TTK4150 Nonlinear Control Systems

Fall 2024 - Assignment 3
Due date: Monday, October 14. at 23:59

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Problem 1 (5%) Using the Lyapunov function candidate $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$, show that the origin of the following system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1 + x_2) - x_1^2 x_2\end{aligned}$$

is globally asymptotically stable.
Hint: See LaSalle's invariance principle (Theorem 4.4 in Khalil).

Problem 2 (20%) Consider again the mass-damper-spring system from Assignment 2:

a Use the transformed system from Assignment 2 (Exercise 1b) given by

$$\dot{\tilde{x}}_1 = \tilde{x}_2 \tag{1}$$
$$\dot{\tilde{x}}_2 = -\frac{f_3}{m}\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - \frac{f_1}{m}\tilde{x}_1 - \frac{d}{m}\tilde{x}_2 + \frac{\tilde{u}}{m} \tag{2}$$

and the Lyapunov function candidate

$$V(\tilde{x}) = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$$

to derive a controller (that is, find $\tilde{u}$) such that

$$\dot{V}(\tilde{x}) = -(d + k_2)\tilde{x}_2^2$$

where $k_2 > 0$ is a controller gain.
Hint: The resulting closed-loop system should be linear.

b Is the closed-loop system locally/globally asymptotically/exponentially stable at the origin? Find the strongest achievable stability result and motivate your answers.

c What happens to the system dynamics as k_2 increases? Explain this physically.

d By using the controller in part **a**, is it possible to place the poles of the system arbitrarily?

e An unknown constant disturbance w is acting on the system, such that the original system equations are changed to

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} + \frac{w}{m}\end{aligned}$$

Does our control input u from before stabilize the desired setpoint $(x_1, x_2) = (x_{1d}, 0)$ (the origin in error coordinates $(\tilde{x}_1, \tilde{x}_2)$) in the presence of the disturbance?

Problem 3 (10%) Consider Liènard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where the functions g, h are continuously differentiable.

a Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation. Find conditions on g and h to ensure that the origin is an isolated equilibrium point.

b Using $V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$, find conditions on g and h to ensure that the origin is asymptotically stable.

c Find conditions ensuring that the origin is globally asymptotically stable.

Problem 4 (10%) Consider the system

$$\begin{aligned}\dot{x}_1 =& x_1(x_1^2 - 1) + 2x_2 \\ \dot{x}_2 =& - x_1^3 - x_2\end{aligned}$$

a Find the equilibrium point(s) of the system.

b Show that the origin is asymptotically stable using the quadratic Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$, with $P = \left[\begin{smallmatrix}1 & 1 \\ 1 & 3\end{smallmatrix}\right]$

c Find an estimate of the region of attraction and sketch it in the (x_1, x_2)-plane.

Problem 5 (15%) Consider the system

$$\begin{aligned}\dot{x}_1 &= 4x_1^2x_2 - f_1(x_1)\left(x_1^2 + 2x_2^2 - 4\right) \\ \dot{x}_2 &= -2x_1^3 - f_2(x_2)\left(x_1^2 + 2x_2^2 - 4\right)\end{aligned}$$

where f_1 and f_2 are continuously differentiable functions satisfying $x_i f_i(x_i) > 0$, $\forall x_i \neq 0$ and $f_i(0) = 0$.

a Show that the origin is an equilibrium point of the system and investigate whether the system has other equilibrium points.

b Show that $\left\{x \in \mathbb{R}^2 \mid x_1^2 + 2x_2^2 - 4 = 0\right\}$ is an invariant set for the system.

c Use the Lyapunov function $V(x) = \left(x_1^2 + 2x_2^2 - 4\right)^2$ and Theorem 4.4 in Khalil to show that every trajectory approaches the set $(0, 0) \cup \left\{x \in \mathbb{R}^2 \mid x_1^2 + 2x_2^2 - 4 = 0\right\}$.

d Why do you think that the set $\left\{x \in \mathbb{R}^2 \mid x_1^2 + 2x_2^2 - 4 = 0\right\}$ is not a limit cycle?

Hint: A limit cycle is a periodic solution, such that the system trajectories cycle through a set of points. What could stop trajectories from cycling through an invariant set?

Problem 6 (15%) Consider the system

$$\begin{aligned}\dot{x}_1 &= (x_1 + 2x_2)(-x_1 - 2) \\ \dot{x}_2 &= -5x_2(2 + 2x_1 + x_2)\end{aligned}$$

a Using the indirect method, i.e. Theorem 4.7, show that the origin is asymptotically stable.

b Using the direct method, i.e. Theorem 4.1, show that the origin is asymptotically stable in the domain $D = \left\{x \in \mathbb{R}^2 \mid 1 + 2x_1 + x_2 \geq 0,\ 1 + x_1 + 2x_2 \geq 0\right\}$.

Hint: Use the Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

c Show that an estimate of the region of attraction can be found as $\Omega_{0.1} = \left\{x \in \mathbb{R}^2 \mid V(x) \leq 0.1\right\}$.

d Draw the domain D and the level sets $\Omega_{0.1}$ and $\Omega_{2.5}$ in the (x_1, x_2)-plane. You may use the script `pplane` and the feature "plot level curves" where you input the Lyapunov function you used in **b**.

Explain why the trajectories in $\Omega_{0.1}$ converge to the origin, while some of the trajectories in $\Omega_{2.5}$ do not. Relate your discussion to the domain D, and explain why we cannot use D as an estimate of the region of attraction.

Problem 7 (5%) Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2x_2 + x_2 \cos x_2\end{aligned}$$

Use the Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$, with $P = \left[\begin{smallmatrix}4 & 1 \\ 1 & 3\end{smallmatrix}\right]$, and Young's inequality:

$$xy \leq \epsilon x^2 + \frac{1}{4\epsilon}y^2, \qquad \epsilon > 0,$$

to prove global asymptotic stability of the origin.

Problem 8 (5%) Consider the system

$$\dot{x}_1 = -\sqrt{x_1^2 + x_2^2} - 11x_2$$
$$\dot{x}_2 = 2x_1 - 5x_2$$

Use the Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$, with $P = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 3 \end{smallmatrix}\right]$, and the Cauchy-Schwarz inequality:

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|,$$

to show that the origin is globally asymptotically stable.

Problem 9 (5%) Exercise 4.35 in Khalil

Let α be a class $\mathcal{K}$ function on $[0, a)$. Show that

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2), \quad \forall\, r_1, r_2 \in [0, a/2) .$$

Hint: See Definition 4.2 in Khalil.

Problem 10 (10%) Suppose that for each initial condition $x(0)$ the solution of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq \beta(\|x(0)\|, t)$$

for $t \geq 0$ where β is of class $\mathcal{KL}$. Show that the origin of the system is globally asymptotically stable, that is

- **a** Show stability for $x = 0$ using the definition of stability (Definition 4.1 in Khalil) and the definition of class $\mathcal{KL}$ functions (Definition 4.3)
- **b** Show that every trajectory of the system converges to the origin.

Optional exercise:

The *variable gradient method* presents a way to search for Lyapunov functions in a backwards manner. Consider a function $g(x) = \nabla V = (\partial V/\partial x)^\top$, where V is our undetermined Lyapunov function. The derivative of V along the trajectories of the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is then

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^\top(x) f(x).$$

We want to choose g such that $\dot{V}$ above is negative definite. At the same time, V must be a scalar, positive definite function. For V to be scalar, the function $g = [g_1, \ldots, g_n]^\top$ must satisfy the symmetry condition:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall i, j = 1, \ldots, n$$

Then, to find V, we integrate g along the solutions of the system:

$$V(x) = \int_0^x g(y)^\top dy = \int_0^{x_i} \sum_{i=1}^{n} g_i(y) dy_i$$

For further details on how to proceed, see Khalil pages 120-122.

Problem 11 Exercise 4.6 in Khalil

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(x_1 + x_2) - h(x_1 + x_2)$$

where h is continuously differentiable and $zh(z) > 0$ for all $z \in \mathbb{R}$. Using the variable gradient method, find a Lyapunov function that shows that the origin is globally asymptotically stable.