Assignment 1 solutions

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Task 1 - Explaining stable matching

Explain, in your own words, what it means for a matching to be *stable*.

Solution: An answer should be compatible with the underlying mathematical notion of stability:

> A matching is considered to be stable when there does *not* exist a pairing (x, y) where x and y prefer each other to their current matching.

An intuitive explanation is subjective, so this is an open-ended question. $\diamond$

Task 2 - Stable matching problem

Prove or disprove the following statement.

> Consider an instance of the stable matching problem in which there exists an element x of the first matched set and an element y of the second matched set such that x is ranked first on the preference list of y and y is ranked first on the preference list of x. Then in every stable matching S for this instance, the pair (x, y) belongs to S.

Solution: Assume the pair (x, y) does not belong to a stable matching S. Then both x and y are matched with some other elements in S. However, since they both prefer each other over their current matches (as they are each other's top choices), the pair (x, y) would be unstable pair in S.

This contradicts the fact that S is a stable matching. Therefore, in every stable matching for this instance, the pair (x, y) must be included. $\diamond$

Task 3 - College admissions with ties

Consider college admissions where applicants have preferences for multiple colleges and are indifferent between them. Similarly, colleges may have multiple applicants they consider equally suitable. The challenge lies in creating a stable matching that reflects these preferences and indifference options, while ensuring no better-off pairs can form.

We define a student-college pair (s, c) as unstable if

1. student s prefers college c to the college c' that is assigned to him or is indifferent between c and c', or

2. college c prefers student s over some other student s' who is currently assigned to c, or is indifferent between the two students s and s' .

Does there always exist a perfect matching with this notion of stability? Justify your answer.

Solution: No. We can show this with a counterexample.

Consider a case with only two students s_1 and s_2, and two colleges c_1 and c_2. Let s_1 be indifferent between c_1 and c_2, and let both colleges prefer s_1 to s_2. The choices of s_2 do not make a difference.

There is no match with this kind of instability in this example, since regardless of which college was matched with s_1, the other college with s_1 would form an unstable pair. ◇