TTK4150 Nonlinear Control Systems	Department of Engineering Cybernetics,
Fall 2024 - Assignment 3 (Solution)	Norwegian University of Science and Technology

Problem 1 With $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ and

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1 + x_2) - x_1^2 x_2 \end{aligned}$$

we have

$$\begin{aligned} \dot{V}(x) &= x_1 x_2 - (x_1 + x_2)x_2 - x_1^2 x_2^2 \\ &= x_1 x_2 - x_1 x_2 - x_2^2 - x_1^2 x_2^2 \\ &= -(1 + x_1^2)x_2^2 \leq 0 \quad \forall x \in \mathbb{R}^2 \backslash \{0\} \end{aligned}$$

Thus $\dot{V}$ is negative semidefinite since $\dot{V} = 0$ on the line $x_2 = 0$ (where $x_1 \in \mathbb{R}$). When $\dot{V} = 0$ we must have $x_2 = 0$ and also $\dot{x}_2 = 0$. From $\dot{x}_2 = 0$ we must have $-(x_1 + x_2) - x_1^2 x_2 = 0$ or equivalently $-x_1 = 0$ since $x_2 = 0$. For $x_1 = x_2 = 0$ we also have $\dot{x}_1 = 0$, such that $(0, 0)$ is an equilibrium point of the system, and it follows that $\dot{V} = 0$ identically on $x = (0, 0)$. Hence, by Corollary 4.1, the origin is asymptotically stable. Furthermore since V is radially unbounded, by Corollary 4.2 we conclude that the origin is globally asymptotically stable.

Problem 2

a Consider the following Lyapunov function candidate

$$V(\tilde{x}) = \frac{1}{2}\left[\tilde{x}_1^2 + m\tilde{x}_2^2\right]$$

Differentiating V along the trajectories of the system yields

$$\begin{aligned} \dot{V}(\tilde{x}) &= \tilde{x}_1 \dot{\tilde{x}}_1 + m\tilde{x}_2 \dot{\tilde{x}}_2 \\ &= \tilde{x}_1 \tilde{x}_2 + \tilde{x}_2\left(-f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u}\right) \end{aligned}$$

Requiring $\dot{V}(\tilde{x}) = -(d + k_2)\tilde{x}_2^2$, we must thus solve

$$-(d + k_2)\tilde{x}_2^2 = \tilde{x}_1 \tilde{x}_2 + \tilde{x}_2\left(-f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u}\right)$$

for the input $\tilde{u}$. This yields

$$\tilde{u} = f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] + f_1\tilde{x}_1 - \tilde{x}_1 - k_2\tilde{x}_2.$$

b The closed-loop system is found by inserting $\tilde{u}$ into the system equations

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ m\dot{\tilde{x}}_2 &= -\tilde{x}_1 - (d + k_2)\tilde{x}_2 \end{aligned} \tag{1}$$

$$\downarrow$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{m} & -\frac{1}{m}(d + k_2) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{2}$$

1. $\dot{V}$ from the previous question is negative semidefinite (that is, $\dot{V} = 0$ on the line $\tilde{x}_2 = 0$). V is a continuously differentiable, positive definite, radially unbounded function such that $\dot{V}(\tilde{x}) \leq 0$ for all $\tilde{x} \in \mathbb{R}^2$ and $V(0) = 0$. Let $S = \left\{\tilde{x} \in \mathbb{R}^2 | \dot{V} = 0\right\}$, i.e. $S = \left\{\tilde{x} \in \mathbb{R}^2 | \tilde{x}_2 = 0\right\}$. Inserting $\tilde{x}_2 = 0$ gives

$$\tilde{x}_2 \equiv 0 \Rightarrow \dot{\tilde{x}}_2 \equiv 0 \Rightarrow \tilde{x}_1 \equiv 0$$

i.e. the only solution that can stay in S is the trivial solution $\tilde{x} = 0$. LaSalle's theorem states that the origin is a globally asymptotically stable equilibrium point of the closed-loop system.

2. Since the closed-loop system is linear, we can analyze its stability properties by considering the eigenvalues. The eigenvalues of the closed loop system can be calculated from

$$\begin{vmatrix} \lambda & -1 \\ \frac{1}{m} & \lambda + \frac{1}{m}(d + k_2) \end{vmatrix} = 0$$

which gives

$$\lambda^2 + \lambda\frac{1}{m}(d + k_2) + \frac{1}{m} = 0$$

$$\lambda = \frac{1}{2}\left(-\left(\frac{d + k_2}{m}\right) \pm \sqrt{\left(\frac{d + k_2}{m}\right)^2 - \frac{4}{m}}\right) \tag{3}$$

Since d, k_2, $m > 0$

$$\frac{d + k_2}{m} > \sqrt{\left(\frac{d + k_2}{m}\right)^2 - \frac{4}{m}}$$

and the eigenvalues will always lie in the left half plane which means that A is Hurwitz. Any linear system on the form

$$\dot{x} = Ax$$

is exponentially stable as long as A is Hurwtitz, and the system (2) is hence globally exponentially stable.

c From (1) it is easy to see that increasing k_2 will introduce more damping into the system. Therefore the overall system dynamics get slower as the controller gain k_2 increases.

d The closed-loop system has two poles, but we only have one control parameter k_2. Trying to place the two poles arbitrarily by choosing only one parameter would give two equations with only one unknown, which is an overdetermined system and has (in general) no solution. If k_2 is found such that the first eigenvalue given by (3) has a desired value, the second eigenvalue is also given by (3) and can no longer be chosen arbitrarily.

e In Assignment 2 the input $u = u_0 + \tilde{u}$ was used to shift the equilibrium point to a desired equilibrium point. Since the disturbance w is not counteracted by the term u_0, the transformed form of the system is now

$$\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 \\
\dot{\tilde{x}}_2 &= -\frac{f_3}{m}\left((\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right) - \frac{f_1}{m}\tilde{x}_1 - \frac{d}{m}\tilde{x}_2 + \frac{w}{m} + \frac{\tilde{u}}{m}
\end{aligned}$$

Inserting for $\tilde{u}$ we get

$$\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 \\
\dot{\tilde{x}}_2 &= -\frac{1}{m}\tilde{x}_1 - \frac{d + k_2}{m}\tilde{x}_2 + \frac{w}{m}
\end{aligned}$$

a system which is not in equilibrium at the origin. Since the origin is not an equilibrium point of the system (in error coordinates), it is also not stable. Hence our previous control input u is unable to make the desired setpoint stable in the presence of an unknown, constant disturbance.

Problem 3 The system is given by

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

a Using $x_1 = y$ and $x_2 = \dot{y}$, the state equation becomes

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(x_1)x_2 - g(x_1)
\end{aligned}$$

To be an equilibrium point of the system, the origin must satisfy $\dot{x} = 0$. We see that $\dot{x}_1 = 0$ implies that $x_2 = 0$. Inserting this in $\dot{x}_2 = 0$ yields

$$0 = -g(x_1)$$

Thus, for $x = 0$ to be an isolated equilibrium point, we must have $g(x_1) = 0$ if and only if $x_1 = 0$. Note that this also makes the origin the unique equilibrium point of the system.

b We first need to find the conditions ensuring that $V = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$ is positive definite. We see that this is the case if the function $g(x_1)$ satisfies the condition $x_1 g(x_1) > 0 \, \forall \, x_1 \neq 0$ since then the term $\int_0^{x_1} g(y)dy$ is positive definite.

The time derivative of V along the trajectories of the system is

$$\begin{aligned}\dot{V}(x) &= x_2\dot{x}_2 + \dot{x}_1 g(x_1)\\ &= -g(x_1)x_2 - h(x_1)x_2^2 + g(x_1)x_2\\ &= -h(x_1)x_2^2\end{aligned}$$

$\dot{V}$ is negative semidefinite in a neighbourhood of the origin D if the function $h(x_1)$ satisfies the condition $h(x_1) > 0 \; \forall \, x_1 \in D\backslash\{0\}$. Let $S = \{x \in \mathbb{R}^2 \,|\, \dot{V} = 0\} = \{x \in \mathbb{R}^2 \,|\, x_2 = 0\}$. To stay identically in S, a solution must satisfy

$$\dot{x}_2 = 0 \Rightarrow -g(x_1) = 0$$

Hence, for the origin to be the only solution, we must have $g(x_1) = 0$ if and only if $x_1 = 0$ (equivalent to what we found in **a**). By Corollary 4.1, the origin is asymptotically stable.

c To show global asymptotic stability, the function $V(x)$ needs to be radially unbounded. This is the case if the function g satisfies $\int_0^z g(y)\,dy \to \infty$ as $|z| \to \infty$. The condition $h(x_1) > 0$ must also hold for all $x_1 \neq 0$, i.e. $D = \mathbb{R}^2$. Then, by the arguments provided in **b** and Corollary 4.2, the origin is globally asymptotically stable.

Problem 4

a To be an equilibrium point of the system

$$\begin{aligned}\dot{x}_1 &= x_1(x_1^2 - 1) + 2x_2\\ \dot{x}_2 &= -x_1^3 - x_2\end{aligned}$$

a solution must satisfy $\dot{x} = 0$, that is

$$\begin{aligned}\dot{x}_1 = 0 &\implies x_2 = -\frac{1}{2}x_1(x_1^2 - 1)\\ \dot{x}_2 = 0 &\implies -x_1^3 + \frac{1}{2}x_1(x_1^2 - 1) = 0\end{aligned}$$

From this, we see that we must have

$$-\frac{1}{2}x_1(x_1^2 + 1) = 0 \implies x_1 = 0$$

as the only real solution, which gives $x_2 = 0$. Hence, the origin is the unique equilibrium point of the system.

b Consider the Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$, where $P = \left[\begin{smallmatrix}1 & 1\\ 1 & 3\end{smallmatrix}\right]$. The derivative of V along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) = x^\top P\dot{x} &= (x_1(x_1^2 - 1) + 2x_2)(x_1 + x_2) - (x_1^3 + x_2)(x_1 + 3x_2)\\ &= x_1^2(x_1^2 - 1) + 2x_1x_2 + x_1x_2(x_1^2 - 1) + 2x_2^2 - x_1^4 - 3x_1^3x_2 - x_1x_2 - 3x_2^2\\ &= -x_1^2 - x_2^2 - 2x_1^3x_2\end{aligned}$$

To show that the origin is asymptotically stable, we can write the above expression as

$$\dot{V} = -x^\top \begin{bmatrix}1 + 2x_1x_2 & 0\\ 0 & 1\end{bmatrix} x = -x^\top Qx$$

where Q is positive definite if $1 + 2x_1x_2 > 0 \implies x_1x_2 > -\frac{1}{2}$. Alternatively,

$$\dot{V} = -x^\top \begin{bmatrix}1 & x_1\\ x_1 & 1\end{bmatrix} x = -x^\top Qx,$$

where Q is positive definite given that $1 - x_1^2 > 0 \implies |x_1| < 1$. This gives the two possible regions $D = \{x \in \mathbb{R}^2 \,|\, x_1 x_2 > -\frac{1}{2}\}$ and $D = \{x \in \mathbb{R}^2 \,|\, |x_1| > 1\}$ such that $\dot{V}(x) < 0,\ \forall x \in D \backslash \{0\}$.

Notice that Theorem 4.1 states only that x is asymptotically stable, but nothing specifically about from where the solution will converge to the origin. So long as there exists *some* region D containing the origin where V is a strict Lyapunov function, then Theorem 4.1 states that there is a (small) δ such that if $\|x(0)\| \leq \delta$, then $x(t)$ will converge to the origin as $t \to \infty$ (Definition 4.1 of asymptotic stability).

Therefore any choices of D are equally good for only showing local asymptotic stability. However, the choice of D will influence the estimate of the region of attraction obtained in the next task.

c Since V is radially unbounded, the level sets $\Omega_c = \left\{x \in \mathbb{R}^2 | V(x) \leq c\right\}$, where c is chosen such that $\dot{V}(x) \leq 0 \ \ \forall x \in \Omega_c$, are positively invariant. We can use this to find an estimate of the region of attraction. To get the biggest possible Ω_c, choose c as the smallest value V along the boundary of D found in **b**. Depending on the choice of D, there are again two alternatives.

Alternative 1, $D = \left\{x \in \mathbb{R}^2 \,\middle|\, |x_1| < 1\right\}$:

$$\begin{aligned} c &= \min_{|x_1|=1} V(x) = \min_{|x_1|=1} x^\top P x \\ &= \min_{|x_1|=1} \left(\frac{1}{2}x_1^2 + x_1 x_2 + \frac{3}{2}x_2^2\right) \\ &= \min \begin{cases} \frac{1}{2} + x_2 + \frac{3}{2}x_2^2, & \text{if } x_1 = 1 \\ \frac{1}{2} - x_2 + \frac{3}{2}x_2^2, & \text{if } x_1 = -1 \end{cases} \end{aligned}$$

The minimum of $V(x)$ along $x_1 = 1$ and $x_1 = -1$ is found through

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(\frac{1}{2} + x_2 + \frac{3}{2}x_2^2\right) &= 1 + 3x_2 = 0 \\ \frac{\partial}{\partial x_2}\left(\frac{1}{2} - x_2 + \frac{3}{2}x_2^2\right) &= -1 + 3x_2 = 0 \end{aligned}$$

which implies that $\left(-1, \frac{1}{3}\right)$ and $\left(-1, -\frac{1}{3}\right)$ are candidates for minimum:

$$\begin{aligned} c &= \min_{|x_1|=1} V(x) \\ &= \min V(x)|_{x \in \left\{\left(-1,\frac{1}{3}\right),\left(1,-\frac{1}{3}\right)\right\}} \\ &= \min \left\{V\left(-1, \frac{1}{3}\right), V\left(1, -\frac{1}{3}\right)\right\} \\ &= \frac{1}{3} \end{aligned}$$

Let $E = \left\{x \in \Omega_{\frac{1}{3}} \,\middle|\, \dot{V}(x) = 0\right\} = (0,0) = M$, since the system cannot stay in any of the other the points where $\dot{V} = 0$ (you can verify this by inserting into the system dynamics). This means that $(0,0)$ is the largest invariant set in E, and with respect to Theorem 4.4 $\Omega_{\frac{1}{3}}$ can be taken as an estimate of the region of attraction. Figure 1 shows a plot of the estimated region of attraction for both alternatives.

Alternative 2, $D = \left\{x \in \mathbb{R}^2 \,\middle|\, x_1 x_2 > -\frac{1}{2}\right\}$: The boundary of D is described by $x_2 = -\frac{1}{2x_1}$. This gives

$$\begin{aligned} c &= \min_{x_2 = -\frac{1}{2x_1}} V(x) \\ &= \min_{x_2 = -\frac{1}{2x_1}} \left(\frac{1}{2}x_1^2 + x_1 x_2 + \frac{3}{2}x_2^2\right) \\ &= \min\left(\frac{1}{2}x_1^2 - \frac{1}{2} + \frac{3}{8x_1^2}\right) \end{aligned}$$

Then we find the minimum through

$$\frac{\partial}{\partial x_1}\left(\frac{1}{2}x_1^2 - \frac{1}{2} + \frac{3}{8x_1^2}\right) = x_1 - \frac{3}{4x_1^3} = 0$$

which gives possible minima at $\left(\pm\sqrt[4]{\frac{3}{4}}, \mp\frac{1}{2}\sqrt[4]{\frac{4}{3}}\right)$.

$$
\begin{aligned}
&V\left(\sqrt[4]{\tfrac{3}{4}}, -\tfrac{1}{2}\sqrt[4]{\tfrac{4}{3}}\right) = V\left(-\sqrt[4]{\tfrac{3}{4}}, \tfrac{1}{2}\sqrt[4]{\tfrac{4}{3}}\right) \\
&= \tfrac{1}{2}\sqrt{\tfrac{3}{4}} - \tfrac{1}{2} + \tfrac{3}{8}\sqrt{\tfrac{4}{3}} = \tfrac{1}{2}\left(\sqrt{\tfrac{3}{4}} + \sqrt{\tfrac{3}{4}}\sqrt{\tfrac{3\cdot 4}{4\cdot 3}} - 1\right) = \tfrac{1}{2}\left(2\sqrt{\tfrac{3}{4}} - 1\right) = \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2} \approx 0.3660
\end{aligned}
$$

Like in the previous alternative, let $E = \left\{x \in \Omega_{0.3660} | \dot{V}(x) = 0\right\} = (0,0) = M$. Then by Theorem 4.4 $\Omega_{0.3660}$ can be taken as an estimate of the region of attraction.

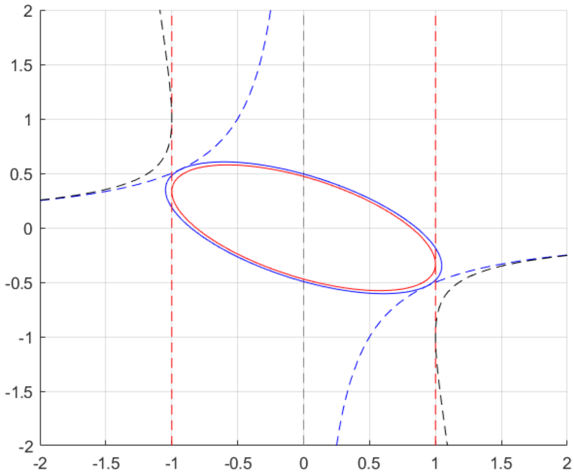

Figure 1: Estimates of the region of attraction, $\Omega_{\frac{1}{3}}$ in red, with corresponding D, and $\Omega_{0.3660}$ in blue. Black dashed line shows the boundary of the true region where $\dot{V} \leq 0$.

In Figure 1, notice that the region where $\dot{V} \leq 0$ is bigger than either of the regions D we found previously. In alternative 1 we disregard all values of x where the full expression $-x^\top Q(x)x$ is negative, even if $Q(x)$ itself is not positive definite. In alternative 2 we disregard the points where x_2^2 is sufficiently large to make up for the term in x_1^2 being positive.

Problem 5

a Inserting $x_1 = x_2 = 0$ into the system dynamics gives $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, confirming that the origin is an equilibrium point.
While we do not know enough about f_1 and f_2 to solve for all possible equilibrium points, it is possible to check if there are equilibrium points on the set $x_1^2 + 2x_2^2 - 4 = 0$. Inserting into the system dynamics gives

$$
\begin{aligned}
\dot{x}_1 &= 4x_1^2 x_2 \\
\dot{x}_2 &= -2x_1^3
\end{aligned}
$$

where we see that $x_1 = 0$ gives $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Since we restricted the dynamics to the set $x_1^2 + 2x_2^2 - 4 = 0$, $x_1 = 0$ gives $x_2 = \pm\sqrt{2}$. Hence the system has at least two more equilibrium points, $(0, \pm\sqrt{2})$.

b A set C is invariant with respect to the system if, for every trajectory x,

$$x(t_0) \in C \implies x(t) \in C \quad \forall t \in \mathbb{R}$$

To show that $C = \left\{x \in \mathbb{R}^2 \,|\, x_1^2 + 2x_2^2 - 4 = 0\right\}$ is an invariant set of the system, define the new variable $z = x_1^2 + 2x_2^2 - 4$. We need to show that $\dot{z} = 0$ when $x \in C$, i.e. when $z = 0$, and $\dot{x}$ satisfies the system equations. The time-derivative of z is found as

$$
\begin{aligned}
\dot{z} &= 2x_1\dot{x}_1 + 4x_2\dot{x}_2 \\
&= 2x_1\left[4x_1^2x_2 - f_1(x_1)\left(x_1^2 + 2x_2^2 - 4\right)\right] \\
&\quad +4x_2\left[-2x_1^3 - f_2(x_2)\left(x_1^2 + 2x_2^2 - 4\right)\right] \\
&= -2x_1f_1(x_1)\left(x_1^2 + 2x_2^2 - 4\right) - 4x_2f_2(x_2)\left(x_1^2 + 2x_2^2 - 4\right) \\
&= -\left(2x_1f_1(x_1) + 4x_2f_2(x_2)\right)\left(x_1^2 + 2x_2^2 - 4\right) \\
&= -2\left(x_1f_1(x_1) + 2x_2f_2(x_2)\right)z
\end{aligned}
$$

where it can be seen that $z = 0$ yields $\dot{z} = 0$. This means that the relationship between x_1 and x_2 as given by z stays constant if $z = 0$ initially. Hence, $\left\{x \in \mathbb{R}^2 | x_1^2 + 2x_2^2 - 4 = 0\right\}$ is an invariant set for the system.

c The function

$$V(x) = \left(x_1^2 + 2x_2^2 - 4\right)^2$$

is radially unbounded. The derivative of V with respect to time is

$$
\begin{aligned}
\dot{V}(x) &= 2\left(x_1^2 + 2x_2^2 - 4\right)\left(2x_1\dot{x}_1 + 4x_2\dot{x}_2\right) \\
&= -4\left(x_1f_1(x_1) + 2x_2f_2(x_2)\right)\left(x_1^2 + 2x_2^2 - 4\right)^2
\end{aligned}
$$

which is negative semidefinite. Let $D = \mathbb{R}^2$. The set $\Omega_c = \left\{x \in \mathbb{R}^2 | V(x) \le c, \dot{V}(x) \le 0\right\} = \left\{x \in \mathbb{R}^2 | V(x) \le c\right\}$ is a compact positively invariant set for any finite $c > 0$ due to the radially unboundedness of V. Let $\Omega = \Omega_c$, the set E is given by

$$
\begin{aligned}
E &= \left\{x \in \Omega | \dot{V}(x) = 0\right\} \\
&= \left\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\ \right\} \cup \left\{x \in \Omega |\ x_1f_1(x_1) + 2x_2f_2(x_2) = 0\right\} \\
&= \left\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\ \right\} \cup (0,0)
\end{aligned}
$$

Since both $\left\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\ \right\}$ and $(0,0)$ are invariant sets (recall that the origin is an equilibrium point), the largest invariant set in E is given by $M = E$, and by Theorem 4.4 it can be concluded that every solution starting in Ω approaches $\left\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\right\}$ or the origin as $t \to \infty$.

d Since the set C contains equilibrium points as shown in **a**, it cannot be a limit cycle (once a trajectory $x(t)$ reaches an equilibrium point, it will stay there for all future time).

Problem 6

a The Jacobian of the system

$$
\begin{aligned}
\dot{x}_1 &= (x_1 + 2x_2)(-x_1 - 2) \\
\dot{x}_2 &= -5x_2(2 + 2x_1 + x_2)
\end{aligned}
$$

is given by

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -(x_1 + 2) - (x_1 + 2x_2) & -2(x_1 + 2) \\ -10x_2 & -5(2 + 2x_1 + x_2) - 5x_2 \end{bmatrix}$$

Evaluation $[\partial f / \partial x]$ at the origin we have

$$A = \frac{\partial f(0)}{\partial x} = \begin{bmatrix} -2 & -4 \\ 0 & -10 \end{bmatrix}$$

and the eigenvalues are $\lambda_{1,2} = -2, -10$. Thus the origin is asymptotically stable.

b The derivative of the Lyapunov function along the trajectory of the system is

$$
\begin{aligned}
\dot{V}(x) &= x_1(x_1 + 2x_2)(-x_1 - 2) - 5x_2^2(2 + 2x_1 + x_2) \\
&= -x_1^2(2 + x_1 + 2x_2) - 4x_1x_2 - 5x_2^2(2 + 2x_1 + x_2)
\end{aligned}
$$

In the domain D, we have

$$\begin{aligned} 2 + x_1 + 2x_2 &\geq 1 \\ 2 + 2x_1 + x_2 &\geq 1 \end{aligned}$$

Hence,

$$\begin{aligned} \dot{V}(x) &\leq -x_1^2 - 4x_1x_2 - 5x_2^2, \quad \forall\, x \in D \\ &= -x^\top Q x \end{aligned}$$

where

$$Q = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

is positive definite (since $q_{11} > 0$ and $q_{11}q_{22} - q_{12}^2 > 0$). Thus $\dot{V}$ is negative definite which implies that the origin is asymptotically stable.

Alternatively, use a simpler completion of squares:

$$\begin{aligned} \dot{V}(x) &\leq -x_1^2 - 4x_1x_2 - 5x_2^2, \quad \forall\, x \in D \\ &= -(x_1 + 2x_2)^2 - x_2^2 < 0 \end{aligned}$$

which yields the same result.

c Since V is radially unbounded, the level sets $\Omega_c = \{x \in \mathbb{R}^2 \,|\, V(x) \leq c\}$, where $\dot{V}(x) \leq 0 \,\forall\, x \in \Omega_c$ are positively invariant sets and can thus be used as estimates of the region of attraction. To get the largest estimate, we choose c as the smallest value of V along the boundary of D. We start by considering the boundary $1 + x_1 + 2x_2 = 0$:

$$\min_{x_1 = -1 - 2x_2} V(x) = \min_{x_2} \frac{1}{2}\left(1 + 4x_2 + 5x_2^2\right)$$

To find the minimum, we solve

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(1 + 4x_2 + 5x_2^2\right) &= 0 \\ 4 + 10x_2 &= 0 \end{aligned}$$

which gives $x_2 = -0.4, x_1 = -0.2$. Next, we consider the boundary $1 + 2x_1 + x_2 = 0$:

$$\min_{x_1 = -\frac{1}{2} - \frac{1}{2}x_2} V(x) = \min_{x_2} \frac{1}{8}\left(1 + 2x_2 + 5x_2^2\right)$$

The minimum is given by

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(1 + 2x_2 + 5x_2^2\right) &= 0 \\ 2 + 10x_2 &= 0 \end{aligned}$$

yielding $x_2 = -0.2, x_1 = -0.4$. Since

$$V(-0.4, -0.2) = V(-0.2, -0.4) = 0.1$$

an estimate of the region of attraction can thus be taken as $\Omega_{0.1} = \{x \in \mathbb{R}^2 \,|\, V(x) \leq 0.1\}$.

d Since $\Omega_{0.1}$ is bounded and contained in D, it is positively invariant. Thus, any trajectory starting in the set cannot leave it. Moreover, by Theorem 4.4, all trajectories starting in the set will converge to the origin, which we can see is the case from Figure 2. On the other hand, the set $\Omega_{2.5}$ is not bounded and contained in D. This implies that it is possible for the trajectories not to converge to the origin in this set. Even though $V(x)$ decreases while x is in D, D itself is not positively invariant, meaning that it might be possible for the trajectories to leave D (as is the case here). Hence, D cannot be taken as an estimate of the region of attraction. Note also that since $\Omega_{0.1}$ is only an estimate of the region of attraction, the true region is bigger and therefore, some trajectories in $\Omega_{2.5} \backslash \Omega_{0.1}$ do in fact converge to the origin.

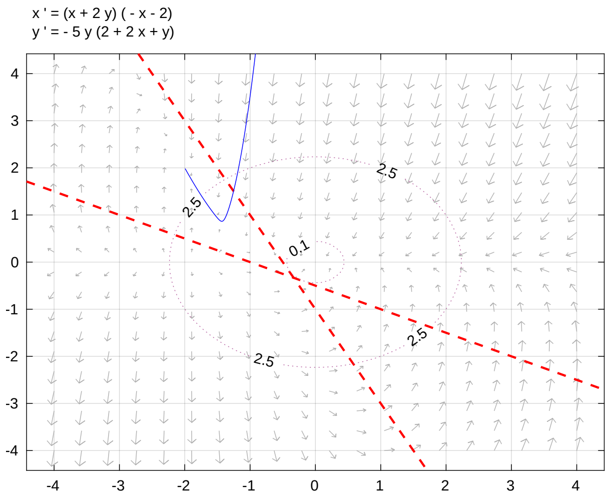

Figure 2: Phase portrait for the system in Exercise 6. The boundaries $1+x_1+2x_2=0$ and $1+2x_1+x_2=0$ are given by the red, dashed lines. The blue line shows a trajectory that escapes $\Omega_{2.5}$ and D.

Problem 7 The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - 2x_2 + x_2\cos x_2\end{aligned}$$

and the Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^\top P x, \quad P = \left[\begin{smallmatrix}4 & 1\\ 1 & 3\end{smallmatrix}\right]$$

which is positive definite and radially unbounded. The derivative of V along the trajectories of the system is given by:

$$\begin{aligned}\dot{V}(x) &= \dot{x}^\top P x\\ &= \begin{bmatrix}x_2\\ -x_1 - 2x_2 + x_2\cos x_2\end{bmatrix}^\top \begin{bmatrix}4 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\\ &= 4x_2x_1 + x_2^2 - x_1^2 - 3x_1x_2 - 2x_2x_1 - 6x_2^2 + x_1x_2\cos x_2 + 3x_2^2\cos x_2\\ &\le -2x_2^2 - x_1^2 - x_1x_2 + x_1x_2\cos x_2\end{aligned}$$

where we use that $|\cos x_2| \le 1$ in the last line. To deal with the cross-term containing $\cos x_2$, we can use Young's inequality:

$$xy \le \epsilon x^2 + \frac{1}{4\epsilon}y^2, \forall \epsilon > 0\ \forall x, y \in \mathbb{R}$$

with $x = x_1$ and $y = x_2\cos x_2$. This gives

$$\begin{aligned}\dot{V}(x) \le& -2x_2^2 - x_1^2 - x_1x_2 + \epsilon x_1^2 + \frac{1}{4\epsilon}x_2^2\cos^2 x_2\\ \le& -x_2^2\left(2 - \frac{1}{4\epsilon}\right) - x_1x_2 - x_1^2(1-\epsilon)\end{aligned}$$

where we once again use that $|\cos x_2| \le 1$, which we now can do since the term has a definite sign. Moreover, $\dot{V}$ is now on the form $\dot{V}(x) = -x^\top Q x$, with

$$Q = \begin{bmatrix}1-\epsilon & \frac{1}{2}\\ \frac{1}{2} & 2-\frac{1}{4\epsilon}\end{bmatrix}.$$

We want to choose $\epsilon > 0$ such that Q is positive definite (as this will make $\dot{V}$ negative definite). We know that $Q > 0$ if and only if all the principal minors are positive, that is

$$1 - \epsilon > 0$$
$$(1-\epsilon)\left(2 - \frac{1}{4\epsilon}\right) - \frac{1}{4} > 0$$

Solving the last inequality, we find that ϵ must satisfy

$$\frac{2-\sqrt{2}}{4} < \epsilon < \frac{2+\sqrt{2}}{4}, \tag{4}$$

which also fulfills the first inequality. Hence, we have shown that the origin of the system is globally asymptotically stable.

Problem 8 The system is given by

$$\begin{aligned}\dot{x}_1 &= -\sqrt{x_1^2 + x_2^2} - 11x_2 \\ \dot{x}_2 &= 2x_1 - 5x_2\end{aligned}$$

and the Lyapunov function candidate is given by $V(x) = \frac{1}{2}x^\top Px$ with $P = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 3\end{smallmatrix}\right]$, which is positive definite as the leading principal minors are positive, and radially unbounded. The derivative of V along the trajectories of the system is given by:

$$\begin{aligned}
\dot{V} &= x^\top P\dot{x} \\
&= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -\sqrt{x_1^2+x_2^2} - 11x_2 \\ 2x_1 - 5x_2 \end{bmatrix} \\
&= (x_1 - x_2)\left(-\sqrt{x_1^2+x_2^2} - 11x_2\right) + (-x_1 + 3x_2)(2x_1 - 5x_2) \\
&= -(x_1 - x_2)\sqrt{x_1^2+x_2^2} - 11x_1x_2 + 11x_2^2 - 2x_1^2 + 5x_1x_2 + 6x_1x_2 - 15x_2^2 \\
&= -(x_1 - x_2)\|x\|_2 - 2x_1^2 - 4x_2^2 \\
&\le -(x_1 - x_2)\|x\|_2 - 2\|x\|_2^2 \\
&\le |x_1 - x_2| \cdot \|x\|_2 - 2\|x\|_2^2
\end{aligned}$$

We can rewrite the term $x_1 - x_2$ as the inner product between the vector $[1\ -1]^\top$ and the vector x, that is $|x_1 - x_2| = |[1\ -1]x|$. The Cauchy-Schwarz inequality then gives:

$$\begin{aligned}
\dot{V} &\le \sqrt{1+1}\|x\|_2 \cdot \|x\|_2 - 2\|x\|_2^2 \\
&= -\left(2 - \sqrt{2}\right)\|x\|_2^2 < 0 \quad \forall x \in \mathbb{R}^2 \backslash \{0\}
\end{aligned}$$

Thus, since $\dot{V}$ is negative definite (on $\mathbb{R}^2$) and V is positive definite and radially unbounded, by Theorem 4.2 the origin is globally asymptotically stable.

Problem 9 Since α is a class $\mathcal{K}$ function, it must be strictly increasing in its argument and positive for positive arguments. Therefore, for any $r' \ge r > 0$, we have $\alpha(r') \ge \alpha(r) > 0$. Now, if $r_1 \ge r_2$, we must consequently have that $r_1 + r_2 \le 2r_1$. Using the properties of α, we thus have

$$\alpha(r_1 + r_2) \le \alpha(2r_1) \le \alpha(2r_1) + \underbrace{\alpha(2r_2)}_{\ge 0} \qquad \forall\, r_1, r_2 \in [0, \alpha/2)$$

The same holds if $r_2 \ge r_1$, that is

$$\alpha(r_1 + r_2) \le \alpha(2r_2) \le \alpha(2r_1) + \alpha(2r_2) \qquad \forall\, r_1, r_2 \in [0, \alpha/2)$$

The proof can be shortened by covering both cases at once:

$$\alpha(r_1 + r_2) \le \alpha(2\max(r_1,\, r_2)) \le \alpha(2r_1) + \alpha(2r_2).$$

Problem 10

a For class $\mathcal{KL}$ functions we know that, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s. Therefore,

$$\beta(r, t) \le \beta(r, 0)$$

Since

$$\|x(t)\| \le \beta(\|x(0)\|, t),$$

we must consequently have

$$\|x(t)\| \leq \beta\left(\|x(0)\|, 0\right),$$

To show that $x = 0$ is stable, we want $\delta > 0$ such that

$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon \quad \forall t \geq 0$$

From the above, we see that if we choose $\epsilon > \beta\left(\delta, 0\right)$, then we will have $\|x(t)\| \leq \epsilon$ for any $\|x(0)\| < \delta$, which is possible because $\lim_{r \to 0} \beta(r, 0) = 0$ and β is strictly increasing with respect to r. Hence, $x = 0$ is stable.

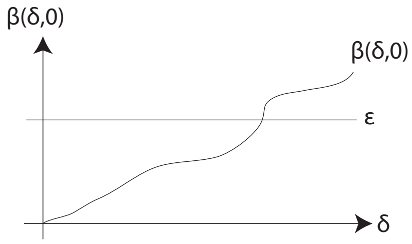

Figure 3: Plot of $\beta\left(\delta, 0\right)$ vs. ϵ

b Since β is of class $\mathcal{KL}$ then $\lim_{t \to \infty} \beta\left(\|x\left(0\right)\|, t\right) = 0$. Thus $\lim_{t \to \infty} \|x\left(t\right)\| = 0$. By definition of the norm, it follows that $\lim_{t \to \infty} x\left(t\right) = 0$. We conclude that the origin of the system is globally asymptotically stable.

Problem 11 We have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(x_1 + x_2\right) - h\left(x_1 + x_2\right) \end{aligned} \tag{5}$$

We want to determine the gradient, $g(x) = \nabla V$, of the Lyapunov function so that

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} \tag{6}$$

and

$$\dot{V}(x) = g(x)^\top f(x) < 0 \quad \forall \quad x \neq 0 \tag{7}$$
$$V(x) = \int_0^x g^\top(y) dy > 0 \quad \forall \quad x \neq 0$$

Looking at the system dynamics (5), we have the function $h(\cdot)$ of which we know very little. In order to satisfy (7) we should choose g_2 such that $-g_2(x)h(x_1 + x_2)$ is negative. Since we know that $zh(z) > 0 \; \forall z$, a smart choice for g_2 is

$$g_2(x) = \beta(x_1 + x_2)$$

with $\beta > 0$ being a constant. Then the symmetry condition (6) gives

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} = \beta$$

Integrating this with respect to x_2 in order to find g_1 yields

$$g_1(x) = \int_0^{x_2} \beta \; dy = \beta x_2 + \phi(x_1)$$

where ϕ is a function of x_1 only, to be found later. We now have

$$\begin{aligned} \dot{V}(x) = g(x)^\top f(x) &= \left(\phi(x_1) + \beta x_2\right) x_2 + \beta\left(x_1 + x_2\right)\left(-(x_1 + x_2) - h(x_1 + x_2)\right) \\ &= \phi(x_1)x_2 + \beta x_2^2 - \beta(x_1 + x_2)^2 - \beta(x_1 + x_2)h(x_1 + x_2) \\ &= \phi(x_1)x_2 + \beta x_2^2 - \beta x_1^2 - 2\beta x_1 x_2 - \beta x_1 x_2 - \beta(x_1 + x_2)h(x_1 + x_2) \\ &= \phi(x_1)x_2 - \beta x_1^2 - 2\beta x_1 x_2 - \beta(x_1 + x_2)h(x_1 + x_2) \end{aligned}$$

Now choose $\phi(x_1) = 2\beta x_1$ to cancel the cross-term and get

$$\dot{V}(x) = -\beta x_1^2 - \beta(x_1 + x_2)h(x_1 + x_2) < 0 \forall x \in \mathbb{R}^2 \backslash \{0\}$$

since $zh(z) > 0\ \forall z \neq 0$ and $\beta > 0$. The function V is constructed by integrating $\dot{V}$ along any path from 0 to x. We integrate along the axes to get

$$\begin{aligned}
V(x) &= \int_0^{x_1} g_1(y_1, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2) dy_2 \\
&= \int_0^{x_1} 2\beta y_1 dy_1 + \int_0^{x_2} \beta(x_1 + y_2)\, dy_2 \\
&= 2\beta \left[\frac{1}{2} y_1^2\right]_0^{x_1} + \beta x_1 [y_2]_0^{x_2} + \beta \left[\frac{1}{2} y_2^2\right]_0^{x_2} \\
&= \beta x_1^2 + \beta x_1 x_2 + \frac{\beta}{2} x_2^2 = x^\top P x
\end{aligned}$$

where

$$P = \begin{bmatrix} \beta & \frac{\beta}{2} \\ \frac{\beta}{2} & \frac{\beta}{2} \end{bmatrix}$$

and

$$\begin{aligned}
\beta &> 0 \\
\frac{\beta^2}{2} - \frac{\beta^2}{4} = \frac{\beta^2}{4} &> 0
\end{aligned}$$

which implies that $P > 0$, and so V is positive definite on $\mathbb{R}^2$ and radially unbounded. By Theorem 4.2 it is concluded that the origin is globally asymptotically stable. Alternatively, g can be found similarly to Example 4.5.