Summary

- If a is a constant and X is a random variable, then

$$E[aX] = a E[X]$$

- If $X_1, X_2, \ldots, X_n$ are random variables, then

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

  This property of expectation allows us to compute the expected value of certain random variables quite easily by defining appropriate indicator variables.

- The *covariance* between random variables X and Y is defined by

$$\mathrm{Cov}(X, Y) = E\Big[(X - E[X])(Y - E[Y])\Big]$$

  and is most easily computed as

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

- If X and Y are independent, then

$$E[XY] = E[X]E[Y]$$

  hence

$$\mathrm{Cov}(X, Y) = 0$$

- We proved the following properties of covariance:

  (i) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$

  (ii) $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$

  (iii) $\mathrm{Cov}(aX, Y) = a\,\mathrm{Cov}(X, Y)$

  (iv) $\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$

- We then used them to prove the following properties of variance:

  (i) $\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$

  (ii) $\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i=1}^{n} \sum_{j \neq i} \mathrm{Cov}(X_i, X_j) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum\sum_{i<j} \mathrm{Cov}(X_i, X_j)$

  (iii) If $X_1, X_2, \ldots, X_n$ are independent, then $\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$

- The *correlation* between X and Y, denoted $\rho(X, Y)$, is defined by

$$\rho(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \ \sigma_Y}$$

- The *conditional expectation* of X, given that $Y = y$, denoted $E[X \mid Y = y]$, is defined as follows.
  - If X and Y are jointly discrete, then

$$E[X \mid Y = y] = \sum_{\text{all } x} x \cdot p_{X|Y}(x,y) = \sum_{\text{all } x} x \cdot \frac{p(x,y)}{p_Y(y)}$$

  - If X and Y are jointly continuous, then

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x,y)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{f_Y(y)}\,dx$$

- We let $E[X \mid Y]$ denote the function of Y whose value at $Y = y$ is $E[X \mid Y = y]$.
- The *Law of Total Expectation* says that

$$E[X] = E\left[\,E[X \mid Y]\,\right]$$

  This result allows us to compute $E[X]$ by *conditioning* on the value of another random variable Y.
  - If Y is discrete, then

$$E[X] = \sum_{\text{all } y} E[X \mid Y = y] \cdot P\{Y = y\}$$

  - If Y is continuous, then

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y = y]\, f_Y(y)\,dy$$

- The *moment generating function* of X, denoted $M_X(t)$, is defined by

$$M_X(t) = E\left[e^{tX}\right]$$

  - For all $n \geq 1$, we have

$$M^{(n)}(0) = E\left[X^n\right]$$

  - If X and Y are independent random variables, then

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

  - The moment generating function of X uniquely determines the probability distribution of X.