TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 2	Norwegian University of Science and Technology
Due date: Monday, September 23. at 23:59	

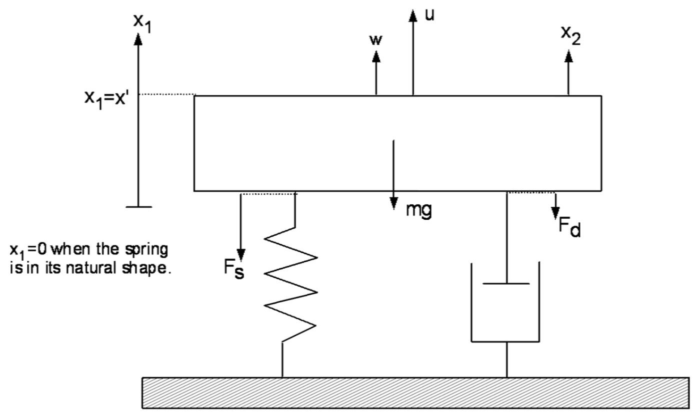

Figure 1: Sketch of the mass-damper-spring system (as indicated by x' the system position is measured at the top of the mass)

Problem 1 (20%) Consider the nonlinear mass-damper-spring system

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} \tag{2}$$

illustrated in Figure 1, where f_1, $f_3 > 0$ are spring constants, $d > 0$ is the damping coefficient, m is the mass of the box attached to the spring, and g is the gravitational constant. Moreover, u is an input force, x_1 represents the vertical position of the box, and x_2 is the velocity of the box.

a Is $(0, 0)$ an equilibrium point of the unforced system (i.e. when $u = 0$)?

Since it is desirable to control the position x_1 to a desired position x_{1d}, the equilibrium point of (1)–(2) should be placed at $(x_{1d}, 0)$.

b Find the constant input force that places the equilibrium point at $(x_{1d}, 0)$. That is, solve the system $\dot{x}|_{x=(x_{1d},0)} = 0$ for the input u.

Split the states and input into stationary values (x_0, u_0) and difference terms $(\tilde{x}, \tilde{u})$. Define the terms $\tilde{x} = x - x_0$ and $\tilde{u} = u - u_0$, where $x_0 = (x_{1d}, 0)$ and u_0 is the input found in **b**.

c Derive the dynamics of $\tilde{x}$ with $\tilde{u}$ as an input. What is the equilibrium point when $\tilde{u} = 0$?

d Calculate the Jacobian of this system (i.e. of the $\tilde{x}$ dynamics) evaluated in $\tilde{x} = 0$, and denote it A. Is A Hurwitz or not? What does this mean related to the stability of the equilibrium point?

Problem 2 (20%) For the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Comment also on the possibility of a global result.

a

$$\dot{x} = -x^5$$

b

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2^2 \\ \dot{x}_2 &= 2x_1x_2 - x_2^3\end{aligned}$$

c

$$\begin{aligned} \dot{x}_1 &= -x_1 + 4x_2^2 \\ \dot{x}_2 &= -x_2^3 \end{aligned}$$

d

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2\left(1 - x_1^2\right) \\ \dot{x}_2 &= -\left(x_1 + x_2\right)\left(1 - x_1^2\right) \end{aligned}$$

Hint: Look at the equilibrium point(s).

Problem 3 (10%)

a Consider

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2^2 \\ \dot{x}_2 &= 2x_1x_2 \end{aligned}$$

Construct the phase portrait of the system (you may use the Matlab script `pplane` as in Assignment 1). Is the origin stable? Provide your argument with respect to Definition 4.1. on page 112 of Khalil (qualitative argument is enough).

b Use Definition 4.1. to show that the origin of the following system

$$\dot{x} = \alpha x, \quad \alpha < 0$$

is asymptotically stable. (Note: in addition to convergence you also have to show quantitatively that for any given ε you could obtain a δ which depends on ε).

Problem 4 (15%) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1^2 - 2x_1x_2 + 2x_1 - x_1^2x_2 \\ \dot{x}_2 &= x_1^3 + 2x_1^2 + x_1^2x_2 + 2x_1x_2 \end{aligned}$$

a Show that the system can be written as

$$\begin{aligned} \dot{x}_1 &= \left(x_1^2 + 2x_1\right)\left(1 - x_2\right) \\ \dot{x}_2 &= \left(x_1^2 + 2x_1\right)\left(x_1 + x_2\right) \end{aligned}$$

b Find the equilibrium point(s) of the system.

c Do a change of variables

$$\begin{aligned} z_1 = x_1 - x_1^* \\ z_2 = x_2 - x_2^* \end{aligned}$$

to shift the equilibrium point $(x_1^*, x_2^*) = (-1, 1)$ to the origin. Show that the new system can be written as

$$\dot{z} = \begin{bmatrix} z_2 \\ -\left(z_1 + z_2\right) \end{bmatrix} \left(1 - z_1^2\right) \tag{3}$$

d By using a quadratic Lyapunov function candidate, show that the equilibrium point is asymptotically stable.

Hint: One way to show this is to write $\dot{V}$ on the form $\dot{V}(z) = -(1 - z_1^2)z^\top Qz$, where Q is a positive definite, symmetric matrix.

Problem 5 (5%) Consider the following pendulum system with friction as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10\sin x_1 - 2x_2\end{aligned}$$

Use the general Lyapunov function

$$V(x) = \frac{1}{2}x^T P x + \gamma(1 - \cos x_1)$$

where $\gamma > 0$ and

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \qquad P = P^T > 0$$

to examine the stability characteristic of the system. In particular, prove that the origin is locally asymptotically stable by an appropriate selection of matrix P and constant γ.
Hint: $z\sin z > 0$ *for* $-\pi < z < \pi$

Problem 6 (5%) Consider again the system

$$\begin{aligned}\dot{x}_1 &= x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right) \\ \dot{x}_2 &= -x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right)\end{aligned}$$

from Assignment 1, where $\alpha, \beta > 0$ are constants. Using Chetaev's theorem (i.e. Theorem 4.3 in Khalil), show that the origin is unstable.
Hint: $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$.

Problem 7 (7.5%) Consider the system

$$\begin{aligned}\dot{x}_1 &= \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 &= \frac{-2(x_1+x_2)}{(1+x_1^2)^2}\end{aligned}$$

and the Lyapunov function candidate $V(x) = \frac{1}{2}\left(\frac{x_1^2}{1+x_1^2} + x_2^2\right)$.

- **a** Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in \mathbb{R}^2 \setminus \{0\}$
- **b** Show that V is not radially unbounded.
- **c** What stability properties can you conclude about the origin of this system using the given V as the Lyapunov function? Explain.

Problem 8 (7.5%) Exercise 4.2 in Khalil

Consider the scalar system

$$\dot{x} = ax^p + g(x)$$

where p is a positive integer and g satisfies $|g(x)| \le k|x|^{p+1}$ in some neighbourhood of the origin. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

Problem 9 (10%) Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2 - x_2x_3 \\ \dot{x}_2 &= -x_2 \\ \dot{x}_3 &= -x_3 + x_1x_2 + x_2^2\end{aligned}$$

Verify that the origin is the only equilibrium point of the system. Using a quadratic Lyapunov function, show that it is globally exponentially stable.

Hint: Use $V(x) = \frac{1}{2}x^T P x$ *with* P *on the form* $P = \begin{bmatrix} p_{11} & p_{12} & 0 \\ p_{12} & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}$.

The two remaining exercises are optional:

In the lectures you will learn some methods for choosing Lyapunov function candidates. The following exercises introduce yet another approach, Krasovskii's method. Consider the system

$$\dot{x} = f(x), \quad f(0) = 0$$

where f is continuously differentiable. Denote the Jacobian as $A(x) \triangleq [\partial f/\partial x]$.

The generalized Krasovskii's theorem states that a sufficient condition for the origin to be asymptotically stable is that the matrix $F(x) = A^T(x)P + PA(x)$ is negative definite in some neighbourhood D of the origin, where P is a positive definite, symmetric matrix. In addition, if $D = \mathbb{R}^n$ and $V(x) \triangleq f^T(x)Pf(x)$ is radially unbounded, then the system is globally asymptotically stable.

Problem 10 Apply Krasovskii's theorem to analyze the stability behaviour of the following system:

$$\begin{aligned}\dot{x}_1 &= -4x_1 + 3x_2 \\ \dot{x}_2 &= x_1 - 2x_2 - x_2^3 \ .\end{aligned}$$

Problem 11 Exercise 4.10 in Khalil

Appendix

The following properties are equivalent to $P = P^T$ being positive definite (denoted by $P > 0$):

- All eigenvalues of P are greater than zero.
- All leading principal minors of P are greater than zero.

Some useful properties if $P = P^T > 0$:

- $\frac{d}{dt}(x^T P x) = 2\dot{x}^T P x$
- $\lambda_{\min}\|x\|^2 \leq x^T P x \leq \lambda_{\max}\|x\|^2$, with $\lambda_{\min}, \lambda_{\max}$ the smallest and largest eigenvalue of P.

Definition: Leading principal minors

Given an $n \times n$ matrix A, a leading principal submatrix of A is a submatrix formed by ignoring all but the first n rows and columns. A leading principal minor is the determinant of a leading principal submatrix. Thus, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then the leading principal minors are

$$a_{11}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

where the operator $|X|$ denotes the determinant of the submatrix X.