy: $x^*_{i,t} = \mu\left(\tilde{x}_1 z_{i,t} + \tilde{x}_2 \sum_{j\neq i} z_j\right)$
  - $\tilde{x}_1$: market size effect (> 0)
  - $\tilde{x}_2$: strategic substitutability (< 0) / complementarity (> 0)

- Growth rate: $g = \underbrace{(n-1)\omega}_{\text{Tech Spillover}} + \underbrace{\mu^2(\tilde{x}_1 + (n-1)\tilde{x}_2)}_{\text{R\&D}}$

- Stability (irreducibility) requires $\omega + \mu^2\tilde{x}_2 > 0$
  - Tech spillover (ω) must be strong relative to strategic substitutability $(\tilde{x}_2 < 0)$

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Intuition Behind BGP

- On the BGP, $\boldsymbol{a}_t$, $\boldsymbol{b}_t$, $\boldsymbol{z}_t$, and $\boldsymbol{q}_t$ grow at the same rate

Technological Choice:	$\zeta a_{i,t} + b_{i,t} = z_{i,t}$
Linear Production Technology:	$q_{i,t} = a_{i,t} l_{i,t}$
Inelastic Labor Supply:	$L = \sum_i l_{i,t}$

- Linear and quadratic terms in $\boldsymbol{q}_t$ of output grow at the same rate:

$$Y_t = \boldsymbol{q}_t^T \boldsymbol{b}_t - \frac{1}{2} \boldsymbol{q}_t^T \boldsymbol{\Sigma} \boldsymbol{q}_t$$

Social Optimum

- Static optimal allocation: $\boldsymbol{q}_t^* = \boldsymbol{N}^* \boldsymbol{z}_t$ where $\boldsymbol{N}^* \equiv \left\{2\frac{\zeta}{L}\mathbf{1}_{n\times n} + \boldsymbol{\Sigma}\right\}^{-1}$
- Optimal output: $Y_t^* = \boldsymbol{z}_t^T \boldsymbol{Q}^* \boldsymbol{z}_t$ where $\boldsymbol{Q}^* = \frac{1}{2}\boldsymbol{N}^*$
- Optimal expected utility:

$$V^*(\boldsymbol{z}_t) \equiv \boldsymbol{E}_t\left[\int_t^\infty \exp(-\rho s)\, C_s ds \middle| \boldsymbol{z}_t\right] = \boldsymbol{z}_t^T \boldsymbol{X}^* \boldsymbol{z}_t,$$

where $\boldsymbol{X}^*$ is the solution of the Riccati equation (obtained from planner's HJB equation):

$$0 = \boldsymbol{Q}^* - \mu^2(\boldsymbol{X}^*)^T\boldsymbol{X}^* + \boldsymbol{X}^*\left(\boldsymbol{\Phi}^* - \frac{1}{2}(\rho - \gamma^2)\boldsymbol{I}\right) + \left(\boldsymbol{\Phi}^* - \frac{1}{2}(\rho - \gamma^2)\boldsymbol{I}\right)\boldsymbol{X}^*$$

- Optimal R&D: $\boldsymbol{x}_t^* = \mu \boldsymbol{X}^* \boldsymbol{z}_t$
- Optimal technology transition matrix: $\boldsymbol{\Phi}^* = \boldsymbol{\Omega} + \mu^2 \boldsymbol{X}^*$

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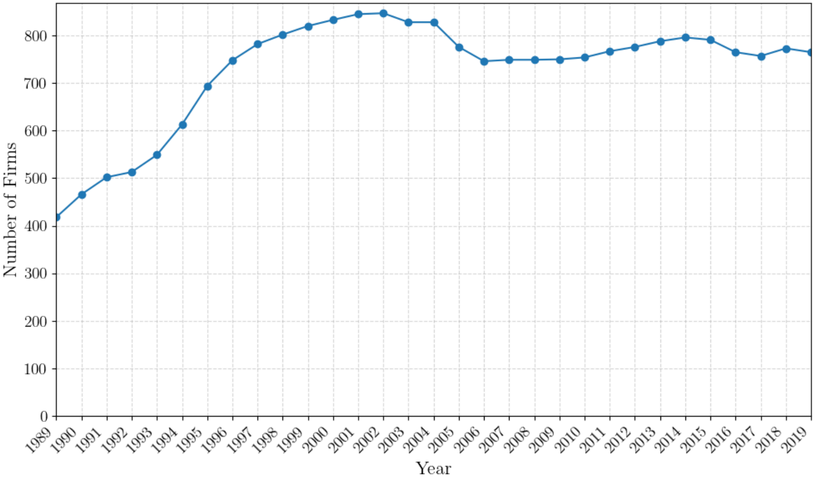
Growth Decomposition

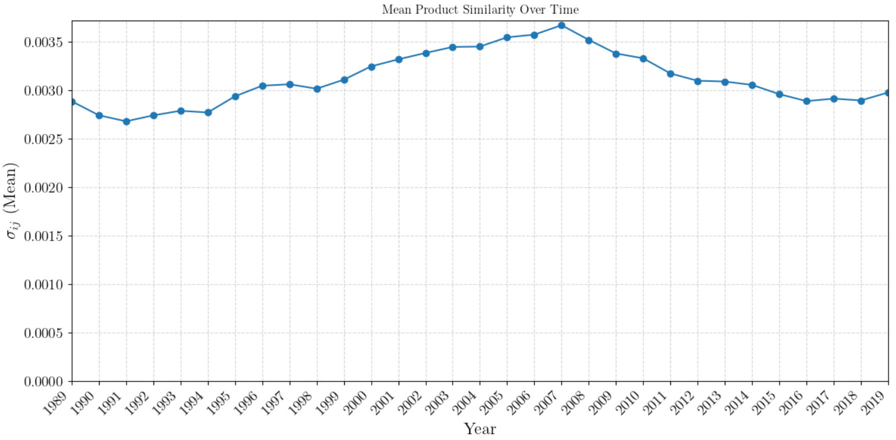
- Aggregate output: $Y_t = z_t^T Q z_t$
- $dz_t/dt = \mathbf{\Phi} z_t$ where $\mathbf{\Phi} = \mathbf{\Omega} + \mu^2 \widetilde{\mathbf{X}} - \delta \mathbf{I}$

$$\frac{d \log Y_t}{dt} = \underbrace{\frac{z_t^T \left(\mathbf{Q}\mathbf{\Omega} + \mathbf{\Omega}\mathbf{Q} \right) z_t}{Y_t}}_{\text{Tech Spillover}} + \underbrace{\frac{\mu^2 z_t^T \left(\mathbf{Q}\widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}^T \mathbf{Q} \right) z_t}{Y_t}}_{\text{R\&D}} - \underbrace{2\delta}_{\text{Depreciation}}$$

Trend of Number of Sample Firms

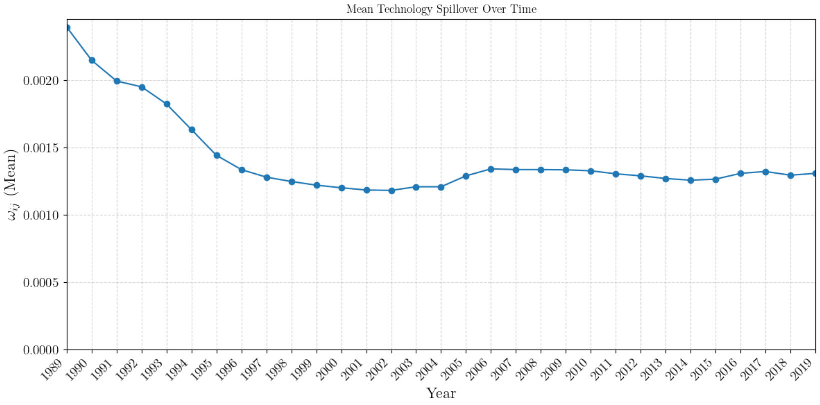


Trend of Product Substitutability



Trend of Technological Proximity

- Merge USPTO data with Compustat firms using DISCERN 2 dataset (Arora et al., 2024)
- Jaffe measure, Group-level patent classification, Stack for 5 years



Microeconometric Estimates vs. GHL (Pellegrino, 2025) (1/2)

Market	Firm i	Firm j	Micro Estimate	GHL
Auto	Ford	Ford	-4.320	-5.197
Auto	Ford	General Motors	0.034	0.056
Auto	Ford	Toyota	0.007	0.017
Auto	General Motors	Ford	0.065	0.052
Auto	General Motors	General Motors	-6.433	-4.685
Auto	General Motors	Toyota	0.008	0.005
Auto	Toyota	Ford	0.018	0.025
Auto	Toyota	General Motors	0.008	0.008
Auto	Toyota	Toyota	-3.085	-4.851

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Microeconometric Estimates vs. GHL (Pellegrino, 2025) (2/2)

Market	Firm i	Firm j	Micro Estimate	GHL
Cereals	Kellogg's	Kellogg's	-3.231	-1.770
Cereals	Kellogg's	Quaker Oats	0.033	0.023
Cereals	Quaker Oats	Kellogg's	0.046	0.031
Cereals	Quaker Oats	Quaker Oats	-3.031	-1.941
Computers	Apple	Apple	-11.979	-8.945
Computers	Apple	Dell	0.018	0.025
Computers	Dell	Apple	0.027	0.047
Computers	Dell	Dell	-5.570	-5.110

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First Stage Back

	R&D (1)
State tax credit component of R&D user cost	-1.16^{***} (0.29)
Federal tax credit component of R&D user cost	-34.29^{***} (3.64)
Firm fixed effects	✓
Year fixed effects	✓
No. of observations	16197

SEs clustered by years and 4-digit NAICS industries are reported in parentheses.

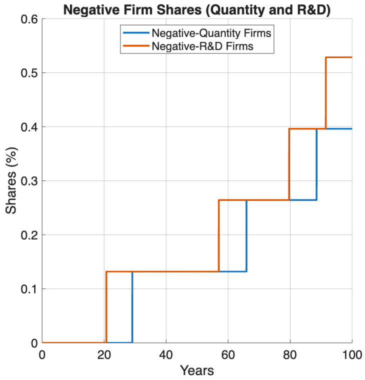
- IV: User cost of R&D, driven by federal and state-specific rules variations (Wilson, 2009; Bloom et al., 2013)

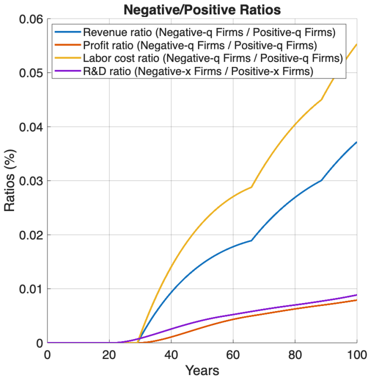
Negative R&D and Output

- Issue with the model: negative output and R&D
  - Inada condition is not satisfied
  - Non-negativity constraint makes model intractable

Negative R&D and Quantity

- Firms with negative values are negligible along the transition path
- The weight on consumption 100 years and beyond is 0.00454% ($\rho = 0.1$)





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