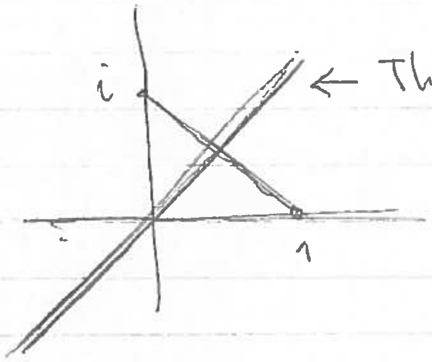
Solutions to exam in TMA4175
31. 05. 2017.

① Geometric description of the set
$\{z \in \mathbb{C} : |z-i| = |z-1|\}$

Answer: Points which are equidistant from 1 and i:



← This straight line:

$\{z = te^{i\frac{\pi}{4}},\ -\infty < t < \infty\}.$

(2) Find all harmonic conjugates to the function

$$u(x,y) = x^2 - y^2 + 2xy.$$

<u>Solution</u> Let $z = x+iy$. We know

$z^2 = x^2 - y^2 + 2ixy$ so $2xy$ is conjugate to $x^2 - y^2$

Respectively $-iz^2 = 2xy + i(y^2 - x^2)$, so $y^2 - x^2$ is conjugate to $2xy$.

Summing up we see that

$2xy + y^2 - x^2$ is conjugate to $x^2 - y^2 + 2xy$

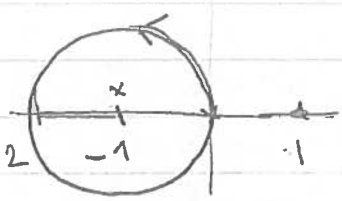
We need <u>all</u> harmonic conjugates so the answer is

$$\| \quad v(x,y) = 2xy + y^2 - x^2 + C, \qquad C \in \mathbb{R}$$

<u>Comment:</u> you may run the standard integration procedure (full credit if doing correctly) but this is a bit redundant.

(3) Evaluate the integral

$$I = \int_{|z+1|=1} \frac{dz}{(z^2-1)(z-1)^2}$$



Solution $\quad f(z) = \dfrac{1}{(z+1)(z-1)^3}$

Poles at the points ± 1

$+1$ is outside the countur.

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{(-2)^3} = -\frac{1}{8}$$

Answer:

$$I = 2i\pi\left(-\frac{1}{8}\right) = \frac{\pi}{4i}$$

(4) Evaluate: $I = \int_0^{\pi} \frac{\cos^4\varphi}{1+\sin^2\varphi}\, d\varphi$

Step 1 $\left.\begin{array}{l}\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi) \\ \cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi)\end{array}\right\} \Rightarrow$

$$\Rightarrow I = \int_0^{2\pi} \frac{(1+\cos\theta)^2}{3-\cos\theta}\, d\theta \qquad (*)$$

Step 2 Think a little: one can make the standard change of variables:

$z = e^{i\theta}$ than obtain the integral

$$I = \frac{1}{i} \int_{|z|=1} \frac{\left(1+\frac{1}{2}(z+1/z)\right)^2}{3-\frac{1}{2}(z+1/z)} \frac{dz}{z}$$

This can be evaluated, but leads us to too cumbersome expressions. Can we simplify the integral before making the standard change of variables.

Step 3 Simplifying the expression $(*)$

$$(1+\cos\theta)^2 = \left(4+(\cos\theta-3)\right)^2 = 16 + 8(\cos\theta - 3) + (\cos\theta - 3)^2$$

Respectively

$$\frac{(1+\cos\theta)^2}{3-\cos\theta} = \frac{16}{3-\cos\theta} - 8 + (3-\cos\theta)$$

and

$$I = \int_0^{2\pi} \frac{16\, d\theta}{3-\cos\theta} - 16\pi + \int_0^{2\pi} (3-\cos\theta)\, d\theta$$

The first integral is now easy to take by the standard change of variables $z = e^{i\theta}$.

The rest is explicit.

The final answer:

$$I = 2\pi\left(\sqrt{2} - \frac{5}{4}\right)$$

(5) Find the linear fractional mapping

$w = w(z)$ such that

$w(i) = 0$, $w(\infty) = 1$, $w(-i) = \infty$.

Well, this is straightforward:

$$w(z) = \frac{z - i}{z + i}$$

(6) Let a function $f(z)$ be analytic in

$\Pi = \{z : |\mathrm{Re}\, z| < \frac{\pi}{4}\}$, $|f(z)| < 1$, $z \in \Pi$, and $f(0) = 0$

Prove that $|f(z)| < |\tan z|$.

Proof #1 Prove that the mapping $w = \tan z$

is a conformal mapping of Π onto the unit disk $\mathbb{D}$. Let $g(w) = f(\arctan w)$

Then we have $|g(w)| < 1$, $w \in \mathbb{D}$; $g(0) = 0$.

By the Schwarz lemma $|g(w)| \le |w|$

This is the desired inequality if return to variable z.

Proof #2 Consider the function

$F(z) = f(z)/\tan z$, $z \in \Pi$,

observe that $|\tan z| = 1$, $|\mathrm{Re}\, z| = \pi/4$

and apply the max principle.

You have to be a bit careful because

Π is an infinite domain!

apply it first in the domain

$\Pi_N = \{ z \in \Pi , \ |\operatorname{Im} z| < N \}$

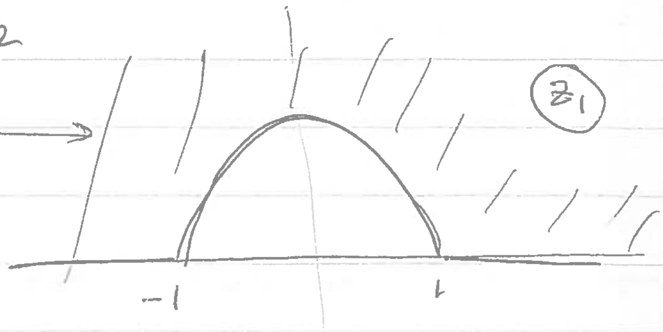
and then let $N \to \infty$. You have to

observe that $|\tan z| \to 1$ as $|\operatorname{Im} z| \to \infty$.

(7) Convergence of the product $\prod_1^\infty (1-z^n)$ is equivalent to convergence of the sum $\sum_1^\infty z^n$. It takes place for all z such that $|z| < 1$.

8 a

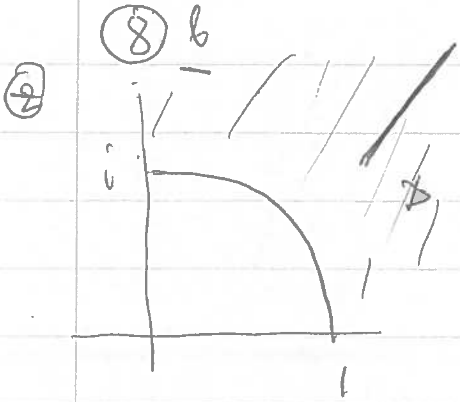
$z_1 = z^2$



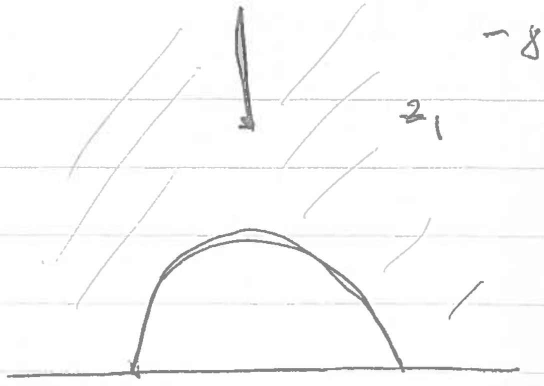
$z_3 = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$ – Zhukovskii mapping

Comment: There are other (more complicated) ways. Full credit if correct.

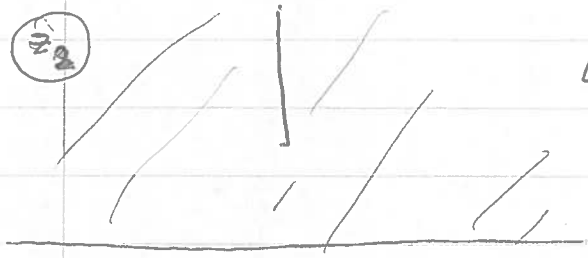
8 b



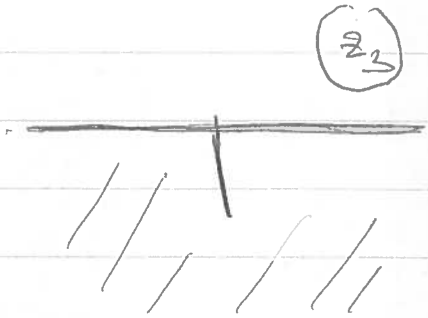
$z_1 = z^2$



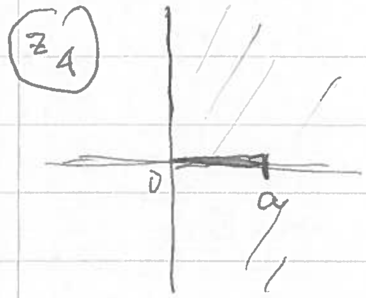
$z_2 = \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$



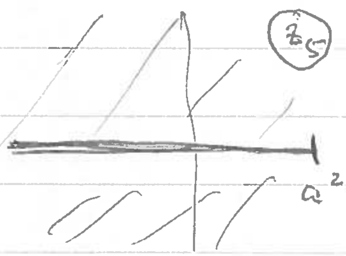
$z_3 = \frac{1}{z_2}$



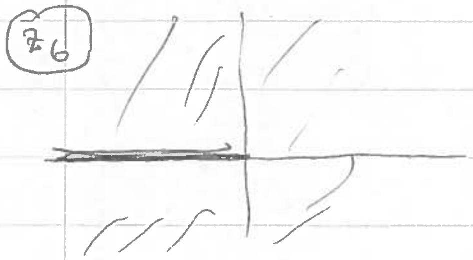
$z_4 = i z_3$



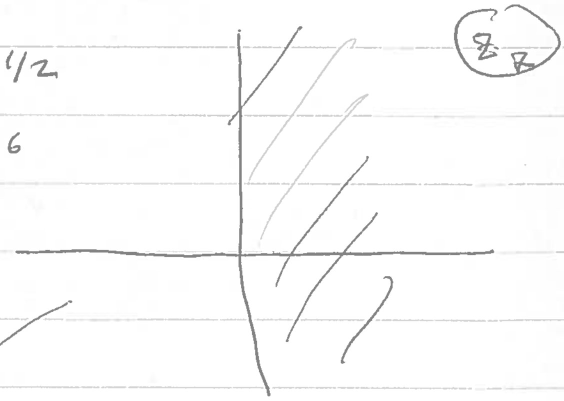
$z_5 = z_4^2$



$z_6 = z_5 - a^2$



$z_7 = z_6^{1/2}$



$z_8 = i z_7$