TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 1	Norwegian University of Science and Technology
Due date: Monday, September 9. at 23:59	

Problem 1 (25%) Norms

a Calculate the vector norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ for the two following vectors:

$$x_1 = \begin{bmatrix} 6 \\ -10 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

b For each vector, explain graphically and in words what the computed norms represent.

c Show that, for any vector $y \in \mathbb{R}^n$, the following statement is true:

$$\|y\|_1 \geq \|y\|_2$$

(If you are not able to show this for arbitrary dimension n, show it specifically for the case $n = 2$.)

d Calculate the matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ for the matrix:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

e The following statements are true for any norm on a finite-dimensional space:

$$\|Ax\|_p \leq \|A\|_p \|x\|_p$$

$$c_1 \|x\|_{q_1} \leq \|x\|_{q_2} \leq c_2 \|x\|_{q_1} \quad \text{where } c_1, c_2 > 0 \text{ are constants}$$

Show that they are true for the given matrix A and vector

$$x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

using $p = 2$, $q_1 = 1$, and $q_2 = \infty$. Specify the constants c_1 and c_2.

f For the two signals:

$$f_1(t) = \sqrt{t}$$

$$f_2(t) = \frac{1}{(t+1)^2}$$

decide if $f_i : [0, \infty) \to \mathbb{R}$ is $\mathcal{L}_\infty$, $\mathcal{L}_2$ and $\mathcal{L}_1$.

Problem 2 (20%) Existence and Uniqueness

a Let

$$f(x) = x^{\frac{1}{3}}$$

Is f globally Lipschitz? Explain. Find the area for which f is locally Lipschitz.

b For the following systems, find whether

f is locally Lipschitz on the whole state space

f is globally Lipschitz

Based on this information, specify for which systems we can conclude about local or global existence and uniqueness of solutions.

The systems are:

(a) The pendulum equation with friction:

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \end{bmatrix}$$

(b) The mass-spring equation with linear spring, linear viscous damping, Coulomb friction and no external force:

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}\eta(x_1, x_2) \end{bmatrix}$$

where η is defined in Section 1.2.3 in Khalil.

(c) The Van der Pol oscillator:

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + \varepsilon\left(1 - x_1^2\right)x_2 \end{bmatrix}$$

All parameters in the above systems can be considered positive.

c Show that any linear, time-invariant system

$$\dot{x} = Ax,\ x \in \mathbb{R}^n$$

is globally Lipschitz. How does the choice of norm affect the Lipschitz property? Explain.

Problem 3 (25%)

Using (nonlinear) differential equations we can not only describe and analyze technical systems, but also biological, economical and other systems. A simplified representation of populations of foxes and rabbits is given by the following equation set:

$$\begin{aligned} \dot{x} &= x(\alpha - \beta y) \\ \dot{y} &= -y(\gamma - \delta x) \end{aligned}$$

where

x	:	Number of rabbits
αx	:	Natural reproduction of rabbits
βxy	:	Rate of predation
y	:	Number of foxes
γy	:	Natural death of foxes
δxy	:	Growth rate for foxes
$\alpha, \beta, \gamma, \delta > 0$	:	Parameters representing the interaction of the two species

We know that $x \geq 0, y \geq 0$, since the number of animals cannot be negative.
Note that the rate of predation is quite similar to the growth rate for foxes, but they contain different parameters (as the fox population growth is not *necessarily* equal to the rate at which it consumes the rabbits).

a If initially $x > 0, y = 0$, what would happen to the rabbit population?

b If initially $x = 0, y > 0$, what would happen to the fox population?

c Find the equilibrium points of the system, and determine the type of each equilibrium point when all parameters are positive.

d Is the existence of several isolated equilibrium points possible for linear systems? Why/why not?

e Construct the phase portrait for $x \geq 0, y \geq 0$ and discuss the qualitative behavior of the system. Choose the values $\alpha = 63, \beta = 5, \gamma = 457, \delta = 6$. Use the Matlab phase portrait program `pplane` attached to the assignment.

f If initially $x > 0$ and $y > 0$, is it possible to arrive at $x < 0$ or $y < 0$ using this model? Explain your answer.

g Using Bendixson's criterion (Lemma 2.2), specify a *simply connected* region in which *no* periodic orbits are lying entirely. You may describe the region using inequalities.
Hint: See Example 2.10 in Khalil on page 68.

Problem 4 (5%)

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= (2x_2 - 1)x_1 \\ \dot{x}_2 &= \cos(x_1) - (x_2 + x_2^2 + x_2^4)x_2\end{aligned}$$

Prove that there are no periodic orbits on $\mathbb{R}^2$.

Problem 5 (10%)

In some cases, nonlinear systems $\dot{x} = f(x)$ with a complicated f can be written in a simpler form by changing the coordinate vector x into a new coordinate vector $z = \psi(x)$, where ψ is a continuously differentiable function (such that $\psi^{-1}(z)$ exists and is also continuously differentiable).
With the new coordinate vector, the corresponding system equations can be obtained by

$$\dot{z} = \frac{\partial \psi}{\partial x}(x)\dot{x} = \frac{\partial \psi}{\partial x}(x)f(x)\bigg|_{x=\psi^{-1}(z)} = \frac{\partial \psi}{\partial x}(\psi^{-1}(z))f(\psi^{-1}(z))$$

Although this expression may look more complicated, in some cases the resulting system can be simpler and more suitable for analysis than the original system, as illustrated in the following exercise.

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right) \\ \dot{x}_2 &= -x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right)\end{aligned}$$

where $\alpha, \beta \geq 0$ are constants.
Show that using polar coordinates ($x_1 = r\cos\theta, x_2 = r\sin\theta$) we can rewrite the original system in terms of the new coordinates, to get a simpler equation set:

$$\begin{aligned}\dot{r} &= \alpha r\left(\beta^2 - r^2\right) \\ \dot{\theta} &= -1\end{aligned}$$

Hint: Show first that $r = \pm\sqrt{x_1^2 + x_2^2}$ *and* $\theta = \tan^{-1}(x_2/x_1)$.

Problem 6 (15%)

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -ax_1 - x_2^2 \\ \dot{x}_2 &= bx_1x_2 + cx_2\end{aligned}$$

where $a, b, c > 0$.

a Find the Jacobian of the system.

b Find all equilibrium points of the system. Determine the type of each isolated equilibrium point for all values of $a, b, c > 0$.

c For each of the following cases, construct the phase portrait and discuss the qualitative behavior of the system.

(a) $a = 4, \quad b = 1, \quad c = 1$

(b) $a = 12, \quad b = 1, \quad c = 1$

Do the phase portraits confirm the placement and type of equilibrium points which you found in **b**?