Assignment 5 solutions

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Task 1 - Interval cover

Given a set $x_1, x_2, \ldots, x_n$ of points on the real line, we want to find the smallest set of unit-length closed intervals that contains all of the given points.

Find a greedy algorithm that solves this problem optimally. Justify your answer.

Solution. Begin finding the smallest point in the set. We place a covering interval exactly one unit-length after said point. Now remove all points in our set contained in said interval; note how this includes the first point in the sorting. In this new scenario, repeat the previous procedure until no points remain. The intervals we placed during the algorithm are our solution.

This algorithm gives a solution to the problem. Indeed, at every iteration of the algorithm we are removing at least a point from the original set, and the algorithm does not terminate until there are no points remaining.

We can check that it is also optimal. Consider the instance of the problem in the original statement in which the points are sorted. Say our algorithm outputs a set of intervals $S = \{s_1, \ldots, s_k\}$, and let $S^* = \{t_1, \ldots, t_m\}$ be the optimal solution.

If $I = [a, b]$ is a unit-length interval we done $L(I) = a$ and $R(I) = b$. Note how if I_1 and I_2 are two unit-length intervals, we know that if $L(a) \leq L(a)$ then we know that $R(a) \leq R(b)$. Without loss of generality, assume that the intervals in S and S^* are sorted in increasing order according to this definition.

We will first prove that $L(s_i) \geq L(t_i)$ for every i by induction on i. If $i = 1$ this is true, since the greedy algorithm picks the highest possible value for $L(s_1)$.

Assume then this holds for some value $i \geq 1$. By the induction hypothesis we know that the intervals $\{s_1, \ldots, s_i\}$ cover the houses covered by the intervals $\{t_1, \ldots, t_i\}$. Thus, the intervals $\{s_1, \ldots, s_i, t_{i+1}\}$ cover all the points between $L(s_i)$ and $R(t_{i+1})$, but the $(i+1)$-th step of our algorithm chooses s_{i+1} to be as far as possible while covering all points from the start of the set. Therefore $s_{i+1} \geq t_{i+1}$, which is what we wanted to prove by induction. $\Diamond$

Task 2 - Unique minimum spanning tree

Let G be a weighted graph in which all the edges have distinct weights. Prove or disprove that G has a unique minimum spanning tree.

Solution. The statement is true.

Indeed, suppose that T and T' are two distinct minimum spanning trees of G. Since T and T' have the same number of edges, but are not equal, there is some edge e' in T' that is not found in T. If we add e' to T, we get a cycle C. Let e be the most expensive edge in C. By the cycle property, we know that e does not belong to any minimum spanning tree, contradicting the fact that it is in at least one of T or T'. $\Diamond$

Task 3 - Interval graph coloring

Given a set of intervals, we define a graph whose vertices are intervals, and whose edges connect intersecting intervals. A coloring of a graph is an assignments of colors to the vertices such that no two adjacent vertices have the same color. Describe a greedy algorithm to color an interval graph with the fewest amount of colours possible. Justify your answer.

Solution. This is equivalent to the interval partitioning problem, covered in class. $\Diamond$