MATH 3030, Winter 2023

Instructor: Lucas Mol

Written Assignment 6

Due: Thursday, March 30

Guidelines:

- Try to explain your reasoning as clearly as possible. Feel free to reference theorems, lemmas, or anything else in the notes by number. For example, you could say "By Theorem 2.1.4,..."
- Please make sure that the work you submit is organized. Complete rough work on a different page, and then write up a good copy of your final solutions. Make sure to indicate each problem number clearly.
- Working with others is allowed (and encouraged!) but copying is strictly prohibited, and is a form of academic misconduct. Every student should write up their own good copy. Your work should not be the same, word-for-word, as any other student's work.
- The more work you put into these assignments, the more you will get out of them. **Please think about the problems before looking for solutions online.**

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1. When Lucas buys a new set of bike tires, he repairs all flat tires for the first s years that he owns them, where s is some constant. After these s years have passed, he replaces the tires the next time he catches a flat.

   (a) Suppose that the time between successive flats is exponential with rate λ per year. What is the rate at which Lucas buys new tires?

   Solution: This is a renewal process where a renewal occurs every time Lucas buys a new set of tires. Let T be the length of a cycle. Then we have

   $$T = s + A,$$

   where A is the additional time after s years until the next flat. By the memoryless property of the exponential distribution, we see that A is exponential with rate λ. Therefore, we have

   $$E[T] = s + E[A] = s + \tfrac{1}{\lambda}.$$

   So by Theorem 7.3.4, the average rate at which Lucas buys new tires is

   $$\frac{1}{E[T]} = \frac{1}{s + \frac{1}{\lambda}} = \frac{\lambda}{\lambda s + 1}.$$

(b) Suppose that repairing a flat costs c, while buying a new set of tires costs C. (Assume that when he buys a new set of tires, he does not repair the old ones.) Find the long run average amount per year that Lucas spends on buying and repairing tires.

Solution: By the renewal reward theorem, the long run average amount per year that Lucas spends on tires is given by

$$\frac{E[\text{amount spent in a cycle}]}{E[\text{length of a cycle}]}.$$

Now, the amount spent in a cycle is given by

$$cN + C,$$

where N is the number of flats that Lucas gets in the first s years. (Note that after the first s years, he will buy new tires the next time he gets a flat, and spend C.) Since the length of time between successive flats is exponential with rate λ, they occur according to a Poisson process with rate λ, and the number of flats in the first s years is Poisson with mean λs. Hence we have

$$E[\text{amount spent in a cycle}] = E[cN + C] = cE[N] + C = c\lambda s + C.$$

Therefore, the long run average amount per year that Lucas spends on tires is

$$\frac{\lambda(c\lambda s + C)}{\lambda s + 1}.$$

(c) Give your answers to parts (a) and (b) in the special case that $s = 3$, $\lambda = 2$, $c = \$6$, and $C = \$160$. Do they seem reasonable?

Solution: The rate at which Lucas buys new tires is

$$\frac{\lambda}{\lambda s + 1} = \frac{2}{2 \cdot 3 + 1} = \frac{2}{7}.$$

So Lucas buys a new set of tires twice every seven years, or once every 3.5 years, on average. This seems reasonable, since the expected time after 3 years until the next flat is $\frac{1}{\lambda} = \frac{1}{2}$ years.
The long run average amount per year that Lucas spends on tires is

$$\frac{\lambda(c\lambda s + C)}{\lambda s + 1} = \frac{2(\$6 \cdot 2 \cdot 3 + \$160)}{2 \cdot 3 + 1} = \$56.$$

This seems reasonable. He will spend \$160 once every 3.5 years on average, and he will also repair an average number of 6 flats every 3.5 years, for a total average cost of $\$160 + 6 \cdot \$6 = \$196$ every 3.5 years, for an average cost of $\$196/3.5 = \$56$ per year.

(d) **Challenge problem (optional):** Suppose instead that when Lucas buys a new set of tires, the time until the first flat is exponential with rate μ per year. After the first flat, the time between successive flats is exponential with rate λ per year. What is the rate at which Lucas buys new tires?

Hint: Condition on whether or not Lucas gets a flat in the first s years that he owns a new set of tires.

2. Bikes are not allowed on the Confederation Bridge between New Brunswick and PEI. Instead, cyclists are taken over the bridge on a bus which drives back and forth across the bridge, picking up and dropping off cyclists. Suppose that on each side of the bridge, cyclists arrive according to a Poisson process with rate $\lambda = 6/\text{hr}$ (or $\frac{1}{10}/\text{min}$). The time that the bus takes to complete a one-way trip across the bridge (in either direction) has mean 10 minutes.

   (a) Suppose that the time it takes to load/unload has mean 5 minutes (regardless of the number of cyclists that are loading/unloading). Suppose that the bus makes trips back and forth as quickly as possible. In the long run, what is the average rate of round trips that the bus makes? What is the average number of cyclists on each trip across the bridge?

   Solution: Say that a renewal occurs every time the bus leaves the NB side. (This is just one of several possible ways to define when the renewals occur! The important thing is that one renewal occurs each time the bus completes a round trip.) Let T be the length of a round trip. By the linearity of expectation, we have

   $$E[T] = 10 + 5 + 10 + 5 = 30 \text{ minutes},$$

   or $E[T] = 0.5$ hours. So by Theorem 7.3.4, the rate at which round trips are completed is

   $$\frac{1}{0.5\,\text{hr}} = 2/\text{hr}.$$

   In order to find the average number of cyclists on each trip across the bridge, note that we can simply find the average number of cyclists that board the bus on the NB side—the same thing is happening (probabilistically) on the PEI side, just offset by 15 minutes. Let N be the number of cyclists that are on the bus when it leaves the NB side in a particular cycle, and let T be the length of the preceding cycle. We find $E[N]$ by conditioning on T. We have

   $$E\left[N \mid T = t\right] = \lambda t,$$

   since cyclists arrive according to a Poisson process with rate λ. So $E\left[N \mid T\right] = \lambda T$, and by the law of total expectation, we have

   $$E[N] = E[E\left[N \mid T\right]] = E[\lambda T] = \lambda E[T] = 6/\text{hr} \cdot 0.5\,\text{hr} = 3.$$

   Therefore, there are an average of 3 cyclists on each trip across the bridge.

   (b) Now suppose that the operator of the bus wants to reduce their costs. They know that the bus can reliably make a round trip every 40 minutes. They want to know if making trips even less frequently would be cheaper. So suppose that instead of making trips back and forth as quickly as possible (as in the previous part), the bus operates on a fixed schedule, and makes a round trip every s minutes, where $s \geq 40$ is some constant. They incur the following costs:

- An average cost of \$15 per one-way trip across the bridge in fuel and maintenance costs for the bus.
- A cost of \$0.50/minute to pay the bus driver (whether or not they are driving).
- A cost of \$0.10 for each minute that each cyclist spends waiting for the bus.

What value of s will minimize the long run average cost?

Hint: Given that n events of a Poisson process have occurred by time s, the times at which the events occur, considered as unordered random variables, are distributed independently and uniformly over $(0, s)$.

Solution: By the renewal reward theorem, the long run average cost is given by

$$\frac{E[\text{cost in a cycle}]}{E[\text{length of a cycle}]}$$

The length of cycle is now simply s—it is constant! The expected cost in a cycle is given by

$$\begin{aligned}&E[\text{cost of fuel and maintenance}] + E[\text{cost of bus driver}] + E[\text{cost of waiting cyclists}]\\ &= 2 \cdot 15 + 0.5s + 0.1 \cdot 2E[W],\end{aligned}$$

where W is the total amount of time that cyclists on *one side* of the bridge spend waiting. (We multiply by 2 so that we count the waiting time of the cyclists on the other side of the bridge as well. Note that the same thing (probabilistically) is going on over in PEI, just at an offset time.) To find $E[W]$, we condition on N, the number of cyclists that arrive over the length s of a round trip. We have

$$E\left[W \mid N = n\right] = n \cdot \frac{s}{2},$$

since given n Poisson arrivals in an interval of length s, the conditional distributions of the arrival times are uniform on $(0, s)$. So $E\left[W \mid N\right] = N \cdot \frac{s}{2}$, and by the law of conditional expectation, we have

$$E[W] = E[E\left[W \mid N\right]] = E[N] \cdot \frac{s}{2} = \lambda s \cdot \frac{s}{2} = \frac{\lambda s^2}{2}$$

Therefore, we have

$$E[\text{cost in a cycle}] = 30 + 0.5s + 0.1 \cdot 2 \cdot \lambda s^2/2 = 30 + 0.5s + 0.01s^2,$$

and the long run average cost is given by

$$C(s) = \frac{30}{s} + 0.5 + 0.01s.$$

Differentiating, we find

$$C'(s) = -30s^{-2} + 0.01.$$

Solving $C'(s) = 0$, we find

$$\begin{aligned} 0.01 = 30s^{-2} \quad &\Rightarrow \quad 0.01s^2 = 30 \\ &\Rightarrow \quad s^2 = 3000 \\ &\Rightarrow \quad s = \pm 54.77. \end{aligned}$$

So $s = 54.77$ is the only critical point in the interval $[40, \infty)$ that we care about. Further, we have

$$C''(s) = 60s^{-3},$$

which is positive for all $s \geq 40$. So the global minimum average cost on $[40, \infty)$ is attained when $s = 54.77$. Therefore, to minimize the average cost, the bus should make a round trip approximately every 55 minutes. ☺

3. Lucas lives exactly 3km from school. His bike ride up the hill to school takes an average of 15 minutes with a standard deviation of 3 minutes, while his bike ride back down the hill at the end of the day takes an average of 5 minutes with a standard deviation of 2 minutes. (Assume that everything is independent, and that this is all of the biking that Lucas does.)

   Note: The first two parts have intuitive answers, but try to frame your solutions in terms of renewal theory!

   (a) What long-run proportion of the time that Lucas is biking is he biking uphill?

   Solution: Say that a renewal occurs every time that Lucas leaves home on his bike. Then we have an alternating renewal process, where Lucas bikes uphill (system on) and then downhill (system off) in each cycle. Let π_{up} and π_{down} denote the long-run proportions of time that Lucas is biking uphill and downhill, respectively. Similarly, let μ_{up} and μ_{down} denote the average times that Lucas bikes uphill and downhill in a cycle, respectively. Then by Proposition 7.5.3, we have

$$\pi_{\text{up}} = \frac{\mu_{\text{up}}}{\mu_{\text{up}} + \mu_{\text{down}}} = \frac{15}{15+5} = \frac{3}{4},$$

   and

$$\pi_{\text{down}} = 1 - \pi_{\text{up}} = \frac{1}{4}.$$

   (b) What is Lucas' long-run average speed?

   Solution: Lucas covers a distance of 6 km in every cycle. Consider this the "reward" for each cycle. Let $D(t)$ be the reward earned by time t, i.e., the distance Lucas has travelled by time t. Then Lucas' long-run average speed is given by

$$\lim_{t \to \infty} \frac{D(t)}{t}.$$

By the renewal reward theorem, we have

$$\lim_{t\to\infty} \frac{D(t)}{t} = \frac{E[\text{distance travelled in a cycle}]}{E[\text{length of a cycle}]} = \frac{6\,\text{km}}{\frac{1}{3}\,\text{hr}} = 18\,\text{km/h}.$$

Note: Another (perhaps more intuitive) solution is to weight Lucas' average uphill and downhill speeds by the respective long-run proportions. We then calculate Lucas' average speed as

$$12\,\text{km/h} \cdot \pi_{\text{up}} + 36\,\text{km/h} \cdot \pi_{\text{down}} = 12\,\text{km/h} \cdot \tfrac{3}{4} + 36\,\text{km/h} \cdot \tfrac{1}{4} = 18\,\text{km/h}.$$

We obtain the same answer this way!

(c) Suppose that there are 60 school days in a semester. Approximate the probability that Lucas spends more than 18 hours biking to school over the course of the semester.

Solution: Recall that S_n denotes the time of the nth renewal, and $N(t)$ denotes the number of renewals by time t. Recall further that for all $n \geq 1$ and $t \geq 0$, we have

$$P\{S_n \geq t\} = P\{N(t) \leq n\}.$$

We work in units of minutes to line up with our parameters. The probability that Lucas spends more than 18 hours, or 1080 minutes, biking to school over the course of 60 days is

$$P\{S_{60} > 1080\} = P\{N(1080) < 60\}$$

By the central limit theorem for renewal processes, the random variable $N(t)$ is approximately normally distributed with mean $\frac{t}{\mu}$ and variance $t\sigma^2/\mu^3$, where μ and σ^2 are the mean and variance of the times between successive renewals. Here we have

$$\mu = 20\,\text{minutes}$$

and

$$\sigma^2 = \text{Var(uphill time)} + \text{Var(downhill time)} = 3^2 + 2^2 = 13\,\text{minutes}^2,$$

where we used the fact that the uphill and downhill times are independent. So in this case, $N(1080)$ is approximately normal with mean

$$\frac{t}{\mu} = \frac{1080}{20} = 54$$

and variance

$$\frac{t\sigma^2}{\mu^3} = 1.755$$

Hence, we can compute the desired probability as follows:

$$\begin{aligned} P\{S_{60} > 1080\} &= P\{N(1080) < 60\} \\ &= P\left\{\frac{N(1080) - 54}{\sqrt{1.755}} < \frac{60 - 54}{\sqrt{1.755}}\right\} \\ &\approx P\{Z < 4.529\} \\ &= 1. \end{aligned}$$

In other words, he's almost certainly going to bike more than 18 hours in a semester. ☺