MATH 3030, Winter 2023

Instructor: Lucas Mol

Written Assignment 3

Due: Thursday, February 16

Guidelines:

- Try to explain your reasoning as clearly as possible. Feel free to reference theorems, lemmas, or anything else in the notes by number. For example, you could say "By Theorem 2.1.4,..."
- Please make sure that the work you submit is organized. Complete rough work on a different page, and then write up a good copy of your final solutions. Make sure to indicate each problem number clearly.
- Working with others is allowed (and encouraged!) but copying is strictly prohibited, and is a form of academic misconduct. Every student should write up their own good copy. Your work should not be the same, word-for-word, as any other student's work.
- The more work you put into these assignments, the more you will get out of them. **Please think about the problems before looking for solutions online.**

1. Recall that the odds of winning when betting on either red or black in roullette are $\frac{18}{38}$. Suppose that a gambler starts with \$10, and plays until they double their money or they lose it all. Find the probability that they double their money if they bet

   - \$1 on each game;
   - \$2 on each game;
   - \$5 on each game; and
   - \$10 on each game.

   Do you see why betting it all is the best strategy? ☺

Solution: From class, if the gambler starts with i units, then the probability that they make it to N units before they lose it all is given by

$$P_i = \frac{1-r^i}{1-r^N},$$

where $r = \frac{q}{p}$. In this problem, we have $r = \frac{20}{18} = \frac{10}{9}$.

If they bet \$1 on each game, then they start with 10 units, and we want the probability that they make it to 20 units before they lose it all, which is given by

$$\frac{1 - \left(\frac{10}{9}\right)^{10}}{1 - \left(\frac{10}{9}\right)^{20}} \approx 0.259.$$

If they bet \$2 on each game, then they start with 5 units, and we want the probability that they make it to 10 units before they lost it all, which is given by

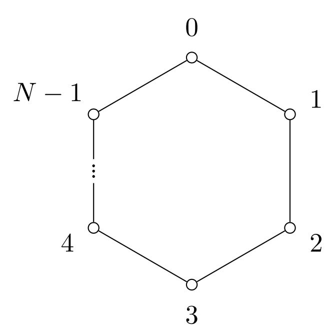
$$\frac{1 - \left(\frac{10}{9}\right)^{5}}{1 - \left(\frac{10}{9}\right)^{10}} \approx 0.371.$$

If they bet \$5 on each game, then they start with 2 units, and we want the probability that they make it to 4 units before they lost it all, which is given by

$$\frac{1 - \left(\frac{10}{9}\right)^{2}}{1 - \left(\frac{10}{9}\right)^{4}} \approx 0.448.$$

Finally, if they bet \$10 on each game, then they double their money before they lose it all if and only if they win the first game. This happens with probability $\frac{18}{38} \approx 0.474$. (One could also compute this using the formula above.)

2. Ernie moves among N vertices that are situated on a circle, as in the following diagram.



At each step, he moves clockwise with probability p, where $p \in (0, 1)$ and $p \neq \frac{1}{2}$, and counter-clockwise with probability $q = 1 - p$, independently of his choices at all other steps. Suppose that Ernie starts at state 0, and let T be the time when Ernie first returns to state 0. Find the probability that Ernie visits every vertex before time T.

Hint: Condition on Ernie's first step, and then use probabilities from the Gambler's Ruin problem.

Solution: Suppose that Ernie's first step is in the clockwise direction. Then the probability that he visits every vertex before he returns to 0 is the same as the probability that, starting with 1 unit, a gambler that wins each round with probability p makes it to $N - 1$

units before before losing it all. (Think about it! Ernie visits every vertex before returning to 0 if and only if he makes it to vertex $N-1$ before he goes back to 0. A clockwise move by Ernie corresponds to a win for the gambler (with probability p), while a counterclockwise move by Ernie corresponds to a loss for the gambler.)

By symmetry, if Ernie's first step is in the counterclockwise direction, then the probability that he vists every vertex before he returns to 0 is the same as the probability that, starting with 1 unit, a gambler that wins each round with probability q makes it to $N-1$ units before losing it all. (So we simply swap p and q in the gambler's ruin probabilities.)

Let A be the event that Ernie visits every vertex before he returns to 0. Let F_1 be the event that Ernie's first step is to vertex 1, and let F_{N-1} be the event that his first step is to vertex $N-1$. Then

$$\begin{aligned} P(A) &= P\left(A \mid F_1\right) \cdot P(F_1) + P\left(A \mid F_{N-1}\right) \cdot P(F_{N-1}) \\ &= \frac{1-q/p}{1-(q/p)^{N-1}} \cdot p + \frac{1-p/q}{1-(p/q)^{N-1}} \cdot q \\ &= \frac{p-q}{1-(q/p)^{N-1}} + \frac{q-p}{1-(p/q)^{N-1}} \end{aligned}$$

3. Recall question 3 from Assignment 1, about Lucas practicing his disc golf putting.

- He starts 4m from the basket, and throws three putts.
- If he makes all three putts, then he moves back one metre.
- If he makes two out of three putts, then he stays the same distance from the basket.
- If he makes less than two of the three putts, then he moves forward one metre.
- He then repeats the same process, throwing three discs in each round.
- He practices until he makes it 6m away from the basket.
- He makes each putt, independently of all others, with probability $\frac{3}{s}$, where $s \geq 3$ is his distance from the basket.

Let X_t be his distance from the basket after t rounds. Then $\{X_t,\ t = 0, 1, 2, \ldots\}$ is a Markov chain with transition probability matrix

$$P = \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{10}{64} & \frac{27}{64} & \frac{27}{64} & 0 \\ 0 & \frac{44}{125} & \frac{54}{125} & \frac{27}{125} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(a) Find the expected number of rounds that he putts from 4m.

Solution: Note that states 3, 4, and 5 are transient, while state 6 is recurrent. So the submatrix P_T corresponding to the transient states is:

$$P_T = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{10}{64} & \frac{27}{64} & \frac{27}{64} \\ 0 & \frac{44}{125} & \frac{54}{125} \end{bmatrix} \end{array}$$

Using Sage, we find

$$S = (I - P_T)^{-1} \approx \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1.97 & 6.23 & 4.63 \\ 0.97 & 6.23 & 4.63 \\ 0.60 & 3.86 & 4.63 \end{bmatrix} \end{array}$$

Starting from 4m, the expected number of rounds that he putts from 4m is $s_{44} \approx 6.23$.

(b) Find the expected number of rounds until he finishes practicing.

Solution: Starting from 4m, the expected number of rounds until he finishes is

$$s_{43} + s_{44} + s_{45} \approx 0.97 + 6.23 + 4.63 = 11.83.$$

4. Consider a branching process $\{X_n,\ n = 0, 1, 2, \ldots\}$ in which the probability mass function for the number of offspring of each individual is given by

$$p_0 = \tfrac{1}{3}, \quad p_1 = \tfrac{1}{3}, \quad p_2 = \tfrac{1}{6}, \quad p_3 = \tfrac{1}{6}.$$

(a) If $X_0 = 1$, find $E[X_n]$, $\mathrm{Var}(X_n)$, and the probability that the population eventually dies out.

Solution: First we find the mean μ and variance σ^2 of the number of offspring of each individual. Let Z be the number of offspring of an individual. Then

$$\mu = E[Z] = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 = \tfrac{7}{6},$$

and

$$E[Z^2] = 0^2 \cdot p_0 + 1^2 \cdot p_1 + 2^2 \cdot p_2 + 3^2 \cdot p_3 = \tfrac{15}{6},$$

so

$$\sigma^2 = \mathrm{Var}(Z) = E[Z^2] - (E[Z])^2 = \tfrac{15}{6} - \tfrac{49}{36} = \tfrac{41}{36}.$$

Now, by results presented in class, we have

$$E[X_n] = \mu^n = \left(\tfrac{7}{6}\right)^n$$

and

$$\mathrm{Var}(X_n) = \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} = \tfrac{41}{36} \left(\tfrac{7}{6}\right)^{n-1} \frac{\left(\frac{7}{6}\right)^n - 1}{\frac{1}{6}} = \tfrac{41}{6} \left(\tfrac{7}{6}\right)^{n-1} \left[\left(\tfrac{7}{6}\right)^n - 1\right]$$

Finally, the probability π_0 that the population eventually dies out is the smallest positive solution of the equation

$$x = \sum_{j=0}^{\infty} p_j x^j.$$

Substituting the given values of p_j, this equation becomes

$$x = \tfrac{1}{3} + \tfrac{1}{3}x + \tfrac{1}{6}x^2 + \tfrac{1}{6}x^3.$$

We solve this equation using Sage, and we find that the smallest positive solution is $\sqrt{3} - 1$, so $\pi_0 = \sqrt{3} - 1 \approx 0.732$.

(b) If $X_0 = 5$ instead, find $E[X_n]$, $\mathrm{Var}(X_n)$, and the probability that the population eventually dies out.

Solution: Suppose that we start with five individuals instead. For all $n \geq 0$ and $i \in \{1, 2, 3, 4, 5\}$, let $X_{n,i}$ denote the size of the nth generation of descendants of the ith initial individual. By the independence of the number of offspring of each individual, each of the $X_{n,i}$'s is distributed exactly as the size of the nth generation in part (a). So we have

$$E[X_n] = E\left[\sum_{i=1}^{5} X_{n,i}\right] = \sum_{i=1}^{n} E[X_{n,i}] = 5\left(\tfrac{7}{6}\right)^n,$$

and by independence,

$$\mathrm{Var}(X_n) = \mathrm{Var}\left(\sum_{i=1}^{5} X_{n,i}\right) = \sum_{i=1}^{5} \mathrm{Var}(X_{n,i}) = 5 \cdot \tfrac{41}{6} \left(\tfrac{7}{6}\right)^{n-1} \left[\left(\tfrac{7}{6}\right)^n - 1\right].$$

Finally, the probability that the population eventually dies out is just the probability that the five individual lines all die out. By independence, this is simply $(\pi_0)^5 \approx 0.210$.

5. Consider a branching process $\{X_n,\ n = 0, 1, 2, \ldots\}$ in which the number of offspring of each individual has mean $\mu < 1$. Show that if $X_0 = 1$, then the expected number of individuals that ever exist in the population is given by $\frac{1}{1-\mu}$. What if we had $\mu \geq 1$ instead?

Solution: The expected number of individuals that ever exist in the population is

$$E\left[\sum_{n=0}^{\infty} X_n\right] = \sum_{n=0}^{\infty} E[X_n] = \sum_{n=0}^{\infty} \mu^n.$$

This is a geometric series, which converges to $\frac{1}{1-\mu}$ when $\mu < 1$ (note that μ must be nonnegative since the number of offspring of each individual is nonnegative), and diverges when $\mu \geq 1$.

We see that if $\mu < 1$, then the expected number of individuals that ever exist in the population is $\frac{1}{1-\mu}$, while if $\mu \geq 1$, then the expected number of individuals that ever exist in the population is infinite. (This makes intuitive sense!)

6. **Bonus Problem:** Complete Example 4.6.6 from the class notes. You should find that

$$M_i = \begin{cases} i(N-i), & \text{if } p = \frac{1}{2}; \\ \dfrac{i}{q-p} - \dfrac{N}{q-p} \cdot \dfrac{1-r^i}{1-r^N}, & \text{if } p \neq \frac{1}{2}; \end{cases}$$

where $r = \frac{q}{p}$.

Solution: Please let me know if you would like to see a solution to this problem.