MATH 3030, Winter 2023

Instructor: Lucas Mol

Written Assignment 1

Due: Tuesday, January 24

Guidelines:

- Try to explain your reasoning as clearly as possible. Feel free to reference theorems, lemmas, or anything else in the notes by number. For example, you could say "By Theorem 2.1.4,..."
- Please make sure that the work you submit is organized. Complete rough work on a different page, and then write up a good copy of your final solutions. Make sure to indicate each problem number clearly.
- Working with others is allowed (and encouraged!) but copying is strictly prohibited, and is a form of academic misconduct. Every student should write up their own good copy. Your work should not be the same, word-for-word, as any other student's work.
- The more work you put into these assignments, the more you will get out of them. **Please think about the problems before looking for solutions online.**

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1. **Optional.** Print an extra copy of Quiz 1 – it is posted on the Moodle page. Resubmit it along with the version of Quiz 1 that you completed in class. You will be given 1/3 of the credit for any extra points that you earn. Please do not write over your completed quiz – print a brand new version, and submit both.

   Important Note: This is a one-time offer. Please don't expect it on every quiz.

2. Consider Example 4.1.4 from the course notes, where the weather conditions are predicted by a four-state Markov chain.

   (a) If it was sunny today and yesterday, what is the probability that it will be sunny three days from now?

   Solution: With the states 0, 1, 2, and 3 defined as in the notes, the transition probability matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.9 & 0 & 0.1 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix} \end{array}$$

Using Sage, we find the three-step transition matrix:

$$P^3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.753 & 0.054 & 0.097 & 0.096 \\ 0.582 & 0.096 & 0.118 & 0.204 \\ 0.324 & 0.190 & 0.096 & 0.390 \\ 0.288 & 0.195 & 0.102 & 0.415 \end{bmatrix} \end{array}$$

Note that it will be sunny three days from now if we are in state 0 or state 1 three days from now. So the desired probability is

$$P\{X_3 = 0 \text{ or } X_3 = 1 \mid X_0 = 0\} = P_{00}^{(3)} + P_{01}^{(3)} = 0.753 + 0.054 = 0.807.$$

(Note that we can simply add the probabilities because the events $X_3 = 0$ and $X_3 = 1$ are mutually exclusive – we are in at most one state at time 3. ☺)

(b) If it was cloudy today but sunny yesterday, what is the probability that it will be sunny three days from now?

Solution: Here, the desired probability is

$$P\{X_3 = 0 \text{ or } X_3 = 1 \mid X_0 = 2\} = P_{20}^{(3)} + P_{21}^{(3)} = 0.324 + 0.190 = 0.514.$$

3. Lucas practices his disc golf putting as follows.

- He starts 4m from the basket, and throws three putts. We'll call this the first round.
- If he makes all three putts, then he moves back one metre.
- If he makes two out of three putts, then he stays the same distance from the basket.
- If he makes less than two of the three putts, then he moves forward one metre.
- He then repeats the same process, throwing three discs in each round.
- He practices until he makes it 6m away from the basket (i.e., until he makes all three putts from 5m in a single round).

Suppose that he makes each putt, independently of all others, with probability $\frac{3}{s}$, where $s \geq 3$ is his distance from the basket.

(a) Let X_t be his distance from the basket after t rounds (so $X_0 = 4$). Explain why the process $\{X_t,\ t = 0, 1, 2, ...\}$ is a Markov chain.

Solution: The process is a Markov chain because Lucas makes each putt independently of all others. This means that whenever Lucas is in a given state i, the probability that he will transition to another state j is the same, regardless of how he got to state i.

(b) Write the transition probability matrix of the process. (Assume that when he reaches 6m, the process stays in state 6 indefinitely.)

Solution: Since Lucas makes putts from 3m with probability 1, we see that the state space will be $\{3, 4, 5, 6\}$. The transition probabilities can be calculated using our knowledge of binomial random variables. We have

$$P = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ \left(\frac{1}{4}\right)^3 + 3 \cdot \frac{3}{4} \cdot \left(\frac{1}{4}\right)^2 & 3 \cdot \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} & \left(\frac{3}{4}\right)^3 & 0 \\ 0 & \left(\frac{2}{5}\right)^3 + 3 \cdot \frac{3}{5} \cdot \left(\frac{2}{5}\right)^2 & 3 \cdot \left(\frac{3}{5}\right)^2 \cdot \frac{2}{5} & \left(\frac{3}{5}\right)^3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{10}{64} & \frac{27}{64} & \frac{27}{64} & 0 \\ 0 & \frac{44}{125} & \frac{54}{125} & \frac{27}{125} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(c) Find the probability that Lucas finishes in at most 4 rounds, i.e., the probability that $X_4 = 6$ given that $X_0 = 4$.

Solution: We use Sage to find the four-step transition matrix. The desired probability is

$$P\{X_4 = 6 \mid X_0 = 4\} = P_{46}^{(4)} \approx 0.247$$

(d) Find the probability that Lucas finishes in exactly 4 rounds.

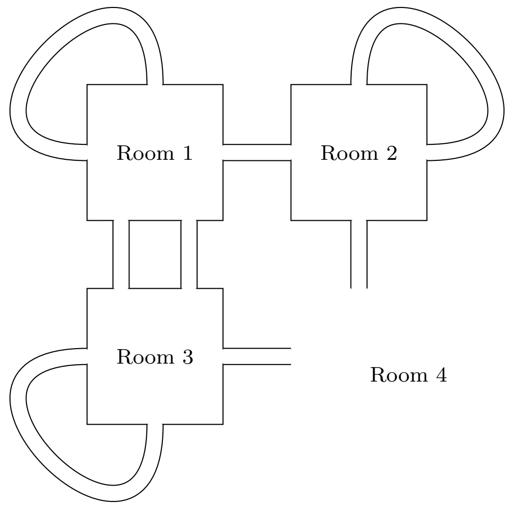
Solution: Since $P_{46}^{(}n)$ is the probability that Lucas finishes in *at most* n rounds, the probability that Lucas finishes in exactly 4 rounds is given by

$$P_{46}^{(4)} - P_{46}^{(3)} \approx 0.247 - 0.169 = 0.078.$$

Alternatively, the probability that Lucas finishes in exactly 4 rounds is equal to the probability that he is in state 5 after 3 rounds, and makes all three putts in the fourth round. This probability is

$$P_{45}^{(3)} P_{56} \approx (0.359) \cdot (0.216) \approx 0.078.$$

4. Lucas has designed a new maze for his small dog Ernie.



Assume as usual that whenever Ernie is in Room i, he randomly chooses an exit and walks through that tunnel to the next room.

(a) Given that Ernie starts in Room 1, find the probability that he escapes the maze in at most 10 steps.

Solution: The transition probability matrix is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

We compute the ten-step transition matrix in Sage to find

$$P_{14}^{(10)} = 0.706.$$

(b) Suppose that Ernie is equally likely to start in Room 1, Room 2, or Room 3. Find the probability that Ernie escapes the Maze in at most 10 steps.

Solution: We let α denote the row vector whose ith entry is the probability that Ernie starts in Room i:

$$\alpha = [\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0].$$

Then the probability that Ernie escapes the Maze in at most 10 steps is the fourth entry in the row vector

$$\alpha P^{10},$$

which, from Sage, is approximately 0.753.

(c) Suppose now that Ernie's friend Bella is in Room 2. If he ever enters Room 2, then he will want to stay there with Bella! Given that Ernie starts in Room 1, find the probability that Ernie escapes in at most 10 steps without ever entering Room 2.

Hint: Modify the transition probability matrix used in parts (a) and (b) appropriately!

Solution: We modify the transition matrix by turning 2 into an absorbing state – once Ernie enters Room 2, he gets distracted by his friend and stays there forever. The new transition matrix is

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The desired probability (which we compute using Sage) is

$$Q_{14}^{(10)} \approx 0.346.$$