MATH 3030, Winter 2023

Instructor: Lucas Mol

Written Assignment 5

Due: Thursday, March 16

Guidelines:

- Try to explain your reasoning as clearly as possible. Feel free to reference theorems, lemmas, or anything else in the notes by number. For example, you could say "By Theorem 2.1.4,..."
- Please make sure that the work you submit is organized. Complete rough work on a different page, and then write up a good copy of your final solutions. Make sure to indicate each problem number clearly.
- Working with others is allowed (and encouraged!) but copying is strictly prohibited, and is a form of academic misconduct. Every student should write up their own good copy. Your work should not be the same, word-for-word, as any other student's work.
- The more work you put into these assignments, the more you will get out of them. **Please think about the problems before looking for solutions online.**

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1. There are N individuals in a population, some of whom have a certain infection that spreads as follows. Contacts between two members of this population occur in accordance with a Poisson process having rate λ. When a contact occurs, it is equally likely to involve any of the $\binom{N}{2}$ pairs of individuals in the population. If a contact involves an infected and a noninfected individual, then with probability p the noninfected individual becomes infected. Once infected, an individual remains infected throughout. Let $X(t)$ denote the number of infected members of the population at time t.

   (a) Explain why $\{X(t), t \geq 0\}$ is a continuous-time Markov chain. What type of continuous-time Markov chain is it? Specify the relevant rates of the chain.

   Solution: The process $\{X(t), t \geq 0\}$ is a continuous-time Markov chain because given that $X(t) = i$, the probability that $X(t+s) = j$ is independent of $X(u)$ for $0 \leq u < t$. In other words, given the present state, any future state is independent of the past states. We will see that whenever we are in state i, the time we spend in state i is exponential.

   Since infected individuals remain infected throughout, this is a *pure birth process.* Whenever we are in state i, the next transition is always to state $i+1$. (This transition occurs when an uninfected individual becomes infected.)

   Now let's find the rate v_i at which we leave state i. First note that when $i = N$, all individuals are infected, and it will remain this way forever. So we have $v_N = 0$. Now suppose that $1 \leq i < N$. So we have i infected individuals and $N - i$ noninfected individuals. Two individuals meet according to a Poisson process with rate λ. Call such a meeting a Type 1 meeting if it results in a transmission of the infection to a noninfected individual, and a Type 2 meeting otherwise. By Proposition 5.3.19, Type 1 meetings occur according to a Poisson process with rate λp_i, where p_i is the probability of a Type 1 meeting when i individuals are infected. So we need to determine the probability p_i. In order for the infection to be transmitted to a noninfected individual, we need an infected and a noninfected individual to meet—this occurs with probability

$$\frac{i \cdot (N-i)}{\binom{N}{2}}.$$

Given that an infected and a noninfected individual meet, the probability that a transmission occurs is p. So we have

$$p_i = \frac{i \cdot (N-i)}{\binom{N}{2}} \cdot p.$$

Thus, when we have i infected individuals (where $1 \leq i < N$), the rate at which a new transmission occurs is

$$v_i = \lambda p_i = \frac{\lambda \cdot p \cdot i \cdot (N-i)}{\binom{N}{2}}.$$

(Note that $P_{i,i+1} = 1$ in this case.)

(b) Starting with a single infected individual, what is the expected time until all N members of the population are infected? (You may leave your answer as a sum.)

Solution: Let T be the total time until all N members of the population are infected. For $1 \leq i < N$, let T_i be the time that it takes for the process to transition from state i to state $i+1$. Since we transition from state i to state $i+1$ with probability 1, we see that T_i is exponential with rate v_i, and mean $\frac{1}{v_i}$. Since we start with one infected individual, we have

$$T = \sum_{i=1}^{N-1} T_i.$$

Hence we have

$$E[T] = \sum_{i=1}^{N-1} E[T_i] = \sum_{i=1}^{N-1} \frac{1}{v_i} = \sum_{i=1}^{N-1} \frac{\binom{N}{2}}{\lambda \cdot p \cdot i \cdot (N-i)}.$$

Phew.

2. Consider the $M/M/3$ queuing system, where customers arrive according to a Poisson process with rate λ, and service times with each of the three servers are exponential with rate μ.

(a) If the system is currently empty, find the expected time until all three servers are busy.

Solution: For the $M/M/3$ queue, we have

- Birth rates: $\lambda_n = \lambda$ for all $n \geq 0$
- Death rates: $\mu_n = \begin{cases} n\mu & \text{if } 1 \leq n \leq 3; \\ 3\mu & \text{if } n > 3. \end{cases}$

For all $i \geq 0$, let T_i be the time, starting form state i, until the process enters state $i+1$. Then we want

$$E[T_0] + E[T_1] + E[T_3].$$

Using the result of Proposition 6.3.7, we have

$$\begin{aligned}
E[T_0] &= \frac{1}{\lambda_0} = \frac{1}{\lambda}, \\
E[T_1] &= \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}E[T_0] = \frac{1}{\lambda} + \frac{\mu}{\lambda}\cdot\frac{1}{\lambda} = \frac{1}{\lambda} + \frac{\mu}{\lambda^2}, \text{ and} \\
E[T_2] &= \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}E[T_1] = \frac{1}{\lambda} + \frac{2\mu}{\lambda}\left(\frac{1}{\lambda} + \frac{\mu}{\lambda^2}\right) = \frac{1}{\lambda} + \frac{2\mu}{\lambda^2} + \frac{2\mu^2}{\lambda^3}.
\end{aligned}$$

Adding these three expressions gives

$$E[T] = \frac{3}{\lambda} + \frac{3\mu}{\lambda^2} + \frac{2\mu^2}{\lambda^3}.$$

(b) What is the expected time from part (a) when $\lambda = 2$/hour and $\mu = 1$/hour? When $\lambda = 4$/hour and $\mu = 1$/hour? (Think about why these answers make sense relative to one another.)

Solution: Substituting $\lambda = 2$ and $\mu = 1$, we find

$$E[T] = \frac{3}{2} + \frac{3}{2^2} + \frac{2}{2^3} = \frac{5}{2},$$

so we expect it to take two and half hours for all three servers to be busy.
Substituting $\lambda = 4$ and $\mu = 1$, we find

$$E[T] = \frac{3}{4} + \frac{3}{4^2} + \frac{2}{4^3} = \frac{31}{32},$$

so we expect it to take just under an hour for all three servers to be busy.
It makes sense that when customers arrive at a faster rate (and the service rate remains the same), the expected time until all three servers are busy goes down.

3. The following concern the Kolmogorov forward equations.

(a) Prove the Kolmogorov forward equations, assuming that we can interchange the limit with summation.

Hint: You can mimic the proof of the backward equations from the notes – just switch the roles of t and h when you use the Chapman-Kolmogorov equations.

Solution: From the Chapman-Kolmogorov equations, we have

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{\text{all } k} P_{ik}(t)P_{kj}(h) - P_{ij}(t).$$

Breaking off the $k = j$ term in the sum, we find

$$\begin{aligned} P_{ij}(t+h) - P_{ij}(t) &= \sum_{k \neq j} P_{ik}(t)P_{kj}(h) + P_{ij}(t)P_{jj}(h) - P_{ij}(t) \\ &= \sum_{k \neq j} P_{ik}(t)P_{kj}(h) - [1 - P_{jj}(h)]P_{ij}(t). \end{aligned}$$

By Lemma 6.4.5, we have

$$\lim_{h \to 0} \frac{P_{kj}(h)}{h} = q_{kj} \quad \text{and} \quad \lim_{h \to 0} \frac{1 - P_{jj}(h)}{h} = v_j.$$

From the definition of the derivative, and assuming that we can interchange limit and sum, we have

$$\begin{aligned} P'_{ij}(t) &= \lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} \\ &= \lim_{h \to 0} \left[\sum_{k \neq j} \left(P_{ik}(t) \cdot \tfrac{P_{kj}(h)}{h} \right) - \tfrac{1 - P_{jj}(h)}{h} \cdot P_{ij}(t) \right] \\ &= \sum_{k \neq j} \left(P_{ik}(t) \lim_{h \to 0} \tfrac{P_{kj}(h)}{h} \right) - \lim_{h \to 0} \tfrac{1 - P_{jj}(h)}{h} \cdot P_{ij}(t) \\ &= \sum_{k \neq j} P_{ik}(t)q_{kj} - v_j P_{ij}(t). \end{aligned}$$

(b) Write the Kolmogorov forward equations for a birth and death process with birth rates $\lambda_n = \lambda$ for all $n \geq 0$ and death rates $\mu_n = \mu$ for all $n \geq 1$.

Solution: Note that we can enter state j only from state $j - 1$ or from state $j + 1$. (Unless $j = 0$; we can only enter state 0 from state 1.) So we have

$$\begin{aligned} P'_{i0}(t) &= P_{i1}(t)q_{10} - v_0 P_{i0}(t) \\ &= \mu P_{i1}(t) - \lambda P_{i0}(t) \end{aligned}$$

and

$$\begin{aligned} P'_{ij}(t) &= P_{i,j-1}(t)q_{j-1,j} + P_{i,j+1}(t)q_{j+1,j} - v_j P_{ij}(t) \\ &= \lambda P_{i,j-1}(t) + \mu P_{i,j+1}(t) - (\lambda + \mu)P_{ij}(t). \end{aligned}$$

4. A small barbershop, operated by a single barber, has room for at most two customers (one getting a haircut, and one waiting for a haircut). Potential customers arrive according to a Poisson process at a rate of four per hour (if the shop is full, they simply pass by), and the length of each haircut is exponential with mean 20 minutes.

   (a) In the long run, what proportion of potential customers enter the shop?

Solution: Let $N(t)$ be the number of customers in the shop at time t. Then $\{N(t),\ t \geq 0\}$ is a continuous-time Markov chain with states 0, 1, and 2. We will let

$$q_{01} = q_{12} = \lambda,$$

and

$$q_{10} = q_{21} = \mu.$$

For this first part, λ will be 4 and μ will be 3, but we solve everything with λ and μ as variables because μ will change to 6 in part (c). ☺

To find the limiting probabilities π_0, π_1, and π_2, we solve the system of equations below:

$$\begin{cases} \lambda\pi_0 = \mu\pi_1 \\ (\lambda+\mu)\pi_1 = \lambda\pi_0 + \mu\pi_2 \\ \mu\pi_2 = \lambda\pi_1 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

(The first three equations are the *balance equations* for states 0, 1, and 2, respectively.) From the first equation, we find

$$\pi_1 = \frac{\lambda}{\mu}\pi_0,$$

and from the third equation, we find

$$\pi_2 = \frac{\lambda}{\mu}\pi_1 = \frac{\lambda^2}{\mu^2}\pi_0.$$

Substituting into the fourth equation, we find

$$\pi_0\left[1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}\right] = 1,$$

so

$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}}.$$

Back-substitution gives

$$\pi_1 = \frac{\frac{\lambda}{\mu}}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}} \quad \text{and} \quad \pi_2 = \frac{\frac{\lambda^2}{\mu^2}}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}}.$$

So with $\lambda = 4$ and $\mu = 3$, the proportion of customers that enter the shop in the long run is

$$\pi_0 + \pi_1 = \frac{1 + \frac{4}{3}}{1 + \frac{4}{3} + \frac{4^2}{3^2}} = \tfrac{21}{37}.$$

(b) What is the average number of customers in the shop?

> **Solution:** The long-run average number of customers in the shop is
>
> $$\sum_{n=0}^{2} n \cdot \pi_n = 0 \cdot \pi_0 + 1 \cdot \pi_1 + 2 \cdot \pi_2 = 0 \cdot \tfrac{9}{37} + 1 \cdot \tfrac{12}{37} + 2 \cdot \tfrac{16}{37} = \tfrac{44}{37}.$$

(c) If the barber could work twice as fast, how much more business would they do?

> **Solution:** We substitute $\lambda = 4$ and $\mu = 6$ in our equations from part (a) to find that the new long-run proportion of customers that enter the shop is now
>
> $$\pi_0 + \pi_1 = \frac{1 + \frac{4}{6}}{1 + \frac{4}{6} + \frac{4^2}{6^2}} = \tfrac{15}{19}.$$
>
> So the ratio of the proportion of customers being served at this new speed compared to the proportion of customers being served at the original speed is
>
> $$\frac{\frac{15}{19}}{\frac{21}{37}} = \tfrac{185}{133}.$$
>
> Therefore, the barber will do $\frac{185}{133}$ times as much business as before. (Note that even though the barber can cut hair faster, we're assuming that the customers are still arriving at the same rate! If the barber cuts twice as fast, they won't necessarily serve twice as many customers – their chairs will just be empty more often!)

5. When Lucas and Ernie are home on the weekend, they usually go out for several walks. Suppose that the length of time between successive walks is an exponential random variable with mean 4 hours. Each time they decide to go for a walk, it is independently a "long" walk with probability $\frac{1}{3}$, which lasts an exponential amount of time with mean 1 hour, and a "short" walk with probability $\frac{2}{3}$, which lasts an exponential amount of time with mean 0.25 hours. In the long run, what proportion of time do they spend out walking?

> **Solution:** We have a continuous-time Markov chain with three states:
>
> - State 0: Lucas and Ernie are home.
> - State 1: Lucas and Ernie are out for a "long" walk.
> - State 2: Lucas and Ernie are out for a "short" walk.
>
> The rates (per hour) that we leave states 0, 1, and 2 are
>
> $$v_0 = \tfrac{1}{4}, \quad v_1 = 1, \quad \text{and} \quad v_2 = 4.$$

The transition probabilities when we leave state 0 are

$$P_{01} = \tfrac{1}{3} \quad \text{and} \quad P_{02} = \tfrac{2}{3},$$

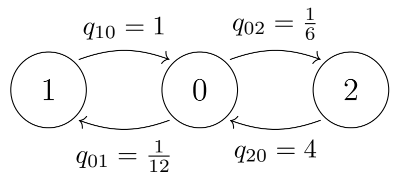
so the transition rates from 0 are

$$q_{01} = v_0 P_{01} = \tfrac{1}{12} \quad \text{and} \quad q_{02} = v_0 P_{02} = \tfrac{1}{6}.$$

Finally, when Lucas and Ernie finish a walk, we assume that they return home with probability 1. So we have

$$q_{10} = v_1 = 1 \quad \text{and} \quad q_{20} = v_2 = 4.$$

To summarize, the following diagram shows the transition rates.



To find the limiting probabilities π_0, π_1, and π_2, we solve the system

$$\begin{cases} \frac{1}{4}\pi_0 = \pi_1 + 4\pi_2 \\ \pi_1 = \frac{1}{12}\pi_0 \\ 4\pi_2 = \frac{1}{6}\pi_0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

(The first three equations are the *balance equations* for states 0, 1, and 2, respectively.) From the second and third equations, respectively, we have

$$\pi_1 = \tfrac{1}{12}\pi_0 \quad \text{and} \quad \pi_2 = \tfrac{1}{24}\pi_0.$$

Substituting into the last equation, we find

$$\pi_0 + \tfrac{1}{12}\pi_0 + \tfrac{1}{24}\pi_0 = 1,$$

which we solve to find

$$\pi_0 = \tfrac{24}{27}.$$

So the long-run proportion of time that they spend out walking is

$$1 - \pi_0 = \tfrac{3}{27} = \tfrac{1}{9}.$$

Hooray!