MATH 3030, Winter 2023

Instructor: Lucas Mol

Written Assignment 2

Due: Tuesday, January 31

Guidelines:

- Try to explain your reasoning as clearly as possible. Feel free to reference theorems, lemmas, or anything else in the notes by number. For example, you could say "By Theorem 2.1.4,..."
- Please make sure that the work you submit is organized. Complete rough work on a different page, and then write up a good copy of your final solutions. Make sure to indicate each problem number clearly.
- Working with others is allowed (and encouraged!) but copying is strictly prohibited, and is a form of academic misconduct. Every student should write up their own good copy. Your work should not be the same, word-for-word, as any other student's work.
- The more work you put into these assignments, the more you will get out of them. **Please think about the problems before looking for solutions online.**

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1. Suppose that Coin 1 comes up heads with probability $p_1 = \frac{1}{2}$, while Coin 2 comes up heads with probability $\frac{1}{5}$. We pick up one of the two coins and flip it repeatedly until it turns up heads. Once it turns up heads, we switch to the other coin, and flip it repeatedly until it turns up heads. We continue in this manner, switching coins whenever the one we are flipping turns up heads. For all $n \geq 0$, let X_n be the number of the coin that we will flip immediately after the nth flip. For example, suppose that we start with coin 1. This means that $X_0 = 1$. If it comes up tails on the first flip, then $X_1 = 1$ as well. If it comes up heads on the second flip, then we will switch to Coin 2 for the next flip, so $X_2 = 2$.

   (a) Explain why the process $\{X_n,\ n = 0, 1, 2, \ldots\}$ is a Markov chain, and find its transition probability matrix.

   Solution: Whenever we are flipping a coin, there is a fixed probability that we will be flipping each of the coins next, no matter what happened in the past. For example, if we are flipping Coin 1, the probability that we will next be flipping Coin 2 is $\frac{1}{2}$. Note that this is because each flip is independent of all other flips!

   The transition probability matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} \end{array}$$

(b) If we start by flipping Coin 1, what is the probability that we will be flipping Coin 2 after 4 flips? After 10 flips?

> **Solution:** We use Sage to compute P^4 and P^{10}, and read off
>
> $$P_{12}^{(4)} \approx 0.709$$
>
> and
>
> $$P_{12}^{(10)} \approx 0.714$$

(c) Find the long-run proportion of time that we will be flipping each coin.

> **Solution:** We solve the system
>
> $$\begin{cases} \pi_1 = 0.5\pi_1 + 0.2\pi_2 \\ \pi_2 = 0.5\pi_1 + 0.8\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$
>
> This can be done by hand or using Sage. We find
>
> $$\pi_1 = \tfrac{2}{7}, \ \pi_2 = \tfrac{5}{7}.$$
>
> Note that this lines up with our answers to part (b).

(d) In the long-run, what proportion of flips land heads?

> **Solution:** The long-run proportion of flips that land heads is equal to
>
> $$\pi_1 \cdot P(\text{Coin 1 lands heads}) + \pi_2 \cdot P(\text{Coin 2 lands heads}) = \tfrac{2}{7} \cdot \tfrac{1}{2} + \tfrac{5}{7} \cdot \tfrac{1}{5} = \tfrac{2}{7}.$$
>
> Note that we are essentially conditioning on which coin is being flipped. ☺

2. Lucas has two sets of crampons (ice spikes for shoes) that he uses to climb the icy hill on the way to school. But he is absent-minded, and sometimes forgets to wear them. So every morning, he has either 0, 1, or 2 pairs of crampons at home, and the others are in his office at school.

   - Every morning when he leaves home, he remembers that he should wear a pair of crampons with probability 0.8 (independently of how many pairs he has at home). If there are one or two pairs at home, then he wears one pair to school.
   - Every afternoon when he leaves school, he remembers that he should wear a pair of crampons with probability 0.5. If there are one or two pairs in his office, then we wears one pair home.

   (a) Find the long-run proportion of time that he has no crampons at home in the morning.

Solution: For $n = 0, 1, 2, \ldots$, let X_n denote the number of pairs of crampons that Lucas has at home when he wakes up on the nth day. Then $\{X_n,\ n = 0, 1, 2, \ldots\}$ is a Markov Chain with transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.8(0.5) & 0.8(0.5) + 0.2(0.5) & 0.2(0.5) \\ 0 & 0.8(0.5) & 0.2 + 0.8(0.5) \end{bmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.6 \end{bmatrix} \end{array}.$$

For example, the probability P_{10} is the probability that he has 0 pairs at home tomorrow morning, given that he had one pair at home this morning. This is the probability that he remembers to wear a pair to school, but forgets to wear a pair home: $P_{10} = 0.8 \cdot 0.5$.

The long-run proportion of time that he has no crampons at home in the morning is our friend π_0. We find all of the long-run proportions by solving the system

$$\begin{cases} \pi_0 = 0.5\pi_0 + 0.4\pi_1 \\ \pi_1 = 0.5\pi_0 + 0.5\pi_1 + 0.4\pi_2 \\ \pi_2 = 0.1\pi_1 + 0.6\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

Using Sage, we find

$$\pi_0 = \tfrac{16}{41},\ \pi_1 = \tfrac{20}{41},\ \pi_2 = \tfrac{5}{41}.$$

(b) Find the long-run proportion of time that he walks to school without crampons.

Solution: Conditioning on the number of pairs that he has at home, the desired long-run proportion is

$$\pi_0 + \pi_1 \cdot 0.2 + \pi_2 \cdot 0.2 = \tfrac{16}{41} + \tfrac{20}{41} \cdot \tfrac{1}{5} + \tfrac{5}{41} \cdot \tfrac{1}{5} = \tfrac{16}{41} + \tfrac{4}{41} + \tfrac{1}{41} = \tfrac{21}{41}.$$

3. In class, we demonstrated that the one-dimensional symmetric random walk is recurrent. Now consider the *two-dimensional symmetric random walk*, where the state space is $\mathbb{Z} \times \mathbb{Z}$, the set of ordered pairs of integers. Starting from any state, the process moves up, down, left, or right with probability $\frac{1}{4}$. It is proven on Pages 210–211 of the textbook that this process is recurrent, i.e., that every state is recurrent. Read that proof and answer the following questions. Short explanations of just one to four sentences should suffice.

(a) Explain why it suffices to show that any one state is recurrent.

Solution: Note that all states communicate – there is a positive probability of transitioning from state (i, j) to state (k, ℓ) in $|i-k|+|j-\ell|$ steps. In fact, this probability is $1/4^{|i-k|+|j-\ell|}$. Therefore, all states are in the same class, so if one state is recurrent, then all states are recurrent.

(b) Explain the statement "Now after $2n$ steps, the chain will be back in its original location if for some i, $0 \leq i \leq n$, the $2n$ steps consist of i steps to the left, i to the right, $n-i$ up, and $n-i$ down."

Solution: In order to return to the original location in $2n$ steps, the number of steps to the left must equal the number of steps to the right, and the number of steps up must equal the number of steps down. So if we take some number of steps to the left, say i, then we must take i steps to the right. This means that i must be between 0 and n. The remaining $2n-2i$ steps must be either up or down, and since we take the same number of steps up as we do down, we must take $n-i$ steps up and $n-i$ steps down.

(c) Prove the combinatorial identity

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i}$$

by counting the number of ways to select a set of n balls from an urn containing n white balls and n black balls in two different ways.

Solution: Let N be the number of ways to select a set of n balls from an urn containing n white balls and n black balls. Then we have

$$N = \binom{2n}{n}.$$

Now let N_i be the number of ways to select n balls from the urn, exactly i of which are white. Then

$$N_i = \binom{n}{i} \cdot \binom{n}{n-i}.$$

Note that

$$N = \sum_{i=0}^{n} N_i,$$

so we conclude that

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \cdot \binom{n}{n-i}.$$

(d) At the end of the proof, it is deduced that $P_{00}^{(2n)} \sim \frac{1}{\pi n}$. Explain why it follows that

$$\sum_{n=1}^{\infty} P_{00}^{(2n)}$$

diverges.

Solution: Recall that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. It follows that the series $\sum_{n=1}^{\infty} \frac{1}{\pi n}$ is also divergent, since it is a constant multiple of the divergent harmonic series. Since $P_{00}^{(2n)} \sim \frac{1}{\pi n}$, we have

$$\lim_{n \to \infty} \frac{P_{00}^{(2n)}}{\frac{1}{\pi n}} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{\pi n}$ diverges, we conclude by the Limit Comparison Test that $\sum_{n=1}^{\infty} P_{00}^{(2n)}$ also diverges.