Q1 (4 points)

1. Recall the hypergeometric random variable $X \sim \mathbf{HYP}(n, M, N)$ where N is the total of balls, M is the number of black balls, n is the number of balls selected without replacement, and X is the number of black balls selected.

   In class, we figured the support set for X as follows:

   $$A = \boldsymbol{max}\{M - N + n, 0\} \leqslant x \leqslant \boldsymbol{min}\{M, n\} = B$$

   where $0 < n < N$ $\quad 0 < M < N$

   and no other assumptions are made about M and n.

   (4 points) Show $A \leqslant B$.

- N ... the total of balls
- M ... # of black balls
- n ... # of balls selected without replacement
- X ... # of black balls selected.

$A = max\{\underbrace{M-N+n}, 0\}$

$M-N+n$ means that A takes the maximum value of either $M-N+n$ or 0.

$B = min\{M, n\}$ means that B is the minimum value of either # of black balls or # of balls selected without replacement.

I will consider 2 cases:

- Case 1: when $M-N+n < 0$

  In this case, the value of A will be 0 since $M-N+n$ is a negative value. Also B is always non-negative value, so $A \leq B$ holds.

- Case 2: when $0 \leq M-N+n$

  In this case, the value of A will be $M-N+n$.
  Because $M-N+n$ is always either greater than or equal to M or less than or equal to n, $A \leq B$ holds.

$\therefore$ $A \leq B$ holds in both cases which means $A \leq B$ is true in this questions.

Q2 (10 points)

$Y < X < 1$

2. (Exercise 20 in Chapter 4 in textbook) Suppose that X and Y have the joint pdf

$$f(x, y) = 8xy \qquad 0 \leqslant x \leqslant y \leqslant 1$$

and zero otherwise. Find each of the following:

$0 \le x \le y$
$x \le y \le 1$

(a) (2 points) The joint CDF $F(x, y)$.
(b) (2 points) $f(y|x)$.
(c) (2 points) $f(x|y)$.
(d) (2 points) $P[X \leqslant 0.5|Y = 0.75]$.
(e) (2 points) $P[X \leqslant 0.5|Y \leqslant 0.75]$.

$4x(1-x^2)$

?

(a) $\int_x^1 \int_0^y 8xy \, dx \, dy$

$= \int_x^1 4x^2 y \Big|_0^y \, dy$

$= \int_x^1 4y^3 \, dy$

$= \frac{4}{4} y^4 \Big|_x^1$

$= 1 - x^4$

(b) We know $f(y|x) = \frac{f(x,y)}{f_X(x)}$

$f_X(x) = \int_x^1 f(x,y) \, dy$

$= \int_x^1 8xy \, dy$

$= 4xy^2 \Big|_x^1$

$= 4x(1-x^2)$

$= 4x(1-x^2)$

$f(y|x) = \frac{f(x,y)}{f_X(x)}$

$= \frac{8xy}{4x(1-x^2)}$

$= \frac{2y}{(1-x^2)}$ $\quad 0 < x < y < 1$

(c) $f(x|y) = \frac{f(x,y)}{f_Y(y)}$

$f_Y(y) = \int_0^y 8xy \, dx$

$= 4x^2 y \Big|_0^y$

$= 4y^3 - 0$

$= 4y^3$

$f(x|y) = \frac{f(x,y)}{f_Y(y)}$

$= \frac{8xy}{4y^3}$

$= \frac{2x}{y^2}$

(d) $P[X \le 0.5 | Y = 0.75]$ use (c)

$= \int_0^{0.5} f(x | Y = 0.75) \, dx$

$= \int_0^{0.5} \frac{2x}{0.75^2} \, dx$

$= \frac{1}{0.75^2} \cdot \frac{2}{2} x^2 \Big|_0^{0.5}$

$= \frac{1}{0.75^2} \cdot 0.5^2 = 0.4444\ldots$

≈ 0.4444

e) $P[X \le 0.50 \mid Y \le 0.75] = \dfrac{\iint f(x,y)\,dy\,dx}{\int f_Y(y)\,dy}$

$$\frac{\int_0^{0.50}\int_x^{0.75} 8xy\,dy\,dx}{\int_0^{0.75} 4y^3\,dy}$$

$$= \frac{\int_0^{0.50} 4xy^2\Big|_x^{0.75}\,dy\,dx}{y^4\Big|_0^{0.75}} = \frac{\int_0^{0.50} 4x(0.75)^2 - 4x^3\,dx}{(0.75)^4}$$

$$= \frac{2x^2(0.75)^2 - x^4\Big|_0^{0.50}}{(0.75)^4}$$

$$= \frac{2\times(0.5)^2(0.75)^2 - (0.50)^4}{(0.75)^4}$$

$$= \underline{\frac{56}{81}}$$

Q3 (10 points)

3. Suppose that X and Y have the joint pdf

$$f(x,y) = \frac{2}{3}(x+1) \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1$$

and zero otherwise. Find each of the following:

(a) (2 points) $f_X(x)$.
(b) (2 points) $f_Y(y)$.
(c) (2 points) Are X and Y independent? Justify.
(d) (4 points) $P[X \leqslant 2Y \leqslant 3X]$.

(a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$
$= \int_0^1 \frac{2}{3}(x+1)\,dy$
$= \frac{2}{3}(x+1)y \Big|_0^1$
$= \frac{2}{3}(x+1)$ if $0 \leq x \leq 1$

(b) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$
$= \int_0^1 \frac{2}{3}(x+1)\,dx$
$= \frac{2}{3}\int_0^1 x\,dx + \frac{2}{3}\int_0^1 dx$
$= \frac{2}{3} \times \frac{1}{2}x^2 \Big|_0^1 + \frac{2}{3}x \Big|_0^1$
$= \frac{2}{6} + \frac{2}{3}$
$= \frac{1}{3} + \frac{2}{3} = 1$ if $0 \leq y \leq 1$

(c) Two random variable X and Y are independent iff

$$f(x,y) = f_X(x) f_Y(y)$$

In this case, I got $f_X(x) = \frac{2}{3}(x+1)$ $f_Y(y) = 1$

$\therefore f(x,y) = \frac{2}{3}(x+1) = f_X(x)f_Y(y) = \frac{2}{3}(x+1)\cdot 1 = \frac{2}{3}(x+1)$
So X and Y are independent ■

(d) The range inside of the probability is:
$X \leq 2Y \leq 3X$
Let's divide all of these variables by 2
$\frac{X}{2} \leq Y \leq \frac{3}{2}X$
this is true when $X = 2$ and $X = \frac{2}{3}$
since $0 \leq X \leq 1$, $X = \frac{2}{3}$.

So the range of Y becomes
$\frac{1}{3} \leq Y \leq 1$

$$= \int_0^{\frac{2}{3}} \int_{\frac{x}{2}}^{\frac{3}{2}x} \frac{2}{3}(x+1)\,dy\,dx \qquad + \int_{\frac{2}{3}}^{1} \int_{\frac{x}{2}}^{1} \frac{2}{3}(x+1)\,dy\,dx$$

$$= \frac{2}{3}\int_0^{\frac{2}{3}} (x+1)y\Big|_{\frac{1}{2}x}^{\frac{3}{2}x} dx \qquad + \frac{2}{3}\int_{\frac{2}{3}}^{1} (x+1)y\Big|_{\frac{x}{2}}^{1} dx$$

$$= \frac{2}{3}\int_0^{\frac{2}{3}} (x+1)\left(\frac{3}{2}x - \frac{1}{2}x\right) dx \qquad + \frac{2}{3}\int_{\frac{2}{3}}^{1} (x+1)\left(1 - \frac{x}{2}\right) dx$$

$$= \frac{2}{3}\int_0^{\frac{2}{3}} (x+1)x\,dx \qquad + \frac{2}{3}\int_{\frac{2}{3}}^{1} \left(x+1 - \frac{1}{2}x^2 - \frac{x}{2}\right) dx$$

$$= \frac{2}{3}\int_0^{\frac{2}{3}} x^2 + x\,dx \qquad + \frac{2}{3}\int_{\frac{2}{3}}^{1} \left(\frac{x}{2} + 1 - \frac{1}{2}x^2\right) dx$$

$$= \frac{2}{3}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\Big|_0^{\frac{2}{3}}\right] \qquad + \frac{2}{3}\left[\frac{x^2}{4} + x - \frac{1}{6}x^3\right]_{\frac{2}{3}}^{1}$$

$$= \frac{2}{3}\left[\frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{2}\cdot\left(\frac{2}{3}\right)^2 - 0\right] \qquad + \frac{2}{3}\left[\left(\frac{1}{4} + 1 - \frac{1}{6}\right) - \left(\frac{1}{4}\cdot\frac{4}{9} + \frac{2}{3} + -\frac{1}{\cancel{6}3}\cdot\frac{\cancel{8}4}{27}\right)\right]$$

$$= \frac{2}{3}\left[\frac{8}{81} + \frac{4}{18}\right] \qquad + \frac{2}{3}\left[\frac{6}{24} + \frac{24}{24} - \frac{4}{24} - \left(\frac{1}{9} + \frac{2}{3} - \frac{4}{81}\right)\right]$$

$$= \frac{2}{3}\left[\frac{8}{81} + \frac{18}{81}\right] \qquad + \frac{2}{3}\left[\frac{26}{24} - \left(\frac{9}{81} + \frac{54}{81} - \frac{4}{81}\right)\right]$$

$$= \frac{2}{3}\left[\frac{26}{81}\right] \qquad + \frac{2}{3}\left[\frac{21}{24} - \left(\frac{59}{81}\right)\right]$$

$$= \frac{2}{3}\left(\frac{26}{81} + \frac{115}{324}\right) \qquad + \frac{2}{3}\left(\frac{115}{324}\right)$$

$$= \frac{2}{3}\left(\frac{104}{324} + \frac{115}{324}\right)$$

$$= \frac{\cancel{2}}{\cancel{3}}\left(\frac{\cancel{219}\,73}{\cancel{324}\,162}\right)$$

$$= \frac{73}{162}$$

Q4 (11 points)

4. A bivariate r.v. (X, Y) is said to be a bivariate standard normal r.v. if its joint pdf is given by:

$$f(x,y) = \frac{1}{2\pi\left(1-\rho^2\right)^{1/2}} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2\left(1-\rho^2\right)}\right]$$

where

$\exp[\,]$ means $e^{[\,]}$.

For part (a) to (c), let $\rho = 0$.

(a) (2 points) Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Note: By showing part (a), you will also show that the pdf for the standard normal distribution integrates to 1.

(b) (3 points) Find $f_X(x)$.

Note: By symmetry, $f_Y(y)$ is the same as $f_X(x)$ with the y's replacing the x's.

(c) (2 points) Use part (b) to justify if X and Y are independent when $\rho = 0$.

For part (d) and (e), let $\rho \in (-1, 0) \cup (0, 1)$.

(d) (2 points) Find $f_Y(y)$.

(e) (2 points) Show that $X|y$ is a normal r.v. and find its mean and variance.

Hint: Find $f(x|y)$ and recognize it as a pdf for a normal r.v. and identify the mean and the variance.

(a) When $\rho = 0$, $f(x,y)$ becomes

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

This is pdf for the standard normal distribution.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx \cdot \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy$$
$$= \frac{1}{2\pi}\sqrt{2\pi}\cdot\sqrt{2\pi}$$
$$= 1$$

(b) $$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,dy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dy$$
$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy$$
$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}}\sqrt{2\pi}$$
$$= \frac{\sqrt{2\pi}}{2\pi} e^{-\frac{x^2}{2}}$$

(c) By symmetry, $f_X(x) = f_Y(y)$

So $f_Y(y) = \frac{\sqrt{2\pi}}{2\pi} e^{-\frac{y^2}{2}}$

When $f(x,y) = f_Y(y)\cdot f_X(x)$, X and Y are independent when $\rho = 0$.

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} = \frac{\sqrt{2\pi}}{2\pi} e^{-\frac{x^2}{2}} \cdot \frac{\sqrt{2\pi}}{2\pi} e^{-\frac{y^2}{2}} = \frac{2\pi}{4\pi^2} e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}}$$
$$= \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

since $f(x,y) = f_X(x) f_Y(y)$,

X and Y are independent.

(d) Find $F_Y(z)$

$\rho \in (-1,0) \cup (0,1)$ ρ is the correlation coefficient of X and Y.

$(-1<\rho<1$ or $0<\rho<1) \rightarrow$ this means X and Y are correlated.

$$F_Y(z) = \int f(x,z)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} e^{\frac{-x^2-2\rho xz+z^2}{2(1-\rho^2)}}\,dx$$

First of all I'll focus on $e^{\frac{-x^2-2\rho xz+z^2}{2(1-\rho^2)}}$

$$\frac{-(x^2+2\rho xz-z^2)}{2(1-\rho^2)} = \frac{-(x^2+2\rho xz+\rho^2z^2)-(\rho^2z^2+z^2)}{(2(1-\rho^2))}$$

$$= \frac{-[(x+\rho z)^2-(1+\rho^2)z^2]}{(2(1-\rho^2))}$$

Now,

$$e^{\frac{-[(x+\rho z)^2-(1+\rho^2)z^2]}{(2(1-\rho^2))}}\,dx$$

let $u = \frac{x+\rho z}{\sqrt{2(1-\rho^2)}}$, when $x=-\infty$, $u=-\infty$

$x=\infty$, $u=\infty$

Now $e^{\frac{-[(x+\rho z)^2-(1+\rho^2)z^2]}{(2(1-\rho^2))}}\,dx$

$$= e^{\frac{(-(1+\rho^2))z^2}{2(1-\rho^2)}} \int_{-\infty}^{\infty} e^{(-u^2)}\sqrt{2(1-\rho^2)}\,du$$

$$= e^{\frac{(-(1+\rho^2))z^2}{2(1-\rho^2)}} \cdot \sqrt{2(1-\rho^2)}\sqrt{\pi}$$

$$= \sqrt{2\pi(1-\rho^2)}\, e^{\frac{(-1+\rho^2)z^2}{2(1-\rho^2)}}$$

So we get

$$F_Y(z) = \frac{1}{2\pi\sqrt{1-\rho^2}}\sqrt{2\pi(1-\rho^2)}\, e^{\frac{(-1+\rho^2)z^2}{2(1-\rho^2)}}$$

$$= \underline{\frac{\sqrt{2\pi}}{2\pi} e^{\frac{(-1+\rho^2)z^2}{2(1-\rho^2)}}}$$

(e) We know $f(x|z) = \frac{f(x,z)}{f_Y(z)}$

from (d) we know $f_Y(z)$

$$\text{So } f(x|z) = \frac{\frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} e^{\frac{-x^2-2\rho xz+z^2}{2(1-\rho^2)}}}{\frac{\sqrt{2\pi}}{2\pi} e^{\frac{(-1+\rho^2)z^2}{2(1-\rho^2)}}}$$

$$= \underline{\frac{e^{-x^2-2\rho xz+z^2}}{\sqrt{2\pi(1-\rho^2}\, e^{\frac{(-1+\rho^2)z^2}{2(1-\rho^2)}}}}$$

mean is

$$E[f(X,Y)] \doteq f(\mu_1,\mu_2) + \frac{d^2f}{dx^2}\sigma_1^2 + \frac{d^2f}{dz^2}\sigma_2^2$$

Variance is

$$Var[f(X,Y)] \doteq \left(\frac{df}{dx}\right)^2\sigma_1^2 + \left(\frac{df}{dz}\right)^2\sigma_2^2 + 2\left(\frac{df}{dx}\right)\left(\frac{df}{dz}\right)\sigma_{12}$$