Testing

8.3 Most powerful tests

Power = P(rejecting when H_0 is false)

- α = P(" " " " true)

We've mostly focusd on LRTs for good reason. Although we can find other tests, that is, some regions we reject a null if a statistic is in them, LRTs maximize the probability of making the correct rejection under some conditions. This is the Neyman-Pearson Lemma. Herein, rejection regions will be defined as $R = \overline{C}$ because it's easier to type.

Theorem 1: Neyman Pearson Lemma

Consider the LRT test, T, where

$$H_0 : \theta \in \omega_0 = \{\theta_0\} \quad \text{vs} \quad H_1 : \theta \in \omega_1 = \{\theta\}$$

H_0 true

with significance level $\alpha = P(\boldsymbol{X} \in R; \theta = \theta_0)$, which rejects if and only if $\exists k, \forall \boldsymbol{X} \in R$

$$\Lambda(\theta_0; \boldsymbol{X}) = \frac{\mathcal{L}(\theta_0; \boldsymbol{X})}{\mathcal{L}(\theta_1; \boldsymbol{X})} \leq k$$

Then T is the most powerful test of all tests with significance at most α.

This says that if the LRT can be used with a single cutoff, k, with maximum type I error α, than no other test with its own comparable rejection between the two simple hypothes H_0 against H_1 can have smaller type II error (or have larger power, however you want to think of it). The proof will be familiar to what we've done several times in the course.

Suppose H_0 ~~tr~~ $\theta = \theta_0$

if $x \in R$

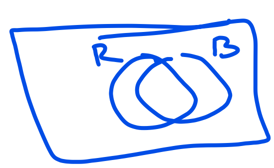
$$\frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\theta_1)} \leq k$$

$$\mathcal{L}(\theta_0) \leq k\mathcal{L}(\theta_1)$$

$$P(x \in R; \theta = \theta_0) \leq k\, P(x \in R; \theta = \theta_1)$$

6 $\quad P_0(R) \leq k\, P_1(R)$

$P_0(BR) \leq k\, P_1(BR) \quad (1)$



$\forall B$

if $x \in \bar{R}$

$$\frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\theta_1)} \ge k$$

$$\vdots$$

$$P_0(\bar{R}) \ge k P_1(\bar{R})$$

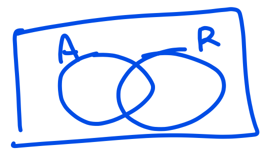
$$-P_0(\bar{R}) \le -k P_1(\bar{R})$$

$$-P_0(B\bar{R}) \le -k P_1(B\bar{R}) \quad ② \quad \forall B$$

pf of NP-Lemma. suppose we have another test where

$$P(x \in A; \theta = \theta_0) \le \alpha = P(x \in R; \theta = \theta_0)$$

$$P_0(A) \le P_0(R)$$



$$P_0(A\bar{R}) + \cancel{P_0(AR)} \le \cancel{P_0(AR)} + P_0(\bar{A}R)$$

$$0 \le P_0(\bar{A}R) - P_0(A\bar{R})$$

$$\le kP_1(\bar{A}R) - P_0(A\bar{R}) \quad \text{by ①}$$

$$\le kP_1(\bar{A}R) - kP_1(A\bar{R}) \quad \text{by ②}$$

$$0 \le \cdots \cdots \le kP_1(\bar{A}R) - kP_1(A\bar{R})$$

$$+kP_1(AR) \quad kP_1(A\bar{R}) \le kP_1(\bar{A}R) \quad +kP_1(AR)$$

$$\cancel{k}P_1(A) \le \cancel{k}P_1(R)$$

$$P_1(A) \le P_1(R)$$

$$P(X \in A; \theta = \theta_1) \le P(X \in R; \theta = \theta_1)$$

$\therefore$ Power of LRT $\ge$ Power of any other test with sig $\le \alpha$.

The Neyman-Pearson Lemma only directly applies to two simple hypotheses. We can use it repeatedly for an alternative that is composite to help us find a uniformly most powerful test.

Definition 1: Uniformly Most Powerful (UMP) test

If $\forall\theta_1 \in \omega_1$, T is the most powerful test between $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ then T is uniformly most powerful. for $\theta_1 \in \omega_1$

It should be surprise that to show a test is UMP, we invoke the Neyman Pearson Lemma for ever instance for a particular θ_1.

Example 1

Find a cutoff for a sufficient statistic for the exponential to be uniformly most powerful to test $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda > \lambda_0$ using the density for $f(x; \lambda) = \lambda e^{-\lambda x}$ (this is an easier form to work with). Hint: start with a single, fixed $\lambda_1 > \lambda_0$.

$\lambda \in \{\lambda_0\}$ all $\lambda > \lambda_0$

Consider 2 simple hypotheses

$H_0: \lambda = \lambda_0$ vs H_1 $\lambda = \lambda_1$ where $\lambda_1 > \lambda_0$

$$\Lambda = \frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\theta_1)} = \frac{\prod \lambda_0 e^{-\lambda_0 x_i}}{\prod \lambda_1 e^{-\lambda_1 x_i}} = \left(\frac{\lambda_0}{\lambda_1}\right)^n e^{(\lambda_1 - \lambda_0)\sum x_i}$$

let $T = \sum x$ be my sufficient stat.

we reject H_0 when $\Lambda \leq k$ for some k.

$$\left(\frac{\lambda_0}{\lambda_1}\right)^n e^{(\lambda_1-\lambda_0)T} \leq k$$

$$0 \leq e^{(\lambda_1-\lambda_0)T} \leq k\left(\frac{\lambda_0}{\lambda_1}\right)^{-n} \quad c_1$$

$$(\lambda_1 - \lambda_0)T \leq \log\left(k\left(\frac{\lambda_0}{\lambda_1}\right)^{-n}\right) \quad c_2$$

$$T \leq \frac{\log\left(k\left(\frac{\lambda_0}{\lambda_1}\right)^{-n}\right)}{\lambda_1 - \lambda_0}$$

$\therefore \lambda_1 > \lambda_0$

$\lambda_1 - \lambda_0 > 0$

$\therefore$ to reject H_0, $\sum x_i \leq k_2$ where $k_2 = $)

we can create a test where

$$\underbrace{\sum x_i \leq k_2} \iff \Lambda \leq k$$

Because $\sum x_i \leq k$ is equivalent to the LRT

this test is most powerful. By the NP Lemma

Because this test works for all $\lambda_1 > \lambda_0$, then

the test stays the same.

$\therefore$ the test $\sum x_i \leq k_2$ works for all $\lambda_1 \in \{\lambda > \lambda_0\}$

and because it's most powerful, then

$\boxed{\sum x_i \leq k_2}$ is a uniformly most powerful.

Example 2

Find a cutoff and show that it is UMP testing $H_0 : p_0$ against the alternative $p > p_0$ in a Bernoulli.