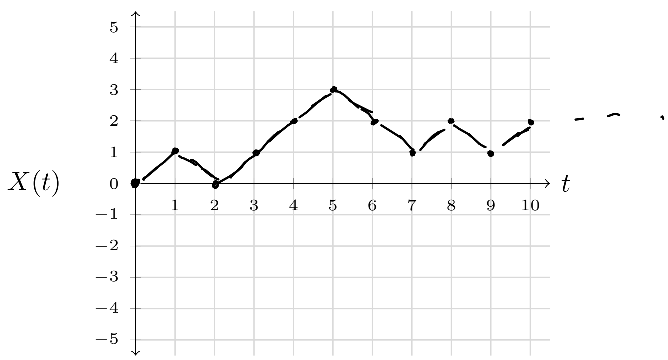
In 1827, a botanist named Robert Brown noticed that microscopic pollen grains suspended in a drop of water would constantly move in a seemingly random manner, changing direction abruptly and zigzagging back and forth. Similar *Brownian motion* was detected by other scientists whenever very small particles were suspended in a fluid. Albert Einstein gave the first satisfactory explanation of this phenomenon in 1905, arguing that the "random" motion is due to the continual bombardment of the small particles by the molecules of the surrounding fluid. The mathematical model of Brownian motion that we will study was developed by Norbert Wiener in a series of papers from 1918 to 1923.

10.1 Brownian Motion

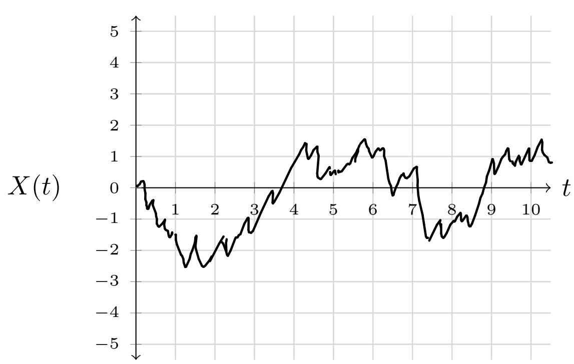
Recall that the symmetric random walk is the discrete-time Markov chain whose state space is $\mathbb{Z}$, and whose transition probabilities are given by

$$P_{i,i+1} = P_{i,i-1} = \tfrac{1}{2} \quad \text{for all } i \in \mathbb{Z}.$$

If we let $X(t)$ denote the state of the process after time t, then the graph of $X(t)$ might look something like this:



Now suppose that we speed up this process by taking smaller and smaller steps in smaller and smaller time intervals. Now the graph of $X(t)$ might look something like this:



We obtain a model of Brownian motion by letting the length Δx of each step and the length of time Δt between successive steps approach 0 in a carefully chosen manner.

Suppose that we start at 0, and that after every (small) unit of time Δt, we take a (small) step of size Δx, either to the left or to the right, with equal probabilities.

- Let $X(t)$ denote our position at time t. Then

$$X(t) = \Delta x \cdot X_1 + \Delta x \cdot X_2 + \cdots + \Delta x \cdot X_{\lfloor t/\Delta t \rfloor}$$

$$= \Delta x \left[X_1 + X_2 + \cdots + X_{\lfloor t/\Delta t \rfloor} \right]$$

  where $X_1, X_2, \ldots,$ are independent with

$$P\{X_i = 1\} = P\{X_i = -1\} = \tfrac{1}{2}.$$

- Note that

$$E[X_i] = 1 \cdot P\{X_i = 1\} + (-1) \cdot P\{X_i = -1\} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = 1^2 \cdot P\{X_i = 1\} + (-1)^2 \cdot P\{X_i = -1\} = 1$$

- It follows that

$$E[X(t)] = 0$$

$$\text{Var}(X(t)) = \text{Var}\left(\Delta x \left[X_1 + X_2 + \cdots + X_{\lfloor t/\Delta t \rfloor} \right] \right) = (\Delta x)^2 \cdot \left\lfloor \frac{t}{\Delta t} \right\rfloor$$

What happens when we let Δx and Δt approach zero?

- If we let $\Delta x = \Delta t$, then we find

$$\lim_{\Delta t \to 0} E[X(t)] = \lim_{\Delta t \to 0} 0 = 0$$

$$\lim_{\Delta t \to 0} \text{Var}(X(t)) = \lim_{\Delta t \to 0} (\Delta t)^{\cancel{2}} \cdot \left\lfloor \frac{t}{\cancel{\Delta t}} \right\rfloor = 0$$

  In this case, we have $X(t) = 0$ with probability 1. This is not a great model of motion!

- If we instead let $\Delta x = \sigma\sqrt{\Delta t}$ for some positive constant σ, then we find

$$\lim_{\Delta t \to 0} E[X(t)] = 0$$

$$\lim_{\Delta t \to 0} \text{Var}(X(t)) = \lim_{\Delta t \to 0} \left(\sigma \sqrt{\Delta t} \right)^2 \cdot \left\lfloor \frac{t}{\Delta t} \right\rfloor = \lim_{\Delta t \to 0} \sigma^2 \Delta t \cdot \left\lfloor \frac{t}{\Delta t} \right\rfloor$$

$$= \lim_{\Delta t \to 0} \sigma^2 t$$

$$= \sigma^2 t$$

  This is more like it!

So suppose now that

$$X(t) = \lim_{\Delta t \to 0} \sigma \sqrt{\Delta t} \left[X_1 + X_2 + \cdots + X_{\lfloor t/\Delta t \rfloor} \right]$$

The following properties seem plausible:

- For all $t > 0$, the random variable $X(t)$ is normal with mean 0 and variance $\sigma^2 t$.
  - This follows from

    the Central Limit Theorem!

- The process $\{X(t), t \geq 0\}$ has independent increments!
  - That is, for all $0 < t_1 < t_2 < \cdots < t_n$

    $X(t_1), X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$

    are independent.
- The process $\{X(t), t \geq 0\}$ has stationary increments!
  - That is, the distribution of $X(t+s) - X(t)$ does not depend on t. (The distances travelled in any two intervals of the same length have the same distribution.)

Definition 10.1.1. A stochastic process $\{X(t), t \geq 0\}$ is said to be a *Brownian motion process* if

(i) $X(0) = 0$

(ii) $\{X(t), t \geq 0\}$ has independent and stationary increments

(iii) $X(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

In the special case that $\sigma = 1$, the process is called standard Brownian motion.

- Note that any Brownian motion process $\{X(t), t \geq 0\}$ can be converted to the standard Brownian motion process by letting

$$B(t) = \frac{X(t)}{\sigma}.$$

$$\begin{aligned} Var(B(t)) &= \frac{1}{\sigma^2} Var(X(t)) \\ &= \frac{1}{\sigma^2} \cdot \sigma^2 t \\ &= t \end{aligned}$$

- Thus, we focus on standard Brownian motion, and suppose throughout this chapter that $\sigma = 1$ unless otherwise stated.

The interpretation of Brownian motion as a limit of random walks suggests that $X(t)$ should be a continuous function of time.

Theorem 10.1.2. *With probability* 1,

$$\lim_{h\to 0} X(t+h) - X(t) = 0,$$

i.e., $X(t)$ is a continuous function of t.

Plausibility argument: $X(t+h) - X(t)$ reprents the distance travelled in an interval of length h, so it is normally distributed with mean 0 and variance h. (recall that we assume $\sigma = 1$) So as $h \to 0$, it seems plausible that $X(t+h) - X(t)$ is normal with mean 0 and variance 0, i.e., $X(t+h) - X(t) = 0$ with probability 1.

Although $X(t)$ is continuous, remember that it models the haphazard zigzag motion of a small particle suspended in a fluid.

> One realises from such examples how near the mathematicians are to the truth in refusing, by a logical instinct, to admit the pretended geometrical demonstrations, which are regarded as experimental evidence for the existence of a tangent at each point of a curve.
>
> – Jean Baptiste Perrin

Theorem 10.1.3. *With probability* 1,

$$\lim_{h\to 0} \frac{X(t+h) - X(t)}{h} \quad \textit{does not exist,}$$

i.e., $X(t)$ is nowhere differentiable.

Plausibility argument: The expected of $\frac{X(t+h) - X(t)}{h}$ is $\frac{0}{h} = 0$;

but

$$\begin{aligned} \mathrm{Var}\left(\frac{X(t+h) - X(t)}{h}\right) &= \frac{1}{h^2} \mathrm{Var}\left(X(t+h) - X(t)\right) \\ &= \frac{1}{h^2} \cdot h \\ &= \frac{1}{h}. \end{aligned}$$

So as $h \to 0$, the variance explodes! It seems plausible that $\lim_{h\to 0} \frac{X(t+h) - X(t)}{h}$ does not exist.

10.2 Some Gross Calculations

Recall that a normal random variable with mean μ and variance σ^2 has probability density function

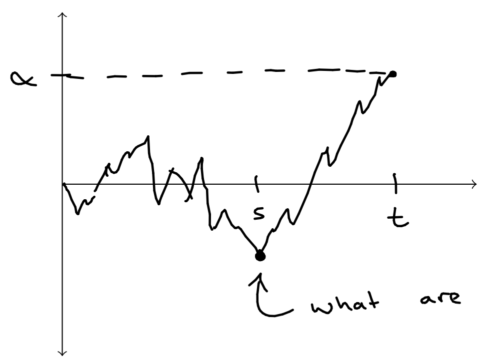
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Let $\{X(t),\, t \geq 0\}$ be the standard Brownian motion process.

- Then $X(t)$ is normal with mean 0 and variance t.
- So the probability density function of $X(t)$, denoted f_t, is given by

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

Example 10.2.1 (Conditional distribution of the past given the present)**.** For $0 < s < t$, find the conditional distribution of $X(s)$ given that $X(t) = \alpha$. What are $E[X(s) \mid X(t) = \alpha]$ and $\mathrm{Var}(X(s) \mid X(t) = \alpha)$?



what are probabilities involving $X(s)$ given that $X(t) = \alpha$

We will show that, given $X(t) = \alpha$, the conditional distribution of $X(s)$ is normal with mean $\alpha \cdot \frac{s}{t}$ and variance $\frac{s}{t}(t-s)$.

We have $f_{s|t}(x \mid \alpha) = \dfrac{f_{s,t}(x, \alpha)}{f_t(\alpha)}$ ← joint density of $X(s)$ and $X(t)$; ← density of $X(t)$

$X(s) = x$ given $X(t) = \alpha$

$$= \frac{f_s(x)\, f_{t-s}(\alpha - x)}{f_t(\alpha)}$$

← by indep. and stationary increments!

normal w/ mean 0 and variance s

normal with mean 0 and variance $t-s$

constant!

$$= K_1 \cdot e^{-x^2/2s} \cdot e^{-(\alpha - x)^2/2(t-s)}$$

$$= K_2 \cdot e^{-\left(x - \underbrace{\alpha \cdot \frac{s}{t}}_{\mu}\right)^2 / \left(2\underbrace{s(t-s)/t}_{\sigma^2}\right)}$$

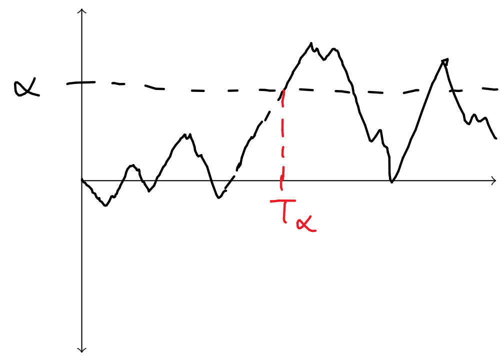
by gross algebra.

K_1 and K_2 are constants that do not depend on x.

So given $X(t) = \alpha$, $X(s)$ is normal with mean $\alpha \cdot \frac{s}{t}$ and variance $\frac{s(t-s)}{t}$.

Example 10.2.2 (Hitting times)**.** For $\alpha > 0$, let T_α denote the first time that the standard Brownian motion process "hits" α, i.e.,

$$T_\alpha = \min\{t \geq 0 : X(t) = \alpha\}.$$



We will compute $P\{T_\alpha \leq t\}$ in a somewhat strange way—we consider $P\{X(t) \geq \alpha\}$, and condition on whether or not $T_\alpha \leq t$.

$$\begin{aligned} P\{X(t) \geq \alpha\} &= P\{X(t) \geq \alpha \mid T_\alpha \leq t\} \cdot P\{T_\alpha \leq t\} \\ &\quad + P\{X(t) \geq \alpha \mid T_\alpha > t\} \cdot P\{T_\alpha > t\}. \\ &= P\{X(t) \geq \alpha \mid T_\alpha \leq t\} \cdot P\{T_\alpha \leq t\}, \end{aligned}$$

since $P\{X(t) \geq \alpha \mid T_\alpha > t\} = 0$.

Note that $P\{X(t) \geq \alpha \mid T_\alpha \leq t\} = \frac{1}{2}$.

This is because we hit α at some time before t. By symmetry, we are equally likely to be above or below α at time t.

So we obtain

$$\begin{aligned} P\{T_\alpha \leq t\} &= 2 \cdot P\{X(t) \geq \alpha\} \\ &= 2 \cdot \frac{1}{\sqrt{2\pi t}} \cdot \int_\alpha^\infty e^{-x^2/2t}\, dx \\ &= \frac{2}{\sqrt{2\pi}} \int_{\alpha/\sqrt{t}}^\infty e^{-y^2/2}\, dy \end{aligned}$$

since $X(t)$ is normal with mean 0 and variance t.

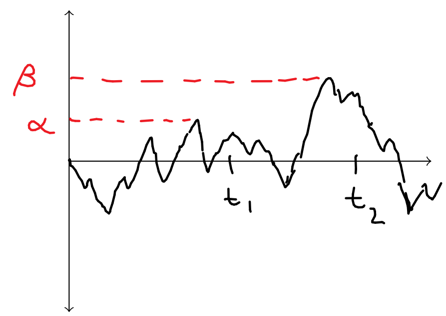
by making the substitution $y = \frac{x}{\sqrt{t}}$.

For $\alpha < 0$, the distribution of T_α is the same as that of $T_{-\alpha}$. Hence, for all $\alpha \neq 0$, we have

$$P\{T_\alpha \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{|\alpha|}{\sqrt{t}}}^\infty e^{-y^2/2}\, dy$$

Example 10.2.3 (Maximum values)**.** For $t > 0$, let M_t denote the maximum value that the Brownian motion process attains in the interval $[0, t]$, i.e.

$$M_t = \max_{0 \leq s \leq t} X(s)$$



$M_{t_1} = \alpha$

$M_{t_2} = \beta$

For $\alpha > 0$, find $P\{M_t \geq \alpha\}$.

We have $P\{M_t \geq \alpha\} = P\{T_\alpha \leq t\}$ (by continuity of $X(t)$)

$$= \frac{2}{\sqrt{2\pi}} \int_{\frac{\alpha}{\sqrt{t}}}^{\infty} e^{-y^2/2}\, dy$$

for all $\alpha \geq 0$.

Example 10.2.4 (Gambler's ruin)**.** For $\alpha, \beta > 0$, what is the probability that the standard Brownian motion process goes up α before it goes down β?

- We make use of the interpretation of Brownian motion as the limit of a symmetric random walk.
- A gambler starting with i units that bets 1 unit at each time step, and is equally likely to win or lose each bet, gets to N before they lose it all with probability $\frac{i}{N}$.
  - In other words, the probability that the symmetric random walk starting at 0 goes up $A = N - i$ before it goes down $B = i$ is

$$\frac{B}{A + B}$$

- Brownian motion is the limit (as $\Delta x \to 0$) of a symmetric random walk starting at 0 that is equally likely to go up or down a distance Δx at each step. So the probability that it goes up α before it goes down β is

$$\frac{\beta \Delta x}{\alpha \Delta x + \beta \Delta x} = \frac{\beta}{\alpha + \beta}$$

e.g. prob. up 2 before down 1 is $\frac{1}{3}$.

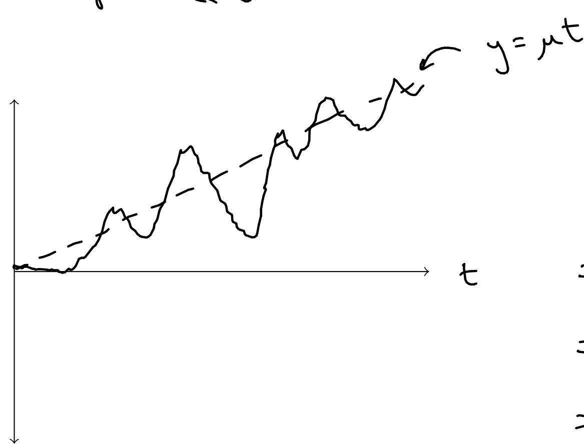
10.3 Variations on Brownian Motion

Definition 10.3.1. Let $\{B(t),\, t \geq 0\}$ be standard Brownian motion. The stochastic process $\{X(t),\, t \geq 0\}$ defined by

$$X(t) = \mu t + \sigma B(t)$$

is called a Brownian motion process with drift.

- μ is called the drift coefficient
- σ^2 is called the variance parameter



$E[X(t)] = \mu t$

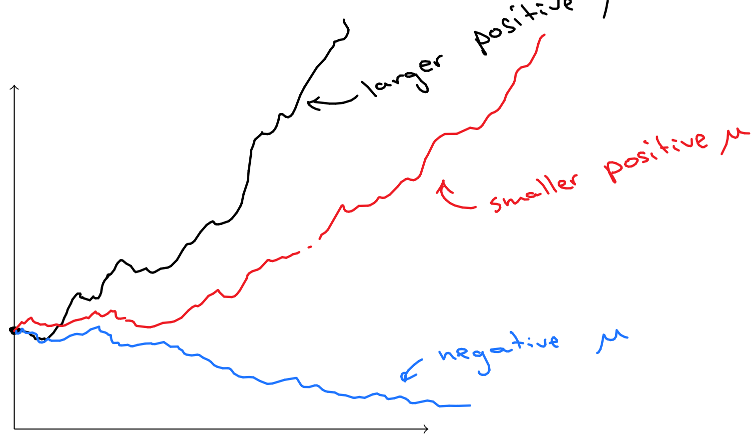
$$\begin{aligned} \mathrm{Var}(X(t)) &= 0 + \mathrm{Var}(\sigma B(t)) \\ &= \sigma^2 \, \mathrm{Var}(B(t)) \\ &= \sigma^2 t \end{aligned}$$

Definition 10.3.2. Let $\{X(t),\, t \geq 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. The stochastic process $\{Z(t),\, t \geq 0\}$ defined by

$$Z(t) = e^{X(t)}$$

is called a geometric Brownian motion process.

$Z(0) = e^{X(0)} = 1$



Recall that the moment generating function of a normal random variable W is given by

$$M_W(r) = E\left[e^{rW}\right] = e^{rE[W]+r^2\,\mathrm{Var}(W)/2}.$$

Observation 10.3.3. Let $\{Z(t),\, t \geq 0\}$ be a geometric Brownian motion process with

$$Z(t) = e^{X(t)},$$

where $X(t)$ is Brownian motion with drift coefficient μ and variance parameter σ^2. Then

$$E[Z(t)] = E\left[e^{X(t)}\right] = e^{\left(\mu + \frac{\sigma^2}{2}\right)t}$$

Proof. We know that $X(t)$ is normal with mean μt and variance $\sigma^2 t$.

So the moment generating function of $X(t)$ is

$$M_{X(t)}(r) = E\left[e^{rX(t)}\right] = e^{r\cdot\mu t + r^2\cdot\sigma^2 t/2}.$$

Substituting $r=1$ gives $E\left[e^{X(t)}\right] = e^{\mu t + \sigma^2 t/2} = e^{\left(\mu + \frac{\sigma^2}{2}\right)t}$

$\square$

For this reason, the parameter

$$\alpha = \mu + \tfrac{1}{2}\sigma^2 \qquad \leftarrow E[Z(t)] = e^{\alpha t}$$

is sometimes called the *drift parameter* of the geometric Brownian motion process. Note that it is possible to have $\alpha > 0$ and $\mu < 0$. Suppose that this is the case. Then we have

$$\lim_{t\to\infty} E[Z(t)] = \lim_{t\to\infty} e^{\alpha t} = \infty$$

Note however, that since $\mu < 0$, we have

$$X(t) = \mu t + \sigma B(t),$$

so $X(t)$ is "drifting" to the negative side. In fact, a consequence of the strong law of large numbers is that with probability 1, we have

$$\lim_{t\to\infty} X(t) = -\infty$$

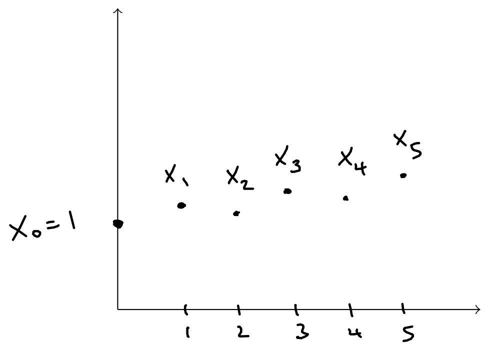
Hence, with probability 1, we have

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} \left[e^{X(t)}\right] = 0$$

So if $\mu < 0$ and $\alpha = \mu + \frac{1}{2}\sigma^2 > 0$, then $Z(t)$ is almost surely drifting towards 0, but its expected value is headed off to ∞. Weird!

Geometric Brownian motion is often used to model stock prices over time. It is a good model whenever one believes that percentage changes in a stock price are independent and identically distributed. We provide some justification for this statement below.

- Suppose for all $n \geq 0$ that X_n is the price of some stock at time n.
- For simplicity, suppose that $X_0 = 1$.



$$1.1 = \frac{X_n}{X_{n-1}} \implies 1.1\, X_{n-1} = X_n$$

- For all $n \geq 1$, let $Y_n = X_n / X_{n-1}$
- Roughly speaking, Y_n represents the percentage change in the price of the stock from time $n-1$ to time n. E.g.,
  - If $Y_n = 1.1$, then the stock price went up by 10% from time n-1 to time n.
  - If $Y_n = 0.9$, then the stock price went down by 10% from time n-1 to time n.
- Assume that $Y_1, Y_2, \ldots$ are independent and identically distributed.
- Since $Y_n = X_n / X_{n-1}$, we obtain

$$X_n = Y_n X_{n-1}$$

$$Y_{n-1} = \frac{X_{n-1}}{X_{n-2}} \implies X_{n-1} = Y_{n-1} X_{n-2}$$

- Iterating gives

$$\begin{aligned} X_n &= Y_n X_{n-1} \\ &= Y_n \cdot Y_{n-1} \cdot X_{n-2} \\ &= Y_n \cdot Y_{n-1} \cdot Y_{n-2} \cdot X_{n-3} \\ &\;\vdots \\ &= Y_n \cdot Y_{n-1} \cdot Y_{n-2} \cdot \ldots Y_1 \cdot X_0 \end{aligned}$$

$$Y_{n-2} = \frac{X_{n-2}}{X_{n-3}} \implies X_{n-2} = Y_{n-2} X_{n-3}$$

$= 1$

- Therefore, we have $\ln(X_n) = \ln(Y_1) + \ln(Y_2) + \ldots + \ln(Y_n)$.
- Note that $\ln(Y_1), \ln(Y_2), \ldots$ are independent and identically distributed.
- If the time between successive steps is small, then the process $\{\ln(X_n),\, n \geq 0\}$ can be approximated by Brownian motion with drift!
- It follows that $\{X_n,\, n \geq 0\}$ can be approximated by geometric Brownian motion.