TTK4115 Linear System Theory

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Solution to homework assignment 6

Problem 1: Kalman-filter derivation

a) From the output equation in (1), it follows that

$$\begin{aligned}
\mathbf{y}_k^- &= E[\mathbf{y}_k] \\
&= E[\mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{H}\mathbf{v}_k] \\
&= \mathbf{C}E[\mathbf{x}_k] + \mathbf{D}\mathbf{u}_k + \mathbf{H}E[\mathbf{v}_k] \\
&= \mathbf{C}\hat{\mathbf{x}}_k^- + \mathbf{D}\mathbf{u}_k.
\end{aligned}$$

b) By substituting the expression for $\hat{\mathbf{x}}_k$ in (2), the *a posteriori* estimation error $\mathbf{x}_k - \hat{\mathbf{x}}_k$ can be written as

$$\mathbf{x}_k - \hat{\mathbf{x}}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k^- - \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k^-).$$

From the output equation in (1) and from (2), it follows that

$$\begin{aligned}
\mathbf{y}_k - \hat{\mathbf{y}}_k^- &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{H}\mathbf{v}_k - \mathbf{C}\hat{\mathbf{x}}_k^- - \mathbf{D}\mathbf{u}_k \\
&= \mathbf{C}\mathbf{x}_k + \mathbf{H}\mathbf{v}_k - \mathbf{C}\hat{\mathbf{x}}_k^- \\
&= \mathbf{C}(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) + \mathbf{H}\mathbf{v}_k.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
\mathbf{x}_k - \hat{\mathbf{x}}_k &= \mathbf{x}_k - \hat{\mathbf{x}}_k^- - \mathbf{K}_k(\mathbf{C}(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) + \mathbf{H}\mathbf{v}_k) \\
&= (\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) - \mathbf{K}_k\mathbf{H}\mathbf{v}_k.
\end{aligned}$$

By substituting this in the definition of $\mathbf{P}_k$, we get

$$\begin{aligned}
\mathbf{P}_k &= E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T] \\
&= E[((\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) - \mathbf{K}_k\mathbf{H}\mathbf{v}_k)((\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) - \mathbf{K}_k\mathbf{H}\mathbf{v}_k)^T] \\
&= E[(\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T] \\
&\quad - E[\mathbf{K}_k\mathbf{H}\mathbf{v}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T] - E[(\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{v}_k^T\mathbf{H}^T\mathbf{K}_k^T] \\
&\quad + E[\mathbf{K}_k\mathbf{H}\mathbf{v}_k\mathbf{v}_k^T\mathbf{H}^T\mathbf{K}_k^T] \\
&= (\mathbf{I} - \mathbf{K}_k\mathbf{C})E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T](\mathbf{I} - \mathbf{K}_k\mathbf{C})^T \\
&\quad - \mathbf{K}_k\mathbf{H}E[\mathbf{v}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T](\mathbf{I} - \mathbf{K}_k\mathbf{C})^T - (\mathbf{I} - \mathbf{K}_k\mathbf{C})E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{v}_k^T]\mathbf{H}^T\mathbf{K}_k^T \\
&\quad + \mathbf{K}_k\mathbf{H}E[\mathbf{v}_k\mathbf{v}_k^T]\mathbf{H}^T\mathbf{K}_k^T \\
&= (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T + \mathbf{K}_k\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_k^T.
\end{aligned}$$

c) The *a posteriori* error covariance matrix $\mathbf{P}_k$ in (4) can be written as

$$\begin{aligned}
\mathbf{P}_k &= (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T + \mathbf{K}_k\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_k^T \\
&= \mathbf{P}_k^- - \mathbf{K}_k\mathbf{C}\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{C}^T\mathbf{K}_k^T + \mathbf{K}_k\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T\mathbf{K}_k^T + \mathbf{K}_k\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_k^T \\
&= \mathbf{P}_k^- - \mathbf{K}_k\mathbf{C}\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{C}^T\mathbf{K}_k^T + \mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T.
\end{aligned}$$

d) By taking the derivative of $\mathbf{P}_k$ with respect to $\mathbf{K}_k$, we obtain

$$\begin{aligned}
\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} &= -\mathbf{C}\mathbf{P}_k^- - \mathbf{C}\mathbf{P}_k^- + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^T)\mathbf{K}_k^T \\
&= -\mathbf{C}\mathbf{P}_k^- - \mathbf{C}\mathbf{P}_k^- + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^T\mathbf{K}_k^T \\
&= -\mathbf{C}\mathbf{P}_k^- - \mathbf{C}\mathbf{P}_k^- + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T \\
&= -2\mathbf{C}\mathbf{P}_k^- + 2(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T.
\end{aligned}$$

e) From d) and $\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} = \mathbf{0}$, we have that

$$\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} = -2\mathbf{C}\mathbf{P}_k^- + 2(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T = \mathbf{0}.$$

By taking the transposed, it follows that

$$-2\mathbf{P}_k^-\mathbf{C}^T + 2\mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^T = -2\mathbf{P}_k^-\mathbf{C}^T + 2\mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T) = \mathbf{0}.$$

From this, we get

$$\mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T) = \mathbf{P}_k^-\mathbf{C}^T.$$

By postmultiplying both sides of the equation by $(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1}$, we obtain

$$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{C}^T(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1}.$$

f) From the state equation in (1), we get

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1}^- &= E[\mathbf{x}_{k+1}] \\
&= E[\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{w}_k] \\
&= \mathbf{A}E[\mathbf{x}_k] + \mathbf{B}\mathbf{u}_k + \mathbf{G}E[\mathbf{w}_k] \\
&= \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k.
\end{aligned}$$

g) From equations (1) and (6), it follows that

$$\begin{aligned}
\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^- &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{w}_k - \mathbf{A}\hat{\mathbf{x}}_k - \mathbf{B}\mathbf{u}_k \\
&= \mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{G}\mathbf{w}_k.
\end{aligned}$$

Substituting this in the definition of $\mathbf{P}_{k+1}^-$ yields

$$\begin{aligned}
\mathbf{P}_{k+1}^- &= E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)^T] \\
&= E[(\mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{G}\mathbf{w}_k)(\mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{G}\mathbf{w}_k)^T] \\
&= E[\mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T\mathbf{A}^T] + E[\mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{w}_k^T\mathbf{G}^T] \\
&\quad + E[\mathbf{G}\mathbf{w}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T\mathbf{A}^T] + E[\mathbf{G}\mathbf{w}_k\mathbf{w}_k^T\mathbf{G}^T] \\
&= \mathbf{A}E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T]\mathbf{A}^T + \mathbf{A}E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{w}_k^T]\mathbf{G}^T \\
&\quad + \mathbf{G}E[\mathbf{w}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T]\mathbf{A}^T + \mathbf{G}E[\mathbf{w}_k\mathbf{w}_k^T]\mathbf{G}^T \\
&= \mathbf{A}\mathbf{P}_k\mathbf{A}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T.
\end{aligned}$$

h) To find the values for $\hat{\mathbf{x}}_0$, $\hat{\mathbf{x}}_1^-$ and $\hat{\mathbf{x}}_1$, we use the formulas in the assignment as follows:

$$\hat{y}_0^- = \mathbf{C}\hat{\mathbf{x}}_0^- + Du_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 4 \cdot 1 = 4,$$

$$\begin{aligned}
\mathbf{K}_0 &= \mathbf{P}_0^-\mathbf{C}^T(\mathbf{C}\mathbf{P}_0^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} 2 \\ 0 \end{bmatrix} (2+2)^{-1} \\
&= \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},
\end{aligned}$$

$$\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_0^- + \mathbf{K}_0(y_0 - \hat{y}_0^-) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (3-4) = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix},$$

$$\begin{aligned}
\mathbf{P}_0 &= (\mathbf{I} - \mathbf{K}_0\mathbf{C})\mathbf{P}_0^-(\mathbf{I} - \mathbf{K}_0\mathbf{C})^T + \mathbf{K}_0\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_0^T \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right)^T \\
&\quad + \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

$$\hat{\mathbf{x}}_1^- = \mathbf{A}\hat{\mathbf{x}}_0 + \mathbf{B}u_0 = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

$$
\begin{aligned}
\mathbf{P}_1^- &= \mathbf{A}\mathbf{P}_0\mathbf{A}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T \\
&= \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 2 \begin{bmatrix} 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -3 \\ -3 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix},
\end{aligned}
$$

$$
\hat{y}_1^- = \mathbf{C}\hat{\mathbf{x}}_1^- + Du_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + 4 \cdot -1 = -\frac{7}{2} = -3.5,
$$

$$
\begin{aligned}
\mathbf{K}_1 &= \mathbf{P}_1^-\mathbf{C}^T(\mathbf{C}\mathbf{P}_1^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1} \\
&= \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} 4 \\ -3 \end{bmatrix} (4+2)^{-1} \\
&= \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \end{bmatrix} \approx \begin{bmatrix} 0.6667 \\ -0.5 \end{bmatrix},
\end{aligned}
$$

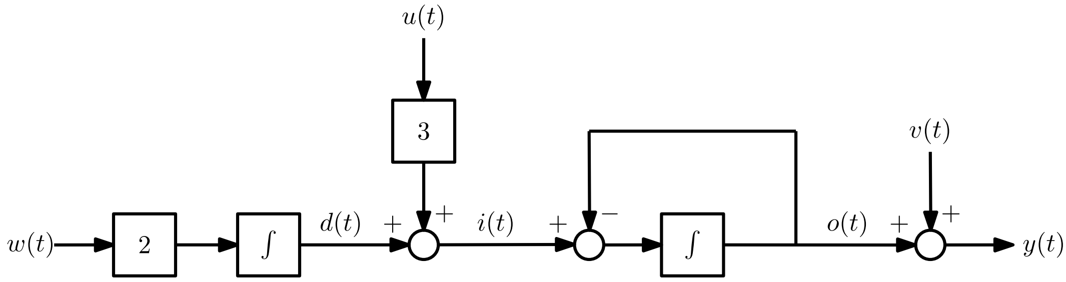
$$
\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_1^- + \mathbf{K}_1(y_1 - \hat{y}_1^-) = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \end{bmatrix} \left(-4 + \frac{7}{2} \right) = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{4} \end{bmatrix} \approx \begin{bmatrix} 0.1667 \\ 1.25 \end{bmatrix}.
$$

Hence, we obtain

$$
\hat{\mathbf{x}}_0 = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{x}}_1^- = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \hat{\mathbf{x}}_1 = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{4} \end{bmatrix}.
$$

Problem 2: Stationary Kalman filter

a) A block diagram is given by



b) From the transfer function $g(s) = \frac{o(s)}{i(s)} = \frac{1}{s+1}$, it follows that

$$(s+1)o(s) = i(s).$$

By taking the inverse Laplace transform, the following dynamics are obtained:

$$\dot{o}(t) + o(t) = i(t).$$

This can be written as

$$\dot{o}(t) = -o(t) + i(t).$$

Substituting $i(t) = 3u(t) + d(t)$, we get

$$\dot{o}(t) = -o(t) + d(t) + 3u(t).$$

From $d(t) = 2\int_0^t w(\tau)d\tau$, it follows that

$$\dot{d}(t) = 2w(t).$$

Combining these two differential equations and the output equation $y(t) = o(t) + v(t)$, we obtain the following system

$$\begin{bmatrix} \dot{o}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} o(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} w(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} o(t) \\ d(t) \end{bmatrix} + v(t).$$

This can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t),$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + Hv(t),$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} o(t) \\ d(t) \end{bmatrix}$ and matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad H = 1.$$

c) Because the matrix

$$\mathbf{P}(t) = \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}$$

is time invariant, we have

$$\dot{\mathbf{P}}(t) = \mathbf{0}.$$

Therefore, to show that $\mathbf{P}(t) = \mathbf{P}$ is a solution of the Riccati differential equation, we must show that

$$\mathbf{AP} + \mathbf{PA}^T - \mathbf{PC}^T(HRH^T)^{-1}\mathbf{CP} + \mathbf{G}Q\mathbf{G}^T = \mathbf{0}.$$

By substituting the values of the various matrices, we obtain

$$\begin{aligned}
&\mathbf{AP} + \mathbf{PA}^T - \mathbf{PC}^T(HRH^T)^{-1}\mathbf{CP} + \mathbf{G}Q\mathbf{G}^T \\
&\quad = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} + \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \\
&\qquad - \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 \cdot 4 \cdot 1)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} 1 \begin{bmatrix} 0 & 2 \end{bmatrix} \\
&\quad = \begin{bmatrix} 4(2-\sqrt{3}) & 4(\sqrt{3}-1) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 4(2-\sqrt{3}) & 0 \\ 4(\sqrt{3}-1) & 0 \end{bmatrix} \\
&\qquad - \begin{bmatrix} 4(\sqrt{3}-1) \\ 4 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 4(\sqrt{3}-1) & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \\
&\quad = \begin{bmatrix} 8(2-\sqrt{3}) - 4(\sqrt{3}-1)^2 & 4(\sqrt{3}-1) - 4(\sqrt{3}-1) \\ 4(\sqrt{3}-1) - 4(\sqrt{3}-1) & -4+4 \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Hence, the matrix $\mathbf{P}(t)$ is a solution of the Riccati differential equation.

d) The corresponding Kalman gain is given by

$$\begin{aligned}
\mathbf{K}(t) &= \mathbf{P}(t)\mathbf{C}^T(HRH^T)^{-1} \\
&= \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 \cdot 4 \cdot 1)^{-1} \\
&= \begin{bmatrix} 4(\sqrt{3}-1) \\ 4 \end{bmatrix} \frac{1}{4} \\
&= \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}.
\end{aligned}$$

e) From the definition of the estimation error $\mathbf{e}(t)$ and the differential equation of

$\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$, it follows that

$$\begin{aligned}
\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\
&= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t) - (\mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\
&= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\
&= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}(\mathbf{C}\mathbf{x}(t) + Hv(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\
&= (\mathbf{A} - \mathbf{K}\mathbf{C})(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}Hv(t) \\
&= (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) + \mathbf{G}w(t) - \mathbf{K}Hv(t).
\end{aligned}$$

f) The poles of the state estimator are equal to the eigenvalues of the matrix $\mathbf{A}-\mathbf{K}\mathbf{C}$, which is given by

$$\mathbf{A} - \mathbf{K}\mathbf{C} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{3} - 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{3} - 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A} - \mathbf{K}\mathbf{C}$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{K}\mathbf{C}$, which is given by

$$\begin{aligned}
\det(\mathbf{A} - \mathbf{K}\mathbf{C} - \lambda\mathbf{I}) &= \begin{vmatrix} -\sqrt{3} - \lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + \sqrt{3}\lambda + 1 \\
&= \left(\lambda + \frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)\left(\lambda + \frac{1}{2}\sqrt{3} - \frac{1}{2}i\right).
\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_{1,2} = -\frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$. Therefore, the estimator poles are given by $-\frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$.

g) From the answer in e), we have that the dynamics of the estimation error dynamics are perturbed by the disturbances $w(t)$ and $v(t)$:

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) + \mathbf{G}w(t) - \mathbf{K}Hv(t).$$

We note that the last term in the right-hand side of this equation implies that the contribution of the disturbance $v(t)$ is proportional to the Kalman gain $\mathbf{K}$ (while the contribution of the disturbance $w(t)$ is not). If the covariance of the disturbance $v(t)$ increases by a factor ten, we can expect a larger contribution of $v(t)$ in the error $\mathbf{e}(t)$. To limit the effect of this increase, we should lower the values of the elements in $\mathbf{K}$.