TTK4115 Linear System Theory

Department of Engineering Cybernetics
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Homework assignment 1

Hand-out time:	Monday, August 26, 2024, at 7:00
Hand-in deadline:	Sunday, September 15, 2024, at 23:59

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard. Any questions regarding the assignment should be directed through Piazza. Exercise hours are on Tuesdays between 16:15-18:00 in S7 (Sentralbygg 2).

Problem 1: State-space equation, transfer function and impulse response

Consider the system described by the differential equation

$$\ddot{y}(t) + 2\dot{y}(t) = \dot{u}(t) + 4u(t).$$

a) For this system, derive a state-space equation of the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t),\end{aligned}$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) - u(t) \end{bmatrix}$.

b) Assume zero initial conditions. Use $\hat{g}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$ to find the transfer function $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ for the system.

c) To check your answer of the previous question, compute $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ by applying the Laplace transform to the differential equation while assuming zero initial conditions.

d) Note that the transfer function $\hat{g}(s)$ can be written as $\hat{g}(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+2}$, where α_1 and α_2 are constants. Use this to find the impulse response $g(t)$ of the system by taking the inverse Laplace transform of $\hat{g}(s)$, i.e. $g(t) = \mathcal{L}^{-1}[\hat{g}(s)]$.

Problem 2: Solutions of state-space equations

Consider the following state-space equation:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t),\end{aligned}$$

with state $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 3 & 0 \end{bmatrix} \qquad \text{and} \qquad D = 5.$$

a) Compute $e^{\mathbf{A}t}$ by taking the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1}$, i.e. compute $e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$.

b) Calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

c) Find a diagonal matrix $\hat{\mathbf{A}}$ and a nonsingular matrix $\mathbf{Q}$ such that $\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1}$.

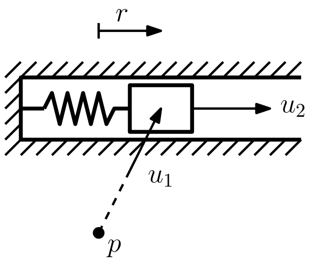
d) Compute $e^{\mathbf{A}t} = \mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1}$ and check your answer at a).

e) Let $u(t) = 1$ for all t. Compute $y(t)$ as a function of the initial conditions $\mathbf{x}(0)$.

f) Assuming that $u(t) = 1$ for all t, show that $\mathbf{x}(0) = [-\frac{1}{3}, 2]^T$ if $y(1) = y(2) = 4$.

Problem 3: Linearization

Consider the slider mechanism as shown in the following figure.



The slider moves inside a rectangular slot. The displacement of the slider is given by r. On one end of the slot, the slider is attached to a linear spring. The slider is excited by the input forces u_1 and u_2. The direction of u_1 is perpendicular to a line drawn from the fixed point p to the center of the slider. Note that this direction is dependent on the displacement of the slider. The direction of u_2 is perpendicular to the displacement of the slider. Due to the force u_1, the slider is pushed against the side wall of the slot resulting in friction between the slider and the wall. The equation of motion of the slider (for positive u_1) is given by

$$\ddot{r}(t) + \frac{2u_1(t)}{\sqrt{r^2(t)+9}}\dot{r}(t) + 3r(t) + 4 = \frac{r(t)}{\sqrt{r^2(t)+9}}u_1(t) + u_2(t).$$

Moreover, the output y is defined as five times the distance of the slider with respect to the fixed point p and is given by

$$y(t) = 5\sqrt{r^2(t)+9}.$$

a) Define $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{u} = [u_1, u_2]^T$, with $x_1 = r$ and $x_2 = \dot{r}$. Show that the equations of the slider mechanism can be written as a system of the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)),$$
$$y(t) = f(\mathbf{x}(t), \mathbf{u}(t)),$$

where $\mathbf{h}$ and f are nonlinear functions.

b) The above system is linearized about the point $\mathbf{x} = \mathbf{x}_0$, $\mathbf{u} = \mathbf{u}_0$, with $\mathbf{x}_0 = [4, 2]^T$ and $\mathbf{u}_0 = [5, 0]^T$, such that the following linearized system is obtained:

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{A}\bar{\mathbf{x}}(t) + \mathbf{B}\bar{\mathbf{u}}(t),$$
$$\bar{y}(t) = \mathbf{C}\bar{\mathbf{x}}(t) + \mathbf{D}\bar{\mathbf{u}}(t),$$

with matrices

$$\mathbf{A} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0), \quad \mathbf{B} = \frac{\partial \mathbf{h}}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0), \quad \mathbf{C} = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0) \quad \text{and} \quad \mathbf{D} = \frac{\partial f}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0).$$

Show that the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Problem 4: Jordan forms

Consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t),$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 0 & -9 \\ 1 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \text{and} \qquad D = 2.$$

a) Calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

b) Can we transform this system into a diagonal form using a similarity transform? Explain why.

c) Transform the system into a Jordan form using a similarity transform.