TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 2 (Solution)	Norwegian University of Science and Technology

Problem 1

a By using the fact that the derivatives are zero in an equilibrium point, the following equations must be true

$$\begin{aligned} 0 &= x_2^* \\ 0 &= -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - \frac{d}{m}x_2^* - g \\ &= -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - g \end{aligned}$$

Inserting $(0,0)$ into the above equations leads to an illegal expression since g is not zero, but $9.81m/s^2$. Therefore, $(0,0)$ is not an equilibrium point.

b With $u = u_0$ we have the equations

$$0 = x_2^* \tag{1}$$

$$0 = -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - g + \frac{u_0}{m} \tag{2}$$

Inserting $(x_1^*, x_2^*) = (x_{1d}, 0)$ into 2 gives

$$0 = -f_3x_{1d}^3 - f_1x_{1d} - mg + u_0$$

thus

$$u_0 = f_3x_{1d}^3 + f_1x_{1d} + mg \tag{3}$$

c In the original system equations we insert $x_1 = \tilde{x}_1 + x_{1d}$, $\tilde{x}_2 = x_2$ and $u = u_0 + \tilde{u}$. We have

$$\begin{aligned} \dot{\tilde{x}}_1 &= x_2 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} \\ &= -\frac{f_3}{m}(\tilde{x}_1 + x_{1d})^3 - \frac{f_1}{m}(\tilde{x}_1 + x_{1d}) - \frac{d}{m}\tilde{x}_2 - g + \frac{\left(f_3x_{1d}^3 + f_1x_{1d} + mg + \tilde{u}\right)}{m} \end{aligned}$$

The resulting system equations are

$$\dot{\tilde{x}}_1 = \tilde{x}_2 \tag{4}$$

$$m\dot{\tilde{x}}_2 = -f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u} \tag{5}$$

In the equilibrium point for $\tilde{u} = 0$ we have

$$\begin{aligned} 0 &= \tilde{x}_2^* \\ 0 &= -f_3\left[(\tilde{x}_1^* + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1^* - d\tilde{x}_2^* \\ &= -f_3\left[(\tilde{x}_1^* + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1^* \end{aligned}$$

The equilibrium point is now in the origin.

d The Jacobian is calculated for (4)–(5).

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial \tilde{x}_1} & \frac{\partial f_1}{\partial \tilde{x}_2} \\ \frac{\partial f_2}{\partial \tilde{x}_1} & \frac{\partial f_2}{\partial \tilde{x}_2} \end{bmatrix}\right|_{\tilde{x}=(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{3f_3x_{1d}^2 + f_1}{m} & -\frac{d}{m} \end{bmatrix} \tag{6}$$

(Note that you may instead calculate the Jacobian for the original system, as long as you use the correct equilibrium point for this system.)
To find out whether A is Hurwitz or not, the eigenvalues of the matrix must be calculated

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{3f_3x_{1d}^2 + f_1}{m} & -\frac{d}{m} \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ \frac{3f_3x_{1d}^2 + f_1}{m} & \lambda + \frac{d}{m} \end{bmatrix} \tag{7}$$

$$
\begin{aligned}
|\lambda I - A| &= \lambda(\lambda + \frac{d}{m}) + \frac{3f_3x_{1d}^2 + f_1}{m} & (8)\\
&= \lambda^2 + \frac{d}{m}\lambda + \frac{3f_3x_{1d}^2 + f_1}{m} & (9)
\end{aligned}
$$

The eigenvalues of A are thus given as

$$
\lambda = \frac{1}{2}\left(-\frac{d}{m} \pm \sqrt{\left(\frac{d}{m}\right)^2 - \frac{4\left(3f_3x_{1d}^2 + f_1\right)}{m}}\right) \tag{10}
$$

Since $f_1, f_3, x_{1d}^2, m > 0$

$$
\frac{d}{m} > \sqrt{\left(\frac{d}{m}\right)^2 - \frac{4\left(3f_3x_{1d}^2 + f_1\right)}{m}} \tag{11}
$$

and the eigenvalues will always lie in the left half plane which means that A is Hurwitz. This means that $(0, 0)$ of (4)–(5) is locally asymptotically stable.

Problem 2 Remark: the answer to this exercise may vary a lot depending on the chosen parameters of Lyapunov function and the domain D. One possible solution set is presented below.

a The scalar system is given by $\dot{x} = -x^5$ with the only equilibrium point at the origin. Consider the Lyapunov function candidate $V(x) = px^2$ where $p > 0$. Then V is positive definite. Taking the derivative along the trajectory we have $\dot{V}(x) = 2px\dot{x} = 2px\left(-x^5\right) = -2px^6$ which is negative definite for all $x \in \mathbb{R}$. Hence the origin is asymptotically stable. Further V is radially unbounded which implies that the origin is globally asymptotically stable.

b The system is given by

$$
\begin{aligned}
\dot{x}_1 &= -x_1 - x_2^2\\
\dot{x}_2 &= 2x_1x_2 - x_2^3
\end{aligned}
$$

where it can be seen that the equilibrium point is given by

$$
(x_1^*, x_2^*) = (0, 0)
$$

A general quadratic Lyapunov function candidate is given by

$$
V(x) = \frac{1}{2}x^T P x, \quad P = P^T = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right]
$$

which is positive definite if and only if all the leading principal minors of P are positive, that is

$$
\begin{aligned}
p_{11} &> 0\\
p_{11}p_{22} - p_{12}^2 &> 0
\end{aligned}
$$

It follows that $p_{22} > 0$. The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$
\begin{aligned}
\dot{V}(x) &= \dot{x}^T P x\\
&= \begin{bmatrix} -x_1 - x_2^2 \\ 2x_2x_1 - x_2^3 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\\
&= -p_{11}x_1^2 - p_{11}x_1x_2^2 + 2p_{12}x_1^2x_2 - p_{12}x_2^3x_1 - p_{12}x_1x_2 - p_{12}x_2^3 + 2p_{22}x_1x_2^2 - p_{22}x_2^4\\
&= -p_{11}x_1^2 - p_{22}x_2^4 + x_1x_2^2(2p_{22} - p_{11}) + p_{12}(2x_1^2x_2 - x_2^3x_1 - x_1x_2 - x_2^3)
\end{aligned}
$$

In order to eliminate the undesirable terms (cross terms with indefinite signs), p_i is chosen according to

$$
\begin{aligned}
2p_{22} - p_{11} &= 0\\
p_{12} &= 0\\
&\Rightarrow p_{11} = 2p_{22}
\end{aligned}
$$

which fulfill the requirements imposed in order to guarantee V to be positive definite. The derivative of V with respect to time is

$$\begin{aligned}\dot{V}(x) &= -p_{11}x_1^2 - p_{22}x_2^4 \\ &< 0 \quad \forall x \in \mathbb{R}^2 \backslash \{0\}\end{aligned}$$

Since V is radially unbounded it follows that the origin is globally asymptotically stable.

c The system is given by

$$\begin{aligned}\dot{x}_1 &= -x_1 + 4x_2^2 \\ \dot{x}_2 &= -x_2^3\end{aligned}$$

where it can be seen that the equilibrium point is $(x_1^*, x_2^*) = (0, 0)$. A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T P x, \quad P = P^T$$

which is positive definite if and only if all the leading principal minors of P are positive

$$\begin{aligned}p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0\end{aligned}$$

(and it follows that $p_{22} > 0$). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x \\ &= \begin{bmatrix} -x_1 + 4x_2^2 \\ -x_2^3 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 + 4x_2^2 \\ -x_2^3 \end{bmatrix}^T \begin{bmatrix} p_{11}x_1 + p_{12}x_2 \\ p_{12}x_1 + p_{22}x_2 \end{bmatrix} \\ &= \left(-x_1 + 4x_2^2\right)(p_{11}x_1 + p_{12}x_2) - x_2^3(p_{12}x_1 + p_{22}x_2) \\ &= -p_{11}x_1^2 + 4p_{11}x_2^2x_1 - p_{12}x_1x_2 + 4p_{12}x_2^3 - p_{12}x_1x_2^3 - p_{22}x_2^4\end{aligned}$$

By choosing $p_{12} = 0$ we eliminate most of the unwanted terms, and we have

$$\dot{V}(x) = -p_{11}x_1^2 + 4p_{11}x_1x_2^2 - p_{22}x_2^4$$

Choosing $p_{22} > 4p_{11}$, we have

$$\begin{aligned}\dot{V}(x) &< -p_{11}\left(x_1^2 - 4x_1x_2^2 + 4x_2^4\right) \\ &= -p_{11}\left(x_1^2 - 2x_1\left(\sqrt{2}x_2\right)^2 + \left(\sqrt{2}x_2\right)^4\right) \\ &= -p_{11}\left(x_1 - 2x_2^2\right)^2 \le 0, \quad \forall x \in \mathbb{R}^2\end{aligned}$$

Hence, the equilibrium point is asymptotically stable. Since V is radially unbounded, the result is global.

Note: It is necessary to choose p_{22} strictly greater than $4p_{11}$ as the term $-p_{11}\left(x_1 - 2x_2^2\right)^2$ is not only zero in the origin. If we allow $p_{22} = 4p_{11}$, this would make $\dot{V}$ negative semidefinite, which does not show asymptotic convergence. This can also be seen by observing that $\dot{V}$ takes the form

$$\dot{V}(x) = -\begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_{11} & -2p_{11} \\ -2p_{11} & p_{22} \end{bmatrix}}_{Q} \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix}$$

which is negative definite if and only if all the leading principal minors of Q are positive, that is

$$p_{11} > 0, \quad p_{11}p_{22} - 4p_{11}^2 > 0$$

We see that the last condition requires that $p_{22} > 4p_{11}$.

d The system is given by

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\left(1 - x_1^2\right)\\ \dot{x}_2 &= -\left(x_1 + x_2\right)\left(1 - x_1^2\right)\end{aligned}$$

To be an equilibrium point of the system, both the following conditions must be satisfied

$$\begin{aligned}\dot{x}_1 = 0 &\implies x_2 = \frac{x_1}{1 - x_1^2}\ \forall x_1 \neq \pm 1\\ \dot{x}_2 = 0 &\implies x_1 = -x_2 \vee x_1 = \pm 1\end{aligned}$$

Thus, we need to solve

$$x_1 = -\frac{x_1}{1 - x_1^2}, \quad x_1 \neq \pm 1 \implies 0 = 2x_1 - x_1^3 \implies 0 = x_1(2 - x_1^2)$$

which gives the equilibrium points

$$(x_1^*, x_2^*) = (0, 0) \wedge (\pm\sqrt{2}, \mp\sqrt{2})$$

This implies that the origin cannot be globally asymptotically stable, since any trajectory starting in $(\pm\sqrt{2}, \mp\sqrt{2})$ will stay there. A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T P x, \quad P = P^T$$

which is positive definite if and only if all leading principal minors of P are positive, that is

$$\begin{aligned}p_{11} &> 0\\ p_{11}p_{22} - p_{12}^2 &> 0\end{aligned}$$

(and it follows that $p_{22} > 0$). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x\\ &= \begin{bmatrix} -x_1 + x_2\left(1 - x_1^2\right)\\ -\left(x_1 + x_2\right)\left(1 - x_1^2\right)\end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix}\\ &= \begin{bmatrix} -x_1 + x_2\left(1 - x_1^2\right)\\ -\left(x_1 + x_2\right)\left(1 - x_1^2\right)\end{bmatrix}^T \begin{bmatrix} p_{11}x_1 + p_{12}x_2\\ p_{12}x_1 + p_{22}x_2\end{bmatrix}\\ &= -p_{11}x_1^2 + p_{12}x_1x_2 + (1 - x_1^2)(p_{11}x_2x_1 + p_{12}(x_2^2 - x_1^2 - x_2x_1) - p_{22}x_1x_2 - p_{22}x_2^2)\end{aligned}$$

Choosing $p_{12} = 0$, the expression is reduced to

$$\dot{V}(x) = -p_{11}x_1^2 + (1 - x_1^2)(x_1x_2(p_{11} - p_{22}) - p_{22}x_2^2)$$

To eliminate the last cross term, we choose $p_{11} = p_{22}$. Taking $D = \{x \in \mathbb{R}^n \mid \|x_1\| \leq 1\}$, we have

$$\dot{V}(x) = -p_{11}x_1^2 - p_{22}(1 - x_1^2)x_2^2 < 0, \quad \forall x \in D\backslash\{0\}$$

It follows that the origin is asymptotically stable.

Problem 3

a The phase portrait can be seen in Figure 1.
The origin is not stable in the sense of Lyapunov. Given any $\varepsilon > 0$, no matter how small a δ we choose for the region of initial condition there always some initial conditions close to the x_1-axis which will exit the ε-region before converging to the origin.

b We need to show two conditions for asymptotic stability. First we need to show that for any given $\varepsilon > 0$ we could always find a $\delta > 0$ such that $\|x(0)\| < \delta \implies \|x(t)\| < \varepsilon$, $\forall t \geq 0$. Furthermore we need to show that when the initial condition is on some domain every trajectory converges to the origin.

The solution is given by $x(t) = e^{\alpha t}x(0)$. We then have $|x(t)| \leq |x(0)|$ for all $t \geq 0$ since $\alpha < 0$. Given any $\varepsilon > 0$, choose $\delta = \varepsilon$ to show that for all $|x(0)| < \delta = \varepsilon$ it follows that $|x(t)| < \varepsilon$, $\forall t \geq 0$. Thus the origin is stable. From the solution it is easy to see that for any δ we have

$$|x(0)| < \delta \implies \lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{\alpha t}x(0) = 0.$$

as $\alpha < 0$.

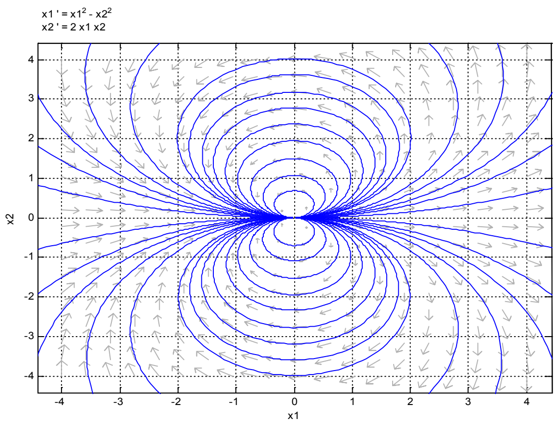

Figure 1: Phase portrait

Problem 4

a We can factor the expressions as follows:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - 2x_1x_2 + 2x_1 - x_1^2x_2\\ &= x_1^2(1-x_2) + 2x_1(1-x_2)\\ &= \left(x_1^2 + 2x_1\right)(1-x_2)\end{aligned}$$

$$\begin{aligned}\dot{x}_1 &= x_1^3 + 2x_1^2 + x_1^2x_2 + 2x_1x_2\\ &= x_1^2(x_1+x_2) + 2x_1(x_1+x_2)\\ &= \left(x_1^2 + 2x_1\right)(x_1+x_2)\end{aligned}$$

b To find the equilibrium point(s) of the system, we must solve

$$\begin{aligned}\dot{x}_1 = 0 &\implies x_1 = 0 \vee x_1 = -2 \vee x_2 = 1\\ \dot{x}_2 = 0 &\implies x_1 = 0 \vee x_1 = -2 \vee x_2 = -x_1\end{aligned}$$

The system has two equilibrium sets, i.e. the lines $x_1 = 0$ and $x_1 = -2$ (where x_2 can take any value). By combining the conditions $x_2 = 1$ and $x_2 = -x_1$, it is seen that the system also has an isolated equilibrium point at $(x_1, x_2) = (-1, 1)$.

c We introduce a change of variables

$$\begin{aligned}z_1 &= x_1 - x_1^* \implies \dot{z}_1 = \dot{x}_1\\ z_2 &= x_2 - x_2^* \implies \dot{z}_2 = \dot{x}_2\end{aligned}$$

where $(x_1^*, x_2^*) = (-1, 1)$. Using that $x_1 = z_1 - 1$ and $x_2 = z_2 + 1$, we have

$$\begin{aligned}\dot{z}_1 &= \left(x_1^2 + 2x_1\right)(1-x_2)\\ &= \left((z_1-1)^2 + 2z_1 - 2\right)(1 - z_2 - 1)\\ &= -z_2\left(z_1^2 - 2z_1 + 1 - 2z_1 - 2\right)\\ &= z_2\left(1 - z_1^2\right)\end{aligned}$$

$$\begin{aligned}\dot{z}_2 &= \left(x_1^2 + 2x_1\right)(x_1+x_2)\\ &= \left((z_1-1)^2 + 2z_1 - 2\right)(z_1 - 1 + z_2 + 1)\\ &= -(z_1+z_2)\left(1 - z_1^2\right)\end{aligned}$$

and the new system becomes

$$\dot{z} = \begin{bmatrix} z_2 \\ -(z_1+z_2)\end{bmatrix}\left(1 - z_1^2\right)$$

d Let our Lyapunov function candidate be in the form $V(z) = z^T P z$ where $z^T = \begin{bmatrix} z_1 & z_2 \end{bmatrix}$ and

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

is a positive definite, symmetric matrix. The derivative of $V(z)$ along the trajectories of the system is

$$\begin{aligned}
\dot{V}(z) &= \left(1 - z_1^2\right) \begin{bmatrix} z_2 \\ -\left(z_1 + z_2\right) \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\
&= \left(1 - z_1^2\right) \left(z_1 z_2 p_{11} + p_{12} z_2^2 - p_{12} z_1^2 - z_1 z_2 p_{22} - p_{12} z_1 z_2 - p_{22} z_2^2\right) \\
&= \left(1 - z_1^2\right) \left(-z_1 z_2 (p_{12} + p_{22} - p_{11}) - z_1^2 p_{12} - z_2^2 (p_{22} - p_{12})\right) \\
&= -\left(1 - z_1^2\right) \frac{1}{2} z^T Q z
\end{aligned}$$

where

$$Q = \begin{bmatrix} 2p_{12} & (-p_{11} + p_{12} + p_{22}) \\ (-p_{11} + p_{12} + p_{22}) & 2(p_{22} - p_{12}) \end{bmatrix}$$

We can thus see that $\dot{V}$ is negative definite on a domain $D = \left\{z \in \mathbb{R}^2 | 1 - z_1^2 > 0\right\}$ under the condition that $z^T Q z > 0 \forall z \neq 0$, i.e. that Q is positive definite. Recall that P also needs to be positive definite for V to be a Lyapunov function.

For P to be positive definite, we need:

$$p_{11} > 0, \quad p_{11} p_{22} - p_{12}^2 > 0 \quad \Longrightarrow p_{22} > 0.$$

For Q to be positive definite, we need:

$$p_{12} > 0, \quad 4p_{12}(p_{22} - p_{12}) - (p_{12} + p_{22} - p_{11})^2 > 0 \Longrightarrow p_{22} > p_{12}.$$

Hence, we need to choose $p_{11}, p_{22}, p_{12} > 0$ and $p_{22} > p_{12}$. From this point on, there are many options that satisfy the last two constraints. For convenience, we here choose $p_{22} = p_{11}$. Since $p_{22} > p_{12}$, we then ensure that $p_{11} p_{22} - p_{12}^2 > 0$. Moreover, the last condition then becomes

$$\begin{aligned}
4p_{12}(p_{22} - p_{12}) - (p_{12} + p_{22} - p_{22})^2 > 0 \\
4p_{12}(p_{22} - p_{12}) > p_{12}^2 \\
\Longrightarrow p_{22} > \frac{5}{4} p_{12}.
\end{aligned}$$

Thus, by using the Lyapunov function $V(z) = z^T P z$, where $p_{11} = p_{22}$, $p_{22} > \frac{5}{4} p_{12}$, and $p_{12} > 0$, we have shown that the equilibrium point is asymptotically stable.

Problem 5 The pendulum system is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -10 \sin x_1 - 2x_2
\end{aligned}$$

A general quadratic Lyapunov function candidate is given by

$$\begin{aligned}
V(x) &= \frac{1}{2} x^T P x + \gamma(1 - \cos x_1) \\
&= \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \gamma(1 - \cos x_1)
\end{aligned}$$

The quadratic form $\frac{1}{2} x^T P x$ is positive definite if and only if all the leading principal minors of P are positive

$$p_{11} > 0 \tag{12}$$

$$p_{11} p_{22} - p_{12}^2 > 0 \Longrightarrow p_{22} > 0 \tag{13}$$

The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}
\dot{V}(x) &= x^T P\dot{x} + \gamma\dot{x}_1 \sin x_1 \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \gamma\dot{x}_1 \sin x_1 \\
&= (x_1 p_{11} + x_2 p_{12})\dot{x}_1 + (x_1 p_{12} + x_2 p_{22})\dot{x}_2 + \gamma \sin x_1 \cdot \dot{x}_1 \\
&= [p_{11}x_1 + p_{12}x_2 + \gamma \sin x_1]\, x_2 + (p_{12}x_1 + p_{22}x_2)\,[-10\sin x_1 - 2x_2] \\
&= x_1 x_2 (p_{11} - 2p_{12}) - 10p_{12}x_1 \sin x_1 - x_2^2(2p_{22} - p_{12}) + x_2 \sin x_1 (\gamma - 10p_{22})
\end{aligned}$$

The elements p_{11}, p_{12}, and p_{22} should be selected such that $\dot{V}$ becomes negative definite. The signs of $x_2 \sin x_1$ and $x_1 x_2$ change based on the quadrant of x_1 and x_2 and therefore, these two elements should be eliminated. This happens by choosing $\gamma = 10p_{22}$ and $p_{11} = 2p_{12}$. Then, from (13), we must choose $p_{12} > 0$ and $p_{22} > \frac{1}{2}p_{12}$ to have a positive definite V. Moreover, this choice will ensure that the quadratic term in $\dot{V}$ is negative definite, and we have

$$\dot{V}(x) = -10p_{12}x_1 \sin x_1 - (2p_{22} - p_{12})x_2^2 < 0, \quad \forall\, x \in B_r \backslash \{0\},$$

where $B_r = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$, since $x_1 \sin x_1 > 0$ for all $-\pi < x_1 < \pi$. Hence, V is positive definite and $\dot{V}$ is negative definite in B_r. Thus, we can conclude that the origin is locally asymptotically stable.

Problem 6 The function $V(x) = 0.5\left(x_1^2 + x_2^2\right)$ is positive definite at all points which are not in the origin and

$$\begin{aligned}
\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1\left(x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right)\right) + x_2\left(-x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right)\right) \\
&= \alpha\left(x_1^2 + x_2^2\right)\left(\beta^2 - x_1^2 - x_2^2\right)
\end{aligned}$$

Defining

$$U \triangleq \left\{x \in \mathbb{R}^2 \mid \|x\|_2 \leq r, 0 < r < \beta\right\}$$

which is nonempty, it follows that V and $\dot{V}$ are positive definite in U. By Chetaev's theorem, the origin is unstable.

Problem 7

a It is seen that $V(x) > 0,\ \forall\, x \in \mathbb{R}^2\backslash\{0\}$ and we have $V(0) = 0$. The derivative of V along the trajectories of the system is

$$\begin{aligned}
\dot{V}(x) &= \frac{x_1}{\left(1 + x_1^2\right)^2}\dot{x}_1 + x_2\dot{x}_2 \\
&= -\frac{6x_1^2}{\left(1 + x_1^2\right)^4} + \frac{2x_1 x_2}{\left(1 + x_1^2\right)^2} - \frac{2x_1 x_2}{\left(1 + x_1^2\right)^2} - \frac{2x_2^2}{\left(1 + x_1^2\right)^2} \\
&= -\frac{6x_1^2}{\left(1 + x_1^2\right)^4} - \frac{2x_2^2}{\left(1 + x_1^2\right)^2} < 0, \quad \forall\, x \in \mathbb{R}^2\backslash\{0\}.
\end{aligned}$$

b Consider the line $x_2 = 0$, where $V(x) = \frac{x_1^2}{1+x_1^2}$. Along this line, it can be seen that $V(x) \to 1$ as $x_1 \to \infty$, which shows that V is not radially unbounded.

c Although $\dot{V}$ is negative definite on the whole state space, the Lyapunov function V is not radially unbounded. This is a necessary condition to conclude global asymptotic convergence. Hence, we can only conclude local asymptotic stability of the origin using this Lyapunov function candidate.

Problem 8 Consider the Lyapunov function candidate $V(x) = \frac{1}{2}x^2$. The derivative of V along the trajectories of the system is

$$\begin{aligned}
\dot{V}(x) &= \dot{x}x \\
&= (ax^p + g(x))x \\
&= ax^{p+1} + g(x)x
\end{aligned}$$

In a neighbourhood of the origin, we have

$$ax^{p+1} - k|x^{p+1}||x| \leq \dot{V}(x) \leq ax^{p+1} + k|x^{p+1}||x|$$

which follows from the bound on g. If $a < 0$ and p is odd, we then have

$$\dot{V}(x) \leq -(|a| - k|x|)\left|x^{p+1}\right| < 0, \quad \forall x \in D\backslash\{0\},$$

where $D = \{x \in \mathbb{R}\,|\,|x| < \left|\frac{a}{k}\right|, |g(x)| \leq k|x^{p+1}|\}$. Hence, since $\dot{V}$ is negative definite in a neighbourhood of the origin, we have shown that the origin is asymptotically stable (by Theorem 4.1). Similarly, if $a > 0$ (p is still odd), we have

$$\dot{V}(x) \geq (|a| - k|x|)\left|x^{p+1}\right| > 0, \quad \forall x \in D\backslash\{0\},$$

which shows that the origin is unstable since $\dot{V}$ is positive definite in a neighbourhood of the origin (by Theorem 4.3).
Finally, if $a \neq 0$ and p is even, the system dynamics satisfy

$$|x^p|(a - k|x|) \leq \dot{x} \leq |x^p|(a + k|x|)$$

We see that if $a > 0$ and $\frac{a}{k} > x > 0$, then we will have $\dot{x} > 0$. Similarly, if $a < 0$ and $\frac{a}{k} < x < 0$, then we will have $\dot{x} < 0$. Hence, depending on the sign of a, on one side of the origin, the trajectories diverge. Hence, the origin is unstable.

Problem 9 The system is given by

$$\begin{aligned}
\dot{x}_1 &= -x_1 + 2x_2 - x_2x_3 \\
\dot{x}_2 &= -x_2 \\
\dot{x}_3 &= -x_3 + x_1x_2 + x_2^2
\end{aligned}$$

By setting $\dot{x}_2 = 0$, we see that $x_2^* = 0$. Inserting this into $\dot{x}_1$ and $\dot{x}_3$ gives $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$ as the only possible equilibrium of the system.

A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^TPx, \quad P = P^T$$

For a positive definite P, we have that

$$\lambda_{\min}\|x\|^2 \leq x^TPx \leq \lambda_{\max}\|x\|^2 \quad \forall x \in \mathbb{R}^n$$

with $\lambda_{\min},\ \lambda_{\max} > 0$ being the smallest and largest eigenvalue of P, respectively.
As given in the hint, let

$$P = \begin{bmatrix} p_{11} & p_{12} & 0 \\ p_{12} & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}$$

For P to be positive definite, the coefficients will have to be chosen such that the principal minors of P are positive, that is

$$\begin{aligned}
p_{11} &> 0 && (14) \\
p_{11}p_{22} - p_{12}^2 &> 0 \quad \Rightarrow p_{22} > 0 && (15) \\
p_{33}\left(p_{11}p_{22} - p_{12}^2\right) &> 0 \quad \Rightarrow p_{33} > 0 && (16)
\end{aligned}$$

The derivative of V along the trajectories of the system is given by

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^TPx \\
&= p_{11}x_1\dot{x}_1 + p_{12}x_1\dot{x}_2 + p_{12}x_2\dot{x}_1 + p_{22}x_2\dot{x}_2 + p_{33}x_3\dot{x}_3 \\
&= -p_{11}x_1^2 + 2p_{11}x_1x_2 - p_{11}x_1x_2x_3 - p_{12}x_1x_2 - p_{12}x_1x_2 + 2p_{12}x_2^2 - p_{12}x_2^2x_3 \\
&\quad - p_{22}x_2^2 - p_{33}x_3^2 + p_{33}x_1x_2x_3 + p_{33}x_3x_2^2 \\
&= -p_{11}x_1^2 - (p_{22} - 2p_{12})x_2^2 - p_{33}x_3^2 + 2(p_{11} - p_{12})x_1x_2 + (p_{33} - p_{11})x_1x_2x_3
\end{aligned}$$

To cancel out the terms with indeterminate sign, choose $p_{12} = p_{11}$ and $p_{33} = p_{11}$. In order for $\dot{V}$ to be negative and for there to be a constant $k_3 > 0$ such that $\dot{V} \leq -k_3\|x\|_2^2$, choose p_{22} such that $p_{22} > 2p_{12}$. Since $p_{12} = p_{11} > 0$, the condition (15) holds for this choice of p_{22}, such that P is positive definite. $\dot{V}$ is then

$$\dot{V}(x) = -p_{11}x_1^2 - (p_{22} - 2p_{11})x_2^2 - p_{11}x_3^2 \leq -\min(p_{11},\ p_{22} - 2p_{11})\|x\|_2^2 \quad \forall x \in \mathbb{R}^3$$

The conditions of Theorem 4.10 are fulfilled with $a = 2$, $k_1 = \lambda_{\min}$, $k_2 = \lambda_{\max}$ and $k_3 = \min(p_{11},\ p_{22} - 2p_{11})$, all positive. Since they hold on all of $\mathbb{R}^3$, the origin is globally exponentially stable.

Problem 10 The system is given as

$$\begin{aligned} \dot{x}_1 &= -4x_1 + 3x_2 \\ \dot{x}_2 &= x_1 - 2x_2 - x_2^3 \end{aligned} \tag{17}$$

The Jacobian matrix of the considered system is

$$A = \left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} -4 & 3 \\ 1 & -2 - 3x_2^2 \end{bmatrix}$$

For simplicity, choose $P = I$. This gives

$$F(x) = A^T(x)P + PA(x) = \begin{bmatrix} -8 & 4 \\ 4 & -4 - 6x_2^2 \end{bmatrix}$$

The matrix F is negative definite iff all its Eigenvalues are negative. The eigenvalues are given by

$$\begin{aligned} \begin{vmatrix} \lambda + 8 & -4 \\ -4 & \lambda + 4 + 6x_2^2 \end{vmatrix} &= 0 \\ (\lambda + 8)(\lambda + 4 + 6x_2^2) - 16 &= 0 \\ \lambda^2 + 4\lambda + 6x_2^2\lambda + 8\lambda + 32 + 48x_2^2 - 16 &= 0 \\ \lambda^2 + (12 + 6x_2^2)\lambda + (16 + 48x_2^2) &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left(-(12 + 6x_2^2) \pm \sqrt{(12 + 6x_2^2)^2 - 4(16 + 48x_2^2)}\right) \\ &= -(6 + 3x_2^2) \pm \sqrt{(6 + 3x_2^2)^2 - (16 + 48x_2^2)} \end{aligned}$$

where

$$0 < \sqrt{(6 + 3x_2^2)^2 - (16 + 48x_2^2)} < (6 + 3x_2^2)$$

which means that

$$\lambda_{1,2} = -(6 + 3x_2^2) \pm \sqrt{(6 + 3x_2^2)^2 - (16 + 48x_2^2)} < 0 \quad \forall\ x_2 \in \mathbb{R}$$

and F is negative definite on $\mathbb{R}^2$. Moreover,

$$\begin{aligned} V(x) &= f^T(x)f(x) \\ &= (-4x_1 + 3x_2)^2 + (x_1 - 2x_2 - x_2^3)^2 \to \infty \text{ as } ||x|| \to \infty \end{aligned}$$

(if $\|x\|$ were to approach ∞ along the line $4x_1 = 3x_2$, the second term of V would still approach ∞, and vice versa). It can therefore be concluded that by Krasovskii's theorem, the origin of the system (17) is globally asymptotically stable.

Problem 11

a Given $f(x) = \int_0^1 \frac{\partial}{\partial x} f(\sigma x) x d\sigma$

$$\begin{aligned} x^T P f(x) + f^T(x) P x &= x^T P \int_0^1 \frac{\partial}{\partial x} f(\sigma x) x d\sigma + \left(\int_0^1 \frac{\partial}{\partial x} f(\sigma x) x d\sigma \right)^T P x \\ &= x^T P \int_0^1 \frac{\partial}{\partial x} f(\sigma x) x d\sigma + \int_0^1 x^T \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T d\sigma P x \\ &= x^T \left(P \int_0^1 \frac{\partial}{\partial x} f(\sigma x) d\sigma + \int_0^1 \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T d\sigma P \right) x \\ &= x^T \int_0^1 \left(P \frac{\partial}{\partial x} f(\sigma x) + \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T P \right) d\sigma x \end{aligned}$$

and by using $P \frac{\partial}{\partial x} f(\sigma x) + \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T P \leq -I$ the expression may be upper bounded by

$$x^T P f(x) + f^T(x) P x \leq x^T (-I) x = -x^T x = -\|x\|_2^2$$

b A Lyapunov function candidate is given as $V(x) = f^T(x) P f(x)$ where P is symmetric and positive definite. To show that $V(x)$ is positive definite, we need to show that $f(x) = 0$ if and only if $x = 0$. In other words we need to show that the origin is a unique equilibrium point. Suppose, to the contrary that there is a $p \neq 0$ such that $f(p) = 0$. Then

$$p^T p \leq -\left(p^T P f(p) + f^T(p) P p \right) = 0 \tag{18}$$

which is a contradiction since $p \neq 0$ (in order to satisfy the inequality (18), p needs to be equal to zero). Hence the origin is a unique equilibrium point.

To see that the function is radially unbounded, we first assume that V bounded such that

$$V(x) = |V(x)| \leq a, \quad \forall x \in \mathbb{R}^n$$

which means that

$$|V(x)| = |f(x)^\top P f(x)| \leq \lambda_{max}(P) ||f(x)||_2^2 \leq a.$$

We can conclude if V is bounded then $||f||_2$ needs to be bounded by

$$||f||_2 \leq \sqrt{\frac{a}{\lambda_{max}(P)}} = c$$

where $c > 0$. Now we see if there exist a $c > 0$ such that $||f||_2 \leq c$. Here,

$$x^\top P f(x) + f(x)^\top P x \leq -x^\top x$$

is used. First we see that

$$x^\top P f(x) + f(x)^\top P x = 2x^\top P f(x) \leq -x^\top x. \tag{19}$$

Next, we have

$$2x^\top P f(x) \leq 2||x||_2 \lambda_{max}(P) ||f(x)||_2 \leq ||x||_2 b, \tag{20}$$

where

$$b = 2\lambda_{max}(P) c.$$

It can be seen that (19) is equivalent to

$$2x^\top P f(x) \leq -||x||_2^2. \tag{21}$$

By summing (21) and (20) together we get

$$\begin{aligned}
4x^\top Pf(x) &\leq ||x||_2(b - ||x||_2) \\
\frac{4x^\top Pf(x)}{||x||_2} &\leq b - ||x||_2 \\
\lim_{x\to\infty} \frac{4x^\top Pf(x)}{||x||_2} &\leq b - \lim_{x\to\infty} ||x||_2 \\
0 &\leq -\infty,
\end{aligned}$$

which is clear contradiction. Thus as $||x||_2 \to \infty$, the magnitude of f must approach ∞, which shows that $V(x) \to \infty$ as $||x||_2 \to \infty$.

c We have shown that V is positive definite and radially unbounded. The time derivative of the function is found as

$$\begin{aligned}
\dot{V}(x) &= \dot{f}^T(x) Pf(x) + f^T(x) P\dot{f}(x) \\
&= \left(\frac{\partial}{\partial x} f(x)\dot{x}\right)^T Pf(x) + f^T(x) P\left(\frac{\partial}{\partial x} f(x)\dot{x}\right) \\
&= \left(\frac{\partial f(x)}{\partial x} f(x)\right)^T Pf(x) + f^T(x) P\left(\frac{\partial f(x)}{\partial x} f(x)\right) \\
&= f^T(x)\left(\frac{\partial f(x)}{\partial x}\right)^T Pf(x) + f^T(x) P\left(\frac{\partial f(x)}{\partial x} f(x)\right) \\
&= f^T(x)\left(P\frac{\partial f(x)}{\partial x} + \left(\frac{\partial f(x)}{\partial x}\right)^T P\right) f(x) \\
&\leq -f^T(x) f(x) \\
&= -\|f(x)\|_2^2
\end{aligned}$$

Since origin is a unique equilibrium point and all of the conditions hold globally, the origin is a globally asymptotically stable equilibrium point.