Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing

Suggested Solutions for Problem Set 3

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

Problem 1

(a) For the RC-filter we have

$$x(t) = Ri + y(t) \quad \text{and} \quad i = C\frac{dy(t)}{dt}$$

and after insertion

$$x(t) = RC\frac{dy(t)}{dt} + y(t).$$

Laplace transforming gives

$$X(s) = RCsY(s) + Y(s)$$

from which we get the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{RCs+1},$$

Now consider the RL-filter. We have

$$y(t) = L\frac{di}{dt} \quad \text{and} \quad x(t) = Ri + y(t).$$

Differentiating the latter equation and substituting in the former equation gives

$$\begin{aligned}\frac{dx(t)}{dt} &= R\frac{di}{dt} + \frac{dy(t)}{dt}\\ &= \frac{R}{L}y(t) + \frac{dy(t)}{dt}.\end{aligned}$$

Taking the Laplace transform of the above equation results in

$$sX(s) = \frac{R}{L}Y(s) + sY(s),$$

and the transfer function is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s+\frac{R}{L}}.$$

(b) The frequency response for the RC-filter is given by

$$H(\Omega) = H(s)|_{s=j\Omega} = \frac{1}{j\Omega RC + 1}.$$

The magnitude response is thus given by

$$|H(\Omega)| = \frac{1}{\sqrt{1 + (\Omega RC)^2}}.$$

We see that

$$|H(0)| = 1 \quad \text{and} \quad |H(\infty)| = 0,$$

and $|H(\Omega)|$ is monotonically decreasing function of Ω, which are the characteristics of a lowpass filter.

The frequency response for the RL-filter is given by

$$H(\Omega) = H(s)|_{s=j\Omega} = \frac{j\Omega}{j\Omega + \frac{R}{L}}.$$

The magnitude response is thus given by

$$|H(\Omega)| = \frac{\Omega}{\sqrt{\frac{R^2}{L^2} + \Omega^2}} = \frac{1}{\sqrt{\frac{R^2}{(\Omega L)^2} + 1}}.$$

We see that

$$|H(0)| = 0 \quad \text{and} \quad |H(\infty)| = 1,$$

and $|H(\Omega)|$ is monotonically increasing function of Ω, which is the characteristics of a highpass filter.

(c) The transfer function of the RC-filter can be written as

$$H(s) = \frac{1/RC}{s + 1/RC}$$

The impulse response can be determined simply from the table of common Laplace-transform pairs

$$h(t) = \frac{1}{RC}\ e^{-\frac{t}{RC}}\, u(t)$$

To find the impulse response of the RL-filter, first note that the transfer function can be written

$$H(s) = 1 - \frac{R/L}{s + R/L}.$$

Then

$$h(t) = \delta(t) - \frac{R}{L}\ e^{-\frac{R}{L}\,t}\, u(t)$$

Alternatively, $h(t)$ can be found in the following way. We have

$$\begin{aligned} H(s) &= s \cdot \frac{1}{s + \frac{R}{L}} = s \cdot G(s) \\ &= [s \cdot G(s) - g(0)] + g(0) \cdot 1 \end{aligned}$$

It follows from the derivation property of the Laplace transform that

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{dg(t)}{dt} + g(0) \cdot \delta(t)$$

Furthermore,

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-\frac{R}{L}\,t}\, u(t),$$

which gives

$$h(t) = -\frac{R}{L}\ e^{-\frac{R}{L}\,t}\, u(t) + \delta(t).$$

Problem 2

(a) Since the system is causal, the region of convergence (ROC) is defined as $|z| > |p_{\max}|$, where $p_{\max}$ denotes the pole in the system with the largest magnitude.

The system has a pole at $z = 2/3$, so the ROC is $|z| > 2/3$.

The impulse response $h[n]$ can be found by taking the inverse z-transform of the transfer function $H(z)$. From Table 3.3 in the textbook we see that

$$\mathcal{Z}^{-1}\left(\frac{1}{1 - az^{-1}}\right) = a^n u[n] \quad \text{for } ROC : |z| > |a|,$$

For $z = \frac{2}{3}$ this gives:

$$h[n] = \left(\frac{2}{3}\right)^n u[n]$$

(b)

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-z^{-1})}$$

Since the system is causal, the region of convergence (ROC) is defined as $|z| > |p_{\max}|$, where $p_{\max}$ denotes the pole in the system with the largest magnitude.

The system has poles at $z = -1/2$ and $z = 1$, so the ROC is $|z| > 1$.

We can decompose $H(z)$ as

$$H(z) = \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B}{1-z^{-1}},$$

where

$$A = H(z)(1+\frac{1}{2}z^{-1})\Big|_{z=-\frac{1}{2}} = \frac{1}{1-z^{-1}}\Big|_{z=-\frac{1}{2}} = \frac{1}{1+2} = \frac{1}{3}$$

and

$$B = H(z)(1-z^{-1})\Big|_{z=1} = \frac{1}{1+\frac{1}{2}z^{-1}}\Big|_{z=1} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

Then

$$H(z) = \frac{1}{3}\cdot\frac{1}{1+\frac{1}{2}z^{-1}} + \frac{2}{3}\cdot\frac{1}{1-z^{-1}}$$

and

$$\begin{aligned} h[n] = \mathcal{Z}^{-1}\{H(z)\} &= \frac{1}{3}\mathcal{Z}^{-1}\{\frac{1}{1+\frac{1}{2}z^{-1}}\} + \frac{2}{3}\mathcal{Z}^{-1}\{\frac{1}{1-z^{-1}}\} \\ &= \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{2}{3}u[n], \end{aligned}$$

where we have used the fact that ROC is $|z| > 1$.

(c)

$$H(z) = \frac{z^{-1}}{(1+\frac{3}{2}z^{-1})(1-3z^{-1})}$$

Since the system is anti-causal, the region of convergence (ROC) is defined as $|z| < |p_{\min}|$, where $p_{\min}$ denotes the pole in the system with the smallest magnitude

The system has a pole at $z = -\frac{3}{2}$ and $z = 3$, so the ROC is $|z| < \frac{3}{2}$.

We can decompose $H(z)$ as

$$H(z) = \frac{A}{1+\frac{3}{2}z^{-1}} + \frac{B}{1-3z^{-1}},$$

where

$$A = H(z)(1+\frac{3}{2}z^{-1})\Big|_{z=-\frac{3}{2}} = \frac{z^{-1}}{1-3z^{-1}}\Big|_{z=-\frac{3}{2}} = -\frac{2}{9}$$

and

$$B = H(z)(1-3z^{-1})\Big|_{z=3} = \frac{z^{-1}}{1+\frac{3}{2}z^{-1}}\Big|_{z=3} = \frac{2}{9}.$$

Then

$$H(z) = \frac{-\frac{2}{9}}{1+\frac{3}{2}z^{-1}} + \frac{\frac{2}{9}}{1-3z^{-1}}$$

and

$$h[n] = \frac{2}{9}\cdot(\frac{-3}{2})^n u[-n-1] - \frac{2}{9}\cdot 3^n u[-n-1],$$

where we have used the fact that ROC is $|z| < \frac{3}{2}$.

(d) A filter is stable if its ROC contains the unit circle ($|z| = 1$). We see that this is satisfied for the filters in a) and c), but not for the filter in b).

Problem 3

(a) The z-transform of $h(n)$ is

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n} z^{-n} \\ &= \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n \\ &= \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{for } |z| > \frac{1}{2} \end{aligned}$$

and the z-transform of $x[n]$ is

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=2}^{\infty} z^{-n} \\ &= \frac{z^{-2}}{1-z^{-1}}, \quad \text{for } |z| > 1. \end{aligned}$$

(b) Start by noting that we can write $h[n] = \frac{1}{2^n}u[n]$ and $x[n] = u[n-2]$. Then

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k]u[n-2-k] \\ &= \sum_{k=0}^{\infty} \frac{1}{2^k} u[n-2-k] \end{aligned}$$

Note that $u[n-2-k] = 0$ for $n-2-k < 0$, i.e. $k > n-2$. Therefore we have

$$y[n] = \begin{cases} \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^k & n-2 \geq 0 \\ 0 & n-2 < 0, \end{cases}$$

this gives

$$y[n] = \begin{cases} \frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{n-2} & n-2 \geq 0 \\ 0 & n-2 < 0, \end{cases}$$

This can be written as

$$y[n] = 2u[n-2] - \left(\frac{1}{2}\right)^{n-2} u[n-2].$$

(c) $X(z)$ and $H(z)$ were computed in 4a.

Then

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \\ &= z^{-2}Y_1(z), \quad \text{for } |z| > 1. \end{aligned}$$

where

$$Y_1(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}, \quad |z| > 1.$$

$y_1[n]$ follows from the result in 2b:

$$y_1[n] = -\left(\frac{1}{2}\right)^n u[n] + 2u[n].$$

Therefore we have

$$\begin{aligned} y[n] &= Z^{-1}\left\{z^{-2}Y_1(z)\right\} = y_1[n-2] \\ &= -\left(\frac{1}{2}\right)^{n-2} u[n-2] + 2u[n-2] \end{aligned}$$

which is the the same as we got in (a).

Problem 4

(a) We can find the transfer function $H(z)$ by taking the z-transform on both sides of the difference equation:

$$\begin{aligned} Y(z) &= X(z) - X(z)z^{-2} - \frac{1}{4}Y(z)z^{-2} \\ Y(z)(1+\frac{1}{4}z^{-2}) &= X(z)(1-z^{-2}) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1-z^{-2}}{1+\frac{1}{4}z^{-2}} \end{aligned}$$

(b) The poles can be found as follows:

$$\left(1+\frac{1}{4}z^{-2}\right) = 0 \quad \Rightarrow \quad p_1 = \frac{1}{2}j, \; p_2 = -\frac{1}{2}j$$

$$|p_1| = |p_2| = \frac{1}{2}$$

The zeros can be found as follows:

$$\left(1-z^{-2}\right) = 0 \quad \Rightarrow \quad z_1 = 1, \; z_2 = -1$$

The pole-zero plot in the z-plane is shown in the following figure:

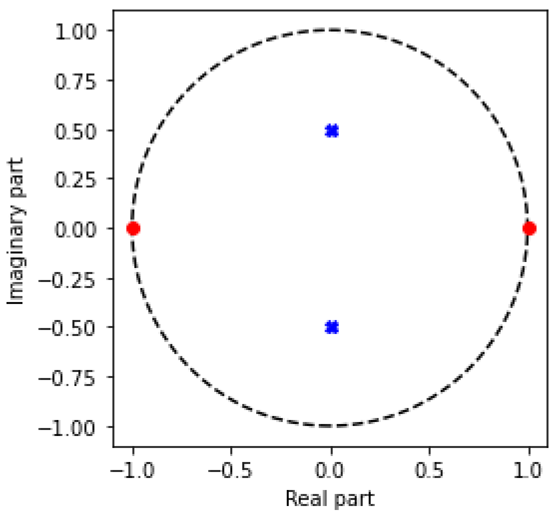
In [2]:
```
num = [1, 0, -1]
den = [1, 0, 1/4]

poles = np.roots(den)
zeros = np.roots(num)

fig, ax = plt.subplots()

# plot circle
theta = np.linspace(-np.pi, np.pi, 1000)
ax.plot(np.sin(theta), np.cos(theta), '--k')
ax.set_aspect(1)

# plot poles and zeros
ax.plot(np.real(poles), np.imag(poles), 'Xb', label = 'Poles')
ax.plot(np.real(zeros), np.imag(zeros), 'or', label = 'Zeros')
ax.set_xlabel('Real part')
ax.set_ylabel('Imaginary part')
plt.show()
```



(c) Since the filter is causal with poles on the circle with radius $1/2$, its ROC is outside of the circle. Since the ROC includes the unit circle, the filter is stable.

(d) For $\omega = 0$ we have the zero on the unit circle, so the amplitude response will be zero. Increasing the ω from 0 to $\frac{\pi}{2}$, the distance from the zero increases, while the distance to the pole p_1 decreases. The amplitude response will thus increase and reach its maximum at $\omega = \frac{\pi}{2}$. As ω increases further from $\frac{\pi}{2}$ to π, the amplitude response decreases and reaches zero again at $\omega = \pi$.

We conclude that this is a bandpass filter with the passband centred around $\omega = \frac{\pi}{2}$.