Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4125 Matematikk 4N
Spring 2017

Solutions to exercise set 4

$\boxed{1}$ **a)** The equation $u_{tt} = u_{xx}$ is linear and homogeneous, whilst $v_{xx} + v_{yy} - \frac{2y}{x^3} = 0$ is linear but inhomogeneous.

b) Let $u = x^2 + t^2$. Then $u_x = 2x$ and $u_{xx} = 2$, and similarly $u_{tt} = 2$, hence the equation is satisfied.
Setting $v = \frac{y}{x}$, we obtain $v_y = \frac{1}{x}$, and hence $v_{yy} = 0$. Moreover, we have $v_x = -\frac{y}{x^2}$ and thus $v_{xx} = \frac{2y}{x^3}$, satisfying the second equation.

$\boxed{2}$ **a)** The characteristic equation takes the form $m^2 - m - 6 = (m-3)(m+2) = 0$, with roots $m = 3$ and $m = -2$. The general solution of the ODE is thus

$$y(t) = A \exp 3t + B \exp -2t,$$

where A and B are arbitrary constants. The PDE can be solved like the above ODE, where the constants A, B are allowed to depend on the variable x for which no derivatives u_x appear in the equation, i.e.

$$u(x,t) = A(x) \exp 3t + B(x) \exp(-2t)$$

b) Here the characteristic equation is $m^2 + 2m + 2 = 0$, with roots

$$m = \frac{-2 \pm \sqrt{(-2)^2 - 4.2.1}}{2} = -1 \pm i$$

The general solution of the ODE is

$$y(t) = e^{-t}\big(A \cos t + B \sin t\big),$$

and the PDE is this solved by $u(x,t) = e^{-t}\big(A(x) \cos t + B(x) \sin t\big)$

$\boxed{3}$ **a)** Writing $u(x,t) = F(x)G(t)$, we obtain $u_{xx} = F''G$ and $u_{tt} = F\ddot{G}$, hence the equation becomes

$$F\ddot{G} - 4F''G = 0$$

We divide the above equation by $4FG$ to obtain

$$\frac{\ddot{G}}{4G} - \frac{F''}{F} = 0$$

Separating the variables gives the pair of ODEs

$$\ddot{G} - 4kG = 0 \qquad \text{and} \qquad F'' - kF = 0,$$

where k is a constant yet to be determined. We solve the equation for $F(x)$ subject to the boundary conditions; the characteristic equation $m^2 - km = 0$ admits 3 kinds of solution: $m = \pm n$ if $k = n^2$ (i.e. $k > 0$), $m = \pm in$ if $k = -n^2$ (i.e. $k < 0$), and a double root $m = 0$ if $k = 0$.
In the case of the double root we have $F = Ax + B$. Then solving for $F(0) = 0$ requires that $A = 0$, and satisfying the remaining boundary condition $F(2) = 0$ requires that $F = 0$, which we can dismiss as giving no contribution to the final solution. Similarly, the case $k = n^2$ results in solutions $F = A\cosh nx + B\sinh nx$; then $A = 0$ as $F(0) = 0$, and satisfying $F(2) = 0$ again requires $B = 0$ and hence $F = 0$.
So we have $k = -n^2$, with solution $F = A\cos nx + B\sin nx$. Satisfying $F(0) = 0$ gives $A = 0$, whilst to obtain $F(2) = 0$ we require that $\sin 2n = 0$, which implies that $n = \frac{m\pi}{2}$ for some positive integer m (negative integers give nothing new as $\sin -nx = \sin nx$). Collecting this, we have

$$F_m(x) = B_m \sin\frac{m\pi x}{2}$$

The equation for G_m is then $\ddot{G}_m + 4\left(\frac{m\pi}{2}\right)^2 G_m = 0$, with solution

$$G_m = \tilde{A}_m \cos m\pi t + \tilde{B}_m \sin m\pi t$$

Summing all the solutions $u_m(x,t) = F_m(x)G_m(t)$ and combining the constants then gives the general solution

$$u = \sum_{m=1}^{\infty} \big(A_m \cos m\pi t + B_m \sin m\pi t\big) \sin\frac{m\pi x}{2}$$

b) We see from the form of the general solution that

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin\frac{m\pi x}{2}$$

Comparing this with the initial condition $u(x,0) = \sin\pi x$, we see that $A_2 = 1$, and $A_m = 0$ for $m \neq 2$. Similarly,

$$u_t(x,0) = \sum_{m=1}^{\infty} m\pi B_m \sin\frac{m\pi x}{2},$$

which we compare with $u_t(x,0) = \cos\pi x$. We use that the expression above (defined for all x) is the Fourier series of an odd, 4-periodic function. In order to satisfy $u_t(x,0) = \cos\pi x$ for $0 \le x \le 2$, we therefore compare the infinite series for $u_t(x,0)$ with the Fourier series of the odd, 4-periodic extension $g^*(x)$ of $g(x) = \cos\pi x,\ 0 \le x \le 2$. This is given by

$$g^*(x) = \sum_{m=1}^{\infty} B_m^* \sin\frac{m\pi x}{2},$$

where

$$
\begin{aligned}
B_m^* &= \frac{2}{2}\int_0^2 \cos\pi x \sin\frac{m\pi x}{2} dx \\
&= \frac{1}{2}\int_0^2 \sin(\frac{m}{2}+1)\pi x + \sin(\frac{m}{2}-1)\pi x \; dx \\
&= \left[-\frac{\cos\left(\frac{m+2}{2}\right)\pi x}{m+2} - \frac{\cos\left(\frac{m-2}{2}\right)\pi x}{m-2} \right]_0^2 \\
&= -\frac{\cos(m+2)\pi - 1}{(m+2)\pi} - \frac{\cos(m-2)\pi - 1}{(m-2)\pi} \\
&= \begin{cases} \frac{2}{(m-2)\pi} + \frac{2}{(m+2)\pi}, & \text{m odd} \\ 0 & \text{m even} \end{cases} \\
&= \begin{cases} \frac{4m}{(m^2-4)\pi}, & \text{m odd} \\ 0 & \text{m even} \end{cases}
\end{aligned}
$$

Comparing the two Fourier series, we have $m\pi B_m = B_m^*$, hence

$$
B_m = \begin{cases} \frac{4}{(m^2-4)\pi^2}, & \text{m odd} \\ 0 & \text{m even} \end{cases}
$$

We therefore have

$$
u(x,t) = \cos 2\pi t \sin \pi x + \sum_{k=1}^{\infty} \frac{4\sin(2k-1)\pi t}{(4k^2-4k-3)\pi^2} \sin\frac{(2k-1)\pi x}{2}
$$

4 **a)** D'Alembert's solution is

$$
u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big),
$$

in accordance with Chapter 12.4 of Kreyszig.

b) We require boundary conditions $u(0,t) = u(L,t) = 0$, as there can be no vertical displacement where the string is fixed. The initial condition $u(x,0) = f(x)$ means that the initial position of the string is given by the function $f(x)$, whilst the condition $u_t(x,0) = 0$ says that the string is initially at rest. The constant c is the speed the displacements (waves) spread along the string.

c) The first condition $u(0,t) = 0$ implies that

$$
\frac{1}{2}\big(f(ct) + f(-ct)\big) = 0,
$$

and hence $f(t) = f(-t)$, i.e. f must be an odd function. The second condition $u(L,t) = 0$ gives

$$
\frac{1}{2}\big(f(L+ct) + f(L-ct)\big) = 0,
$$

and as f is odd we have $f(L-ct) = f(-L+ct)$. We therefore have

$$
f(ct+L) = f(ct-L)
$$

for all t, i.e. f must be $2L$-periodic.