1.1 Introduction

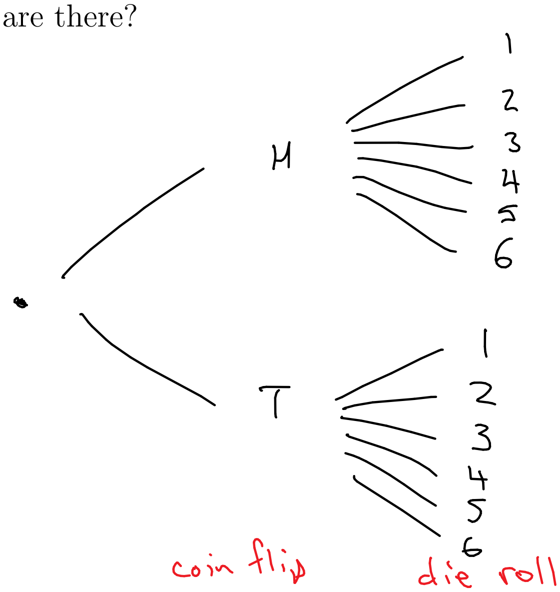
Counting is at the heart of probability – many problems in probability theory, such as the following, can be solved by counting the number of different ways that a certain event can occur.

- If two six-sided dice are rolled, what is the probability that the numbers on the dice sum to 8?
- A bag contains five red balls and seven blue balls. If three balls are drawn from the bag randomly without replacement, what is the probability that all three balls are blue?
- If a room contains 30 people, what is the probability that at least two of the people have the same birthday?

In this chapter, we review the basics of counting.

1.2 The Basic Principle of Counting

Example 1.2.1. If a coin is flipped and then a six-sided die is rolled, how many possible outcomes are there?



total # of possible outcomes:

$2 \times 6 = 12$.

Fact 1.2.2 (The basic principle of counting)**.** Suppose that two experiments are to be performed one after another. If the first experiment has m possible outcomes, and for each outcome of the first experiment, the second experiment has n possible outcomes, then the total number of possible outcomes of the two experiments is: mn

More generally, suppose that $r \geq 2$ experiments are to be performed one after another. Roughly speaking, if the ith experiment has n_i possible outcomes, then the total number of possible outcomes of the r experiments is:

$$n_1 n_2 n_3 \cdots n_r$$

Example 1.2.3. How many different license plates, consisting of three numbers (0–9) followed by three letters (A–Z), are possible?

10 choices, 10, 10, 26, 26, 26

The # of different license plates is $10^3 \cdot 26^3 =$

How many of these license plates have no repeated numbers or letters?

10 9 8 26 25 24

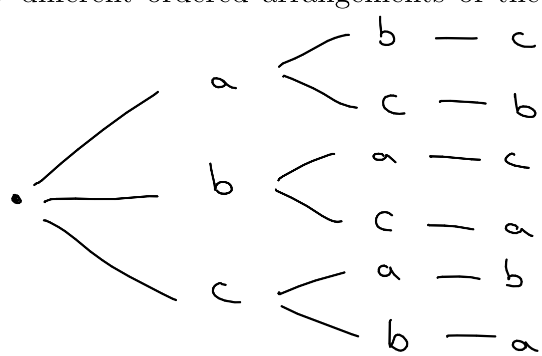
The # of license plates with no repeated numbers or letters is

$10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 =$

1.3 Permutations

Example 1.3.1. How many different ordered arrangements of the letters a, b, and c are possible?

abc
acb
bac
bca
cab
cba



3 2 1

# of different arrangements is $3 \cdot 2 \cdot 1 = 6$.

Each of these ordered arrangements is called a *permutation* of the set $\{a, b, c\}$.

Fact 1.3.2. The number of different permutations of a set of n distinct objects is:

$n \cdot (n-1) \cdot (n-2) \cdots \cdot 2 \cdot 1$

Notation 1.3.3. When n is a positive integer, we let $n! = n(n-1)(n-2)\cdots 2 \cdot 1$

- $3! = 3 \cdot 2 \cdot 1 = 6$
- $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

It is also convenient to define $0! = 1$

Example 1.3.4. Lucas has 19 books that he wants to arrange on a shelf:

- 7 books on graph theory;
- 5 books on combinatorics on words;
- 4 books on probability; and
- 3 books on number theory.

If he wants to keep all books on the same subject matter together on the shelf, then how many different arrangements are possible?

There are:
- $7!$ ways to arrange the books on graph theory
- $5!$ " " " " " " CoW
- $4!$ " " " " " " probability
- $3!$ " " " " " " # theory
- $4!$ ways to arrange the four subjects.

$\therefore$ Lucas can arrange the books in

$7! \cdot 5! \cdot 4! \cdot 3! \cdot 4!$ different ways.

Example 1.3.5. Lucas has 3 pieces of carrot, 4 pieces of apple, and 2 pieces of cheese. He gives his small dog Ernie a single one of these treats each time Ernie does a trick, until all of the treats are gone. In how many different orders can Lucas give Ernie his treats?

Let τ be the number of different arrangements of the 9 treats, where treats of the same type are indistinguishable.

For each arrangement in τ, I can arrange the carrots in $3!$ ways, the apples in $4!$ ways, and the cheese in $2!$ ways. Doing this for all arrangements in τ, we get all permutations of 9 distinct objects.

So we have $3! \cdot 4! \cdot 2! \cdot |\tau| = 9!$

$$\Rightarrow |\tau| = \frac{9!}{3!\,4!\,2!}$$

1.4 Combinations

Example 1.4.1. If a pizza place offers 8 different vegetarian toppings, then how many ways can a vegetarian order a 3-topping pizza?

$$\_\ \_\ \_\ |\ \_\ \_\ \_\ \_\ \_ \qquad \frac{8!}{3!\,5!} = \binom{8}{3}$$

Notation 1.4.2. For integers r and n with $0 \leq r \leq n$, we let

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

We say that this is the number of *combinations* of n objects taken r at a time.

- $\binom{7}{2} = \frac{7!}{2!\,5!} = \frac{7 \cdot 6}{2!} = 21$
- $\binom{3}{0} = 1$
- $\binom{8}{3} = \frac{8!}{3!\,5!} =$
- $\binom{5}{5} = 1$

When $r < 0$ or $r > n$, it is convenient to define $\binom{n}{r} = 0$

Example 1.4.3. The Kamloops SPCA currently has 7 dogs and 9 cats available for adoption. How many ways can a person adopt...

- 5 animals total?

$$\binom{16}{5} = \frac{16!}{5!\,11!} =$$

- 3 dogs and 2 cats?

$$\binom{7}{3} \cdot \binom{9}{2} =$$

- 5 animals total, at least one of which is a dog?

$$\underbrace{\binom{7}{1}}_{\text{one dog}} \cdot \underbrace{\binom{15}{4}}_{\text{4 other animals}} = 9555$$

Whoops! There is some overcounting here!

or

$$\binom{16}{5} - \binom{9}{5} = 4242$$

This is the correct answer.

Example 1.4.4. Pippin is shocked that Gandalf does not seem to know about several of the seven daily meals that hobbits normally observe, in the following order: breakfast, second breakfast, elevenses, luncheon, afternoon tea, dinner, and supper. Pippin and Gandalf eventually reach an agreement that they will eat four meals a day, in such a way that no two consecutive meals are skipped on a given day. How many ways can this be done?

Let $\circledcirc$ denote a meal that gets eaten, and let $\times$ denote a meal that gets skipped.

One possible arrangement is $\times \circledcirc \circledcirc \times \circledcirc \times \circledcirc$

Consider the 4 meals that we eat:

$$\times \quad \circledcirc \quad \times \quad \circledcirc \quad \times \quad \circledcirc \quad \times \quad \circledcirc \quad \times$$

We can skip at most one meal between each of these four meals. (A red $\times$ denotes a possible skip.)

There are $\binom{5}{3}$ ways to choose the positions of the meals that we skip.

So there are $\binom{5}{3}$ ways to dine in the agreed upon manner.

The values $\binom{n}{r}$ are often called *binomial coefficients* because of their role in the following theorem.

Theorem 1.4.5 (The binomial theorem)**.** *For every $n \geq 0$, we have*

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \dots + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n} x^n y^0$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Example 1.4.6. Expand $(3x+y^2)^4$.

$$(3x+y^2)^4 = \binom{4}{0}(3x)^0(y^2)^4 + \binom{4}{1}(3x)^1(y^2)^3 + \binom{4}{2}(3x)^2(y^2)^2$$

$$+ \binom{4}{3}(3x)^3(y^2)^1 + \binom{4}{4}(3x)^4(y^3)^0$$

$$= y^8 + 12xy^6 + 54x^2y^4 + 108x^3y^2 + 81x^4$$

1.5 Multinomial Coefficients

Example 1.5.1. Ten distinct books are to be awarded as prizes in a math competition: four to the first place winner, three to second place, two to third place, and one to fourth place. How many different ways can the prizes be awarded?

The first-place winner can pick their books in $\binom{10}{4}$ ways,
then the second-place " " " " " " $\binom{6}{3}$ ways,
the third-place " " " " " " $\binom{3}{2}$ ways,
and the fourth-place " " " " " " $\binom{1}{1}$ ways

So the number of different ways that the prizes can be awarded is:

$$\binom{10}{4}\cdot\binom{6}{3}\cdot\binom{3}{2}\cdot\binom{1}{1} = \frac{10!}{4!\,\cancel{6!}}\cdot\frac{\cancel{6!}}{3!\,\cancel{3!}}\cdot\frac{\cancel{3!}}{2!\,\cancel{1!}}\cdot\frac{\cancel{1!}}{1!\,\cancel{0!}}$$

$$= \frac{10!}{4!\,3!\,2!\,1!}$$

$$=$$

Notation 1.5.2. If $n_1 + n_2 + \ldots + n_r = n$, then we let

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

This represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Note: When $r = 2$, we have

but $n_1 + n_2 = n$, so $n_2 = n - n_1$

$$\binom{n}{n_1, n_2} = \frac{n!}{n_1!\, n_2!} = \frac{n!}{n_1!\,(n-n_1)!} = \binom{n}{n_1}$$

which is just a binomial coefficient. The values $\binom{n}{n_1,n_2,\ldots,n_r}$ are often called *multinomial coefficients*.

Theorem 1.5.3 (The multinomial theorem). *For every $n \geq 0$, we have*

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, n_2, \ldots, n_r): \\ n_1 + n_2 + \ldots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Example 1.5.4. The 4-player card game Euchre uses a deck of 24 distinct cards. Before play starts, each player is dealt 5 cards, and then the top card among the remaining 4 cards is turned face up. How many different deals are possible?

Hand 2
5 cards

Hand 1
5 cards

Kitty
4 cards,
1 face up

Hand 3
5 cards

Hand 4
5 cards

We are sorting 24 cards into 5 distinct groups of sizes 5, 5, 5, 5, and 4. We can do this in

$$\binom{24}{5,5,5,5,4} = \frac{24!}{(5!)^4\, 4!}$$

For each division, there are 4 different ways that the card in the middle can be flipped face up.

So the total number of different deals in the game of Euchre is

$$\frac{24!}{(5!)^4\, 4!} \cdot 4 \approx 4.98 \times 10^{14}$$

Example 1.5.5. A class of 25 students is to be divided into four groups as evenly as possible. How many ways can this be done? (Assume that groups of the same size are interchangeable.)

The groups will have sizes 6, 6, 6, and 7.

There are $\binom{25}{6,6,6,7} = \frac{25!}{(6!)^3\, 7!}$ ways to divide the class into distinct groups of sizes 6, 6, 6, and 7, say

$$\underbrace{\boxed{\text{Group 1}}_{6} \quad \boxed{\text{Group 2}}_{6} \quad \boxed{\text{Group 3}}_{6}} \quad \boxed{\text{Group 4}}_{7}$$

But we can reorder these groups in $3! = 6$ ways.

So the total number of divisions is

$$\frac{25!}{(6!)^3\, 7! \cdot 3!}$$

Summary

- **The Basic Principle of Counting.** If $r \geq 2$ experiments are to be performed one after another, and the ith experiment has n_i possible outcomes, then the total number of possible outcomes of the r experiments is

$$n_1 \cdot n_2 \cdot n_3 \cdot \dots \cdot n_r$$

- **Permutations.** The number of ways to order n distinct objects is $n!$.

- **Combinations/Binomial Coefficients.** The number of ways to choose a subset of r objects from among n distinct objects is

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

- **The Binomial Theorem.**

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

- **Multinomial Coefficients.** The number of ways to divide n distinct objects into r distinct groups of sizes $n_1, n_2, \ldots, n_r$ (where $n_1 + n_2 + \cdots + n_r = n$) is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n}{n_1!\, n_2! \cdots n_r!}$$

General Tips

- Permutations are for lists (order matters), while combinations are for groups (order doesn't matter).

- One can often count things in more than one way, but some ways are easier than others.

  - If you are struggling to count arrangements of a certain type, especially if you are having to deal with many different cases, then try counting the arrangements that are NOT of the type you want, and subtracting that number from the total number of arrangements.

- Read through Chapter 1 of the textbook. There are *many* excellent examples.

- Try lots of practice problems. Work with others and ask questions.