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TMA4145 Linear Methods
Fall 2017

Exercise set 9

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Use the Banach fixed point theorem to solve:

$$\begin{aligned} 7x_1 - x_2 + 2x_3 &= 1 \\ -x_1 + 3x_2 + x_3 &= 2 \\ x_1 - x_2 + 5x_3 &= 1 \end{aligned}$$

Hint: Pick appropriate norms on $\mathbb{R}^3$ to get a contraction.

2 We denote by c_f the set of all sequences with only finitely many non-zero entries.

a) For $1 \leq p < \infty$ show that c_f is dense in ℓ^p.

b) For $1 \leq p < \infty$ show that ℓ^p is separable.

3 Let M be a subspace of a Hilbert space X. Show that the orthogonal complement $M^\perp = \{x \in X : \langle x, y \rangle = 0 \text{ for all } y \in M\}$ is a subspace of X.

4 Consider the integral operator $T : (C[0,1], \|.\|_\infty) \to (C[0,1], \|.\|_\infty)$

$$Tf(x) = \int_0^1 k(x,y) f(y) dy,$$

where k is given by

$$k(x,y) = \sum_{i=1}^n g_i(x) h_i(y)$$

for $g_1, ..., g_n$ and $h_1, ..., h_n$ are continuous functions on $[0,1]$. We assume that $\{g_1, ..., g_n\}$ are linearly independent.

a) Determine the kernel and the range of T.

b) Investigate if the range of T is closed.

$\boxed{5}$ Suppose $\|.\|_a$ and $\|.\|_b$ are equivalent norms on X. Then $(X, \|.\|_a)$ is a Banach space if and only if $(X, \|.\|_b)$ is a Banach space.

$\boxed{6}$ Let $\|.\|_a$ and $\|.\|_b$ be two norms on a vector space X. Show that the following statements are equivalent:

1. $\|.\|_a$ and $\|.\|_b$ are equivalent norms.
2. For a set $U \subseteq X$ we have that U is open in $(X, \|.\|_a)$ if and only if U is open in $(X, \|.\|_b)$.