Q2 (2 points)

Consider the word "ASSETS". Assuming the letter S can be used at most three times, but the other letters at most once. How many four-letter words can be formed? For example, SSET, SAET and SASS are all valid words.

Hint: Consider different cases.

- We need to consider different cases which is based on the number of S in the word.

Case 1 : S is used 3 times

S O S S

"4 ways to arrange this letter position

$$\underbrace{\binom{3}{1}}_{\text{Choose 1 letter from "AET"}} \times \underbrace{4}_{\text{The position of one letter which isn't "S"}} = 12$$

Case 2 : S is used 2 times (A~~S~~ETA~~S~~)

S S O O (not S)

We need to choose 2 letters from the remaining 3 letters ("A", "E", "T")

$$\underbrace{\binom{3}{2}}_{\text{Choose 2 letters from 3 distinct letters}} \times \underbrace{\frac{4!}{2!}}_{\text{4! ways to arrange letters and there are two same letters}} = 36$$

Let's say If there are 2 S $S = S_1 = S_2$

$S_1 S_2 O O = S_2 S_1 O O$ since $S_1 = S_2 = S$

$S_1 O S_2 O = S_2 O S_1 O$

Therefore we need to divide the number of ways to arrange letters by 2!.

Case 3 : S is used one time

S O O O (not S)

$$\underbrace{\binom{3}{3}}_{\text{chose 3 letters}} \times \underbrace{4!}_{\text{4! ways to arrange letters}} = 4 \times 3 \times 2 \times 1 = 24$$

$12 + 36 + 24 = 72$ $\therefore$ There are 72 four-letters words

Q3 (2 points)

There are 10 balls numbered 1 to 10.

In how many ways can we put them in 4 different boxes so that there are 3 balls in box 1, 1 ball in box 2, 4 balls in box 3, and 2 balls in box 4?

[1 2 3 4 5 6 7 8 9 10] we need to put these balls into boxes below.

3 balls	1 ball	4 balls	2 balls
box 1	box 2	box 3	box 4

So I can write

$$\binom{10}{3,1,4,2} = \frac{10!}{3!\times 4!\times 2!\times 1!} = \frac{10\times \overset{3}{\cancel{9}}\times \overset{4}{\cancel{8}}\times 7\times \overset{3}{\cancel{6}}\times 5\times \cancel{4\times 3\times 2\times 1}}{\cancel{3}\times 2\times 1\times \cancel{4!}\times \cancel{2}\times 1\times 1}$$

$$= 120\times 105$$

$$= \underline{12600 \text{ ways}}$$

Q4 (2 points)

There are 10 balls numbered 1 to 10.

If we draw 5 balls, one at a time with replacement, How many distinct **samples** are possible? **Note** (1,2,2,3,6) and (2,3,2,6,1) are considered the same.

1 2 3 4 5 6 7 8 9 10

↓ select 5 balls with replacement

○ ○ ○ ○ ○

$\bullet | \bullet | \bullet | \bullet \bullet | \bullet$

$n = 10 \quad k = 5$

From the formula

$$\binom{k+n-1}{k} = \binom{5+9}{5} = \binom{14}{5} = 2002.$$

∴ There are 2002 possible distinct samples.

Q5 (3 points)

Working with summation notation:

$$\left(\sum_{k=0}^{n-1} k^2\right) + 2\left(\sum_{k=0}^{n-1} k\right) + n = \sum_{k=1}^{n}\left(\quad A \quad \right)$$

What is the expression A? Answer without work will receive zero.

$\left(\sum_{k=0}^{n-1} k^2\right) + 2\left(\sum_{k=0}^{n-1} k\right) + n$

Now, I would like to change $\sum_{k=0}^{n-1}$ to $\sum_{k=1}^{n}$

$= \left(\sum_{k=1}^{n} (k-1)^2\right) + 2\left(\sum_{k=1}^{n}(k-1)\right) + n$

increase by 1 decrease by 1 increase by 1 decrease by 1

we can also write n as $\sum_{k=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ times}}$,

and put 2 inside of $\sum_{k=1}^{n}(k-1)$

$= \left(\sum_{k=1}^{n}(k-1)^2 + \sum_{k=1}^{n} 2(k-1) + \sum_{k=1}^{n} 1\right)$

$= \sum_{k=1}^{n}(k^2 - 2k + 1 + 2k - 2 + 1)$

$= \sum_{k=1}^{n}\left((k^2 + (-2k + 2k) + (1 + 1 - 2)\right)$

$= \underline{\sum_{k=1}^{n} k^2}$

$\therefore$ A is k^2 ∎

Q6 (6 points)

Two sport teams, A and B, play each other twice. The probability that team A winning the first game is 0.5. The probability of team A winning the second game is 0.4. The probability that team A winning either one (or both) game is 0.6.

a. [2 marks] Are events "~~team A wins the first game~~ A" and "~~team A wins the second game~~ B" independent? Explain.

b. [2 marks] Find the probability that team A wins exactly one game.

c. [2 marks] If team A wins at least one game, what is the probability that it wins the first game?

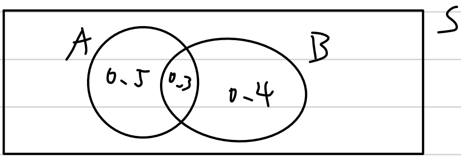
	First game	
team	A	A'
probability of winning	0.5	0.5

	Second game	
team	A	A'
probability of winning	0.4	0.6

	wins either one or both games	
team	A	A'
probability of winning	0.6	?

?

$P(A \cup B)$

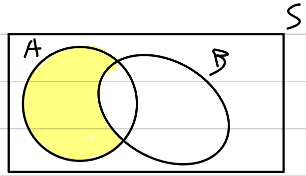


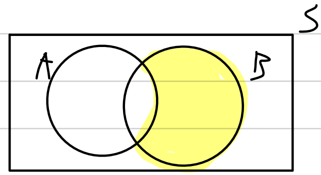
a) Events A and B are independent if A occured does not change the probability of B.
Mathmatically, I can write $P(A \cap B) = P(A) \times P(B)$ or $P(B|A) = P(B)$

Let's say $P(A) = 0.5$ $P(B) = 0.4$ based on the given Probabilities

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.5 + 0.4 - 0.6 = 0.3$$
$$P(A) \times P(B) = 0.5 \times 0.4 = 0.2 \qquad P(A \cap B) \neq P(A) \times P(B)$$

$\therefore$ Events A and B are not independent we want yellow parts





b) I can think 2 cases.

Case 1

team A wins only the first game

$$P(A \cap B') = P(A) - P(A \cap B)$$
$$= 0.5 - 0.3$$
$$= 0.2$$

Case 2

team A wins only the second game

$$P(A' \cap B) = P(B) - P(A \cap B)$$
$$= 0.4 - 0.3$$
$$= 0.1$$

· The probability that the team A wins exactly once is $0.2 + 0.1 = 0.3$

c.) Let's say $P(C)$ is the probability that team A wins at least once.
Now I'd like to know the probability that the team A wins first game, given team A wins at least once.
So I use conditional probability

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0.5 \times 0.6}{0.6} = \frac{0.3}{0.6} = \frac{1}{2} = 0.50$$

The probability that the team A wins first game, given team A wins at least once is 0.50

Q6 (6 points)

Two sport teams, A and B, play each other twice. The probability that team A winning the first game is 0.5. The probability of team A winning the second game is 0.4. The probability that team A winning either one (or both) game is 0.6.

a. [2 marks] Are events "t~~eam A wins the first game~~" A and "~~team A wins the second game~~" B independent? Explain.

b. [2 marks] Find the probability that team A wins exactly one game.

c. [2 marks] If team A wins at least one game, what is the probability that it wins the first game?

	First game	
team	A	A'
probability of winning	0.5	0.5

	Second game	
team	A	A'
probability of winning	0.4	0.6

	wins either one or both games	?
team	A	A'
probability of winning	0.6	?

a) Events A and B are independent if A occured does not change the probability of B.
Mathmatically, I can write $P(A \cap B) = P(A) \times P(B)$ or $P(B|A) = P(B)$

Let's say $P(A) = 0.5$ $P(B) = 0.4$ based on the given Probabilities

$$P(B|A) = \frac{0.5 \times 0.4}{0.5} = 0.4 = P(B)$$

$\therefore$ Events A and B are independent

b) I can think 2 cases.

Case 1

team A wins only the first game

$\underbrace{0.5}_{\text{A wins the first game}} \times \underbrace{0.6}_{\text{A lose second game}} = 0.30$

Case 2

team A wins only the second game

$\underbrace{0.5}_{\text{A loses the first game}} \times \underbrace{0.4}_{\text{A wins the second game}} = 0.20$

$\therefore$ The probability that the team A wins exactly once is $0.30 + 0.20 = 0.50$

C.) Let's say P(C) is the probability that team A wins at least once.
Now I'd like to know the probability that the team A wins first game, given team A wins at least once.
So I use conditional probability.

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0.5 \times 0.6}{0.6} = \frac{0.3}{0.6} = \frac{1}{2} = 0.50$$

The probability that the team A wins first game, given team A wins at least once is 0.50

Q7 (5 points)

Let A, B, C be events. In class, we showed that:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$. (Theorem 1.4.3 in text.)

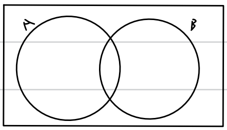
Use the above to show that:

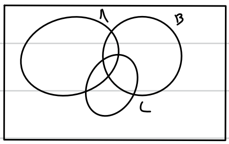
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

(Theorem 1.4.4 in text.)

Hint: Write $A \cup B \cup C = (A \cup B) \cup C$ and apply Theorem 1.4.3.

Your proof should be written such that your fellow students in Stat 350 will understand.





We know $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ... ①

First of all, let $A \cup B = X$ $C = Y$ to simplify the equation

$p(A \cup B \cup C) = p((A \cup B) \cup C) = p(X \cup Y)$

From ①, we know that

$p(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

Now let X and Y change back to $A \cup B$ and C

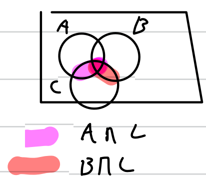
$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$

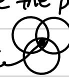
$= P(A \cup B) + P(C) - \underline{P((A \cap C) \cup (B \cap C))}$ Now we can use ① for this

$= \underline{P(A \cup B)} + P(C) - (P(A \cap C) + P(B \cap C) - P((A \cap C) \cap P(B \cap C)))$

We can also apply ① to $P(A \cup B)$

$= P(A) + P(B) - P(A \cap B) + P(C) - P(A \cap C) - P(B \cap C) + \boxed{P(A \cap C) \cap P(B \cap C))}$



From this graph we can see the part which is painted twice in the middle.  this part is $(A \cap C) \cap (B \cap C)$

So we can rewrite this as $P(A \cap B \cap C)$

Therefore

$P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Q8 (4 points)

Here is a calculus review question. Find:

$\int_0^3 t^2 e^{-3t}\,dt$

Show your steps. Do not look up integral tables or formula. You are expected to be able to do something similar in tests.

Let's call $f = x^2$ $g' = e^{-3x}$

I'll integrate this by parts

$\int fg' = fg - \int f'g$

$f' = 2t$ $g = -\frac{e^{-3t}}{3}$

$\int t^2 e^{-3t}\,dt$

$= -\frac{t^2 e^{-3t}}{3} - \int -\frac{2te^{-3t}}{3}\,dt \quad \cdots ①$

$\int -\frac{2te^{-3t}}{3}\,dt$

$= -\frac{2}{3}\int te^{-3t}\,dt$

$f = t \quad g' = e^{-3t}$

$f' = 1 \quad g = -\frac{e^{-3t}}{3}$

$= -\frac{te^{-3t}}{3} - \int -\frac{e^{-3t}}{3}\,dt \quad \cdots ②$

solve this : $\int -\frac{e^{-3t}}{3}\,dt$

let $u = -3t$

$\frac{du}{dt} = -3$

$\int -\frac{e^{-3t}}{3}\,dt = \frac{1}{9}\int e^u\,du$

$= \frac{1}{9}e^u$

substitute $u = -3t$

$= \frac{1}{9}e^{-3t} \quad \cdots ③$

Now plug ③ into ②

$-\frac{te^{-3t}}{3} - \int -\frac{e^{-3t}}{3}\,dt$

$= -\frac{te^{-3t}}{3} - \frac{e^{-3t}}{9}$

$-\frac{2}{3}\int te^{-3t}\,dt$

$= \frac{2te^{-3t}}{9} + \frac{2e^{-3t}}{27} \quad \cdots ④$

plug ④ into ①

$-\frac{t^2e^{-3t}}{3} - \int -\frac{2te^{-3t}}{3}\,dt$

$= -\frac{t^2e^{-3t}}{3} - \frac{2te^{-3t}}{9} - \frac{2e^{-3t}}{27}$

$= -\frac{(9t^2+6t+2)e^{-3t}}{27}$

Therefore $\int_0^3 t^2e^{-3t}\,dt$

$= \left[-\frac{(9t^2+6t+2)e^{-3t}}{27}\right]_0^3$

$= -\frac{(81+18+2)e^{-9}}{27} - \left(-\frac{2\times e^0}{27}\right)$

$= \frac{2}{27} - \frac{101}{27}e^{-9}$

200

$\binom{3}{2}$ 1 · 4! 33 $\binom{3}{2}$

$\binom{3}{2}2! \cdot \binom{4}{2}$

30 33

Q2 Image/PDF question | 2 points

Consider the word "ASSETS". Assuming the letter S can be used at most three times, but the other letters at most once. How many four-letter words can be formed? For example, SSET, SAET and SASS are all valid words.

Hint: Consider different cases.

$SS\binom{3}{2}\binom{4}{2}\,2!$

Case 1 : 0 's

\# of ways = 0 (must use at least 1 S to form a four-letter word)

Case 2 : 1 's

\# of ways = $4!$ (all 4 letters are different)

Case 3 : 2 's

\# of ways = $\binom{4}{2} \cdot {}_3P_2$ or $\binom{4}{2}\binom{3}{2} \cdot 2!$

select 2 locations for the S's

arrange the remaining 2 letters from 3 available letters

Case 4 : 3 's

\# of ways = $\binom{4}{3} \cdot {}_3P_1$ or $\binom{4}{3} \cdot \binom{3}{1}$

select 3 locations for the S's

fill in the last letter.

Total # of 4-letter word $= 0 + 4! + \binom{4}{2}{}_3P_2 + \binom{4}{3}\binom{3}{1}$

$= 72$

Q3 Image/PDF question | 2 points

There are 10 balls numbered 1 to 10.

In how many ways can we put them in 4 different boxes so that there are 3 balls in box 1, 1 ball in box 2, 4 balls in box 3, and 2 balls in box 4?

This is a simple multinomial counting question.

$$\# \text{ of ways} = \binom{10}{3,1,2,4}$$

$$= \frac{10!}{3!\,1!\,2!\,4!} = 12{,}600 \;//$$

$k=2$

Q4 Image/PDF question | 2 points

There are 10 balls numbered 1 to 10.

If we draw 5 balls, one at a time with replacement, How many distinct **samples** are possible? **Note** (1,2,2,3,6) and (2,3,2,6,1) are considered the same.

This is an ordered selection with replacement question

$n = 10$ objects

$k = 5$ objects to be selected

($n-1$ bars and k $*$'s)

$$\# \text{ of samples} = \binom{n-1+k}{k} \text{ or } \binom{n-1+k}{n-1}$$

$$= \binom{10-1+5}{5} \text{ or } \binom{10-1+5}{10-1}$$

$$= \binom{14}{5} \text{ or } \binom{14}{9}$$

$$= 2002$$

Q5 Image/PDF question | 3 points

Working with summation notation:

$$\left(\sum_{k=0}^{n-1} k^2\right) + 2\left(\sum_{k=0}^{n-1} k\right) + n = \sum_{k=1}^{n}\left(A \right)$$

What is the expression A? Answer without work will receive zero.

$$\left(\sum_{k=0}^{n-1} k^2\right) + 2\left(\sum_{k=0}^{n-1} k\right) + n$$

$$= \left(\sum_{k=0}^{n-1} k^2\right) + \left(\sum_{k=0}^{n-1} 2k\right) + \left(\sum_{k=0}^{n-1} 1\right)$$

$$= \sum_{k=0}^{n-1} \left(k^2 + 2k + 1\right)$$

$$= \sum_{k=0}^{n-1} (k+1)^2$$

$$= \sum_{k=1}^{n} k^2$$

limit shift up by 1

expression shift down by 1

$\therefore \quad A = k^2$ //

$P(A - P(A \cap B)$

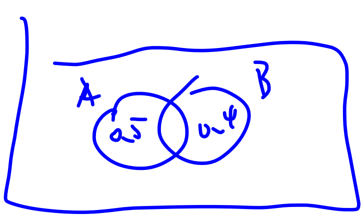
$0.5 - 0.3 - 0.2$

$0.6 = P(A) \cap P(B) - P(A \cap B)$

$= 0.6 - 0.9$

$0.3 = P(A \cap B$ $\cancel{0.2}$ 0.1

$P(A) \times P(B) = P(A \cap B)$



$A \cup B = 0.6$

$P($

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.6 = 0.5 + 0.4 - \bigcirc$

$P(A) \cdot P(B) = P(A \cap B)$

Q6 Image/PDF question | 6 points

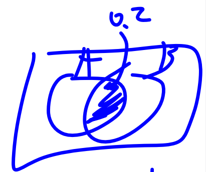
$P(A \cap B)$

Two sport teams, Red and Blue, play each other twice. The probability that team Red winning the first game (event A) is 0.5. The probability of team Red winning the second game (event B) is 0.4. The probability that team Red winning either (or both) game (event $A \cup B$) is 0.6.

a. [2 marks] Are events A and B independent? Explain using definition of independence.

b. [2 marks] Find the probability that team Red wins exactly one game. A venn diagram may help.

c. [2 marks] If team Red wins at least one game, what is the probability that it wins the first game?



$P(A \cap B') + P(A' \cap B)$

$= 0.2 + 0.2$

a. You can use a Venn diagram or formula.

Tree diagram is not appropriate since we don't have (or need) any conditional probabilities.

Given
$$P(A) = 0.5$$
$$P(B) = 0.4$$
$$P(A \cup B) = 0.6$$

We can find $P(A \cap B)$ using the inclusion/exclusion formula.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

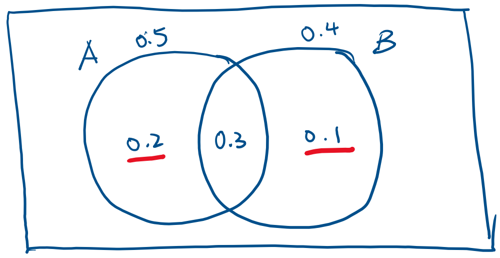
$$= 0.5 + 0.4 - 0.6 = 0.3$$

Since

$$P(A \cap B) = 0.3$$
$$P(A)\,P(B) = (0.5)(0.4) = 0.2$$

They are not equal, so A and B are not independent.

b.



We are looking for

P[Team Red wins first game but loses the second win or Team Red loses first game but wins the second game]

$$= P\left[(A \cap B') \cup (A' \cap B) \right]$$

$$= P(A \cap B') + P(A' \cap B)) \quad \Leftarrow \text{mutually exclusive events}$$

$$= \left[P(A) - P(A \cap B) \right] + \left[P(B) - P(A \cap B) \right]$$

$$= (0.5 - 0.3) + (0.4 - 0.3) = 0.3$$

c. We want $P(A \mid A \cup B)$

$$= \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$

$$= \frac{P(A)}{P(A \cup B)}$$

$A \cap (A \cup B) = A$ since $A \subseteq A \cup B$

$$= \frac{0.5}{0.6} = \frac{5}{6}$$

Let $A \cup B = D$

$P(D \cup C) = P(D) + P(C) - P(D \cap C)$

$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$

$- P(A \cap C)$

Q7 Image/PDF question | 5 points

Let A, B, C be events. In class, we showed that:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$. (Theorem 1.4.3 in text.)

Use the above to show that:

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$. (Theorem 1.4.4 in text.)

Hint: Write $A \cup B \cup C$ = $(A \cup B) \cup C$ and apply Theorem 1.4.3.

Your proof should be written such that your fellow students in Stat 350 will understand.

$$P(A \cup B \cup C) = P[\underbrace{(A \cup B)}_{\text{one event}} \cup C]$$

Thm 1.4.3 $\quad = P(A \cup B) + P(C) - P[(A \cup B) \cap C]$

Thm 1.43 $\quad = P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cup B) \cap C]$ $\quad$ (1)

Now:

$$P[(A \cup B) \cap C]$$

$$= P[(A \cap C) \cup (B \cap C)]$$

Distributive property of set union and intersection

$$= P(A \cap C) + P(B \cap C) - P[(A \cap C) \cap (B \cap C)]$$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$ $\quad$ (2)

Sub ② into ①

$$P(A \cup B \cup C) = P(A) + P(B) - P(A \cap B) + P(C) - \left[P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) \right]$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

∎

Note: This formula can be extended to k events.

Q8 Image/PDF question | 4 points

Here is a calculus review question. Find:

$\int_0^3 t^2 e^{-3t} dt$

Show your steps. Do not look up integral tables or formula. You are expected to be able to do something similar in tests.

You can solve this integral by applying integration by parts twice.

$$\int_0^3 t^2 e^{-3t} dt$$

$$= \left[\frac{-t^2 e^{-3t}}{3} - \frac{2t e^{-3t}}{9} - \frac{2e^{-3t}}{27} \right]\Bigg|_0^3$$

$$= \left[\frac{-3^2 e^{-9}}{3} - \frac{6e^{-9}}{9} - \frac{2e^{-9}}{27} \right] - \left[0 - 0 - \frac{2}{27} \right]$$

$$= \frac{2}{27} - \left(3 + \frac{2}{3} + \frac{2}{27} \right) e^{-9}$$

$$= \frac{2}{27} - \frac{101}{27} e^{-9} = \frac{2 - 101 e^{-9}}{27} = 0.0736$$

This is fine.

diff.		int.
t^2	$+$	e^{-3t}
$2t$	$-$	$-\frac{e^{-3t}}{3}$
2	$+$	$\frac{e^{-3t}}{9}$
0		$-\frac{e^{-3t}}{27}$