Boolean Algebra

A non-empty set B on which two binary operations **+** and **.** and one unary operation **'** are defined is said to be Boolean algebra if the following postulates are satisfied :

1. Commutative laws :

$$a + b = b + a \quad and \quad a.b = b.a \quad \forall\, a, b \in B$$

2. Distributive laws :

$$a.(b + c) = (a.b) + (a.c) \quad and \quad a + (b.c) = (a + b).(a + c) \quad \forall\, a, b, c \in B$$

3. Identity Laws :

$$a + 0 = a \quad and \quad a.1 = a \quad \forall\, a \in B$$

4. Complement laws :

For every element $'a'$ there is an element $a^{/}$ in B such that

$$a + a^{/} = 1 \quad and \quad a.a^{/} = 0$$

Notes :

(a) The element '1' is called unit element and the element '0' is called the zero element.

(b) $a^{/}$ is called complement of a.

(c) above postulates are known as Huntington's postulates in Boolean algebra.

(d) If a theorem hold in a Boolean algebra then another theorem is obtained by interchanging **+** and **.** ; 0 and 1 in the original statement. Later one is said to be dual of the former. (Principle of Duality)

Examples :

1. The set of all positive divisors of 70 is a Boolean Algebra w.r.t the **+, .** and $'$, where
   $a + b = lcm\ (a, b);\quad a.b = \gcd(a, b)\ \ and\quad a' = 70/a$.

   [Here, $B = \{1, 2, 5, 7, 10, 14, 35, 70\}$. Then commutative law and distributive law are satisfied from the properties of lcm and gcd.
   Let, x, y be the zero element and unit elements respectively.
   Then, $a + x = a\ \forall\ a \in B\quad \Rightarrow lcm(a, x) = a\quad \Rightarrow x = 1$
   and $a.y = a\ \forall\ a \in B\quad \Rightarrow gcd(a, y) = a\quad \Rightarrow y = 70.$
   Thus identity law is satisfied with zero element 1 and unit element 70.
   Also, $1' = 70, 2' = 35, 5' = 14, 7' = 10$ and vice versa.
   So, every element of B has it's inverses in B. hence B is a Boolean algebra.]

2. The set of all positive divisors of 48 is not a Boolean Algebra w.r.t the **+, .** and $'$, where $a + b = lcm\ (a, b);\quad a.b = \gcd(a, b)\ \ and\quad a' = 48/a$.

   [Here, $B = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$. Then commutative law, distributive law and identity law are satisfied with zero element 1 and unit element 48.
   Also, $8' = 6$ but $8 + 8' = 8 + 6 = lcm(8, 6) = 24(\neq unit\ element\ 48)$.
   Hence B is not a Boolean algebra.]

Problem :

Simplify the Boolean expression : $y'x' + y'x + yx$

Solution :

$$y'x' + y'x + yx = y'(x' + x) + xy \quad \text{[Distributive \& Commutative law]}$$

$$= y'.1 + xy = y' + xy \quad \text{[Identity \& Complement law]}$$

Different laws in Boolean algebra :

In a Boolean algebra B,

- Idempotent laws :
  $a + a = a \quad and \quad a.a = a \quad \forall a \in B$
- Boundedness laws :
  $a + 1 = 1 \quad and \quad a.0 = 0 \quad \forall a \in B$
- Absorption laws :
  $a + (a.b) = a \quad and \quad a.(a + b) = a \quad \forall a, b \in B$
- Associative laws :
  $(a + b) + c = a + (b + c) \quad and \quad (a.b).c = a.(b.c) \quad \forall a, b, c \in B$
- De Morgan's laws :
  $(a + b)' = a'.b' \quad and \quad (a.b)' = a' + b' \quad \forall a, b \in B$
- Involution law :
  $(a')' = a \quad \forall a \in B$

Note :

In a Boolean algebra B, $0' = 1 \; and \; 1' = 0$

Problem :

In a Boolean algebra B, prove that

$(a + b).(b + c).(c + a) = a.b + b.c + c.a \quad \forall a, b, c \in B$

Solution :

$LHS = (a + b).(b + c).(c + a)$

$= (a.b + a.c + b.b + b.c).(c + a)$ [Distributive law]

$= (a.b + a.c + b + b.c).(c + a)$ [Idempotent law]

$= a.b.c + a.b.a + a.c.c + a.c.a + b.c + b.a + b.c.c + b.c.a$

[Distributive law]

$= a.b.c + a.b + a.c + a.c + b.c + a.b + b.c + a.b.c$

[Commutative & Idempotent law]

$= a.b.c + a.b + b.c + c.a$ [Commutative & Idempotent law]

$= a.b.(c + 1) + b.c + c.a$ [Identity & Distributive law]

$= a.b.1 + b.c + c.a$ [Boundedness law]

$= a.b + b.c + c.a = RHS$

Problem :

In a Boolean algebra B, prove that if

$a + b = a + c$ and $a.b = a.c \quad \forall a, b, c \in B$, then $b = c$.

Solution :

Since, $a + b = a + c \;\Rightarrow\; (a + b).b = (a + c).b$

$\Rightarrow\; a.b + b.b = a.b + c.b$ [Distributive law]

$\Rightarrow\; a.b + b = a.b + c.b$ [Idempotent law]

$\Rightarrow (a + 1).b = a.b + b.c$

[Distributive & Commutative law]

$\Rightarrow 1.b = a.b + b.c$ [Boundedness law]

$\Rightarrow b = a.c + b.c$ [$a.b = a.c$ & Identity law]

$\Rightarrow b = (a + b).c$ [Distributive law]

$\Rightarrow b = (a + c).c$ [$a + b = a + c$]

$\Rightarrow b = a.c + c.c = a.c + c = (a + 1).c$

[Distributive & Idempotent law]

$\Rightarrow b = c$ [Boundedness & Identity law]