Norwegian University of Science and Technology
Department of Electronics and Telecommunications

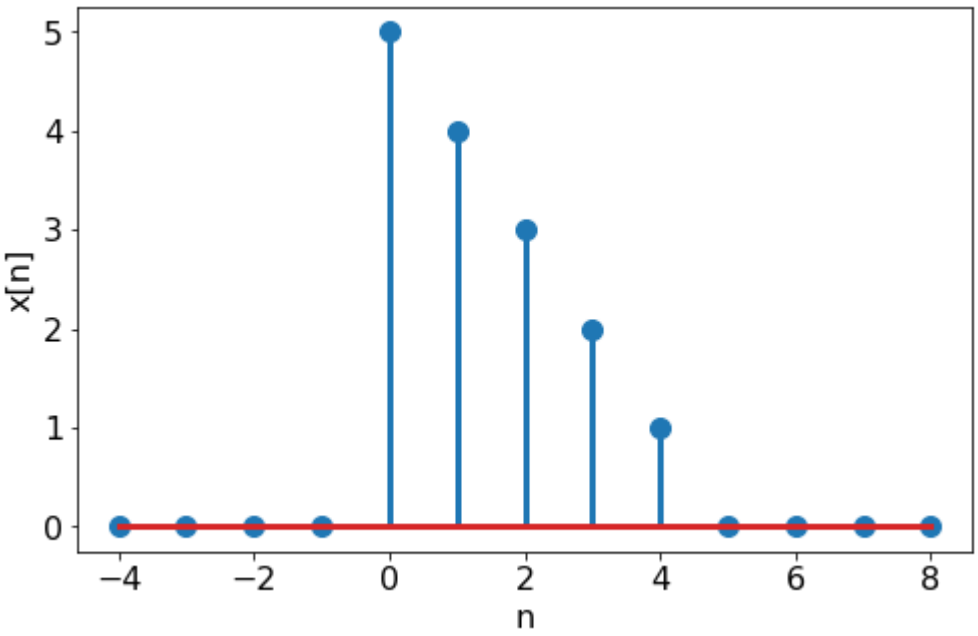
# TTT4120 Digital Signal Processing

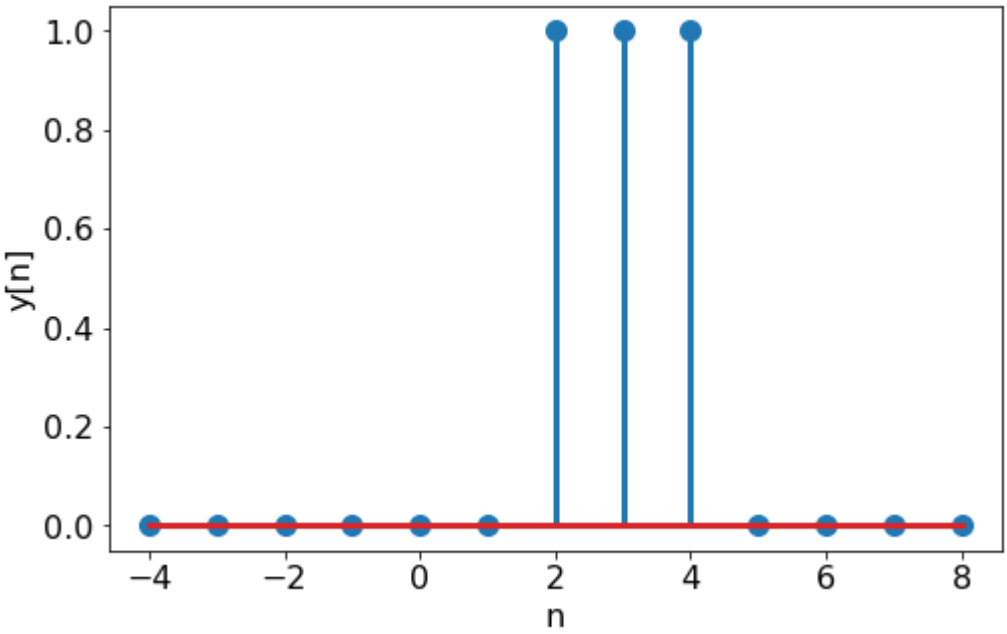
# Suggested Solutions for Problem Set 1

In [2]:
```
import numpy as np
import sounddevice as sd
import matplotlib.pyplot as plt
%matplotlib inline
```

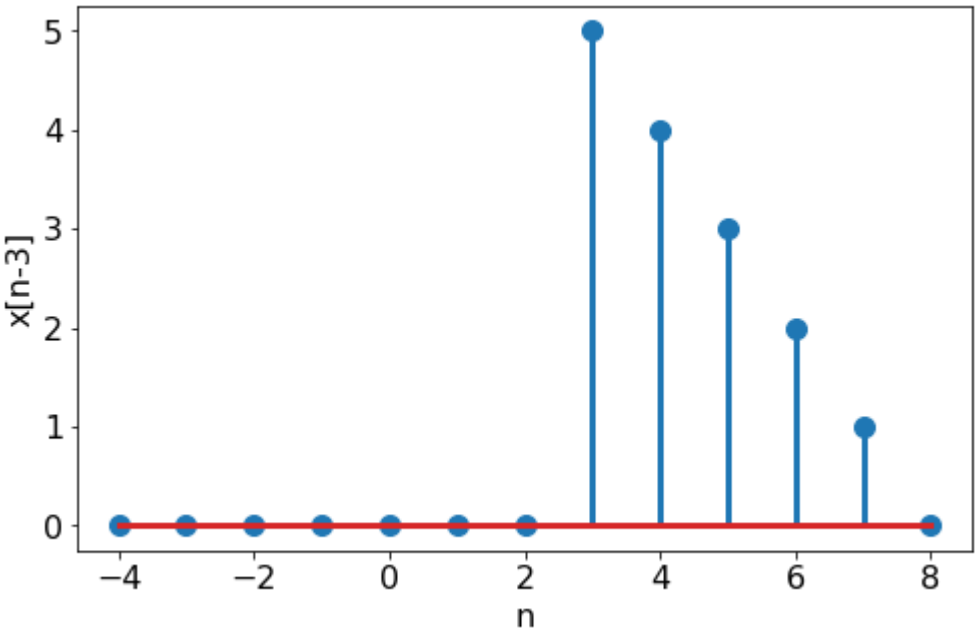
## Problem 1

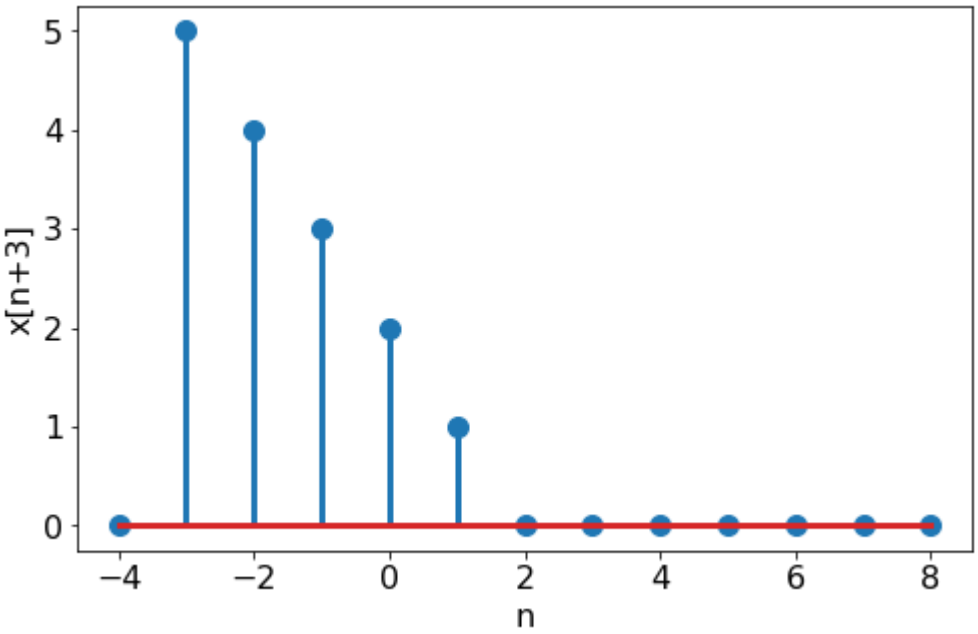
(a) The signals $x[n]$ and $y[n]$ are shown in the following figure.



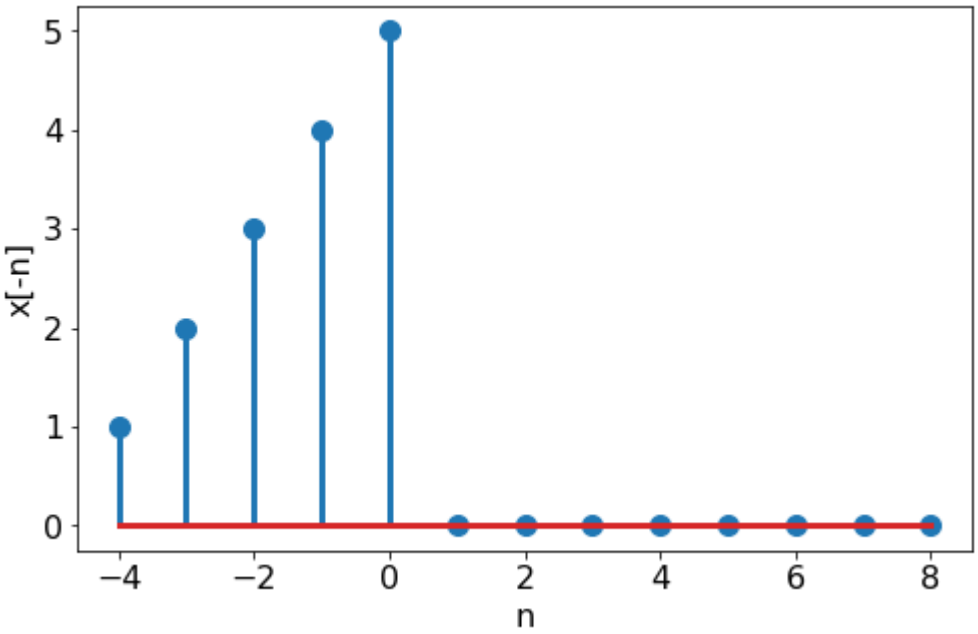



(b) When $k$ is positive, the signal will be shifted to the right, and for negative $k$, the signal will be shifted left. Thus, we get the sketches below.

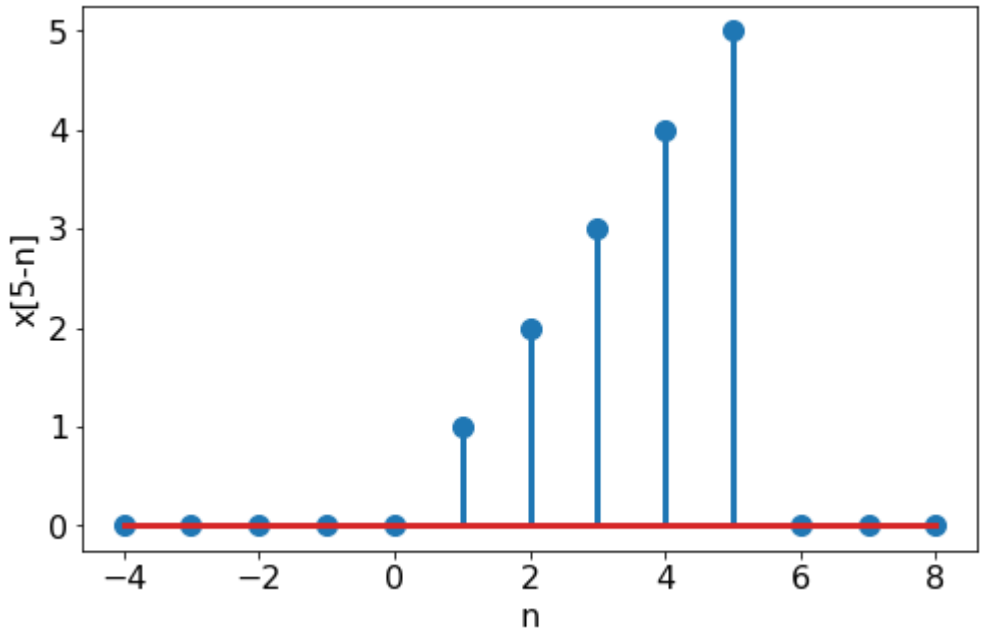





(c) The signal $x[-n]$ will be $x[n]$ flipped about $n = 0$. The resulting sketch is shown in the following figure.



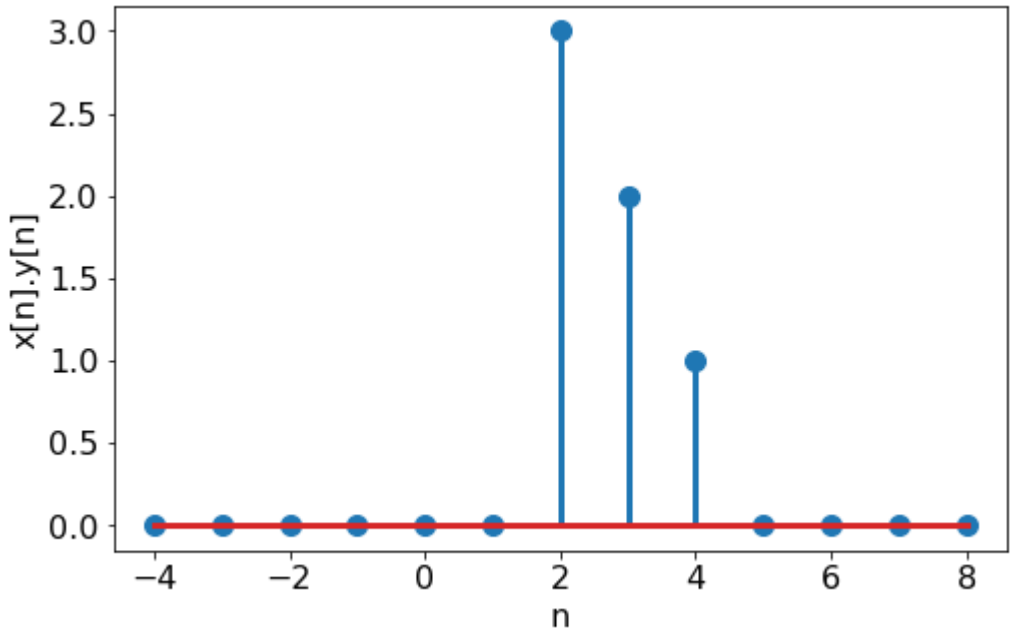

(d) The signal $x[5 - n]$ will be a flipped version of $x[n]$ shifted to the right. The sketch is shown in the following figure.




(e) The signal $y[n]$ is a window signal. When multiplying $x[n]$ by $y[n]$, the two first samples of $x[n]$ will be removed. Thus, we get

$$z[n] = \begin{cases} 5-n & 2 \leq n \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

The sketch of the resulting signal $z[n]$ is shown in the following figure.



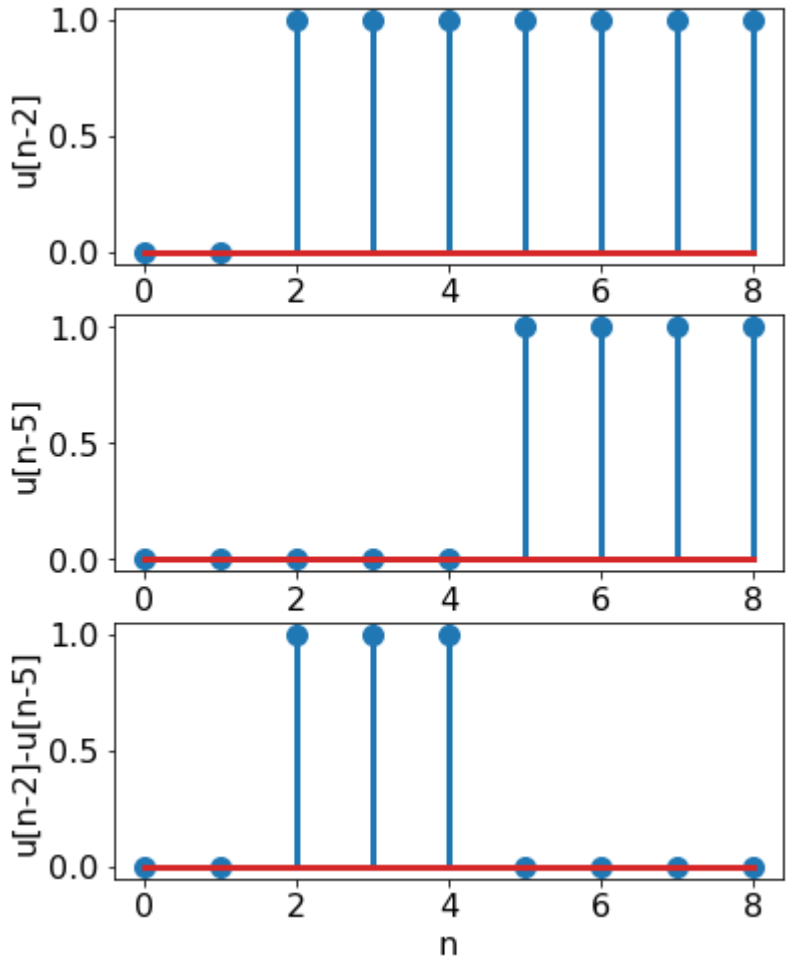

(f) The signal $x[n]$ can be expressed as:

$$x[n] = 5\delta[n] + 4\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$$

(g) $y[n]$ can be expressed as the difference between two unit step signals:

$$y[n] = u[n-2] - u[n-5]$$

which is shown in the following figure.




(h) The energy of $x[n]$ can be found as:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = 25 + 16 + 9 + 4 + 1 = 55.$$

# Problem 2

(a) The normalized frequency is used to represent discrete time signals in the frequency domain. Discrete time signals have a periodic structure in the frequency domain. The period is $[-.5, 0.5)$ (or $[0, 1)$). Using the first alternative we must have that $f_1 \in [-0.5, 0.5)$ which corresponds to $F_1 = F_s * f_1 \in [-3000, 3000)$ Hz for $F_s = 6000$ Hz.

(b) A sampled sinusoidal signal of 4 seconds can be generated as:

In [13]:
```
duration = 4 # in seconds
f1 = 0.3     # normalized frequency

Fs = 1000    # sampling frequency
n = np.arange(duration*Fs)
signal = np.sin(2*np.pi*f1*n)
```

(c) For $F_s = 1000/3000/12000$Hz the normalized frequency $f_1 = 0.3$ corresponds to $F_1 = f_1 * F_s = 300/900/3600$Hz, respectively. Thus we will hear a higher tone when we increase the sampling rate. Thus a constant normalized frequency can correspond to any physical frequency depending on the chosen sampling rate. Especially for filter design we will see that this is an advantage. The signal can be played as:

In [14]:
```
sd.play(signal,Fs)
```

In [15]:
```
Fs = 3000    # sampling frequency
n = np.arange(duration*Fs)
signal = np.sin(2*np.pi*f1*n)

sd.play(signal,Fs)
```

In [16]:
```
Fs = 12000    # sampling frequency
n = np.arange(duration*Fs)
signal = np.sin(2*np.pi*f1*n)

sd.play(signal,Fs)
```

(d) Now we use the formula $f_1 = F_1/F_s$. Thus for a sampling rate of $F_2 = 8000$Hz the physical frequencies $F_1 = 1000/3000/6000$Hz correspond to $f_1 = F_1/F_s = 0.125/0.375/0.75$, respectively. Logically one should expect a higher tone as the physical frequency $F_1$ increases. However, for $F_1 > F_2/2 = 4000$Hz, we violate the Nyquist sampling theorem. This applies for $F_1 = 6000$Hz, i.e. $f_1 = 0.75 > 0.5$. Due to the periodicity of one this frequency will be converted to $1 - f_1 = 0.25$ which corresponds to that we hear the physical frequency $F_1 = 0.25 * 8000 = 2000$Hz.

In [17]:
```
Fs = 8000    # fixed sampling frequency
duration = 4 # in seconds
n = np.arange(duration*Fs)
```

In [18]:
```
F1 = 1000    # physical frequency
f1 = F1/Fs   # normalized frequency
signal = np.sin(2*np.pi*f1*n)
sd.play(signal,Fs)
```

In [19]:
```
F1 = 3000    # physical frequency
f1 = F1/Fs   # normalized frequency
signal = np.sin(2*np.pi*f1*n)
sd.play(signal,Fs)
```

In [20]:
```
F1 = 6000    # physical frequency
f1 = F1/Fs   # normalized frequency
signal = np.sin(2*np.pi*f1*n)
sd.play(signal,Fs)
```

# Problem 3

(a) Since this system involves the quadratic term $x^2[n-1]$, it is not linear. However, since the difference equation has constant coefficients (independent of $n$), the system is time-invariant. It is also causal, since $y[n]$ only depends on present and past samples of $x[n]$.

To show the time-invariance property from the definition, we excite the system with a delayed signal $x_1[n] = x[n-k]$, and find the output signal $y_1[n]$. If $y_1[n] = y[n-k]$, the system is time-invariant.

$$\begin{aligned} y_1[n] &= x_1[n-k] - x_1^2[n-k-1] \\ &= y[n-k] \end{aligned}$$

Thus, we have shown that the system is time-invariant.

Now, to show that it is not linear from the definition, we excite the system with two different signals $x_1[n]$ and $x_2[n]$. We call the output signals $y_1[n]$ and $y_2[n]$ respectively.

$$\begin{aligned} y_1[n] &= x_1[n] - x_1[n-1]^2 \\ y_2[n] &= x_2[n] - x_2[n-1]^2 \end{aligned}$$

Then, we excite the system with another signal, $x_3[n] = a_1x_1[n] + a_2x_2[n]$. If the system is linear then the corresponding output signal should be $y_3[n] = a_1y_1[n] + a_2y_2[n]$.

$$\begin{aligned} y_3[n] &= x_3[n] - x_3^2[n-1] \\ &= a_1x_1[n] + a_2x_2[n] - (a_1x_1[n-1] + a_2x_2[n-1])^2 \\ &= a_1x_1[n] + a_2x_2[n] - ((a_1x_1[n-1])^2 + 2a_1a_2x_1[n-1]x_2[n-1] + (a_2x_2[n-1])^2) \\ &= a_1(x_1[n] - a_1x_1^2[n]) + a_2(y_2[n] - a_2y_2^2[n]) - 2a_1a_2x_1[n]x_2[n-1] \\ &\neq a_1y_1[n] + a_2y_2[n] \end{aligned}$$

Thus, we have shown that the system is not linear.

(b) Since $y[n]$ is now a linear combination of samples from $x[n]$, this system is linear. However, since one of the coefficients is dependent on $n$, the system is not time-invariant. Finally, since $y[n]$ only depends on present and past samples of $x[n]$, the system is causal.

We now check time-invariance and linearity by the definitions. First time-invariance. Let $x_1[n] = x[n-k]$. Then

$$\begin{aligned} y_1[n] &= nx_1[n] + 2x_1[n-2] \\ &= [n-k]x[n-k] + 2x[n-k-2] \\ &\neq y[n-k] \end{aligned}$$

Now, we check linearity. Let $x_3[n] = a_1x_1[n] + a_2x_2[n]$

$$\begin{aligned} y_1[n] &= nx_1[n] + 2x_1[n-2] \\ y_2[n] &= nx_2[n] + 2x_2[n-2] \\ y_3[n] &= nx_3[n] + 2x_3[n-2] \\ y_3[n] &= a_1(nx_1[n] + 2x_1[n-2]) + a_2(nx_2[n] + x_2[n-2]) \\ &= a_1y_1[n] + a_2y_2[n] \end{aligned}$$

Thus, the system is linear.

(c) In this system $y[n]$ is a simple linear combination of present and past samples of $x[n]$ with constant coefficients. Thus, this system is time-invariant, linear, and causal. Again, we can check this by the definitions.

$$\begin{aligned} y_1[n] &= x_1[n] - x_1[n-1] \\ &= x[n-k] - x[n-k-1] \\ &= y[n-k] \end{aligned}$$

Thus, we have shown time-invariance. Then we show that the system is linear.

$$\begin{aligned} y_1[n] &= x_1[n] - x_1[n-1] \\ y_2[n] &= x_2[n] - x_2[n-1] \\ y_3[n] &= x_3[n] - x_3[n-1] \\ &= a_1x_1[n] + a_2x_2[n] - a_1x_1[n-1] - a_2x_2[n-1] \\ &= a_1y_1[n] + a_2y_2[n] \end{aligned}$$

(d) This system is both linear and time-invariant for the same reasons as the system in (c). However, in this system $y[n]$ depends on a future sample of $x[n]$. Thus, the system is not causal.

# Problem 4

(a) The unit sample response is obtained at the output of the system when the system is excited by a unit sample $\delta[n]$. Thus, if we replace the signal $x[n]$ in the difference equation by the $\delta$ signal, we can replace the output signal $y[n]$ by the unit sample response $h[n]$. For the first system, we get

$$\begin{aligned} h[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \\ &= \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 1 & n = 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For the second system we have

$$h[n] = -0.9h[n-1] + \delta[n]$$

In this case we have a recursive equation. An iterative method can be used to find the unit sample response. Note that $h[n] = 0$ for $n < 0$ since the system is causal. So we only have to find $h[n]$ for $n \geq 0$. We start by determining $h[0]$.

$$h[0] = 0.9h[-1] + 1 = -0.9 \cdot 0 + 1 = 1$$

Now, for $n \neq 0$, we have

$$h[n] = -0.9h[n-1].$$

Now, we do some iterations.

$$\begin{aligned} h[1] &= -0.9h[0] = -0.9 \\ h[2] &= -0.9h[1] = (-0.9)^2 \\ h[3] &= -0.9h[2] = (-0.9)^3 \\ &\vdots \\ h[n] &= (-0.9)^n \text{ for } n \geq 0 \\ &= (-0.9)^n u[n] \end{aligned}$$

(b) As we saw in (a), the first system has a finite length unit sample response, while the unit sample response of the other system was of infinite length. Thus, the two systems are FIR and IIR, respectively.

(c) To check whether the systems are stable, we need to check whether

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

For the first system, we get

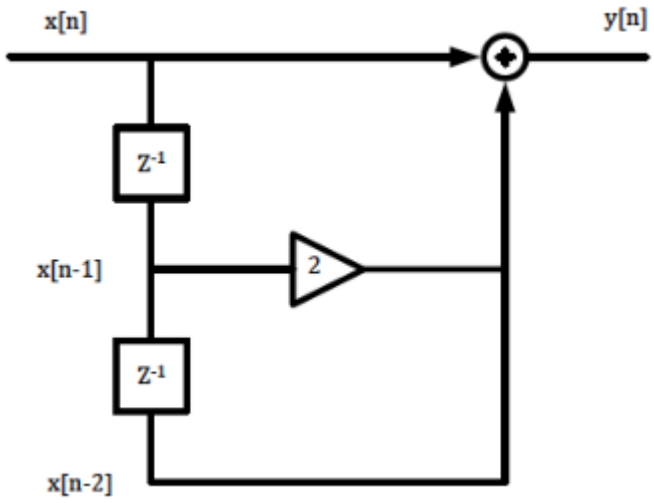
$$\sum_{n=-\infty}^{\infty} |h[n]| = 1 + 2 + 1 = 4$$

so this system is stable. Note that all FIR systems are stable. For the second system we get

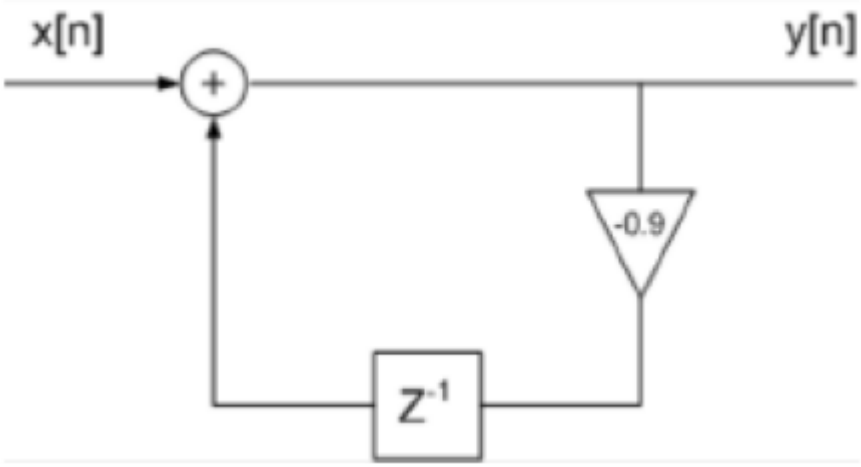
$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h[n]| &= \sum_{n=0}^{\infty} |(-0.9)^n| \\ &= \sum_{n=0}^{\infty} 0.9^n \\ &= \frac{1}{1-0.9} \\ &= 10\end{aligned}$$

so this system is also stable.

(d) Filter structure for the first system:




Filter structure for the second system:

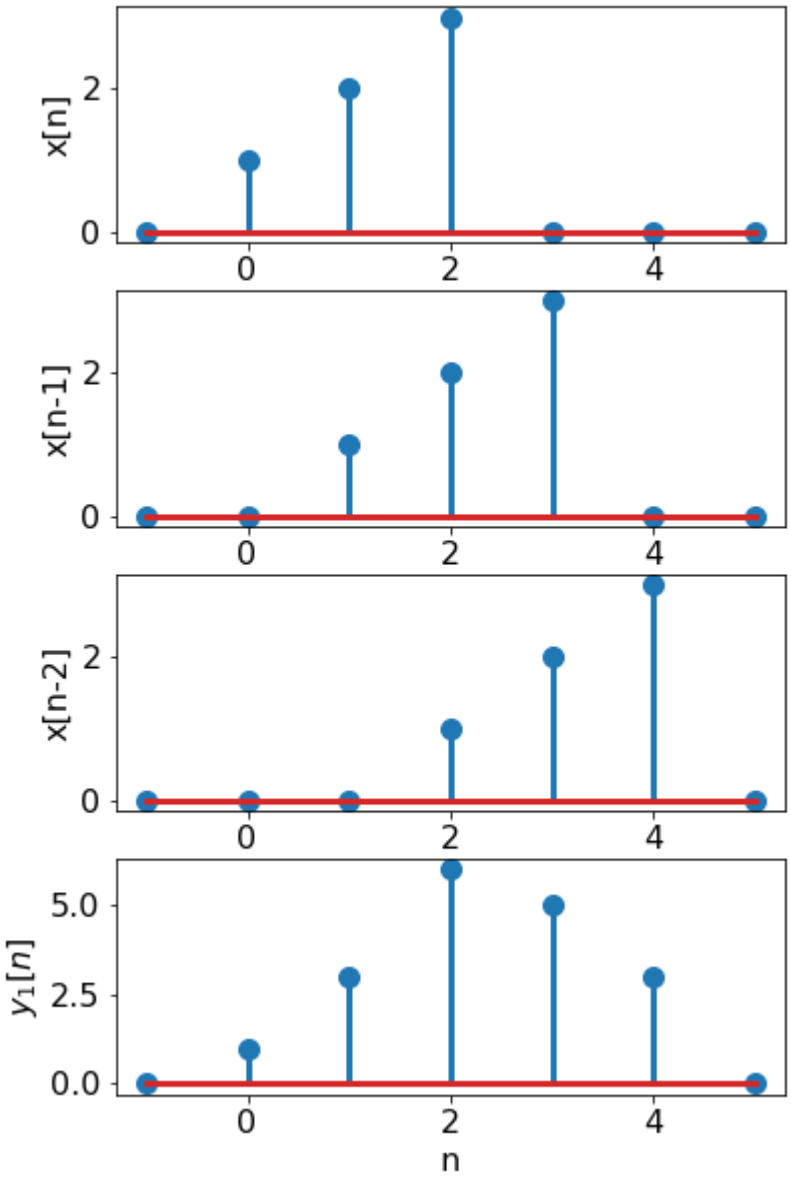



# Problem 5

(a) The signal $y_1[n]$ can be computed as follows

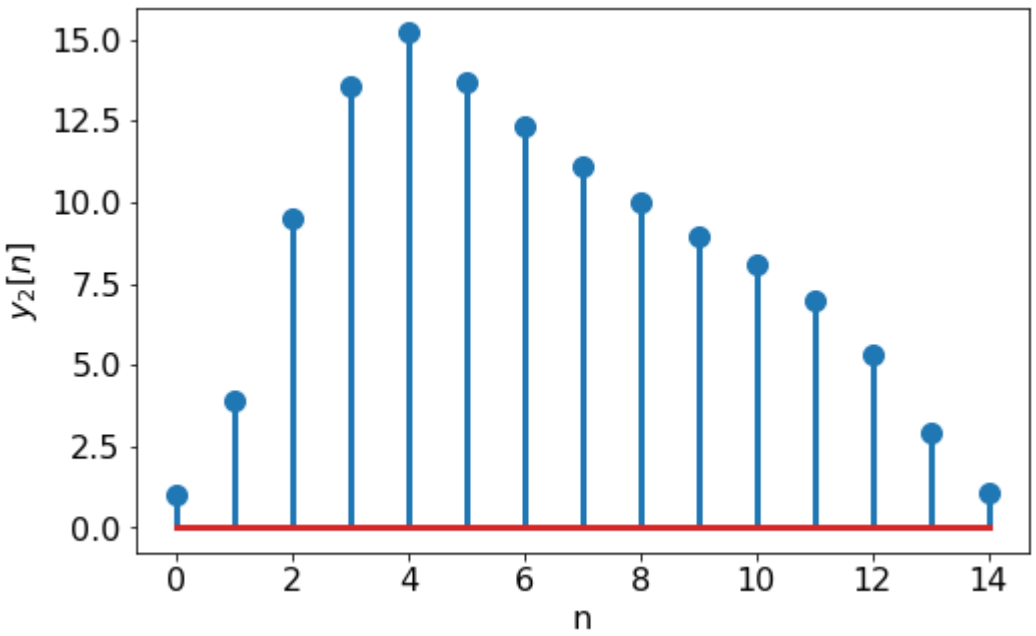
$$\begin{aligned} y_1[n] &= x[n] * h_1[n] = x[n] * (\delta[n] + \delta[n-1] + \delta[n-2]) \\ &= x[n] * \delta[n] + x[n] * \delta[n-1] + x[n] * \delta[n-2] \\ &= x[n] + x[n-1] + x[n-2] \end{aligned}$$

To get the final result we can use a graphical computation method, which is displayed in the following.



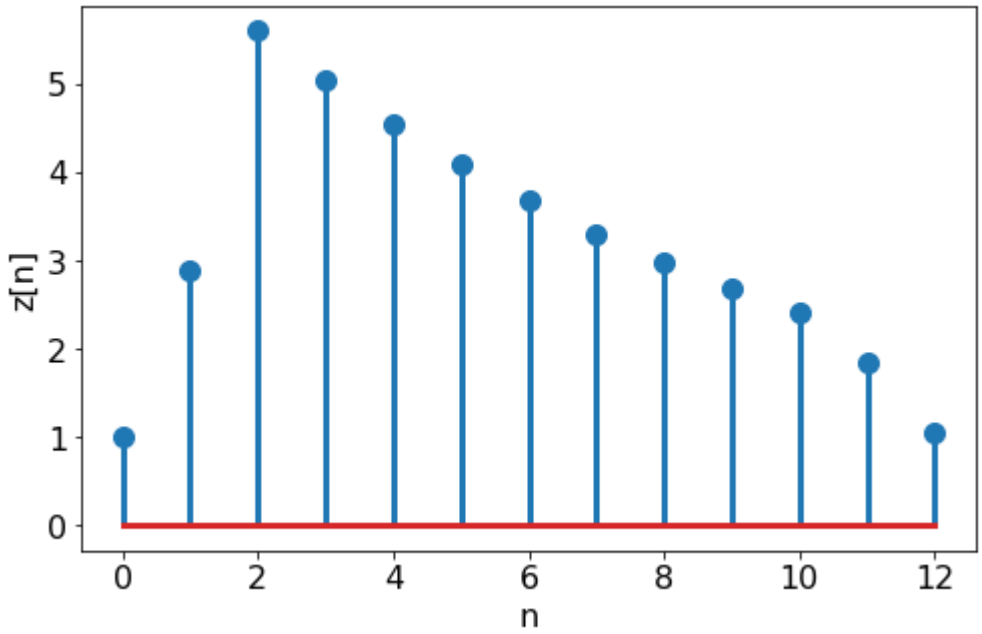

(b) The second ouput is shown below and it can be computed as follows:

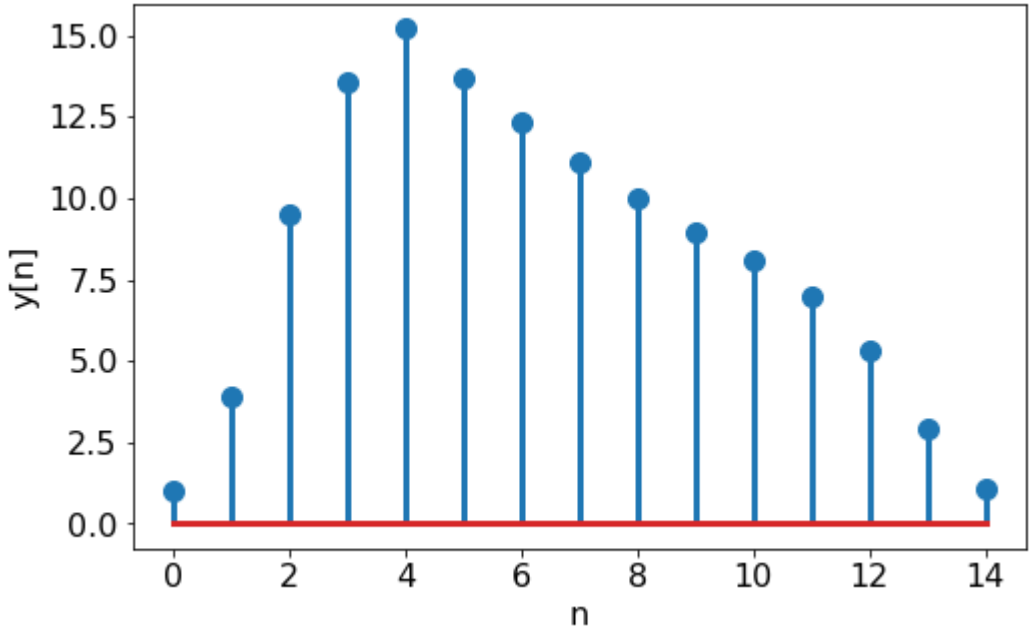
In [26]:
```
y1n = [1, 3, 6, 5, 3]
h2n = 0.9**np.arange(11)
y2n = np.convolve(h2n, y1n)

plt.stem(range(len(y2n)), y2n)
plt.ylabel('$y_2[n]$')
plt.xlabel('n')
plt.show()
```




(c) The length of an output signal $y[n]$ is $L_x + L_h - 1$, where $L_x$ and $L_h$ are the length of the input signal and the unit sample response of the filter. In our problem, $y_1[n]$ has length $3 + 3 - 1 = 5$ and $y_2[n]$ has length $5 + 11 - 1 = 15$.

(d) Since the convolution operation is commutative, it does not matter which filter comes first. Thus, the plot of the output signal after the second filter, $h_1[n]$ in this case, is exactly equal to the figure in (b). However, the output of the first filter, $h_2[n]$ in this case, is different than before and it is shown below.


z[n]
n
0
1
2
3
4
5
0
2
4
6
8
10
12


y[n]
n
0.0
2.5
5.0
7.5
10.0
12.5
15.0
0
2
4
6
8
10
12
14