Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2017

Solutions exercise 12

$\boxed{11.8}$ Show that $\ddot{x} + \beta(x^2 - 1)\dot{x} + x^3 = 0$ has one and only one periodic solution. (We have to assume $\beta > 0$ even though this was not mentioned in the exercise. For $\beta < 0$ the system will have an unstable spiral at the origin.)

We can write the equation as

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x) = \beta(x^2 - 1)$ and $g(x) = x^3$. If we define

$$F(x) = \int_0^x f(u)\mathrm{d}u = \beta x\left(\frac{x^2}{3} - 1\right),$$

we see that F is odd, $F(x) = 0$ if and only if $x = 0$ and $x = \pm\sqrt{3}$, F tends to infinity when x tends to ∞, and g is an odd function satisfying $g(x) > 0$ for $x > 0$. The conditions in theorem 11.4 are satisfied, so the equation has a unique periodic solution.

$\boxed{11.9}$ Show that $\ddot{x} + (|x| + |\dot{x}| - 1)\dot{x} + x|x| = 0$ has at least one periodic solution.

We can write the equations as

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$$

with $f(x, \dot{x}) = |x| + |\dot{x}| - 1$ and $g(x) = x|x|$.

We see that $f(x, y) = |x| + |y| - 1 > 0$ for $|x| + |y| > 1$, that is for $\sqrt{x^2 + y^2} > 1$. Further, $f(0, 0) = -1 < 0$, $g(0) = 0$, $g(x) = x|x| > 0$ for $x > 0$ and $g(x) = x|x| < 0$ for $x < 0$. We also have

$$\lim_{x\to\infty} G(x) = \lim_{x\to\infty} \int_0^x g(u)\mathrm{d}u = \lim_{x\to\infty} \mathrm{sgn}(x)\frac{x^3}{3} = \infty.$$

The conditions of theorem 11.2 are satisfied, so there exists at least one periodic solution.

$\boxed{11.10}$ Show that the origin is a centre for $\ddot{x} + (k\dot{x} + 1)\sin x = 0$.

We write the equation as $\ddot{x} + f(x)\dot{x} + g(x) = 0$ with $f(x) = k\sin x$ and $g(x) = \sin x$.

We see that f and g are odd functions, and f does not change sign for positive x in a neighborhood of the origin. Further, $g(x) > 0$ for $x > 0$ in a neighborhood of the origin.

Finally, we verify

$$g(x) = \sin(x) > \alpha k^2 \sin x(1 - \cos x)$$

for x small enough holds for an $\alpha > 1$ since the term $(1 - \cos x)$ can be made arbitrary small enough close to the origin. By theorem 11.3, the origin is a centre for the equation.

$\boxed{12.1 \text{ (ii)}}$ We are asked to find the bifurcation points of the system $\dot{x} = A(\lambda)x$ where

$$A(\lambda) = \begin{bmatrix} \lambda & 1-\lambda \\ 1 & \lambda \end{bmatrix}$$

We find the eigenvalues of A, μ_1 and μ_2 by solving

$$(\lambda - \mu)^2 - (1 - \lambda) = 0,$$

which has solution

$$\mu = \lambda \pm \sqrt{1 - \lambda}.$$

If $\lambda > 1$ we have complex conjugated eigenvalues, which give us an unstable spiral. If $\lambda < 1$ we have real eigenvalues. We check for which values of $0 < \lambda < 1$ gives positive eigenvalues. For $\mu_2 = \lambda - \sqrt{1 - \lambda}$ to be positive we need

$$\begin{aligned} &\lambda > \sqrt{1-\lambda} > 1 - \lambda, \\ &\lambda^2 > 1 - 2\lambda + \lambda^2, \\ &\lambda > \frac{1}{2}. \end{aligned}$$

Thus, for $\frac{1}{2} < \lambda < 1$ we have an unstable node. Similarly, we find that A has both a positive and a negative eigenvalue for $0 < \lambda < \frac{1}{2}$, which is a saddle point. Finally, for $\lambda < 0$, the eigenvalues have opposite sign, so we have a saddle point for $\lambda < 0$. We summarize in the following table.

$\lambda > 1$	Unstable spiral
$\frac{1}{2} < \lambda < 1$	Unstable node
$0 < \lambda < \frac{1}{2}$	Saddle
$\lambda < 0$	Saddle

The only bifurcation point is the saddle-node bifurcation, for $\lambda = \frac{1}{2}$.

Exam 1992, 3 Give an example of an n-dimensional, dynamical system (n given and $n \geq 2$)

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

such that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, $\lim_{t\to\infty} x(t) = 0$ for all solutions, and not all eigenvalues of its linearisation at 0 have strictly negative real part.

The system $\dot{x} = f(x)$ must have a linearization such that some eigenvalues has real-part zero, while the remaining eigenvalues are less than zero. For the linearization, put

$$\dot{x} = Ax$$

where $A = a_{ij}$ is given by

$$\begin{aligned} a_{ii} &= \lambda_i \quad \text{for } i \geq 3 \\ a_{12} &= 1 \\ a_{21} &= -1 \\ a_{ij} &= 0 \quad \text{otherwise.} \end{aligned}$$

That is, we solve the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \\ \dot{x}_i &= \lambda_i x_i. \end{aligned}$$

for $i \geq 3$. We manipulate the first two equations to get a nonlinear system. Put

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 f(r) \\ \dot{x}_2 &= -x_1 + x_2 f(r). \end{aligned}$$

We have

$$\left(\dot{r}^2\right) = 2r^2 f(r).$$

We now choose $f(r)$ so that $\dot{r} < 0$, that is $f(r) < 0$ for all r. This is needed to ensure $\lim_{t\to\infty} x(t) = 0$. A suitable choice is $f(r) = -r^2$, and we get the system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2) \\ \dot{x}_i &= \lambda_i x_i \end{aligned}$$

for $i \geq 3$.