TTK4115 Linear System Theory
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Solution to homework assignment 2

Problem 1: Discretization

The discretized linearized system is given by

$$\begin{aligned}\bar{\mathbf{x}}[k+1] &= \mathbf{A}_d\bar{\mathbf{x}}[k] + \mathbf{B}_d\bar{\mathbf{u}}[k],\\ \bar{y}[k] &= \mathbf{C}_d\bar{\mathbf{x}}[k] + \mathbf{D}_d\bar{\mathbf{u}}[k].\end{aligned}$$

The corresponding matrices $\mathbf{A}_d$, $\mathbf{B}_d$, $\mathbf{C}_d$ and $\mathbf{D}_d$ are calculated next.

To obtain the matrix $\mathbf{A}_d = e^{\mathbf{A}T}$, we compute $e^{\mathbf{A}t}$. To compute $e^{\mathbf{A}t}$, we first determine the matrices $\hat{\mathbf{A}}$ and $\mathbf{Q}$, such that $\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1}$, where $\hat{\mathbf{A}}$ is in Jordan form. The characteristic polynomial of $\mathbf{A}$ is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -2 & -\lambda - 2 \end{vmatrix} = \lambda^2 + 2\lambda + 2 = (\lambda + 1 - j)(\lambda + 1 + j).$$

From this, it is easy to see that the roots of the characteristic polynomial of $\mathbf{A}$, and therefore the eigenvalues of $\mathbf{A}$, are given by $\lambda_1 = -1 + j$ and $\lambda_1 = -1 - j$. The corresponding eigenvectors $\mathbf{q}_i$ can be obtained from the kernel of the matrix $(\lambda_i\mathbf{I} - \mathbf{A})$ for $i = 1, 2$:

$$\ker(\lambda_1\mathbf{I} - \mathbf{A}) = \ker\left(\begin{bmatrix} 1-j & 1 \\ -2 & -1-j \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 2 & 1+j \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q}_1 = \begin{bmatrix} -1-j \\ 2 \end{bmatrix},$$

$$\ker(\lambda_2\mathbf{I} - \mathbf{A}) = \ker\left(\begin{bmatrix} 1+j & 1 \\ -2 & -1+j \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 2 & 1-j \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q}_2 = \begin{bmatrix} -1+j \\ 2 \end{bmatrix}.$$

Therefore, $\hat{\mathbf{A}}$ and $\mathbf{Q}$ are given by

$$\hat{\mathbf{A}} = \begin{bmatrix} -1+j & 0 \\ 0 & -1-j \end{bmatrix}, \qquad \text{and} \qquad \mathbf{Q} = \begin{bmatrix} -1-j & -1+j \\ 2 & 2 \end{bmatrix}.$$

We note that $\mathbf{Q}$ is not unique. Next, we compute $e^{\mathbf{A}t} = \mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1}$, with

$$e^{\hat{\mathbf{A}}t} = \begin{bmatrix} e^{(-1+j)t} & 0 \\ 0 & e^{(-1-j)t} \end{bmatrix} = \begin{bmatrix} e^{-t}(\cos(t) + j\sin(t)) & 0 \\ 0 & e^{-t}(\cos(t) - j\sin(t)) \end{bmatrix},$$

where we used $e^{(-1+j)t} = e^{-t}e^{jt}$ and $e^{(-1-j)t} = e^{-t}e^{-jt}$, with Euler's formula $e^{jt} = \cos(t) + j\sin(t)$. We obtain,

$$
\begin{aligned}
e^{\mathbf{A}t} &= \mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1} \\
&= \begin{bmatrix} -1-j & -1+j \\ 2 & 2 \end{bmatrix} \begin{bmatrix} e^{-t}(\cos(t)+j\sin(t)) & 0 \\ 0 & e^{-t}(\cos(t)-j\sin(t)) \end{bmatrix} \frac{1}{4} \begin{bmatrix} 2j & 1+j \\ -2j & 1-j \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(t)+\sin(t)) & e^{-t}\sin(t) \\ -2e^{-t}\sin(t) & e^{-t}(\cos(t)-\sin(t)) \end{bmatrix}.
\end{aligned}
$$

Substituting $t = T = \frac{\pi}{2}$ in $e^{\mathbf{A}t}$, we have

$$
\begin{aligned}
\mathbf{A}_d = e^{\mathbf{A}T} &= \begin{bmatrix} e^{-T}(\cos(T)+\sin(T)) & e^{-T}\sin(T) \\ -2e^{-T}\sin(T) & e^{-T}(\cos(T)-\sin(T)) \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{\pi}{2}}(\cos(\frac{\pi}{2})+\sin(\frac{\pi}{2})) & e^{-\frac{\pi}{2}}\sin(\frac{\pi}{2}) \\ -2e^{-\frac{\pi}{2}}\sin(\frac{\pi}{2}) & e^{-\frac{\pi}{2}}(\cos(\frac{\pi}{2})-\sin(\frac{\pi}{2})) \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{\pi}{2}} & e^{-\frac{\pi}{2}} \\ -2e^{-\frac{\pi}{2}} & -e^{-\frac{\pi}{2}} \end{bmatrix} \approx \begin{bmatrix} 0.2079 & 0.2079 \\ -0.4158 & -0.2079 \end{bmatrix}.
\end{aligned}
$$

Because $\mathbf{A}$ is nonsingular, it follows that

$$
\begin{aligned}
\mathbf{B}_d = \left(\int_0^T e^{\mathbf{A}\tau}d\tau\right)\mathbf{B} = \mathbf{A}^{-1}(\mathbf{A}_d - \mathbf{I})\mathbf{B} &= \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{\pi}{2}}-1 & e^{-\frac{\pi}{2}} \\ -2e^{-\frac{\pi}{2}} & -e^{-\frac{\pi}{2}}-1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{2}-\frac{1}{2}e^{-\frac{\pi}{2}} \\ 0 & e^{-\frac{\pi}{2}} \end{bmatrix} \approx \begin{bmatrix} 0 & 0.4991 \\ 0 & 0.0019 \end{bmatrix}.
\end{aligned}
$$

The matrices $\mathbf{C}_d$ and $\mathbf{D}_d$ are given by

$$
\mathbf{C}_d = \mathbf{C} = \begin{bmatrix} 4 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D}_d = \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}.
$$

Problem 2: Similarity transforms and equivalent state-space equations

a) Using the equations of the coordinate transformation (2) and the system (1), we obtain

$$
\dot{\bar{\mathbf{x}}} = \mathbf{T}\dot{\mathbf{x}} = \mathbf{T}\mathbf{A}\mathbf{x} + \mathbf{T}\mathbf{B}u = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\bar{\mathbf{x}} + \mathbf{T}\mathbf{B}u
$$

and

$$
y = \mathbf{C}\mathbf{x} + Du = \mathbf{C}\mathbf{T}^{-1}\bar{\mathbf{x}} + Du.
$$

Hence, we get

$$
\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}u \\
y &= \bar{\mathbf{C}}\bar{\mathbf{x}} + \bar{D}u,
\end{aligned}
$$

with

$$
\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \quad \bar{\mathbf{B}} = \mathbf{T}\mathbf{B}, \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} \quad \text{and} \quad \bar{D} = D.
$$

Substituting the values for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, D and $\mathbf{T}$ yields

$$\begin{aligned}
\bar{\mathbf{A}} &= \mathbf{TAT}^{-1} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} \\
\bar{\mathbf{B}} &= \mathbf{TB} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \\
\bar{\mathbf{C}} &= \mathbf{CT}^{-1} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \\
\bar{D} &= D = 2.
\end{aligned}$$

b) Because $\bar{\mathbf{x}} = \mathbf{Tx}$ is a similarity transformation, the systems (1) and (3) are algebraically equivalent. Because the systems (1) and (3) are algebraically equivalent, they are also zero-state equivalent.

c) Because the dimensions of the states of the systems (1) and (4) are different, there exists no similarity transform for the systems, i.e. there exists no invertible matrix $\mathbf{S}$ such that $\tilde{x} = \mathbf{Sx}$. Therefore, the systems (1) and (4) are not algebraically equivalent. To check if the systems (1) and (4) are zero-state equivalent, we have to check if the systems have the same transfer function (or impulse response). The transfer function of system (1) is given by

$$\begin{aligned}
\hat{G}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s+2 & -4 \\ 1 & s-3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 2 \end{bmatrix} + 2 \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{s-3}{s^2-s-2} & \frac{4}{s^2-s-2} \\ \frac{-1}{s^2-s-2} & \frac{s+2}{s^2-s-2} \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} + 2 \\
&= \frac{6}{s+1} + 2 = \frac{2s+8}{s+1}.
\end{aligned}$$

The transfer function of system (4) is given by

$$\hat{\tilde{G}}(s) = \tilde{C}(s - \tilde{A})^{-1}\tilde{B} + \tilde{D} = 3(s+1)^{-1}2 + 2 = \frac{6}{s+1} + 2 = \frac{2s+8}{s+1}.$$

Hence, because the systems (1) and (4) have the same transfer function, they are zero-state equivalent.

Problem 3: Controllability tests

a) The controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 & -6 \\ 2 & 2 & -10 & -10 \end{bmatrix}.$$

Because the controllability matrix has full row rank, i.e. $\text{rank}(\mathcal{C}) = 2 = n$, we conclude that the system is controllable.

b) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & -3 \\ 4 & -5-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2).$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

c) For $\lambda = \lambda_1 = -1$, we have

$$\operatorname{rank}\begin{bmatrix}\mathbf{A} - \lambda_1\mathbf{I} & \mathbf{B}\end{bmatrix} = \operatorname{rank}\begin{bmatrix} 3 & -3 & 0 & 0 \\ 4 & -4 & 2 & 2 \end{bmatrix} = 2.$$

Similarly, for $\lambda = \lambda_2 = -2$, we have

$$\operatorname{rank}\begin{bmatrix}\mathbf{A} - \lambda_2\mathbf{I} & \mathbf{B}\end{bmatrix} = \operatorname{rank}\begin{bmatrix} 4 & -3 & 0 & 0 \\ 4 & -3 & 2 & 2 \end{bmatrix} = 2.$$

Because the matrix $\begin{bmatrix}\mathbf{A} - \lambda\mathbf{I} & \mathbf{B}\end{bmatrix}$ has full row rank for every eigenvalue λ of $\mathbf{A}$, i.e. $\operatorname{rank}\begin{bmatrix}\mathbf{A} - \lambda\mathbf{I} & \mathbf{B}\end{bmatrix} = n = 2$ for every eigenvalue λ of $\mathbf{A}$, we conclude that the system is controllable.

d) To find the matrix $\mathbf{W}$, we solve the Lyapunov equation

$$\mathbf{AW} + \mathbf{WA}^T = -\mathbf{BB}^T.$$

Note that $\mathbf{W}$ is a symmetric matrix, i.e. $\mathbf{W} = \mathbf{W}^T$. Let $\mathbf{W}$ be given by

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix},$$

where w_1, w_2 and w_3 are constant that are yet to be determined. Substituting the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{W}$ in the Lyapunov equation, we obtain

$$\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix} + \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} 2w_1 - 3w_2 & 2w_2 - 3w_3 \\ 4w_1 - 5w_2 & 4w_2 - 5w_3 \end{bmatrix} + \begin{bmatrix} 2w_1 - 3w_2 & 4w_1 - 5w_2 \\ 2w_2 - 3w_3 & 4w_2 - 5w_3 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}.$$

From this, we obtain the equations

$$\begin{aligned} 4w_1 - 6w_2 &= 0, \\ 4w_1 - 3w_2 - 3w_3 &= 0, \\ 8w_2 - 10w_3 &= -8, \end{aligned}$$

which can be written in the following form:

$$\begin{bmatrix} 4 & -6 & 0 \\ 4 & -3 & -3 \\ 0 & 8 & -10 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}.$$

Solving for w_1, w_2 and w_3 yields $w_1 = 6$, $w_2 = 4$ and $w_3 = 4$. Hence, we obtain the matrix

$$\mathbf{W} = \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}.$$

The matrix $\mathbf{W}$ is positive definite if and only if all its leading principle minors are positive. The leading principle minors of $\mathbf{W}$ are

$$w_1 = 6 \qquad \text{and} \qquad \det(\mathbf{W}) = \begin{vmatrix} 6 & 4 \\ 4 & 4 \end{vmatrix} = 8.$$

Because all leading principle minors of $\mathbf{W}$ are positive, the matrix $\mathbf{W}$ is positive definite. In addition, from b), we know that the eigenvalues of $\mathbf{A}$ are given by $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence, the eigenvalues of $\mathbf{A}$ have strictly negative real parts. Because the eigenvalue of $\mathbf{A}$ have strictly negative real parts and $\mathbf{W}$ is positive definite, we conclude that the system is controllable.

Problem 4: State feedback

a) The controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 4 & 32 \\ 1 & 5 & 25 \end{bmatrix}.$$

Because the controllability matrix has full row rank, i.e. $\text{rank}(\mathcal{C}) = 3 = n$, we conclude that the system is controllable.

b) The characteristic polynomial of $\bar{\mathbf{A}}$ is given by

$$\begin{aligned} \det(\bar{\mathbf{A}} - \lambda\mathbf{I}) &= \det(\mathbf{A} - \mathbf{BK} - \lambda\mathbf{I}) \\ &= \det\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) \\ &= \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 4 \\ -k_1 & -k_2 & 5-k_3-\lambda \end{vmatrix} \\ &= -\lambda^3 + (9-k_3)\lambda^2 + (4k_3 - 4k_2 - 23)\lambda - 8k_1 + 4k_2 - 3k_3 + 15. \end{aligned}$$

c) The characteristic polynomial of $\bar{\mathbf{A}}$ should be equal to

$$\begin{aligned} \det(\bar{\mathbf{A}} - \lambda\mathbf{I}) &= (\bar{\lambda}_1 - \lambda)(\bar{\lambda}_2 - \lambda)(\bar{\lambda}_3 - \lambda) \\ &= (-1-\lambda)(-2-\lambda)(-3-\lambda) \\ &= -\lambda^3 - 6\lambda^2 - 11\lambda - 6. \end{aligned}$$

Comparing this to the characteristic polynomial obtained in a), we obtain the equations

$$\begin{aligned} 9 - k_3 &= -6 \\ 4k_3 - 4k_2 - 23 &= -11 \\ -8k_1 + 4k_2 - 3k_3 + 15 &= -6. \end{aligned}$$

By solving these equalities, we obtain the feedback matrix $\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, with

$$k_1 = 3, \qquad k_2 = 12 \qquad \text{and} \qquad k_3 = 15.$$