Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing

Solutions for Problem Set 8

Problem 1:

(a).

The filter $H(z)$ is a first order all-pole filter. The output of the filter is therefore an $AR[1]$ process.

(b).

The first order predictor is defined as

$$\hat{x}[n] = -a_1 x[n-1]$$

The prediction coefficient $-a_1$ can be found by minimizing the prediction error power $\sigma_f^2 = E\{f^2[n]\}$, where $f[n] = x[n] - \hat{x}[n]$ is the prediction error. We have that

$$\sigma_f^2 = E\{(x[n] - \hat{x}[n])^2\} = E\{(x[n] + a_1 x[n-1])^2\}$$

The optimal value for a_1 can be found by minimizing the prediction error power, i.e $\frac{\partial \sigma_f^2}{\partial a_1} = 0$. Thus, we get

$$E\{2(x[n] + a_1 x[n-1])x[n-1]\} = 0,$$

$$\gamma_{xx}(1) + a_1 \gamma_{xx}(0) = 0 \Rightarrow a_1 = -\frac{\gamma_{xx}(1)}{\gamma_{xx}(0)}.$$

In the previous problem set the autocorrelation function of the signal $x[n]$ was found as

$$\gamma_{xx}(m) = \left(-\frac{1}{2}\right)^{|m|}.$$

Thus, we have $a_1 = \frac{1}{2}$. By repeating the procedure for the second order predictor, we get $a_1 = \frac{1}{2}$ and $a_2 = 0$. This means thet we can not obtain further reduction of the prediction error by using a higher order peredictor.

The above results could be expected, since $x[n]$ is an $AR[1]$ process. The optimal predictor is thus the first order predictor with prediction coefficient equal to the filter coefficient.

Problem 2:

(a).

This is a $MA(1)$-process, as only the current and the former value of the input signal are used in forming the output signal.

(b).

We have

$$\begin{aligned} x[n]x[n-1] &= \Big(w[n]-0.5w[n-1]\Big)\Big(w[n-l]-0.5w[n-l-1]\Big) \\ &= w[n]w[n-l]-0.5w[n]w[n-l-1]-0.5w[n-1]w[n-l]+0.25w[n-1]w[n-l-1]. \end{aligned}$$

Taking the expectation of this leads to

$$\begin{aligned} \gamma_{xx}(l) &= \gamma_{ww}(l)-0.5\gamma_{ww}(l+1)-0.5\gamma_{ww}(l-1)+0.25\gamma_{ww}(l) \\ &= 1.25\gamma_{ww}(l)-0.5\Big(\gamma_{ww}(l+1)+\gamma_{ww}(l-1)\Big) \\ &= 1.25\sigma_w^2\delta(l)-0.5\sigma_w^2\Big(\delta(l+1)+\delta(l-1)\Big) \\ &= \begin{cases} 1.25 & l=0, \\ -0.5 & l=\pm 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The expression for the power density spectrum $\Gamma_{xx}(f)$ can be found as follows

$$\Gamma_{xx}(f)=\sum_{l=-1}^{1}\gamma_{xx}(l)e^{-j\omega l}=-\frac{1}{2}e^{j\omega}+1.25-\frac{1}{2}e^{-j\omega}=1.25-\cos(2\pi f).$$

(c).

The optimal predictor of order p is given by:

$$\hat{x}[n]=-\sum_{k=1}^{p}a_k x[n-k].$$

To obtain simple matrix equations that we can solve using Matlab, we use the version of the Yule-Walker that does not contain σ_f^2, i.e.

$$\gamma_{xx}(0)a_1=-\gamma_{xx}(-1)$$

$$\begin{bmatrix}\gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(-1) & \gamma_{xx}(0)\end{bmatrix}\begin{bmatrix}a_1 \\ a_2\end{bmatrix}=\begin{bmatrix}-\gamma_{xx}(-1) \\ -\gamma_{xx}(-2)\end{bmatrix}$$

$$\begin{bmatrix}\gamma_{xx}(0) & \gamma_{xx}(1) & \gamma_{xx}(2) \\ \gamma_{xx}(-1) & \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(-2) & \gamma_{xx}(-1) & \gamma_{xx}(0)\end{bmatrix}\begin{bmatrix}a_1 \\ a_2 \\ a_3\end{bmatrix}=\begin{bmatrix}-\gamma_{xx}(-1) \\ -\gamma_{xx}(-2) \\ -\gamma_{xx}(-3)\end{bmatrix}$$

for order one, two, and three respectively. Then we calculate σ_f^2 as

$$\sigma_f^2=\sum_{k=0}^{p}a_k\gamma_{xx}(k).$$

The following Matlab code can be used to find the coefficients, and σ_f^2 for each AR order.

```
gamma_xx=[1.25 -0.5 0 0];
R1 = [1.25];
R2 = [1.25 -.5; -.5 1.25];
R3 = [1.25 -.5 0 ; -.5 1.25 -.5 ; 0 -.5 1.25];
a_1 = R1^(-1)*-gamma_xx(2)'
sigma_f1 = sum([1 a_1'].*gamma_xx(1:2))
a_2 = R2^(-1)*-gamma_xx(2:3)'
sigma_f2 = sum([1 a_2'].*gamma_xx(1:3))
a_3 = R3^(-1)*-gamma_xx(2:4)'
sigma_f3 = sum([1 a_3'].*gamma_xx(1:4))
```

Approximated values are summarized in the following table.

AR order p	Coefficients	σ_f^2
1	$a_1 = 0.40$	1.05
2	$a_1 = 0.48$, $a_2 = 0.19$	1.01
3	$a_1 = 0.49$, $a_2 = 0.24$, $a_3 = 0.09$	1.00

We can see that the mean square error decreases as we increase the model order. This means that the AR model is a better approximation of the $MA[1]$ process.

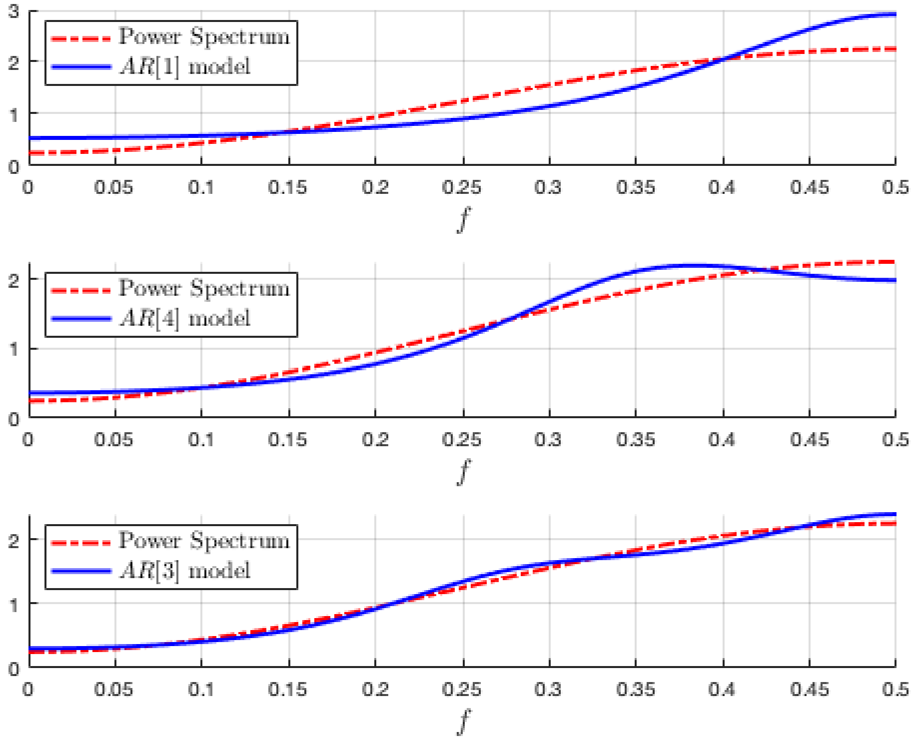
(d).

The power density spectrum estimate obtained by using an $AR[p]$ model is given by

$$\hat{\Gamma}_{ff}(f) = \Gamma_{ff}(f)|H(f)|^2 = \sigma_f^2\left|\frac{1}{A(f)}\right|^2 = \frac{\sigma_f^2}{\left|1+\sum_{k=1}^{p} a_k e^{-j2\pi fk}\right|^2}$$

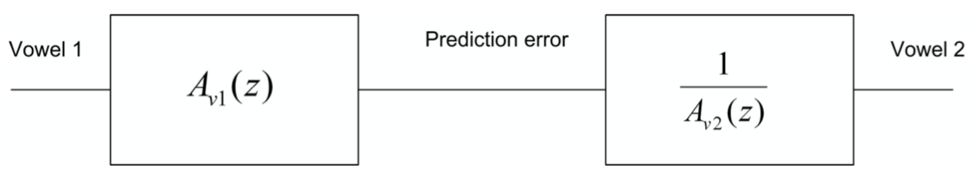
Fig.1 shows the power density spectrum estimates based on AR models of different order compared to the power density spectrum of the $MA[1]$ process. It can be seen that the estimates become closer to the correct value as the model order increases. Thus, the $AR[3]$ model is the best approximation of these three.

Fig.1: Power density spectrum estimates



Problem 3:

- First, the vowel sample files are loaded with `audioread`.
- Then each vowel is modelled as an $AR[10]$ process which the coefficients can be found with `lpc` (needs to install the signal processing toolbox).
- To transform a vowel v_i into another vowel v_j we need the prediction error signal from v_i by useing `filter` and the AR coefficients of that vowel which is shown in the first part of Fig.2.



- Generate the new vowel v_j by filtering the prediction error using the inverse prediction-error filter (with the coefficients that you found for v_j) which represents in the second part of Fig.2.
- The output signal can be played using `sound`.