TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Homework assignment 5

Hand-out time:	Monday, October 21, 2024, at 7:00
Hand-in deadline:	Sunday, November 10, 2024, at 23:59

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard. Any questions regarding the assignment should be directed through Piazza. Exercise hours are on Tuesdays between 16:15-18:00 in S7 (Sentralbygg 2).

Problem 1: Process classification

Consider the following process:

$$X(t) = a\sin(\omega t + \Phi),$$

where a and ω are constants and the variable Φ is uniformly distributed in the interval $[-\pi, \pi]$ (i.e. $\Phi \sim \mathcal{U}(-\pi, \pi)$).

a) Show that the mean $\mu_X(t) = E[X(t)]$ is given by $\mu_X(t) = 0$.

b) Show that the variance $\sigma_X^2(t) = E[(X(t) - \mu_X(t))^2]$ is given by $\sigma_X^2(t) = \frac{a^2}{2}$.

Note that for any real numbers b and c, we have

$$\sin(b)\sin(c) = \frac{1}{2}\cos(b-c) - \frac{1}{2}\cos(b+c).$$

c) Show that the autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$ is given by $R_X(t_1, t_2) = \frac{a^2}{2}\cos(\omega(t_1 - t_2))$. Write the autocorrelation function as $R_X(\tau) = E[X(t)X(t+\tau)]$ if possible.

d) Is the process deterministic? Motivate your answer.

e) Is the process wide-sense stationary? Motivate your answer.

f) Is the process ergodic (in wide sense)? Motivate your answer.

Problem 2: Linear systems with white input noise

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}w(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

with state $\mathbf{x}(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The disturbance $w(t)$ is a white noise process with autocorrelation function

$$R_w(\tau) = 4\delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta function. Assume zero initial conditions for the state $\mathbf{x}(t)$ (i.e. $\mathbf{x}(0) = \mathbf{0}$).

a) Calculate the mean μ_w of the disturbance $w(t)$.

b) Calculate the variance σ_w^2 of the disturbance $w(t)$.

c) Show that the power spectral density function $S_w(j\omega)$ of the disturbance $w(t)$ is given by $S_w(j\omega) = 4$.

d) Show that the transfer function $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{w}(s)}$ is given by $\hat{g}(s) = \frac{s+8}{s^2+6s+8}$.

Note that the transfer function can be written as $\hat{g}(s) = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2}$, where λ_1 and λ_2 are the poles of the system and α_1 and α_2 are constants.

e) Calculate the impulse response of the system $g(t) = \mathcal{L}^{-1}\{\hat{g}(s)\}$, where $\mathcal{L}^{-1}$ is the inverse Laplace transform. Show that $g(t) = -2e^{-4t} + 3e^{-2t}$.

Note that (for zero initial conditions) the output of the system is given by $y(t) = \int_0^t g(\tau)w(t-\tau)d\tau$.

f) Calculate the stationary mean $\bar{\mu}_y$ of the output $y(t)$ (i.e. $\bar{\mu}_y = \lim_{t\to\infty} \mu_y(t)$). Show that $\bar{\mu}_y = 0$

g) Calculate the stationary variance $\bar{\sigma}_y^2$ of the output $y(t)$ (i.e. $\bar{\sigma}_y^2 = \lim_{t\to\infty} \sigma_y^2(t)$). Show that $\bar{\sigma}_y^2 = 3$.

h) Show that the power spectral density function $S_y(j\omega)$ of the output $y(t)$ is given by $S_y(j\omega) = \frac{20}{\omega^2+4} - \frac{16}{\omega^2+16}$.