TTK4115 Linear System Theory

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Solution to homework assignment 3

Problem 1: Linear quadratic regulator and tracking control

a) Using the differential equation, we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ -2y(t) - \dot{y}(t) + 2u(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -2x_1(t) - x_2(t) + 2u(t) \end{bmatrix}.$$

Therefore, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t).$$

Note that $y(t) = x_1(t)$. Therefore, we have

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

b) From the first equation in (2), we obtain that

$$\mathbf{A}\mathbf{x}_{eq} = -\mathbf{B}u_{eq}.$$

Because $\mathbf{A}$ is invertible, we get the following expression for $\mathbf{x}_{eq}$:

$$\mathbf{x}_{eq} = -\mathbf{A}^{-1}\mathbf{B}u_{eq} = -\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} u_{eq} = -\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} u_{eq} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{eq}.$$

By substituting this in the second equation in (2), we obtain

$$r = \mathbf{C}\mathbf{x}_{eq} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{eq} = u_{eq}.$$

Hence, we have

$$u_{eq} = Gr,$$

with

$$G = 1.$$

Substituting this in the expression for $\mathbf{x}_{eq}$ leads to

$$\mathbf{x}_{eq} = \mathbf{F}r,$$

with

$$\mathbf{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

c) From the definition of the state error, the control input and the system equations, we have

$$\dot{\tilde{\mathbf{x}}}(t) = \dot{\mathbf{x}}(t) - \underbrace{\dot{\mathbf{x}}_{eq}}_{=\mathbf{0}} = \mathbf{A}\mathbf{x}(t) - \mathbf{B}u(t) = \mathbf{A}(\mathbf{x}_{eq} + \tilde{\mathbf{x}}(t)) + \mathbf{B}(u_{eq} + \tilde{u}(t)).$$

By combining this with the first equation in (2), we obtain

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(t) &= \underbrace{\mathbf{A}\mathbf{x}_{eq} + \mathbf{B}u_{eq}}_{=\mathbf{0}} + \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t) \\ &= \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t). \end{aligned}$$

The output error is given by

$$\tilde{y}(t) = y(t) - r = \mathbf{C}\mathbf{x}(t) - r = \mathbf{C}(\mathbf{x}_{eq} + \tilde{\mathbf{x}}(t)) - r.$$

From the second equation in (2), we have that $r = \mathbf{C}\mathbf{x}_{eq}$, which implies that

$$\begin{aligned} \tilde{y}(t) &= \mathbf{C}\mathbf{x}_{eq} + \mathbf{C}\tilde{\mathbf{x}}(t) - r \\ &= \mathbf{C}\mathbf{x}_{eq} + \mathbf{C}\tilde{\mathbf{x}}(t) - \mathbf{C}\mathbf{x}_{eq} \\ &= \mathbf{C}\tilde{\mathbf{x}}(t). \end{aligned}$$

d) Let the positive-definite matrix $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix},$$

where the constants p_1, p_2 and p_3 are obtained from the algebraic Riccati equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{B}^T R^{-1}\mathbf{P} + \mathbf{Q} = \mathbf{0}$. Substituting $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, R and $\mathbf{P}$ in the algebraic Riccati equation yields

$$\begin{aligned} &\begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2p_2 & -2p_3 \\ p_1 - p_2 & p_2 - p_3 \end{bmatrix} + \begin{bmatrix} -2p_2 & p_1 - p_2 \\ -2p_3 & p_2 - p_3 \end{bmatrix} - \begin{bmatrix} p_2^2 & p_2p_3 \\ p_2p_3 & p_3^2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4p_2 - p_2^2 + 5 & p_1 - p_2 - 2p_3 - p_2p_3 \\ p_1 - p_2 - 2p_3 - p_2p_3 & 2p_2 - 2p_3 - p_3^2 + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Hence, we obtain the equations

$$\begin{aligned} -4p_2 - p_2^2 + 5 &= 0, \\ p_1 - p_2 - 2p_3 - p_2 p_3 &= 0, \\ 2p_2 - 2p_3 - p_3^2 + 1 &= 0. \end{aligned}$$

From the first equation, it follows that

$$p_2 = -2 \pm 3.$$

From the third equation, we obtain

$$p_3 = -1 \pm \sqrt{2(p_2 + 1)}.$$

From the second equation, we get

$$p_1 = p_2 + 2p_3 + p_2 p_3.$$

Because there are two different solutions for p_2 and an additional two solutions for p_3, there are four solutions that satisfy the algebraic Riccati equation:

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -8 & 1 \\ 1 & -3 \end{bmatrix},$$

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -2 - 6\sqrt{2}j & -5 \\ -5 & -1 + 2\sqrt{2}j \end{bmatrix}, \quad \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -2 + 6\sqrt{2}j & -5 \\ -5 & -1 - 2\sqrt{2}j \end{bmatrix}.$$

The matrix $\mathbf{P}$ is a real and positive-definite solution of the algebraic Ricatti equation. A real matrix is positive definite if all of its leading principal minors are positive. (Positive definiteness of a real matrix can also be checked by looking at its eigenvalues, which should be real and positive.) The leading principle minors of $\mathbf{P}$ are

$$p_1 \qquad \text{and} \qquad \det(\mathbf{P}) = \begin{vmatrix} p_1 & p_2 \\ p_2 & p_3 \end{vmatrix} = p_1 p_3 - p_2^2.$$

Because $\mathbf{P}$ is positive definite, we must have that

$$p_1 > 0 \qquad \text{and} \qquad p_1 p_3 - p_2^2 > 0.$$

The only solution of the algebraic Riccati equation that is real and positive definite is $\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$. Therefore, we have

$$\mathbf{P} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

The corresponding gain matrix $\mathbf{K}$ is given by

$$\mathbf{K} = R^{-1}\mathbf{B}^T\mathbf{P} = \frac{1}{4}\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{1}$$

e) The controller is given by

$$\begin{aligned} u(t) &= u_{eq} + \tilde{u}(t) \\ &= Gr - \mathbf{K}\tilde{\mathbf{x}}(t) \\ &= Gr - \mathbf{K}(\mathbf{x}(t) - \mathbf{x}_{eq}) \\ &= Gr - \mathbf{K}(\mathbf{x}(t) - \mathbf{F}r). \end{aligned}$$

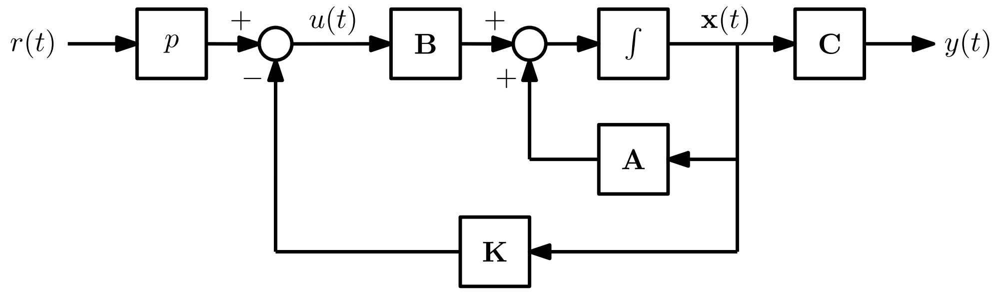
Hence, we have

$$u(t) = -\mathbf{K}\mathbf{x}(t) + pr,$$

with

$$p = G + \mathbf{KF} = 1 + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{3}{2}.$$

f) The block diagram of the system with controller is given by:



g) In order to obtain a faster convergence of the output, we need to penalize the deviation of the output $y(t)$ with respect to the setpoint value r more (i.e. we need to penalize $\tilde{y}(t)$ more). We can do this by increasing the first element on the diagonal of $\mathbf{Q}$ (which currently has the value 5). This element corresponds to the state $\tilde{x}_1(t) = \tilde{y}(t)$. Alternatively, we can penalize $\tilde{u}(t)$ less by decreasing the value of R.
By increasing the effort to get a small deviation of the output $y(t)$ with respect to the setpoint value r, we decrease the effort of having a small feedback input $\tilde{u}(t)$. Therefore, the signal $\tilde{u}(t)$ will be larger. Hence, we can expect a larger input signal $u(t) = u_{eq} + \tilde{u}(t)$, especially when the deviation of $y(t)$ with respect to r is large.

Problem 2: Realizations

a) A transfer matrix is realizable if and only if it is proper and rational.

- **Proper:** A transfer function $\hat{G}(s) = \frac{n(s)}{d(s)}$ is proper if the degree of its denominator $d(s)$ is larger than or equal to the degree of its numerator $n(s)$, i.e. deg $d(s) \geq$ deg $n(s)$. A transfer matrix is proper if all its elements (i.e. transfer functions) are proper.
- **Rational:** A transfer function $\hat{G}(s) = \frac{n(s)}{d(s)}$ is rational if the degrees of the numerator $n(s)$ and the denominator $d(s)$ are finite. A transfer matrix is rational if all its elements (i.e. transfer functions) are rational.

b) The transfer matrix $\hat{\mathbf{G}}(s)$ can be written as

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \hat{G}_{11}(s) & \hat{G}_{12}(s) \\ 0 & \hat{G}_{22}(s) \end{bmatrix},$$

with

$$\begin{aligned} \hat{G}_{11}(s) &= \frac{n_{11}(s)}{d_{11}(s)} = \frac{s^2+4s+2}{s^2+2s}, \\ \hat{G}_{12}(s) &= \frac{n_{12}(s)}{d_{12}(s)} = \frac{3}{s+2}, \\ \hat{G}_{22}(s) &= \frac{n_{22}(s)}{d_{22}(s)} = \frac{2s^2}{s^2-4}. \end{aligned}$$

The degrees of the numerator and denominator polynomials of the transfer functions are

$$\begin{aligned} \deg n_{11}(s) &= 2, & \deg d_{11}(s) &= 2, \\ \deg n_{12}(s) &= 0, & \deg d_{12}(s) &= 1, \\ \deg n_{22}(s) &= 2, & \deg d_{22}(s) &= 2. \end{aligned}$$

Because the degrees of the denominators of the transfer functions $\hat{G}_{11}(s)$, $\hat{G}_{12}(s)$ and $\hat{G}_{22}(s)$ are larger than or equal to the degrees of the corresponding numerators, the transfer matrix is proper. Moreover, because the degrees of the numerators and denominators of each transfer function are finite, the transfer matrix is rational. Hence, it follows that the transfer matrix is realizable.

c) The constant matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \lim_{s\to\infty} \hat{\mathbf{G}}(s) = \begin{bmatrix} \lim\limits_{s\to\infty} \frac{s^2+4s+2}{s^2+2s} & \lim\limits_{s\to\infty} \frac{3}{s+2} \\ 0 & \lim\limits_{s\to\infty} \frac{2s^2}{s^2-4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now, the strictly proper transfer matrix $\hat{\mathbf{G}}_{sp}(s)$ can be calculated as

$$\begin{aligned} \hat{\mathbf{G}}_{sp}(s) = \hat{\mathbf{G}}(s) - \mathbf{D} &= \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} - 1 & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} - 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} - \frac{s^2+2s}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} - \frac{2s^2-8}{s^2-4} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{8}{s^2-4} \end{bmatrix}. \end{aligned}$$

Hence, we obtain $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{sp}(s) + \mathbf{D}$, with $\hat{\mathbf{G}}_{sp}(s)$ and $\mathbf{D}$ defined above.

d) For notational convenience, we write

$$\hat{\mathbf{G}}_{sp}(s) = \begin{bmatrix} \hat{G}_{sp11}(s) & \hat{G}_{sp12}(s) \\ 0 & \hat{G}_{sp22}(s) \end{bmatrix},$$

with transfer functions

$$\hat{G}_{sp11}(s) = \frac{n_{sp11}(s)}{d_{sp11}(s)} = \frac{2s+2}{s^2+2s},$$
$$\hat{G}_{sp12}(s) = \frac{n_{sp12}(s)}{d_{sp12}(s)} = \frac{3}{s+2},$$
$$\hat{G}_{sp22}(s) = \frac{n_{sp22}(s)}{d_{sp22}(s)} = \frac{8}{s^2-4}.$$

To find the least common denominator for the transfer functions of the transfer matrix $\hat{\mathbf{G}}_{sp}(s)$, we write the denominator of each transfer function as a product of first-order factors:

$$d_{sp11}(s) = s^2 + 2s = s(s+2),$$
$$d_{sp12}(s) = s + 2,$$
$$d_{sp22}(s) = s^2 - 4 = (s+2)(s-2).$$

The least common denominator is given by

$$d(s) = s(s+2)(s-2) = s^3 - 4s.$$

From this, we obtain that

$$d(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3,$$

with $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = -4$.

Next, the transfer matrix $\hat{\mathbf{G}}_{sp}(s)$ is written as

$$\begin{aligned}
\hat{\mathbf{G}}_{sp}(s) &= \begin{bmatrix} \frac{2s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{8}{s^2-4} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s(s+2)} & \frac{3}{s+2} \\ 0 & \frac{8}{(s+2)(s-2)} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s(s+2)} \cdot \frac{s-2}{s-2} & \frac{3}{s+2} \cdot \frac{s(s-2)}{s(s-2)} \\ 0 & \frac{8}{(s+2)(s-2)} \cdot \frac{s}{s} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2s^2-2s-4}{s^3-4s} & \frac{3s^2-6s}{s^3-4s} \\ 0 & \frac{8s}{s^3-4s} \end{bmatrix} = \frac{1}{s^3-4s} \begin{bmatrix} 2s^2-2s-4 & 3s^2-6s \\ 0 & 8s \end{bmatrix} \\
&= \frac{1}{s^3-4s} \left\{ \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} -2 & -6 \\ 0 & 8 \end{bmatrix} s + \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \right\}.
\end{aligned}$$

Hence, we obtain

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)} \left[\mathbf{N}_1 s^2 + \mathbf{N}_2 s + \mathbf{N}_3 \right],$$

with

$$\mathbf{N}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} -2 & -6 \\ 0 & 8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{N}_3 = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}.$$

e) Substitution of $\mathbf{D}$, α_1, α_2, α_3, $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ yields

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} 2 & 3 & -2 & -6 & -4 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{u}(t).$$

Problem 3: State estimator

a) The observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}.$$

Because the observability matrix has full column rank, i.e. $\text{rank}(\mathcal{O}) = 2 = n$, we conclude that the system is observable.

b) From the state-space equation of the system, the equation of the state estimator and $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, it immediately follows that

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}u(t) - \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t) - Du(t)) \\ &= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) - \mathbf{L}(\mathbf{C}\mathbf{x}(t) + Du(t) - \mathbf{C}\hat{\mathbf{x}}(t) - Du(t)) \\ &= (\mathbf{A} - \mathbf{LC})(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \\ &= (\mathbf{A} - \mathbf{LC})\mathbf{e}(t). \end{aligned}$$

c) Let the estimator-gain matrix $\mathbf{L}$ be given by

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix},$$

where l_1 and l_2 are constants that have to be determined. The eigenvalues of $\mathbf{A} - \mathbf{LC}$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{LC}$, which is given by

$$\det(\mathbf{A} - \mathbf{LC} - \lambda\mathbf{I}) = \begin{vmatrix} -4 - \lambda & -l_1 \\ 3 & -1 - l_2 - \lambda \end{vmatrix} = \lambda^2 + (l_2 + 5)\lambda + 3l_1 + 4l_2 + 4.$$

If the eigenvalues of $\mathbf{A} - \mathbf{LC}$ are equal to -8 and -10, the characteristic polynomial is given by

$$\begin{aligned} \det(\mathbf{A} - \mathbf{LC} - \lambda\mathbf{I}) &= (-8 - \lambda)(-10 - \lambda) \\ &= \lambda^2 + 18\lambda + 80. \end{aligned}$$

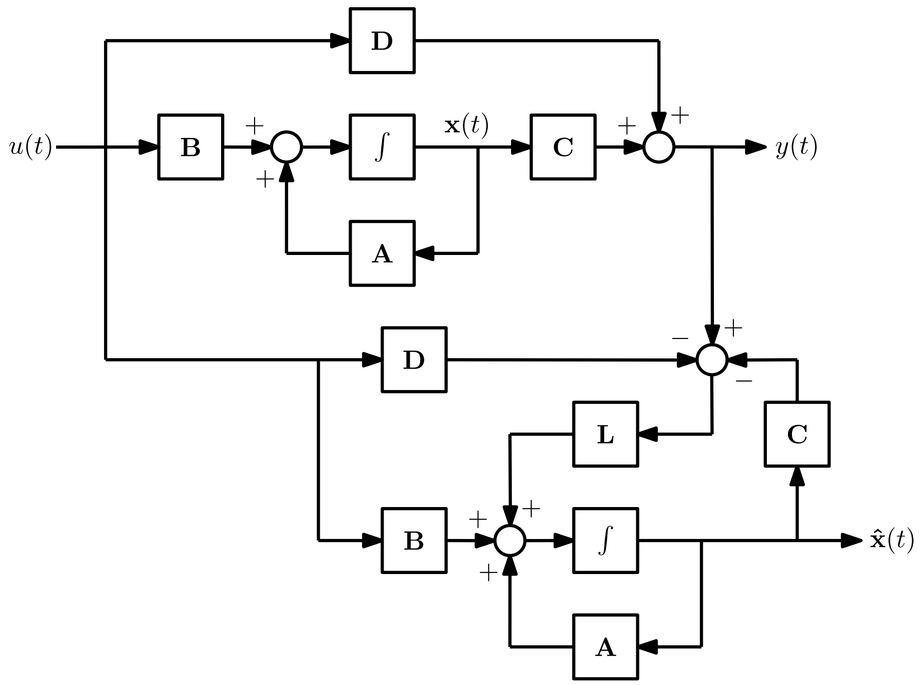
By combining these two expressions for the characteristic polynomial, we obtain the equations

$$l_2 + 5 = 18,$$
$$3l_1 + 4l_2 + 4 = 80.$$

Solving for l_1 and l_2 yields $l_1 = 8$ and $l_2 = 13$. Hence, we obtain

$$\mathbf{L} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}.$$

d) The block diagram of the system with state estimator is given by:



Problem 4: Separation principle

a) By combining the equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$, we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t).$$

From $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$, it follows that

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}(\mathbf{e}(t) + \mathbf{x}(t)) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{e}(t).$$

Taking the time derivative of $\mathbf{e}(t)$ yields

$$\dot{\mathbf{e}}(t) = \dot{\hat{\mathbf{x}}}(t) - \dot{\mathbf{x}}(t).$$

By substituting the equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t))$, we get

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)) - \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{u}(t)\\ &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)).\end{aligned}$$

By combining this with $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$, we obtain

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}(\mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t))\\ &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}\mathbf{C}(\mathbf{x}(t) - \hat{\mathbf{x}}(t))\\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).\end{aligned}$$

From $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{e}(t)$ and $\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t)$, we get

$$\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\mathbf{e}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K}\\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{e}(t)\end{bmatrix}.$$

b) The characteristic polynomial of the matrix $\mathbf{H}$ is given by

$$\begin{aligned}\det(\mathbf{H} - \lambda\mathbf{I}) &= \det\left(\begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I} & -\mathbf{B}\mathbf{K}\\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}\end{bmatrix}\right)\\ &= \det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I})\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}).\end{aligned}$$

For any eigenvalue λ of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$, we have $\det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I}) = 0$ or $\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}) = 0$. This implies that $\det(\mathbf{H} - \lambda\mathbf{I}) = \det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I})\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}) = 0$. Because $\det(\mathbf{H} - \lambda\mathbf{I})$ is zero, λ must be an eigenvalue of $\mathbf{H}$. Hence, any eigenvalue of the matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$ or the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$ is an eigenvalue of the matrix $\mathbf{H}$. Moreover, because $\det(\mathbf{H} - \lambda\mathbf{I})$ is only zero if λ is an eigenvalue of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$, we have that all eigenvalues of $\mathbf{H}$ are eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$. Hence, the eigenvalues of $\mathbf{H}$ are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$.

c) If the system is controllable, then the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ can be assigned arbitrarily by choosing $\mathbf{K}$. Moreover, if the system is observable, then the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be assigned arbitrarily by choosing $\mathbf{L}$. Because the poles of the closed-loop system (i.e. the eigenvalues of $\mathbf{H}$) are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$, we conclude that the poles of the closed-loop system can be assigned arbitrarily.