TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 5	Norwegian University of Science and Technology
Due date: Monday, November 11. at 23:59	

Problem 1 (7.5%) Consider the system from Exercise 6.4 in Khalil:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(x_1) - ax_2 + u \\
y &= kx_2 + u
\end{aligned}
$$

where $a > 0$, $k > 0$, and h is locally Lipschitz and fulfills the sector condition $h \in [\alpha_1, \infty]$ with $\alpha_1 > 0$. Let $V(x) = k \int_0^{x_1} h(s)ds + \frac{1}{2}x^T Px$, where P is a positive definite and symmetric matrix. Using V as the storage function, show that the system is strictly passive.

Problem 2 (7.5%) Exercise 6.11 in Khalil

The Euler equations for a rotating rigid spacecraft are given by

$$
\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3
\end{aligned}
$$

where ω_1 to ω_3 are the components of the angular velocity vector along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of the inertia.

a Show that the map from $u = [u_1, u_2, u_3]^T$ to $\omega = [\omega_1, \omega_2, \omega_3]^T$ is lossless.

b Let $u = -K\omega + v$, where K is a positive definite symmetric matrix. Show that the map from v to ω is finite-gain $\mathcal{L}_2$ stable.

c Show that when $v = 0$, the origin $\omega = 0$ is globally asymptotically stable.

Hint: Use $V(\omega) = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$.

For linear, time-invariant systems with proper transfer functions $h(s)$, where all poles have negative real parts, the system is[1]

1. Passive $\Leftrightarrow \Re[h(j\omega)] \geq 0$
2. Input strictly passive $\Leftrightarrow$ There exists $\delta > 0$ such that $\Re[h(j\omega)] \geq \delta$
3. Output strictly passive $\Leftrightarrow$ There exists $\epsilon > 0$ such that $\Re[h(j\omega)] \geq \epsilon|h(j\omega)|^2$

where the conditions must hold for all $\omega \in \mathbb{R}$.

Problem 3 (10%) Consider the system given by the transfer function

$$
h(s) = \frac{\gamma s}{(\alpha + s)(\beta + s)}
$$

where $\gamma, \alpha, \beta > 0$ are constants.

a Show that the system is passive.

Hint: $(a + j\omega)(a - j\omega) = (a^2 + \omega^2)$. *Use this to split* $h(j\omega)$ *into one real and one imaginary part.*

b Is the system input strictly passive?

c Is the system output strictly passive?

Hint: $|h(j\omega)|^2 = h(j\omega)h(-j\omega)$

[1] Lozano et al. (2007). *Dissipative Systems Analysis and Control.*

d Show that the system is zero-state observable.

Hint: Transform the system to the time domain. This is easier to do if you set $u(s) = 0$ first.

Problem 4 (10%) Exercise 6.14 in Khalil

Consider the feedback connection of Figure 1 with

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - h_1(x_2) + e_1 \\ y_1 = x_2 \end{cases} \qquad H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = h_2(x_3) \end{cases}$$

where h_1 and h_2 are locally Lipschitz functions which satisfy $h_1 \in (0, \infty]$, $h_2 \in (0, \infty]$ and $|h_2(z)| \geq |z|/(1 + z^2)$ for all z.

a Show that the feedback connection is passive.

b Show that the origin of the unforced system is globally asymptotically stable.

Hint: Use $V_1(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ *and* $V_2(x_3) = \int_0^{x_3} h_2(z)\, dz$.

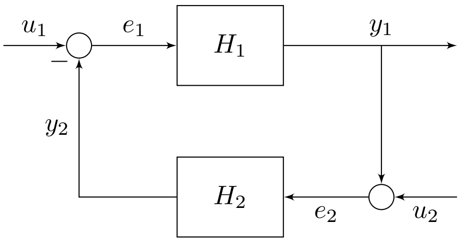

Figure 1: Feedback connection.

Problem 5 (5%) Exercise 6.15 in Khalil

Repeat the previous exercise for the system

$$H_1 : \begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2 + e_1 \\ y_1 = x_2 \end{cases} \qquad H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = x_3^3 \end{cases}$$

Hint: Use $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ *and* $V_2(x_3) = \frac{1}{4}x_3^4$.

Problem 6 (5%) Exercise 14.43 in Khalil

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + \psi(u) \end{aligned}$$

where ψ is a locally Lipschitz function that satisfies $\psi(0) = 0$ and $u\psi(u) > 0$ for all $u \neq 0$. Design a globally stabilizing state feedback controller.
Hint: See Theorem 14.4 in Khalil.

Problem 7 (10%) Exercise 13.2 in Khalil

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - x_3 \\ \dot{x}_2 &= -x_1 x_3 - x_2 + u \\ \dot{x}_3 &= -x_1 + u \\ y &= x_3 \end{aligned}$$

a Is the system input-output linearizable?

b Transform it into the normal form and specify the region over which the transformation is valid.

c Is the system minimum phase?

Problem 8 (15%) Exercise 13.1 in Khalil

Consider the third-order model of a synchronous generator connected to an infinite bus:

$$M\ddot{\delta} = P - D\dot{\delta} - \eta_1 E_q \sin(\delta)$$
$$\tau\dot{E}_q = -\eta_2 E_q + \eta_3 \cos(\delta) + E_{FD}$$

where δ is an angle in radians, E_q is the voltage, P is the mechanical input power (constant), E_{FD} is the field voltage (input), D is the damping coefficient, M is the inertial coefficient, τ is a time constant and η_1, η_2 and η_3 are constant parameters. Consider two possible choices of the output:

a $y = \delta$

b $y = \delta + \gamma\dot{\delta} \quad \gamma \neq 0$

In each case, study the relative degree of the system and transform it into the normal form. Specify the region over which transformation is valid. If there are nontrivial zero dynamics, find whether or not the system is minimum phase.

Problem 9 (20%) Consider the system

$$\dot{x}_1 = -x_1 + e^{x_2}u$$
$$\dot{x}_2 = x_1 x_2 + u$$
$$\dot{x}_3 = x_2$$
$$y = x_3$$

a Find the relative degree of the system and specify the region on which this relative degree holds.

b Show that the system is input-output linearizable. Specify the region on which it is input-output linearizable.

c Find a coordinate transformation $z = T(x)$ that transforms the system into the normal form. (Note: T must be a *diffeomorphism*[2] over the region of interest and $T(0) = 0$)

d Express the system in normal form. Determine all functions and constants involved in the normal form. Which part of the normal form counts for the internal dynamics?

e Find the zero dynamics and show that it has a globally asymptotically stable equilibrium at the origin.

f Choose an input u that makes the origin of the complete system asymptotically stable.

g Choose an input u that makes the output track a reference signal $r(t)$ asymptotically.

Hint: See Section 13.4.2 in Khalil.

h In order to prove global tracking of the reference, it is sufficient to show that the internal dynamics $f_0(\eta, \xi)$ are ISS with respect to ξ. Analyze the found internal dynamics and conclude whether or not this is the case.

(If you did not manage to find the internal dynamics, you can instead use

$$f_0(\eta, \xi) = (1 - 2\eta - e^{\xi_2})(1 + e^{\xi_1}\xi_1)$$

which are not the correct internal dynamics but have the same properties.)

Hint: $|1 - e^x| \leq e^{|x|} - 1$ *and* $xe^x \geq -\frac{1}{e}$

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[2]Let $\mathcal{D} \subset \mathbb{R}^n$ be an open set. A function $T : \mathcal{D} \to \mathbb{R}^n$ is a *diffeomorphism* if T and its inverse mapping $T^{-1} : T(\mathcal{D}) \to \mathcal{D}$ are continuously differentiable, where $T(\mathcal{D}) = \{y \in \mathbb{R}^n : \exists x \in \mathcal{D} \text{ such that } y = T(x)\}$.

If $\mathcal{D} = \mathbb{R}^n$, then $T : \mathbb{R}^n \to \mathbb{R}^n$ is a global diffeomorphism if T is proper ($\lim_{\|x\|\to\infty} \|T(x)\| = \infty$) and the Jacobian $\frac{\partial T}{\partial x}$ is non-singular for all $x \in \mathcal{D}$.

Problem 10 (10%) Exercise 13.25 in Khalil

Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 + 2x_1^2 \\
\dot{x}_2 &= x_3 + u \\
\dot{x}_3 &= x_1 - x_3 \\
y &= x_1
\end{aligned}$$

Design a state feedback control law such that the output y asymptotically tracks the reference signal $r(t) = \sin(t)$. Comment also on the possibility of a global result.