MATH 3030, Winter 2023

Instructor: Lucas Mol

Written Assignment 4

Due: Monday, February 27

Guidelines:

- Try to explain your reasoning as clearly as possible. Feel free to reference theorems, lemmas, or anything else in the notes by number. For example, you could say "By Theorem 2.1.4,..."
- Please make sure that the work you submit is organized. Complete rough work on a different page, and then write up a good copy of your final solutions. Make sure to indicate each problem number clearly.
- Working with others is allowed (and encouraged!) but copying is strictly prohibited, and is a form of academic misconduct. Every student should write up their own good copy. Your work should not be the same, word-for-word, as any other student's work.
- The more work you put into these assignments, the more you will get out of them. **Please think about the problems before looking for solutions online.**

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1. Suppose that you arrive at a bank at which two clerks are working. Both clerks are busy, and there are three people in line in front of you. If the service times of all individuals are independent exponential random variables with mean 4 minutes (from either clerk), find the expected time until you leave the bank.

Solution: There are two people currently being helped, and three more people in line in front of you.

- Let W_1 be the time until the first of these people leaves the bank.
- For $i \geq 2$, let W_i be the additional time, after the $i-1$st person leaves, until the ith person leaves.
- Then the total waiting time W, until we are served, can be written as

$$W = W_1 + W_2 + W_3 + W_4.$$

- Let S be the length of our service.
- Then the total time T that we spend at the bank is given by $T = W + S$.

First we find W_1. When we enter the bank, there are two customers being served. By the memoryless property, their remaining service times are independent exponential random variables with common parameter $\frac{1}{4}$. Since W_1 is the minimum of these two service times, we see that W_1 is an independent exponential random variable with parameter $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. So $E[W_1]=2$.

At the moment when the first customer's service is complete, the next customer in line starts being served, and by the memoryless property, the remaining service time of the other person being served is exponential with parameter $\frac{1}{4}$. So we see that W_2 is again the minimum of two independent exponential random variables with parameter $\frac{1}{4}$. So W_2 is exponential with parameter $\frac{1}{2}$, hence $E[W_2]=2$. By the same reasoning, we have $E[W_3]=E[W_4]=2$.

Finally, as soon as the fourth customer leaves the bank, our service will start. Since the service time with either clerk is exponential with mean 4, we have $E[S]=4$.

We conclude that $E[T]=2+2+2+2+4=E[W]+E[S]=12$.

2. Suppose that you arrive at a bank at which two clerks are working. Both clerks are busy, but you are next in line. If the service times of the clerks are independent exponential random variables with rates λ_1 and λ_2, what is the probability that you leave the bank before one of the customers currently being served?

Solution: Let A be the event that we leave the bank before one of the customers currently being served. We determine $P(A)$ by conditioning on the clerk that serves us. For $i=1,2$, let A_i be the event that we are served by Clerk i. Then

$$P(A)=P\left(A \mid A_1\right)\cdot P(A_1)+P\left(A \mid A_2\right)\cdot P(A_2)$$

Note that $P(A_1)$ is the probability that the customer currently being served by Clerk 1 leaves before the customer currently being served by Clerk 2. But since the service times with Clerks 1 and 2 are independent exponential random variables with rates λ_1 and λ_2, respectively, we have

$$P(A_1)=\frac{\lambda_1}{\lambda_1+\lambda_2}.$$

Now consider $P\left(A \mid A_1\right)$. If A_1 occurs, then at the moment Clerk 1's first customer leaves the bank, our service with Clerk 1 will start, and by the memoryless property, the remaining time that Clerk 2 will spend helping their customer is exponentially distributed with rate λ_2. So the conditional probability that we leave the bank before the customer being served by Clerk 2 is given by

$$P\left(A \mid A_1\right)=\frac{\lambda_1}{\lambda_1+\lambda_2}.$$

By symmetric reasoning, we find that

$$P(A_2)=\frac{\lambda_2}{\lambda_1+\lambda_2} \quad \text{and} \quad P\left(A \mid A_2\right)=\frac{\lambda_2}{\lambda_1+\lambda_2}.$$

Therefore, we conclude that

$$P(A) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^2.$$

3. Suppose that Lucas leaves for vacation and forgets to ask anyone to water his plants. He has twelve plants. Assume that without water, the lifetimes of the plants are independent exponential random variables with common mean 14 days.

   (a) Let T be the time that the second plant dies. Find $E[T]$ and $\mathrm{Var}(T)$.
   Hint: Let T_1 be the time that the first plant dies, and let T_2 be the additional time until the second plant dies.

   Solution: For $i = 1, 2, \ldots, 12$, let X_i be the lifetime of the ith plant. Then $T_1 = \min\{X_1, X_2, \ldots, X_{12}\}$. Since the X_i's are independent exponential random variables with common rate $\frac{1}{14}$, this means that T_1 is an exponential random variable with rate $12 \cdot \frac{1}{14} = \frac{12}{14}$. Therefore, we have $E[T_1] = \frac{14}{12}$ and $\mathrm{Var}(T_1) = \left(\frac{14}{12}\right)^2 = \frac{49}{36}$.
   Now at time T_1, the first plant dies. For $i = 1, 2, \ldots, 11$, let Y_i be the remaining lifetime of the ith plant. By the memoryless property, the Y_i's are exponential random variables with common rate $\frac{1}{14}$. Hence T_2, being the minimum of the Y_i's, is exponential with rate $\frac{11}{14}$, and we see that $E[T_2] = \frac{14}{11}$ and $\mathrm{Var}(T_2) = \frac{196}{121}$.
   Therefore, since the total time T until the second plant dies is the sum of T_1 and T_2, we have

   $$E[T] = E[T_1] + E[T_2] = \tfrac{14}{12} + \tfrac{14}{11} \approx 2.44 \text{ days}$$

   and since T_1 and T_2 are independent (as the lifetimes of all twelve plants are independent), we have

   $$\mathrm{Var}(T) = \mathrm{Var}(T_1) + \mathrm{Var}(T_2) = \tfrac{49}{36} + \tfrac{196}{121} \approx 2.98.$$

   (b) If he is gone for one week, what is the probability that at most two of the twelve plants are dead when he returns?

   Solution: Consider any particular $i \in \{1, 2, \ldots, 12\}$. The probability that the ith plant dies in the week that Lucas is gone is given by

   $$P\{X_i < 7\} = 1 - e^{-1/14 \cdot 7} \approx 0.393.$$

   Now let Z be the number of dead plants when Lucas returns. Since each plant dies while Lucas is gone independently with probability 0.393, note that Z is a binomial random variable with parameter $p = 0.393$. Thus the desired probability is

   $$P\{Z \le 2\} = 0.607^{12} + \binom{12}{1} \cdot 0.393^1 \cdot 0.607^{11} + \binom{12}{2} \cdot 0.393^2 \cdot 0.607^{10} \approx 0.0911.$$

4. Between 8:30am and 4:30pm, emails arrive to Lucas' inbox according to a Poisson process with rate 2 per hour.

   (a) What is the probability that Lucas receives no emails between 8:30am and 4:30pm? (One can dream...)

   Solution: Since the emails arrive according to a Poisson process at a rate of 2 per hour, the number of emails that he receives in an interval of length 8 hours is a Poisson random variable with mean $2 \cdot 8 = 16$. So the probability that he receives no emails between 8:30am and 4:30pm is

   $$\frac{e^{-16}(16)^0}{0!} =\approx 1.125 \times 10^{-7}.$$

   Looks like Lucas will have to keep dreaming.

   (b) Starting from 8:30am, at what time should Lucas expect to receive his fourth email?

   Solution: Since the interarrival times $T_1, T_2, \ldots$ are exponential random variables with common rate 2, the expected length of time (after 8:30am) at which we receives his fourth email is given by

   $$E[T_1] + E[T_2] + E[T_3] + E[T_4] = 4 \cdot \tfrac{1}{2} = 2.$$

   Therefore, he should expect to receive his fourth email at 10:30am.

   (c) What is the probability that Lucas receives exactly 5 emails between 9:30am and 11:30am?

   Solution: The number of emails that he receives in an interval of length 2 hours is a Poisson random variable with mean $2 \cdot 2 = 4$. So the probability that he receives exactly 5 emails between 9:30am and 11:30am is

   $$\frac{e^{-4}4^5}{5!} = 0.156.$$

   (d) Lucas checks his email before class at 12:30pm. When he returns to his office at 2:00pm, how many new emails should he expect to have?

   Solution: The number of emails that he receives in an interval of length 1.5 hours is a Poisson random variable with mean $2 \cdot 1.5 = 3$. So he should expect to have 3 new emails when he gets back to his office.

5. A train has just left a subway station. Suppose that the number of minutes until the next train arrives is exponentially distributed with mean 10 minutes. Suppose further that passengers arrive at the station according to a Poisson process at a rate of 6 per minute. Let T be the time that the next train arrives, and let X be the number of passengers that board the next train.

(a) Find $E[X \mid T = t]$ and $\text{Var}(X \mid T = t)$.

Solution: Since passengers arrive according to a Poisson process at a rate of 6 per minute, the number of passengers that arrive in the next t minutes is a Poisson random variable with parameter $6t$. Since the mean and variance of a Poisson random variable are equal to its parameter, we have

$$E[X \mid T = t] = 6t \quad \text{and} \quad \text{Var}(X \mid T = t) = 6t.$$

(b) Use your answers to part (a) to find $E[X]$ and $\text{Var}(X)$.

Solution: From part (a), we have $E[X \mid T] = 6T$ and $\text{Var}(X \mid T) = 6T$. Since T is exponential with mean 10, we have

$$E[T] = 10 \quad \text{and} \quad \text{Var}(T) = 10^2 = 100.$$

So the law of total expectation gives

$$E[X] = E[E[X \mid T]] = E[6T] = 6E[T] = 6 \cdot 10 = 60,$$

and the law of total variance gives

$$\begin{aligned}
\text{Var}(X) &= E[\text{Var}(X \mid T)] + \text{Var}(E[X \mid T]) \\
&= E[6T] + \text{Var}(6T) \\
&= 6E[T] + 6^2 \,\text{Var}(T) \\
&= 6 \cdot 10 + 36 \cdot 100 \\
&= 3660.
\end{aligned}$$

(c) Repeat part (b) if the number of minutes until the next train arrives is uniformly distributed on $(5, 15)$.

Solution: Since T is uniformly distributed on $(5, 15)$, we have

$$E[T] = \tfrac{5+15}{2} = 10 \quad \text{and} \quad \text{Var}(T) = \tfrac{(15-5)^2}{12} = \frac{25}{3}.$$

But notice that $E[X \mid T]$ and $\text{Var}(X \mid T)$ do not change! So the law of total expectation gives

$$E[X] = E[E[X \mid T]] = E[6T] = 6E[T] = 6 \cdot 10 = 60,$$

and the law of total variance gives

$$\begin{aligned}\operatorname{Var}(X) &= E[\operatorname{Var}(X \mid T)] + \operatorname{Var}(E[X \mid T]) \\ &= E[6T] + \operatorname{Var}(6T) \\ &= 6E[T] + 6^2 \operatorname{Var}(T) \\ &= 6 \cdot 10 + 36 \cdot \frac{25}{3} \\ &= 360.\end{aligned}$$

6. Two individuals, A and B, both require kidney transplants. Without a new kidney, A will die after an exponential time with rate μ_A, and B will die after an exponential time with rate μ_B. New kidneys arrive in accordance with a Poisson process with rate λ. The first kidney will go to A (or to B if A is dead but B is still alive), and the second kidney will go to B (if B is still alive).

   (a) Find the probability that A receives a new kidney.

Solution: Let L_A be the lifetime of A without a new kidney, and let T_1 be the arrival time of the first kidney. Then we want $P\{T_1 < L_A\}$. Since T_1 and L_A are independent exponential random variables with rates λ and μ_A, respectively, we have

$$P\{T_1 < L_A\} = \frac{\lambda}{\lambda + \mu_A}.$$

   (b) Find the probability that B receives a new kidney.

Solution: Let L_B be the lifetime of B without a new kidney, let T_2 by the interarrival time between the first and second kidneys, and let $S_2 = T_1 + T_2$ be the arrival time of the second kidney. There are two ways in which B could get a new kidney:

- A dies before the first kidney arrives, and the first kidney arrives before B dies. Call this event E.
- The first kidney arrives before either A or B dies (so that A gets the first kidney), and then the second kidney arrives before B dies. Call this event F.

Since these events are mutually exclusive (since A dies in the first case and not in the second), the probability that B gets a new kidney is given by $P(E) + P(F)$.
First we compute $P(E)$. Notice that the event E occurs exactly when

$$L_A < T_1 < L_B.$$

Thus we have

$$\begin{aligned}P(E) &= P\{L_A < T_1 < L_B\} \\ &= P\{L_A = \min\{L_A, T_1, L_B\}\} \cdot P\{T_1 < L_B \mid L_A = \min\{L_A, T_1, L_B\}\}.\end{aligned}$$

Now

$$P\{L_A = \min\{L_A, T_1, L_B\}\} = \frac{\mu_A}{\lambda + \mu_A + \mu_B},$$

and by the memoryless property,

$$P\left\{T_1 < L_B \mid L_A = \min\{L_A, T_1, L_B\}\right\} = P\{T_1 < L_B\} = \frac{\lambda}{\lambda + \mu_B}.$$

So

$$P(E) = \frac{\mu_A}{\lambda + \mu_A + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_B}.$$

Now we compute $P(F)$. The event F occurs if and only if both of the following occur:

- The first kidney arrives before either A or B dies, i.e., $T_1 = \min\{T_1, L_A, L_B\}$. (When the first kidney arrives, it will go to A.)
- The second kidney arrives before B dies.

Thus, by the multiplication rule, we have

$$P(F) = P\{T_1 = \min\{T_1, L_A, L_B\}\} \cdot P\left\{S_2 < L_B \mid T_1 = \min\{T_1, L_A, L_B\}\right\}.$$

Now

$$P\{T_1 = \min\{T_1, L_A, L_B\}\} = \frac{\lambda}{\lambda + \mu_A + \mu_B}.$$

By the memoryless property, given that B survives until the first kidney arrives, the remaining lifetime of B after the time when the first kidney arrives is still exponentially distributed with rate μ_B. So we have

$$P\left\{S_2 < L_B \mid T_1 = \min\{T_1, L_A, L_B\}\right\} = P\{T_2 < L_B\} = \frac{\lambda}{\lambda + \mu_B}.$$

Therefore, we conclude that the probability that B gets a new kidney is

$$\begin{aligned} P(E) + P(F) &= \frac{\mu_A}{\lambda + \mu_A + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_B} + \frac{\lambda}{\lambda + \mu_A + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_B} \\ &= \frac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_B}. \end{aligned}$$