Summary

- A random variable X is *continuous* if there is a nonnegative function f, called the *probability density function* of X, such that for any set B of real numbers, we have

$$P\{X \in B\} = \int_B f(x)\,dx$$

Let X be a continuous random variable.

- The cumulative distribution function of X is given by

$$F(x) = P\{X \leq x\} = \int_0^x f(t)\,dt$$

- By the Fundamental Theorem of Calculus, we have

$$\frac{d}{dx}F(x) = f(x)$$

- The *expected value* of X is given by

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

- If g is any function, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

- The *variance* of X is defined by

$$\mathrm{Var}(X) = E\left[(X-\mu)^2\right], \quad \text{where} \quad \mu = E[X].$$

  and can be computed as

$$\mathrm{Var}(X) = E\left[X^2\right] - \left(E[X]\right)^2$$

We studied three different types of continuous random variables that arise frequently in practice.

- We say that X is a *uniform* random variable on the interval (α, β) if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if} \quad \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$

  - We have

$$E[X] = \frac{\alpha+\beta}{2} \qquad \mathrm{Var}(X) = \frac{(\beta-\alpha)^2}{12}$$

- The random variable X is "uniform" in the sense that the value of X is equally likely to lie in any two subintervals of (α, β) of the same length.

- We say that X is a *normal* random variable with parameters μ and σ^2 if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \text{for all } x \in \mathbb{R}$$

  - We have

$$E[X] = \mu \qquad \text{Var}(X) = \sigma^2$$

  - If X is normal with mean μ and variance σ^2, then the random variable Z, defined by

$$Z = \frac{X-\mu}{\sigma},$$

    is a *standard* normal random variable – it is normally distributed with mean 0 and variance 1.
  - Probabilities involving X can be expressed in terms of the standard normal variable Z. One can use a table to find values of the cdf of Z.
  - When n is large, a binomial random variable with parameters (n, p) is approximated well by a normal random variable with mean $\mu = np$ and variance $\sigma^2 = np(1-p)$.
  - Many random quantities are (approximately) normally distributed. The *Central Limit Theorem*, which we will prove towards the end of the course, provides a theoretical basis for this empirical observation.

- We say that X is an *exponential* random variable with parameter $\lambda > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

  - We have

$$E[X] = \frac{1}{\lambda} \qquad \text{Var}(X) = \frac{1}{\lambda^2}$$

  - If X is an exponential random variable, then X is *memoryless*, i.e., we have

$$P\{X > s+t \mid X > t\} = P\{X > s\} \quad \text{for all } s,t \geq 0$$

  - Exponential random variables are often used to model the length of time until a certain event occurs.