TTK4115 Linear System Theory
Department of Engineering Cybernetics
NTNU

Homework assignment 6

Hand-out time:	Monday, November 4, 2024, at 7:00
Hand-in deadline:	Sunday, November 24, 2024, at 23:59

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard. Any questions regarding the assignment should be directed through Piazza. Exercise hours are on Tuesdays between 16:15-18:00 in S7 (Sentralbygg 2).

Problem 1: Kalman-filter derivation

Consider the following discrete-time system:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{w}_k, \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{H}\mathbf{v}_k, \end{aligned} \tag{1}$$

with state $\mathbf{x}_k$, input $\mathbf{u}_k$ and output $\mathbf{y}_k$, where $\mathbf{w}_k$ and $\mathbf{v}_k$ are white noise disturbances that satisfy

$$E[\mathbf{w}_k] = \mathbf{0},$$

$$E[\mathbf{v}_k] = \mathbf{0},$$

$$E[\mathbf{w}_k\mathbf{w}_i^T] = \begin{cases} \mathbf{Q}, & \text{if } k = i, \\ \mathbf{0}, & \text{if } k \neq i, \end{cases}$$

$$E[\mathbf{v}_k\mathbf{v}_i^T] = \begin{cases} \mathbf{R}, & \text{if } k = i, \\ \mathbf{0}, & \text{if } k \neq i, \end{cases}$$

$$E[\mathbf{w}_k\mathbf{v}_i^T] = \mathbf{0}.$$

Although the input $\mathbf{u}_k$ is known and the output $\mathbf{y}_k$ is available via measurement, the state $\mathbf{x}_k$ is unknown due to the unknown disturbances. The measurement $\mathbf{y}_k$ contains information about the state $\mathbf{x}_k$. The state estimate that is generated without information of the measurement $\mathbf{y}_k$ is called the *a priori* state estimate and is denoted by $\hat{\mathbf{x}}_k^-$. Hence, we have $\hat{\mathbf{x}}_k^- = E[\mathbf{x}_k]$ if the measurement $\mathbf{y}_k$ is not taken into account. Without the actual measurement $\mathbf{y}_k$, the corresponding output estimate is given by $\hat{\mathbf{y}}_k^- = E[\mathbf{y}_k]$.

a) Use the output equation in (1) to determine $\hat{\mathbf{y}}_k^- = E[\mathbf{y}_k]$, where $E[\mathbf{x}_k] = \hat{\mathbf{x}}_k^-$. Show that

$$\hat{\mathbf{y}}_k^- = \mathbf{C}\hat{\mathbf{x}}_k^- + \mathbf{D}\mathbf{u}_k. \tag{2}$$

We use the measurement $\mathbf{y}_k$ to update the state estimate $\hat{\mathbf{x}}_k^-$. The *a posteriori* (or updated) state estimate is denoted $\hat{\mathbf{x}}_k$. We use the following linear blending to extract the information about the state $\mathbf{x}_k$ that is hidden in the measurement $\mathbf{y}_k$:

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k^-), \tag{3}$$

where $\mathbf{K}_k$ is a blending factor that is yet to be determined. Let the *a priori* and *a posteriori* error covariance matrices be denoted by

$$\mathbf{P}_k^- = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T]$$

and

$$\mathbf{P}_k = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T],$$

respectively.

b) Use the equations (1)-(3) to show that

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T + \mathbf{K}_k\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_k^T, \tag{4}$$

where it should be noted that the error $\mathbf{x}_k - \hat{\mathbf{x}}_k^-$ is uncorrelated with the disturbance $\mathbf{v}_k$ (i.e. $E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{v}_k^T] = \mathbf{0}$).

For later convenience, we write the expression for $\mathbf{P}_k$ in a different form.

c) Show that $\mathbf{P}_k$ in (4) can be rewritten as

$$\mathbf{P}_k = \mathbf{P}_k^- - \mathbf{K}_k\mathbf{C}\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{C}^T\mathbf{K}_k^T + \mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T. \tag{5}$$

To obtain an accurate state estimate $\hat{\mathbf{x}}_k$, we want to find the blending factor $\mathbf{K}_k$ that minimizes the expected value of $\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2$ (i.e. the squared magnitude of the estimation error $\mathbf{x}_k - \hat{\mathbf{x}}_k$). Here, we note that

$$E[\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2] = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T(\mathbf{x}_k - \hat{\mathbf{x}}_k)] = \mathrm{tr}(E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T]) = \mathrm{tr}(\mathbf{P}_k),$$

where tr is the matrix trace. To obtain the blending factor $\mathbf{K}_k$ that minimizes $\mathrm{tr}(\mathbf{P}_k)$, we determine the derivative $\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k}$ and set it to zero. Note that for any matrices $\mathbf{M}$ and $\mathbf{N}$, such that $\mathbf{MN}$ is a square matrix, we have[1]

$$\frac{d\,\mathrm{tr}(\mathbf{MN})}{d\mathbf{M}} = \frac{d\,\mathrm{tr}(\mathbf{N}^T\mathbf{M}^T)}{d\mathbf{M}} = \mathbf{N}.$$

Moreover, for any matrices $\mathbf{M}$ and $\mathbf{S}$, we have

$$\frac{d\,\mathrm{tr}(\mathbf{M}\mathbf{S}\mathbf{M}^T)}{d\mathbf{M}} = (\mathbf{S} + \mathbf{S}^T)\mathbf{M}^T.$$

[1]Here, we use a different matrix layout than in Brown & Hwang.

d) Use the expression for $\mathbf{P}_k$ in (5) to show that

$$\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} = -2\mathbf{C}\mathbf{P}_k^- + 2(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T.$$

e) Assuming that $\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T$ is an invertible matrix, show that from $\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} = \mathbf{0}$, it follows that

$$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{C}^T(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1}.$$

The *a priori* state estimate for the next time step $\hat{\mathbf{x}}_{k+1}^-$ is the expected value of $\mathbf{x}_{k+1}$ without information of the measurement $\mathbf{y}_{k+1}$.

f) Use the state equation in (1) to determine $\hat{\mathbf{x}}_{k+1}^- = E[\mathbf{x}_{k+1}]$, where $E[\mathbf{x}_k] = \hat{\mathbf{x}}_k$. Show that

$$\hat{\mathbf{x}}_{k+1}^- = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k. \tag{6}$$

We note that

$$\mathbf{P}_{k+1}^- = E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)^T].$$

g) Use the equations (1) and (6) to show that

$$\mathbf{P}_{k+1}^- = \mathbf{A}\mathbf{P}_k\mathbf{A}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T,$$

where it should be noted that the error $\mathbf{x}_k - \hat{\mathbf{x}}_k$ is uncorrelated with the disturbance $\mathbf{w}_k$ (i.e. $E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{w}_k^T] = \mathbf{0}$).

Let the matrices of the system (1) be given by

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 4 \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} -1 & 1 \end{bmatrix},$$

with covariance matrices

$$Q = 2 \qquad \text{and} \qquad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consider the initial conditions

$$\hat{\mathbf{x}}_0^- = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P}_0^- = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

the inputs

$$u_0 = 1 \qquad \text{and} \qquad u_1 = -1,$$

and the outputs

$$y_0 = 3 \qquad \text{and} \qquad y_1 = -4.$$

h) Show that the corresponding *a priori* and *a posteriori* state estimates are given by

$$\hat{\mathbf{x}}_0 = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{x}}_1^- = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \hat{\mathbf{x}}_1 = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{4} \end{bmatrix}.$$

Problem 2: Stationary Kalman filter

Consider a plant where the relation between the input $i(t)$ and the output $o(t)$ is given by the transfer function

$$\hat{g}(s) = \frac{\hat{o}(s)}{\hat{i}(s)} = \frac{1}{s+1}.$$

The input $i(t)$ consists of the control input $u(t)$ and an input disturbance $d(t)$ and is given by

$$i(t) = 3u(t) + d(t).$$

The disturbance $d(t)$ is the output of the following Wiener process:

$$d(t) = 2\int_0^t w(\tau)d\tau,$$

where $w(t)$ is Gaussian white noise. The output $o(t)$ of the plant is measured. The corresponding output measurement is given by

$$y(t) = o(t) + v(t),$$

where $v(t)$ is Gaussian white noise.

a) Draw a block diagram of the system.

b) Show that the system can be written as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Hv(t),\end{aligned}$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} o(t) \\ d(t) \end{bmatrix}$, and matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad H = 1.$$

Let the covariance of the white noise processes $w(t)$ and $v(t)$ be given by

$$E[w(t)w(\tau)] = Q\delta(t-\tau)$$

and

$$E[v(t)v(\tau)] = R\delta(t-\tau),$$

with $Q = 1$ and $R = 4$, where δ is the Dirac delta function. Moreover, assume that the processes are independent, such that

$$E[w(t)v(\tau)] = 0$$

for all t and τ. For the initial expected value $\hat{\mathbf{x}}(0) = E[\mathbf{x}(0)]$ and the initial covariance matrix $\mathbf{P}(0) = E[(\mathbf{x}(0) - \hat{\mathbf{x}}(\mathbf{0}))(\mathbf{x}(0) - \hat{\mathbf{x}}(\mathbf{0}))^T]$, the state estimate $\hat{\mathbf{x}}(t)$ that minimizes the expected value of $\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\|^2$ is given by the solution of the differential equation

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}(t)(y(t) - \mathbf{C}\hat{\mathbf{x}}(t)),$$

with Kalman gain

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{C}^T(HRH^T)^{-1},$$

where $\mathbf{P}(t)$ is the solution of the Riccati differential equation

$$\dot{\mathbf{P}}(t) = \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T - \mathbf{P}(t)\mathbf{C}^T(HRH^T)^{-1}\mathbf{C}\mathbf{P}(t) + \mathbf{G}Q\mathbf{G}^T.$$

c) Show that the constant matrix

$$\mathbf{P}(t) = \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}$$

is a solution of the Riccati differential equation.

d) Show that the corresponding Kalman gain is given by

$$\mathbf{K}(t) = \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}.$$

Because the matrix $\mathbf{P}(t)$ and the Kalman gain $\mathbf{K}(t)$ are time invariant for these values, we drop the time index and write $\mathbf{P}$ and $\mathbf{K}$ instead. We define the estimation error

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t).$$

e) Show that the following differential equation holds

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) + \mathbf{G}w(t) - \mathbf{K}Hv(t).$$

f) Determine the poles of the state estimator (i.e. the eigenvalues of the matrix $\mathbf{A} - \mathbf{K}\mathbf{C}$). Show that the poles are given by $-\frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$, with $i = \sqrt{-1}$.

g) Suppose that the variance of $v(t)$ becomes ten times larger (the value of R changes from 4 to 40). Reason if the elements of $\mathbf{K}$ should be increased or decreased to minimize the expected value of $\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\|^2$.