Norwegian University of Science and Technology
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TMA4125 Matematikk 4N
Spring 2017

Exercise set 2

$\boxed{1}$ Let $f : [0, \pi] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{for } 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2} & \text{for } \frac{\pi}{2} < x \le \pi \end{cases}$$

a) Let $g : \mathbb{R} \to \mathbb{R}$ be the odd 2π-periodic extension of f. Sketch its graph and compute its Fourier series.

b) Let $h : \mathbb{R} \to \mathbb{R}$ be the even 2π-periodic extension of f. Sketch its graph and compute its Fourier series.

$\boxed{2}$ The motion of a spring subject to an external force is described by the equation

$$y'' + 0.02y' + 25y = r(t),$$

where $r(t)$ is the 2π-periodic driving force

$$r(t) = \begin{cases} t + \frac{\pi}{2} & \text{for } -\pi < t < 0 \\ -t + \frac{\pi}{2} & \text{for } 0 \le t < \pi \end{cases}$$

a) The Fourier series for $r(t)$ is given by

$$r(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)t$$

Use this to find the steady state solution describing the motion, given in the form

$$y = \sum_{n=1}^{\infty} A_n \cos nt + B_n \sin nt$$

b) We wish to approximate the solution by a function of the form

$$\hat{y} = \sum_{n=1}^{5} \hat{A}_n \cos nt + \hat{B}_n \sin nt$$

Find the values of $\hat{A}_n$ and $\hat{B}_n$ that minimize the mean-square error of the approximation.

$\boxed{3}$ Let f be the 2π-periodic function given on $[-\pi, \pi]$ by $f(x) = x$. The Fourier series for this function is

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$$

a) Compute $\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2 dx$ by using Parseval's theorem.

b) Now evaluate the integral $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx$ by standard integration techniques.

c) Compare the two results above, and thereby evaluate the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$

d) Find the approximation of f of the form

$$\hat{f}(x) = \hat{b}_1 \sin x + \hat{b}_2 \sin 2x + \hat{b}_3 \sin 3x$$

that minimizes the mean square error $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(x) - \hat{f}(x)\right)^2 dx$, and compute this error.

e) (Optional) The series of Fourier coefficients $\sum_{n=1}^{\infty} b_n = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ does not converge absolutely. Why does this not contradict the proof of convergence of Fourier series given in Kreyszig, Chapter 11.1, which showed the absolute summability of the Fourier coefficients (The same proof was given in the lectures)?

$\boxed{4}$ Compute the complex Fourier series of the 2π-periodic function

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

i.e. find the Fourier series for f in the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$