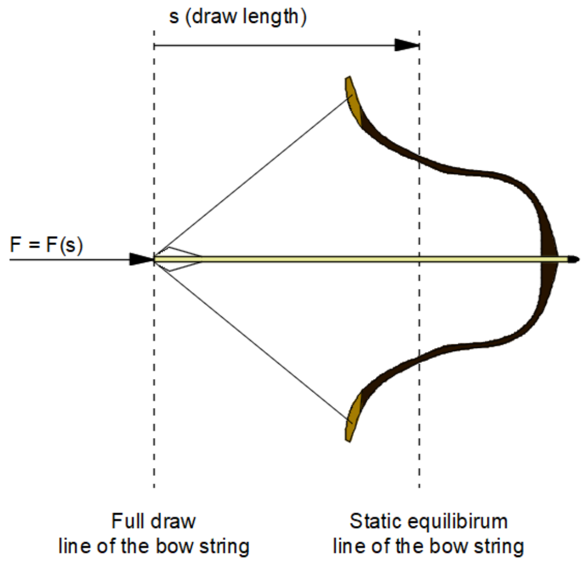
## Motivation

This homework includes a very limited knowledge used/given in your coursework. However, it is planned to give you an insight to solve real engineering problems and show you the difference between the analytical and numerical approaches to those problems. Analytical approach is a more direct approach and it enlightens the real physics behind a phenomenon however, numerical approach provides more flexibility and complexity to the problem while making its solution possible. The homework will also force you to use at least MS Excel or other programming/scripting languages such as Python or MATLAB which will be very helpful for your next course's assignments too.

Since all of those above, most of the knowledge, equations, definitions etc. are already given in the "**Problem Definition**" part. Your work is to implement this knowledge to a simple computer program which will have around 100-120 code lines.

Good luck!

## Problem Definition



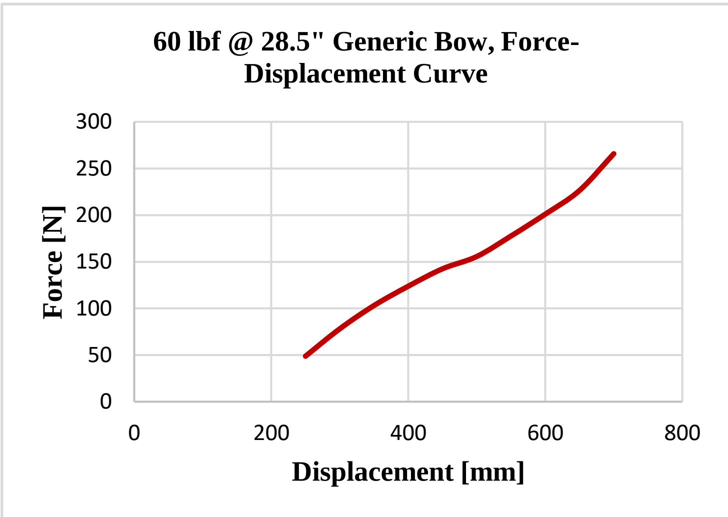




**Figure 1:** Schematic of a generic traditional bow (left) and its force-displacement curve (right).

Composite bow's history goes back to 2000 BC [1] as these bows utilized nonwooden materials alongside the wooden structure such as sinews and animal horns with animal based natural adhesives to increase the stiffness, mobility and decrease the weight of this weapon. The ballistic investigation of an arrow released from a bow is a very difficult task, as it includes many disciplines of mechanical engineering such as fluid mechanics, solid mechanics and thermodynamics etc. however, one can model the dynamics of a simple arrow flight motion with a low fidelity yet with a fair accuracy up to some extent.

**Figure 1** shows the schematic of a traditional composite bow at a full draw position and the force-displacement curve of a generic recurved composite war bow (This data is provided in "generic_bow_draw_weight.xlsx").

The kinematic and the dynamic FBD (free body diagram) of a flying arrow is shown in **Figure 2**. As it is seen on the diagram, the motion has 3 DOF (degree of freedom) which are the longitudinal (**x**), the vertical (**y**) and the angular motion (**θ**) of the arrow body which means; a system of three second order differential equation is enough to solve this motion. Since the mass moment of inertia ($\mathbf{J_P}$) of the arrow changes wrt. the global frame (OXY) as the arrow moves, it is convenient to use the body fixed coordinate system for the differential equations. This requires to derive the translational acceleration values $\mathbf{a_x}$ and $\mathbf{a_y}$ (wrt. the body fixed system) as in (1) and (2):

$$a_x(t) = \frac{d[v_x(t)]}{dt} - v_y \frac{d[\dot{\theta}(t)]}{dt} \quad (1)$$

$$a_y(t) = \frac{d[v_y(t)]}{dt} + v_x \frac{d[\dot{\theta}(t)]}{dt} \quad (2)$$

The arrow undertakes the gravitational force, the aerodynamic drag (**$F_D$**) and lift forces (**$F_L$**) and the aerodynamic pitching moment (**$M_P$**) during its flight. Aerodynamic effects are modelled as second order equations given in (3), (4) and (5):

$$F_D = \frac{1}{2}\rho C_D A_s v^2 \quad (3)$$

$$F_L = \frac{1}{2}\rho C_L A_s v^2 \quad (4)$$

$$M_P = \frac{1}{2}\rho C_{PM} A_s v^2 L \quad (5)$$

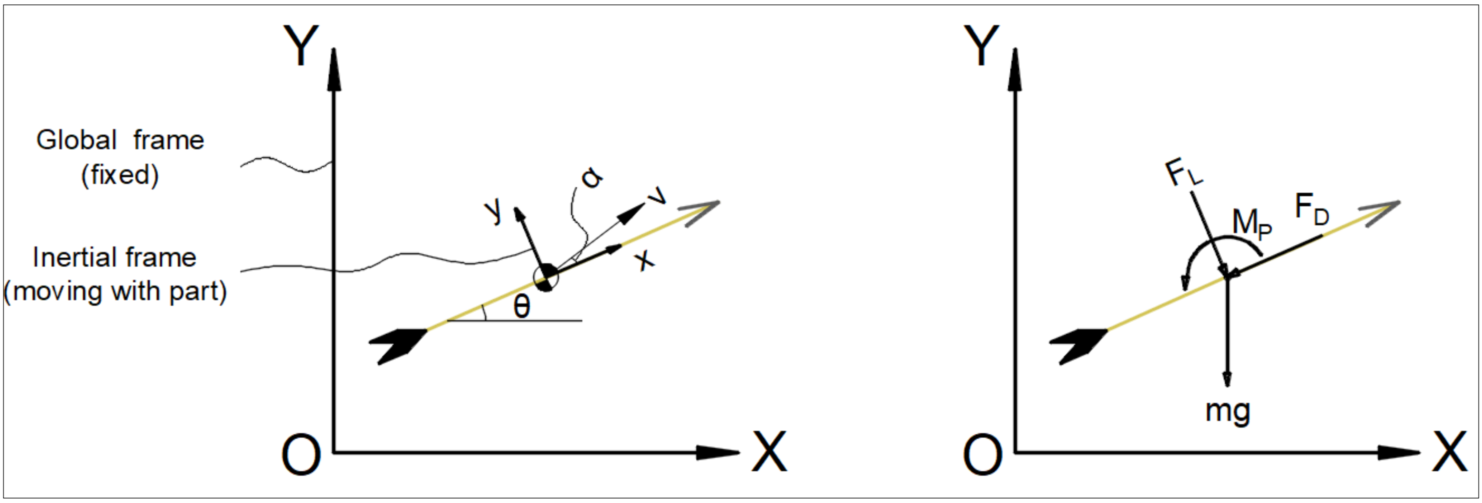



**Figure 2:** The kinematic and the dynamic FBD (free body diagram) of planar motion of an arrow.

For the used arrow, whose aerodynamic properties are measured and modeled, the coefficient $\boldsymbol{C_D}$ (drag) can be taken constant for this problem's arrow flight speed and angle of attack (**$\alpha$**) interval, however $\boldsymbol{C_L}$ (lift) and the $\boldsymbol{C_{PM}}$ (pitch) coefficients will be modeled as an approximated linear function of **$\alpha$** which is the angle between the arrow longitudinal axis (x) and the arrow velocity vector, as following [2]:

$$C_L(\alpha) = 0.73\alpha \quad (6)$$

$$C_{PM}(\alpha) = 0.325\alpha \quad (7)$$

Although some physical phenomenon is included in the model described till here, a lot of them are also neglected for the sake of simplicity and possibility of the solution:

- The air is windless for the duration of motion.
- In reality, because of the fletches at the back, the arrow rotates around its shaft axis, but this motion or DOF is neglected for this problem.
- In reality, the elasticity of the arrow shaft and the uneven weight distribution causes the shaft to undertake axial and bending vibrations which affects the flight. For this problem, the shaft is accepted as a rigid body.
- Archer's paradox is neglected, arrow makes a pure planar motion in XY plane.
- The transient dynamics during the arrow and string's joint motion is not considered, the differential equation's initial values start from the string's static equilibrium line where the arrow gains an initial speed which is calculated from the potential energy transferred from the bow to the arrow with an efficiency of **$\eta_{bow}$**.

## Question

An archer is making range shooting practice on an open field, holding the arrow on the position at $\mathbf{X_0}$ and $\mathbf{Y_0}$ and an initial releasing angle $\boldsymbol{\theta_0}$ coordinates wrt. the fixed global frame. The needed parameter definitions and their values are given in **Table 1**. Solve the following questions:

**a)** **(10%)** Calculate the arrow's initial velocity at release if the bow is fully drawn (@ 27.5 [inches] ≅ 700 [mm]), using the given force-displacement curve and the bow efficiency value.

**b)** **(20%)** **Write down** and **classify** the equations of motion of an arrow neglecting the angular motion (θ) of the arrow shaft and all the aerodynamic effects. Solve the problem analytically. [**Hint**: A short trip back to high school!]

**c)** **(20%)** Derive the equation for the releasing angle $\boldsymbol{\theta_{0,\,max}}$ which maximizes the bow's range and find the value of it and the range using your analytical solution.

**d)** **(30%)** **Write down** and **classify** the equations of 3 DOF motion of the arrow including the aerodynamic effects. Use **Euler backward formula (**see pg. 25 of the course's textbook**)** to solve the system, determine your own time step for your solver program**.**

**e)** **(20%)** Calculate and plot the trajectory of the arrow for your analytical solution **(b)** and the numerical solution **(d)** on the same graph. Compare the results and elaborate on them. Coordinate transformation from the body fixed and global inertia frame is used in trajectory calculation. (These are continuous integrals and need to be transformed to discrete summation):

$$X(t) = X_0 + \int (v_x \cos(\theta) - v_y \sin(\theta))dt \tag{8}$$

$$Y(t) = Y_0 + \int (v_x \sin(\theta) - v_y \cos(\theta))dt \tag{9}$$

**f)** **Bonus (40%)** Write a program that calculates numerically the releasing angle $\boldsymbol{\theta_{0,\,max}}$ for the maximum range.

➢ **Attention 1:** Parts **d, e** and **f** will be solved using computer programs written by yourselves. Coding/programming is a highly personal work and copying someone else's code can be easily spotted. Using copy-paste codes will be graded zero for those parts!

➢ **Attention 2:** Uploading written programs/excel files for part **d,e** and **f** is mandatory for taking points from those parts.

**Table 1:** Parameters and initial conditions for the solution.

| Parameter | Definition | Value | Unit |
|---|---|---|---|
| $\mathbf{m_{arrow}}$ | Arrow weight | 30 + id_1* | g |
| $\mathbf{J_p}$ | Arrow mass moment of inertia around its z axis. | 0,003 | $kgm^2$ |
| $\mathbf{A_{shaft}}$ | Arrow projectional area | 5,4106E-5 | $m^2$ |
| $\mathbf{C_D}$ | Arrow drag coefficient | 1.5 | [-] |
| **L** | Arrow length (characteristic length) | 800 | mm |
| $\mathbf{X_0}$ | Starting coordinates of the arrow wrt. Global frame. | 0 | m |
| $\mathbf{Y_0}$ | | 1.5 | m |
| $\boldsymbol{\theta_0}$ | Initial releasing angle of the arrow. | 35 | [deg] |
| $\boldsymbol{\rho}$ | Air density | 1.225 | $kg/m^3$ |
| $\boldsymbol{\eta_{bow}}$ | Bow energy efficiency | 0.70 | [-] |

* id_1 : last digit of your student id number

**References**

**[1] Url-1** <*https://en.wikipedia.org/wiki/Composite_bow*>, date retrieved 16.09.2021.

**[2]** Miyazaki, T., Mukaiyama, K., Komori, Y. *et al.* Aerodynamic properties of an archery arrow. *Sports Eng* **16,** 43–54 (2013). https://doi.org/10.1007/s12283-012-0102-y