TTK4115 Linear System Theory
Department of Engineering Cybernetics
NTNU

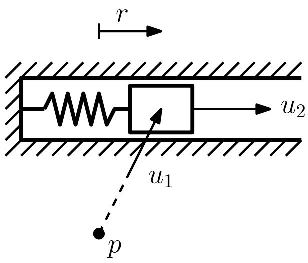
Homework assignment 2

Hand-out time:	Monday, September 9, 2024, at 7:00
Hand-in deadline:	Sunday, September 29, 2024, at 23:59

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard. Any questions regarding the assignment should be directed through Piazza. Exercise hours are on Tuesdays between 16:15-18:00 in S7 (Sentralbygg 2).

Problem 1: Discretization

Consider the slider mechanism as shown in the following figure.



In Assignment 1 it was shown that the system could be linearized about $\mathbf{x} = \mathbf{x}_0$ and $\mathbf{u} = \mathbf{u}_0$, with $\mathbf{x}_0 = [4, 2]^T$ and $\mathbf{u}_0 = [5, 0]^T$, by the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

The linearized system is sampled with sampling time $T = \frac{\pi}{2}$. Assuming that the input $\bar{\mathbf{u}}$ remains constant between subsequent samples, we use *exact discretization* to discretize the linearized system. Show that the discretized linearized system can be written in the following form:

$$\bar{\mathbf{x}}[k+1] = \mathbf{A}_d\bar{\mathbf{x}}[k] + \mathbf{B}_d\bar{\mathbf{u}}[k],$$
$$\bar{y}[k] = \mathbf{C}_d\bar{\mathbf{x}}[k] + \mathbf{D}_d\bar{\mathbf{u}}[k],$$

with $\bar{\mathbf{x}}[k] = \bar{\mathbf{x}}(t)$ for all $t = kT$, where k is an integer and the matrices $\mathbf{A}_d$, $\mathbf{B}_d$, $\mathbf{C}_d$ and $\mathbf{D}_d$ are given by

$$\mathbf{A}_d = \begin{bmatrix} e^{-\frac{\pi}{2}} & e^{-\frac{\pi}{2}} \\ -2e^{-\frac{\pi}{2}} & -e^{-\frac{\pi}{2}} \end{bmatrix}, \quad \mathbf{B}_d = \begin{bmatrix} 0 & \frac{1}{2} - \frac{1}{2}e^{-\frac{\pi}{2}} \\ 0 & e^{-\frac{\pi}{2}} \end{bmatrix}, \quad \mathbf{C}_d = \begin{bmatrix} 4 & 0 \end{bmatrix}, \quad \mathbf{D}_d = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Problem 2: Similarity transforms and equivalent state-space equations

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t),\end{aligned} \tag{1}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix}-2 & 4\\ -1 & 3\end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix}8\\ 2\end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix}1 & -1\end{bmatrix} \qquad \text{and} \qquad D = 2.$$

Consider the coordinate transformation

$$\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}, \tag{2}$$

with

$$\mathbf{T} = \begin{bmatrix}0 & -2\\ 1 & -1\end{bmatrix}.$$

By applying the coordinate transformation (2), the system (1) can be written in the following form:

$$\begin{aligned}\dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}u(t),\\ y(t) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(t) + \bar{D}u(t).\end{aligned} \tag{3}$$

a) Show that the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$ and $\bar{D}$ are given by

$$\bar{\mathbf{A}} = \begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix}-4\\ 6\end{bmatrix}, \quad \bar{\mathbf{C}} = \begin{bmatrix}0 & 1\end{bmatrix} \quad \text{and} \quad \bar{D} = 2.$$

b) Are the systems (1) and (3) algebraically equivalent? Are the systems (1) and (3) zero-state equivalent? Motivate your answer.

Consider the system

$$\begin{aligned}\dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t),\\ y(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}u(t),\end{aligned} \tag{4}$$

with

$$\tilde{A} = -1, \qquad \tilde{B} = 2, \qquad \tilde{C} = 3 \qquad \text{and} \qquad \tilde{D} = 2.$$

c) Are the systems (1) and (4) algebraically equivalent? Are the systems (1) and (4) zero-state equivalent? Motivate your answer.

Problem 3: Controllability tests

Consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$

with state $\mathbf{x}(t)$, input $\mathbf{u}(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}.$$

a) Calculate the controllability matrix of the system and determine if the system is controllable.

b) Calculate the eigenvalues of $\mathbf{A}$.

The system is controllable if and only if

$$\text{rank} \begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B} \end{bmatrix} = n = 2$$

for every eigenvalue λ of $\mathbf{A}$. This is known as the Popov-Belevitch-Hautus test for controllability.

c) Use the Popov-Belevitch-Hautus test for controllability to determine if the system is controllable.

We want to use the Lyapunov test for controllability to determine if the system is controllable. The corresponding Lyapunov equation is given by

$$\mathbf{AW} + \mathbf{WA}^T = -\mathbf{BB}^T,$$

where $\mathbf{W}$ is a symmetric matrix. If the eigenvalues of $\mathbf{A}$ have strictly negative real parts, then the system is controllable if and only if the matrix $\mathbf{W}$ is positive definite.

d) Calculate the matrix $\mathbf{W}$ from the Lyapunov equation and determine if the system is controllable.

Problem 4: State feedback

Consider the following system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

with state $\mathbf{x}(t)$, input $u(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

a) Calculate the controllability matrix of the system and determine if the system is controllable.

Consider a state-feedback controller of the following form:

$$u(t) = -\mathbf{K}\mathbf{x}(t),$$

where $\mathbf{K} = [k_1, k_2, k_3]$ is a feedback matrix. The closed-loop system (i.e. the system with controller) can be written as

$$\dot{\mathbf{x}}(t) = \bar{\mathbf{A}}\mathbf{x}(t),$$

with $\bar{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$.

b) Calculate the characteristic polynomial of the closed-loop system matrix $\bar{\mathbf{A}}$ as a function of k_1, k_2 and k_3.

We aim to find the values of k_1, k_2 and k_3 such that the poles of the closed-loop system, which are equal to the eigenvalues of $\bar{\mathbf{A}}$, are given by

$$\bar{\lambda}_1 = -1, \qquad \bar{\lambda}_2 = -2 \qquad \text{and} \qquad \bar{\lambda}_3 = -3.$$

Note that if the eigenvalues of $\bar{\mathbf{A}}$ are given by $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$, the characteristic polynomial of $\bar{\mathbf{A}}$ is given by

$$\det(\bar{\mathbf{A}} - \lambda \mathbf{I}) = (\bar{\lambda}_1 - \lambda)(\bar{\lambda}_2 - \lambda)(\bar{\lambda}_3 - \lambda). \tag{5}$$

c) Determine the feedback matrix $\mathbf{K} = [k_1, k_2, k_3]$ such that the eigenvalues of $\bar{\mathbf{A}}$ are given by $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ by comparing your result in b) with the characteristic polynomial (5). Show that $\mathbf{K} = [3, 12, 15]$.