5.1 Introduction

Remember that a stochastic process is a collection of random variables

$$\{X(t),\ t \in T\},$$

where the index T represents *time*. In Chapter 4 we considered certain *discrete-time* stochastic processes called Markov chains. In this chapter, we consider certain *continuous-time* stochastic processes called *counting processes*.

A stochastic process $\{N(t),\ t \geq 0\}$ is called a *counting process* if $N(t)$ represents the total number of "events" that have occurred by time t. For example, $N(t)$ could be:

- the number of phone calls received by the WestJet customer service line by time t; or
- The number of goals scored by time t in a hockey game.

We will see that when one makes certain simple assumptions about $N(t)$, several familiar distributions pop up!

5.2 The Exponential Distribution

In this section we review some of the important properties of exponential random variables.

5.2.1 Definition

A continuous random variable X is said to have an *exponential distribution* (or to be *exponential*) with parameter $\lambda > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0; \\ 0, & \text{otherwise}. \end{cases}$$

If X is an exponential random variable, then

- $F(x) = P\{X \leq x\} = \int_{-\infty}^{x} f(t)\,dt = 1 - e^{-\lambda x}$ when $x \geq 0$
- $P\{X > x\} = 1 - F(x) = e^{-\lambda x}$
- $E[X] = \int_0^\infty x f(x)\,dx = \frac{1}{\lambda}$
- $\text{Var}(X) = \frac{1}{\lambda^2}$
- $M_X(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}$ for $t < \lambda$.

5.2.2 Properties of the Exponential Distribution

Definition 5.2.1. A nonnegative random variable X is *memoryless* if

$$P\{X > s+t \mid X > s\} = P\{X > t\} \quad \text{for all } s,t \geq 0.$$

As a specific example, let X be the length of time that Lucas spends on a phone call with his mom.

- If X is memoryless, and Lucas is still on the phone after 30 minutes, then the remaining time that he spends on the phone has the same distribution as the original random variable X.
- In other words, Lucas doesn't "remember" that he has been talking to his mom for 30 minutes already!

Note that the memoryless property is equivalent to

$$P\{X > s+t\} = P\{X > s\} \cdot P\{X > t\} \quad \text{for all } s,t \geq 0.$$

If X is an exponential random variable, then

$$P\{X > s+t\} = e^{-\lambda(s+t)}, \quad \text{and} \quad P\{X > s\} = e^{-\lambda s}, \quad \text{and} \quad P\{X > t\} = e^{-\lambda t}$$

Therefore, exponential random variables are memoryless!

Example 5.2.2. Suppose that the length of Lucas and Ernie's afternoon walk (in minutes) is an exponential random variable X with mean 15. so $\lambda = \frac{1}{15}$.

(a) Find the probability that their walk is longer than 20 minutes.

$$P\{X > 20\} = e^{-\lambda(20)} = e^{-\frac{1}{15}\cdot 20} = e^{-\frac{4}{3}} \approx 0.264.$$

(b) If they left ten minutes ago and they aren't back yet, how much longer do you expect the walk to last?

By the memoryless property, we expect the walk to last 15 more minutes.

Definition 5.2.3. For a continuous nonnegative random variable X, the *failure rate* (or *hazard rate*) of X, denoted $r(t)$, is defined by

$$r(t) = \frac{f(t)}{1 - F(t)}$$

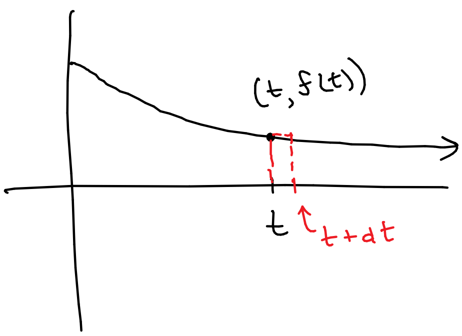
What does the failure rate represent? Let X be the length of time that a battery has been working. Suppose that the battery has already been working for t hours. What is the probability that the battery will stop working in the next dt hours, where dt is some small amount?

$$P\{X < t+dt \mid X > t\} = \frac{P\{X < t+dt,\ X > t\}}{P\{X > t\}}$$

$$= \frac{P\{X \in (t, t+dt)\}}{1 - F(t)}$$ exact area under the pdf from t to $t+dt$

$$\approx \frac{f(t)\,dt}{1 - F(t)}$$ area of an approximating rectangle

$$= r(t)\,dt$$



That is, $r(t)$ represents the conditional probability density that a t-hour-old battery will fail.

Proposition 5.2.4. *If X is an exponential random variable, then $r(t)$ is constant.*

Proof. Suppose that X is exponential with parameter λ.

Then
$$r(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda e^{-\lambda t}}{1 - (1 - e^{-\lambda t})} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda.$$

$\therefore$ The failure rate is constant. (In fact, it is equal to the parameter λ.)

Note: The fact that $r(t)$ is constant makes sense since exponential random variables are memoryless.

Sometimes, λ is called the rate of the exponential random variable.

Proposition 5.2.5. *The failure rate function of a nonnegative random variable X completely determines the distribution of X.*

Proof. Suppose that X is a nonnegative random variable with failure rate function $r(t)$.

Then by definition, $r(t) = \dfrac{f(t)}{1-F(t)} = \dfrac{\frac{d}{dt}[F(t)]}{1-F(t)}$

Integrating both sides, we find

$$\int_0^x r(t)\,dt = \int_0^x \frac{f(t)}{1-F(t)}\,dt.$$

Let $u = 1-F(t)$
then $du = -f(t)dt$

$$\Rightarrow \int_0^x r(t)\,dt = \int_{t=0}^{t=x} \frac{-1}{u}\,du \Rightarrow -\int_0^x r(t)\,dt = \Big[\ln\big(1-F(t)\big)\Big|_{t=0}^{t=x}$$

Thus we have $\ln(1-F(x)) = -\int_0^x r(t)\,dt$

$$\Rightarrow 1-F(x) = e^{-\int_0^x r(t)dt}$$

$$\Rightarrow F(x) = 1 - e^{-\int_0^x r(t)dt}$$

$\therefore$ The failure rate function completely determines the cumulative distribution function. $\square$

Proposition 5.2.6. *If a nonnegative random variable X is memoryless, then X is exponentially distributed.*

Proof. Suppose that X is a nonnegative, memoryless random variable. Since X is memoryless, its failure rate must be constant, say $r(t) = \lambda$.

By the proof of the previous result, the cdf of X is given by $F(x) = 1 - e^{-\int_0^x \lambda\,dt} = 1 - e^{-\lambda x}$.

Since this is the cdf of an exponential r.v. $\bar{w}$ parameter λ, we conclude that X is exponential $\bar{w}$ parameter λ. $\square$

5.2.3 Further Properties of the Exponential Distribution

In this subsection, we prove several results about sums and minimums of independent exponential random variables. For the first result, recall that if X and Y are independent continuous random variables, then

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_Y(t-s)\, f_X(s)\, ds$$

Proposition 5.2.7. *Let $X_1, X_2, \ldots,$ be independent exponential random variables with common parameter λ, and let $Y_n = \sum_{i=1}^{n} X_i$. Then Y_n has density*

$$f_n(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \qquad t \geq 0.$$

Note: A random variable with density $f_n(t)$ is called a *gamma* random variable with parameters n and λ.

Proof. We proceed by induction on n.

Base Case: $n = 1$. Then $Y_n = X_1$.

So the density of $Y_n = Y_1$ is

$$f_{Y_1}(t) = f_{X_1}(t) = \lambda e^{-\lambda t}, \quad t \geq 0.$$

and this is equal to $f_1(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^0}{0!} = \lambda e^{-\lambda t}$.

Inductive Hypothesis: Suppose for some $n > 1$ that $Y_{n-1} = X_1 + \ldots + X_{n-1}$ has density $f_{n-1}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-2}}{(n-2)!}, \; t \geq 0$.

Inductive Step: Consider $Y_n = X_1 + \ldots + X_n = Y_{n-1} + X_n$.

The density of Y_n is given by

$$\begin{aligned}
f_{Y_{n-1}+X_n}(t) &= \int_{-\infty}^{\infty} f_{X_n}(t-s)\, f_{Y_{n-1}}(s)\, ds \\
&= \int_0^t \lambda e^{-\lambda(t-s)} \lambda e^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!}\, ds \\
&= \int_0^t \lambda^2 e^{-\lambda t} \frac{(\lambda s)^{n-2}}{(n-2)!}\, ds \\
&= \left[\lambda e^{-\lambda t} \frac{(\lambda s)^{n-1}}{(n-1)!} \right]_0^t = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} = f_n(t)
\end{aligned}$$

$\therefore$ The statement holds for all $n \geq 1$ by math induction.

Proposition 5.2.8. *Let X_1 and X_2 be independent exponential random variables with parameters λ_1 and λ_2, respectively. Then*

$$P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

Proof. We condition on X_1 as follows.

$$P\{X_1<X_2\} = \int_0^\infty P\{X_1<X_2 \mid X_1=x\}\cdot f_{X_1}(x)\,dx$$

$$= \int_0^\infty P\{x<X_2 \mid X_1=x\}\cdot \lambda_1 e^{-\lambda_1 x}\,dx$$

$$= \int_0^\infty P\{X_2>x\}\cdot \lambda_1 e^{-\lambda_1 x}\,dx \qquad \text{(by independence)}$$

$$= \int_0^\infty e^{-\lambda_2 x}\cdot \lambda_1 e^{-\lambda_1 x}\,dx$$

$$= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x}\,dx$$

$$= \lim_{t\to\infty} \int_0^t \lambda_1 e^{-(\lambda_1+\lambda_2)x}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)x} \right|_0^t$$

$$= \lim_{t\to\infty} \left[\overbrace{-\frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}}^{0} + \frac{\lambda_1}{\lambda_1+\lambda_2} e^{0} \right]$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2}. \qquad \square$$

On the next page, we will generalize this result by showing that if $X_1, X_2, \ldots, X_n$ are independent exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, then

$$P\{X_i = \min_j X_j\} = \frac{\lambda_i}{\lambda_1+\lambda_2+\cdots+\lambda_n}$$

$1 \le j \le n$

Proposition 5.2.9. *Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Then $X = \min_j X_j$ is exponential with parameter $\lambda = \sum_{i=1}^{n} \lambda_i$.*

Proof. We find $P\{X > x\}$ as follows:

$$\begin{aligned} P\{X > x\} &= P\{\min_j X_j > x\} \\ &= P\{X_1 > x, X_2 > x, \ldots, X_n > x\} \\ &= P\{X_1 > x\} \cdot P\{X_2 > x\} \cdot \ldots \cdot P\{X_n > x\} \quad \text{(by independence)} \\ &= e^{-\lambda_1 x} \cdot e^{-\lambda_2 x} \cdot \ldots \cdot e^{-\lambda_n x} \\ &= e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n) x} \end{aligned}$$

But this means that $X = \min_j X_j$ is exponential w/ parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$. □

Corollary 5.2.10. *Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Then*

$$P\{X_i = \min_j X_j\} = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}.$$

Proof. We can write

$$P\{X_i = \min_j X_j\} = P\{X_i < \min_{j \neq i} X_j\}$$

exponential with parameter $\sum_{j \neq i} \lambda_j$ by Prop 5.2.10

$$= \frac{\lambda_i}{\lambda_i + \sum_{j \neq i} \lambda_j}$$

by Prop. 5.2.8

$$= \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}.$$

□

Example 5.2.11. Suppose that you arrive at your bank, and that three clerks are working. They are all busy with another customer when you arrive, but you are next in line, and will be helped as soon as one of the three clerks becomes available. Suppose that for $i = 1, 2, 3$, the length of time that clerk i spends helping each customer is exponentially distributed with parameter λ_i. Find the expected length of time that you spend in the bank.

Let T be the time we spend in the bank.

Let R_i be the length of time that clerk i spends helping their current customer.

Let S be the length of our service.

Then $T = \min\{R_1, R_2, R_3\} + S$.

By linearity of expectation, we have

$$E[T] = E[\min\{R_1, R_2, R_3\}] + E[S].$$

By the memoryless property of exp. r.v.s, R_1, R_2, and R_3 are exponential with parameters λ_1, λ_2, and λ_3.

So $\min\{R_1, R_2, R_3\}$ is exponential w/ parameter $\lambda_1 + \lambda_2 + \lambda_3$.

$\therefore E[\min\{R_1, R_2, R_3\}] = \dfrac{1}{\lambda_1 + \lambda_2 + \lambda_3}$.

To find $E[S]$, we condition on which clerk first becomes available.

$$\begin{aligned}
E[S] &= E[S \mid \text{served by clerk } 1] \cdot P\{\text{served by clerk } 1\} \\
&\quad + E[S \mid \text{" " " } 2] \cdot P\{\text{" " " } 2\} \\
&\quad + E[S \mid \text{" " " } 3] \cdot P\{\text{" " " } 3\} \\
&= \frac{1}{\lambda_1} \cdot P\{R_1 = \min\{R_1, R_2, R_3\}\} + \frac{1}{\lambda_2} \cdot P\{R_2 = \min\{R_1, R_2, R_3\}\} \\
&\quad + \frac{1}{\lambda_3} \cdot P\{R_3 = \min\{R_1, R_2, R_3\}\} \\
&= \frac{1}{\lambda_1} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2} \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_3} \cdot \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \\
&= \frac{3}{\lambda_1 + \lambda_2 + \lambda_3}
\end{aligned}$$

$\therefore E[T] = \dfrac{4}{\lambda_1 + \lambda_2 + \lambda_3}$

5.3 The Poisson Process

A Poisson Process is a continuous-time stochastic process that arises naturally from some simple assumptions, and ties together several different probability distributions that are familiar to us.

5.3.1 Counting Processes

A stochastic process $\{N(t),\ t \geq 0\}$ is called a *counting process* if $N(t)$ represents the total number of "events" that occur by time t. The following are some examples of counting processes.

- If $N(t)$ is the number of buses that have stopped at a certain bus stop by time t, then $\{N(t),\ t \geq 0\}$ is a counting process.
  - An "event" of this process corresponds to

    a bus arriving at the stop.

- If $N(t)$ is the number of runners who have finished a marathon by time t, then $\{N(t),\ t \geq 0\}$ is a counting process.
  - An "event" of this process corresponds to

    a runner crossing the finish line.

- If $N(t)$ is the number of wolves born on Vancouver Island by time t, then $\{N(t),\ t \geq 0\}$ is a counting process.
  - An "event" of this process corresponds to

    a wolf being born.

Observation 5.3.1. If $\{N(t),\ t \geq 0\}$ is a counting process, then the following conditions are satisfied for all $s, t \geq 0$.

(i) $N(t)$ takes on only nonnegative integer values.

(ii) If $s < t$, then $N(t) \geq N(s)$

and $N(t) - N(s)$ represents the number of events in the interval $(s, t]$.

Example 5.3.2. If $P(t)$ is the number of wolves living on Vancouver Island at time t, is $P(t)$ a counting process?

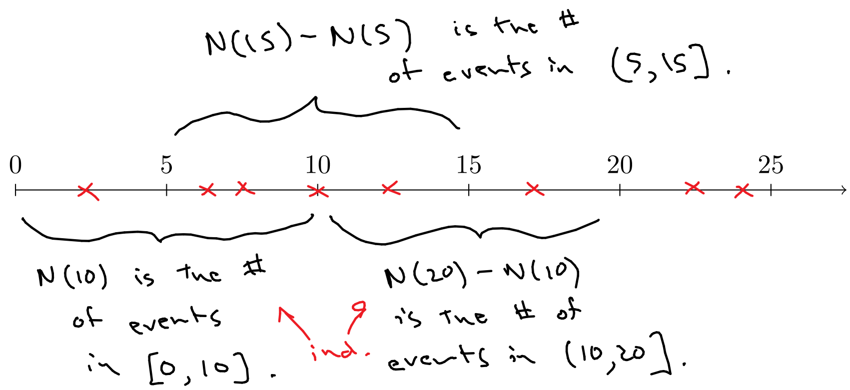
No, it is not a counting process, since when a wolf diss, the population decreaces.

Definition 5.3.3. A counting process is said to possess *independent increments* if

the numbers of events that occur in disjoint (i.e. non-overlapping) time intervals are independent.

For instance, if the counting process $\{N(t),\ t \geq 0\}$ has independent increments, then

- $N(10)$ is independent of $N(20) - N(10)$, but
- $N(10)$ and $N(20) - N(10)$ are not necessarily independent of $N(15) - N(5)$.



Definition 5.3.4. A counting process is said to possess *stationary increments* if

the distribution of the number of events that occur in any interval depends only on the length of the interval.

In other words, a counting process has stationary increments if

the number of events that occur in the interal (s, s+t) is the same for all s.

For instance, if the counting process $\{N(t),\ t \geq 0\}$ has stationary increments, then

- $N(5)$ has the same distribution as $N(10) - N(5)$, and as $N(7) - N(2)$.

Example 5.3.5. Let $N(t)$ be the number of customers that have entered a certain grocery store by time t.

(a) Does it seem reasonable to assume that $\{N(t),\ t \geq 0\}$ has independent increments?

It seems somewhat, or even mostly, reasonable.

(b) Does it seem reasonable to assume that $\{N(t),\ t \geq 0\}$ has stationary increments?

It depends how long the window of time under consideration is. Over a whole day or a whole week, probably not! But if we consider just one hour, all subintervals of that hour probably have similar distributions.

5.3.2 Definition of the Poisson Process

In order to describe the Poisson process, we make use of little-o notation.

Definition 5.3.6. A function f is said to be $o(h)$ (as $h \to 0$), and we write $f(h) = o(h)$, if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

Example 5.3.7. Determine whether or not the given function is $o(h)$.

(a) $f(h) = h^2$

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0$$

So $h^2 = o(h)$

(b) $f(h) = \frac{h}{2}$

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{\frac{h}{2}}{h} = \lim_{h \to 0} \frac{1}{2} = \frac{1}{2}$$

So $\frac{h}{2}$ is <u>not</u> $o(h)$.

Essentially, $f(h) = o(h)$ means that $f(h)$ is *really small* compared to h for all h sufficiently close to 0. That is, as h tends to 0, the quantity $f(h)$ tends to 0 *significantly faster* than h.

Observation 5.3.8. Let f and g be $o(h)$, and let c be a constant. Then

(i) $f(h) + g(h) = o(h)$

(ii) $cf(h) = o(h)$

Proof of (i). Since $f(h) = o(h)$ and $g(h) = o(h)$, we have

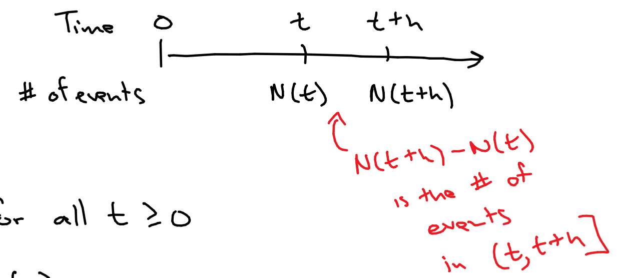
$$\lim_{h \to 0} \frac{f(h) + g(h)}{h} = \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0} \frac{g(h)}{h}$$

$$= 0 + 0.$$

$\therefore$ $f(h) + g(h) = o(h)$. $\square$

Remember: The notation $o(h)$ represents a function whose values are *extremely small* relative to h for all h sufficiently close to 0.

Definition 5.3.9. The counting process $\{N(t),\ t \geq 0\}$, is said to be a *Poisson process with rate* $\lambda > 0$ if the following axioms hold:



(i) $N(0) = 0$

(ii) $\{N(t),\ t \geq 0\}$ has independent increments

(iii) $P\{N(t+h) - N(t) = 1\} = \lambda h + o(h)$, for all $t \geq 0$

(iv) $P\{N(t+h) - N(t) \geq 2\} = o(h)$, for all $t \geq 0$

Note that Axioms (iii) and (iv) say that at any time t, we have the following.

- The probability that exactly one event occurs in the next h units of time is

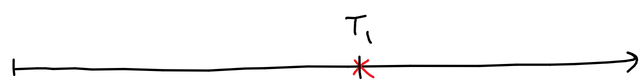
  approximately proportional to the length of time h.

- The probability that more than one event occurs in the next h units of time is

  basically negligible when h is small.

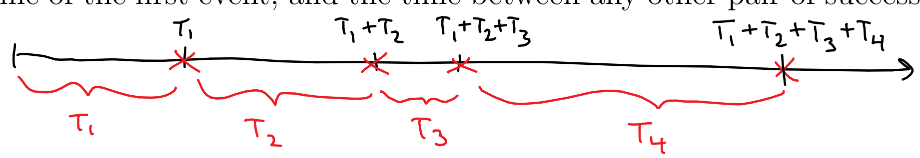
Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. In the remainder of this subsection, we will prove the following results.

- The time of the first event is an exponential random variable with rate λ.



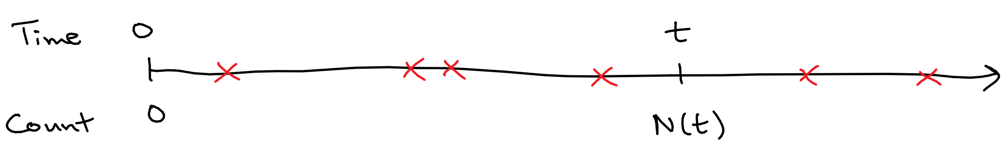
T_1 is an exponential r.v. with rate λ.

- The time between any pair of successive events is exponential with rate λ, independent of the time of the first event, and the time between any other pair of successive events.



$T_1, T_2, T_3, \ldots$ are independent exp. r.v.'s with rate λ.

- For all $t \geq 0$, the random variable $N(t)$ is Poisson with parameter λt.



Throughout the remainder of this subsection, let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ.

- Let T_1 be the time of the first event of this process.
- For all $n \geq 2$, let T_n be the time between the $(n-1)$st event and the nth event.

Lemma 5.3.10. *T_1 is an exponential random variable with rate λ.*

Proof. It suffices to show that for all $t \geq 0$, we have $P\{T_1 > t\} = e^{-\lambda t}$.

- For ease of notation, let $P_0(t) = P\{N(t)=0\} = P\{T_1 > t\}$
- So we must show that $P_0(t) = e^{-\lambda t}$.
- We first obtain an expression for $P_0'(t) = \lim_{h \to 0} \frac{P_0(t+h) - P_0(t)}{h}$

We have

$$\begin{aligned} P_0(t+h) &= P\{N(t+h)=0\} = P\{N(t)=0,\ N(t+h)-N(t)=0\} \\ &= P\{N(t)=0\} \cdot P\{N(t+h)-N(t)=0\} \\ &= P_0(t) \cdot \left(1 - P\{N(t+h)-N(t) \geq 1\}\right) \\ &= P_0(t) \cdot \left(1 - \lambda h + o(h)\right) \end{aligned}$$

by ind. increments (Axiom (ii))

By Axioms (iii) and (iv).

Therefore,

$$\begin{aligned} P_0'(t) &= \lim_{h \to 0} \frac{P_0(t) \cdot (1 - \lambda h + o(h)) - P_0(t)}{h} \\ &= P_0(t) \cdot \lim_{h \to 0} \frac{-\lambda h + o(h)}{h} \\ &= P_0(t) \cdot \left[\lim_{h \to 0} -\frac{\lambda \cancel{h}}{\cancel{h}} + \underbrace{\lim_{h \to 0} \frac{o(h)}{h}}_{0}\right] = -\lambda P_0(t) \end{aligned}$$

Hence we have $\dfrac{P_0'(t)}{P_0(t)} = -\lambda$

Integrating both sides gives $\ln(P_0(t)) = -\lambda t + C \implies P_0(t) = e^{-\lambda t + C}$

Finally, note that $P_0(0) = P\{N(0)=0\} = 1$ ← by Axiom (i)

so subbing $t=0$ into the above, we find $P_0(0) = e^{-\lambda(0) + C}$

$$\implies 1 = e^{C}$$

$$\implies C = 0$$

$\therefore\ P_0(t) = e^{-\lambda t}$, as desired. □

For $s > 0$, let

$$N_s(t) = N(s+t) - N(s)$$

= # of events in the interval $(s, s+t]$.

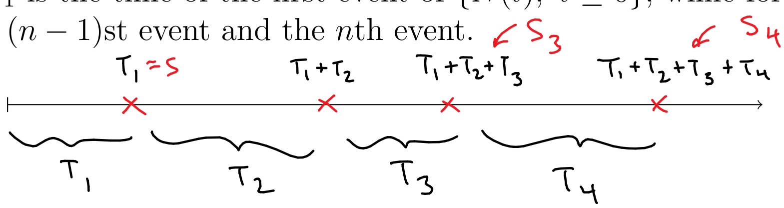
Lemma 5.3.11. *$\{N_s(t),\ t \geq 0\}$ is a Poisson process with rate λ.*

Proof. We check that all of the axioms of a Poisson process hold for $N_s(t)$. $\square$

What does this lemma mean?

- Imagine that we "reset" our counter at time s. That is, at time s, we start counting from 0 again.
- Then this new count, which we've called $N_s(t)$, is another Poisson process.
- This should remind you of the memoryless property!

Remember that T_1 is the time of the first event of $\{N(t),\ t \geq 0\}$, while for $n \geq 2$, T_n is the time between the $(n-1)$st event and the nth event.



Theorem 5.3.12. *$T_1, T_2, \ldots$ are independent exponential random variables with rate λ.*

Proof. We've already shown (in Lemma 5.3.10) that T_1 is exponential with rate λ. Now

$$\begin{aligned} P\{T_2 > t \mid T_1 = s\} &= P\{0 \text{ events in } (s, s+t] \mid T_1 = s\} \\ &= P\{0 \text{ events in } (s, s+t]\} \\ &= P\{N_s(t) = 0\} \\ &= e^{-\lambda t} \end{aligned}$$

by axiom (ii) (ind. increments)

by Lemma 5.3.10 and 5.3.11

- Since $P\{T_2 > t \mid T_1 = s\}$ does not depend on s, we conclude that

  T_1 and T_2 are independent.
- Hence $P\{T_2 > t\} = P\{T_2 > t \mid T_1 = s\} = e^{-\lambda t}$.

A similar argument shows that T_n is independent of $T_1, T_2, \ldots, T_{n-1}$, and is exponential with rate λ, and the theorem statement follows by induction. $\square$

Now for $n \geq 1$, let S_n be the time of the nth event. So

$$S_n = T_1 + T_2 + \cdots + T_n$$

The following is an immediate corollary of Proposition 5.2.7 and Theorem 5.3.12.

Corollary 5.3.13. *S_n is a* gamma random variable with parameters n and λ.

$$f_{S_n}(s) = \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!}, \quad s \geq 0.$$

Theorem 5.3.14. *For all $t \geq 0$, the random variable $N(t)$ is Poisson with parameter λt. That is,*

$$P\{N(t) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad t = 0, 1, 2, \ldots$$

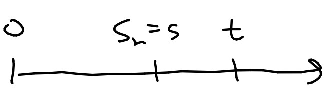
Proof. Let $t \geq 0$. It was shown in Lemma 5.3.10 that

$$P\{N(t) = 0\} = e^{-\lambda t},$$ which confirms the theorem statement for $n = 0$.

Now let $n > 0$. We compute $P\{N(t) = n\}$ by conditioning on S_n, the time of the nth event. This gives

$$P\{N(t) = n\} = \int_0^\infty P\{N(t) = n \mid S_n = s\} f_{S_n}(s)\, ds$$

since $P\{N(t) = n \mid S_n = s\} = 0$ if $s > t$.



$$= \int_0^t P\{N(t) = n \mid S_n = s\} \cdot f_{S_n}(s)\, ds$$

$$= \int_0^t P\{N(t) = n \mid S_n = s\} \cdot \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!}\, ds$$

Now for $0 < s < t$, we have

$$\begin{aligned} P\{N(t) = n \mid S_n = s\} &= P\{0 \text{ events in } (s,t] \mid S_n = s\} \\ &= P\{0 \text{ events in } (s,t]\} \\ &= P\{N_s(t-s) = 0\} \\ &= e^{-\lambda(t-s)} \end{aligned}$$

by ind. increments

by Lemma 5.3.10 and Lemma 5.3.11

Substituting into our expression for $P\{N(t) = n\}$, we find

$$P\{N(t) = n\} = \int_0^t e^{-\lambda(t-s)} \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!}\, ds$$

$$= \lambda e^{-\lambda t} \int_0^t \frac{(\lambda s)^{n-1}}{(n-1)!}\, ds$$

$$= \lambda e^{-\lambda t} \left[\frac{1}{\lambda} \frac{(\lambda s)^n}{n!}\right]_0^t$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$\therefore$ Since $P\{N(t) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$ for all $n = 0, 1, 2, \ldots$, we conclude that $N(t)$ is Poisson with parameter λt. □

Corollary 5.3.15. *The number of events in any fixed interval of length t is Poisson with parameter λt.*

Note: In particular, this means that $\{N(t),\ t \geq 0\}$ has *stationary increments.*

Proof. Let $(s, s+t]$ be any interval of length t.

Then $P\{n \text{ events in } (s, s+t]\}$

$$= P\{N(s+t) - N(s) = \boxed{n}\}$$

$$= P\{N_s(t) = \boxed{n}\}$$

$$= P\{N(t) = \boxed{n}\}. \quad \text{By Lemma 5.3.11.}$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad \text{By Theorem 5.3.14.}$$

$$, \quad \text{for all } n = 0, 1, 2, \ldots.$$

Example 5.3.16. Lucas and Ernie are going for a walk in Kenna-Cartwright Nature Park. Suppose that Ernie receives treats according to a Poisson process at a rate of 5 per hour.

(a) Find the expected length of time until Ernie receives his first treat.

The time until he receives his first treat is exponential with rate 5, so the expected time is $\frac{1}{5}$ hours, or 12 minutes.

(b) Find the expected length of time until Ernie receives his tenth treat.

This is the expected value of S_{10}, which is given by

$$E[S_{10}] = E[T_1 + T_2 + \cdots + T_{10}] = E[T_1] + E[T_2] + \cdots + E[T_{10}]$$

all exp. with rate 5

$$= 10 \cdot \frac{1}{5}$$

$$= 2 \text{ hours.}$$

(c) If they walk for ~~2~~ 3 hours, find the expected number of treats that Ernie receives.

The number of treats that he receives in 3 hours is a Poisson r.v. with parameter $\lambda t = 5 \cdot 3$. So the expected number of treats is 15.

(d) Find the probability that Ernie gets at least 3 treats in the last half hour of the walk.

The number of treats that Ernie gets in any interval of length 0.5 hours is Poisson with rate $\lambda t = 5 \cdot 0.5 = 2.5$. Let X be the # of treats that he gets in the last half hour. Then we want

$$P\{X \geq 3\} = 1 - P\{X=0\} - P\{X=1\} - P\{X=2\}$$

$$= 1 - \frac{e^{-2.5}(2.5)^0}{0!} - \frac{e^{-2.5}(2.5)^1}{1!} - \frac{e^{-2.5}(2.5)^2}{2!}$$

Example 5.3.17. A chicken wants to cross a road at a certain point, where cars pass by according to a Poisson process with rate λ per second. The chicken waits until she can see that no cars will come by in the next T seconds.

(a) Find the probability that her waiting time is 0.

This is exactly the probability that the first car arrives after time T. Since the time T_1 that the first car arrives is exponential with parameter λ, we have

$$P\{T_1 > T\} = e^{-\lambda T}.$$

(b) Find her expected waiting time.

Let W be her waiting time. We condition on the time that the first car passes.

$$E[W] = \int_0^\infty E[W \mid T_1 = t]\, f_{T_1}(t)\, dt$$

$$= \int_0^\infty E[W \mid T_1 = t] \cdot \lambda e^{-\lambda t}\, dt$$

$$= \int_0^T E[W \mid T_1 = t] \cdot \lambda e^{-\lambda t}\, dt + \int_T^\infty \underbrace{E[W \mid T_1 = t]}_{=0} \cdot \lambda e^{-\lambda t}\, dt$$

$$= \int_0^T (t + E[W]) \cdot \lambda e^{-\lambda t}\, dt$$

since she must wait t seconds for the first car to pass, at which point, the process restarts.

$$= \int_0^T \lambda t e^{-\lambda t}\, dt + E[W] \int_0^T \lambda e^{-\lambda t}\, dt$$

Now one can evaluate these two integrals, and solve for $E[W]$.

Example 5.3.18. Consider a two-server parallel queuing system where customers arrive according to a Poisson process with rate λ, and where the service times are exponential with rate μ. Moreover, suppose that arrivals finding both servers busy immediately depart without receiving any service (they are *lost*), while those finding at least one free server immediately enter service and then depart when their service is completed.

(a) If both servers are presently busy, find the expected time until the next customer (who isn't lost) starts being served.

Let T be the time until the next customer starts being served. Let S be the time unil one of the servers becomes free. Let R be the time after a server becomes free until the next arrival. Then $E[T] = E[S+R] = E[S] + E[R]$

$$= \frac{1}{\mu+\mu} + \frac{1}{\lambda}$$

Since the minimum of two ind. exponentials with rates μ_1 and μ_2 is exp. with rate $\mu_1+\mu_2$

reset the count at time S, and the first arrival is exp. with rate λ.

(b) If both servers start empty, find the expected time until both servers are busy.

Let B_0 be the time until both servers are busy when 0 servers are busy at the start, and let B_1 be the time until both servers are busy when 1 server is busy at the start.

Note that we have

$$E[B_0] = \frac{1}{\lambda} + E[B_1]$$

reset the count when the first customer arrives, but now one server is busy.

Let X be the time until either a departure or an arrival occurs, and let Y be the additional time until both servers are busy. To determine $E[Y]$, we will condition on whether a departure or arrival occurs first.

Then $E[B_1] = E[X+Y] = E[X] + E[Y]$

$$= \frac{1}{\mu+\lambda} + E[Y \mid \text{arrival first}] \cdot P\{\text{arrival first}\} + E[Y \mid \text{departure first}] \cdot P\{\text{departure first}\}$$

$$= \frac{1}{\mu+\lambda} + 0 \cdot P\{\text{arrival first}\} + E[B_0] \cdot \frac{\mu}{\mu+\lambda}$$

$$= \frac{1}{\mu+\lambda} + E[B_0] \cdot \frac{\mu}{\mu+\lambda}.$$

Now solve
$$\begin{cases} E[B_0] = \frac{1}{\lambda} + E[B_1] \\ E[B_1] = \frac{1}{\mu+\lambda} + E[B_0] \cdot \frac{\mu}{\mu+\lambda}. \end{cases}$$

Solving gives:
$$E[B_0] = \frac{2\lambda+\mu}{\lambda^2}$$
$$E[B_1] = \frac{\lambda+\mu}{\lambda^2}$$

(c) Find the expected time between two successive lost customers.

Suppose that a customer was just lost, and let L denote the time until the next lost customer. Let M denote the time until the next departure or arrival, and let N denote the additional time until the next lost customer. (So $L = M + N$.)

Then $E[L] = E[M] + E[N]$

M is the min. of ind. exponentials with rates μ, μ, and λ.

$$= \frac{1}{2\mu+\lambda} + E[N \mid \text{arrival first}] \cdot P\{\text{arrival first}\} + E[N \mid \text{departure first}] \cdot P\{\text{departure first}\}$$

$$= \frac{1}{2\mu+\lambda} + 0 \cdot \frac{\lambda}{\lambda+2\mu} + \left(E[B_1] + E[L]\right) \cdot \frac{2\mu}{\lambda+2\mu}$$

time until both servers are busy again

once servers are both busy, time until we lose the next customer.

Now substitute $E[B_1] = \frac{\lambda+\mu}{\lambda^2}$, and solve for $E[L]$.

Should find $E[L] = \frac{1}{\lambda} + \frac{2\mu(\lambda+\mu)}{\lambda^3}$. :)

5.3.3 Further Properties of Poisson Processes

Consider a Poisson process $\{N(t),\ t \geq 0\}$ having rate λ, ~~and that~~ where the events can be classified into two different types.

- Suppose that each event will be classified independently as:
  - a type 1 event with probability p, and
  - a type 2 event with probability $1-p$.
- For example, if $N(t)$ represents the number of goals scored by time t in a hockey game, a type 1 event could correspond to the home team scoring, while a type 2 event corresponds to the away team scoring.

Let $N_1(t)$ and $N_2(t)$ denote the number of type 1 and type 2 events that occur by time t. Note that

$$N(t) = N_1(t) + N_2(t) \quad \text{for all } t \geq 0.$$

Proposition 5.3.19. *$\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ are Poisson processes with rates λp and $\lambda(1-p)$, respectively. Furthermore, the two processes are independent.*

Proof. We check that all of the axioms of a Poisson process hold for $N_1(t)$ (and $N_2(t)$ is similar). For independence, see Example 3.5.3. $\square$

Example 5.3.20. Suppose that goals are scored in a one-hour long hockey game according to a Poisson process at a rate of 4.5 per hour. Suppose that each goal is scored by the Montreal Canadiens with probability 0.6, and by the Toronto Maple Leafs with probability 0.4, independently of all others. Find the probability that Montreal leads 2–0 after the first period (which lasts 20 minutes).

Type 1 prob. $p=0.6$

Type 2 prob. $1-p=0.4$

Let $N_1(t)$ be the number of goals scored by Montreal by time t, and let $N_2(t)$ be the number of goals scored by Toronto by time t. Then by Proposition 5.3.19, $\{N_1(t), t \geq 0\}$ is a Poisson process with rate $4.5(0.6) = 2.7$, and $\{N_2(t), t \geq 0\}$ is a Poisson process with rate $4.5(0.4) = 1.8$.

So $N_1(\frac{1}{3})$ is Poisson with rate $2.7(\frac{1}{3}) = 0.9$, and $N_2(\frac{1}{3})$ is Poisson with rate $1.8(\frac{1}{3}) = 0.6$.

So the probability that Montreal leads 2–0 after the first period is

$$P\{N_1(\tfrac{1}{3}) = 2\} \cdot P\{N_2(\tfrac{1}{3}) = 0\}$$

$$= \frac{e^{-0.9}(0.9)^2}{2!} \cdot \frac{e^{-0.6}(0.6)^0}{0!}$$

Suppose now that each time an event occurs, its type is classified according to a probability that *depends on the time at which the event occurs.*

- Specifically, suppose that if an event occurs at time s, then it will be classified as:
  - a type 1 event with probability ~~$p(s)$, and~~ $p_1(s)$
  - a type 2 event with probability ~~$1 - p(s)$.~~ $p_2(s) = 1 - p_1(s)$

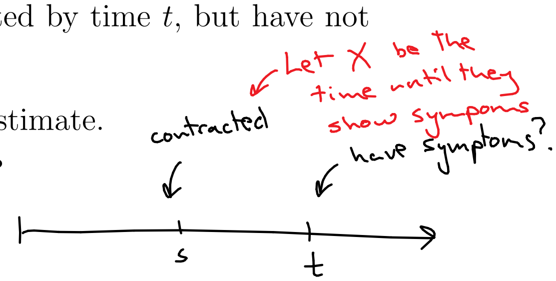
The following result turns out to be very useful in practice. We'll omit its proof for now.

Theorem 5.3.21. *If $N_i(t)$ represents the number of type i events occurring by time t, then the $N_i(t)$ are independent Poisson random variables with means*

$$E[N_i(t)] = \lambda \int_0^t p_i(s)\, ds$$

Example 5.3.22. Suppose that individuals contract a virus according to a Poisson process whose rate λ is unknown. Suppose that the time from when an individual becomes infected until symptoms of the disease appear is a random variable having a known distribution F, and that these "incubation times" are independent for all infected individuals. If the average incubation time is relatively long, then it is difficult to determine the total number of infected individuals at a given time t – that is our goal here.

- Let $N_1(t)$ denote the number of individuals who are infected and have shown symptoms by time t.
  - This quantity is easier for public health officials to estimate.
  - Suppose that we have a reasonable estimate of $N_1(t)$.
- Let $N_2(t)$ denote the number of individuals who are infected by time t, but have not yet shown symptoms.
  - This quantity is harder for public health officials to estimate.
  - Can we come up with a reasonable estimate of $N_2(t)$?



An individual who contracts the virus at time s will

- have symptoms by time t with probability: $P\{X \le t - s\} = F(t-s)$
- not yet have symptoms at time t with probability: $P\{X > t-s\} = 1 - F(t-s)$

Therefore, by Theorem 5.3.21, we see that $N_1(t)$ and $N_2(t)$ are Poisson random variables with

$$E[N_1(t)] = \lambda \int_0^t p_1(s)\, ds = \lambda \int_0^t F(t-s)\, ds = \lambda \int_0^t F(x)\, dx$$

and

$$E[N_2(t)] = \lambda \int_0^t p_2(s)\, ds = \lambda \int_0^t 1 - F(t-s)\, ds = \lambda \int_0^t 1 - F(x)\, dx$$

So we have

$$E[N_1(t)] = \lambda \int_0^t F(x)\,dx$$

and

$$E[N_2(t)] = \lambda \int_0^t 1 - F(x)\,dx.$$

Remember that λ is unknown, but that we have a reasonable estimate of $N_1(t)$. How can we estimate $N_2(t)$?

Let our estimate of $N_1(t)$ be $\hat{N}_1(t)$. Then we can estimate λ by

$$\hat{\lambda} = \frac{\hat{N}_1(t)}{\int_0^t F(x)\,dx}$$

(assuming that $\hat{N}_1(t) \approx E[N_1(t)]$)

So we can estimate $N_2(t)$ with

$$\hat{N}_2(t) \approx E[N_2(t)] \approx \hat{\lambda} \int_0^t 1 - F(x)\,dx = \frac{\hat{N}_1(t) \cdot \int_0^t 1-F(x)\,dx}{\int_0^t F(x)\,dx}$$

Suppose that F is exponential with rate μ, so $F(x) = 1 - e^{-\mu x}$ for all $x \geq 0$.

Then $\int_0^t 1-F(x)\,dx = \int_0^t e^{-\mu x}\,dx = \left[-\frac{1}{\mu} e^{-\mu x} \right]_0^t$

$$= -\frac{1}{\mu} e^{-\mu t} + \frac{1}{\mu} = \frac{1}{\mu}\left(1 - e^{-\mu t}\right)$$

and $\int_0^t F(x)\,dx = \int_0^t 1 - e^{-\mu x}\,dx = t - \frac{1}{\mu}\left(1 - e^{-\mu t}\right)$

Therefore

$$\hat{N}_2(t) = \frac{\hat{N}_1(t) \cdot \frac{1}{\mu}\left(1 - e^{-\mu t}\right)}{t - \frac{1}{\mu}\left(1 - e^{-\mu t}\right)}$$

As a specific example, consider HIV/AIDS. Suppose that $t = 15$ years, the mean incubation period is 10 years, and we estimate that the number of people with the disease is 200,000.

mean is 10 years, so the rate is $\frac{1}{10}$, i.e. $\mu = \frac{1}{10}$.

So

$$\hat{N}_2(t) = \frac{200{,}000 \cdot 10\left(1 - e^{-\frac{1}{10}\cdot 15}\right)}{15 - 10\left(1 - e^{-\frac{1}{10}\cdot 15}\right)}$$

$$\approx 214{,}863$$

5.3.4 Conditional Distribution of the Arrival Times

Let $Y_1, Y_2, \ldots, Y_n$ be random variables. We say that

$Y_1, Y_2, \ldots, Y_n$

put them in increasing order

$Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$

$$Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$$

are the *order statistics* corresponding to $Y_1, Y_2, \ldots, Y_n$ if

$$Y_{(k)} \text{ is the } k^{th} \text{ smallest of } Y_1, Y_2, \ldots, Y_n.$$

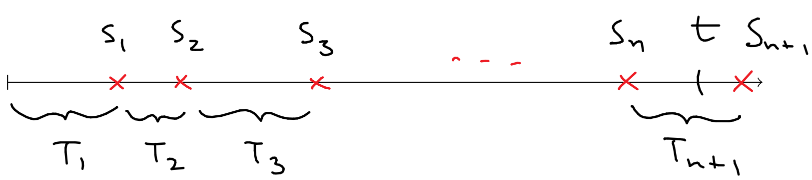
If $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed with density f, then the joint density of the order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is given by

$$f(y_1, y_2, \ldots, y_n) = n! \cdot \prod_{i=1}^{n} f(y_i) \quad \text{for} \quad y_1 < y_2 < \cdots < y_n$$

In particular, if $Y_1, Y_2, \ldots, Y_n$ are uniformly distributed over $(0, t)$, then the joint density of the order statistics is given by

$$f(y_1, y_2, \ldots, y_n) = n! \cdot \prod_{i=1}^{n} \frac{1}{t} = \frac{n!}{t^n} \quad \text{for} \quad 0 < y_1 < y_2 < \cdots < y_n < t.$$

Theorem 5.3.23. *Let $\{N(t),\ t \geq 0\}$ be a Poisson process. Given that $N(t) = n$, the n arrival times $S_1, S_2, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.*



Proof. Note that for $0 < s_1 < s_2 < \ldots < s_n < t$, the event that

$$S_1 = s_1,\ S_2 = s_2,\ \ldots\ S_n = s_n,\ N(t) = n,$$

is equivalent to the event that

$$T_1 = s_1,\ T_2 = s_2 - s_1,\ T_3 = s_3 - s_2,\ \ldots,\ T_n = s_n - s_{n-1},\ T_{n+1} > t - s_n$$

Thus the conditional joint density of $S_1, S_2, \ldots, S_n$ given $N(t) = n$ is given by

$$f(s_1, s_2, \ldots, s_n \mid n) = \frac{f(s_1, s_2, \ldots, s_n, n)}{P\{N(t) = n\}}$$

$N(t)$ is Poisson with parameter λt

$$= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \cdot \lambda e^{-\lambda(s_3 - s_2)} \cdots \lambda e^{-\lambda(s_n - s_{n-1})} \cdot e^{-\lambda(t - s_n)}}{e^{-\lambda t} \frac{(\lambda t)^n}{n!}}$$

$$= \frac{\cancel{\lambda^n} \cancel{e^{-\lambda t}}\, n!}{\cancel{e^{-\lambda t}} (\cancel{\lambda} t)^n} = \frac{n!}{t^n}$$

$\square$

What does this mean? Given that n events have occurred in the interval $(0, t)$, the times $S_1, S_2, \ldots, S_n$ at which events occur, considered as unordered random variables, are distributed independently and uniformly over $(0, t)$.

Example 5.3.24. Suppose that people arrive at a subway station in accordance with a Poisson process with rate λ. The train departs at a fixed time t. Let X denote the total amount of waiting time of all those who get on the train at time t. Find $E[X]$ and $\mathrm{Var}(X)$.

We condition on the number of people that have arrived by time t.

We have $E[X \mid N(t)=n] = E\left[\sum_{i=1}^{n}(t-S_i) \;\middle|\; N(t)=n\right]$

$$= E\left[\sum_{i=1}^{n}(t-u_i)\right],$$

where the u_i's are independent and uniform on $(0,t)$.

Thus we have $E[X \mid N(t)=n] = \sum_{i=1}^{n} E[t-u_i]$

$$= \sum_{i=1}^{n} t - E[u_i]$$

$$= \sum_{i=1}^{n} t - \frac{t}{2}$$

$$= \sum_{i=1}^{n} \frac{t}{2} = \frac{nt}{2}.$$

So $E[X \mid N(t)] = N(t)\cdot\frac{t}{2}$.

$$\therefore\ E[X] = E\big[E[X \mid N(t)]\big] = E\left[N(t)\cdot\frac{t}{2}\right]$$

$$= \frac{t}{2}\, E[N(t)]$$

$$= \frac{t}{2}\cdot\boxed{\lambda t}$$

$$= \boxed{\frac{\lambda t^2}{2}}.$$

Since $N(t)$ is Poisson with parameter λt.

For the variance, see Exercise 64 in the book.

Example 5.3.25. Suppose that items arrive at a processing plant in accordance with a Poisson process with rate λ. At a fixed time T, all items are dispatched from the system. We want to choose an intermediate time $t \in (0, T)$ at which all items in the system are dispatched, so as to minimize the total expected wait of all items. What time t should we choose?

Let W be the total wait of all items.
Let W_1 " " " " " " " dispatched at time t.
Let W_2 " " " " " " " " " " T.

Then
$$E[W] = E[W_1 + W_2]$$
$$= E[W_1] + E[W_2]$$

By a calculation similar to the one in the previous example, we find
$$E[W_1] = \frac{\lambda t^2}{2}$$
and
$$E[W_2] = \frac{\lambda (T-t)^2}{2}.$$

So
$$E[W] = \frac{\lambda t^2}{2} + \frac{\lambda (T-t)^2}{2} = \frac{\lambda}{2}\left[t^2 + (T-t)^2\right]$$

We wish to find the value of t that minimizes $E[W]$, so we use calculus.

$$\frac{d}{dt} E[W] = \frac{\lambda}{2}\left[2t + 2(T-t)\cdot(-1)\right]$$
$$= \frac{\lambda}{2}\left[2t - 2T + 2t\right]$$
$$= \lambda\left[2t - T\right]$$

The only critical point is $t = \frac{T}{2}$, and since $\frac{d^2}{dt^2} E[W] = 2\lambda > 0$, we see that this point corresponds to a minimum.

$\therefore$ We should choose the intermediate time $t = \frac{T}{2}$.

5.4 Generalizations of the Poisson Process

In this section we very briefly describe two generalizations of the Poisson process.

5.4.1 The Nonhomogeneous Poisson Process

In a *nonhomogeneous* or *nonstationary* Poisson process, the rate of arrivals at time t is allowed to be a function of t.

Definition 5.4.1. The counting process $\{N(t),\ t \geq 0\}$ is said to be a *nonhomogeneous Poisson process with intensity function* $\lambda(t)$ if

(i) $N(0) = 0$

(ii) $\{N(t),\ t \geq 0\}$ has independent increments.

(iii) $P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h)$ for all $t \geq 0$

(iv) $P\{N(t+h) - N(t) \geq 2\} = o(h)$ for all $t \geq 0$.

the rate now depends on t. If $\lambda(t) = \lambda$ (constant), then we get the usual Poisson process.

The function $m(t)$ defined by

$$m(t) = \int_0^t \lambda(s)\, ds$$

Note that if $\lambda(s) = \lambda$ (constant), as in the original Poisson process, then this is just λt.

is called the *mean value function* of the nonhomogeneous Poisson process.

The following result generalizes Theorem 5.3.14, and can be proven in a similar (though slightly more involved) manner.

Theorem 5.4.2. *If $\{N(t),\ t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t)$, then*

$$P\{N(t) = n\} = e^{-m(t)} \frac{(m(t))^n}{n!}$$

i.e., the number of events that occur by time t is a Poisson random variable with rate $m(t)$.

5.4.2 The Compound Poisson Process

Definition 5.4.3. A stochastic process $\{X(t),\ t \geq 0\}$ is said to be a *compound Poisson process* if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

where $\{N(t),\ t \geq 0\}$ is a Poisson process, and $Y_1, Y_2, \ldots$ are independent and identically distributed random variables that are also independent of the process $\{N(t),\ t \geq 0\}$.

Examples:

- Suppose that customers leave a store according to a Poisson process, and let Y_i be the amount that the ith customer spends.
- Suppose that buses arrive at a tourist attraction according to a Poisson process, and let Y_i be the number of tourists on the ith bus.

Note: If $Y_i = 1$ for all $i \geq 1$, then $X(t) = N(t)$ for all $t \geq 0$, i.e., $X(t)$ is the usual Poisson process.