Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing

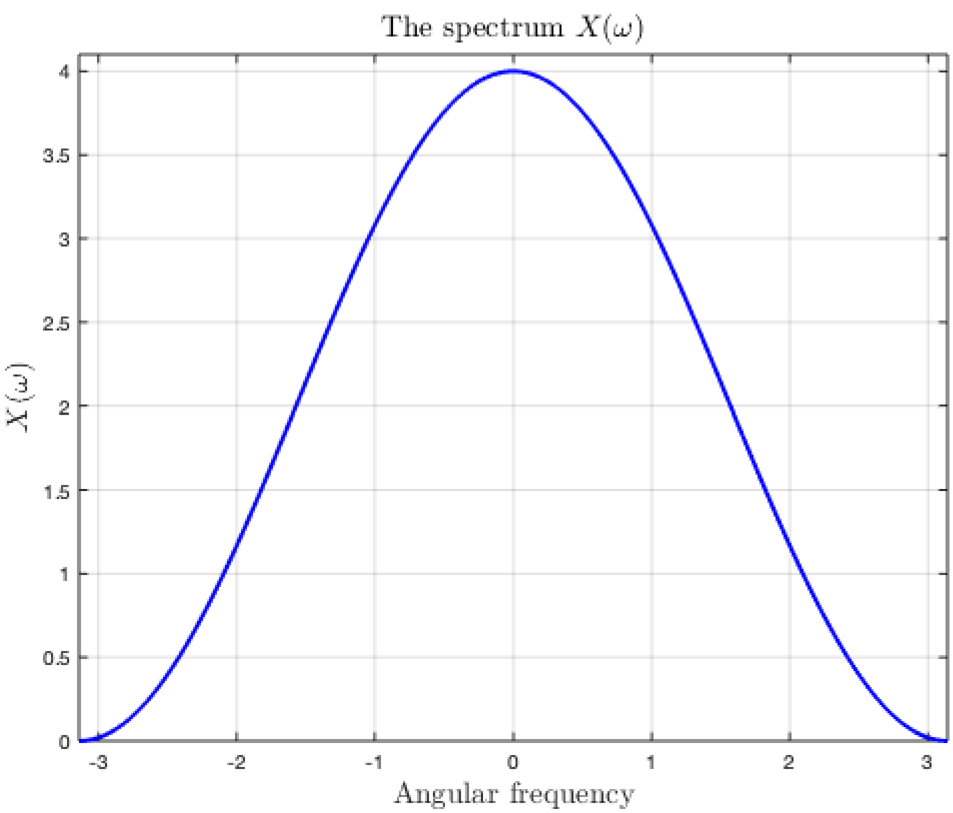
Solutions for Problem Set 2

Problem 1:

(a).

The spectrum $X(\omega)$ can be found and plotted as follows.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = e^{j\omega} + 2 + e^{-j\omega} = 2 + 2\cos\omega$$

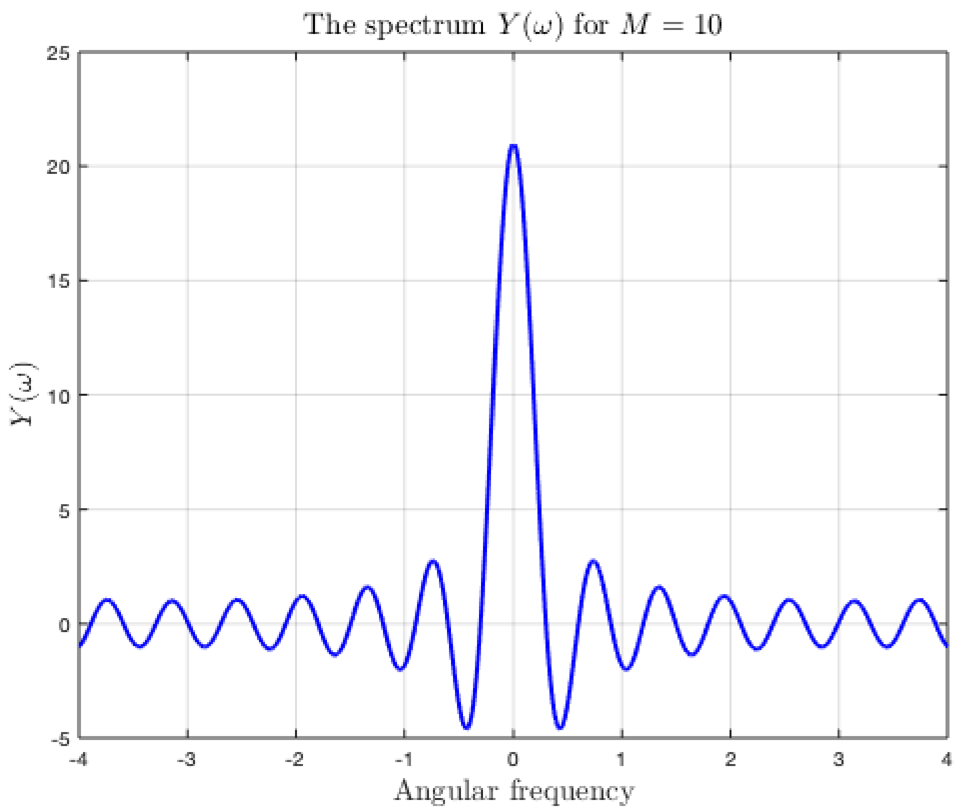


(b).

The spectrum $Y(\omega)$ can be found as follows.

$$
\begin{aligned}
Y(\omega) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} \quad l = n + M = \sum_{l=0}^{2M} e^{-j\omega(l-M)} \\
&= e^{j\omega M} \sum_{l=0}^{2M} e^{-j\omega l} = e^{j\omega M} \frac{1 - e^{-j\omega(2M+1)}}{1 - e^{-j\omega}} = \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} \\
&= \frac{e^{\frac{-j\omega}{2}}}{e^{\frac{-j\omega}{2}}} \frac{\left(e^{j\omega(M+\frac{1}{2})} - e^{-j\omega(M+\frac{1}{2})}\right)}{\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)} = \frac{\sin\left(\omega(M+\frac{1}{2})\right)}{\sin(\frac{\omega}{2})}
\end{aligned}
$$

The sketch is shown in the following Figure.

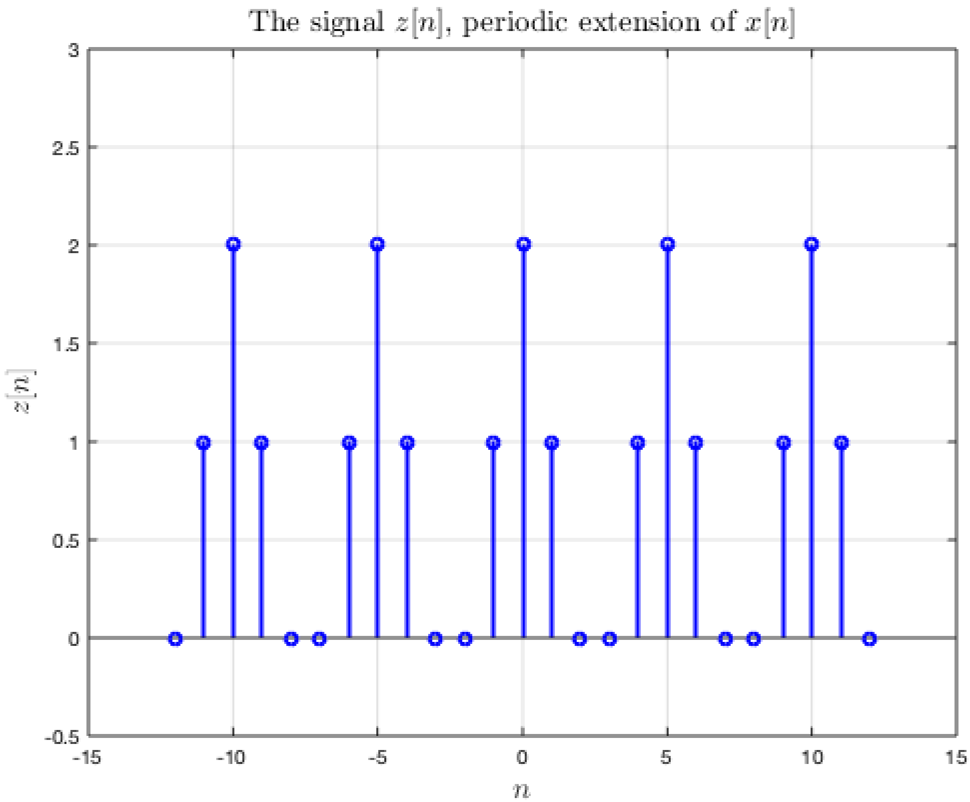


(c).

Because they are even signals.

(d).

A sketch of $z[n]$ for $N = 5$ is shown as follows.



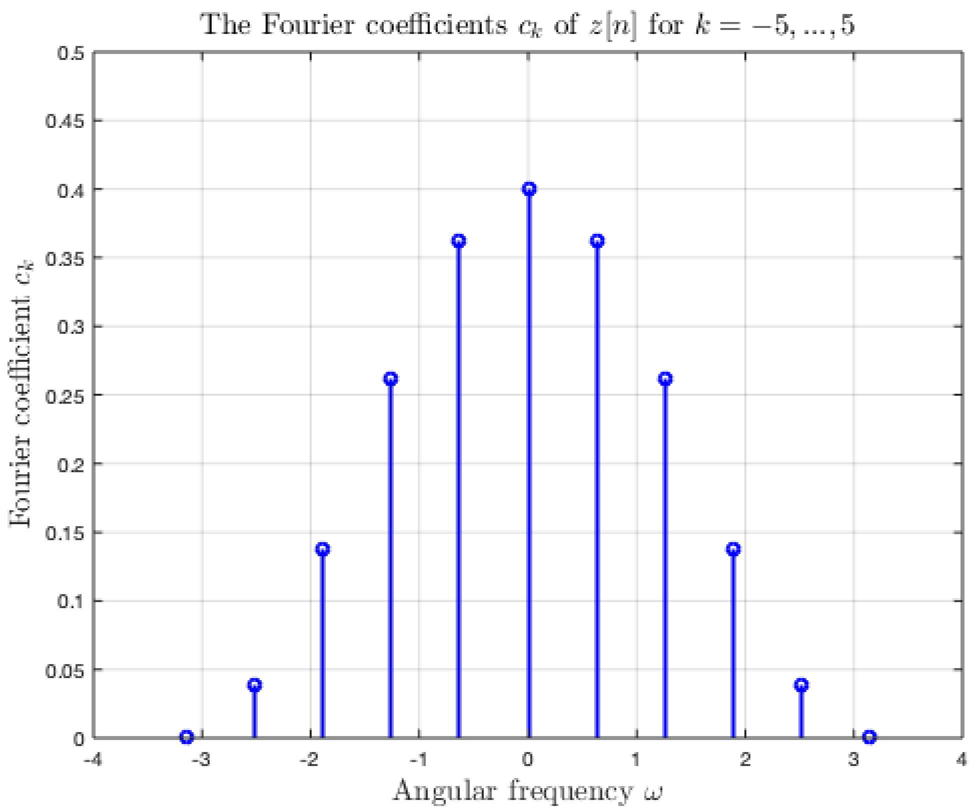
The Fourier coefficients are given by:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} z[n]e^{-j2\pi kn/N},\ k = 0, \cdots, N-1.$$

Note that we sum from 0 up to $N-1$. Thus, the first two samples are 2 and 1 respectively, and the last sample is 1. All other samples are 0. The coefficients could be calculated over any other period.

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{n=0}^{N-1} z[n]e^{-j2\pi kn/N} = \frac{1}{N}\left(2 + e^{-j2\pi k/N} + e^{-j2\pi k(N-1)/N}\right) \\ &= \frac{1}{N}\left(2 + e^{-j2\pi k/N} + e^{-j2\pi k}e^{j2\pi k/N}\right) = \frac{1}{N}\left(2 + e^{-j2\pi k/N} + e^{j2\pi k/N}\right) \\ &= \frac{1}{N}\left(2 + 2\cos(2\pi k/N)\right) \end{aligned}$$

The Fourier coefficients are displayed as follows.



(e).

We have the following.

$$X(f) = 2 + 2\cos(2\pi f)$$
$$c_k = \frac{1}{N}(2 + 2\cos(2\pi k/N))$$

Thus, we see that

$$c_k = \frac{1}{N} X\left(\frac{k}{N}\right).$$

This means that the Fourier coefficients are (scaled) samples of the continuous spectrum $X(f)$. This always holds true: a periodic extension in the time domain equals sampling in the frequency domain.

Problem 2:

(a).

For the first case, we use the time-shift property of the DTFT, and get

$$X_1(\omega) = e^{j3\omega} X(\omega)$$

(b).

For the second case, we use the time-reversal property of the DTFT, and it follows that

$$X_2(\omega) = X(-\omega)$$

(c).

For the third case notice that:

$$x_3[n] = x[3-n] = x[-(n-3)] = x_2[n-3]$$

so that by the time-reversal and time-shift properties, it follows that

$$X_3(\omega) = e^{-j3\omega}X_2(\omega) = e^{-j3\omega}X(-\omega)$$

(d).

For the last case, we have that

$$X_4(\omega) = \text{DTFT}\{x[n] * w[n]\} = X(\omega)W(\omega).$$

Problem 3:

(a).

By taking the DTFT of both sides of the first difference equation, we get

$$\begin{aligned} Y(\omega) &= X(\omega) + 2e^{-j\omega}X(\omega) + e^{-2j\omega}X(\omega) \\ H_1(\omega) &= \frac{Y(\omega)}{X(\omega)} = 1 + 2e^{-j\omega} + e^{-2j\omega} = e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega}) = e^{-j\omega}(2 + 2\cos\omega). \end{aligned}$$

And for the second case, we get

$$\begin{aligned} Y(\omega) &= -0.9Y(\omega)e^{-j\omega} + X(\omega) \\ H_2(\omega) &= \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + 0.9e^{-j\omega}}. \end{aligned}$$

(b).

We already have the frequency response $H_1(\omega)$ on polar form. Thus, the magnitude is simply

$$|H_1(\omega)| = 2 + 2\cos\omega.$$

Since $2 + 2\cos\omega \geq 0$ for all ω, the phase is simply

$$\Theta_1(\omega) = \angle H_1(\omega) = -\omega.$$

The magnitude response of the second system can be found as follows.

$$\begin{aligned} |H_2(\omega)| &= \left|\frac{1}{1 + 0.9e^{-j\omega}}\right| = \frac{1}{|1 + 0.9e^{-j\omega}|} \\ &= \frac{1}{\sqrt{(1 + 0.9\cos\omega)^2 + (0.9\sin\omega)^2}} = \frac{1}{\sqrt{1 + 1.8\cos\omega + 0.81}} \end{aligned}$$

To find the phase, we can write $H_2(\omega)$ as

$$H_2(\omega) = \frac{1}{W(\omega)},$$

where $W(\omega) = 1 + 0.9e^{-j\omega}$. Then, the phase is given by

$$\Theta_2(\omega) = \angle H_2(\omega) = -\angle W(\omega).$$

Since $\text{Re}\{W(\omega)\} > 0$ for all ω, we have

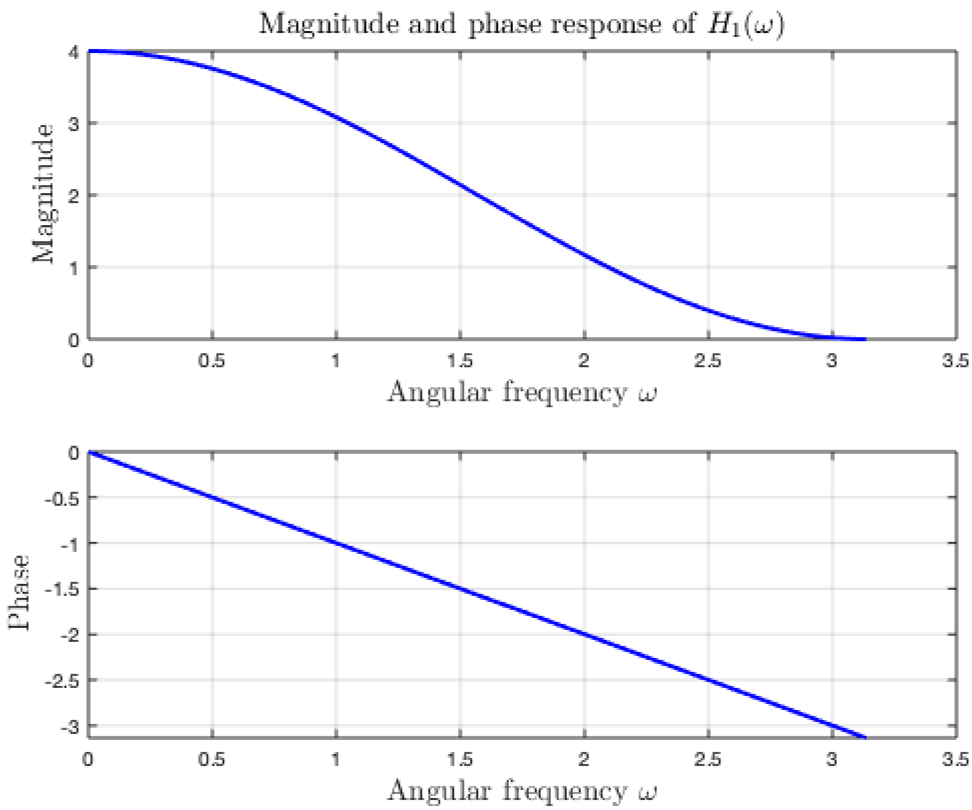
$$\angle H_2(\omega) = -\tan^{-1}\left(\frac{-0.9\sin\omega}{1 + 0.9\cos\omega}\right) = \tan^{-1}\left(\frac{0.9\sin\omega}{1 + 0.9\cos\omega}\right).$$

We notice that all magnitude functions are even and that all phase functions are odd. This is a property of real signals.

(c).

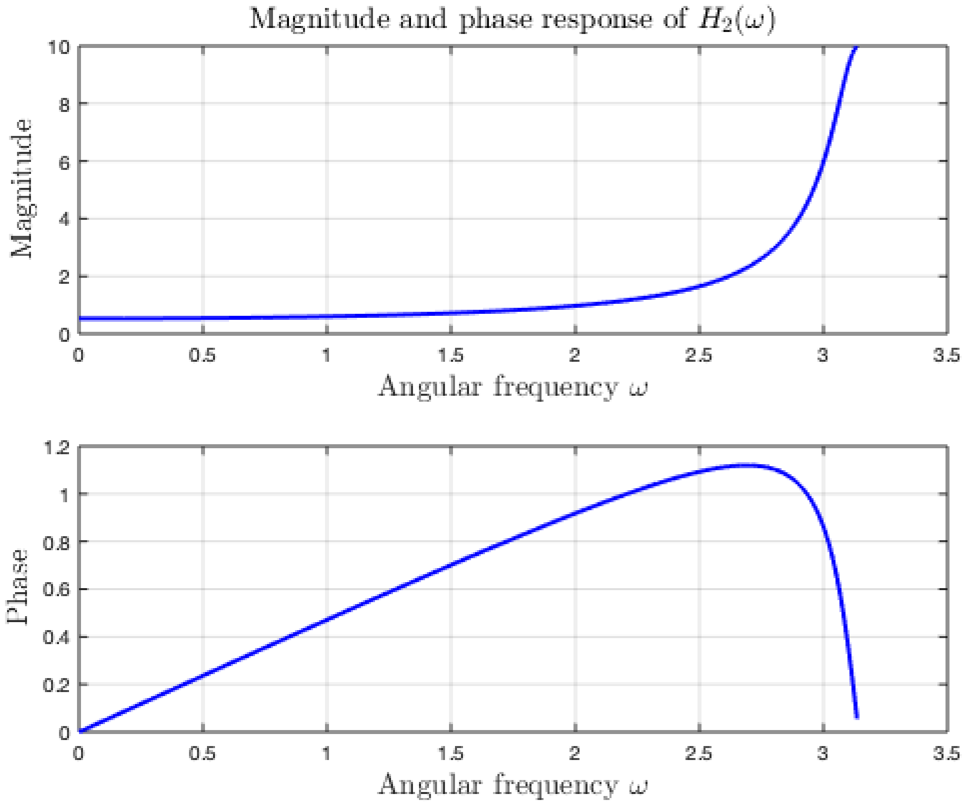
The frequency response of the first filter can be found and plotted by as following.

```
[H_1, w] = freqz([1 2 1], [1]);
subplot(2, 1, 1);
plot(w, abs(H_1));
subplot(2, 1, 2);
plot(w, angle(H_1));
```



For the second filter, we change the `freqz` command as follows which gives the frequency response of the second filter.

```
[H_2, w] = freqz([1], [1 0.9]);
```



(d).

From the plots of the magnitude responses, we can see that the first filter attenuates high frequencies more than low frequencies. Thus, this is a lowpass filter. The second filter attenuates low frequencies more than high frequencies. Thus, this is a highpass filter.

(e).

The response of a LTI-system $H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$ to a sinusoidal input signal $x[n] = A\cos(\omega_0 n + \theta)$ equals

$$y[n] = A|H(\omega_0)|\cos(\omega_0 n + \theta + \Theta(\omega_0)).$$

Thus, the output of the first system is

$$y_1[n] \quad = \frac{1}{2}|H_1(\frac{\pi}{2})|\cos(\frac{\pi}{2}n + \frac{\pi}{4} + \Theta_1(\frac{\pi}{2})) = \frac{1}{2}\cdot 2\cos(\frac{\pi}{2}n + \frac{\pi}{4} - \frac{\pi}{2}) = \cos(\frac{\pi}{2}n - \frac{\pi}{4}).$$

Likewise, the output of the second system is

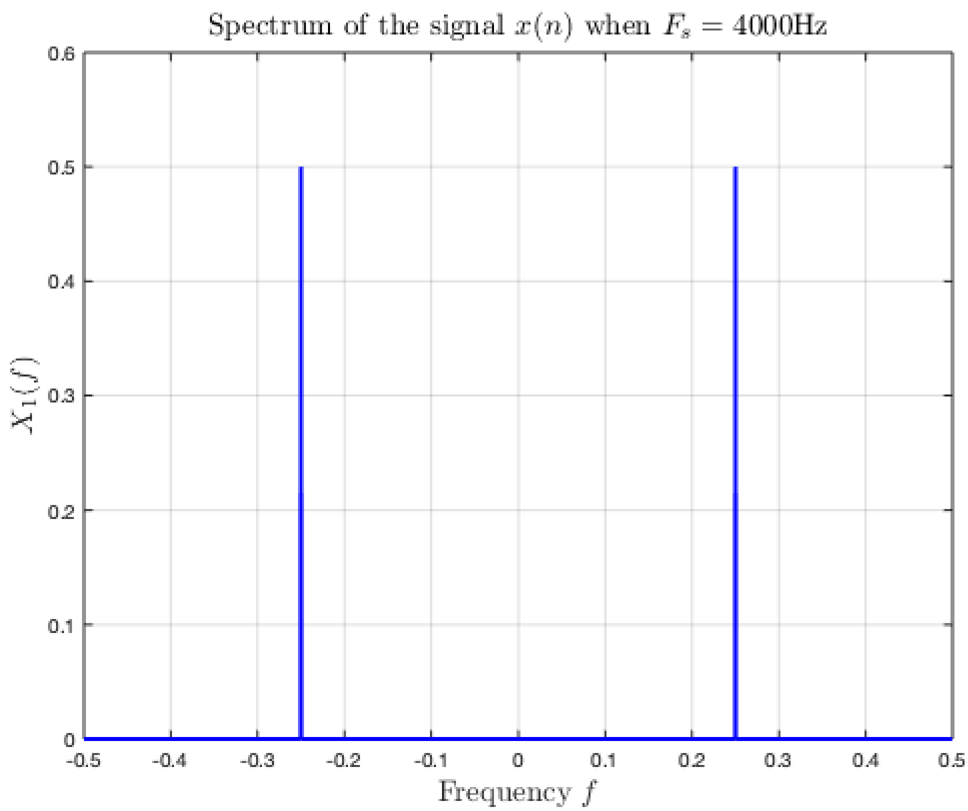
$$\begin{aligned} y_2[n] \quad &= \frac{1}{2}|H_2(\frac{\pi}{2})|\cos(\frac{\pi}{2}n + \frac{\pi}{4} + \Theta_2(\frac{\pi}{2})) \\ &= \frac{1}{2}\frac{1}{\sqrt{1.81 + 1.8\cos(\frac{\pi}{2})}}\cos(\frac{\pi}{2}n + \frac{\pi}{4} + \tan^{-1}(\frac{0.9\sin(\frac{\pi}{2})}{1 + 0.9\cos(\frac{\pi}{2})}) \\ &= \frac{1}{2}\frac{1}{\sqrt{1.81}}\cos(\frac{\pi}{2}n + \frac{\pi}{4} + \tan^{-1}(\frac{9}{10})) \approx \frac{1}{2}\frac{1}{\sqrt{1.81}}\cos(\frac{\pi}{2}n + 1.52)). \end{aligned}$$

Problem 4:

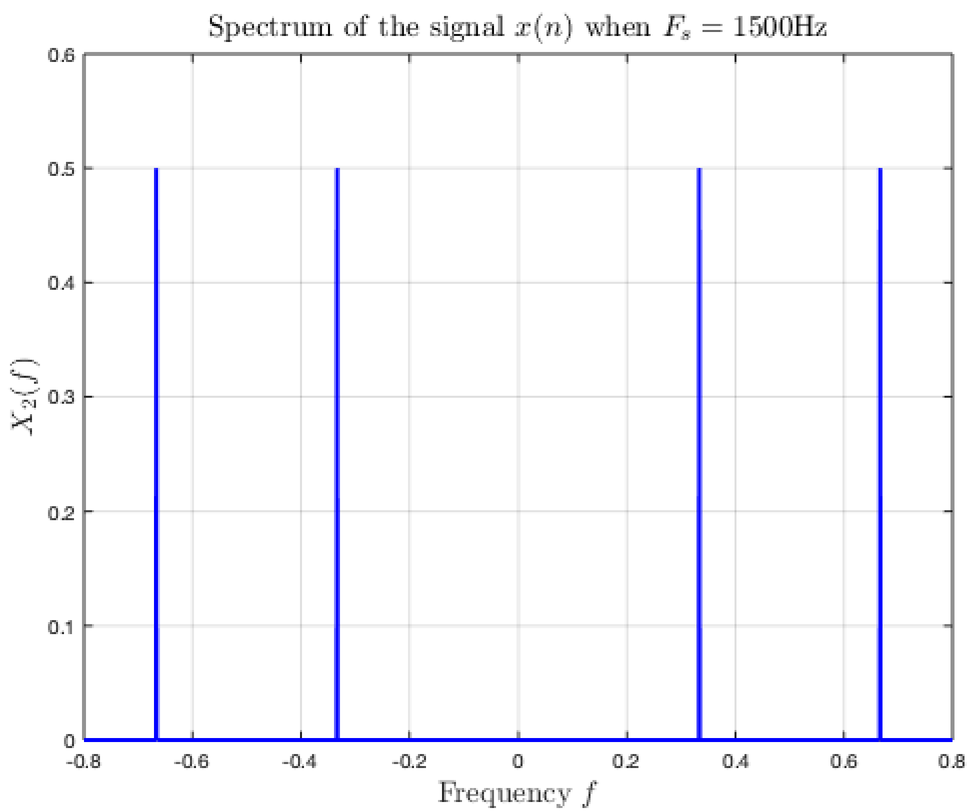
(a).

The spectra of the sampled signals are shown in the following Figures. The latter has a wider range of frequencies than the required $f \in [-\frac{1}{2}, \frac{1}{2}]$ to help making difference between alias components and signal components. The theory behind this is in ch.6.

```
F_s=4000
```




```
F_s=1500
```



(b).

Matlab-code for generating the signal corresponding to $F_s = 4000$:

```
t = [0:1/4000:1-1/4000];
cos4000 = cos(1000*2*pi*t);
```

And for the signal corresponding to $F_s = 1500$:

```
t = [0:1/1500:1-1/1500];
cos1500 = cos(1000*2*pi*t);
```

The sounds can be played with the commands:

```
sound(cos4000,4000);
pause(1)
sound(cos1500,1500);
```

They sound different because the signal incurred aliasing in the sampling. To be able to reconstruct $x_a(t)$ from a sampled signal, the sampling theorem requires that $F_s > 2F_{\max}$, where $F_{\max}$ is the highest frequency component of the signal. In this case,the signal has only one frequency component, at 1000Hz. Thus, we require $F_s > 2000\text{Hz}$.