TDT4121 Introduction to algorithms

Assignment 2

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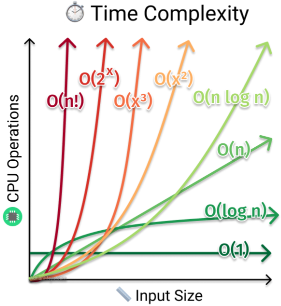
September 2024

Asymptotic growth rate

- The goal is quantifying how fast a function $f(n)$ grows.
- We do this in relation to other functions we know well, such as
  1. Constants k
  2. Logarithms $\log n$
  3. Polynomials n^k
  4. Exponentials k^n
  5. Factorials $n!$

  amongst others.



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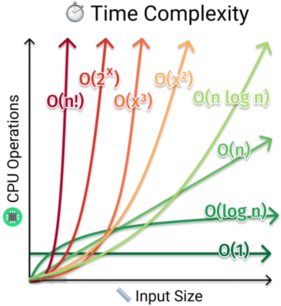

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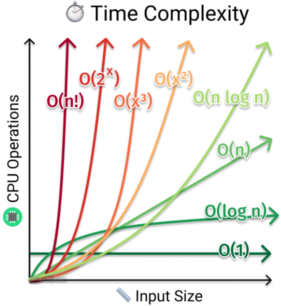

- Intuitively, polynomial runtime (or better) is *efficient* and to be celebrated, and we should be wary of things under that in the previous list.
  Keep in mind that this is a **huge** overgeneralization.

big-O notation

Definition (Asymptotic upper bound)

Consider a function $f(n)$ that, after a sufficiently large n, is bounded *above* by a constant multiple of a function $M(n)$.

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Example

For example, take $f(n) = 5n^2 + 10n + 10$ and $M(n) = n^2$. Note how, although $f(n)$ is always greater than $M(n)$, this does not hold for $6M(n)$. In fact, for $n \geq 11$ we will have $f(n) < 6M(n)$. Therefore, we have that $f(n)$ is $O(M(n))$.

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- An upper bound does not have to be tight.
- We could also have said that $f(n) = O(n^3)$, or $f(n) = O(n!)$.

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Definition (Asymptotic lower bound)

Consider a function $f(n)$ that, after a sufficiently large n, is bounded from *below* by a constant multiple of a function $M(n)$.

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Definition (Asymptotic lower bound)

Consider a function $f(n)$ that, after a sufficiently large n, is bounded from *below* by a constant multiple of a function $M(n)$. Then we will say that "$f(n)$ is $\Omega(M(n))$", and we will write $f(n) = \Omega(M(n))$.

Example

Let's take the same functions as before, $f(n) = 5n^2 + 10n + 10$ and $M(n) = n^2$.
As we pointed out before, we will always have $f(n) < M(n)$.
Then, we have that $f(n)$ is $\Omega(M(n))$, without the need on introducing any constant multiples, or any "sufficiently large n".

- Similarly, this does not have to be tight.
- We could also have said that $f(n) = \Omega(n)$, or $f(n) = \Omega(0)$.

big-O notation

Definition (Asymptotically tight bound)

Consider a function $f(n)$ such that a function $M(n)$ exists satisfying $f(n) = O(M(n))$ and $f(n) = \Omega(M(n))$.

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Consider a function $f(n)$ such that a function $M(n)$ exists satisfying $f(n) = O(M(n))$ and $f(n) = \Omega(M(n))$.
Then we will say that "$f(n)$ is $\Theta(M(n))$", and we will write $f(n) = \Theta(M(n))$.

Example

In the previous examples we saw how $f(n) = 5n^2 + 10n + 10$ satisfied both $f(n) = O(M(n))$ and $f(n) = \Omega(M(n))$, for $M(n) = n^2$.
Therefore, we can say that $f(n) = \Theta(M(n))$.

- This DOES have to be tight (hence the name).
- This gives us a good notion for $f(n)$ behaving like $M(n)$.

How do we prove a function behaves like another?

- This is useful for simple cases, but how do we prove it for more complex situations?
- Ideally, without going through two cumbersome inequalities, finding appropriate multiple factors,...

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Theorem (Page 38 of the course book)

Let f and g be two functions such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

exists and is equal to some number $c > 0$. Then $f(n) = \Theta(g(n))$.

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We first devise a candidate $n^2 \log n$, and we begin to calculate

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= \lim_{n\to\infty} \left(\frac{n^2 \log(2n+1)}{n^2 \log n} + \frac{6n \log(2n+1)}{n^2 \log n} + \frac{9 \log(2n+1)}{n^2 \log n} \right) =
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Therefore, $f(n) = \Theta(n^2 \log n)$.