TTK4150 Nonlinear Control Systems

Fall 2024 - Assignment 4
Due date: Monday, October 28. at 23:59

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Problem 1 (5%) For each of the functions

$$
\begin{aligned}
V_1(x,t) &= x_1^2 + \left(1+e^t\right) x_2^2 \\
V_2(x,t) &= \frac{x_1^2 + x_2^2}{1+t} \\
V_3(x,t) &= \left(1+\cos^4 t\right)\left(x_1^2 + x_2^2\right)
\end{aligned}
$$

determine if V_i is positive definite and decrescent.

Problem 2 (5%) Consider the system

$$
\begin{aligned}
\dot{x}_1 &= -\phi(t)x_1 + a\phi(t)x_2 \\
\dot{x}_2 &= b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3
\end{aligned}
$$

where a, b and c are positive constants and $t \mapsto \phi(t)$ and $t \mapsto \psi(t)$ are nonnegative, continuous, bounded functions that satisfy

$$
\phi(t) \geq \phi_0 > 0, \quad \psi(t) \geq \psi_0 > 0, \quad \forall t \geq 0
$$

Show that the origin is globally uniformly asymptotically stable.
Hint: Use the Lyapunov function candidate $V(x) = \frac{1}{2}\left(bx_1^2 + ax_2^2\right)$.

Problem 3 (10%) Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - c(t)x_2
\end{aligned}
$$

where the function $t \mapsto c(t)$ is continuously differentiable and satisfies

$$
k_1 \leq c(t) \leq k_2, \quad \forall t \geq 0
$$

where k_i are constants and $k_1 > 0$. Using the Lyapunov function candidate $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$, show that the origin is globally uniformly stable and that every solution $t \mapsto x(t)$ satisfies $x_2(t) \to 0$ as $t \to \infty$.
Hint: See the slides from Lecture 6 on Barbalat's lemma and/or the LaSalle-Yoshizawa Theorem.

Problem 4 (10%) Exercise 4.45 in Khalil

Consider the system

$$
\begin{aligned}
\dot{x}_1 &= h(t)x_2 - g(t)x_1^3 \\
\dot{x}_2 &= -h(t)x_1 - g(t)x_2^3
\end{aligned}
$$

where $t \mapsto h(t)$ and $t \mapsto g(t)$ are bounded, continuously differentiable functions and $g(t) \geq k > 0$, for all $t \geq 0$.

- **a** Is the equilibrium point $x = 0$ uniformly asymptotically stable?
- **b** Is it exponentially stable?

  Hint 1: See Theorem 4.15.
  Hint 2: Read p. 157-158 in Khalil, and consider carefully which methods of stability analysis can and cannot be applied to the problem at hand.
- **c** Is it globally uniformly asymptotically stable?
- **d** Is it globally exponentially stable?

Problem 5 (10%) Exercise 4.54 in Khalil

For each of the following scalar systems, investigate input-to-state stability:

(1) $\dot{x} = -(1+u)x^3$

(2) $\dot{x} = -(1+u)x^3 - x^5$

(3) $\dot{x} = -x + x^2 u$

(4) $\dot{x} = x - x^3 + u$

Hint: If a system is ISS, then

- ***a*** *for $u = 0$, the origin is globally asymptotically stable.*
- ***b*** *for a bounded input $t \mapsto u(t)$, every solution $t \mapsto x(t)$ is bounded.*

If one of these is not satisfied, the system can ***not*** *be ISS.*

Problem 6 (15%) Exercise 4.55 in Khalil

For each of the following scalar systems, investigate input-to-state stability:

$$\dot{x}_1 = -x_1 + x_1^2 x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + u \tag{1}$$
$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + u \tag{2}$$
$$\dot{x}_1 = (x_1 - x_2 + u)(x_1^2 - 1), \qquad \dot{x}_2 = (x_1 + x_2 + u)(x_1^2 - 1) \tag{3}$$
$$\dot{x}_1 = -x_1 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2 + x_1 + u \tag{4}$$

Hint for (2): Read example 4.27 before doing this exercise.
Hint for (3): For $u(t) \equiv 0$ an ISS system needs to have a globally asymptotically stable origin. This requires the absence of other equilibria.

Problem 7 (5%) Consider the system

$$\dot{x}_1 = -x_1 + x_2^2$$
$$\dot{x}_2 = -x_2$$

Use a quadratic Lyapunov function candidate to show that the origin is globally asymptotically stable.
Hint: In order to prove this globally, notice that the system can viewed as a cascade.

Problem 8 (10%) Exercise 5.3 in Khalil

Consider a system defined by the memoryless function $y = u^{1/3}$.

- **a** Show that the system is $\mathcal{L}_\infty$ stable with zero bias.
- **b** For any positive constant a, show that the system is finite-gain $\mathcal{L}_\infty$ stable with $\gamma = a$ and $\beta = (1/a)^{1/2}$.
- **c** Compare the two statements.

Problem 9 (5%) Exercise 5.4 in Khalil

Consider a system defined by the memoryless function by $y = h(u)$ where $h : R^m \to R^q$ is globally Lipschitz. Investigate $\mathcal{L}_p$ stability for each $p \in [1, \infty]$ when

(1) $h(0) = 0$.

(2) $h(0) \neq 0$.

Problem 10 (10%) Exercise 5.20 in Khalil

Consider the feedback connection of Figure 1, where H_1 and H_2 are linear time-invariant systems represented by the transfer function $H_1(s) = (s-1)/(s+1)$ and $H_2(s) = 1/(s-1)$. Find the closed-loop transfer function from (u_1, u_2) to (y_1, y_2) and from (u_1, u_2) to (e_1, e_2). Consider these transfer functions and discuss why we have to consider both inputs (u_1, u_2) and both outputs (e_1, e_2) (or (y_1, y_2)) in studying the stability of the feedback connection.

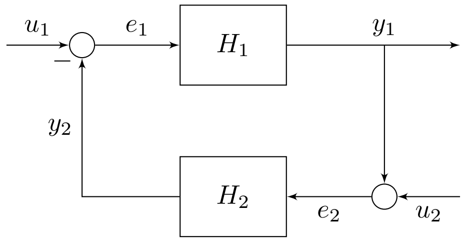

Figure 1: Feedback connection.

Problem 11 (5%) Exercise 6.2 in Khalil

Consider the system

$$
\begin{aligned}
a\dot{x} &= -x + \frac{1}{k}h(x) + u \\
y &= h(x)
\end{aligned}
$$

where a and k are positive constants and h belongs to the sector $[0, k]$. Show that the system is passive with $V(x) = a\int_0^x h(\sigma)\, d\sigma$ as the storage function.

Problem 12 (10%) Consider again the spring-mass-damper system from the previous assignments, but this time without damping, i.e. $d = 0$. The transformed system is now:

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 \\
m\dot{\tilde{x}}_2 &= -f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 + \tilde{u}
\end{aligned}
\tag{5}
$$

a Define the output

$$y = \tilde{x}_2$$

and use $V(\tilde{x}) = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$ as a storage function for the system. Outline a control law (find an input $\tilde{u}$) that makes the system passive from a new control input v to the output y (in Khalil this technique is described as feedback passivation). The new control input v should be introduced as a term of $\tilde{u}$.

b Is the system zero-state observable?

c Derive a controller (find an input v) that globally stabilizes the origin of the system (5).

Note: *Passivity-based control will be covered in the lectures later on, however, this problem serves as a gentle introduction which you should be able to solve already.*

Problem 13 (Optional) Exercise 6.1 in Khalil.