TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 1 (Solution)	Norwegian University of Science and Technology

Problem 1

a

$$
\begin{aligned}
\|x_1\|_1 &= |6| + |-10| = 16 \\
\|x_1\|_2 &= \sqrt{6^2 + (-10)^2} = \sqrt{136} = 2\sqrt{34} \\
\|x_1\|_\infty &= \max(|6|, |-10|) = 10 \\
\|x_2\|_1 &= |-5| + |-4| + |3| = 12 \\
\|x_2\|_2 &= \sqrt{(-5)^2 + (-4)^2 + 3^2} = \sqrt{50} = 5\sqrt{2} \\
\|x_2\|_\infty &= \max(|-5|, |-4|, |3|) = 5
\end{aligned}
$$

b As shown in Figure 1, the 2-norm represents the shortest distance from the origin to the point $x_1 \in \mathbb{R}^2$, while the 1-norm represents the distance if one was to travel in a rectangular grid (known as the Manhattan distance). The ∞-norm represents the vector element with the largest absolute value. In all cases, the norm represents a certain property (that is, size) of that vector. The same analogy applies in the 3D case (when considering the vector x_2).

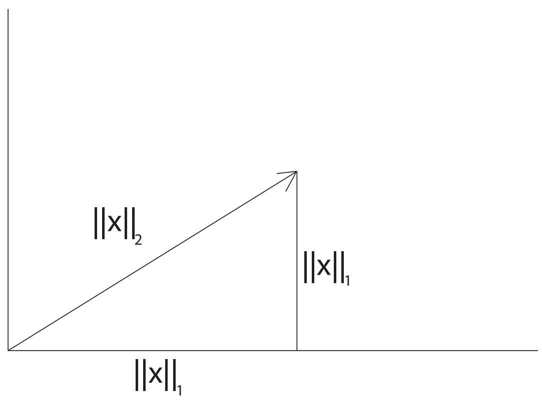

Figure 1: Illustration of 2-norm vs 1-norm in the 2D case.

c With

$$
\begin{aligned}
&\|y\|_2 = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2} \\
\implies &\|y\|_2^2 = y_1^2 + y_2^2 + \cdots + y_n^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\|y\|_1 = |y_1| + |y_2| + \cdots + |y_n| \\
\implies &\|y\|_1^2 = (|y_1| + |y_2| + \cdots + |y_n|)^2 \\
&\qquad = y_1^2 + y_2^2 + \cdots + y_n^2 + 2|y_1||y_2| + \cdots + 2|y_{n-1}||y_n|
\end{aligned}
$$

we see that

$$\|y\|_1^2 = \|y\|_2^2 + 2|y_1||y_2| + \cdots + 2|y_{n-1}||y_n| \geq \|y\|_2^2$$

Hence,

$$\|y\|_1 \geq \|y\|_2$$

d

$$\begin{aligned}
\|A\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}| = \max(4,5,1) = 5\\
\|A\|_\infty &= \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}| = \max(4,3,3) = 4\\
\|A\|_2 &= \left[\lambda_{\max}\left(A^TA\right)\right]^{\frac{1}{2}} = \sqrt{14}
\end{aligned}$$

e From the above, we have

$$\begin{aligned}
\|A\|_2 &= \sqrt{14}, \quad \|x\|_2 = \sqrt{14} \quad \Rightarrow \|A\|_2\,\|x\|_2 = 14\\
Ax &= \begin{bmatrix}-3\\3\\3\end{bmatrix}\\
\|Ax\|_2 &= 3\sqrt{3} \;\le\; 14 = \|A\|_2\,\|x\|_2
\end{aligned}$$

Next, we have

$$\|x\|_\infty = 3, \quad \|x\|_1 = 6$$

Choosing $c_1 \le \frac{1}{2}$ and $c_2 \ge \frac{1}{2}$ ensures that

$$c_1\,\|x\|_1 \le \|x\|_\infty \le c_2\,\|x\|_1$$

f

$$\begin{aligned}
\|f_1\|_{\mathscr{L}_\infty} &= \sup_{0\le t\le\infty}\left|\sqrt{t}\right| \to \infty \Rightarrow f_1 \notin \mathscr{L}_\infty\\
\|f_1\|_{\mathscr{L}_2} &= \left(\int_0^\infty \left|\sqrt{t}\right|^2 dt\right)^{\frac{1}{2}} \to \infty \Rightarrow f_1 \notin \mathscr{L}_2\\
\|f_1\|_{\mathscr{L}_1} &= \left(\int_0^\infty \left|\sqrt{t}\right| \mathrm{dt}\right) \to \infty \Rightarrow f_1 \notin \mathscr{L}_1
\end{aligned}$$

$$\begin{aligned}
\|f_2\|_{\mathscr{L}_\infty} &= \sup_{0\le t\le\infty}\left|\frac{1}{(t+1)^2}\right| = 1 \Rightarrow f_2 \in \mathscr{L}_\infty\\
\|f_2\|_{\mathscr{L}_2} &= \left(\int_0^\infty \left|\frac{1}{(t+1)^2}\right|^2 dt\right)^{\frac{1}{2}} = \left(\frac{1}{3} - \lim_{t\to\infty}\frac{1}{3(t+1)^3}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \Rightarrow f_2 \in \mathscr{L}_2\\
\|f_2\|_{\mathscr{L}_1} &= \left(\int_0^\infty \left|\frac{1}{(t+1)^2}\right| dt\right) = 1 - \lim_{t\to\infty}\frac{1}{t+1} = 1 \Rightarrow f_2 \in \mathscr{L}_1
\end{aligned}$$

Problem 2

a If f is globally Lipschitz, the following inequality should be true for any x and y:

$$\frac{|f(x)-f(y)|}{|x-y|} \le L$$

With $f(x) = x^{\frac{1}{3}}$, we have

$$\frac{|f(x)-f(y)|}{|x-y|} = \frac{\left|x^{\frac{1}{3}} - y^{\frac{1}{3}}\right|}{|x-y|}$$

If we for instance set $y = 0$ and let $x \to y$, i.e. $x \to 0^+$, we will see that

$$\lim_{y=0,x\to 0}\frac{\left|x^{\frac{1}{3}}\right|}{|x|} = \lim_{y=0,x\to 0}\left|x^{-\frac{2}{3}}\right| \to \infty \not\le L$$

Hence, f is not globally Lipschitz.

b We see that $f'(x) = x^{-\frac{2}{3}}$, which is continuous on the whole state space except at the origin $x = 0$. Hence, by Lemma 3.1, f is locally Lipschitz for the area $D = \{x \in \mathbb{R} \,|\, x \neq 0\}$.

c (a) The pendulum equation with friction is given by

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \end{bmatrix} \tag{1}$$

The partial derivative of f with respect to x is found as

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -\frac{k}{m} \end{bmatrix} \tag{2}$$

From (1) and (2) it can be seen that f and $\frac{\partial f}{\partial x}$ are continuous on $\mathbb{R}^2$. Using Lemma 3.1 or Lemma 3.2 it can be concluded that f is locally Lipschitz on the whole state space. Further it can be seen that $\frac{\partial f}{\partial x}$ is bounded on $\mathbb{R}^2$ ($\cos(x_1) \leq 1 \; \forall \; x_1$), and hence by Lemma 3.3, f is globally Lipschitz.
By Theorem 3.2 in Khalil, we can then conclude that the system $\dot{x} = f(x)$ has a unique solution for any starting point $x_0 \in \mathbb{R}^2$ (and in any time interval, since the system dynamics are time-invariant).

(b) The mass-spring equation with linear spring, linear viscous damping, Coulomb friction and zero external force (Section 1.2.3) is given by

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}\eta(x_1, x_2) \end{bmatrix}$$

where η is discontinuous at $x_2 = 0$ (see p. 11 in Khalil). This discontinuity implies that f is not locally Lipschitz at $x_2 = 0$ (any discontinuous function is not locally Lipschitz at the point of discontinuity). Thus f is also not globally Lipschitz, and we cannot use Theorem 3.2 to conclude global existence of solutions. However away from $x_2 = 0$ we have

$$f(x) = \begin{cases} \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 - \mu_k g \end{bmatrix}, & \text{for } x_2 > 0 \\ \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \mu_k g \end{bmatrix}, & \text{for } x_2 < 0 \end{cases}$$

Since f is linear in both of the above cases, we can easily see that it is also locally Lipschitz in these areas. Hence by Theorem 3.1 we can conclude local existence of solutions: for initial values x_0 such that f is locally Lipschitz in a neighbourhood around x_0, the system will have a unique solution over some interval of time $[t_0, t_0 + \delta]$.

(c) The Van der Pol oscillator is given by

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + \varepsilon\left(1 - x_1^2\right)x_2 \end{bmatrix} \tag{3}$$

The partial derivative of f with respect to x is found as

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & \varepsilon\left(1 - x_1^2\right) \end{bmatrix} \tag{4}$$

From (3) and (4) it can be seen that f and $\frac{\partial f}{\partial x}$ are continuous on $\mathbb{R}^2$. Using Lemma 3.1 or Lemma 3.2 it can be concluded that f is locally Lipschitz on the whole state space. Further it can be seen that $\frac{\partial f}{\partial x}$ is not globally bounded. It follows from Lemma 3.3 that f is not globally Lipschitz.
We can therefore only conclude local existence and uniqueness of solutions by Theorem 3.1.

d With $f(x) = Ax$ we have

$$\begin{aligned} \frac{\|f(x) - f(y)\|}{\|x - y\|} &= \frac{\|A(x - y)\|}{\|x - y\|} \\ &\leq \|A\| \frac{\|x - y\|}{\|x - y\|} \\ &= \|A\| \end{aligned}$$

Thus, the system is globally Lispchitz, and the Lipschitz constant depends on the choice of norm. In general, the choice of norm will not affect the Lipschitz property, it only affects the Lipschitz constant as demonstrated here.

Problem 3

a $\dot{x} = \alpha x \quad \rightarrow \quad x(t) = C_1 e^{\alpha t}$, where C_1 is a constant. The rabbit population will increase exponentially.

b $\dot{y} = -\gamma y \quad \rightarrow \quad y(t) = -C_2 e^{\gamma t}$, where C_2 is a constant. The fox population will decrease exponentially.

c Set

$$\begin{aligned} x(\alpha - \beta y) &= 0 \\ -y(\gamma - \delta x) &= 0 \end{aligned}$$

This system has two isolated equilibriums: at $(0, 0)$ (which means an extinction of both species) and at $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$. The last solution represents a fixed point at which both populations sustain their current, non-zero numbers, and, in the simplified model, do so indefinitely. The levels of population at which this equilibrium is achieved depend on the chosen values of the parameters.
We may calculate the Jacobian:

$$\begin{aligned} A &= \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix} \end{aligned}$$

In the equilibrium points:

$$\begin{aligned} A|_{(x,y)=(0,0)} &= \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix} \\ \left| \lambda I - A|_{(x,y)=(0,0)} \right| &= (\lambda - \alpha)(\lambda + \gamma) = 0 \\ &\Rightarrow \quad \lambda_1 = \alpha, \lambda_2 = -\gamma \\ A|_{(x,y)=(\frac{\gamma}{\delta},\frac{\alpha}{\beta})} &= \begin{bmatrix} \alpha - \beta\frac{\alpha}{\beta} & -\beta\frac{\gamma}{\delta} \\ \delta\frac{\alpha}{\beta} & \delta\frac{\gamma}{\delta} - \gamma \end{bmatrix} = \begin{bmatrix} 0 & -\beta\frac{\gamma}{\delta} \\ \delta\frac{\alpha}{\beta} & 0 \end{bmatrix} \\ \left| \lambda I - A|_{(x,y)=(\frac{\gamma}{\delta},\frac{\alpha}{\beta})} \right| &= \lambda^2 + \delta\frac{\alpha}{\beta}\beta\frac{\gamma}{\delta} = \lambda^2 + \gamma\alpha = 0 \\ &\Rightarrow \quad \lambda_{1,2} = \pm i\sqrt{\gamma\alpha} \end{aligned}$$

From the eigenvalues, we see that $(0, 0)$ is a saddle point and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$ is a center point.

d A linear system can not have several isolated equilibrium points, because for linear systems of the form $\dot{x} = Ax + b$ there are only the following options:

(a) if $|A| \neq 0$, then there is one unique isolated equilibrium point, which is the solution of the equation $Ax + b = 0$

(b) if $|A| = 0$, then the equation $Ax + b = 0$ may have either no solutions (in this case there are no equilibrium points), or it may have a whole subspace of solutions (i.e. the equilibrium points are not isolated).

e See the phase portrait in Figure 2. A center point is lying in $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}) \approx (76.2, 12.6)$, and a saddle point is lying in the origin, both as expected.

f No, it is not possible to cross the lines $x = 0$ and $y = 0$, because they are invariant sets. If the states end up in these sets they will stay there forever, as we discussed in exercise (a)–(b).

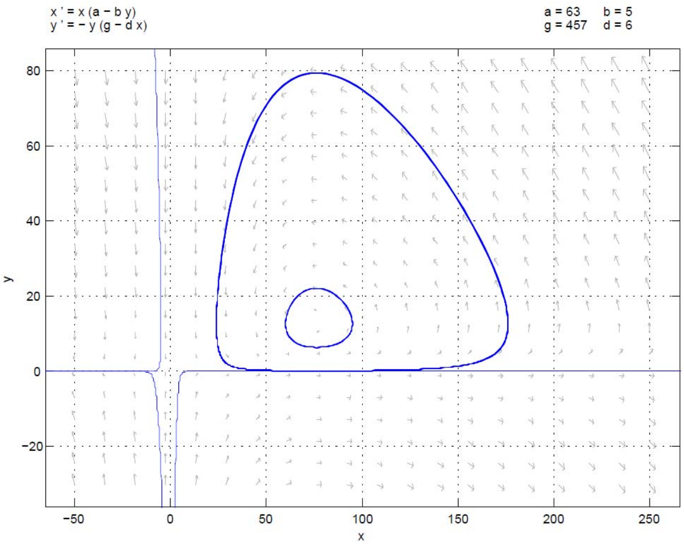

Figure 2: Phase portrait.

g Bendixson's criterion states that the system has no periodic orbits lying entirely in a simply connected region D if $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$ is not identically zero and does not change sign in D. It follows that such sets are given by

$$\begin{aligned} D_1 &= \left\{\mathbb{R}^2 | \alpha - \beta y - (\gamma - \delta x) > 0\right\} \\ D_2 &= \left\{\mathbb{R}^2 | \alpha - \beta y - (\gamma - \delta x) < 0\right\} \end{aligned}$$

Problem 4 It is seen that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -1 - 3x_2^2 - 5x_2^4 < 0, \quad \forall x \in \mathbb{R}^2$$

Since $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign for any $x \in \mathbb{R}^2$, there are no period orbits on $\mathbb{R}^2$ by Lemma 2.2.

Problem 5 We first observe that

$$\begin{aligned} & x_2^2 + x_1^2 = r^2(\cos^2\theta + \sin^2\theta) \implies r = \pm\sqrt{x_2^2 + x_1^2} \\ & \frac{x_2}{x_1} = \frac{r\sin\theta}{r\cos\theta} \implies \theta = \tan^{-1}\frac{x_2}{x_1} \end{aligned}$$

Alternative 1: Using the above relations, we have

$$\begin{aligned} \dot{r} &= \pm\frac{1}{2}\frac{2x_1\dot{x}_1 + 2x_2\dot{x}_2}{\sqrt{x_1^2 + x_2^2}} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} \\ &= \frac{x_1\left[x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right)\right] + x_2\left[-x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right)\right]}{r} \\ &= \frac{\alpha\left(x_1^2 + x_2^2\right)\left(\beta^2 - x_1^2 - x_2^2\right)}{r} = \frac{\alpha r^2\left(\beta^2 - r^2\right)}{r} \\ &= \alpha r\left(\beta^2 - r^2\right) \end{aligned}$$

$$\begin{aligned} \dot{\theta} &= \frac{1}{1 + \left(\frac{x_2}{x_1}\right)^2}\left[\frac{\dot{x}_2 x_1 - x_2\dot{x}_1}{x_1^2}\right] = \frac{1}{1 + \left(\frac{x_2}{x_1}\right)^2}\left[\frac{\dot{x}_2 x_1 - x_2\dot{x}_1}{x_1^2}\right] \\ &= \frac{x_1^2}{x_1^2 + x_2^2}\left[\frac{\left[-x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right)\right]x_1 - x_2\left[x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right)\right]}{x_1^2}\right] \\ &= \frac{-x_1^2 - x_2^2}{x_1^2 + x_2^2} = -1 \end{aligned}$$

Thus the transformed system is

$$\begin{cases} \dot{r} = \alpha r\left(\beta^2 - r^2\right) \\ \dot{\theta} = -1 \end{cases}$$

(Extra note:) By setting $r = \beta$ we have $\dot{r} = 0$ which implies that whenever $r\left(t_1\right) = \beta$ it follows that $r\left(t\right) = \beta$ for all $t > t_1$. Thus $x_1^2 + x_2^2 = \beta^2$ is a periodic orbit which only depends on β and is independent of α.

Alternative 2: The new system equations can be found using the expression given in the assignment. Let

$$z = \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ \tan^{-1}\left(x_2/x_1\right) \end{bmatrix} = \psi(x)$$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} = \psi^{-1}(z)$$

The partial derivative of ψ is given as

$$\frac{\partial \psi}{\partial x}(x) = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{bmatrix}$$

which leads to

$$\frac{\partial \psi}{\partial x}\left(\psi^{-1}(z)\right) = \begin{bmatrix} \frac{r\cos(\theta)}{\sqrt{r^2\cos^2(\theta)+r^2\sin^2(\theta)}} & \frac{r\sin(\theta)}{\sqrt{r^2\cos^2(\theta)+r^2\sin^2(\theta)}} \\ \frac{-r\sin(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)} & \frac{r\cos(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)} \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

and

$$\begin{aligned} \dot{z} &= \frac{\partial \psi}{\partial x}\left(\psi^{-1}(z)\right) f\left(\psi^{-1}(z)\right) \\ &= \frac{1}{r}\begin{bmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} r\sin(\theta) + \alpha r\cos(\theta)\left(\beta^2 - r^2\right) \\ -r\cos(\theta) + \alpha r\sin(\theta)\left(\beta^2 - r^2\right) \end{bmatrix} = \begin{bmatrix} \alpha r\left(\beta^2 - r^2\right) \\ -1 \end{bmatrix} \end{aligned}$$

Alternative 3: It is possible to use that

$$\dot{x}_1 = \dot{r}\cos\theta - \dot{\theta}r\sin\theta \implies \dot{r} = \frac{\dot{x}_1 + \dot{\theta}r\sin\theta}{\cos\theta}$$
$$\dot{x}_2 = \dot{r}\sin\theta + \dot{\theta}r\cos\theta \implies \dot{r} = \frac{\dot{x}_2 - \dot{\theta}r\cos\theta}{\sin\theta}$$

must be equal. We have $\dot{x}_1$ and $\dot{x}_2$ in the new coordinates as

$$\begin{aligned} \dot{x}_1 &= x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right) \\ &= r\sin\theta + \alpha r\cos\theta\left(\beta^2 - r^2\right) \end{aligned}$$

$$\begin{aligned} \dot{x}_2 &= -x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right) \\ &= -r\cos\theta + \alpha r\sin\theta\left(\beta^2 - r^2\right) \end{aligned}$$

To find $\dot{\theta}$, we can then solve

$$\begin{aligned} \dot{r} = \frac{\dot{x}_1 + \dot{\theta}r\sin\theta}{\cos\theta} &= \frac{\dot{x}_2 - \dot{\theta}r\cos\theta}{\sin\theta} \\ \frac{r\sin\theta + \alpha r\cos\theta(\beta^2 - r^2) + \dot{\theta}r\sin\theta}{\cos\theta} &= \frac{-r\cos\theta + \alpha r\sin\theta(\beta^2 - r^2) - \dot{\theta}r\cos\theta}{\sin\theta} \\ r\tan\theta\left(1 + \dot{\theta}\right) + \alpha r\left(\beta^2 - r^2\right) &= -\frac{r}{\tan\theta}\left(1 + \dot{\theta}\right) + \alpha r\left(\beta^2 - r^2\right) \end{aligned}$$

which reduces to

$$\left(\tan^2\theta + 1\right)\left(1 + \dot{\theta}\right) = 0$$

Since $\tan^2\theta + 1 > 0$, the only solution is thus $\dot{\theta} = -1$, which yields $\dot{r} = \alpha r\left(\beta^2 - r^2\right)$.

Problem 6

a

$$\begin{aligned} f_1(x_1,x_2) &= -ax_1 - x_2^2 \\ f_2(x_1,x_2) &= bx_1x_2 + cx_2 \\ A &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} -a & -2x_2 \\ bx_2 & bx_1 + c \end{bmatrix} \end{aligned}$$

b

$$\begin{aligned} -ax_1 - x_2^2 &= 0 \\ bx_1x_2 + cx_2 &= 0 \end{aligned}$$

The second equation gives equilibrium points for $x_2 = 0$ or $x_1 = -\frac{c}{b}$. Insertion of $x_2 = 0$ into the first equation gives the point $(0,0)$ and insertion of $x_1 = -\frac{c}{b}$ gives $x_2 = \pm\sqrt{\frac{ac}{b}}$. In total we have three equilibrium points $(0,0), (-\frac{c}{b}, \sqrt{\frac{ac}{b}})$ and $(-\frac{c}{b}, -\sqrt{\frac{ac}{b}})$. To find the type of point, we calculate the Jacobian in each of these points.

$$\begin{aligned} A|_{x=(0,0)} &= \begin{bmatrix} -a & 0 \\ 0 & c \end{bmatrix} \\ A|_{x=(-\frac{c}{b}, \pm\sqrt{\frac{ac}{b}})} &= \begin{bmatrix} -a & -2\sqrt{\frac{ac}{b}} \\ \sqrt{abc} & 0 \end{bmatrix} \end{aligned}$$

For the point $(0,0)$ the eigenvalues are $\lambda_1 = -a < 0$ and $\lambda_2 = c > 0$, thus it is a saddle point. For the points $(-\frac{c}{b}, \pm\sqrt{\frac{ac}{b}})$ the eigenvalues are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 8ac}}{2}$$

thus $(-\frac{c}{b}, \pm\sqrt{\frac{ac}{b}})$ will be stable nodes if $8c < a$ and stable foci if $8c > a$.

c See phase portraits in Figures 3 and 4. We see that $(0,0)$ is a saddle point (as expected) and that $(-\frac{c}{b}, \pm\sqrt{\frac{ac}{b}})$ are stable nodes or foci depending on the values a and c.

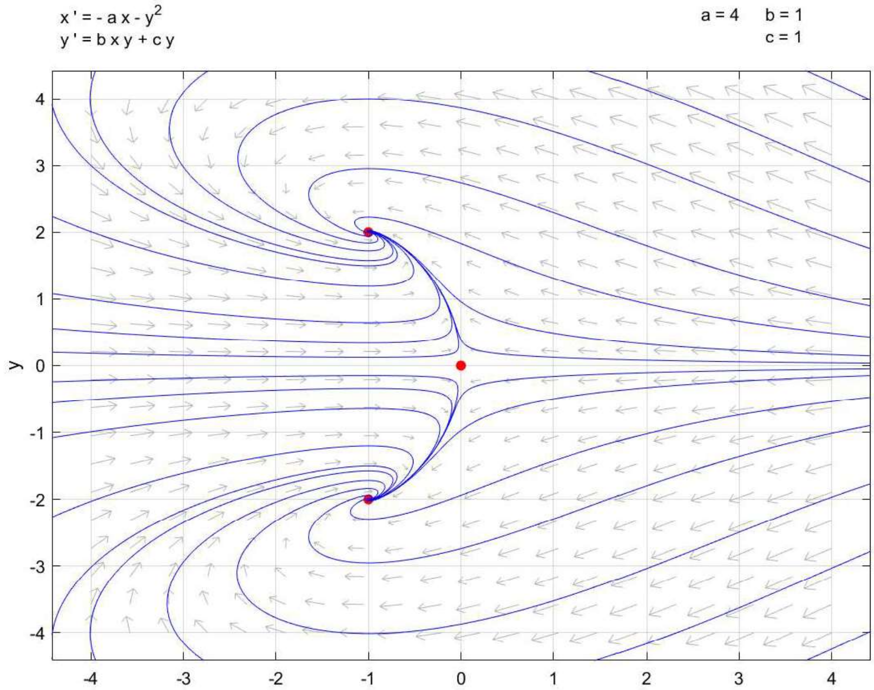

Figure 3: Phase portrait for $a = 4, \quad b = 1, \quad c = 1.$

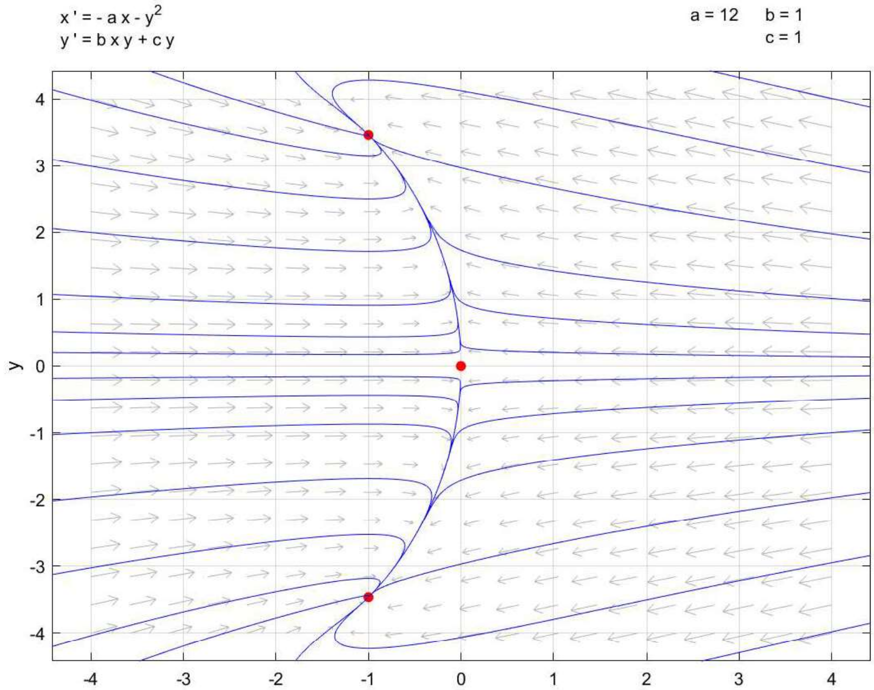

Figure 4: Phase portrait for $a = 12, \quad b = 1, \quad c = 1.$