Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing

Suggested Solutions for Problem Set 2

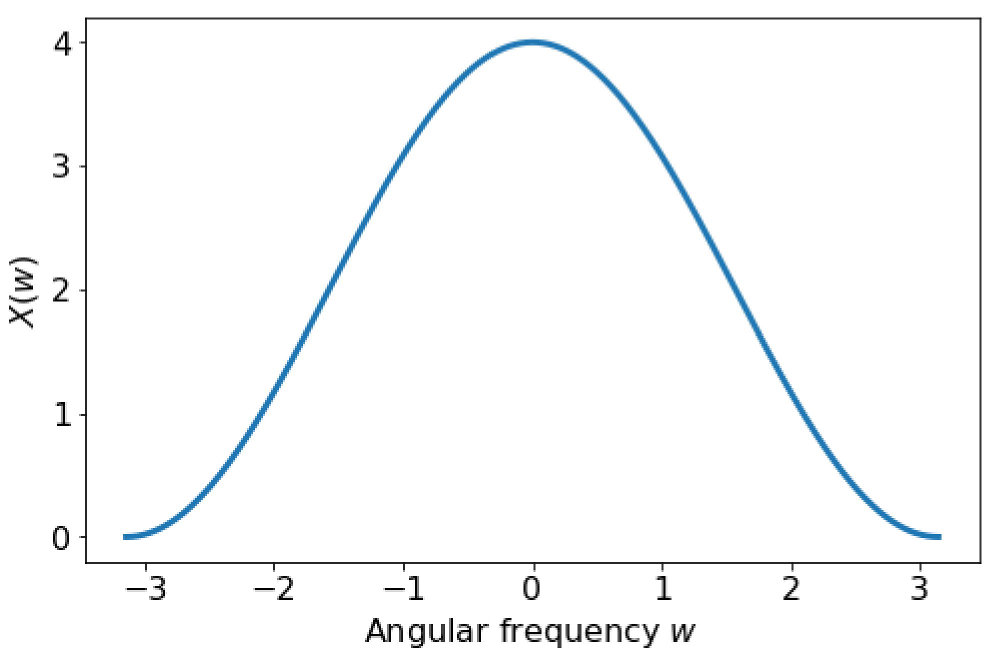
In [2]:
```
import numpy as np
from scipy import signal
import sounddevice as sd
import matplotlib.pyplot as plt
%matplotlib inline
```

Problem 1

(a) The spectrum $X(\omega)$ can be found as follows.

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= e^{j\omega} + 2 + e^{-j\omega} \\ &= 2 + 2\cos\omega \end{aligned}$$

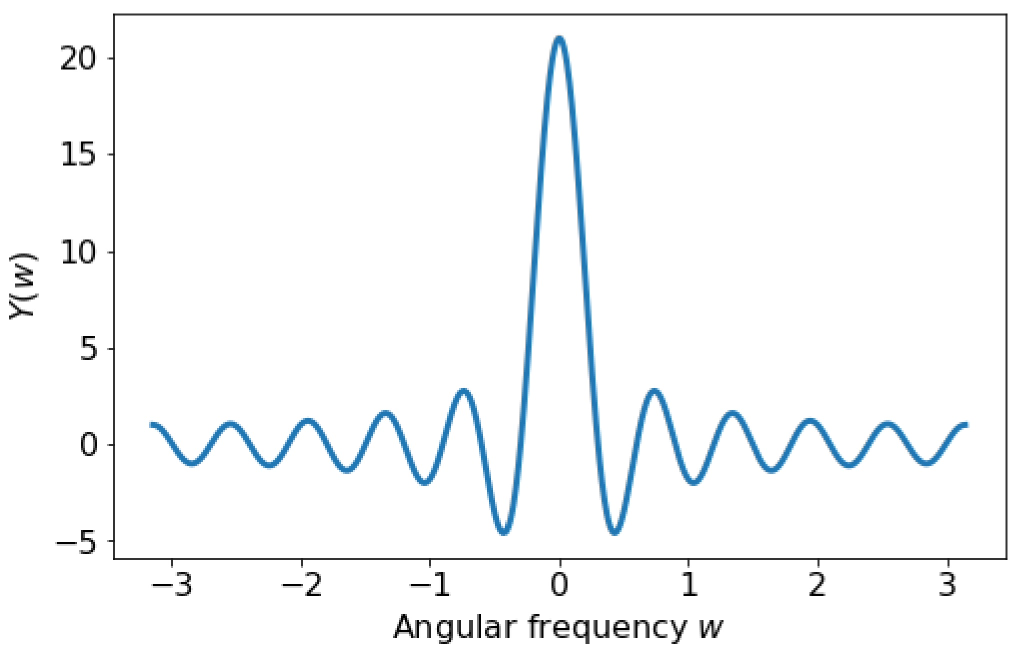
It is shown in the following figure.



(b) The spectrum $Y(\omega)$ can be found as follows.

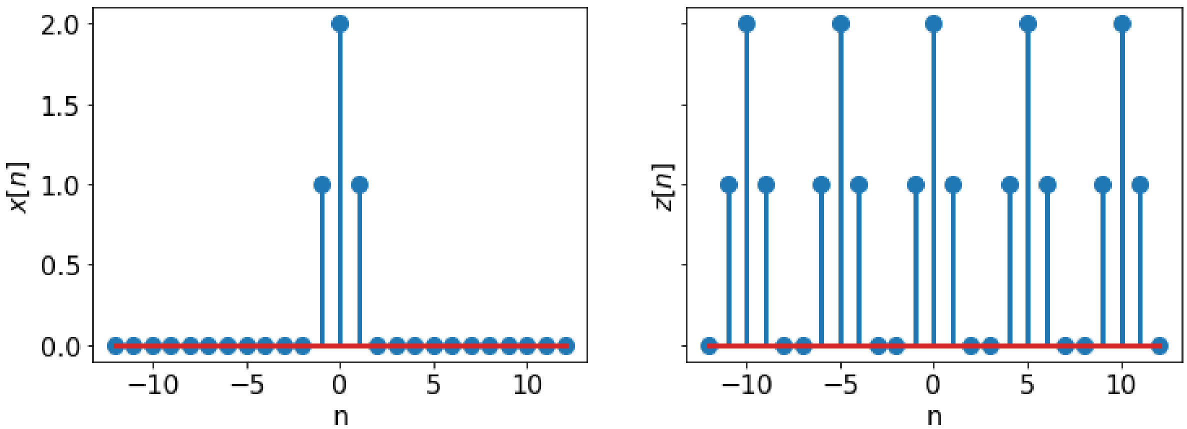
$$\begin{aligned}
Y(\omega) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \\
&= \sum_{n=-M}^{M} e^{-j\omega n} \quad l = n + M \\
&= \sum_{l=0}^{2M} e^{-j\omega(l-M)} \\
&= e^{j\omega M} \sum_{l=0}^{2M} e^{-j\omega l} \\
&= e^{j\omega M} \frac{1 - e^{-j\omega(2M+1)}}{1 - e^{-j\omega}} \\
&= \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} \\
&= \frac{e^{-\frac{j\omega}{2}}}{e^{-\frac{j\omega}{2}}} \frac{\left(e^{j\omega(M+\frac{1}{2})} - e^{-j\omega(M+\frac{1}{2})}\right)}{\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)} \\
&= \frac{\sin\left(\omega(M+\frac{1}{2})\right)}{\sin(\frac{\omega}{2})}
\end{aligned}$$

The sketch is shown in the following figure.



(c) Because they are even signals.

(d) A sketch of $x[n]$ and $z[n]$ for N=5 is shown below.



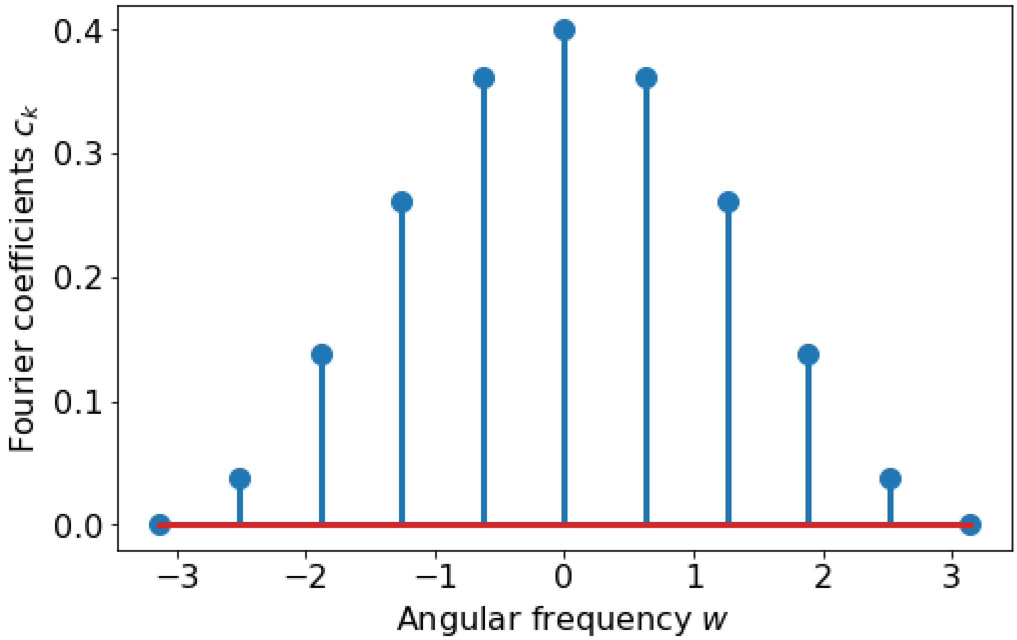
The Fourier coefficients are given by:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} z[n]e^{-j2\pi kn/N}, \quad k = 0, \cdots, N-1.$$

Note that we sum from 0 up to $N-1$. Thus, the first two samples are 2 and 1 respectively, and the last sample is 1. All other samples are 0. The coefficients could be calculated over any other period.

$$\begin{aligned}
c_k &= \frac{1}{N}\sum_{n=0}^{N-1} z[n]e^{-j2\pi kn/N} \\
&= \frac{1}{N}(2 + e^{-j2\pi k/N} + e^{-j2\pi k(N-1)/N}) \\
&= \frac{1}{N}(2 + e^{-j2\pi k/N} + e^{-j2\pi k}e^{j2\pi k/N}) \\
&= \frac{1}{N}(2 + e^{-j2\pi k/N} + e^{j2\pi k/N}) \\
&= \frac{1}{N}(2 + 2\cos(2\pi k/N))
\end{aligned}$$

The Fourier coefficients are displayed below.



(e) We have the following.

$$X(f) = 2 + 2\cos(2\pi f)$$
$$c_k = \frac{1}{N}(2 + 2\cos(2\pi k/N))$$

Thus, we see that

$$c_k = \frac{1}{N} X\left(\frac{k}{N}\right).$$

This means that the Fourier coefficients are (scaled) samples of the continuous spectrum $X(f)$. This always holds true: a periodic extension in the time domain equals sampling in the frequency domain.

Problem 2

(a) For the first case, we use the time-shift property of the DTFT, and get

$$X_1(\omega) = e^{j3\omega} X(\omega)$$

(b) For the second case, we use the time-reversal property of the DTFT, and it follows that

$$X_2(\omega) = X(-\omega)$$

(c) For the third case notice that:

$$x_3[n] = x[3-n] = x[-(n-3)] = x_2[n-3]$$

so that by the time-reversal and time-shift properties, it follows that

$$X_3(\omega) = e^{-j3\omega} X_2(\omega) = e^{-j3\omega} X(-\omega)$$

(d) For the last case, we have that

$$X_4(\omega) = \mathrm{DTFT}\{x[n] * w[n]\} = X(\omega) W(\omega).$$

Problem 3

(a) By taking the DTFT of both sides of the first difference equation, we get

$$\begin{aligned} Y(\omega) &= X(\omega) + 2e^{-j\omega}X(\omega) + e^{-2j\omega}X(\omega) \\ H_1(\omega) &= \frac{Y(\omega)}{X(\omega)} = 1 + 2e^{-j\omega} + e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega}) \\ &= e^{-j\omega}(2 + 2\cos\omega). \end{aligned}$$

And for the second case, we get

$$\begin{aligned} Y(\omega) &= -0.9Y(\omega)e^{-j\omega} + X(\omega) \\ H_2(\omega) &= \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + 0.9e^{-j\omega}}. \end{aligned}$$

(b) We already have the frequency response $H_1(\omega)$ on polar form. Thus, the magnitude is simply

$$|H_1(\omega)| = 2 + 2\cos\omega.$$

Since $2 + 2\cos\omega \geq 0$ for all ω, the phase is simply

$$\Theta_1(\omega) = \measuredangle H_1(\omega) = -\omega.$$

The magnitude response of the second system can be found as follows.

$$\begin{aligned} |H_2(\omega)| &= \left|\frac{1}{1 + 0.9e^{-j\omega}}\right| \\ &= \frac{1}{|1 + 0.9e^{-j\omega}|} \\ &= \frac{1}{\sqrt{(1 + 0.9\cos\omega)^2 + (0.9\sin\omega)^2}} \\ &= \frac{1}{\sqrt{1 + 1.8\cos\omega + 0.81}} \end{aligned}$$

To find the phase, we can write $H_2(\omega)$ as

$$H_2(\omega) = \frac{1}{W(\omega)},$$

where $W(\omega) = 1 + 0.9e^{-j\omega}$. Then, the phase is given by

$$\Theta_2(\omega) = \measuredangle H_2(\omega) = -\measuredangle W(\omega).$$

Since Re$\{W(\omega)\} > 0$ for all ω, we have

$$\begin{aligned} \measuredangle H_2(\omega) &= -\tan^{-1}\left(\frac{-0.9\sin\omega}{1 + 0.9\cos\omega}\right) \\ &= \tan^{-1}\left(\frac{0.9\sin\omega}{1 + 0.9\cos\omega}\right). \end{aligned}$$

We notice that all magnitude functions are even and that all phase functions are odd. This is a property of real signals.

(c) The frequency response of the first filter can be found and plotted by the following code.

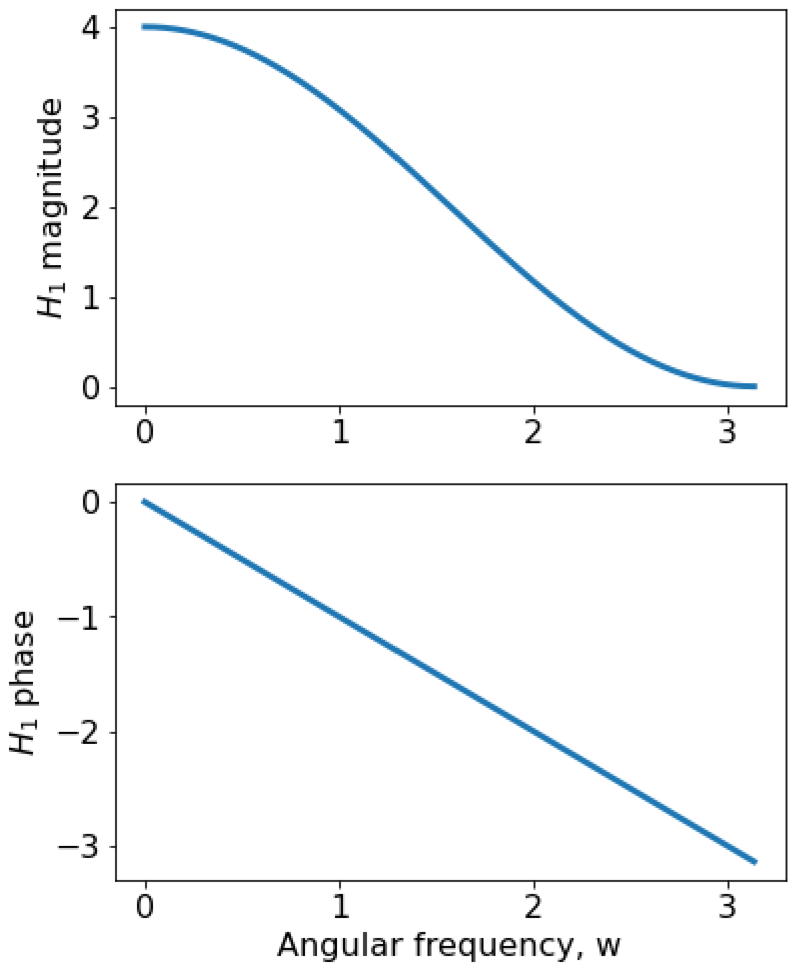
In [8]:
```
[w, H1] = signal.freqz([1, 2, 1], [1]);

plt.subplots(2,1,figsize=(6,8),sharex=True)

plt.subplot(2, 1, 1)
plt.plot(w, np.abs(H1))
plt.ylabel('$H_1$ magnitude')

plt.subplot(2, 1, 2)
plt.plot(w, np.angle(H1))
plt.xlabel('Angular frequency, w')
plt.ylabel('$H_1$ phase')

plt.show()
```



For the second filter, we change the *freqz* command as follows.

In [9]:
```
[w, H2] = signal.freqz([1], [1, 0.9])
```

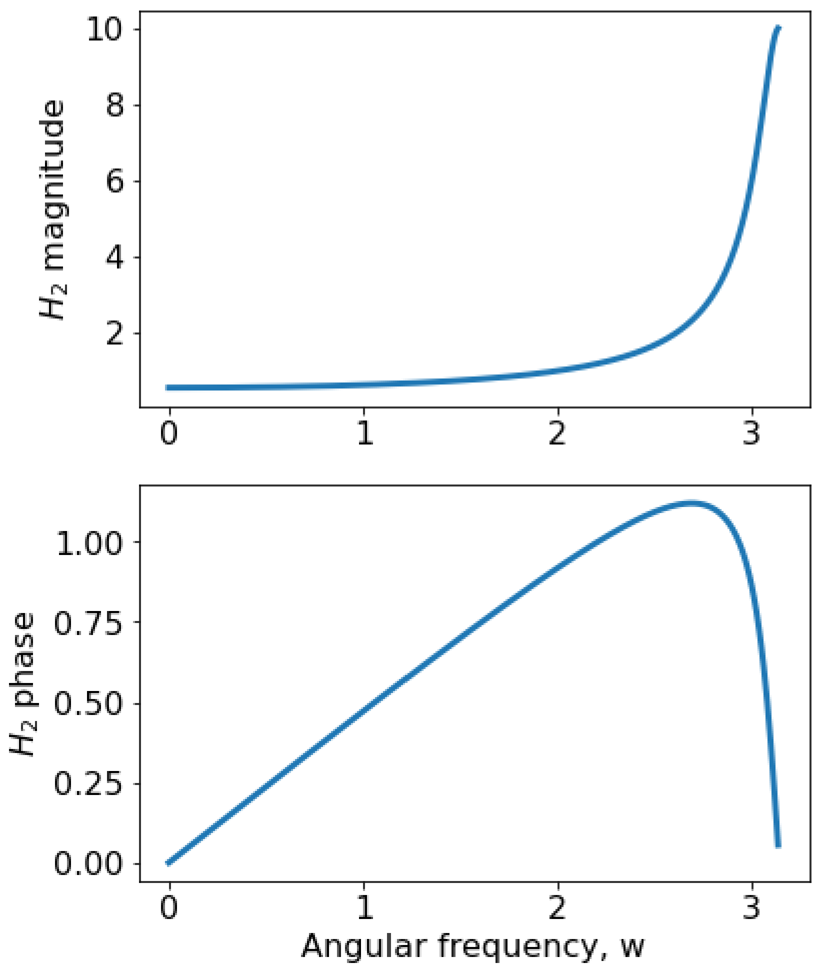
This gives the plots shown in the following figures.

In [10]:
```
plt.subplots(2,1,figsize=(6,8),sharex=True)

plt.subplot(2, 1, 1)
plt.plot(w, np.abs(H2))
plt.ylabel('$H_2$ magnitude')

plt.subplot(2, 1, 2)
plt.plot(w, np.angle(H2))
plt.xlabel('Angular frequency, w')
plt.ylabel('$H_2$ phase')

plt.show()
```



(d) From the plots of the magnitude responses, we can see that the first filter attenuates high frequencies more than low frequencies. Thus, this is a lowpass filter. The second filter attenuates low frequencies more than high frequencies. Thus, this is a highpass filter.

(e) The response of a LTI-system $H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$ to a sinusoidal input signal $x(n) = A\cos(\omega_0 n + \theta)$ equals

$$y(n) = A|H(\omega_0)|\cos(\omega_0 n + \theta + \Theta(\omega_0)).$$

Thus, the output of the first system is

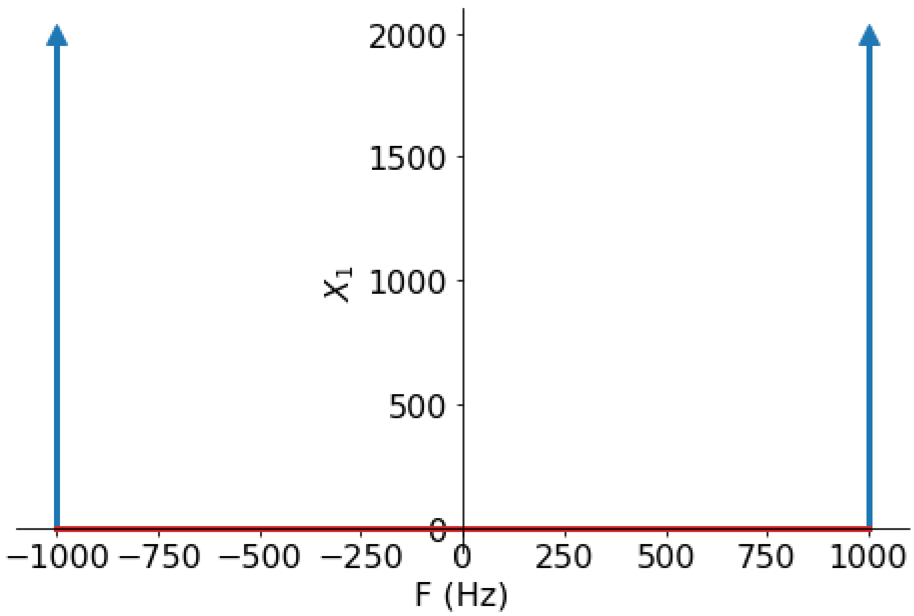
$$\begin{aligned} y_1(n) &= \frac{1}{2}|H_1\left(\frac{\pi}{2}\right)|\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} + \Theta_1\left(\frac{\pi}{2}\right)\right) \\ &= \frac{1}{2}\cdot 2\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} - \frac{\pi}{2}\right) \\ &= \cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right). \end{aligned}$$

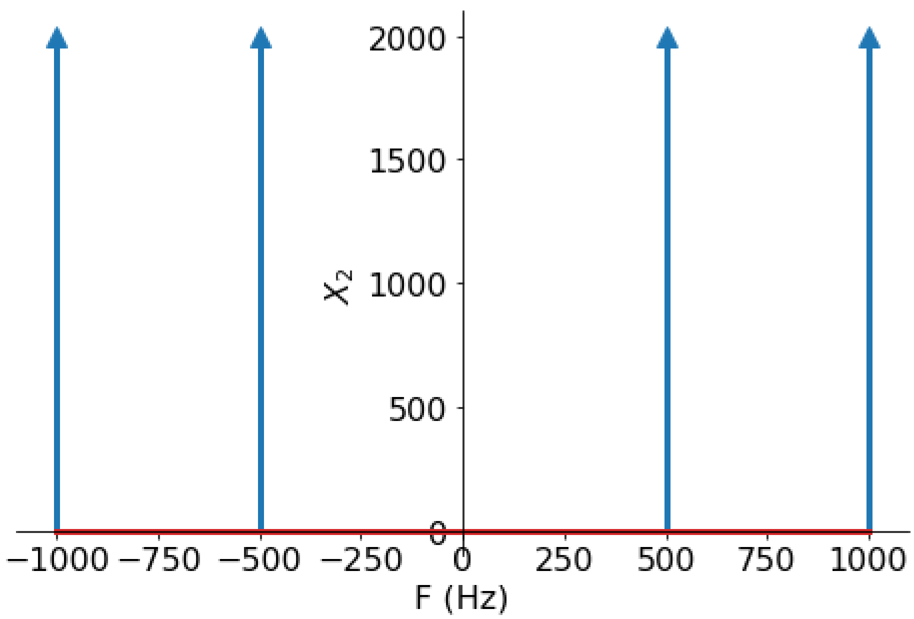
Likewise, the output of the second system is

$$\begin{aligned} y_2(n) &= \frac{1}{2}|H_2\left(\frac{\pi}{2}\right)|\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} + \Theta_2\left(\frac{\pi}{2}\right)\right) \\ &= \frac{1}{2}\frac{1}{\sqrt{1.81 + 1.8\cos\left(\frac{\pi}{2}\right)}}\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} + \tan^{-1}\left(\frac{0.9\sin\left(\frac{\pi}{2}\right)}{1 + 0.9\cos\left(\frac{\pi}{2}\right)}\right)\right) \\ &= \frac{1}{2}\frac{1}{\sqrt{1.81}}\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} + \tan^{-1}\left(\frac{9}{10}\right)\right) \\ &\approx \frac{1}{2}\frac{1}{\sqrt{1.81}}\cos\left(\frac{\pi}{2}n + 1.52\right). \end{aligned}$$

Problem 4

(a) The spectra of the sampled signals are shown in the following figures.





The latter has a wider range of frequencies than the required $f \in [-\frac{1}{2}, \frac{1}{2}]$ to help making difference between alias components and signal components. The theory behind this is in ch.6.

(b) Python-code for generating the signal corresponding to $F_s = 4000$ and $F_s = 1500$:

In [13]:
```
t4000 = np.arange(0,1,1/4000)
cos4000 = np.cos(1000*2*np.pi*t4000)

t1500 = np.arange(0,1,1/1500)
cos1500 = np.cos(1000*2*np.pi*t1500)
```

The sounds can be played with the commands:

In [14]:
```
sd.play(cos4000,4000)
```

In [15]:
```
sd.play(cos1500,1500)
```

They sound different because the signal incurred aliasing in the sampling. To be able to reconstruct $x_a(t)$ from a sampled signal, the sampling theorem requires that $F_s > 2F_{\max}$, where $F_{\max}$ is the highest frequency component of the signal. In this case,the signal has only one frequency component, at 1000Hz. Thus, we require:

$$F_s > 2000\mathrm{Hz}$$