TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 5 (Solution)	Norwegian University of Science and Technology

Problem 1 The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h(x_1) - ax_2 + u\\ y &= kx_2 + u\end{aligned}$$

where $a,\ k,\ \alpha_1 > 0$. The sector condition $h \in [\alpha_1, \infty]$ gives $zh(z) \geq \alpha_1 z^2$. A storage function is given by

$$V(x) = k\int_0^{x_1} h(z)\,dz + \frac{1}{2}x^T P x$$

The time-derivative of V along the trajectories of the system is found as

$$\begin{aligned}\dot{V}(x) &= k\frac{\partial}{\partial x_1}\left(\int_0^{x_1} h(z)\,dz\right)\dot{x}_1 + \dot{x}^T P x\\ &= kh(x_1)\dot{x}_1 + \begin{bmatrix}\dot{x}_1 & \dot{x}_2\end{bmatrix}\begin{bmatrix}p_{11}x_1 + p_{12}x_2\\ p_{12}x_1 + p_{22}x_2\end{bmatrix}\\ &= h(x_1)x_2(k - p_{22}) - p_{12}h(x_1)x_1 + x_1x_2(p_{11} - ap_{12}) + x_2^2(p_{12} - ap_{22}) + u(p_{12}x_1 + p_{22}x_2)\end{aligned}$$

By choosing $p_{22} = k$, $p_{11} = ap_{12}$, and $ak > p_{12} > 0$, then $\dot{V}$ is positive definite and

$$\dot{V}(x) \leq -p_{12}h(x_1)x_1 - \beta_1 x_2^2 + u(p_{12}x_1 + kx_2)$$

where $\beta_1 = ak - p_{12} > 0$. Inserting $x_2 = \frac{1}{k}(y-u)$ into this expression yields

$$yu \geq \dot{V}(x) + p_{12}h(x_1)x_1 + \beta_1 x_2^2 - ux_1p_{12} + u^2$$

Using the sector condition, we have

$$\begin{aligned}yu &\geq \dot{V}(x) + p_{12}\alpha_1 x_1^2 + \beta_1 x_2^2 - ux_1p_{12} + u^2\\ &= \dot{V}(x) + \beta_1 x_2^2 + \frac{1}{2}(u - p_{12}x_1)^2 + u^2\left(1 - \frac{1}{2}\right) + p_{12}x_1^2\left(\alpha_1 - \frac{1}{2}p_{12}\right)\end{aligned}$$

Thus, choosing $\min(2\alpha_1, ak) > p_{12} > 0$, we finally have

$$yu \geq \dot{V}(x) + \psi(x),$$

where

$$\psi(x) = \beta_1 x_2^2 + \beta_2 x_1^2,$$

and $\beta_2 = \alpha_1 - \frac{1}{2}p_{12} > 0$. Since $\psi(x)$ is positive definite, by Definition 6.3 the system is strictly passive.

Problem 2 The system is given by

$$\begin{aligned}J_1\dot{\omega}_1 &= (J_2 - J_3)\,\omega_2\omega_3 + u_1\\ J_2\dot{\omega}_2 &= (J_3 - J_1)\,\omega_3\omega_1 + u_2\\ J_3\dot{\omega}_3 &= (J_1 - J_2)\,\omega_1\omega_2 + u_3\end{aligned}$$

Define $u = [u_1\ u_2\ u_3]^T$ and $\omega = [\omega_1\ \omega_2\ \omega_3]^T$.

a Let $V(\omega) = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$ be a candidate for a storage function. The time-derivative along the trajectories of the system is found as

$$\begin{aligned}\dot{V}(\omega) &= J_1\dot{\omega}_1\omega_1 + J_2\dot{\omega}_2\omega_2 + J_3\dot{\omega}_3\omega_3\\ &= ((J_2 - J_3)\,\omega_2\omega_3 + u_1)\,\omega_1 + ((J_3 - J_1)\,\omega_3\omega_1 + u_2)\,\omega_2\\ &\quad + ((J_1 - J_2)\,\omega_1\omega_2 + u_3)\,\omega_3\\ &= (J_2 - J_2 - J_3 + J_3 - J_1 + J_1)\,\omega_1\omega_2\omega_3 + u_1\omega_1 + u_2\omega_2 + u_3\omega_3\\ &= u^T\omega\end{aligned}$$

which shows that the map from u to ω is lossless with the storage function $V(\omega)$.

b With $u = -K\omega + v$, where $K = K^T > 0$, we have that

$$\begin{aligned}\dot{V}(\omega) &= u^T\omega \\ &= (-K\omega + v)^T\omega \\ &= -\omega^T K^T \omega + v^T\omega \\ &= v^T\omega - \omega^T K\omega \\ &\leq v^T\omega - \lambda_{\min}(K)\,\omega^T\omega \\ \implies v^T\omega &\geq \dot{V}(\omega) + \lambda_{\min}(K)\,\omega^T\omega\end{aligned}$$

From the last equation it can be seen that the system is output strictly passive from v to ω with $v^T\omega \geq \dot{V}(\omega) + \lambda_{\min}(K)\,\omega^T\omega$. Hence, the map from v to ω is finite gain $\mathscr{L}_2$-stable with $\mathscr{L}_2$-gain less than or equal to $\frac{1}{\lambda_{\min}(K)}$ by Lemma 6.5.

c With $u = -K\omega$, we have that

$$\dot{V}(\omega) \leq -\lambda_{\min}(K)\,\omega^T\omega$$

Since $V(\omega)$ is positive definite and radially unbounded and $\dot{V}(\omega)$ is negative definite, by Theorem 4.2 we conclude that the system is globally asymptotically stable.

Alternatively, we can notice that the system is state strictly passive with a radially unbounded storage function, and hence by Lemma 6.7 in Khalil the origin is GAS with $v = 0$.

Problem 3 The system is given by

$$h(s) = \frac{\gamma s}{(\alpha + s)(\beta + s)}$$

with constants $\gamma, \alpha, \beta > 0$.

a We have

$$\begin{aligned}h(j\omega) &= \frac{\gamma j\omega}{(\alpha + j\omega)(\beta + j\omega)} \\ &= \frac{\gamma j\omega(\alpha - j\omega)(\beta - j\omega)}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)} \\ &= \frac{\gamma j\omega(\alpha\beta - j\omega(\beta + \alpha) - \omega^2)}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)} \\ &= \frac{\gamma(\alpha + \beta)\omega^2 + \gamma\omega(\alpha\beta - \omega^2)j}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)}\end{aligned}$$

Since $\Re[h(j\omega)] = \dfrac{\gamma(\alpha + \beta)\omega^2}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)} \geq 0$ for all $\omega \in \mathbb{R}$, the system is passive.

b The system is not input strictly passive since for $\omega = 0$, $\Re[h(j\omega)] = 0$, implying that there exists no $\delta > 0$ such that $\Re[h(j\omega)] \geq \delta$ for all $\omega \in \mathbb{R}$.

c We compute

$$\begin{aligned}|h(j\omega)|^2 &= h(j\omega)h(-j\omega) \\ &= \frac{\gamma j\omega(-\gamma j\omega)}{(\alpha + j\omega)(\beta + j\omega)(\alpha - j\omega)(\beta - j\omega)} \\ &= \frac{\gamma^2\omega^2}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)}\end{aligned}$$

The system is output strictly passive if

$$\Re[h(j\omega)] \geq \epsilon|h(j\omega)|^2, \quad \epsilon > 0, \quad \forall\omega \in \mathbb{R}$$

We can see that this is true if $0 < \epsilon \leq \frac{\alpha+\beta}{\gamma}$, which can be achieved since $\frac{\alpha+\beta}{\gamma}$ is a positive value. Hence, the system is output strictly passive.

d The system is given by

$$h(s) = \frac{y(s)}{u(s)}$$

where u is the input and y is the output. When investigating if a system is zero-state observable, the system is analyzed with inputs set to zero, $u = 0$. This leads to the equation

$$\begin{aligned} u(s)\gamma s = 0 &= y(s)(\alpha + s)(\beta + s) \\ &= y(s)\left(\alpha\beta + s(\alpha + \beta) + s^2\right) \\ &= \alpha\beta y + (\beta + \alpha)\dot{y} + \ddot{y} \end{aligned}$$

Define the states $x_1 = y$ and $x_2 = \dot{y}$, such that the control law with zero input can be expressed as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha\beta x_1 - (\beta + \alpha)x_2 \\ y &= x_1 \end{aligned}$$

To show that the system is zero-state observable we require that no solution can stay identically in $y = 0$ other than the trivial solution $x \equiv 0$, that is

$$y \equiv 0 \implies x_1 \equiv 0 \implies \dot{x}_1 = 0 \implies x_2 \equiv 0 \implies \dot{x}_2 = 0 \implies x_1 = 0$$

Hence, the system is zero-state observable.

Problem 4 We consider the feedback interconnected systems

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - h_1(x_2) + e_1 \\ y_1 = x_2 \end{cases} \qquad H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = h_2(x_3) \end{cases}$$

where functions $h_i(\cdot)$ are locally Lipschitz, $h_i(\cdot) \in (0, \infty]$, and $h_2(z)$ satisfies $|h_2(z)| \geq \frac{|z|}{(1+z^2)}$.

a The passivity properties of H_1 is investigated first. Let $V_1(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ be a candidate for a storage function. The time-derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}_1(x_1, x_2) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2 + x_2(-x_1 - h_1(x_2) + e_1) \\ &= x_1x_2 - x_1x_2 - h_1(x_2)x_2 + e_1x_2 \\ &= -h_1(x_2)x_2 + e_1x_2 \\ &= -h_1(y_1)y_1 + e_1y_1 \\ e_1y_1 &= \dot{V}_1(x_1, x_2) + y_1h_1(y_1) \end{aligned}$$

Since $h_1 \in (0, \infty]$, we know that $y_1h_1(y_1) > 0$, $\forall y_1 \neq 0$. Thus, H_1 is output strictly passive.

Next, the passivity properties of H_2 is investigated by using $V_2(x_3) = \int_0^{x_3} h_2(z)\,dz$ as a candidate for a storage function. The time-derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}_2(x_3) &= \frac{\partial}{\partial x_3}\left(\int_0^{x_3} h_2(z)\,dz\right)\dot{x}_3 \\ &= h_2(x_3)(-x_3 + e_2) \\ &= -x_3h_2(x_3) + h_2(x_3)e_2 \\ &= -x_3h_2(x_3) + y_2e_2 \\ y_2e_2 &= \dot{V}_2(x_3) + x_3h_2(x_3) \end{aligned}$$

Since $h_2 \in (0, \infty]$, $x_3h_2(x_3) > 0$, $\forall x_3 \neq 0$. Thus, H_2 is strictly passive.

By Theorem 6.1 we conclude that the feedback connection is passive.

b Asymptotic stability of the unforced system is shown by using Theorem 6.3 from Khalil. Since we have one strictly passive system and one output strictly passive system, we need to show that the output strictly passive system is zero-state observable. It can be recognized that no solution can stay identically in $S = \left\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 = 0\right\}$ other than the trivial solution $(x_1, x_2) = (0, 0)$. That is

$$\begin{aligned} y_1 &\equiv 0 \Leftrightarrow x_2 \equiv 0 \\ \dot{x}_2 &= 0 \Rightarrow x_1 = -h_1(x_2) = 0 \end{aligned}$$

Hence, the unforced system is asymptotically stable. To prove this globally, we need to show that the storage functions are radially unbounded. The first storage function is given by

$$V_1(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right) = \frac{1}{2}\|(x_1, x_2)\|_2^2$$

which is radially unbounded. The second storage function is given by

$$V_2(x_3) = \int_0^{x_3} h_2(z)\, dz$$

We consider the case $x_3 \geq 0$ and $x_3 < 0$ separately. With $x_3 \geq 0$ we have

$$\begin{aligned} V_2(x_3) = \int_0^{x_3} h_2(z)\, dz &= \int_0^{x_3} |h_2(z)|dz \\ &\geq \int_0^{x_3} \frac{|z|}{(1+z^2)} dz = \int_0^{x_3} \frac{z}{(1+z^2)} dz \end{aligned}$$

With $x_3 < 0$ we have

$$\begin{aligned} V_2(x_3) = \int_0^{x_3} h_2(z)\, dz = -\int_{x_3}^{0} h_2(z)\, dz &= \int_{x_3}^{0} |h_2(z)|dz \\ \geq \int_{x_3}^{0} \frac{|z|}{(1+z^2)} dz = -\int_{x_3}^{0} \frac{z}{(1+z^2)} dz &= \int_0^{x_3} \frac{z}{(1+z^2)} dz \end{aligned}$$

since $h_2(z)$ and z have the same sign due to the sector condition. Either way, we thus have

$$V_2(x_3) \geq \int_0^{x_3} \frac{z}{(1+z^2)} dz = \frac{1}{2}\ln\left(1 + x_3^2\right)$$

such that $V_2(x_3) \to \infty$ as $|x_3| \to \infty$. Hence, the unforced system is globally asymptotically stable.

Problem 5 We consider the feedback interconnected systems

$$H_1 : \begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2 + e_1 \\ y_1 = x_2 \end{cases} \qquad H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = x_3^3 \end{cases}$$

a The passivity properties of H_1 is investigated. Let $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ be a candidate for a storage function. The time-derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}_1(x_1, x_2) &= x_1^3\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1^3(-x_1 + x_2) + x_2\left(-x_1^3 - x_2 + e_1\right) \\ &= -x_1^4 + x_1^3x_2 - x_1^3x_2 - x_2^2 + x_2e_1 \\ &= -x_1^4 - x_2^2 + e_1y_1 \end{aligned}$$

Hence, H_1 is strictly passive with storage function $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.
The passivity properties of H_2 is investigated by using $V_2(x_3) = \frac{1}{4}x_3^4$ as a candidate for a storage function. The time-derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}_2(x_3) &= x_3^3\dot{x}_3 \\ &= x_3^3(-x_3 + e_2) \\ &= -x_3^4 + x_3^3e_2 \\ &= -x_3^4 + e_2y_2 \end{aligned}$$

Hence, H_2 is strictly passive with storage function $V_2(x_3) = \frac{1}{4}x_3^4$, and the feedback connection is passive.

b Since both systems are strictly passive with radially unbounded storage functions, it follows from Theorem 6.3 that the origin of the unforced system is asymptotically stable.

Problem 6 Consider $v = \psi(u)$ as a virtual input to the system. We then have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + v \end{aligned}$$

Let the output be $y = x_2$ and consider the storage function $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, which gives

$$\dot{V} = x_1^3 x_2 - x_1^3 x_2 + x_2 v = yv$$

Hence, the system is passive. With $v = 0$,

$$y \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$$

Thus the system is zero-state observable. Therefore, by Theorem 14.4 in Khalil, we can globally stabilize the system by $v = -\phi(y)$ for any $\phi(y)$ satisfying

$$\begin{aligned} &\phi(y) \text{ is locally Lipschitz} \\ &\phi(0) = 0 \\ &y\phi(y) > 0 \text{ for all } y \neq 0 \end{aligned}$$

Since $v = \psi(u)$, we get

$$\phi(y) = -\psi(u)$$

which ensures that ϕ is locally Lipschitz and $\psi(0) = 0$. In order to fulfill the last condition, we need

$$y\phi(y) > 0 \quad \forall y \neq 0 \quad \Leftrightarrow \quad -y\psi(u) > 0 \quad \forall y \neq 0$$

which is fulfilled by choosing $u = -y$ since $z\psi(z) > 0 \forall z \neq 0$. Thus $v = \psi(u)$ is a globally stabilizing feedback controller with $u = -y = -x_2$.

Problem 7 The system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - x_3 \\ \dot{x}_2 &= -x_1 x_3 - x_2 + u \\ \dot{x}_3 &= -x_1 + u \\ y &= x_3 \end{aligned}$$

Rewriting this model on the form $\dot{x} = f(x) + g(x)u$ results in

$$\begin{aligned} f(x) &= \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix} \\ g(x) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

a The relative degree is obtained from

$$\begin{aligned} y &= x_3 \\ \dot{y} &= \dot{x}_3 \\ &= -x_1 + u \end{aligned}$$

where we see that $L_g h(x) = L_g L_f^{\rho-1} h(x) = 1$, which shows that the system has relative degree $\rho = 1$, which is defined in all of $\mathbb{R}^3$. Hence, the system is input-output linearizable.

b The external part of the normal form is given by

$$\xi_1 = h(x) = x_3$$

To find the internal dynamics we start by considering the requirements on $\frac{\partial \phi_i}{\partial x}$, given by

$$\begin{aligned}
\frac{\partial \phi_1}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_1}{\partial x_3} \end{bmatrix} g(x) \\
&= \frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_1}{\partial x_3} \\
&= 0 \\
\frac{\partial \phi_2}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x_3} \end{bmatrix} g(x) \\
&= \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_2}{\partial x_3} \\
&= 0
\end{aligned}$$

This gives

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & -\frac{\partial \phi_1}{\partial x_2} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & -\frac{\partial \phi_2}{\partial x_2} \\ 0 & 0 & 1 \end{bmatrix}$$

where $T(x) = [\phi_1(x), \phi_2(x), \xi_1(x)]^T$. The internal states ϕ_1, ϕ_2 must be chosen such that $\frac{\partial T}{\partial x}$ is nonsingular. By looking at the above expression, we see that one such choice is

$$\begin{aligned}
\phi_1(x) &= x_1 \\
\phi_2(x) &= x_2 - x_3
\end{aligned}$$

which gives a global diffeomorphism

$$\begin{aligned}
T(x) &= \begin{bmatrix} x_1 \\ x_2 - x_3 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} x
\end{aligned}$$

since T is proper and the Jacobian matrix is nonsingular. The system on normal form is

$$\begin{aligned}
\dot{\eta}_1 &= \dot{x}_1 \\
&= -\eta_1 + \eta_2 \\
\dot{\eta}_2 &= \dot{x}_2 - \dot{x}_3 \\
&= -x_1 x_3 - x_2 + u + x_1 - u \\
&= -\eta_1 \xi_1 - (\eta_2 + x_3) + \eta_1 \\
&= \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 \\
\dot{\xi}_1 &= -\eta_1 + u
\end{aligned}$$

Since $L_g L_f^0 h(x) = L_g h(x) = 1$, the transformation is valid in $\mathbb{R}^3$.

c To investigate if the system is minimum phase, we analyze the zero dynamics

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi)|_{\xi=0} \\
&= \left[\begin{bmatrix} -\eta_1 + \eta_2 \\ \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 \end{bmatrix} \right]\Bigg|_{\xi=0} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \eta = A\eta
\end{aligned}$$

where it can be seen that the system has an eigenvalue with zero real part. Hence, the origin is not asymptotically stable, and the system is therefore not minimum phase.

Problem 8 The system is given by

$$\begin{aligned} M\ddot{\delta} &= P - D\dot{\delta} - \eta_1 E_q \sin(\delta) \\ \tau \dot{E}_q &= -\eta_2 E_q + \eta_3 \cos(\delta) + E_{FD} \end{aligned}$$

Define the states $x_1 = \delta$, $x_2 = \dot{\delta}$, $x_3 = E_q$, and $u = E_{FD}$ which results in the system

$$\dot{x} = f(x) + gu,$$

with $x = [x_1, x_2, x_3]^T$, $g = \left[0, 0, \frac{1}{\tau}\right]^T$, and

$$f(x) = \begin{bmatrix} \frac{1}{M}(P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ \frac{1}{\tau}(-\eta_2 x_3 + \eta_3 \cos(x_1)) \end{bmatrix}$$

a The output is given by $y = \delta = x_1 = h(x)$. The relative degree is found by

$$\begin{aligned} &y = x_1 \\ &\dot{y} = \dot{x}_1 = x_2 \\ &\ddot{y} = \dot{x}_2 = \frac{1}{M}(P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ &\dddot{y} = -\frac{D}{M}\frac{1}{M}(P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ &\quad - \frac{\eta_1}{\tau M}\sin(x_1)(-\eta_2 x_3 + \eta_3 \cos(x_1) + u) - \frac{\eta_1}{M} x_3 \cos(x_1) x_2 \end{aligned}$$

This gives relative degree of the system $\rho = 3$, which is defined if $\sin(x_1) \neq 0$.

We wish to transform the system into the form

$$\begin{aligned} &\dot{\eta} = f_0(\eta, \xi) \\ &\dot{\xi} = A_c \xi + B_c \gamma(x)[u - \alpha(x)] \\ &y = C_c \xi \end{aligned}$$

where η is the internal dynamics and ξ the external dynamics. They are both given through the diffeomorphism

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \end{bmatrix} = \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{1}$$

where

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \text{ for } 1 \leq i \leq n - \rho, \ \forall \ x \in D_0$$

Since $\rho = n$, one only needs the external part of the system on normal form, ξ. External variables of the normal form is given by computing the Lie Derivative of h with respect to f

$$\begin{aligned} \xi_1 &= h(x) \\ &= x_1 \\ \xi_2 &= L_f h(x) \\ &= x_2 \\ \xi_3 &= L_f^2 h(x) \\ &= \frac{1}{M}(P - Dx_2 - \eta_1 x_3 \sin(x_1)) \end{aligned}$$

The system can therefore be written on normal form as

$$\dot{\xi} = A_c \xi + B_c \gamma(x)[u - \alpha(x)] \tag{2}$$

$$y = C_c \xi \tag{3}$$

where

$$\gamma(x) = L_g L_f^{\rho-1} h(x) \quad \text{and} \quad \alpha(x) = -\frac{L_f^{\rho} h(x)}{L_g L_f^{\rho-1} h(x)} = -\frac{L_f^{\rho} h(x)}{\gamma(x)} \tag{4}$$

This transformation is therefore only valid when $\gamma(x) \neq 0$, which means that

$$\begin{aligned} L_g L_f^{\rho-1} h(x) &= L_g L_f^2 h(x) \\ &= -\frac{\eta_1}{\tau M} \sin(x_1) \\ &\neq 0 \;\forall\, x \in D_0 \end{aligned}$$

where $D_0 = \left\{ x \in R^3 \mid \sin(x_1) \neq 0 \right\}$. Since the relative degree equals the dimension of the system, we have no internal dynamics and the system is minimum phase.

b The output is given by $y = \delta + \zeta\dot{\delta} = x_1 + \zeta x_2 = h(x)$ where $\zeta \neq 0$. The relative degree is obtained from

$$\begin{aligned} y &= x_1 + \zeta x_2 \\ \dot{y} &= \left(1 - \frac{\zeta D}{M}\right) x_2 - \frac{\zeta\eta_1}{M} x_3 \sin(x_1) + \zeta P \frac{1}{M} \\ \ddot{y} &= \frac{\zeta\eta_1}{M} x_3 \cos(x_1) x_2 - \frac{\zeta\eta_1}{\tau M} \sin(x_1)\left(-\eta_2 x_3 + \eta_3 \cos(x_1) + u\right) \\ &\quad + \left(1 - \frac{\zeta D}{M}\right) \frac{1}{M} \left(P - D x_2 - \eta_1 x_3 \sin(x_1)\right) \end{aligned}$$

Thus, the system has relative degree $\rho = 2$. The region D_0 where the transformation is valid is found by requiring

$$L_g L_f^{\rho-1} h(x) = L_g L_f^1 h(x) = -\frac{\gamma\eta_1}{\tau M} \sin(x_1) \neq 0$$

Thus, $D_0 = \left\{ x \in R^3 \mid \sin(x_1) \neq 0 \right\}$. Since $\rho < n$, both internal and external dynamics are needed. The external variables of the normal form is found by computing the Lie Derivative of h with respect to f

$$\begin{aligned} \xi_1 &= h(x) = x_1 + \zeta x_2 \\ \xi_2 &= L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) \\ &= x_2 + \frac{\zeta}{M}\left(P - D x_2 - \eta_1 x_3 \sin(x_1)\right) \end{aligned}$$

The internal dynamics $\phi(x)$ must be chosen to satisfy $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ and the existence of $T^{-1}(x)$ in D_0. The coordinate transformation $T(x)$ is invertible in D_0 if $\frac{\partial T}{\partial x}$ has full rank for all $x \in D_0$. Moreover, the condition $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ implies that $\frac{\partial \phi}{\partial x_3} = 0$. With $T(x) = [\phi(x),\, \xi_1(x),\, \xi_2(x)]^T$, we also have

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_1} & 0 \\ 1 & \zeta & 0 \\ -\frac{\zeta\eta_1}{M} x_3 \cos(x_1) & 1 - \frac{\zeta D}{M} & -\frac{\zeta\eta_1}{M} \sin(x_1) \end{bmatrix}$$

From this we can see that one choice of $\phi(x)$ which ensures that $\frac{\partial T}{\partial x}$ has full rank is $\phi(x) = x_1$. With this choice, we have that

$$\dot{\phi}(x) = \dot{x}_1 = x_2 = \frac{1}{\zeta}\left(\xi_1 - \phi\right) = f_0(\phi, \xi)$$

The system on normal form is thus

$$\begin{aligned} \dot{\phi} &= f_0(\phi, \xi) \\ \dot{\xi} &= A_c \xi + B_c \gamma(x)\left[u - \alpha(x)\right] \\ y &= C_c \xi \end{aligned}$$

The system is said to be minimum phase if the zero dynamics, $\dot{\phi} = f_0(\phi, 0)$, has an asymptotically stable equilibrium point in the domain of interest. From $\dot{\phi} = f_0(\phi, 0) = -\frac{1}{\zeta}\phi$ it can be verified that the origin is asymptotically stable.

Problem 9 The system is rewritten as

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

where

$$f(x) = \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix}, \quad h(x) = x_3$$

a The relative degree is found by differentiating y with respect to time:

$$\begin{aligned} y &= x_3 \\ \dot{y} &= \dot{x}_3 = x_2 \\ \ddot{y} &= \dot{x}_2 = x_1 x_2 + u \end{aligned}$$

where it can be seen that the system has a relative degree $\rho = 2$ in $x \in \mathbb{R}^3$. The relative degree holds as long as $L_g L_f^{\rho-1} h(x) \neq 0$.

$$L_g L_f^{\rho-1} h(x) = L_g L_f h(x) = 1 \neq 0 \ \forall \ x \in \mathbb{R}^2$$

The relative degree thus holds on $\mathbb{R}^3$.

b The system has a well defined relative degree ρ in $\mathbb{R}^3$, and is therefore input-output linearizable in $\mathbb{R}^3$.

c The variables for the external dynamics are found according to

$$\begin{aligned} \xi_1 &= h(x) = x_3 \\ \xi_2 &= L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = x_2 \end{aligned}$$

The coordinates for the internal dynamics is chosen such that $T(x)$ is diffeomorphism on $\mathbb{R}^3$ and $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ on $\mathbb{R}^3$, where$\left[\eta, \xi^T\right]^T = [\phi(x), \psi(x)] = T(x)$. In addition to this we require $\phi(0) = 0$ in order to have the origin as equilibrium. We start by calculating

$$\begin{aligned} \frac{\partial \phi(x)}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi(x)}{\partial x_1} & \frac{\partial \phi(x)}{\partial x_2} & \frac{\partial \phi(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{\partial \phi(x)}{\partial x_1} e^{x_2} + \frac{\partial \phi(x)}{\partial x_2} = 0 \end{aligned}$$

and based on these calculations we try

$$\begin{aligned} \frac{\partial \phi(x)}{\partial x_1} &= 1 \\ \frac{\partial \phi(x)}{\partial x_2} &= -e^{x_2} \end{aligned}$$

which implies that

$$\phi(x) = x_1 - e^{x_2} + c$$

where c is some constant (c can also be a function of x_3, but this is not necessary, so we can choose it constant for simplicity). This constant is chosen to satisfy our requirement $\phi(0) = 0$, that is

$$\begin{aligned} \phi(0) &= -e^0 + c \\ &= -1 + c \\ &\Rightarrow c = 1 \end{aligned}$$

Our resulting coordinate transformation is now given by

$$\begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x_1 - e^{x_2} + 1 \\ x_3 \\ x_2 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \eta + e^{\xi_2} - 1 \\ \xi_2 \\ \xi_1 \end{bmatrix}$$

Consequently, the inverse transformation exists. It follows that $T(x)$ and $T^{-1}(x)$ are continuously differentiable. Hence, $T(x)$ is a diffeomorphism on $\mathbb{R}^3$ and $T(0) = T^{-1}(0) = 0$.

Alternatively, we can use the external dynamics and the condition $\frac{\partial \phi}{\partial x} g(x) = 0$ to find $\frac{\partial T}{\partial x}$ and choose $\phi(x)$ such that the matrix $\frac{\partial T}{\partial x}$ has full rank and T is radially unbounded. Following this procedure we get

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial \phi}{\partial x_1} & -\frac{\partial \phi}{\partial x_1} e^{x_2} & \frac{\partial \phi}{\partial x_3} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

where we see that it is sufficient to ensure that $\frac{\partial \phi}{\partial x_1}$ is nonzero for all x for $\frac{\partial T}{\partial x}$ to have full rank. Choosing $\frac{\partial \phi}{\partial x_1} = 1$ gives us $\frac{\partial \phi}{\partial x_2} = -e^{x_2}$. Then choosing $\frac{\partial \phi}{\partial x_3} = 0$ for simplicity and integrating brings us again to $\phi(x) = x_1 - e^{x_2} + c$, where c is a constant which has to be chosen such that $\phi(0) = 0$. Since we know that $T(x)$ is a diffeomorphism from $\frac{\partial T}{\partial x}$ being invertible and radially undboundedness of T, it is not necessary to find $T^{-1}(x)$ and inspect if it is continuously differentiable.

d The system may be rewritten as

$$\begin{aligned} \dot{\eta} &= \dot{x}_1 - \frac{\partial e^{x_2}}{\partial x_2} \dot{x}_2 \\ &= -x_1 + e^{x_2} u - e^{x_2} (x_1 x_2 + u) \\ &= -x_1 - e^{x_2} x_1 x_2 \\ &= -(\eta + e^{x_2} - 1) - e^{x_2} (\eta + e^{x_2} - 1) x_2 \\ &= \left(1 - \eta - e^{\xi_2}\right) + \left(1 - \eta - e^{\xi_2}\right) e^{\xi_2} \xi_2 \\ &= \left(1 - \eta - e^{\xi_2}\right) \left(1 + e^{\xi_2} \xi_2\right) \end{aligned}$$

which constitues the internal dynamics, and

$$\begin{aligned} \dot{\xi} &= A_c \xi + B_c \gamma(x) (u - \alpha(x)) \\ y &= C_c \xi \end{aligned}$$

where

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_c = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\begin{aligned} \gamma(x) &= L_g L_f h(x) \\ &= 1 \\ \alpha(x) &= -\frac{L_f^2 h(x)}{L_g L_f h(x)} \\ &= -\frac{x_1 x_2}{1} \\ &= -x_1 x_2 \end{aligned}$$

e The zero dynamics is given by

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi)\big|_{\xi=0} \\ &= \left(1 - \eta - e^{\xi_2}\right) \left(1 + e^{\xi_2} \xi_2\right)\Big|_{\xi=0} \\ &= (1 - \eta - 1)(1 + 0) \\ &= -\eta \end{aligned}$$

which has a globally asymptotically stable equilibrium at the origin.

f The external dynamics are given by

$$\dot{\xi} = A_c \xi + B_c \gamma(x) (u - \alpha(x))$$

By choosing

$$u = \gamma^{-1}(x)\, v + \alpha(x)$$

the external dynamics are given by

$$\dot{\xi} = A_c \xi + B_c v$$

Since the system is controllable, $rank([B,\ AB]) = 2$, it can be stabilized by a control input $v = -K\xi$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. The input u is now given by

$$u = -\gamma^{-1}(x)\, K\xi + \alpha(x)$$

Since $\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}$ is asymptotically stable, by Lemma 13.1, the origin of the complete system is asymptotically stable.

g Let

$$\mathcal{R} = \begin{bmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{bmatrix} = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}, \quad e = \begin{bmatrix} \xi_1 - r \\ \vdots \\ \xi_\rho - r^{(\rho-1)} \end{bmatrix} = \xi - \mathcal{R}$$

In order for the output y to track the reference r, the external states ξ must track $\mathcal{R}$. The time-derivative of the error e is computed as

$$\begin{aligned} \dot{e} &= \dot{\xi} - \dot{\mathcal{R}} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v - \begin{bmatrix} \dot{r} \\ \ddot{r} \end{bmatrix} \\ &= (A_c \xi + B_c v) - (A_c \mathcal{R} + B_c \ddot{r}) \\ &= A_c(\xi - \mathcal{R}) + B_c(v - \ddot{r}) \\ &= A_c e + B_c(v - \ddot{r}) \\ &= A_c e + B_c(\gamma(x)[u - \alpha(x)] - \ddot{r}) \end{aligned}$$

The state feedback control

$$u = \gamma^{-1}(x)\,(v + \ddot{r}) + \alpha(x)$$

reduces the system to

$$\begin{aligned} \dot{\eta} &= f_0(\eta, e + \mathcal{R}) \\ \dot{e} &= A_c e + B_c v \end{aligned}$$

and since (A_c, B_c) is controllable, the loop is closed with $v = -Ke$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. This makes the external dynamics for e exponentially stable. Since $\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}$ is asymptotically stable, for sufficiently small $e(0), \eta(0)$, and $\mathcal{R}(t)$, all solutions $(\eta(t), e(t))$ of the closed-loop system are bounded and $e(t) \to 0$ as $t \to \infty$. Thus, the chosen state feedback control ensures that the reference is asymptotically tracked.

h The internal dynamics are given by

$$f_0(\eta, \xi) = (1 - \eta - e^{\xi_2})(1 + \xi_2 e^{\xi_2})$$

which has a globally asymptotically stable equilibrium in the origin when $\xi = 0$. If $\xi \neq 0$, we can observe that the term $\xi_2 e^{\xi_2}$ is bounded from below by $-\frac{1}{e}$, such that

$$\dot{\eta} = k - k\eta - ke^{\xi_2}$$

where $k = 1 + \xi_2 e^{\xi_2} > 0$. Consider the Lyapunov function candidate $V(\eta) = \frac{1}{2}\eta^2$, such that

$$\begin{aligned} \dot{V}(\eta) &= -k\eta^2 + k\eta(1 - e^{\xi_2}) \\ &\leq -k\eta^2 + k\|\eta\|_1 \|1 - e^{\xi_2}\|_1 \\ &\leq -k\eta^2 + k\|\eta\|_1 (e^{\|\xi\|_1} - 1) \\ &= -k\eta^2(1-\theta) - k\eta^2\theta + k\|\eta\|_1(e^{\|\xi\|_1} - 1) && 0 < \theta < 1 \\ &\leq -k\eta^2(1-\theta) && \forall \|\eta\|_1 \geq \frac{e^{\|\xi\|_1} - 1}{\theta} \end{aligned}$$

where we use that $|1-e^{\xi_2}| \leq e^{|\xi_2|}-1 \leq e^{|\xi_1|+|\xi_2|}-1$. Since $\gamma : [0,\infty) \to [0,\infty)$ is strictly increasing in its argument (with $\gamma'(r) = \frac{e^r}{\theta} > 0$) and satisfies $\gamma(0) = 0$, it is a class $\mathcal{K}$ function. Hence, all conditions of Theorem 4.19 are satisfied with $\gamma(r) = \frac{e^r-1}{\theta}$, which shows that $f_0(\eta, \xi)$ is ISS with respect to ξ. We can thus conclude that the tracking is global.

Problem 10 The system is given by

$$
\begin{aligned}
\dot{x}_1 &= x_2 + 2x_1^2 \\
\dot{x}_2 &= x_3 + u \\
\dot{x}_3 &= x_1 - x_3 \\
y &= x_1
\end{aligned}
$$

The system has relative degree $\rho = 2$ as seen by

$$y = x_1 \Rightarrow \dot{y} = x_2 + 2x_1^2 \Rightarrow \ddot{y} = x_3 + u + 4x_1(x_2 + 2x_1^2)$$

which is defined on all of $\mathbb{R}^3$. We first find the external dynamics; let

$$
\begin{aligned}
\xi_1 &= h(x) = x_1 \\
\xi_2 &= L_f h(x) = x_2 + 2x_1^2
\end{aligned}
$$

We need one internal state ϕ, which must fulfill

$$\frac{\partial \phi(x)}{\partial x} g(x) = \begin{bmatrix} \frac{\partial \phi(x)}{\partial x_1} & \frac{\partial \phi(x)}{\partial x_2} & \frac{\partial \phi(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{\partial \phi(x)}{\partial x_2} = 0$$

This gives

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & 0 & \frac{\partial \phi_1}{\partial x_3} \\ 1 & 0 & 0 \\ 4x_1 & 1 & 0 \end{bmatrix}$$

In order for $\frac{\partial T}{\partial x}$ to be invertible, let $\phi(x) = x_3$. Since T is also proper, we can conclude that it is a global diffeomorphism. The internal dynamics are then

$$\dot{\phi} = f_0(\phi, \xi) = \xi_1 - \phi$$

To check if the system is minimum-phase, let $\xi = 0$ such that

$$\dot{\phi} = f_0(\phi, 0) = -\phi$$

which is globally exponentially stable. Hence, the system is minimum phase. Moreover, since $f_0(\phi, \xi)$ is linear, it is also globally Lipschitz in ϕ, ξ. Hence, by Lemma 4.6, the internal dynamics are ISS with respect to ξ. Define the tracking error $e = y - r$ such that

$$\ddot{e} = \ddot{y} - \ddot{r} = x_3 + u + 4x_1(x_2 + 2x_1^2) - \ddot{r}$$

We choose

$$
\begin{aligned}
u &= -x_3 - 4x_1(x_2 + 2x_1^2) + \ddot{r} - k_1 e - k_2 \dot{e} \\
&= -x_3 - 4x_1(x_2 + 2x_1^2) + \ddot{r} - k_1(y - r) - k_2(x_2 + 2x_1^2 - \dot{r}),
\end{aligned}
$$

where k_1 and k_2 are positive constants, which ensures that the tracking dynamics

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0$$

are globally exponentially stable. Since the internal dynamics are ISS, we can conclude that the reference is globally asymptotically tracked by the output.