CHAPTER 7 Renewal Theory and its Applications

7.1 Introduction

Recall that a stochastic process $\{N(t),\ t \geq 0\}$ is called a *counting process* if $N(t)$ represents the total number of "events" that have occurred by time t.

- A Poisson process is a counting process for which the times between successive events are

  independent and identically distributed exponential r.v.s

- In this chapter, we study a generalization of the Poisson process, where the times between successive events are

  independent and identically distributed r.v.s with an arbitrary distribution.

Definition 7.1.1. Let $\{N(t),\, t \geq 0\}$ be a counting process.

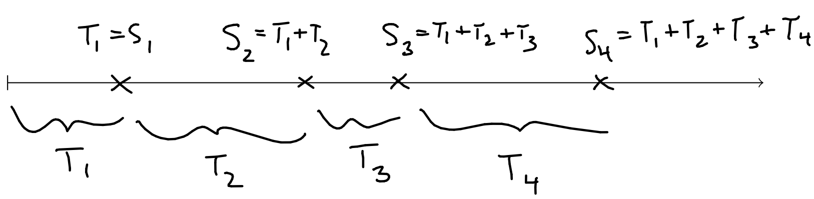
- Let T_1 denote the time of the first event.
- For all $n \geq 2$, let T_n denote the time between the $(n-1)$st event and the nth event.

We say that $\{N(t),\, t \geq 0\}$ is a *renewal process* if $T_1, T_2, T_3, \ldots$ are independent and identically distributed with a common distribution, say F.

Throughout this chapter, we let $\{N(t),\, t \geq 0\}$ be a renewal process. We also fix the following notation and terminology.

- Whenever an event occurs, we say that a renewal has taken place.
- The random variables $T_1, T_2, \ldots$ are called the interarrival times.
  - We let F denote the common cumulative distribution function of the interarrival times.
  - We let μ denote the common mean of the interarrival times.
  - We will assume that $F(0) < 1$, so that $\mu > 0$.
- For $n \geq 1$, we let S_n be the time of the n^{th} renewal.
- We let F_n denote the cumulative distribution function of S_n.

the "n-fold convolution" of F.



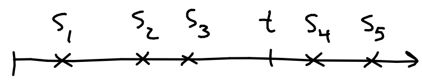
We begin with two simple observations:

- With probability 1, the number of renewals in any finite amount of time is finite.
- With probability 1, the number of renewals approaches ∞ in the long run.

For the proof of the first observation, we use the strong law of large numbers: If $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with common mean μ, then with probability 1, we have
$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mu .$$

Observation 7.1.2. $N(t)$ is finite for all $t \geq 0$ with probability 1.

Proof. First note that we can write



$$(*) \qquad N(t) = \max\{n : S_n \leq t\}$$

Now by the strong law of large numbers, with probability 1, we have
$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{T_1 + T_2 + \cdots + T_n}{n} = \mu > 0$$

Since $\mu > 0$, this means that $\lim_{n\to\infty} S_n = \infty$, hence S_n can be less than t for at most finitely many values of n. So by $(*)$, $N(t)$ is finite. $\square$

For notational convenience, we let $N(\infty) = \lim_{t\to\infty} N(t)$.

Observation 7.1.3. $N(\infty) = \infty$ with probability 1.

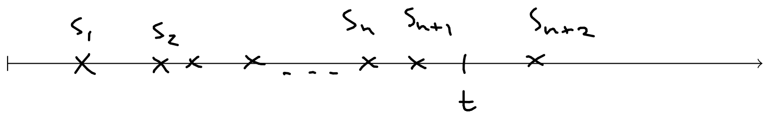
Proof. Note that the only way for $N(\infty)$ to be finite is for one of the interarrival times to be infinite. So we have

$$\begin{aligned}
P\{N(\infty) \text{ is finite}\} &= P\{\text{some } T_n \text{ is infinite}\} \\
&= P\left\{\bigcup_{n=1}^{\infty} T_n \text{ is infinite}\right\} \\
&\leq \sum_{n=1}^{\infty} P\{T_n \text{ is infinite}\} \\
&= 0
\end{aligned}$$

$\therefore N(\infty) = \infty$ with probability 1. $\square$

7.2 The Distribution of N(t)

The distribution of $N(t)$, the number of renewals by time $t \geq 0$, can be obtained (at least theoretically) as follows.



- Note that
$$N(t) \geq n \quad \Leftrightarrow \quad S_n \leq t$$

- Therefore,
$$\begin{aligned} P\{N(t) = n\} &= P\{N(t) \geq n\} - P\{N(t) \geq n+1\} \\ &= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

where F_n is the distribution of S_n, the n-fold convolution of F with itself.

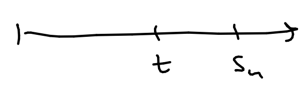
But of course, finding the distribution of S_n can be quite challenging. ☹

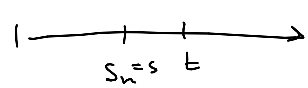
- For example, if the interarrival times are exponentially distributed with rate λ, as in the Poisson process, then S_n is a gamma random variable with parameters n and λ.

Another expression for $P\{N(t) = n\}$ can be obtained by conditioning on S_n. (This is what we did in the proof of Theorem 5.3.14 in order to show that for the Poisson process, $N(t)$ is Poisson with rate λt.)

$$\begin{aligned} P\{N(t) = n\} &= \int_0^\infty P\{N(t)=n \mid S_n = s\} \cdot f_{S_n}(s)\, ds \\ &= \int_0^t P\{T_{n+1} > t-s\} f_{S_n}(s)\, ds \\ &= \int_0^t [1 - F(t-s)] f_{S_n}(s)\, ds \\ &= \int_0^t \bar{F}(t-s) f_{S_n}(s)\, ds, \quad \text{where } \bar{F}(x) = 1 - F(x) \end{aligned}$$

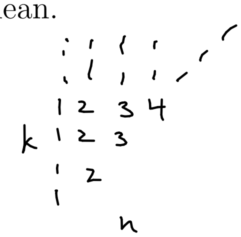
When $s > t$, $P\{N(t)=n\}=0$





Again, finding the distribution of S_n may present a challenge. ☹

Instead of trying to find the distribution of $N(t)$, we could simply try to find its mean.



- For all $t \geq 0$, let $m(t) = E[N(t)]$.
- The function $m(t)$ is called the *mean-value function* or the *renewal function.*
- E.g., a Poisson process with rate λ has $m(t) = \lambda t$

Note that if X is nonnegative and integer-valued, then

$$E[X] = \sum_{n=1}^{\infty} n \cdot P\{X=n\} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} P\{X=n\} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P\{X=n\}$$

$$= \sum_{k=1}^{\infty} P\{X \geq k\}$$

Thus, using our earlier observation that

$$N(t) \cancel{\leq} \geq n \quad \Leftrightarrow \quad S_n \leq t \, .$$

we find

$$m(t) = E[N(t)] = \sum_{k=1}^{\infty} P\{N(t) \geq k\} = \sum_{k=1}^{\infty} P\{S_k \leq t\}$$

$$= \sum_{k=1}^{\infty} F_k(t)$$

Again, this will only help us to find the mean-value function $m(t)$ if we can find the distribution of S_n. ☹

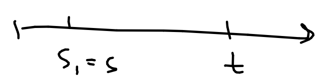
If the interarrival distribution F is continuous with density f, then we can find an integral equation satisfied by $m(t)$ by conditioning on the first arrival time as follows:

i.e. renewal

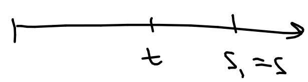
$$m(t) = E[N(t)] = \int_0^{\infty} E[N(t) \mid S_1 = s] \cdot f_{S_1}(s)\, ds$$

- If $s \leq t$, then

$$E[N(t) \mid S_1 = s] = 1 + E[N(t-s)]$$
$$= 1 + m(t-s)$$



- If $s > t$, then $\quad E[N(t) \mid S_1 = s] = 0$



Therefore,

$$m(t) = \int_0^t [1 + m(t-s)] \cdot f(s)\, ds = \int_0^t f(s) + m(t-s) f(s)\, ds$$

This is called the *renewal equation.* It can sometimes be used to solve for the renewal function $m(t)$ without having to find the distribution of S_n. ☺

$$m(t) = F(t) + \int_0^t m(t-s) f(s)\, ds \, .$$

Example 7.2.1. Suppose that the interarrival distribution is uniform on $(0, 1)$, so that

- $f(s) = 1$ for all $0 \le s \le 1$
- $F(t) = t$

Solve the renewal equation to find $m(t)$ in the case that $t \le 1$.

The renewal equation is $m(t) = t + \int_0^t m(t-s) \cdot f(s)\,ds$

$$\Rightarrow m(t) = t + \int_0^t m(t-s)\,ds$$

Let $u = t - s$

Then $du = -ds$

$$\Rightarrow m(t) = t - \int_t^0 m(u)\,du$$

$$\Rightarrow m(t) = t + \int_0^t m(u)\,du$$

Differentiating both sides, we find

$$m'(t) = 1 + m(t).$$

Let $h(t) = 1 + m(t)$.

Then $h'(t) = m'(t) = h(t)$, so $h'(t) = h(t)$.

So we have $\frac{h'(t)}{h(t)} = 1$

Integrating gives $\ln(h(t)) = t + c$

$$\Rightarrow h(t) = e^{t+c}$$

$$\Rightarrow h(t) = Ke^t, \text{ where } K = e^c.$$

So $m(t) = h(t) - 1 = Ke^t - 1$.

Since $m(0) = 0$, we have $0 = Ke^0 - 1$

$$\Rightarrow K = 1.$$

$\therefore$ $m(t) = e^t - 1$ when $t \le 1$.

Finally, we note that the following can be shown:

- The mean-value function $m(t)$ uniquely determines the distribution F of the renewal process.
- The mean-value function $m(t)$ is finite for all $t \geq 0$.
  - One might think that the finiteness of $m(t)$ should follow directly from our observation that $N(t)$ is finite with probability 1. But it is actually more complicated than this, as we show in the next example.

Example 7.2.2. Consider the random variable Y that takes on the value 2^n with probability $\frac{1}{2^n}$ for all $n \geq 1$.

(a) Verify that Y is finite with probability 1.

We want to make sure that $\displaystyle\sum_{\substack{\text{all finite} \\ \text{values } k \text{ of } Y}} P\{Y=k\} = 1$

$$\sum_{n=1}^{\infty} P\{Y=2^n\} = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

(b) Show that $E[Y] = \infty$.

$$\begin{aligned} E[Y] &= \sum_{n=1}^{\infty} 2^n \cdot P\{Y=2^n\} \\ &= \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} 1, \end{aligned}$$

which diverges to ∞.

7.3 Limit Theorems and their Applications

For a renewal process $\{N(t),\, t \geq 0\}$, recall that

$m(t)$ is called the mean-value function, or the renewal function.

- $N(t)$ is the number of renewals by time t.
- $m(t) = E[N(t)]$ is the expected # of renewals by time t.
- The times between successive renewals are independent and identically distributed with cdf F and mean μ.

In the special case of a Poisson process with rate λ, the times between successive renewals are exponential with rate λ (hence mean $\frac{1}{\lambda}$)

In this case, we know that $N(t)$ is Poisson with rate λt.

which means that

$$P\{N(t) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!} \quad \text{for all } n \geq 0.$$

and

$$m(t) = E[N(t)] = \lambda t$$

Given an arbitrary renewal process, it would be nice if we could find the distribution of $N(t)$, or even just the mean-value function $m(t)$. But this problem seems hard in general. ☹

In this section, we focus on determining how $N(t)$ grows in the long run. We have observed that

$$\lim_{t\to\infty} N(t) = \infty$$

- But how quickly does $N(t)$ approach infinity?
- In the long run, how many events happen per unit of time? $\frac{N(100)}{100}$, $\frac{N(1000)}{1000}$, ...
- That is, what is... $\lim_{t\to\infty} \frac{N(t)}{t}$???

We will prove that this limit is $\frac{1}{\mu}$ with probability 1.

- That is, in the long run, we will have 1 event/renewal every μ units of time, on average.
- This is quite intuitive, since μ is the average time between renewals.
- We call $\frac{1}{\mu}$ the rate of the renewal process. :)

Before proceeding with the proof, we consider some examples.

Example 7.3.1. While Lucas works on his notes at home, he listens to records. The mean length of a record is 20 minutes. When a record ends, the mean length of time until Lucas notices that the music is stopped, and puts on a new record, is 10 minutes. In the long run, how often does he get up to change the music? (Assume that everything is independent.)

This is a renewal process, where a renewal takes place whenever Lucas puts on a new record. The mean time between renewals is $\mu = 30$ minutes, or $\mu = 0.5$ hours. So the rate at which Lucas puts on new records is

$$\frac{1}{\mu} = \frac{1}{0.5\,hr} = 2/hr.$$

Example 7.3.2. Suppose that Tom is out soliciting donations to charity again (without Lewis this time). People that are willing to talk to Tom pass by according to a Poisson process with rate λ. However, they only stop if Tom is not currently talking to anyone else. Suppose that when someone stops to talk to Tom, the amount of time that they spend talking to him has distribution G with mean μ_G.

(a) What is the rate at which people actually stop to talk to Tom?

Suppose that the process starts when someone first stops to talk to Tom, and that a renewal takes place whenever a new person stops to talk to Tom. Then the mean time between successive renewals is:

$$\mu_G + \frac{1}{\lambda}$$

mean time the last person spends talking to Tom

mean time, after the last person finishes, until the next person stops (by memoryless property)

$\therefore$ The rate of renewals is $\dfrac{1}{\mu_G + \frac{1}{\lambda}} = \dfrac{\lambda}{\lambda\mu_G + 1}$.

(b) What proportion of people that are willing to talk to Tom actually stop?

Since people willing to talk to Tom pass by at rate λ, the proportion of people that actually stop (in the long run) is

$$\frac{\frac{\lambda}{\lambda\mu_G + 1}}{\lambda} = \frac{1}{\lambda\mu_G + 1}$$

For example, if λ is 10/hr and μ_G is 12min (or $\frac{1}{5}$hr), then only

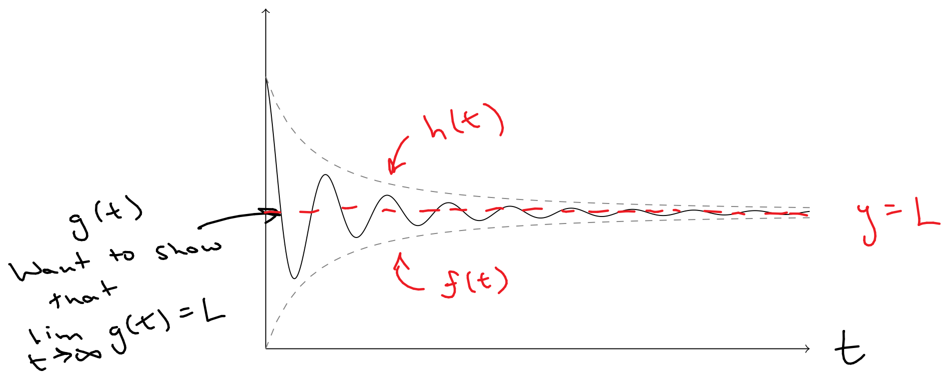
$$\frac{1}{10 \cdot \frac{1}{5} + 1} = \frac{1}{3}$$ of all people that are willing to talk actually stop.

In order to prove that $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$, we use the following result from analysis.

Theorem 7.3.3 (The Squeeze Theorem)**.** *Suppose that f, g, and h are functions satisfying*

$$f(t) \le g(t) \le h(t) \quad \text{for all } t \text{ sufficiently large.}$$

If $\lim_{t\to\infty} f(t) = L$ and $\lim_{t\to\infty} h(t) = L$, then $\lim_{t\to\infty} g(t) = L$.



Theorem 7.3.4. *With probability 1, we have*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$$

Proof. We will show that with probability 1, we have

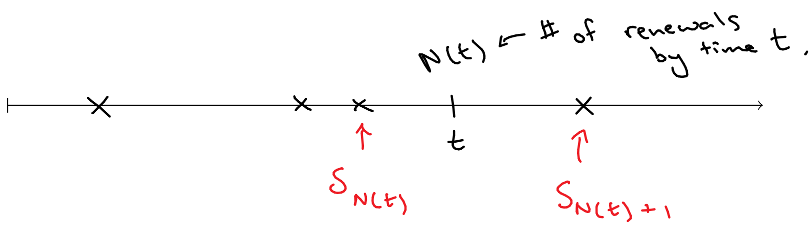
$$\lim_{t\to\infty} \frac{t}{N(t)} = \mu \, ,$$

from which the theorem statement follows immediately.

$S_n \leftarrow$ time of the n^{th} renewal

We make use of the random variables $S_{N(t)}$ and $S_{N(t)+1}$.

- $S_{N(t)}$ represents the time of the last renewal at or before time t.
- $S_{N(t)+1}$ represents the time of the first renewal after time t.



Note that $S_{N(t)} \le t \le S_{N(t)+1}$ for all $t \ge S_1$

So we have $\dfrac{S_{N(t)}}{N(t)} \le \dfrac{t}{N(t)} \le \dfrac{S_{N(t)+1}}{N(t)}$ for all $t \ge S_1$.

So for all $t \geq \cancel{?}$ S_1, we have

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)}.$$

We will find the limit at infinity of the functions on the left and the right, and then apply the Squeeze Theorem.

- First consider the function on the left.

By the strong law of large numbers, with probability 1, we have

$$\frac{S_{N(t)}}{N(t)} = \frac{T_1 + T_2 + \cdots + T_{N(t)}}{N(t)} \longrightarrow \mu \quad \text{as} \quad N(t) \longrightarrow \infty,$$

since $T_1, T_2, \ldots$ are ind. and identically dist. with mean μ.

Recall that we have $\lim_{t\to\infty} N(t) = \infty$, so it follows that

$$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \mu .$$

- Now consider the function on the right.

We have

$$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)} = \lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}$$

$$= \lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1} \cdot \lim_{t\to\infty} \left(1 + \cancelto{0}{\frac{1}{N(t)}}\right)$$

by an argument much like the one above for $\frac{S_{N(t)}}{N(t)}$

$$= \mu \cdot 1$$

$$= \mu$$

$\therefore$ By the Squeeze Theorem, $\lim_{t\to\infty} \frac{t}{N(t)} = \mu$

with probability 1.

We have shown if $\{N(t),\, t \geq 0\}$ is a renewal process, then with probability 1, we have

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}.$$

- What about the *expected* average renewal rate?
- Recall that $m(t) = E[N(t)]$
- Do we have $\displaystyle\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$.

It might seem like this should follow from the statement above. But consider the next example.

Example 7.3.5. Let U be a random variable which is uniformly distributed on $(0, 1)$. For all $n \geq 1$, define

$$Y_n = \begin{cases} 0, & \text{if } U > \frac{1}{n}; \\ n, & \text{if } U \leq \frac{1}{n}. \end{cases}$$

Show that $\displaystyle\lim_{n\to\infty} Y_n = 0$, while $\displaystyle\lim_{n\to\infty} E[Y_n] = 1$.

Note that $U > 0$ with probability 1, so for all n sufficiently large, we have $Y_n = 0$.

Hence $\displaystyle\lim_{n\to\infty} Y_n = 0$ with probability 1.

Note, however, that for all $n \geq 1$, we have

$$\begin{aligned} E[Y_n] &= 0 \cdot P\{Y_n = 0\} + n \cdot P\{Y_n = n\} \\ &= 0 + n \cdot P\{U \leq \tfrac{1}{n}\} \\ &= n \cdot \tfrac{1}{n} \\ &= 1 \end{aligned}$$

Therefore, $\displaystyle\lim_{n\to\infty} E[Y_n] = \lim_{n\to\infty} 1 = 1$.

Theorem 7.3.6 (Elementary Renewal Theorem)**.** $\displaystyle\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$

Recall that the Central Limit Theorem says that if $X_1, X_2, \ldots$ are independent and identically distributed random variables, each having mean μ and variance σ^2, then for large n, the sum

$$X_1 + X_2 + \cdots + X_n$$

is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$. That is, we have

$$\lim_{n\to\infty} P\left\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n\sigma^2}} < z\right\} = \Phi(z).$$

cdf of standard normal.

Theorem 7.3.7 (Central Limit Theorem for Renewal Processes)**.** *Suppose that* $\{N(t),\, t \geq 0\}$ *is a renewal process, and let* μ *and* σ^2 *be the common mean and variance of the times between successive renewals. Then*

$$\lim_{t\to\infty} P\left\{\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < z\right\} = \Phi(z)$$

In other words, for large t*, the random variable* $N(t)$ *is approximately normally distributed with mean* t/μ *and variance* $t\sigma^2/\mu^3$*.*

Example 7.3.8. Lucas is marking an unending number of MATH 1700 quizzes. Suppose that the times $T_1, T_2, \ldots$ to mark the quizzes are independent and identically distributed with mean $\mu = 5$ and variance $\sigma^2 = 9$. Approximate the probability that Lucas can mark 105 quizzes by time $t = 500$.

We know from the Central Limit Theorem for Renewal Processes that $N(t)$ is approximately normally distributed with mean t/μ and variance $t\sigma^2/\mu^3$ when t is large.

$t = 500$ seems large enough. ☺

So $N(500)$ is approx. normally distributed w/ mean $\frac{500}{5} = 100$ and variance $500 \cdot 9 / 5^3 \approx 36$

$\therefore$ We have $P\{N(500) \geq 105\} = P\left\{\frac{N(500) - 100}{\sqrt{36}} \geq \frac{105 - 100}{\sqrt{36}}\right\}$

$\approx P\{Z \geq \frac{5}{6}\}$

$= 1 - P\{Z \leq 0.83333\}$

$= 1 - 0.7967$

≈ 0.2033

could use a continuity correction and put 104.5 here instead!

We will not complete a formal proof of the Central Limit Theorem. But we do give the following heuristic argument for it. Recall that it says

$$\lim_{t\to\infty} P\left\{\frac{N(t)-t/\mu}{\sqrt{t\sigma^2/\mu^3}} < z\right\} = \Phi(z).$$

We have

$$P\{N(t)<n\} = P\{S_n > t\}$$

i.i.d. w/ mean μ and variance σ^2

$$= P\{T_1+T_2+\cdots+T_n > t\}$$

$$= P\left\{\frac{T_1+T_2+\cdots+T_n-n\mu}{\sqrt{n\sigma^2}} > \frac{t-n\mu}{\sqrt{n\sigma^2}}\right\}$$

$$\approx P\left\{Z > \frac{t-n\mu}{\sqrt{n\sigma^2}}\right\}$$

where Z is a standard normal r.v.

Now

$$P\left\{\frac{N(t)-t/\mu}{\sqrt{t\sigma^2/\mu^3}} < z\right\} = P\left\{N(t) < \frac{t}{\mu} + z\sqrt{t\sigma^2/\mu^3}\right\}$$

Pretending that $\frac{t}{\mu} + z\sqrt{t\sigma^2/\mu^3}$ is an integer (☺)

we have

$$P\left\{N(t) < \frac{t}{\mu} + z\sqrt{t\sigma^2/\mu^3}\right\} \approx P\left\{Z > \frac{\cancel{t} - \left(\cancel{t} + z\mu\sqrt{t\sigma^2/\mu^3}\right)}{\sqrt{\left(\frac{t}{\mu} + z\sqrt{t\sigma^2/\mu^3}\right)\sigma^2}}\right\}$$

by algebra & and using the fact that t is large.

$$\approx P\{Z > -z\}$$

$$= P\{Z < z\} = \Phi(z)$$

Finally, we note that all of the limit theorems that we have studied in this section also hold for *delayed renewal processes*, where

- the time T_1 until the first renewal has cumulative distribution function G, and
- the times $T_2, T_3, \ldots$ between successive renewals are independent and have common cumulative distribution function F.

Essentially, this means that the process doesn't necessarily need to "start" with a renewal in order for all of our limit theorems to work.

7.4 Renewal Reward Processes

Consider a renewal process $\{N(t),\, t \geq 0\}$ having interarrival times $T_1, T_2, \ldots$ with common mean μ_T.

- Suppose that there is a "reward" associated with each renewal. For example, this reward might be...
  - The donation that Tom receives from a person that stops to talk to him.
  - The number of passengers that board an arriving train, or the combined length of time that they had to wait.
- Unlike the interarrival times, the rewards may be positive or negative.
  - At an ATM, some customers make deposits, while others make withdrawals.
  - In a basketball game, points scored by the home team might be positive, while points scored by the away team are negative.
- Let R_n denote the reward earned at the time of the nth renewal.
  - We will assume that $R_1, R_2, \ldots$ are independent and identically distributed with common mean μ_R.
  - However, we allow the possibility that the reward R_n may depend on T_n .
  - For example, if a train takes longer than usual to arrive, then we expect more passengers to be waiting when it arrives!
- Let $R(t) = \sum_{i=1}^{N(t)} R_i$
  - $R(t)$ represents the total reward earned by time t.
  - The process $\{R(t),\, t \geq 0\}$ is called a renewal reward proess.

In the long run, what is the average reward per unit time? In other words, what is...

$$\lim_{t \to \infty} \frac{R(t)}{t}$$

Well...

- The expected reward at the time of each renewal is $E[R_n] = \mu_R$
- And we expect to receive this reward every $\frac{1}{\mu_T}$ units of time.

So it seems reasonable to conjecture that

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{\mu_R}{\mu_T} = \frac{E[\text{reward per renewal}]}{E[\text{time between renewals}]}$$

Theorem 7.4.1. *Let $\{R(t),\, t \geq 0\}$ be a renewal reward process with notation as above. If μ_T and μ_R are finite, then the following hold.*

(i) *With probability* 1, $\displaystyle\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mu_R}{\mu_T}$

(ii) $\displaystyle\lim_{t\to\infty} \frac{E[R(t)]}{t} = \frac{\mu_R}{\mu_T}$

Proof of (i). Suppose that μ_T and μ_R are finite. We write

$$\frac{R(t)}{t} = \frac{R(t)}{N(t)} \cdot \frac{N(t)}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \cdot \frac{N(t)}{t}.$$

Now as $t \to \infty$, we have $N(t) \to \infty$, so by the strong law of large numbers, we have

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} = \mu_R.$$

By Thm 7.3.4, $\displaystyle\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu_T}$

$$\therefore \lim_{t\to\infty} \frac{R(t)}{t} = \mu_R \cdot \frac{1}{\mu_T} = \frac{\mu_R}{\mu_T}. \qquad \square$$

Example 7.4.2. Suppose that Tom is out soliciting donations to charity again. People that are willing to talk to Tom pass by according to a Poisson process with rate $\lambda = 10/\text{hr}$. However, they only stop if Tom is not currently talking to anyone else. Suppose that when someone stops to talk to Tom,

- the mean time that they spend talking to Tom is $\mu_G = 12\,\text{min} = \frac{1}{5}\,\text{hr}$; and
- the mean amount that they donate to chairty after speaking to Tom is \$20.

Find the average rate at which Tom collects donations.

This is a renewal process where each renewal corresponds to someone stopping to talk to Tom. The reward associated with each renewal has mean \$20, i.e., $\mu_R = 20$.

The mean time between renewals is given by

$$\mu_T = 12 + \frac{1}{\lambda} = 12 + 6 = 18$$

$\therefore$ The average rate at which Tom collects donations (in the long run) is

$$\frac{\mu_R}{\mu_T} = \frac{\$20}{18\text{ min}} = \frac{\$20}{0.3\text{ hr}} = \$66.66/\text{hr}$$

Although we supposed in Theorem 7.4.1 that the reward is earned at the time of a renewal, the result remains valid when the reward is earned gradually throughout the renewal cycle.

Example 7.4.3. Suppose that orders at a pizza shop are ready for delivery according to a renewal process with mean interarrival time $\mu = 5$ (measured in minutes). Whenever there are N orders waiting to be delivered, the delivery driver leaves. In this problem, we want to determine which value of N will minimize the average cost.

(a) If the shop incurs a cost at the rate of $c = 0.2$ dollars per minute for each order waiting to be delivered, what is the average cost per unit time?

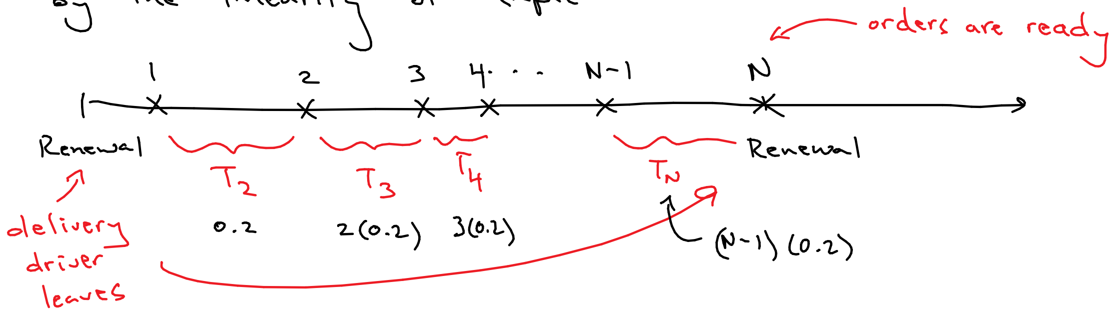
Note: This "cost" can be thought of as the net negative impact on the shop of letting the pizza get cold.

This is a renewal process where a renewal takes place each time the delivery driver leaves.

The average cost per unit time is given by

$$\frac{E[\text{cost per renewal}]}{E[\text{time between renewals}]}.$$

Note that the expected time between renewals as $N\mu$, by the linearity of expectation.



Note that when i orders are ready and waiting to be delivered, the cost per unit time is $0.2i = ci$.

$$\therefore \quad E[\text{cost per renewal}] = E\left[cT_2 + 2cT_3 + 3cT_4 + \dots + (N-1)cT_N\right]$$

$$= cE[T_2] + 2cE[T_3] + 3cE[T_4] + (N-1)cE[T_N]$$

$$= c\mu + 2c\mu + 3c\mu + \dots + (N-1)c\mu$$

$$= c\mu\left[1 + 2 + 3 + \dots + (N-1)\right]$$

$$= c\mu\,\frac{N(N-1)}{2}$$

So the long-run average cost is $\dfrac{c\cancel{\mu}\,\frac{\cancel{N}(N-1)}{2}}{\cancel{N}\cancel{\mu}} = \dfrac{c(N-1)}{2}$.

(b) Suppose now that the average cost of delivering a group of N orders is $5 + 5\ln(N)$. What is the average cost per unit time?

The average cost (in the long run) per unit time will be

$$\frac{E[\text{cost per renewal}]}{E[\text{time of a renewal}]}$$

cost of delivery

cost due to customers waiting (from part (a))

$$= \frac{5 + 5\ln(N) + c\mu\, N(N-1)/2}{N\mu}$$

sub $c = \$0.2/\text{min}$ and $\mu = 5$ min

$$= \frac{5 + 5\ln(N) + N(N-1)/2}{5N}$$

$$= \frac{1}{N} + \frac{\ln(N)}{N} + \frac{(N-1)}{10}$$

(c) What value of N minimizes the shop's long-run average cost?

Using a computer, it looks like the cost is minimized when $N=3$ or $N=4$.

(You can do more investigation into this if you like.)

Example 7.4.4. Suppose that passengers arrive at a subway station according to a Poisson process with rate λ. Suppose also that trains arrive (independently from passengers) according to a renewal process with distribution F. When a train arrives, it picks up all waiting passengers. Find the average number of people who are waiting for a train, averaged over all time.

Solution. A new cycle begins every time a train arrives.

- Let T be a random variable with distribution F, so that T represents

  the time between successive trains, or the length of a cycle.

- Suppose that each passengers pays us money at a rate of 1 per unit time while they wait for a train.

- Then the reward rate at any time is the number of people who are waiting at that time.

- Let R be the reward earned in a given cycle.

- Thus, by the renewal reward theorem, we have the average number of people waiting is the average reward, which is equal to $\dfrac{E[R]}{E[T]}$.

- Let N be the number of passengers that arrive in a given cycle.

- To find expectations involving the reward R earned in a cycle, we kind of need to know the number of passengers N that arrive in a cycle.

- To find expectations involving the number of passengers N that arrive in a cycle, we kind of need to know the length T of a cycle.

So to determine $E[R]$, we condition on the values of both T and N.

We have $E[R \mid N=n, T=t] = \dfrac{nt}{2}$, since, given n Poisson arrivals in t units of time, the arrival times are distributed like uniform random variables. (In other words, we expect each of these n people to wait $\frac{t}{2}$ units of time.)

$$\therefore \quad E[R \mid N, T] = \frac{NT}{2}.$$

Taking expectation on both sides, we find

$$E[R] = \frac{1}{2} E[NT].$$

Now to find $E[NT]$, we condition on T.

$$E[NT \mid T=t] = E[Nt \mid T=t] = t\,E[N \mid T=t]$$
$$= t \cdot \lambda t$$
$$= \lambda t^2$$

Given $T=t$, the number of arrivals is Poisson with mean λt.

So as a function of T, we have $E[NT \mid T] = \lambda T^2$.

By the law of total expectation, $E[NT] = E[E[NT \mid T]] = E[\lambda T^2]$
$= \lambda E[T^2]$.

Putting all of this together, we have

$$\text{average \# of people waiting} = \frac{E[R]}{E[T]} = \frac{\frac{1}{2}E[NT]}{E[T]} = \boxed{\frac{\lambda E[T^2]}{2E[T]}}$$

For example, if T is exponential with rate μ, then

- $E[T] = \frac{1}{\mu}$
- $E[T^2] = \frac{2}{\mu^2}$

So in this case, the average number of people who are waiting for a train is:

$$\lambda \cdot \frac{\frac{2}{\mu^2}}{2 \cdot \frac{1}{\mu}} = \lambda \cdot \frac{\cancel{2}}{\mu^{\cancel{2}}} \cdot \frac{\cancel{\mu}}{\cancel{2}} = \frac{\lambda}{\mu}$$

If T is uniformly distributed on $(0, s)$, then

- $E[T] = \frac{s}{2}$
- $E[T^2] = \frac{s^2}{3}$

So in this case, the average number of people who are waiting for a train is:

$$\frac{\lambda \cdot E[T^2]}{2E[T]} = \frac{\lambda \cdot \frac{s^2}{3}}{2 \cdot \frac{s}{2}} = \frac{\lambda s}{3}$$

Note: One can also show the following:

- The average amount of time that a passenger waits, averaged over all passengers, is
$$\frac{E[T^2]}{2E[T]}$$
if trains exp. w/ rate μ: $\frac{1}{\mu}$ if trains uniformly on $(0,s)$: $\frac{s}{3}$
- The average amount of time until the next train arrives, averaged over all time, is
$$\frac{E[T^2]}{2E[T]}$$

The fact that these quantities are equal is a special case of a general result called the PASTA principle (**P**oisson **A**rrivals **S**ee **T**ime **A**verages).

7.5 Regenerative Processes

A *regenerative process* is a stochastic process $\{X(t),\, t \geq 0\}$ with countable state space with the property that there are time points at which the process (probabilistically) restarts itself.

- That is, with probability 1, there is a time S_1 such that the process starting from S_1 is a probabilistic replica of the process starting from 0.
- This implies the existence of times $S_2, S_3, \ldots$ with the same property as S_1.
- The times $S_1, S_2, S_3 \ldots$ constitute the arrival times of a renewal process.
- We say that a *cycle* is completed every time a renewal/regeneration occurs.

For example, every recurrent Markov chain (discrete-time or continuous-time) is regenerative. In this case,

- S_1 represents the first time that we reenter the initial state.
- S_2 represents the second time that we " " " ".
- And so on!

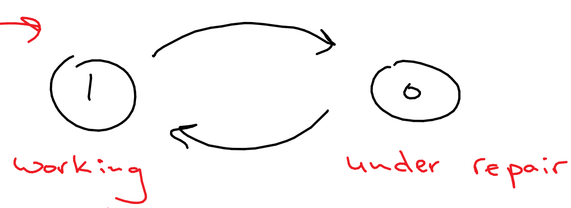
Regenerative processes can be thought of as a generalization of recurrent Markov chains!

- We can transition between states as in a recurrent Markov chain, but the amount of time that we spend in each state can have an arbitrary distribution.
- The only real restriction is that we eventually return to the initial state, and that when this happens, everything is the same (probabilistically) as when we started.

Regenerative processes are lurking behind the following scenarios:

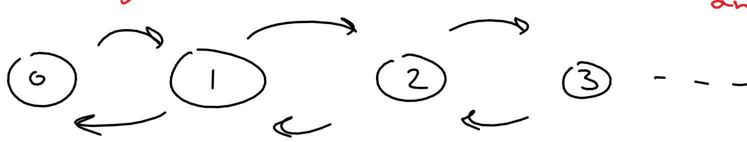
- A machine works for a time, and then breaks down (and takes time to be repaired), but is good as new as soon as it is repaired.

the system "resets" whenever the machine is repaired and put back into operation



- Customers arrive at a queue in accordance with an arbitrary renewal process, and are served one at a time by a single server with an arbitrary service distribution.

the system "resets" whenever a new customer arrives and finds the server free.



What is the long-run proportion of time π_j that a regenerative process spends in state j?

- Suppose that we earn a reward at rate 1 per unit time when the process is in state j, and at rate 0 otherwise.
- Let $R(t)$ be the reward earned by time t.
- Note that $\{R(t),\, t \geq 0\}$ is a renewal reward process, with renewals corresponding to the times $S_1, S_2, S_3, \ldots$ at which the process regenerates.
- By the renewal reward theorem, we have

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[\text{reward in a cycle}]}{E[\text{time of a cycle}]} = \frac{E[\text{time spent in state } j \text{ in a cycle}]}{E[\text{time of a cycle}]}$$

- But by definition, the average reward per unit time is equal to the long-run proportion π_j.
- Thus, we have obtained the following proposition.

Proposition 7.5.1. *For a regenerative process, the long-run proportion of time that the process spends in state j is given by*

$$\pi_j = \frac{E[\text{time spent in state } j \text{ in a cycle}]}{E[\text{time of a cycle}]}$$

Example 7.5.2. Consider a positive recurrent continuous-time Markov chain that is initially in state i. What is the long-run proportion of time that the process spends in state i?

Remember that the process "resets" each time we reenter state i.

So by the proposition above, we have

$$\pi_i = \frac{E[\text{time spent in state } i \text{ in a cycle}]}{E[\text{time of a cycle}]} = \frac{E[\text{time in state } i \text{ before we reenter state } i]}{E[\text{time to reenter state } i]}$$

$$= \frac{\frac{1}{v_i}}{\mu_{ii}}$$

If the cycle time in a regenerative process $\{X(t),\, t \geq 0\}$ is a continuous random variable, then it can be shown that π_j is also equal to the limiting probability that the process is in state j at time t.

- That is, if the cycle time is a continuous random variable, then

$$\lim_{t\to\infty} P\{X(t) = j\} = \pi_j.$$

- The proof of this fact requires an advanced theorem called the *key renewal theorem*.

7.5.1 Alternating Renewal Processes

An *alternating renewal process* is a regenerative process that describes a system that is in one of two states: on or off.

- Suppose that the system is initially on, and remains on for a time Z_1; it then goes off and remains off for a time Y_1.
- The system then goes back on for a time Z_2; then off for a time Y_2.
- And so on!
- Suppose that the random variables $Z_1, Z_2, \ldots$ are independent and identically distributed, and so are the random variables $Y_1, Y_2, \ldots$
- We do allow Y_n to depend on Z_n.
- In other words, each time the system switches back on, the process restarts probabilistically, but when the system turns off, the length of the off time may depend on the length of the previous on time.

The following might be examples of alternating renewal processes. Think about whether the process really "restarts" when the system switches back on. Why might the length of the off time depend on the length of the previous on time?

- A machine that works for a time, and then breaks down (and takes time to be repaired).
  - The system is on when the machine is working, and off when it is under repair.
- A person checking their phone.
  - The system is on when the person is checking their phone, and off otherwise.
- Lucas playing his records.
  - The system is on when a record is playing, and off otherwise.

Proposition 7.5.3. *Consider an alternating renewal process with mean on time μ_{on} and mean off time μ_{off}. Then*

- *the long-run proportion of time π_{on} that the system is on is given by*

$$\pi_{\text{on}} = \frac{\mu_{\text{on}}}{\mu_{\text{on}} + \mu_{\text{off}}}$$

- *the long-run proportion of time π_{off} that the system is off is given by*

$$\pi_{\text{off}} = \frac{\mu_{\text{off}}}{\mu_{\text{on}} + \mu_{\text{off}}}$$

Example 7.5.4. A certain insurance company charges its policyholders a rate that alternates between r_1 and r_0. A new policyholder is initially charged at a rate of r_1 per unit time. When a policyholder paying at rate r_1 has made no claims for s consecutive units of time, then the rate charged switches to r_0 per unit time. The rate charge remains at r_0 until a claim is made, at which time it reverts to r_1. Suppose that a certain policyholder makes claims according to a Poisson process with rate λ.

(a) For $i \in \{0, 1\}$, find the proportion of time π_i that the policyholder pays at rate r_i.

This is an alternating renewal process, where the system is "on" when the customer pays at rate r_1, and "off" when the customer pays at rate r_0. A new cycle starts every time the customer makes a claim.

We have $\pi_1 = \pi_{on} = \dfrac{\mu_{on}}{\mu_{on}+\mu_{off}}$ expected length of time that they pay at rate r_1 in a given cycle

expected length of a given cycle

Since claims are made according to a Poisson process with rate λ, the expected length of a cycle is $\frac{1}{\lambda}$.

To find μ_{on}, suppose the process just restarted (i.e., a claim was just made) and let X be the time until the next claim. Let T be the length of on time before the next claim.

So $\underbrace{E[T]}_{\mu_{on}} = E[E[T|X]] = \int_0^\infty E[T|X=x] \cdot f_X(x)\,dx$

$$= \int_0^s x\, f_X(x)\,dx + \int_s^\infty s\, f_X(x)\,dx$$

$$= \int_0^s x \lambda e^{-\lambda x}\,dx + s\int_s^\infty f_X(x)\,dx$$

use integration by parts!

$$= \frac{1}{\lambda}\left(1-e^{-\lambda s}\right) - se^{-\lambda s} + s\,P\{X>s\}$$

$$= \frac{1}{\lambda}\left(1-e^{-\lambda s}\right) - se^{-\lambda s} + se^{-\lambda s}$$

$$= \frac{1}{\lambda}\left(1-e^{-\lambda s}\right)$$

So $\pi_{on} = \dfrac{\mu_{on}}{\mu_{on}+\mu_{off}} = \dfrac{\frac{1}{\lambda}\left(1-e^{-\lambda s}\right)}{\frac{1}{\lambda}} = 1-e^{-\lambda s}$

and $\pi_{off} = e^{-\lambda s}$

(b) Find the long-run average amount paid per unit time.

This will be

$\pi_{on}\cdot r_1 + \pi_{off}\cdot r_0$

$= \left(1-e^{-\lambda s}\right) r_1 + e^{-\lambda s} r_0$