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Solutions to exercise set 3

1 **a)** We have

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw}\right]_a^b = \frac{e^{-iwa} - e^{-iwb}}{iw\sqrt{2\pi}}$$

b)

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-b}^b e^{iax} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-b}^b e^{ix(a-w)} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ix(a-w)}}{i(a-w)}\right]_{-b}^b = \frac{1}{\sqrt{2\pi}} \frac{2}{a-w} \frac{e^{ib(a-w)} - e^{-ib(a-w)}}{2i} \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin b(a-w)}{a-w}
\end{aligned}$$

c) Here we recognize $f(x)$ as the inverse Fourier transform of $\frac{\sin w}{w}$. It follows that $\hat{f}(w) = \frac{\sin w}{w}$

2 **a)** For the first part see Kreyszig, chapter 11.9. We write $f(x)$ as sum $g(x) + \frac{1}{2}p(x) + \frac{1}{2}q(x)$, where

$$g(x) = \begin{cases} 1 & \text{for } |x| \le \pi \\ 0 & \text{otherwise} \end{cases} \qquad p(x) = \begin{cases} e^{ix} & \text{for } |x| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

$$q(x) = \begin{cases} e^{-ix} & \text{for } |x| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

We then have $\hat{f}(w) = \hat{g}(w) + \frac{1}{2}\hat{p}(w) + \frac{1}{2}\hat{q}(w)$, and hence

$$\begin{aligned}
\hat{f}(w) &= \sqrt{\frac{2}{\pi}} \left(\frac{\sin \pi w}{w} - \frac{1}{2}\frac{\sin \pi(w-1)}{w-1} - \frac{1}{2}\frac{\sin \pi(w+1)}{w+1}\right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{w} + \frac{1}{w-1} + \frac{1}{w+1}\right) \sin \pi w \\
&= \frac{1}{\sqrt{2\pi}} \frac{2w^2 - 2 + w^2 + w + w^2 - w}{w(w^2-1)} \sin \pi w \\
&= \sqrt{\frac{2}{\pi}} \frac{2w^2 - 1}{w(w^2-1)} \sin \pi w
\end{aligned}$$

b) For the first part, see Kreyszig, chapter 11.9. For the next we recognize that

$$f(x) = \left(\frac{\sin x}{x}\right)'$$

It follows that

$$\mathcal{F}\{f(x)\} = iw\mathcal{F}\left\{\frac{\sin x}{x}\right\} = \begin{cases} iw\sqrt{\frac{\pi}{2}} & \text{for } |w| < 1 \\ 0 & \text{otherwise} \end{cases}$$

3 **a)**

$$(f \star f)(x) = \begin{cases} \int_0^x e^{-u}e^{-(x-u)}\,du & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

We then compute

$$\int_0^x e^{-u}e^{-(x-u)} = \int_0^x e^{-x}du = \left[ue^{-x}\right]_0^x = xe^{-x}$$

It follows that

$$(f \star f)(x) = \begin{cases} xe^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

b) By the previous exercise, we have $g = f \star f$. It follows that

$$\hat{g}(w) = \sqrt{2\pi}\hat{f}(w)\hat{f}(w) = \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}(1+iw)}\right)^2 = \frac{1}{\sqrt{2\pi}(1+iw)^2}$$

4 **a)** We begin by writing

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-iwx}dx$$

Making the substitution $y = ax$, so that $dy = a\,dx$, we have (assuming $a > 0$)

$$\hat{g}(w) = \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-iy\frac{w}{a}}\,dy = \frac{1}{a}\hat{f}\left(\frac{w}{a}\right)$$

The case of $a < 0$ is similar, where we pick up a negative sign due to the resultant interchange of the integration limits.

b) If $f(x) = e^{-x^2}$, we have $\hat{f}(w) = \frac{1}{\sqrt{2}}e^{\frac{-w^2}{4}}$. Using the above formula, it follows that $g(x) = f(\frac{1}{\sqrt{2}}x) = \exp(-\frac{x^2}{2})$ is the desired function.

c) Letting $g(x) = \exp(-\frac{x^2}{2})$, we have $g(x) = \hat{g}(x)$, and hence

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \hat{g}(0) = 1$$

5 **a)** We compute $f_{nk} = \exp\left(-\frac{2\pi nk}{3}\right)$. Accordingly, $f_{0k} = f_{n0} = 1$. Moreover

$$\begin{aligned} f_{11} &= e^{-\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ f_{12} = f21 &= e^{-\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ f_{22} &= e^{-\frac{8\pi}{3}} = e^{-\frac{2\pi}{3}} = f_{11} \end{aligned}$$

The discrete Fourier transform matrix F is then

$$F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix}$$

We then have

$$\begin{aligned} \hat{f} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 1 - 1 - i\sqrt{3} - \frac{3}{2} + \frac{3\sqrt{3}}{2} i \\ 1 - 1 + i\sqrt{3} - \frac{3}{2} - \frac{3\sqrt{3}}{2} i \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -\frac{3}{2} + \frac{\sqrt{3}}{2} i \\ -\frac{3}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix} \end{aligned}$$

b) We use the Fast Fourier transform to write (for $n = 0, 1, 2$)

$$\begin{aligned} \hat{f}_n &= \hat{f}_{ev} + w_6^n \hat{f}_{od,n} \\ \hat{f}_{3+n} &= \hat{f}_{ev} - w_6^n \hat{f}_{od,n} \end{aligned}$$

Here by the previous question we have

$$\hat{f}_{ev} = \hat{f}_{od} = \begin{pmatrix} 6 \\ -\frac{3}{2} + \frac{\sqrt{3}}{2} i \\ -\frac{3}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix}$$

We compute

$$w_6^n = e^{\frac{-2\pi i n}{6}} = \begin{pmatrix} 1 \\ e^{-\frac{\pi}{3}} \\ e^{-\frac{2\pi}{3}} \end{pmatrix}$$

Hence

$$w_6^n \hat{f}_{od,n} = \begin{pmatrix} 6 \\ -\frac{3}{4} + \frac{3}{4} + \frac{3\sqrt{3}}{4} i + \frac{\sqrt{3}}{4} i \\ \frac{3}{4} - \frac{3}{4} + \frac{3\sqrt{3}}{4} i + \frac{\sqrt{3}}{4} i \end{pmatrix} = \begin{pmatrix} 6 \\ \sqrt{3} i \\ \sqrt{3} i \end{pmatrix}$$

Then

$$\hat{f}_n = \begin{pmatrix} 6 \\ -\frac{3}{2} + \frac{\sqrt{3}}{2} i \\ -\frac{3}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix} + \begin{pmatrix} 6 \\ \sqrt{3} i \\ \sqrt{3} i \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{3}{2} + \frac{3\sqrt{3}}{2} i \\ -\frac{3}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix}$$

$$\hat{f}_{n+3} = \begin{pmatrix} 6 \\ -\frac{3}{2} + \frac{\sqrt{3}}{2} i \\ -\frac{3}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix} - \begin{pmatrix} 6 \\ \sqrt{3} i \\ \sqrt{3} i \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{3}{2} - \frac{\sqrt{3}}{2} i \\ -\frac{3}{2} - \frac{3\sqrt{3}}{2} i \end{pmatrix}$$

Combining these, we have

$$\hat{f} = \begin{pmatrix} 12 \\ -\frac{3}{2} + \frac{3\sqrt{3}}{2}i \\ -\frac{3}{2} + \frac{\sqrt{3}}{2}i \\ 0 \\ -\frac{3}{2} - \frac{\sqrt{3}}{2}i \\ -\frac{3}{2} - \frac{3\sqrt{3}}{2}i \end{pmatrix}$$