TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 6 (Solution)	Norwegian University of Science and Technology

Problem 1 The roll dynamics of an aircraft, represented by the scalar first-order ODE:

$$\dot{x} = a_p x + b_p u \tag{1}$$

a We want to find u on the form

$$u = a_x x + a_r r$$

such that the aircraft is forced to roll like the reference model, given by the dynamics

$$\dot{x}_m = a_m x_m + b_m r,$$

when $a_p = -0.8$, $b_p = 1.6$, $a_m = -2$ and $b_m = 2$.
To achieve this, we want x to track $x_m(t)$. We define the tracking error $e = x - x_m(t)$. The tracking error dynamics are then

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{x}_m \\ &= a_p x + b_p u - (a_m x_m + b_m r) \end{aligned}$$

In order to achieve tracking dynamics

$$\dot{e} = a_m e$$

we choose

$$u = \frac{a_m - a_p}{b_p} x + \frac{b_m}{b_p} r$$

which gives the ideal gains $a_x^* = \frac{a_m - a_p}{b_p}$ and $a_r^* = \frac{b_m}{b_p}$. Filling in the numerical values, this gives

$$\begin{aligned} u &= \frac{-2 - (-0.8)}{1.6} x + \frac{2}{1.6} r \\ &= -\frac{3}{4} x + \frac{5}{4} r \end{aligned}$$

b Initial values are chosen to be $x(0) = 0$ and $x_m(0) = 0$ for both the reference system and for the plant. The system is modelled as shown in Figure 1.

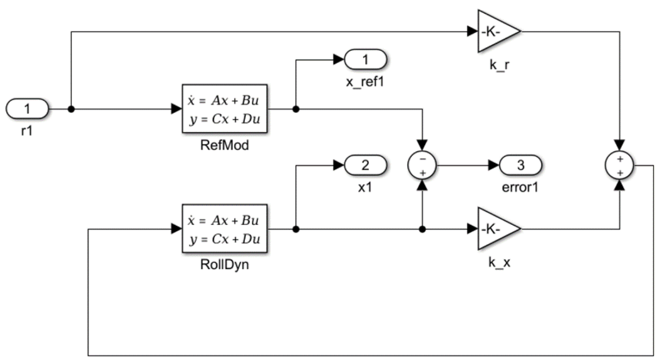
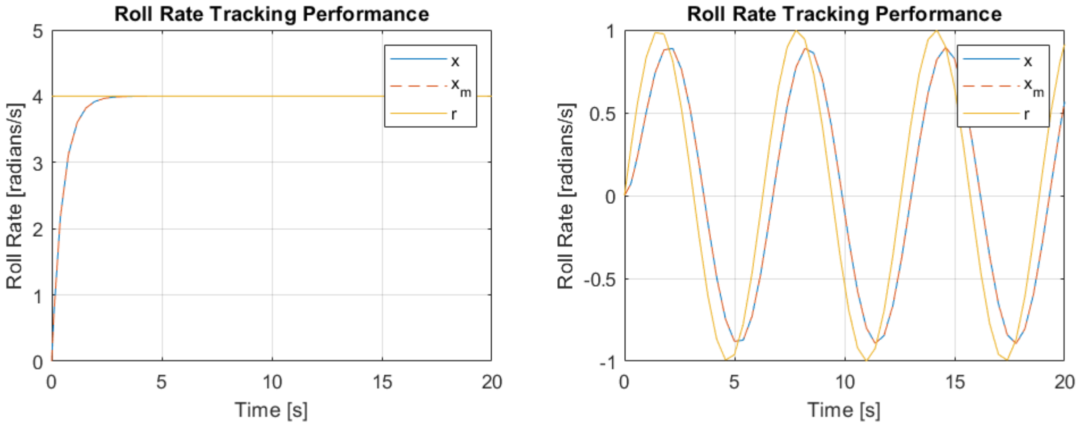


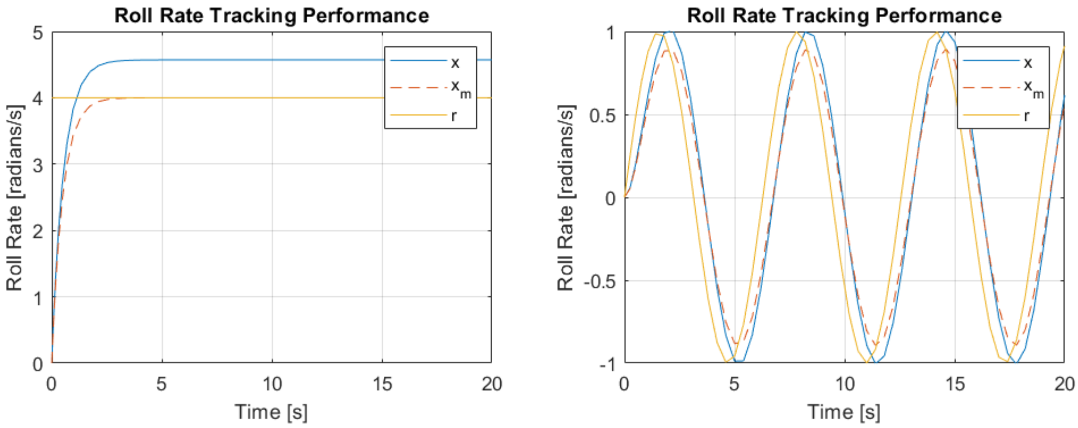
Figure 1: Block diagram of the closed-loop roll dynamics with fixed-gain model reference controller.

With $r = 4$, the tracking performance is shown in Figure 2a. With $r = \sin t$, the tracking performance is shown in Figure 2b. The reference model tracking is perfect with both of the reference signals after a stable transient. Had the initial value been different in the reference model from the plant, then the plant roll angle would have needed some time to converge to the value of the reference roll angle. Note that the model reference does not track the



(a) Tracking performance for $r(t) = 4$. **(b)** Tracking performance for $r(t) = \sin t$.

Figure 2: Simulations showing tracking error using fixed-gain model reference controller.



(a) Tracking performance for $r(t) = 4$. **(b)** Tracking performance for $r(t) = \sin t$.

Figure 3: Simulations showing tracking error using fixed-gain model reference controller with imperfect knowledge of system parameters.

commanded roll rate perfectly, as expected since the reference model is a first order dynamic system with corresponding amplitude and phase shift, as we know from linear frequency response analysis.

Simulation results with imperfect knowledge of parameters, i.e. $a_x^* = \frac{a_m-(-1)}{1.5}$ and $a_r^* = \frac{b_m}{1.5}$ are shown in Figure 3. Even though values close to the true system parameters are used in computing a_x^*, a_r^*, the resulting controller gives an offset compared to the reference model. (While it may appear from Figure 3b that x is now closer to the reference r, keep in mind that this is not the objective when using a model reference controller.)

c The only knowledge about a_p and b_p is that they are constant and $b_p > 0$. Using an adaptive controller in the form

$$u = \hat{a}_x x + \hat{a}_r r,$$

we can insert this controller into (1) to achieve the closed-loop system

$$\dot{x} = (a_p + \hat{a}_x b_p)\, x + \hat{a}_r b_p r$$

The tracking error is defined as $e = x - x_m$. Our error variables are

$$\begin{aligned} e &= x - x_m \\ \tilde{a}_x &= \hat{a}_x - a_x^* = \hat{a}_x - \frac{a_m - a_p}{b_p} \\ \tilde{a}_r &= \hat{a}_r - a_r^* = \hat{a}_r - \frac{b_m}{b_p} \end{aligned}$$

Differentiating e and substituting these relations then gives

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{x}_m \\
&= (a_p + \hat{a}_x b_p)\, x + \hat{a}_r b_p r - (a_m x_m + b_m r) \\
&= a_m x - a_m x + a_m(-x_m) + (a_p + \hat{a}_x b_p)\, x + (b_p \hat{a}_r - b_m)\, r \\
&= a_m (x - x_m) + (a_p - a_m + \hat{a}_x b_p)\, x + (b_p \hat{a}_r - b_m)\, r \\
&= a_m e + \left(-b_p \frac{a_m - a_p}{b_p} + \hat{a}_x b_p\right) x + \left(b_p \hat{a}_r - b_p \frac{b_m}{b_p}\right) r \\
&= a_m e + b_p (\tilde{a}_r r + \tilde{a}_x x)\,,
\end{aligned}$$

and knowing that

$$\begin{aligned}
a_x^* &= \frac{a_m - a_p}{b_p} \\
a_r^* &= \frac{b_m}{b_p},
\end{aligned}$$

are constant, the parameter error dynamics are

$$\begin{aligned}
\dot{\tilde{a}}_x &= \dot{\hat{a}}_x \\
\dot{\tilde{a}}_r &= \dot{\hat{a}}_r
\end{aligned}$$

which will be determined by the choice of adaptation law in the next step.

d We want to show that the tracking error e tend to zero. We are provided with a Lyapunov function candidate,

$$V(e, \tilde{a}_r, \tilde{a}_x) = \frac{e^2}{2} + \frac{|b_p|}{2\gamma_x}\tilde{a}_x^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2$$

with constant and positive scalar weights γ_x and γ_r. Since this function is C^1 and positive definite, we need to find a negative definite (or negitive semidefinite) $\dot{V}$. The time derivative of V is

$$\dot{V}(e, \tilde{a}_r, \tilde{a}_x) = e\dot{e} + \frac{|b_p|}{\gamma_x}\tilde{a}_x \dot{\tilde{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r \dot{\tilde{a}}_r$$

Substituting the expressions for the error dynamics $\dot{e}$, $\dot{\tilde{a}}_x$ and $\dot{\tilde{a}}_r$ into the previous equation, we get

$$\begin{aligned}
\dot{V}(e, \tilde{a}_r, \tilde{a}_x) &= e\left(a_m e + b_p (\tilde{a}_r r + \tilde{a}_x x)\right) + \frac{|b_p|}{\gamma_x}\tilde{a}_x \dot{\tilde{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r \dot{\tilde{a}}_r \\
&= a_m e^2 + e b_p (\tilde{a}_r r + \tilde{a}_x x) + \frac{|b_p|}{\gamma_x}\tilde{a}_x \dot{\tilde{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r \dot{\tilde{a}}_r \\
&= a_m e^2 + \tilde{a}_x |b_p| \left(\operatorname{sgn}(b_p)\, x e + \frac{\dot{\tilde{a}}_x}{\gamma_x}\right) + \tilde{a}_r |b_p| \left(\operatorname{sgn}(b_p)\, r e + \frac{\dot{\tilde{a}}_r}{\gamma_r}\right)
\end{aligned}$$

By choosing the adaptation laws

$$\begin{aligned}
\dot{\hat{a}}_x &= -\gamma_x x e \operatorname{sgn}(b_p) \\
\dot{\hat{a}}_r &= -\gamma_r r e \operatorname{sgn}(b_p)
\end{aligned}$$

we obtain

$$\dot{V}(e, \tilde{a}_x, \tilde{a}_r) = a_m e^2 = -W(e)$$

where W is positive semidefinite since $a_m < 0$. Since V is positive definite and radially unbounded in the states $e, \tilde{a}_x$, and $\tilde{a}_r$, and $\dot{V} \leq 0$ in the whole state space, the origin $(e, \tilde{a}_x, \tilde{a}_r) = (0, 0, 0)$ is globally uniformly stable, ensuring that the error state e and the estimation errors $\tilde{a}_x, \tilde{a}_r$ are uniformly bounded. Because the system is time varying, we cannot use LaSalle's invariance theorem to prove asymptotic convergence. Instead, we can use Barbalat's Lemma (Lemma 8.2): We want to show that $\dot{W}(t, e)$ is uniformly bounded, where

$$\begin{aligned}
\dot{W}(t, e) &= -2 a_m e \dot{e} \\
&= -2 a_m e \left(a_m e + b_p \tilde{a}_x x + b_p \tilde{a}_r r\right)
\end{aligned}$$

Using the triangle inequality, we have

$$|\dot{W}(t,e)| \quad \leq \quad |2a_m^2 e^2| + |2ea_m b_p \tilde{a}_x (e + x_m)| + |2a_m e b_p \tilde{a}_r r|$$

where we use that $x = e + x_m$. It only remains to show that x_m is bounded. Since the system $\dot{x}_m = a_m x_m + b_m r$ is linear in x_m, r, it is globally Lipschitz in x_m, r. The origin $x_m = 0$ is globally exponentially stable with $r = 0$. Hence, the system is input-to-state stable (ISS) with respect to r, by Lemma 4.6. Since r is uniformly bounded by assumption, this entails that x_m is uniformly bounded. Hence, for every $k > 0$ there exists $c > 0$ such that $\| \left[e, \tilde{a}^\top\right]^\top \| \leq k$ implies that $|\dot{W}(t,e)| \leq c$ for all $t \geq 0$, and all solutions approach the set $E = \{(e, \tilde{a}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, W(e) = 0\} = \{(e, \tilde{a}) \in \mathbb{R} \times \mathbb{R}^2, \,|\, e = 0\}$, with $\tilde{a} = [\tilde{a}_x, \tilde{a}_r]^\top$. That is, $e(t) \to 0$ as $t \to \infty$. Thus, we have shown that the adaptive controller ensures that x globally, asymptotically tracks the reference signal x_m, and the estimation errors $\tilde{a}$ are uniformly bounded.

e The system is modelled as shown in Figure 4. The simulations of the tracking performance

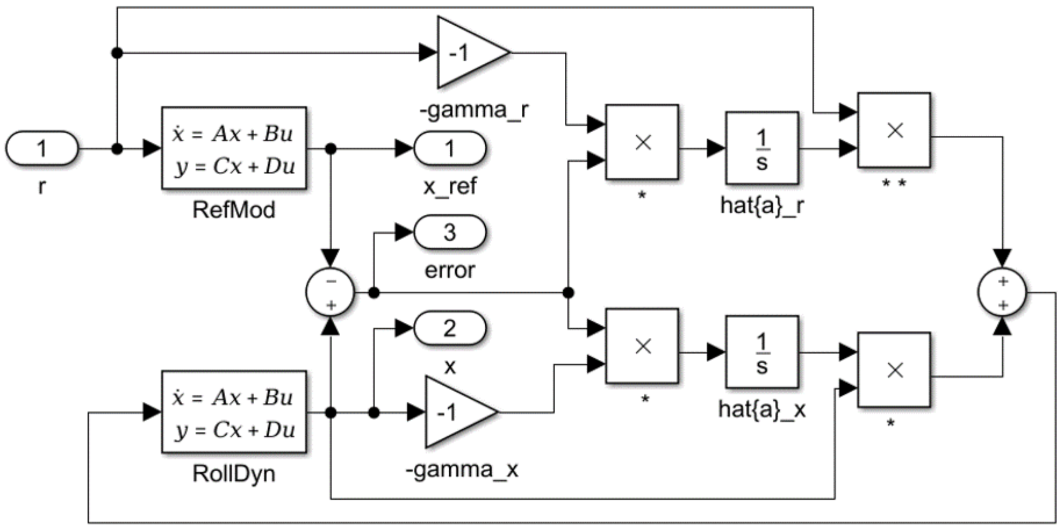
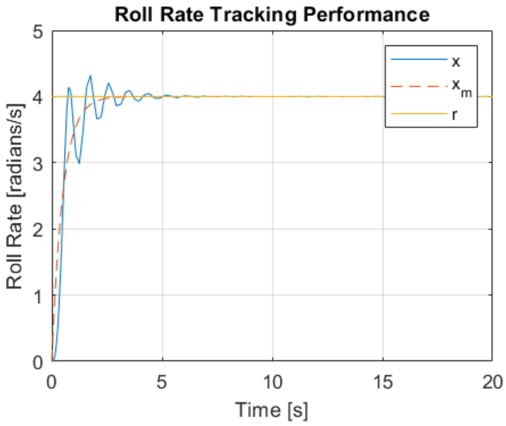


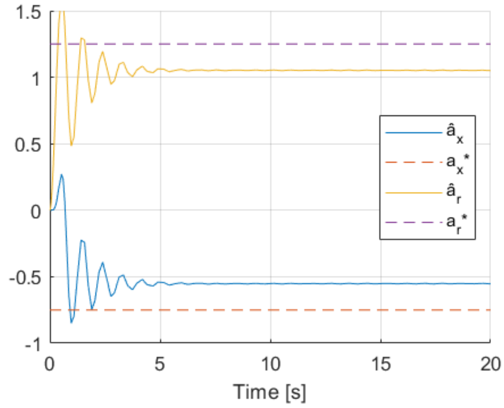
Figure 4: Model reference adaptive controller.

and parameter estimations are shown in Figures 5 and 6 for commanded roll rate $r = 4$, and in Figure 7 for commanded roll rate $r = \sin t$. In these figures, the initial values of the parameters $\hat{a}_x$, $\hat{a}_r$ are set to 0. Figure 5 shows that the model reference controller is able to track the commanded roll rate after the controller parameters have converged to stationary values, and that the tracking error for the adaptive controller converges to zero. In Figure 6, the tracking performance is improved by assuming some rough a priori knowledge of parameters, allowing to initiate the estimates $\hat{a}_x$, $\hat{a}_r$ to -1 and 1, respectively, which are closer to the ideal parameters values. In Figures 5 and 6 γ_x and γ_r are both set to 1. For both sets of initial estimate values the estimates do not converge to the ideal values, but converge instead to other constant values that give zero tracking error.

Figure 7 shows tracking performance and parameter estimates when the reference is set to $r(t) = \sin t$. Again we see that the trajectory converges to the trajectory of the reference model, resulting in zero tracking error. The initial value of both $\hat{a}_x$ and $\hat{a}_r$ is 0, and γ_x and γ_r were set to 10 to achieve faster parameter convergence. Unlike the case of a constant reference, we see in Figure 7b that the controller parameters now converge to the ideal values a_x^*, a_r^*. This is because the signal vector $[r(t)\ x(t)]^T$ is now persistently exciting (for further explanation, see pages 331-332 of Slotine & Li in the compendium).

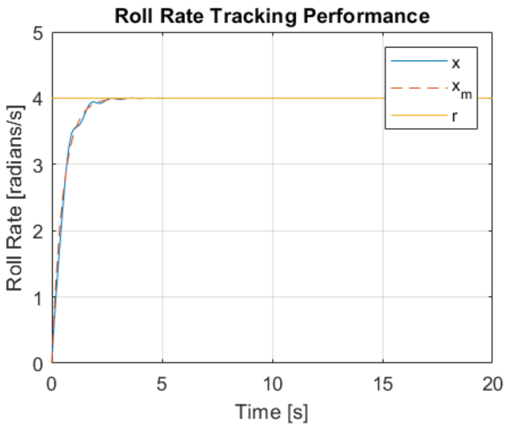


(a) Tracking performance.

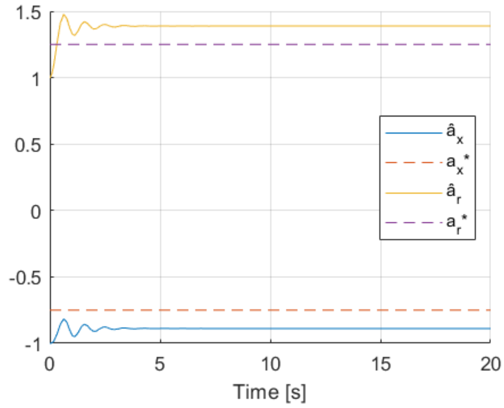


(b) Parameter estimates.

Figure 5: Simulation results using the model reference controller with adaptive gains for $r(t) = 4$, assuming no prior knowledge of parameters.

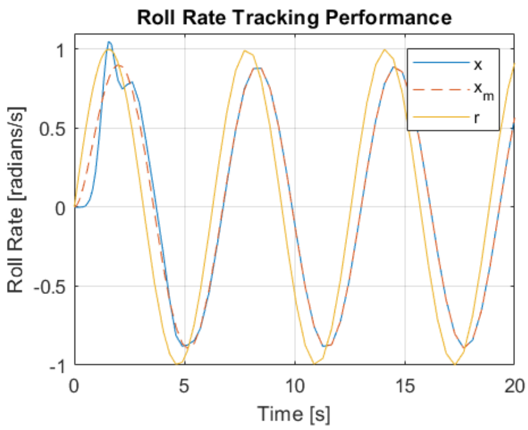


(a) Tracking performance.

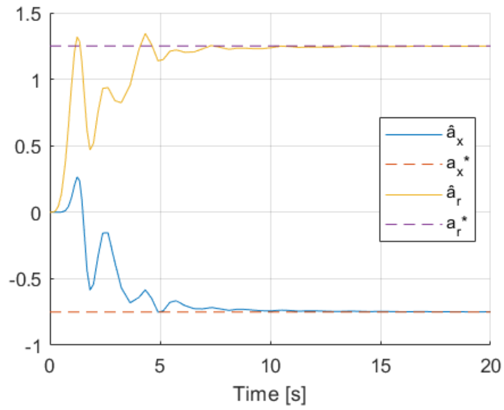


(b) Parameter estimates.

Figure 6: Simulation results using the model reference controller with adaptive gains for $r(t) = 4$, exploiting some a priori knowledge of the system parameters.



(a) Tracking performance.



(b) Parameter estimates.

Figure 7: Simulation results using the model reference controller with adaptive gains for $r(t) = \sin t$.

Problem 2 The roll dynamics are now represented by the nonlinear equation

$$\dot{x} = a_p x + c_p x^3 + b_p u$$

a A linear reference model for the plant could be

$$\dot{x}_m = a_m x + b_m r$$

The tracking error dynamics are then

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{x}_m \\ &= a_p x + c_p x^3 + b_p u - (a_m x + b_m r) \end{aligned}$$

In order to achieve the tracking error dynamics

$$\dot{e} = a_m e,$$

we choose

$$u = \frac{a_m - a_p}{b_p} x - \frac{c_p}{b_p} x^3 + \frac{b_m}{b_p} r$$

such that $f(x) = -x^3$ and the ideal control gains are

$$a_x^* = \frac{a_m - a_p}{b_p}, \; a_f^* = -\frac{c_p}{b_p}, \; a_r^* = \frac{b_m}{b_p}$$

b Since the parameters are unknown, we use estimates for the parameters in the control law found in the previous exercise:

$$u = \hat{a}_x x + \hat{a}_f x^3 + \hat{a}_r r$$

and the closed-loop system becomes

$$\dot{x} = (a_p + \hat{a}_x b_p)\, x + (c_p + \hat{a}_f b_p)\, x^3 + \hat{a}_r b_p r$$

The parameter errors are then

$$\begin{aligned} \tilde{a}_x &= \hat{a}_x - a_x^* = \hat{a}_x - \frac{a_m - a_p}{b_p} \\ \tilde{a}_f &= \hat{a}_f - a_f^* = \hat{a}_f + \frac{c_p}{b_p} \\ \tilde{a}_r &= \hat{a}_r - a_r^* = \hat{a}_r - \frac{b_m}{b_p} \end{aligned}$$

The tracking error dynamics are found by inserting the controller into the plant dynamics and subtracting the reference model

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{x}_m \\ &= (a_p + \hat{a}_x b_p)\, x + (c_p + \hat{a}_f b_p)\, x^3 + \hat{a}_r b_p r - (a_m x_m + b_m r) \\ &= a_m x - a_m x + a_m\,(-x_m) + (a_p + \hat{a}_x b_p)\, x + (c_p + \hat{a}_f b_p)\, x^3 + (\hat{a}_r b_p - b_m)\, r \\ &= a_m\,(x - x_m) + \left(-\frac{a_m - a_p}{b_p} b_p + \hat{a}_x b_p\right) x + \left(\frac{c_p}{b_p} b_p + \hat{a}_f b_p\right) x^3 + \left(\hat{a}_r b_p - \frac{b_m}{b_p} b_p\right) r \\ &= a_m e + b_p \tilde{a}_x x + b_p \tilde{a}_f x^3 + b_p \tilde{a}_r r \\ &= a_m e + b_p \left(\tilde{a}_x x + \tilde{a}_f x^3 + \tilde{a}_r r\right) \end{aligned}$$

The parameter error dynamics are

$$\begin{aligned} \dot{\tilde{a}}_x &= \dot{\hat{a}}_x \\ \dot{\tilde{a}}_f &= \dot{\hat{a}}_f \\ \dot{\tilde{a}}_r &= \dot{\hat{a}}_r \end{aligned}$$

which will be determined by the choice of adaptation law in the next step.

c We want to show that the tracking error converges asymptotically to zero by using the Lyapunov function candidate

$$V\left(e, \tilde{a}_x, \tilde{a}_r, \tilde{a}_f\right) = \frac{e^2}{2} + \frac{|b_p|}{2\gamma_x}\tilde{a}_x^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2 + \frac{|b_p|}{2\gamma_f}\tilde{a}_f^2$$

which is positive definite and radially unbounded. The time-derivative of V along the trajectories of the system is

$$\dot{V}\left(e, \tilde{a}_r, \tilde{a}_x, \tilde{a}_f\right) = e\dot{e} + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\hat{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\hat{a}}_r + \frac{|b_p|}{\gamma_f}\tilde{a}_f\dot{\hat{a}}_f$$

Substituting the expression for $\dot{e}$ gives

$$\begin{aligned}
\dot{V}\left(e, \tilde{a}_r, \tilde{a}_x\right) &= e\left(a_m e + b_p\left(\tilde{a}_x x + \tilde{a}_f x^3 + \tilde{a}_r r\right)\right) + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\hat{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\hat{a}}_r + \frac{|b_p|}{\gamma_f}\tilde{a}_f\dot{\hat{a}}_f \\
&= a_m e^2 + e b_p\left(\tilde{a}_x x + \tilde{a}_f x^3 + \tilde{a}_r r\right) + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\hat{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\hat{a}}_r + \frac{|b_p|}{\gamma_f}\tilde{a}_f\dot{\hat{a}}_f \\
&= a_m e^2 + \tilde{a}_x|b_p|\left(\operatorname{sgn}\left(b_p\right) x e + \frac{\dot{\hat{a}}_x}{\gamma_x}\right) + \tilde{a}_r|b_p|\left(\operatorname{sgn}\left(b_p\right) r e + \frac{\dot{\hat{a}}_r}{\gamma_r}\right) \\
&\quad + \tilde{a}_f|b_p|\left(\operatorname{sgn}\left(b_p\right) x^3 e + \frac{\dot{\hat{a}}_f}{\gamma_f}\right)
\end{aligned}$$

Selecting the following adaptive laws

$$\begin{aligned}
\dot{\hat{a}}_x &= -\gamma_x x e \operatorname{sgn}\left(b_p\right) \\
\dot{\hat{a}}_r &= -\gamma_r r e \operatorname{sgn}\left(b_p\right) \\
\dot{\hat{a}}_f &= -\gamma_f f\left(x\right) e \operatorname{sgn}\left(b_p\right),
\end{aligned}$$

we are left with $\dot{V} = a_m e^2 = -W(e)$, where W is positive semidefinite since $a_m < 0$. This ensures that the origin is globally uniformly stable due to radially unboundedness of V. Using the same arguments as in Exercise 1, we can show that

$$\dot{W} = -2a_m e\dot{e}$$

is uniformly bounded. This gives us the result that W tends to zero asymptotically, which again means that the error e tend to zero and that the adaptive controller ensures that x globally, asymptotically tracks the reference x_m.

d The system is implemented as shown in Figure 8. The simulations for commanded roll rate signals $r = 4$ and $r = \sin t$ are shown in Figures 9a and 9b. The initial values of the controller parameters are all 0. The γ-values are all chosen to be 1 in Figure 9a, and 10 in Figure 9b. In Figure 9a, the system converges to the reference model trajectory. We see that the performance is worse in Figure 9b, which is due to slow parameter convergence. This can be improved by increasing the γ-values further. Parameter convergence for the nonlinear system now depends upon the persistent excitation of the signal vector $[r(t),\, x(t),\, f(x(t))]^T$- The nonlinearity leads to more oscillatory behavior with a sinusoidal reference, and makes it more complicated to find a reference signal that ensures persistent excitation. (Those that are curious are encouraged to try $r(t) = \sin t + \sin\frac{t}{2}$ and increase simulation time.)

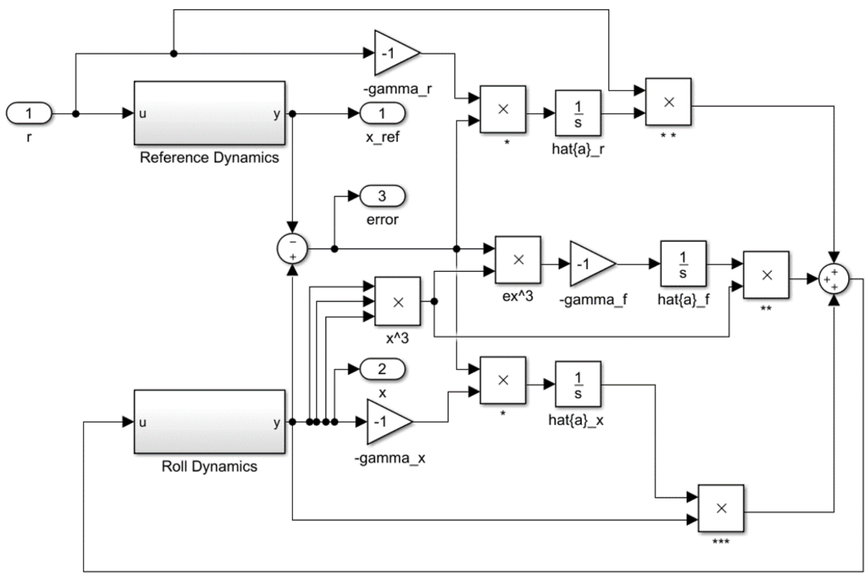


Figure 8: Model reference adaptive controller.

Problem 3 We are given the nonlinear MIMO system

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where

$$\begin{aligned} C_{11} &= -h\dot{\theta}_2 \\ C_{12} &= -h\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ C_{21} &= h\dot{\theta}_1 \\ C_{22} &= 0 \\ H_{11} &= a_1 + 2a_3\cos\theta_2 + 2a_4\sin\theta_2 \\ H_{12} &= H_{21} = a_2 + a_3\cos\theta_2 + a_4\sin\theta_2 \\ H_{22} &= a_2 \\ h &= a_3\sin\theta_2 - a_4\cos\theta_2 \end{aligned}$$

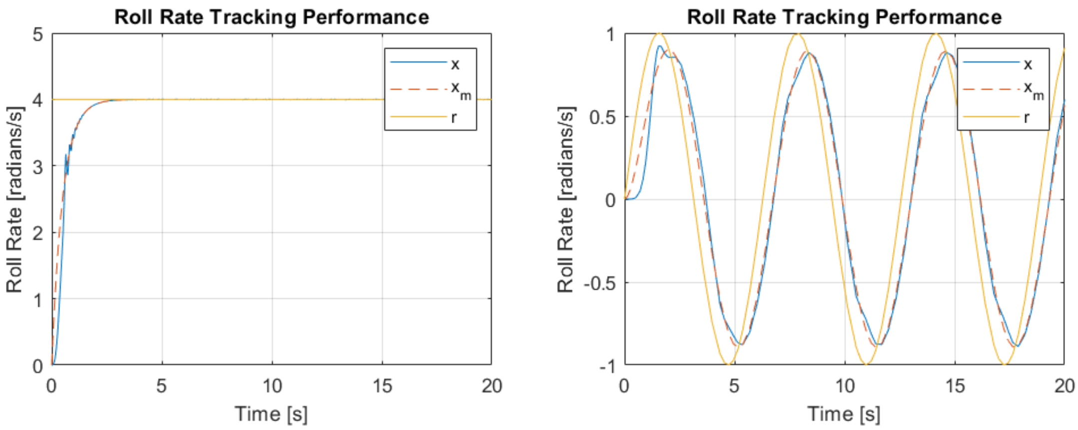
a First, we want to find a linear parametrization to be able to express the system as

$$Ya = u$$

We start by including the full expression for C as

$$\begin{aligned} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{\theta}_2 & -h\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ h\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 \\ H_{21}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ h\dot{\theta}_1^2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ H_{21}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - 2h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2^2 \\ H_{21}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

Then, substituting the expression for H, we have



(a) Tracking performance with $r = 4$. **(b)** Tracking performance with $r = \sin t$.

Figure 9: Tracking using controller with adaptive parameters for the nonlinear system.

$$\begin{bmatrix} (a_1 + 2a_3\cos\theta_2 + 2a_4\sin\theta_2)\,\ddot{\theta}_1 + (a_2 + a_3\cos\theta_2 + a_4\sin\theta_2)\,\ddot{\theta}_2 - 2h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2^2 \\ (a_2 + a_3\cos\theta_2 + a_4\sin\theta_2)\,\ddot{\theta}_1 + a_2\ddot{\theta}_2 + h\dot{\theta}_1^2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} (a_1 + 2a_3\cos\theta_2 + 2a_4\sin\theta_2)\,\ddot{\theta}_1 + (a_2 + a_3\cos\theta_2 + a_4\sin\theta_2)\,\ddot{\theta}_2\,[...] \\ -2\,(a_3\sin\theta_2 - a_4\cos\theta_2)\,\dot{\theta}_1\dot{\theta}_2 - (a_3\sin\theta_2 - a_4\cos\theta_2)\,\dot{\theta}_2^2 \\ \\ (a_2 + a_3\cos\theta_2 + a_4\sin\theta_2)\,\ddot{\theta}_1 + a_2\ddot{\theta}_2 + (a_3\sin\theta_2 - a_4\cos\theta_2)\,\dot{\theta}_1^2 \end{bmatrix} = \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1\ddot{\theta}_1 + a_2\ddot{\theta}_2 + a_3\left(\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos\theta_2 - \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\sin\theta_2\right)[...] \\ +a_4\left(\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\sin\theta_2 + \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\cos\theta_2\right) \\ a_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + a_3\left(\ddot{\theta}_1\cos\theta_2 + \dot{\theta}_1^2\sin\theta_2\right) + a_4\left(\ddot{\theta}_1\sin\theta_2 - \dot{\theta}_1^2\cos\theta_2\right) \end{bmatrix} = \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}$$

Choosing the vector of parameters to be

$$a = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}^T$$

and the input vector to be

$$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$$

we get the regression matrix

$$\begin{aligned} Y &= \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \end{bmatrix} \\ y_{11} &= \ddot{\theta}_1 \\ y_{12} &= \ddot{\theta}_2 \\ y_{13} &= \left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos\theta_2 - \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\sin\theta_2 \\ y_{14} &= \left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\sin\theta_2 + \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\cos\theta_2 \\ y_{21} &= 0 \\ y_{22} &= \ddot{\theta}_1 + \ddot{\theta}_2 \\ y_{23} &= \ddot{\theta}_1\cos\theta_2 + \dot{\theta}_1^2\sin\theta_2 \\ y_{24} &= \ddot{\theta}_1\sin\theta_2 - \dot{\theta}_1^2\cos\theta_2 \end{aligned}$$

Using this notation, we get the model expressed with linear parametrization.

b We want to derive the dynamics of the error variables s and e when the system is in closed-loop with the control law

$$u = H\ddot{\theta}_r + C\dot{\theta}_r - K_p\,(\theta - \theta_d) - K_d\left(\dot{\theta} - \dot{\theta}_d\right)$$

where

$$\dot{\theta}_r = \dot{\theta}_d - \Lambda\,(\theta - \theta_d)$$

We first find the expression for the dynamics of e, given by

$$\begin{aligned}
\dot{e} &= \dot{\theta} - \dot{\theta}_d \\
&= \dot{\theta} - \left(\dot{\theta}_r + \Lambda(\theta - \theta_d)\right) \\
&= -\Lambda e + s
\end{aligned} \tag{2}$$

The dynamic of s are then found by substituting for the control law into the original system as

$$\begin{aligned}
H\dot{s} &= H\ddot{\theta} - H\ddot{\theta}_r \\
&= -C\dot{\theta} + u - H\ddot{\theta}_r \\
&= -C\dot{\theta} + H\ddot{\theta}_r + C\dot{\theta}_r - K_p e - K_d \dot{e} - H\ddot{\theta}_r
\end{aligned}$$

Substituting $\dot{\theta} = s + \dot{\theta}_r$ and $K_p = K_d \Lambda$, we get

$$\begin{aligned}
H\dot{s} &= -C\left(s + \dot{\theta}_r\right) + H\ddot{\theta}_r + C\dot{\theta}_r - K_p e - K_d \dot{e} - H\ddot{\theta}_r \\
&= -Cs - K_d \Lambda e - K_d \dot{e} \\
&= -Cs - K_d (\Lambda e + \dot{e}) \\
&= -Cs - K_d s
\end{aligned} \tag{3}$$

c We are to show that the closed-loop system is globally asymptotically stable. The closed-loop error dynamics (2) and (3) can be recognized as a cascaded system, see Figure 10, where Σ_2 is given by

$$H\dot{s} = -Cs - K_d s$$

with s as an output, and Σ_1 is given by

$$\dot{e} = -\Lambda e + s$$

with s as an input, and e as an output. Lemma 4.7 in Khalil states that if Σ_2 is GUAS and Σ_1 is ISS, then the origin of the total cascaded system is GUAS. We start by investigating whether Σ_1 is ISS:
Alternative 1:

$$\dot{e} = -\Lambda e + s = f(e, s)$$

where $f(e, s)$ is C^1, and it is globally Lipschitz in (e, s) since it is linear. The unforced system

$$\dot{e} = -\Lambda e$$

has a globally exponentially stable origin. By Lemma 4.6 in Khalil, the system Σ_1 is ISS with s as input.

Alternative 2:
Using the Lyapunov function candidate

$$V = \frac{1}{2} e^T e,$$

which is C^1 and satisfies

$$\alpha_1(||e||) \leq V(e) \leq \alpha_2(||e||)$$

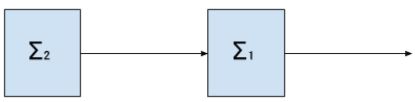


Figure 10: Cascaded system

with $\alpha_1(||e||) = \alpha_2(||e||) = \frac{1}{2}\|e\|^2$. Moreover,

$$\begin{aligned}
\dot{V} &= e^T\dot{e} = -e^T\Lambda e + e^T s \\
&\leq -\lambda_{\max}(\Lambda)\|e\|^2 + \|e\|\|s\| \\
&= -\left(\lambda_{\max}(\Lambda) - \theta\right)\|e\|^2 - \theta\|e\|^2 + \|e\|\|s\|, \quad 0 < \theta < \lambda_{\max}(\Lambda)
\end{aligned}$$

where $\lambda_{\max}(\Lambda)$ is the largest eigenvalue of the matrix Λ. Thus,

$$\dot{V} \leq -\left(\lambda_{\max}(\Lambda) - \theta\right)\|e\|^2 < 0 \quad \forall\|e\| \geq \frac{|s\|}{\theta} > 0$$

By Theorem 4.19 in Khalil the system is ISS with $\gamma(r) = \frac{r}{\theta}$. Next, we investigate whether Σ_2 is GUAS. We use the provided Lyapunov function candidate

$$V(s, \tilde{a}) = \frac{1}{2}\left[s^T H s\right]$$

This is continuous and positive definite. Moreover,

$$\begin{aligned}
\dot{V} &= s^T H\dot{s} + \frac{1}{2}s^T\dot{H}s \\
&= s^T\left(-Cs - K_d s\right) + \frac{1}{2}s^T\dot{H}s \\
&= -s^T K_d s - s^T C s + \frac{1}{2}s^T\dot{H}s
\end{aligned}$$

Using the fact that $\dot{H} - 2C$ is skew-symmetric, $z^T\left(\dot{H} - 2C\right)z = 0,\ \forall z \in \mathbb{R}^2$, and thus

$$\dot{V} = -s^T K_d s < 0, \forall s \in \mathbb{R}^2 \backslash \{0\}.$$

By Theorem 4.9 in Khalil, the system Σ_2 is GUAS, which together with Σ_1 being ISS entails that the full (cascaded) system is GUAS.

d The plant parameters a_i are unknown, but assumed to be constant. We adjust the control law by replacing the parameters with their estimates:

$$u = \hat{H}(\theta)\ddot{\theta}_r + \hat{C}\left(\theta, \dot{\theta}\right)\dot{\theta}_r - K_p(\theta - \theta_d) - K_d\left(\dot{\theta} - \dot{\theta}_d\right)$$

Since

$$\hat{H}(\theta)\ddot{\theta}_r + \hat{C}\left(\theta, \dot{\theta}\right)\dot{\theta}_r = Y\left(\theta, \dot{\theta}, \theta_r, \dot{\theta}_r, \ddot{\theta}_r\right)\hat{a}$$

the control law can be written as

$$u = Y\hat{a} - K_p e - K_d\dot{e}$$

where the error variables are

$$\begin{aligned}
e &= \theta - \theta_d \\
s &= \dot{\theta} - \dot{\theta}_r \\
\tilde{a} &= \hat{a} - a
\end{aligned}$$

The closed-loop error dynamics are then given by

$$\begin{aligned}
H\dot{s} &= H\ddot{\theta} - H\ddot{\theta}_r \\
&= -C\dot{\theta} + u - H\ddot{\theta}_r \\
&= -C\dot{\theta} + Y\hat{a} - K_p e - K_d\dot{e} - H\ddot{\theta}_r
\end{aligned}$$

Substituting $\dot{\theta} = s + \dot{\theta}_r$ and $K_p = K_d\Lambda$, we get

$$\begin{aligned}
H\dot{s} &= -C\left(s + \dot{\theta}_r\right) + Y\hat{a} - K_d(\Lambda e + \dot{e}) - H\ddot{\theta}_r \\
&= -Cs - \left(H\ddot{\theta}_r + C\dot{\theta}_r\right) + Y\hat{a} - K_d s \\
&= -Cs - K_d s + Y(\hat{a} - a) \\
&= -(C + K_d)s + Y\tilde{a}
\end{aligned}$$

The closed-loop error dynamics are finally

$$
\begin{aligned}
\dot{e} &= -\Lambda e + s &(4)\\
H\dot{s} &= -Cs - K_d s + Y\tilde{a} &(5)\\
\dot{\tilde{a}} &= \dot{\hat{a}} &(6)
\end{aligned}
$$

which can be recognized as a cascaded system, where Σ_2 is given by

$$
\begin{aligned}
H\dot{s} &= -\left(C + K_d\right)s + Y\tilde{a}\\
\dot{\tilde{a}} &= \dot{\hat{a}}
\end{aligned}
$$

and Σ_1 is given by

$$\dot{e} = -\Lambda e + s$$

which we already know is ISS. To prove that $e(t) \to 0$, it is enough to show that $s(t) \to 0$ as $t \to \infty$. This is also the case if we can show that Σ_2 is GUAS, which is an even stronger property. We consider the Lyapunov function candidate

$$V\left(s, \tilde{a}\right) = \frac{1}{2}\left[s^T H s + \tilde{a}^T \Gamma^{-1} \tilde{a}\right]$$

which is continuous and positive definite, where $\Gamma = \Gamma^T > 0$ is constant. The time-derivative of V along the system is

$$
\begin{aligned}
\dot{V} &= s^T\left(-\left(C + K_d\right)s + Y\tilde{a}\right) + \frac{1}{2}s^T\dot{H}s + \tilde{a}^T\Gamma^{-1}\dot{\tilde{a}}\\
&= s^T\left(\frac{1}{2}\dot{H} - C\right)s - s^T K_d s + \tilde{a}^T\left(Y^T s + \Gamma^{-1}\dot{\hat{a}}\right)
\end{aligned}
$$

where $s^T\left(\frac{1}{2}\dot{H} - C\right)s = 0$ due to skew-symmetry. Using that $s^T Y\tilde{a} = \tilde{a}^T Y^T s$ since $s^T Y\tilde{a}$ is a scalar and by choosing

$$\dot{\hat{a}} = -\Gamma Y^T s$$

we obtain

$$\dot{V} = -s^T K_d s = -W(s) \leq 0$$

Since $\dot{V}$ is only negative semidefinite, and the system is time-varying through the matrix Y, we must use either Barbalat's lemma or the LaSalle-Yoshizawa Theorem to prove convergence. However, we can conclude that the origin $(s, \tilde{a}) = (0, 0)$ is globally uniformly stable due to radially unboundedness of V, which implies that the estimation error is uniformly bounded. This also implies that e is bounded by the ISS property. Considering Lemma 8.2, we have that

$$
\begin{aligned}
\dot{W}(t, s) &= 2s^T K_d \dot{s}\\
&= 2s^T K_d\left(-H^{-1}\left(C + K_d\right)s + H^{-1}Y\tilde{a}\right)\\
&= -2s^T K_d H^{-1}\left(C + K_d\right)s + 2s^T K_d H^{-1} Y\tilde{a}
\end{aligned}
$$

By the triangle inequality,

$$|\dot{W}| \leq |2s^T K_d H^{-1}\left(C + K_d\right)s| + |2s^T K_d H^{-1} Y\tilde{a}|$$

which is bounded by some constant $c > 0$, for all time $t \geq 0$, if H, C, and Y are uniformly bounded. Since $\cos\theta_2$ and $\sin\theta_2$ are trivially bounded, it follows that H is bounded. Moreover, $\dot{\theta} = -\Lambda e + s + \dot{\theta}_d$ is bounded since $\dot{\theta}_d$ is bounded by assumption, and it follows that C and $\dot{\theta}_r = \dot{\theta} - s$ are uniformly bounded. Since $\ddot{\theta}_d$ is also bounded, $\ddot{\theta}_r = \ddot{\theta}_d - \Lambda(-\Lambda e + s)$ is bounded. It follows that Y is uniformly bounded and we can conclude by Barbalat's lemma that $W(t) \to 0$ as $t \to \infty$. This implies that $s(t) \to 0$, since $W(s) = 0 \Leftrightarrow s = 0$. Moreover, since Σ_1 is ISS with respect to s, we have $e(t) \to 0$ as well.

e With $V(e,s,\tilde{a}) = \frac{1}{2}e^Te + \frac{\beta}{2}\left(s^THs + \tilde{a}^T\Gamma^{-1}\tilde{a}\right)$, we compute

$$\begin{aligned}\dot{V} =& e^T\dot{e} + \beta\left(s^T\left(-\left(C+K_d\right)s + Y\tilde{a}\right) + \frac{1}{2}s^T\dot{H}s + \tilde{a}^T\Gamma^{-1}\dot{\hat{a}}\right)\\ =& -e^T\Lambda e + e^Ts - \beta s^TK_ds + \beta s^T\left(\frac{1}{2}\dot{H} - C\right)s + \beta\tilde{a}^T\left(Y^Ts + \Gamma^{-1}\dot{\hat{a}}\right)\end{aligned}$$

By skew-symmetry, $s^T\left(\frac{1}{2}\dot{H} - C\right)s = 0$. Choosing the adaptation law

$$\dot{\hat{a}} = -\Gamma Y^Ts$$

which is the same as we found previously, we are left with

$$\dot{V} = -e^T\Lambda e + e^Ts - \beta s^TK_ds$$

Recall that for a positive definite matrix $P \in \mathbb{R}^{n\times n}$, we have that $\|x\|_2^2\lambda_{\min}(P) \leq x^TPx \leq \|x\|_2^2\lambda_{\max}(P)$ for all $x \in \mathbb{R}^n$, where $\lambda_{\min}, \lambda_{\max}$ are the smallest and largest eigenvalue of P. Thus, since Λ and K_d are positive definite matrices,

$$\begin{aligned}\dot{V} &\leq -\|e\|_2^2\lambda_{\max}(\Lambda) + e^Ts - \beta\|s\|_2^2\lambda_{\max}(K_d)\\ &= -\begin{bmatrix}e^T & s^T\end{bmatrix}\begin{bmatrix}\lambda_{\max}(\Lambda) & -\frac{1}{2}\\ -\frac{1}{2} & \beta\lambda_{\max}(K_d)\end{bmatrix}\begin{bmatrix}e\\ s\end{bmatrix}\\ &= -e_s^TQe_s = -W(e_s)\end{aligned}$$

with $e_s = \left[e^T, s^T\right]^T$. We know that W is positive semidefinite if the principal minors of Q are positive, that is

$$\begin{aligned}\lambda_{\max}(\Lambda) &> 0\\ \beta\lambda_{\max}(\Lambda)\lambda_{\max}(K_d) - \frac{1}{4} &> 0\end{aligned}$$

Thus, for any $\beta > \frac{1}{4\lambda_{\max}(\Lambda)\lambda_{\max}(K_d)}$, we have $\dot{V} \leq 0$, which entails that the origin $(e_s, \tilde{a}) = (0,0)$ is globally uniformly stable. It follows that the estimation errors are bounded. To show that $e(t) \to 0$ as $t \to \infty$, we must use Barbalat's lemma or the LaSalle-Yoshizawa theorem since the system is time-varying. We have

$$\dot{W}(t, e_s) = 2e_s^TQ\dot{e}_s$$

due to symmetry of Q. Inserting for $\dot{e}$ and $\dot{s}$, we see that $|\dot{W}|$ can be bounded similarly as before, i.e. it relies on boundedness of Y, H, and C. It thus follows from the same arguments as provided in the previous problem that $W(t) \to 0$ as $t \to \infty$, and since $W(0) = 0 \Leftrightarrow (e,s) = (0,0)$, the error states e, s converge globally asymptotically to the origin. Note that the main difference from before is that we have shown that e is bounded and asymptotically converging without needing the ISS property, since we investigate the whole error system in this case, and this does not change the adaptation law.

Problem 4 The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2 + a + \left(x_1 - a^{1/3}\right)^3\\ \dot{x}_2 &= x_1 + u\end{aligned}$$

where the first system equation has x_2 as virtual input. Take $V_1 = \frac{1}{2}x_1^2$, which is positive definite and radially unbounded, as a Lyapunov function candidate for this first part of the system. Differentiating V_1 gives

$$\dot{V}_1 = x_1\dot{x}_1 = x_1\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$

Choose

$$x_2 = -a - x_1 - \left(x_1 - a^{1/3}\right)^3 = \varphi(x_1) \tag{7}$$

where $\varphi(x_1)$ is chosen such that $\varphi(0) = 0$. Inserting this for x_2 gives $\dot{x}_1 = -x_1$ and $\dot{V}_1 = x_1\dot{x}_1 = -x_1^2$, which is negative definite. Hence $x_2 = \varphi(x_1)$ as chosen above is a stabilizing input for the first variable.
Let z now be the error between x_2 and its desired value $\varphi(x_1)$, such that

$$z = x_2 - \varphi(x_1) = x_2 - \left(-a - x_1 - \left(x_1 - a^{1/3}\right)^3\right) = x_1 + x_2 + a + \left(x_1 - a^{1/3}\right)^3$$

and

$$\begin{aligned}
\dot{z} &= \dot{x}_1 + \dot{x}_2 + 3\left(x_1 - a^{1/3}\right)^2 \dot{x}_1 \\
&= \left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) + (x_1 + u) + 3\left(x_1 - a^{1/3}\right)^2 \left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\
&= x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)
\end{aligned}$$

Consequently x_2 is now given as

$$x_2 = -a - x_1 - \left(x_1 - a^{1/3}\right)^3 + z$$

which gives

$$\dot{V}_1 = x_1\dot{x}_1 = x_1\left(-x_1 + z\right) = -x_1^2 + x_1 z$$

We may choose a Lyapunov function candidate for the overall system as $V_c = V_1 + \frac{1}{2}z^2$. We then have

$$\begin{aligned}
\dot{V}_c &= \dot{V}_1 + z\dot{z} \\
&= -x_1^2 + x_1 z + z\left\{x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\right\} \\
&= -x_1^2 + z\left\{2x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\right\}
\end{aligned}$$

Choose u to be

$$\begin{aligned}
u &= -z - 2x_1 - \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\
&= -\overbrace{\left(x_1 + x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)}^{z} - 2x_1 - \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\
&= -3x_1 - \left(2 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)
\end{aligned}$$

This gives

$$\dot{V}_c = -x_1^2 - z^2$$

meaning that $\dot{V}_c$ is negative definite. We already know that V_c is positive definite and radially unbounded. Hence, the origin in coordinates (x_1, z) is globally asymptotically stable (GAS). Since $\varphi(0) = 0$, the origin in new coordinates (x_1, z) coincides with the origin in original coordinates (x_1, x_2), which consequently is also GAS.

Alternative solution:
The system is in the form of (14.53)-(14.54) in Khalil (p. 594) with

$$\begin{aligned}
f &= a + \left(x_1 - a^{1/3}\right)^3 \\
g &= 1 \\
f_a &= x_1 \\
g_a &= 1
\end{aligned}$$

Take

$$\begin{aligned}
\phi\left(x_1\right) &= -a - \left(x_1 - a^{1/3}\right)^3 - x_1 \\
V &= \frac{1}{2}x_1^2
\end{aligned}$$

and use (14.56) in Khalil.

Problem 5 Consider $\dot{x}_1 = x_1x_2 + x_1^2$ with x_2 as a virtual input. Choose a Lyapunov function candidate $V_1(x) = \frac{1}{2}x_1^2$ and calculate

$$\dot{V}_1 = x_1\dot{x}_1 = x_1\left(x_1x_2 + x_1^2\right)$$

We can enforce $\dot{V}_1 = -x_1^4$ which is negative definite, by choosing the input $x_2 = -x_1 - x_1^2 = \varphi(x_1)$ (actually, any choice $x_2 = -x_1 - x_1^{2k}$, $k = 1, 2, 3, \ldots$ will be possible, to get a negative definite $\dot{V}_1$, but for simplification we choose k=1). Notice that $\varphi(0) = 0$, such that the origin in new coordinates (x_1, z) will coincide with the origin in original coordinates (x_1, x_2).
Augment the input with z, such that we have $x_2 = -x_1 - x_1^2 + z$ (i.e. $z = x_2 - \varphi(x_1) = x_2 + x_1 + x_1^2$), which gives

$$\dot{V}_1 = x_1\left(x_1x_2 + x_1^2\right) = x_1\left(x_1\left(-x_1 - x_1^2 + z\right) + x_1^2\right) = -x_1^4 + x_1^2 z$$

Now choose a Lyapunov function candidate for the complete system $V_c = V_1 + \frac{1}{2}z^2$, which is positive definite and radially unbounded. Then

$$\begin{aligned}
\dot{V}_c &= \dot{V}_1 + z\dot{z} \\
&= -x_1^4 + x_1 z + z\left(u + (2x_1 + 1)\left(x_1x_2 + x_1^2\right)\right) \\
&= -x_1^4 + z\left(x_1 + u + (2x_1 + 1)\left(x_1x_2 + x_1^2\right)\right)
\end{aligned}$$

We can enforce $\dot{V}_c = -x_1^2 - z^2$ (such that $\dot{V}_c$ is negative definite), by choosing

$$u = -x_1 - (2x_1 + 1)\left(x_1x_2 + x_1^2\right) - z$$

By inserting for z, we get the expression for the stabilizing input

$$\begin{aligned}
u &= -x_1 - (2x_1 + 1)\left(x_1x_2 + x_1^2\right) - x_2 - x_1 - x_1^2 \\
&= -\left(2x_1 + 1\right)\left(x_1x_2 + x_1^2\right) - x_2 - 2x_1 - x_1^2
\end{aligned}$$

Since $V_c(x_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_c(x_1, z)$ is negative definite, u asymptotically stabilizes x_1 and z at the origin. Since $z = 0$ means that $x_2 = \varphi(x_1)$, and $\varphi(0) = 0$, if $(x_1, z) = (0, 0)$ is asymptotically stabilized, this means that also (x_1, x_2) is asymptotically stabilized at the origin. In addition, since $V_c(x_1, z)$ is radially unbounded and there are no singularities in u, the equilibrium point $x = (0, 0)$ is globally asymptotically stable.