TTK4115 Linear System Theory
Department of Engineering Cybernetics
NTNU

Homework assignment 4

Hand-out time:	Monday, October 7, 2024, at 7:00
Hand-in deadline:	Sunday, October 27, 2024, at 23:59

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard. Any questions regarding the assignment should be directed through Piazza. Exercise hours are on Tuesdays between 16:15-18:00 in S7 (Sentralbygg 2).

Problem 1: Input-output stability of discrete-time systems

Consider the following discrete-time system:

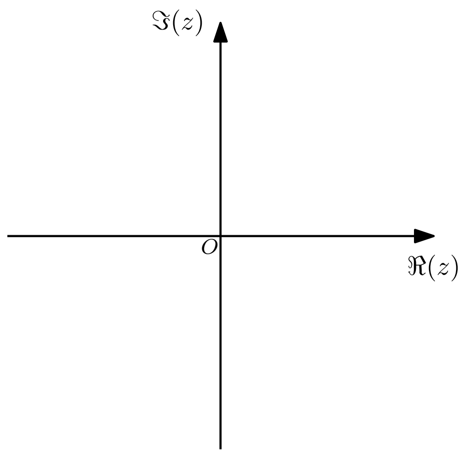
$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}u[k]\\ y[k] &= \mathbf{C}\mathbf{x}[k] + Du[k],\end{aligned}$$

with state $\mathbf{x}[k]$, input $u[k]$, output $y[k]$ and matrices

$$\mathbf{A} = \begin{bmatrix}-0.7 & -0.6\\ 0.4 & 0.7\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}0.1\\ 0.3\end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix}1 & 0\end{bmatrix} \quad \text{and} \quad D = -1.$$

a) Show that the discrete transfer function $\hat{g}(z) = \frac{\hat{y}(z)}{\hat{u}(z)}$ is given by $\hat{g}(z) = \frac{-z^2+0.1z}{z^2-0.25}$.

A graphical representation of the z-plane is given by:



where $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z and O is the origin (i.e. the point $z = 0$).

b) Draw the boundary of the region of the z-plane in which the poles of the discrete transfer function $\hat{g}(z)$ must be for the system to be BIBO stable. Moreover, draw the poles of the discrete transfer function $\hat{g}(z)$ in the z-plane. Determine if the system is BIBO stable.

Problem 2: Stability of continuous-time systems

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t),\end{aligned}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad D = 1.$$

a) Determine from the system matrix $\mathbf{A}$ if the system is marginally stable (or Lyapunov stable), asymptotically stable and/or unstable.

b) Show that the transfer function $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ of the system is given by $\hat{g}(s) = \frac{s+2}{s+1}$.

c) Determine from the transfer function if the system is BIBO stable.

d) Show that the impulse response of the system is given by $g(t) = e^{-t} + \delta(t)$, where $\delta(t)$ is the Dirac delta function.

e) Determine from the impulse response if the system is BIBO stable.

Problem 3: Internal stability

Consider the autonomous system described by the state-space equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t), \quad \text{with} \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}.$$

a) Calculate the eigenvalues and eigenvectors of $\mathbf{A}_1$.

b) Determine if the system is marginally stable (or Lyapunov stable), asymptotically stable and/or unstable.

Now, consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}_2\mathbf{x}(t), \quad \text{with} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

c) Calculate the eigenvalues and eigenvectors of $\mathbf{A}_2$.

d) Determine if the system is marginally stable (or Lyapunov stable), asymptotically stable and/or unstable.

Next, consider the system matrix:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_3\mathbf{x}(t), \quad \text{with} \quad \mathbf{A}_3 = \begin{bmatrix} -4 & -2 \\ 1 & -2 \end{bmatrix}.$$

e) Compute the symmetric matrix $\mathbf{P}$ from the Lyapunov equation

$$\mathbf{A}_3^T\mathbf{P} + \mathbf{P}\mathbf{A}_3 = -\mathbf{Q},$$

where $\mathbf{Q} = \mathbf{I}$ is the identity matrix. Conclude from the matrix $\mathbf{P}$ if the system is asymptotically stable.

Problem 4: Controllable decompositions

Consider the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $\mathbf{y}(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} -4 & -4 & -10 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

a) Use the inverse matrix

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s+4} & \frac{-4}{(s+2)(s+4)} & \frac{-10}{(s+2)(s-3)} \\ 0 & \frac{1}{s+2} & \frac{5}{(s+2)(s-3)} \\ 0 & 0 & \frac{1}{s-3} \end{bmatrix}$$

to show that the transfer matrix of the system is given by

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} 0 \\ \frac{1}{s-3} \end{bmatrix}.$$

b) Calculate the controllability matrix and determine if the system is controllable.

We want to use an equivalence transformation to obtain a controllable canonical decomposition of the system. Consider the following transformation matrix:

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix},$$

where the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are the first and second column of the controllability matrix, respectively. The vector $\mathbf{p}_3$ is chosen such that the matrix $\mathbf{P}$ is invertible. For simplicity, we define

$$\mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

c) Use the similarity transform $\mathbf{x}(t) = \mathbf{P}\hat{\mathbf{x}}(t)$ to transform the system to the controllable canonical decomposition

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}u(t)\\ \mathbf{y}(t) &= \hat{\mathbf{C}}\hat{\mathbf{x}}(t),\end{aligned}$$

with matrices

$$\hat{\mathbf{A}} = \left[\begin{array}{c:c} \hat{\mathbf{A}}_c & \hat{\mathbf{A}}_{12} \\ \hdashline \mathbf{0} & \hat{A}_{\bar{c}} \end{array}\right] = \left[\begin{array}{cc:c} 0 & 6 & 0 \\ 1 & 1 & 0 \\ \hdashline 0 & 0 & -4 \end{array}\right], \qquad \hat{\mathbf{B}} = \left[\begin{array}{c} \hat{\mathbf{B}}_c \\ \hdashline 0 \end{array}\right] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\hat{\mathbf{C}} = \left[\begin{array}{c:c} \hat{\mathbf{C}}_c & \hat{\mathbf{C}}_{\bar{c}} \end{array}\right] = \left[\begin{array}{cc:c} 0 & 0 & 1 \\ 1 & 3 & 0 \end{array}\right],$$

where the subscripts c and $\bar{c}$ indicate the controllable and uncontrollable parts of the matrices, respectively.

d) Show that the transfer matrix $\hat{\mathbf{G}}(s)$ in a) is equal to $\hat{\mathbf{C}}_c(s\mathbf{I} - \hat{\mathbf{A}}_c)^{-1}\hat{\mathbf{B}}_c$.

Problem 5: Minimal realizations

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t),\end{aligned}$$

with matrices

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 5 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} -5 & 1 \\ 10 & -2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

a) Determine the eigenvalues of the system.

From the Popov-Belevitch-Hautus test for controllability, it follows that the system is controllable if and only if

$$\operatorname{rank}\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B} \end{bmatrix} = n = 2$$

for every eigenvalue λ of $\mathbf{A}$.

b) Use the Popov-Belevitch-Hautus test for controllability to determine if the system is controllable.

From the Popov-Belevitch-Hautus test for observability, it follows that the system is observable if and only if

$$\operatorname{rank}\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} \\ \mathbf{C} \end{bmatrix} = n = 2$$

for every eigenvalue λ of $\mathbf{A}$.

c) Use the Popov-Belevitch-Hautus test for observability to determine if the system is observable.

d) Determine from your answers in b) and c) if the system is a minimal realization.

e) Show that the transfer matrix of the system is given by

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s-1}{s+6} \\ \frac{-2s+2}{s+6} \end{bmatrix}.$$

f) Determine from the transfer matrix in e) if the system is a minimal realization.