TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Homework assignment 3

Hand-out time:	Monday, September 23, 2024, at 7:00
Hand-in deadline:	Sunday, October 13, 2024, at 23:59

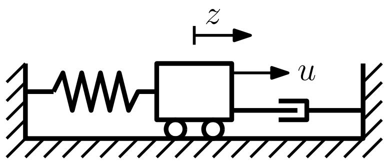
The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard. Any questions regarding the assignment should be directed through Piazza. Exercise hours are on Tuesdays between 16:15-18:00 in S7 (Sentralbygg 2).

Problem 1: Linear quadratic regulator and tracking control

Consider the following mass-spring-damper system:

$$\ddot{y}(t) + \dot{y}(t) + 2y(t) = 2u(t),$$

where $y(t)$ is the displacement of the mass and where the input $u(t)$ is an external force applied to the mass.



a) Derive a state-space equation of the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$.

We aim to design a controller that regulates the displacement of the mass to a desired setpoint value r, i.e. we want to design a controller such that the displacement $y(t)$ converges to the constant value r. The controller can be written as

$$u(t) = u_{eq} + \tilde{u}(t),$$

where u_{eq} correspond to the feedforward part of the controller, and $\tilde{u}(t)$ corresponds to the feedback part of the controller. We design the feedforward part of the controller such that the following equations are satisfied:

$$\begin{aligned} \mathbf{0} &= \mathbf{A}\mathbf{x}_{eq} + \mathbf{B}u_{eq} \\ r &= \mathbf{C}\mathbf{x}_{eq}. \end{aligned} \tag{1}$$

Hence, we choose u_{eq} such that the state $\mathbf{x}(t) = \mathbf{x}_{eq}$ is an equilibrium point of the system that satisfies the condition $y(t) = r$.

b) Determine the state $\mathbf{x}_{eq}$ and the input u_{eq} such that the equations in (1) are satisfied. Show that the state $\mathbf{x}_{eq}$ and the input u_{eq} can be written in the following form:

$$\mathbf{x}_{eq} = \mathbf{F}r \qquad \text{and} \qquad u_{eq} = Gr,$$

with

$$\mathbf{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad G = 1.$$

We define the state error $\tilde{\mathbf{x}}(t) = [\tilde{x}_1(t), \tilde{x}_2(t)]^T$ and the output error $\tilde{y}(t)$ as follows:

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}_{eq} \qquad \text{and} \qquad \tilde{y}(t) = y(t) - r.$$

c) Show that the state error $\tilde{\mathbf{x}}(t)$ and the output error $\tilde{y}(t)$ satisfy the equations

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(t) &= \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t) \\ \tilde{y}(t) &= \mathbf{C}\tilde{\mathbf{x}}(t). \end{aligned}$$

Note that the state error $\tilde{\mathbf{x}}(t)$ is only influenced by the feedback part $\tilde{u}(t)$ of the controller. If the state error $\tilde{\mathbf{x}}(t)$ is at the origin (i.e. if $\tilde{\mathbf{x}}(t) = \mathbf{0}$), then the output error $\tilde{y}(t)$ is equal to zero, which implies that $y(t) = r$. To control the state error to the origin, we design a linear quadratic regulator for the feedback part of the controller. The linear quadratic regulator minimizes the cost function

$$J = \int_0^\infty \left[\tilde{\mathbf{x}}^T(t)\mathbf{Q}\tilde{\mathbf{x}}(t) + R\tilde{u}^2(t)\right] dt.$$

The control input that minimizes this cost function is given by

$$\tilde{u}(t) = -\mathbf{K}\tilde{\mathbf{x}}(t), \qquad \text{with} \qquad \mathbf{K} = R^{-1}\mathbf{B}^T\mathbf{P},$$

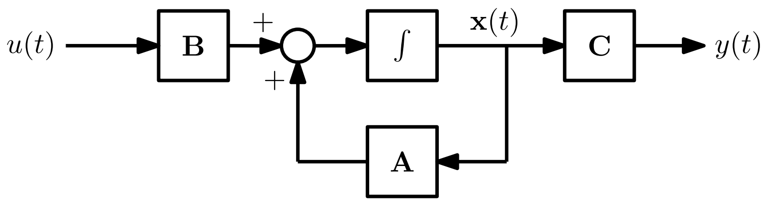
where the real, symetric, and positive-definite matrix $\mathbf{P}$ is the solution of the algebraic Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}R^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}.$$

The weighting matrices are given by

$$\mathbf{Q} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad R = 4.$$

d) Calculate the matrix $\mathbf{P}$ and show that $\mathbf{K} = [\frac{1}{2}, \frac{1}{2}]$.

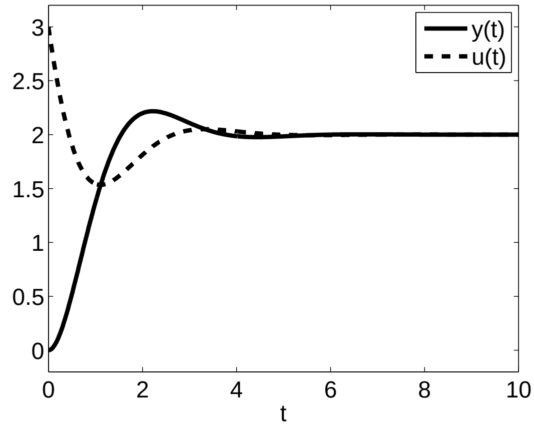


e) Rewrite the controller as
$$u(t) = -\mathbf{K}\mathbf{x}(t) + pr(t)$$
and determine the value of the constant p.

A block diagram of the system (without controller) is given by:

f) Extent the block diagram by adding the controller.

The response of the closed-loop system (i.e. the system with controller) for $r = 2$ and $\mathbf{x}(0) = [0, 0]^T$ is given by:



g) How can the weighting matrices $\mathbf{Q}$ and R be altered such that the output $y(t)$ converges faster to the setpoint value $r = 2$? How will these new values of $\mathbf{Q}$ and R change the input $u(t)$?

Problem 2: Realizations

a) Give conditions under which a transfer matrix is realizable.

Consider the following transfer matrix:

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} \end{bmatrix}.$$

b) Use the conditions in a) to show that the transfer matrix $\hat{\mathbf{G}}(s)$ is realizable.

c) Show that the transfer matrix $\hat{\mathbf{G}}(s)$ can be written as $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{sp}(s) + \mathbf{D}$, where $\hat{\mathbf{G}}_{sp}(s)$ is a strictly proper transfer matrix and $\mathbf{D}$ is a constant matrix.

d) Find the least common denominator

$$d(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$

for the transfer functions of the transfer matrix $\hat{\mathbf{G}}(s)$, where n is the degree of the denominator and $\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$ are constants. Moreover, show that $\hat{\mathbf{G}}_{sp}(s)$ can be written in the following form:

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)}\left[\mathbf{N}_1 s^{n-1} + \mathbf{N}_2 s^{n-2} + \cdots + \mathbf{N}_{n-1}s + \mathbf{N}_n\right],$$

with matrices

$$\mathbf{N}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} -2 & -6 \\ 0 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_3 = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}.$$

e) Find a realization of $\hat{\mathbf{G}}(s)$ using the set of equations

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\alpha_1\mathbf{I} & -\alpha_2\mathbf{I} & \cdots & -\alpha_{n-1}\mathbf{I} & -\alpha_n\mathbf{I} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \cdots & \mathbf{N}_{n-1} & \mathbf{N}_n \end{bmatrix} \mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

where $\mathbf{I}$ is the identity matrix.

Problem 3: State estimator

Consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t),$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 3 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \text{and} \qquad D = -2.$$

a) Calculate the observability matrix and determine if the system is observable.

Consider the following state estimator for the system:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t) - Du(t)),$$

where $\mathbf{L}$ is a gain matrix that will be determined later.

b) Define the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ and show that

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).$$

c) Determine the estimator-gain matrix $\mathbf{L}$ such that the poles of the state estimator (i.e. the eigenvalues of the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$) are equal to -8 and -10, respectively. Show that $\mathbf{L} = [8, 13]^T$.

d) Draw a block diagram of the system with state estimator, similar to f) of Problem 2.

Problem 4: Separation principle

Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

the feedback controller

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$$

and the state estimator

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)).$$

We define the estimation error $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$.

a) Show that the closed-loop system (i.e. the system with feedback controller and state estimator) can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix}}_{=\mathbf{H}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}.$$

Note that for any square matrix $\mathbf{F}$, its characteristic polynomial $\det(\mathbf{F} - \lambda\mathbf{I})$ is equal to zero if and only if λ is an eigenvalue of $\mathbf{F}$. Moreover, for any block-triangular matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{0} & \mathbf{G}_{22} \end{bmatrix},$$

it holds that $\det(\mathbf{G}) = \det(\mathbf{G}_{11})\det(\mathbf{G}_{22})$.

b) Show that any eigenvalue of $\mathbf{H}$ is an eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$ (i.e. the eigenvalues of $\mathbf{H}$ are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$).

c) Assuming that the system is controllable and observable, argue that the poles of the closed-loop system (i.e. the eigenvalues of $\mathbf{H}$) can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs).