TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 6	Norwegian University of Science and Technology
Due date: Monday, November 25. at 23:59	

Problem 1 (25%) Design a Model Reference Adaptive Controller (MRAC) for SISO Linear Systems

Consider the roll dynamics of a conventional aircraft using differential motion of ailerons and spoilers, approximated by a scalar first-order ordinary differential equation on the form

$$\dot{x} = a_p x + b_p u$$

with the following notations:

x	Aircraft roll rate in stability axes	[radians/s]
u	Total differential aileron-spoiler deflection	[radians]
a_p	Roll damping derivative	[s^{-1}]
b_p	Dimensional rolling moment derivative	[s^{-1}]

a Given the roll damping $a_p = -0.8$ and the aileron effectiveness $b_p = 1.6$, find a fixed-gain model reference controller on the form

$$u = a_x x + a_r r$$

that forces x to track the roll rate x_m, i.e. such that

$$\dot{e} = a_m e,$$

where $e = x - x_m$, and x_m is given by the reference model:

$$\dot{x}_m = a_m x_m + b_m r, \qquad a_m < 0$$

with $a_m = -2$, $b_m = 2$.

b For each of the bounded reference signals

(a) $r = 4$
(b) $r = \sin(t)$

simulate the closed-loop system response.

Repeat the simulations, but calculate the controller parameters a_x and a_r using the wrong system parameters, specifically use $a_p = -1$ and $b_p = 1.5$. Note that you should keep a_p and b_p in the roll dynamics unchanged. Include the plots of the tracking performances and comment on the results.

c Assume that the parameters a_p and b_p are unknown but constant, and the sign of b_p is known to be positive. Using the same reference model parameters and an adaptive controller on the form of $u = \hat{a}_x x + \hat{a}_r r$, show that the error dynamics take the form

$$\dot{e} = a_m e + b_p \left(\tilde{a}_x x + \tilde{a}_r r \right)$$

where $\hat{a}_x$ and $\hat{a}_r$ are the estimated values and

$$\tilde{a}_x = \hat{a}_x - a_x^*, \quad \tilde{a}_r = \hat{a}_r - a_r^*$$

is the difference between the estimated gain and the ideal gains a_x^*, a_r^*.

d Use the Lyapunov function candidate

$$V\left(e, \tilde{a}_x, \tilde{a}_r\right) = \frac{e^2}{2} + \frac{|b_p|}{2\gamma_x}\tilde{a}_x^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2$$

with the adaptation laws

$$\begin{aligned} \dot{\hat{a}}_x &= -\gamma_x x e \ \mathrm{sgn}\left(b_p\right) \\ \dot{\hat{a}}_r &= -\gamma_r r e \ \mathrm{sgn}\left(b_p\right) \end{aligned}$$

and Barbalat's lemma to show that the tracking error e tends to zero globally and asymptotically. You can assume that r is continuous and bounded.

e For each of the bounded reference signals:

(a) $r = 4$

(b) $r = \sin(t)$

simulate the closed-loop system response with the MRAC. Experiment and find rates of adaptation that gives the wanted performance. Compare the fixed-gain versus MRAC controller tracking performances and comment on your results.
Include plots of the estimated controller gains as well. Do the estimates converge to the ideal values found in (a) for each of the two references?

Problem 2 (25%) Design an MRAC for SISO Nonlinear Systems

Consider the roll dynamics with nonlinear damping:

$$\dot{x} = a_p x + c_p x^3 + b_p u$$

where:

x	Aircraft roll rate in stability axes	[radians/s]
u	Total differential aileron-spoiler deflection	[radians]
a_p	Roll damping derivative	
b_p	Dimensional rolling moment derivative	
c_p	Damping constant	

a Assume that the constant roll dynamics parameters (a_p,b_p,c_p) are *known* and that the sign of b_p is positive. Specify a desired closed-loop behaviour by a linear reference model and derive a control law $u = a_x x + a_r r + a_f f(x)$ that leads to perfect tracking of the reference model. Find the expressions for a_x, a_r and a_f.

b Assume that the parameters are *unknown*, but the sign of b_p is known to be positive. Replace the parameters in u with their estimates $\hat{a}_x, \hat{a}_r$ and $\hat{a}_f$, and show that the tracking dynamics take the form

$$\dot{e} = a_m e + b_p \left(\tilde{a}_x x + \tilde{a}_f x^3 + \tilde{a}_r r \right)$$

Find the expressions for $\dot{\hat{a}}_f$, $\dot{\hat{a}}_r$ and $\dot{\hat{a}}_x$.

c Using the Lyapunov function candidate

$$V(e, \tilde{a}_x, \tilde{a}_r, \tilde{a}_f) = \frac{e^2}{2} + \frac{|b_p|}{2\gamma_x}\tilde{a}_x^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2 + \frac{|b_p|}{2\gamma_f}\tilde{a}_f^2$$

and Barbalat's lemma, find the adaptation laws $\dot{\hat{a}}_x, \dot{\hat{a}}_r, \dot{\hat{a}}_f$ such that the tracking error e tends to zero asymptotically and globally.

Hint: The adaptation laws will be on the similar form as in Problem 1d. Remember that you cannot include unknown states. Although it is tempting to include the errors $\tilde{a}_x, \tilde{a}_r$, and $\tilde{a}_f$, we do not know the ideal gains, and thus, the estimation errors are unknown to us.

d For each of the bounded reference signals:

(a) $r = 4$

(b) $r = \sin(t)$

simulate the closed-loop system response and comment on the results. As in Exercise 1, $a_p = -0.8$, $b_p = 1.6$ and $a_m = -2$, $b_m = 2$. Let $c_p = -1.2$. Include plots of the tracking performance.

Problem 3 (25%) Design of Adaptive Tracking Controller for a Class of MIMO Nonlinear Systems

Consider the two-link manipulator illustrated in Figure 1, with the dynamics

$$\underbrace{\begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) \\ H_{21}(\theta) & H_{22}(\theta) \end{bmatrix}}_{H(\theta)} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} C_{11}(\theta,\dot{\theta}) & C_{12}(\theta,\dot{\theta}) \\ C_{21}(\theta,\dot{\theta}) & C_{22}(\theta,\dot{\theta}) \end{bmatrix}}_{C(\theta,\dot{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1}$$

where

$$\begin{aligned}
H(\theta) &= \text{Inertia matrix, uniformly positive definite} \\
\theta &= \text{Vector of joint angles} \\
u &= \text{Vector of torques applied at the manipulator joints} \\
C(\theta,\dot{\theta}) &= \text{Coriolis and centripetal matrix} \\
C_{11}(\theta,\dot{\theta}) &= -h(\theta)\dot{\theta}_2 \\
C_{12}(\theta,\dot{\theta}) &= -h(\theta)\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\
C_{21}(\theta,\dot{\theta}) &= h(\theta)\dot{\theta}_1 \\
C_{22}(\theta,\dot{\theta}) &= 0 \\
H_{11}(\theta) &= a_1 + 2a_3\cos\theta_2 + 2a_4\sin\theta_2 \\
H_{12}(\theta) &= H_{21}(\theta) = a_2 + a_3\cos\theta_2 + a_4\sin\theta_2 \\
H_{22}(\theta) &= a_2 \\
h(\theta) &= a_3\sin\theta_2 - a_4\cos\theta_2
\end{aligned}$$

where a_i depends on the length, mass and inertia of the two links.

a Find a regression matrix $Y \in \mathbb{R}^{2\times 4}$ for the model such that the dynamics can be expressed as

$$Ya = u$$

where $a = [a_1,\, a_2,\, a_3,\, a_4]^T$ are the system parameters that may be unknown.

b The system is to follow a bounded trajectory $\theta_d(t) = \begin{bmatrix} \theta_{1d} \\ \theta_{2d} \end{bmatrix}$ of desired joint angles where $\theta_d(t)$, $\dot{\theta}_d(t)$ and $\ddot{\theta}_d(t)$ are bounded. The tracking control law

$$u = H\ddot{\theta}_r + C\dot{\theta}_r - K_p(\theta - \theta_d) - K_d\left(\dot{\theta} - \dot{\theta}_d\right)$$

will be applied to the system (1), with

$$\dot{\theta}_r = \dot{\theta}_d - \Lambda(\theta - \theta_d)$$

where $\Lambda = K_d^{-1}K_p$, K_p and K_d are symmetric, positive definite matrices. Define the error variables

$$\begin{aligned} e &= \theta - \theta_d, \\ s &= \dot{\theta} - \dot{\theta}_r \end{aligned}$$

Derive the error dynamics, i.e. the dynamics of e and s.

c Show that the above control law makes the system track the desired trajectory, i.e. show that the the origin $(e,s) = (0,0)$ is globally uniformly asymptotically stable. Use the Lyapunov function candidate

$$V(s) = \frac{1}{2}s^T H s$$

Note that $\dot{H} = C + C^T$, making $\dot{H} - 2C$ *skew-symmetric.* A matrix $S \in \mathbb{R}^{n\times n}$ is skew-symmetric if and only if $S = -S^T$. Moreover, $x^\top S x = 0$ for all $x \in \mathbb{R}^n$.

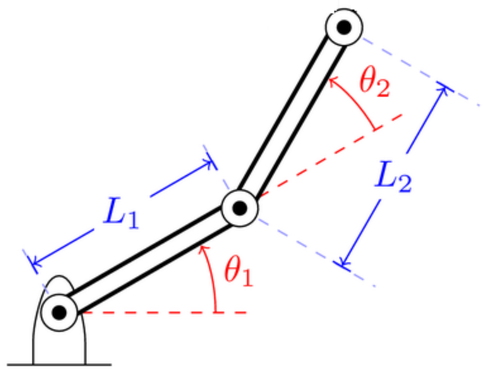

Figure 1: Two-link robotic manipulator.

d The plant parameters a_i are all uncertain (unknown), but can be assumed constant. Replace the parameters in the control law u with an estimate $\hat{a}$. Find an adaptation law $\dot{\hat{a}}$ that ensures global, asymptotic tracking of the reference trajectory $\theta_d(t)$ and renders the estimation errors $\tilde{a} = \hat{a} - a$ bounded. Use the Lyapunov function candidate

$$\begin{aligned} V(s, \tilde{a}) &= \frac{1}{2}\left(s^T H s + \tilde{a}^T \Gamma^{-1} \tilde{a}\right), \\ s &= \dot{\theta} - \dot{\theta}_r, \\ \tilde{a} &= \hat{a} - a \end{aligned}$$

where Γ is a symmetric, positive definite matrix.

e **Optional exercise:** Complete the previous task, but using the Lyapunov function

$$V(e, s, \tilde{a}) = \frac{1}{2} e^T e + \frac{\beta}{2}\left(s^T H s + \tilde{a}^T \Gamma^{-1} \tilde{a}\right),$$

where $\beta > 0$ is a parameter you can choose. Does the adaptation law differ from what you found in the previous problem?

Problem 4 (15%) Exercise 14.31 in Khalil

Using backstepping, design a state feedback controller to globally stabilize the system

$$\begin{aligned} \dot{x}_1 &= x_2 + a + (x_1 - a^{1/3})^3 \\ \dot{x}_2 &= x_1 + u \end{aligned}$$

where a is a known constant.

Problem 5 (10%) Use the backstepping method to stabilize the system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + x_1^2 \\ \dot{x}_2 &= u \end{aligned}$$