TTK4115 Linear System Theory

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Solution to homework assignment 5

Problem 1: Process classification

a) The probability density function of the variable Φ is given by

$$f_\Phi(\phi) = \begin{cases} \dfrac{1}{2\pi}, & \text{if } -\pi \le \phi < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

The mean $\mu_X(t) = E[X(t)]$ is calculated as follows:

$$\begin{aligned}
\mu_X(t) &= E[X(t)] = E[a\sin(\omega t + \Phi)] = aE[\sin(\omega t + \Phi)] \\
&= a\int_{-\infty}^{\infty} \sin(\omega t + \phi) f_\Phi(\phi) d\phi = \frac{a}{2\pi}\int_{-\pi}^{\pi} \sin(\omega t + \phi) d\phi \\
&= \frac{a}{2\pi}\left[-\cos(\omega t + \phi)\right]_{-\pi}^{\pi} = \frac{a}{2\pi}\left[-\cos(\omega t + \pi) + \cos(\omega t + -\pi)\right] \\
&= \frac{a}{2\pi}\left[\cos(\omega t) - \cos(\omega t)\right] = 0.
\end{aligned}$$

b) The variance $\sigma_X^2(t) = E[(X(t) - \mu_X(t))^2]$ is given by

$$\begin{aligned}
\sigma_X^2(t) &= E[X^2(t)] = E[(a\sin(\omega t + \Phi))^2] = a^2 E[\sin^2(\omega t + \Phi)] \\
&= a^2 E\left[\frac{1 - \cos(2\omega t + 2\Phi)}{2}\right] = \frac{a^2}{2}\left(1 - E\left[\cos(2\omega t + 2\Phi)\right]\right) \\
&= \frac{a^2}{2}\left(1 - \int_{-\infty}^{\infty} \cos(2\omega t + 2\phi) f_\Phi(\phi) d\phi\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\omega t + 2\phi) d\phi\right) = \frac{a^2}{2}\left(1 - \frac{1}{2\pi}\left[\frac{\sin(2\omega t + 2\phi)}{2}\right]_{-\pi}^{\pi}\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{4\pi}\left[\sin(2\omega t + 2\pi) - \sin(2\omega t - 2\pi)\right]\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{4\pi}\left[\sin(2\omega t) - \sin(2\omega t)\right]\right) = \frac{a^2}{2},
\end{aligned}$$

where we used the probability density function f_Φ in a).

c) Using the probability density function f_Φ in a), we obtain the following autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$:

$$\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X(t_2)] = E[(a\sin(\omega t_1 + \Phi))(a\sin(\omega t_2 + \Phi))] \\
&= a^2 E[\sin(\omega t_1 + \Phi)\sin(\omega t_2 + \Phi)] \\
&= a^2 E\left[\frac{1}{2}\cos(\omega t_1 + \Phi - (\omega t_2 + \Phi)) - \frac{1}{2}\cos(\omega t_1 + \Phi + (\omega t_2 + \Phi))\right] \\
&= \frac{a^2}{2} E\left[\cos(\omega(t_1 - t_2)) - \cos(\omega(t_1 + t_2) + 2\Phi)\right] \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1 - t_2)) - E[\cos(\omega(t_1 + t_2) + 2\Phi)]\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1 - t_2)) - \int_{-\infty}^{\infty}\cos(\omega(t_1 + t_2) + 2\phi) f_\Phi(\phi)d\phi\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1 - t_2)) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega(t_1 + t_2) + 2\phi)d\phi\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1 - t_2)) - \frac{1}{2\pi}\left[\frac{\sin(\omega(t_1 + t_2) + 2\phi)}{2}\right]_{-\pi}^{\pi}\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1 - t_2)) - \frac{1}{4\pi}\left[\sin(\omega(t_1 + t_2) + 2\pi) - \sin(\omega(t_1 + t_2) - 2\pi)\right]\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1 - t_2)) - \frac{1}{4\pi}\left[\sin(\omega(t_1 + t_2)) - \sin(\omega(t_1 + t_2))\right]\right) \\
&= \frac{a^2}{2}\cos(\omega(t_1 - t_2)).
\end{aligned}$$

Substituting $t_1 = t$ and $t_2 = t + \tau$, we get

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{a^2}{2}\cos(\omega(t - (t+\tau))) = \frac{a^2}{2}\cos(-\omega\tau) = \frac{a^2}{2}\cos(\omega\tau).$$

d) The process is deterministic. With $\Phi = \Phi_1$ the process becomes $X(t, \Phi_1) = a\sin(\omega t + \Phi_1)$. Knowledge about the process for $t \leq t_0$ makes identification of Φ_1, ω and a possible, and the process is uniquely defined $\forall t > t_0$.

e) Because the mean $\mu_X(t)$ is not dependent on the time origin (i.e. $\mu_X(t)$ is independent of t, see a)) and the autocorrelation function $R_X(t_1, t_2)$ in c) is only dependent on the time difference between sample points (i.e. $R_X(t_1, t_2)$ is dependent only on the time difference $t_2 - t_1$, since we can write $R_X(t_1, t_2) = R_X(\tau)$ for $t_1 = t$ and $t_2 = t + \tau$, see c)), the process is wide-sense stationary. In fact, it can be shown that all density functions associated with the process are independent of time, which implies that the process is stationary, which is a stronger property than wide-sense stationarity.

f) While ergodicity applies to all density functions associated with the process, ergodicity in wide sense only applies to the mean and autocorrelation function of the process. For a process to be ergodic in wide sense, the time mean and the time autocorrelation function must be equivalent to the ensemble mean (i.e. μ_X) and the ensemble autocorrelation function (i.e. $R_X(\tau)$), respectively.

The time mean is given by

$$\begin{aligned}
\mathfrak{m}_X &= \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T a\sin(\omega t+\Phi)dt \\
&= \lim_{T\to\infty} \frac{a}{T}\left[\frac{-\cos(\omega t+\Phi)}{\omega}\right]_0^T = \lim_{T\to\infty} \frac{a}{\omega T}\left[-\cos(\omega T+\Phi)+\cos(\Phi)\right] = 0.
\end{aligned}$$

The time autocorrelation function is given by

$$\begin{aligned}
\mathfrak{R}_X(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)X(t+\tau)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T (a\sin(\omega t+\Phi))(a\sin(\omega(t+\tau)+\Phi))dt \\
&= \lim_{T\to\infty} \frac{a^2}{T}\int_0^T \sin(\omega t+\Phi)\sin(\omega(t+\tau)+\Phi)dt \\
&= \lim_{T\to\infty} \frac{a^2}{T}\int_0^T \left(\frac{1}{2}\cos(\omega t+\Phi-(\omega(t+\tau)+\Phi))\right. \\
&\qquad \left. -\frac{1}{2}\cos(\omega t+\Phi+\omega(t+\tau)+\Phi)\right)dt \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\int_0^T \left(\cos(-\omega\tau)-\cos(2\omega t+\omega\tau+2\Phi)\right)dt \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\left[\cos(\omega\tau)t-\frac{\sin(2\omega t+\omega\tau+2\Phi)}{2\omega}\right]_0^T \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\left[\cos(\omega\tau)T-\frac{\sin(2\omega T+\omega\tau+2\Phi)}{2\omega}+\frac{\sin(\omega\tau+2\Phi)}{2\omega}\right] \\
&= \frac{a^2}{2}\cos(\omega\tau).
\end{aligned}$$

Because the time mean $\mathfrak{m}_X$ and time autocorrelation function $\mathfrak{R}_X(\tau)$ are equal to the ensemble mean μ_X in a) and the ensemble autocorrelation function $R_X(\tau)$ in c), respectively, we conclude that the process is ergodic in wide sense. In fact, it can be shown that process is ergodic (not only in wide sense).

Problem 2: Linear system with white noise

a) White noise processes have a zero mean. Because the disturbance $w(t)$ is a white noise process, we have $\mu_w = 0$.

The reasoning behind this follows next. Let $v(t)$ be a white noise process. By definition, white noise has a flat spectrum. Therefore, the power spectrum density function associated with $v(t)$ is given by $S_v(j\omega) = \alpha_v$, where α_v is a nonnegative constant. Using the inverse Fourier transform, we obtain the corresponding autocorrelation function

$$R_v(\tau) = \mathcal{F}^{-1}\{S_v(j\omega)\} = \alpha_v \delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta function. We can define the zero-mean white-noise process $\bar{v}(t) = v(t) - \mu_v$, where $\mu_v = E[v(t)]$ is the mean of $v(t)$. Note that because $\bar{v}(t)$ is a white noise process, we have $S_{\bar{v}}(j\omega) = \alpha_{\bar{v}}$ for some nonnegative constant $\alpha_{\bar{v}}$. Similar as for $v(t)$, the autocorrelation function associated with $\bar{v}(t)$ is given by

$$R_{\bar{v}}(\tau) = \mathcal{F}^{-1}\{S_{\bar{v}}(j\omega)\} = \alpha_{\bar{v}} \delta(\tau).$$

Now, note that from the definition of the autocorrelation function, it follows that

$$\begin{aligned} R_{\bar{v}}(\tau) &= E[\bar{v}(t)\bar{v}(t+\tau)] = E[(v(t) - \mu_v)(v(t+\tau) - \mu_v)] \\ &= E[v(t)v(t+\tau) - \mu_v v(t) - \mu_v v(t+\tau) + \mu_v^2] \\ &= E[v(t)v(t+\tau)] - \mu_v E[v(t)] - \mu_v E[v(t+\tau)] + \mu_v^2 \\ &= R_v(\tau) - \mu_v^2 - \mu_v^2 + \mu_v^2 = R_v(\tau) - \mu_v^2. \end{aligned}$$

Substituting $R_v(\tau) = \alpha_v \delta(\tau)$ and $R_{\bar{v}}(\tau) = \alpha_{\bar{v}} \delta(\tau)$, we obtain

$$\alpha_{\bar{v}} \delta(\tau) = \alpha_v \delta(\tau) - \mu_v^2.$$

This is only valid for all τ if $\alpha_{\bar{v}} = \alpha_v$ and $\mu_v = 0$. Because the mean μ_v of $v(t)$ is equal to zero and $v(t)$ is an arbirtrary white noise process, we conclude that all white noise processes must have a zero mean.

b) The variance σ_w^2 can directly be obtained from the autocorrelation function $R_w(\tau)$:

$$\sigma_w^2 = E[w^2(t)] = R_w(0) = 4\delta(0) = \infty.$$

c) The power spectral density function $S_w(j\omega)$ of the disturbance $w(t)$ is obtained by taking the Fourier transform of the autocorrelation function $R_w(\tau)$:

$$S_w(j\omega) = \mathcal{F}\{R_w(\tau)\} = \mathcal{F}\{4\delta(\tau)\} = 4\mathcal{F}\{\delta(\tau)\} = 4.$$

d) The transfer function $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{w}(s)}$ can be obtained from $\hat{g}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, where $\mathbf{I}$ is the identity matrix. Hence, we get

$$\begin{aligned} \hat{g}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + 6s + 8} \begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{s+8}{s^2 + 6s + 8}. \end{aligned}$$

e) The poles of the system are equal to the roots of the denominator polynomial of the transfer function $\hat{g}(s)$ (i.e. the roots of $s^2 + 6s + 8$) and are given by $\lambda_1 = -4$ and $\lambda_2 = -2$. Given that $\hat{g}(s) = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2}$, we obtain

$$\begin{aligned}\hat{g}(s) &= \frac{\alpha_1}{s+4} + \frac{\alpha_2}{s+2} = \frac{\alpha_1(s+2)}{(s+2)(s+4)} + \frac{\alpha_2(s+4)}{(s+2)(s+4)} \\ &= \frac{(\alpha_1+\alpha_2)s + 2\alpha_1 + 4\alpha_2}{s^2+6s+8} = \frac{s+8}{s^2+6s+8}.\end{aligned}$$

From this, we conclude that

$$\alpha_1 + \alpha_2 = 1 \qquad \text{and} \qquad 2\alpha_1 + 4\alpha_2 = 8.$$

Solving for α_1 and α_2 yields $\alpha_1 = -2$ and $\alpha_2 = 3$. Hence, the transfer function $g(s)$ can be written as

$$\hat{g}(s) = \frac{-2}{s+4} + \frac{3}{s+2}.$$

By taking the inverse Laplace transform of the transfer function $g(s)$, we obtain the impulse response $g(t)$, which is given by

$$\begin{aligned}g(t) &= \mathcal{L}^{-1}\{\hat{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{s+4} + \frac{3}{s+2}\right\} \\ &= -2\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = -2e^{-4t} + 3e^{-2t}.\end{aligned}$$

f) Using $y(t) = \int_0^t g(\tau)w(t-\tau)d\tau$, the mean $\mu_y(t)$ is calculated as follows:

$$\begin{aligned}\mu_y(t) &= E[y(t)] = E\left[\int_0^t g(\tau)w(t-\tau)d\tau\right] = \int_0^t g(\tau)E[w(t-\tau)]d\tau \\ &= \int_0^t g(\tau)\mu_w d\tau = \mu_w \int_0^t g(\tau)d\tau = \mu_w \int_0^t (-2e^{-4\tau} + 3e^{-2\tau})d\tau \\ &= \mu_w \left[\frac{1}{2}e^{-4\tau} - \frac{3}{2}e^{-2\tau}\right]_0^t = \mu_w\left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} - \frac{1}{2} + \frac{3}{2}\right) \\ &= \mu_w\left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} + 1\right).\end{aligned}$$

The stationary mean $\bar{\mu}_y$ is given by

$$\bar{\mu}_y = \lim_{t\to\infty} \mu_y(t) = \lim_{t\to\infty} \mu_w \left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} + 1\right) = \mu_w.$$

From a), we have $\mu_w = 0$. Hence, we obtain $\bar{\mu}_y = \mu_w = 0$.

g) Note that the variance $\sigma_y^2(t)$ is equal to the mean-square value of $y(t)$, i.e. $\sigma_y^2(t) = E[y^2(t)]$. It follows that

$$\begin{aligned}
\sigma_y^2(t) &= E[y^2(t)] = E\left[\int_0^t g(\tau_1)w(t-\tau_1)d\tau_1 \int_0^t g(\tau_2)w(t-\tau_2)d\tau_2\right] \\
&= E\left[\int_0^t g(\tau_2)\int_0^t g(\tau_1)w(t-\tau_1)w(t-\tau_2)d\tau_1 d\tau_2\right] \\
&= \int_0^t g(\tau_2)\int_0^t g(\tau_1)E\left[w(t-\tau_1)w(t-\tau_2)\right]d\tau_1 d\tau_2 \\
&= \int_0^t g(\tau_2)\int_0^t g(\tau_1)R_w(\tau_2-\tau_1)d\tau_1 d\tau_2 \\
&= 4\int_0^t g(\tau_2)\int_0^t g(\tau_1)\delta(\tau_2-\tau_1)d\tau_1 d\tau_2 \\
&= 4\int_0^t g(\tau_2)g(\tau_2)d\tau_2 = 4\int_0^t g^2(\tau_2)d\tau_2 \\
&= 4\int_0^t (-2e^{-4\tau_2}+3e^{-2\tau_2})^2 d\tau_2 = 4\int_0^t (4e^{-8\tau_2}-12e^{-6\tau_2}+9e^{-4\tau_2})d\tau_2 \\
&= 4\left[-\frac{1}{2}e^{-8\tau_2}+2e^{-6\tau_2}-\frac{9}{4}e^{-4\tau_2}\right]_0^t \\
&= 4\left(-\frac{1}{2}e^{-8t}+2e^{-6t}-\frac{9}{4}e^{-4t}+\frac{1}{2}-2+\frac{9}{4}\right) \\
&= -2e^{-8t}+8e^{-6t}-9e^{-4t}+3.
\end{aligned}$$

The stationary variance $\bar{\sigma}_y^2$ is given by

$$\bar{\sigma}_y^2 = \lim_{t\to\infty}\sigma_y^2(t) = \lim_{t\to\infty}\left(-2e^{-8t}+8e^{-6t}-9e^{-4t}+3\right) = 3.$$

h) The power spectral density function $S_y(j\omega)$ of the output $y(t)$ is given by

$$S_y(j\omega) = |g(j\omega)|^2 S_w(j\omega) = g(j\omega)g(-j\omega)S_w(j\omega).$$

From c), we have that $S_w(j\omega) = 4$. In addition, using the transfer function $g(s) = \frac{s+8}{s^2+6s+8}$ in d), we obtain

$$\begin{aligned}
S_y(j\omega) &= \frac{j\omega+8}{(j\omega)^2+6(j\omega)+8}\cdot\frac{(-j\omega)+8}{(-j\omega)^2+6(-j\omega)+8}\cdot 4 \\
&= \frac{j\omega+8}{-\omega^2+6j\omega+8}\cdot\frac{-j\omega+8}{-\omega^2-6j\omega+8}\cdot 4 \\
&= \frac{4\omega^2+256}{\omega^4+20\omega^2+64} = \frac{20}{\omega^2+4}-\frac{16}{\omega^2+16}.
\end{aligned}$$