Supply Chains and R&D Allocation

Yasutaka Koike-Mori
UCLA
5th Year

Koki Okumura
UCLA
4th Year

Supply Chains and R&D Allocation

- How do supply chains affect R&D (mis)allocation?
- e.g., Toyota v.s. Tesla
  1. Given its established supply chains, Toyota would have had a stronger incentive to invest in R&D for gas vehicles than for EV
  2. Tesla does not internalize the impact of EV on Denso (supplier to Toyota) as well as on Toyota

Empirical Analysis of Supply Chains and Innovation

- Firm-level supply chain and patent data in Japan
- A firm with more suppliers and buyers engages more in **internal** innovation compared to **external** innovation
  - Internal innovation: improving firm's own technology/product
  - External innovation: developing technology/product that are new to the firm

Endogenous Growth Model with Supply Chain Formation

Tractable dynamic model of innovation and supply chain

- Innovation and firm dynamics
  - Klette and Kortum (2004)
  - Internal & external innovation
- Supply chain formation and destruction

Implications

- R&D allocation
  - When a firm has more suppliers or buyers per product, the firm makes more internal R&D effort relative to external R&D effort
- R&D misallocation
  1. Interaction b/w knowledge spillover and supply chain
  2. Creative supply chain destruction

Linked Database of Supply Chain $\times$ Patent from Japan

- Teikoku Databank (TDB)
  - A major credit reporting company
  - Panel data on 800,000 firms with suppliers and buyers
  - Different from TSR database used in Carvalho et. al. (2020)
- Intellectual Property Patent Database from the Japan Patent Office
- Linked data between innovation and production networks is new
  - Different from "innovation network" in Liu and Ma (2023)

Empirical Specification

$$\text{Future Internal Innovation Rate}_{it}$$
$$= \beta_0 + \beta_1 \times \log\left(\text{N of Suppliers}\right)_{it} + \beta_2 \times \log\left(\text{N of Buyers}\right)_{it} + \text{Controls}_{it} + \varepsilon_{it}$$

- Time period: 2009–2019
- Patent filing firms: 6,500
- Relationship observed firms: 800,000

Two Measurements of Internal Innovation Rate

1. $1-$ patents in new technology classes / total patents patent class
   - New technology class: The classes (“sub groups”) the firm has not applied in the past ten years
2. Self citation patents / total patents
   - Self citation patent: More than 10% of patents that the focal patent cites are filed by the focal firm

- Both measures are based on the number of patents (3-year forward ave)
  - Robust when using patent citation weighed measures patent citation

Technology Class Measurement

	Internal Innovation Rate (3-year forward ave)						
log N of Suppliers	0.0633***				0.0455***		0.0222***
	(0.0033)				(0.0098)		(0.0082)
log N of Buyers		0.0560***				0.0333***	0.0226***
		(0.0036)				(0.0066)	(0.0050)
log Firm Size			0.0609***		0.0272***	0.0419***	0.0315***
			(0.0035)		(0.0089)	(0.0057)	(0.0084)
Firm Age				0.0011***	-0.0006***	-0.0005**	-0.0006***
				(0.0002)	(0.0002)	(0.0002)	(0.0002)
Year × Industry FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes
N	46,628	46,628	46,628	46,628	46,628	46,628	46,628

Note. Weighted by firm size. Clustering by year×industry. *$p < 0.10$, **$p < 0.05$, ***$p < 0.01$

Self Citation Measurement

	Internal Innovation Rate (3-year forward ave)						
log N of Suppliers	0.0365***				0.0149***		0.0115**
	(0.0023)				(0.0046)		(0.0053)
log N of Buyers		0.0296***				0.0088***	0.0034
		(0.00321)				(0.0030)	(0.0033)
log Firm Size			0.0365***		0.0225***	0.0285***	0.0231***
			(0.0022)		(0.0043)	(0.0030)	(0.0043)
Firm Age				0.0012***	0.0004***	0.0005***	0.0004***
				(0.0001)	(0.0001)	(0.0001)	(0.0001)
Year $\times$ Industry FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes
N	40486	40486	40486	40486	40486	40486	40486

Note. Weighted by firm size. Clustering by year$\times$industry. *$p < 0.10$, **$p < 0.05$, ***$p < 0.01$

Households

$$U_0 = \int_0^{\infty} \exp\left(-\rho t\right) \frac{Y(t)^{1-\theta} - 1}{1-\theta} dt$$

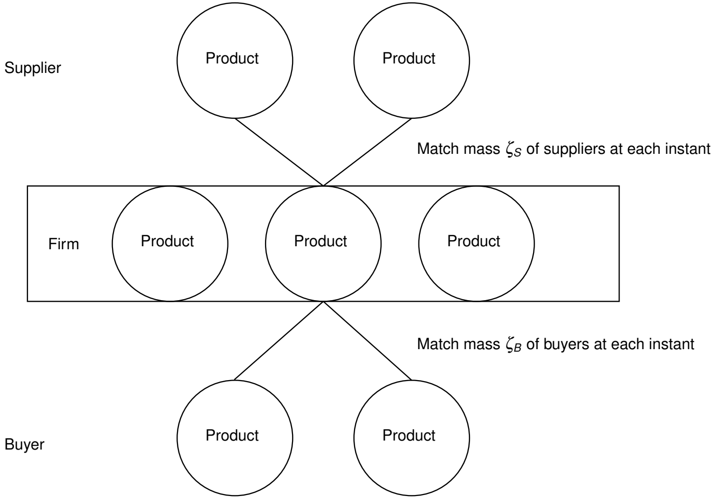
$$\dot{A}(t) \leq r(t)A(t) + w_L(t) + L_H w_H(t) - Y(t)$$

- Unskilled labor for production
- Skilled labor for R&D
- Inelastically supplied, respectively

Supply Chain Formation and Destruction

- Continuum mass of products
- A firm can own multiple product lines
- A product match randomly at each instant with (i) mass ζ_S of suppliers and (ii) mass ζ_B of buyers
- Each match is terminated (i) at exogenous rate δ and (ii) when products on either side exit

Supply Chain Formation and Destruction



Each match is terminated (i) at exogenous rate δ and (ii) when products on either side exit

Production Structure Given Supply Chains

- $\Omega(t)$: the set of product
- $\mathcal{S}(\omega, t)$: the set of suppliers for product ω

$$Y(t) = \left(\int_{\omega' \in \Omega(t)} q^F(\omega', t)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}}$$

$$\frac{1}{\beta^\beta (1-\beta)^{1-\beta}} (z(\omega,t) l(\omega,t))^\beta \left(\int_{\omega' \in \mathcal{S}(\omega,t)} q(\omega', \omega, t)^{\frac{\sigma-1}{\sigma}} d\omega' \right)^{\frac{\sigma}{\sigma-1}(1-\beta)}$$

- Suppliers charge monopolistic prices:

$$p^F(\omega', t) = p(\omega', \omega, t) = \frac{\sigma}{\sigma - 1} c(\omega', t)$$

Internal R&D

- Average productivity

$$Z(t) \equiv \left(\int z(\omega)^{\beta(\sigma-1)} dF(\omega, t) \right)^{\frac{1}{\beta(\sigma-1)}}$$

- Product line productivity growth

$$\frac{d}{dt} \log z(\omega, t) = \mu(\omega, t)$$

- Cost for internal R&D

$$w_H(t) \left(\frac{z}{Z(t)} \right)^{\beta(\sigma-1)} c_I(\mu) \quad \text{where} \quad c_I(\mu) \equiv \frac{1}{\varphi_I} \mu^{\gamma_I}$$

External R&D

- $n(f)$: the number of product lines of firm f
- $X(f)$: external innovation flow rate of firm f
- $x(f) \equiv X(f)/n(f)$
- Cost for external R&D (Klette Kortum)

$$w_H(t) n c_X(x) \quad \text{where} \quad c_X(\mathrm{x}) \equiv \frac{1}{\varphi_X} x^{\gamma_X}$$

- Variety creation with prob. $1-\alpha$: $z' = \varepsilon Z(t)$, $\varepsilon \sim \Phi$
- Creative destruction with prob. α: $z' = \lambda z$
- We assume creative destruction is *patent violation*:
  - The new producer makes take-it-or-leave-it offer to get patent license
  - If the older producer accepts the offer, it stops producing the product

Entry and Exit

- Potential entrant:
  - Hiring a skilled worker generates external innovation at rate φ_E
- A product line dies at an exogenous rate δ_N
- A firm that loses all product lines exits the economy

Reduce State Space

- In general, for each product, we need to track the matched buyer and seller distributions
- Product age *a* is sufficient statistic of these distributions due to
  1. Random matching b/w continuum mass of product lines
  2. Deterministic internal R&D
- Firm's state variable is reduced to $\mathcal{O} = \{(z_1, a_1), (z_2, a_2), ..., (z_n, a_n)\}$
- Guess and verify firm value firm's HJB

$$V(\mathcal{O}, t) = \sum_{(z,a)\in\mathcal{O}} \left\{ Z(t) V^P + z^{\beta(\sigma-1)} Z(t)^{1-\beta(\sigma-1)} V^A(a) \right\}$$

Product Value and Internal Innovation Are Increasing in a

$$\left(r+\delta_N-\left(1-\beta\left(\sigma-1\right)\right)g\right)V^A(a)$$
$$=\frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\hat{c}(a)\right)^{1-\sigma}\hat{D}(a)+V_a^A(a)+\max_{\mu\geq 0}\left[\mu\beta\left(\sigma-1\right)V^A(a)-\hat{w}_H c_I(\mu)\right]$$

- $\hat{c}(a)$: cost function, decreasing in a
  - more supplier $\rightarrow$ lower the cost “love of input variety effect”
- $\hat{D}(a)$: demand shifter, increasing in a
  - more buyer $\rightarrow$ lager the demand “market size effect”
- $\rightarrow$ Product value $V^A(a)$ and optimal internal innovation $\mu(a)$ are increasing in a

Same External Innovation Rate across Incumbents

- Free entry condition

$$\text{Entry Value} = w_H / \varphi_E$$

- Incumbent firms choose external innovation rate x to maximize

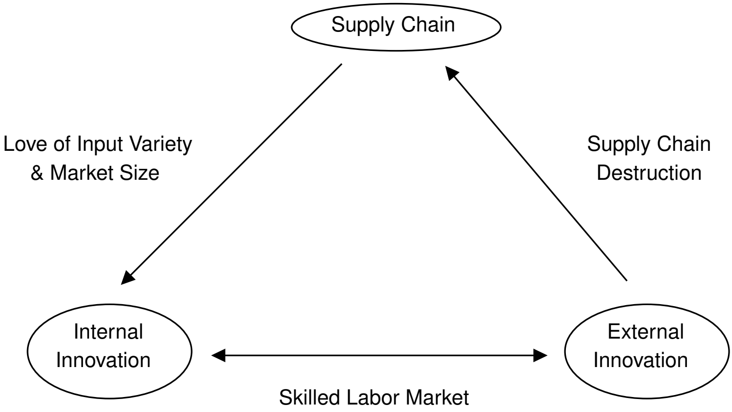
$$nx \times \text{Entry Value} - n w_H c_X(x)$$

- These conditions pin down x for all firms:

$$x = \left(\frac{1}{\gamma_X} \frac{\varphi_X}{\varphi_E} \right)^{1/(\gamma_X - 1)}$$

- $\rightarrow$ When a firm has more suppliers and buyers per product, the firm makes more internal R&D effort relative to external R&D effort

Interaction b/w Innovation and Supply Chain



Sources of R&D Misallocation

1. Interaction b/w knowledge spillover and supply chains (internal)
   - Firms consider the importance of its product in production network
   - Planner consider both the importance of products in production network and knowledge spillover

2. Creative supply chain destruction (external)
   - A firm does not take into account that external innovation destroys supply chains

Parameters

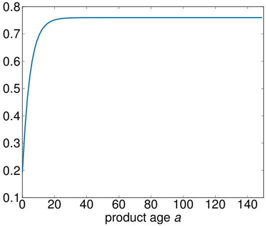
Parameter	Description	Value
ρ	Discount Rate	0.05
θ	Inverse of Intertemporal Elasticity of Substitution	2
σ	Elasticity of Substitution for Intermediate Goods	4
β	Labor Share	0.5
L_H	Measure of High-Skilled Worker	0.166
γ_I	Curvature of Internal Innovation Cost	2
γ_E	Curvature of External Innovation Cost	2
ζ_B	Matching Rate with Buyers	0.14
ζ_S	Matching Rate with Suppliers	0.14
δ	Link Death Rate	0.06
δ_N	Product Death Rate	0.06
α	Share of Creative Destruction	0.6
ϕ_I	Efficiency of Internal Innovation	0.1
ϕ_X	Efficiency of External Innovation by Incumbent	0.04
ϕ_E	Efficiency of External Innovation by Entrant	10
λ	Step Size on Quality Ladder	1.3
$\bar{\varepsilon}$	Average Relative Efficiency of New Products	1.3
$\bar{\sigma}$	Standard Deviation of Entrant Productivity	0.1

Steady State Equilibrium 1

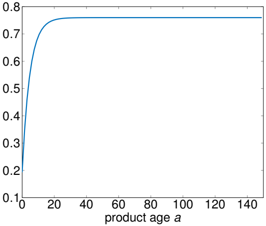
Variables	Description	Value
w_H	Wage for High Skilled	1.43
w_L	Wage for Low Skilled	0.78
N	Product Mass	10.0
Y	Aggregate Consumption	1.30
g	Economic Growth Rate (%)	6.1
V_P	Product Value	0.0008
x	External Innovation Rate by Incumbent (%)	0.2
x_E	Flow of Entry per Product by Entrant (%)	14.8
ν	Total Flow Rate of Entry (%)	15
ν_N	Variety Creation Rate (%)	6
ν_D	Creative Destruction Rate (%)	9

Steady State Equilibrium 2

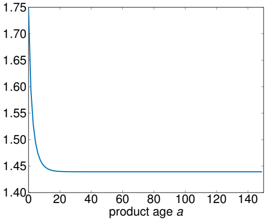
(a) Mass of Suppliers



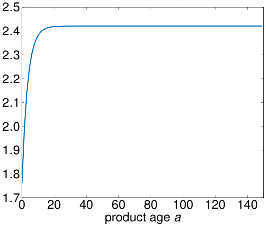
(b) Mass of Buyers



(c) Cost Function $c(a)$

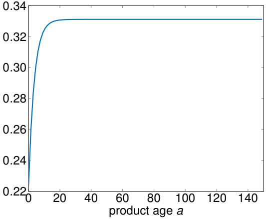


(d) Demand Shifter $D(a)$

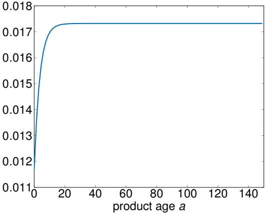


Steady State Equilibrium 3

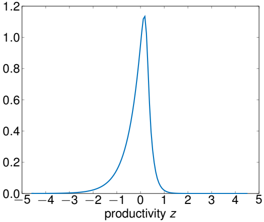
(a) Value Function $V^A(a)$



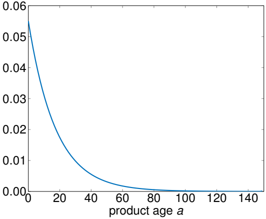
(b) Internal R&D $\mu(a)$



(c) Marginal Density of z



(d) Marginal Density of a



Next Steps

- Analytical characterization for optimal R&D allocation
- Calibration using Japanese supply chain and patent data
- Numerical exercise

International Patent Classification Example

- (Section) H: Electricity
- (Class) H01: Electric elements
- (Sub Class) H01L: Semiconductor devices
- (Group) H01L 21: Processes or apparatus specially adapted for the manufacture or treatment of semiconductor or solid state devices or of parts thereof
- (Sub Group) H01L 21/16: The devices having semiconductor bodies comprising cuprous oxide or cuprous iodide

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Patents in New Technology Class / Total Patents

	Internal Innovation Rate Weighted by Patent Citation (3-year Forward Ave)						
log N of Suppliers	0.0638***				0.0446***		0.0228***
	(0.0033)				(0.0291)		(0.0079)
log N of Buyers		0.0559***				0.0321***	0.0212***
		(0.0036)				(0.0064)	(0.0050)
log Firm Size			0.0619***		0.0291***	0.0437***	0.0331***
			(0.0034)		(0.0086)	(0.0056)	(0.0081)
Firm Age				0.0011***	-0.0007***	-0.0005**	-0.0006***
				(0.0002)	(0.0002)	(0.0002)	(0.0002)
Year $\times$ Industry FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes
N	46,628	46,628	46,628	46,628	46,628	46,628	46,628

Note. Weighted by firm size. Clustering by year$\times$industry. *$p < 0.10$, **$p < 0.05$, ***$p < 0.01$

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Self Citation Patents / Total Patents

	Internal Innovation Rate Weighted by Patent Citation (3-year Forward Ave)						
log N of Suppliers	0.0370***				0.0165***		0.0134**
	(0.0022)				(0.0047)		(0.0053)
log N of Buyers		0.0300***				0.0094***	0.0030
		(0.0021)				(0.0031)	(0.0033)
log Firm Size			0.0366***		0.0213***	0.0282***	0.0219***
			(0.0023)		(0.0045)	(0.0031)	(0.0044)
Firm Age				0.0012***	0.0004***	0.0005***	0.0004***
				(0.0001)	(0.0001)	(0.0001)	(0.0001)
Year $\times$ Industry FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes
N	40,486	40,486	40,486	40,486	40,486	40,486	40,486

Note. Weighted by firm size. Clustering by year$\times$industry. *$p < 0.10$, **$p < 0.05$, ***$p < 0.01$

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Firm Value

$$
\begin{aligned}
& r(t)V(\mathcal{O},t) - V_t(\mathcal{O},t) \\
& = \max_{x \geq 0,\ \{\mu(z,a)\geq 0\}_{\mathcal{O}}} \sum_{(z,a)\in\mathcal{O}} \left[\begin{array}{c} \frac{1}{\sigma}\left(\tilde{\sigma} z^{-\beta} \tilde{c}(a,t)\right)^{1-\sigma} D(a,t) - (z/Z(t))^{\beta(\sigma-1)} w_H(t) c_I(\mu) \\ +\mu z V_z(\mathcal{O},t) + V_a(\mathcal{O},t) \\ +\delta_N \{V(\mathcal{O} - \{(z,a)\},t) - V(\mathcal{O},t)\} \end{array} \right] \\
& \quad + nx \left[\begin{array}{c} \alpha \int [V(\mathcal{O} + \{(\lambda z',0)\},t) - \{V(\mathcal{O}',t) - V(\mathcal{O}' - \{(z',a')\},t)\}] \, dF(z',a',t) \\ (1-\alpha)\int [V(\mathcal{O} + \{(\varepsilon Z(t),0)\},t)] \, d\Phi(\varepsilon) \\ -V(\mathcal{O},t) \end{array} \right] \\
& \quad - n w_H(t) c_X(x)
\end{aligned}
$$

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Flow Rate of Entry, Creative Destruction, and New Variety

- Flow of entry per product by entrant: x_E
- Flow of entry per product by incumbent:

$$x = \left(\frac{1}{\gamma_X} \frac{\varphi_X}{\varphi_E} \right)^{1/(\gamma_X - 1)}$$

- Variety creation rate: $\nu_N = (1-\alpha)(x + x_E)$
- Creative destruction rate: $\nu_D = \alpha (x + x_E)$
- Variety growth rate:

$$g_N \equiv \dot{N}(t)/N(t) = \nu_N - \delta_N = (1-\alpha)(x_E + x) - \delta_N$$

- $g_N = 0$ in SS implies

$$x_E = \frac{\delta_N}{1-\alpha} - x = \frac{\delta_N}{1-\alpha} - \left(\frac{1}{\gamma_X} \frac{\varphi_X}{\varphi_E} \right)^{1/(\gamma_X - 1)}$$

BGP Equilibrium (1)

A *balanced growth path equilibrium* consists of (i) the value of product $V^A(a)$ and V^P; (ii) the internal innovation intensity $\mu(a)$; (iii) the distribution of products $\hat{F}(\hat{z},a)$; (iv) the distribution of matched products $\hat{M}_B(\hat{z}',a';a)$ and $\hat{M}_S(\hat{z}',a';a)$; (v) the cost function $\hat{c}(a)$; (vi) the demand shifter for an age a product $\hat{D}(a)$; (vii) the R&D worker wage $\hat{w}_H$ (viii) the production worker wage $\hat{w}_L$; (ix) the number of variety N; (x) the economic growth rate g; (xi) and the aggregate output $\hat{Y}$, such that;

- The value of products $V^A(a)$ satisfies the HJB

$$\begin{aligned}&\{\rho+\delta_N+(\theta-1+\beta(\sigma-1))g\}V^A(a)\\&=\frac{1}{\sigma}(\tilde{\sigma}\hat{c}(a))^{1-\sigma}\hat{D}(a)+V_a^A(a)+\mu(a)\beta(\sigma-1)V^A(a)-\frac{\hat{w}_H}{\varphi_I}\mu(a)^{\gamma_I}\end{aligned} \tag{1}$$

  and V^P is given by

$$V^P=\frac{1}{\rho+\delta_N+(\theta-1)g}\frac{\hat{w}_H}{\varphi_E}\left(1-\frac{1}{\gamma_X}\right)\left(\frac{1}{\gamma_X}\frac{\varphi_X}{\varphi_E}\right)^{\frac{1}{\gamma_X-1}} \tag{2}$$

- The internal innovation intensity $\mu(a)$ satisfies the FOC

$$\mu(a)=\left(\frac{\beta(\sigma-1)\varphi_I}{\gamma_I\hat{w}_H}V^A(a)\right)^{\frac{1}{\gamma_I-1}} \tag{3}$$

BGP Equilibrium (2)

- The distribution of products $\hat{f}(\hat{z}, a)$ satisfies the KFE

$$0 = -\frac{\partial}{\partial \hat{z}} \left\{ \left(\mu(a) - g \right) \hat{z} \hat{f}(\hat{z}, a) \right\} - \frac{\partial \hat{f}(\hat{z}, a)}{\partial a} + \frac{\delta_N}{1-\alpha} \left[\alpha \hat{f}\left(\frac{\hat{z}}{\lambda}, a \right) + (1-\alpha) \hat{\varphi}(\hat{z}) - \hat{f}(\hat{z}, a) \right] \quad (4)$$

- The distributions of matched buyers $\hat{M}_B(\hat{z}', a'; a)$ and sellers $\hat{M}_S(\hat{z}', a'; a)$ are given by

$$\hat{M}_B(\hat{z}', a'; a) = \zeta_B \int_0^{\min\{a, a'\}} \exp\left(-\left(\delta + \frac{\delta_N}{1-\alpha} \right) \tau \right) \hat{F}\left(\exp\left(-\int_{a'-\tau}^{a'} \mu(s)\, ds + \tau g \right) \hat{z}', a' - \tau \right) d\tau \quad (5)$$

$$\hat{M}_S(\hat{z}', a'; a) = \zeta_S \int_0^{\min\{a, a'\}} \exp\left(-\left(\delta + \frac{\delta_N}{1-\alpha} \right) \tau \right) \hat{F}\left(\exp\left(-\int_{a'-\tau}^{a'} \mu(s)\, ds + \tau g \right) \hat{z}', a' - \tau \right) d\tau \quad (6)$$

BGP Equilibrium (3)

- The product cost function $\hat{c}(a)$ satisfies

$$\hat{c}(a)^{1-\sigma} = \hat{w}_L^{\beta(1-\sigma)} \left(\int \hat{z}'^{\beta(\sigma-1)} \left(\tilde{\sigma}\hat{c}(a')\right)^{1-\sigma} d\hat{M}_S(\hat{z}', a'; a) \right)^{1-\beta} \tag{7}$$

- The demand shifter for an age a product $\hat{D}(a)$ satisfies

$$\hat{D}(a) = \hat{Y} + (1-\beta)\,\tilde{\sigma}^{-\sigma}\,\hat{w}_L^{-\frac{\beta}{1-\beta}(\sigma-1)} \int \hat{z}'^{\beta(\sigma-1)} \left(\tilde{c}\left(a'\right)^{1-\sigma}\right)^{-\frac{\beta}{1-\beta}} \hat{D}(a') d\hat{M}_B(\hat{z}', a'; a) \tag{8}$$

BGP Equilibrium (4)

- The R&D worker wage $\hat{w}_H$ satisfies the free entry condition

$$\frac{\hat{w}_H}{\varphi_E} = \alpha \left\{ \lambda^{\beta(\sigma-1)} V^A(0) - \int \hat{z}^{\beta(\sigma-1)} V^A(a) d\hat{F}(\hat{z},a) \right\} + (1-\alpha) \left\{ V^P + \bar{\varepsilon}^{\beta(\sigma-1)} V^A(0) \right\} \tag{9}$$

- The production worker wage $\hat{w}_L$ satisfies the labor market clearing conditions for production workers

$$\hat{w}_L = \beta \tilde{\sigma}^{-\sigma} N \int \hat{z}^{\beta(\sigma-1)} \hat{c}(a)^{1-\sigma} \hat{D}(a) d\hat{F}(\hat{z},a) \tag{10}$$

- The number of variety N satisfies the labor market clearing conditions for R&D workers

$$L_H = N \left[\frac{1}{\varphi_I} \int \hat{z}^{\beta(\sigma-1)} \mu(a)^{\gamma_I} d\hat{F}(\hat{z},a) + \frac{1}{\varphi_X} \left(\frac{1}{\gamma_X} \frac{\varphi_X}{\varphi_E} \right)^{\gamma_X/(\gamma_X-1)} + \frac{1}{\varphi_E} \left(\frac{\delta_N}{1-\alpha} - \left(\frac{1}{\gamma_X} \frac{\varphi_X}{\varphi_E} \right)^{1/(\gamma_X-1)} \right) \tag{11}$$

BGP Equilibrium (5)

- The economic growth rate g is given by

$$g = \int \hat{z}^{\beta(\sigma-1)} \mu(a) d\hat{F}(\hat{z}, a) + \frac{\alpha \lambda^{\beta(\sigma-1)} + (1-\alpha)\,\bar{\varepsilon}^{\beta(\sigma-1)} - 1}{\beta(\sigma-1)} \frac{\delta_N}{1-\alpha} \tag{12}$$

- The aggregate output $\hat{Y}$ is equal to the total value added:

$$\hat{Y} = \{\tilde{\sigma} - (1-\beta)\} \frac{\hat{w}_L}{\beta} \tag{13}$$

- The aggregate consumption $\hat{Y}$ satisfies the household budget constraint

$$\hat{Y} = (\rho + (\theta - 1)\, g)\, N \left\{ V^P + \int \hat{z}^{\beta(\sigma-1)} V^A(a) d\hat{F}(\hat{z}, a) \right\} + \hat{w}_L + L_H \hat{w}_H \tag{14}$$

Solution Algorithm

- Guess $V^A(a)$
  - Guess $\hat{w}_H$
  - Solve $\mu(a)$ using (3)
    - Guess g
    - Solve $\hat{f}(\hat{z}, a)$ using (4)
    - Update g using (12)
  - Solve V_P using (2)
  - Update $\hat{w}_H$ using (9)
- Solve $\hat{M}_B(\hat{z}', a'; a)$, $\hat{M}_S(\hat{z}', a'; a)$ using (5) and (6)
- Solve N using (9)
- Solve $\hat{w}_L$ and $\hat{Y}$ using (14) and (13)
- Solve $\hat{c}(a)$ using (7)
- Solve $\hat{D}(a)$ using (8)
- Update $V^A(a)$ using (1)

SS Comparison Planner 1

$$\max \frac{1}{\rho-(1-\theta)g}\hat{Y}^{1-\theta} \tag{15}$$

$$x=\left(\frac{1}{\gamma_X}\frac{\varphi_X}{\varphi_E}\right)^{1/(\gamma_X-1)}, \quad x_E=\frac{\delta_N}{1-\alpha}-\left(\frac{1}{\gamma_X}\frac{\varphi_X}{\varphi_E}\right)^{1/(\gamma_X-1)}$$

$$\hat{Y}=\{\tilde{\sigma}-(1-\beta)\}\frac{\hat{w}_L}{\beta} \tag{16}$$

$$\hat{w}_L=\beta\frac{\sigma-1}{\sigma}N\int \hat{z}^{\beta(\sigma-1)}\left(\tilde{\sigma}\hat{c}(a)\right)^{1-\sigma}\hat{D}(a)d\hat{F}\left(\hat{z},a\right) \tag{17}$$

$$\hat{c}\left(a\right)^{1-\sigma}=\hat{w}_L^{\beta(1-\sigma)}\left(\int \hat{z}'^{\beta(\sigma-1)}\left(\tilde{\sigma}\hat{c}(a')\right)^{1-\sigma}d\hat{M}_S(\hat{z}',a';a)\right)^{1-\beta} \tag{18}$$

$$\hat{D}(a)=\hat{Y}+(1-\beta)\tilde{\sigma}^{-\sigma}\hat{w}_L^{-\frac{\beta}{1-\beta}(\sigma-1)}\int \hat{z}'^{\beta(\sigma-1)}\hat{c}\left(a'\right)^{\frac{\beta}{1-\beta}(\sigma-1)}\hat{D}(a')d\hat{M}_B(\hat{z}',a';a) \tag{19}$$

SS Comparison Planner 2

$$\hat{M}_B(\hat{z}',a';a)=\zeta_B\int_0^{\min\{a,a'\}}\exp\left(-\left(\delta+\delta_N+\alpha\left(x_E+x\right)\right)\tau\right)\hat{F}\left(\exp\left(-\int_{a'-\tau}^{a'}\mu\left(s\right)ds+\tau g\right)\hat{z}',a'-\tau\right)d\tau \tag{20}$$

$$\hat{M}_S(\hat{z}',a';a)=\zeta_S\int_0^{\min\{a,a'\}}\exp\left(-\left(\delta+\delta_N+\alpha\left(x_E+x\right)\right)\tau\right)\hat{F}\left(\exp\left(-\int_{a'-\tau}^{a'}\mu\left(s\right)ds+\tau g\right)\hat{z}',a'-\tau\right)d\tau \tag{21}$$

$$0=-\left(\mu(a)-g\right)\hat{z}\frac{\partial}{\partial\hat{z}}\left\{\hat{f}\left(\hat{z},a\right)\right\}-\frac{\partial\hat{f}\left(\hat{z},a\right)}{\partial a}+\frac{\delta_N}{1-\alpha}\left[\alpha\hat{f}\left(\frac{\hat{z}}{\lambda},a\right)+\left(1-\alpha\right)\hat{\varphi}\left(\hat{z}\right)-\hat{f}\left(\hat{z},a\right)\right] \tag{22}$$

$$N\left[\int\hat{z}^{\beta(\sigma-1)}c_l(\mu(a))d\hat{F}(\hat{z},a)+\frac{x_E}{\varphi_E}+c_X(x)\right]=L_H \tag{23}$$

$$g=\int\hat{z}^{\beta(\sigma-1)}\mu(a)d\hat{F}(\hat{z},a)+\frac{\alpha\lambda^{\beta(\sigma-1)}+\left(1-\alpha\right)\bar{\varepsilon}^{\beta(\sigma-1)}-1}{\beta\left(\sigma-1\right)}\left(x_E+x\right) \tag{24}$$

Planner Solution Algorithm

- Find $\mu(a)$ that maximize (15) using projection method
- Guess g
  - Solve $\hat{f}(\hat{z}, a)$ using (22)
  - Update g using (24)
- Solve $\hat{M}_B(\hat{z}', a'; a)$, $\hat{M}_S(\hat{z}', a'; a)$ using (20) and (21)
- Solve N using (23)
- Guess w_L
  - Solve $\hat{c}(a)$ using (18)
  - Solve $\hat{D}(a)$ using (19)
  - Update w_L using (17)
- Solve $\hat{Y}$ using (16)