LØSNINGSSKISSER Øving 6

Oppgave 1

$g(t) = f(r(t))$ der $r(t) = (x(t), y(t))$, tilstr. glatte, og vi anvender kjerneregelen to ganger (bl.a.)!

$g(t) = f(x, y)$ der $x = x(t)$, $y = y(t)$

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\boxed{\frac{d^2 g}{dt^2}} = \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d^2 x}{dt^2} + \text{tilsv. med } x \text{ og } y \text{ byttet}$$

Kont. 2.ord. part.der.

$$= \boxed{\frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial x}\frac{d^2 x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2 y}{dt^2}}$$

Bmk Jeg synes TL's notasjon blir litt tung og „blandet" med både Leibniz og Newton - symboler for derivert! KH

Oppgave 2

Vi skal regne ut $\int_C f\, ds$ når

a) $f(x, y) = xy$, $C: r(t) = (3t, 4t)$, $t \in [0, 2]$ slik at

$$\int_C f\, ds = \int_0^2 3t \cdot 4t \sqrt{3^2 + 4^2}\, dt = 60\left[\frac{t^3}{3}\right]_0^2 = 20 \cdot 8 = \underline{\underline{160}}$$

b) $f(x, y, z) = z$, $C: r(t) = (t\sin t, t\cos t, t)$, $t \in [0, 2\pi]$ slik at

$$\int_C f\, ds = \int_0^{2\pi} t\sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + 1}\, dt$$

$$= \int_0^{2\pi} t\sqrt{\underbrace{\sin^2 t + \cos^2 t + t^2(\cos^2 t + \sin^2 t) + 1}_{2 + t^2}}\, dt$$

$$\boxed{\begin{array}{l} u = 2 + t^2 \\ du = 2t\,dt \end{array}} \quad \int_2^{2+4\pi^2} \frac{1}{2}\sqrt{u}\, du = \frac{1}{3}\left[u^{3/2}\right]_2^{2+4\pi^2}$$

$$= \underline{\underline{\frac{1}{3}\left[(2 + 4\pi^2)^{3/2} - 2\sqrt{2}\right]}}$$

Oppgave 3

Vi kan følge beviset s.186 helt til $|\phi'(t)|$ som nå er $-\phi'(t)$ siden $\phi'(t)<0$. Vi har da

$$I_1 = \dots = -\int_a^b f(r_2(\phi(t)) v_2(\phi(t))\phi'(t)\,dt \quad .$$

Innfører vi en ny variabel $u = \phi(t)$ får vi som før $du = \phi'(t)\,dt$. Siden $\phi(a) = d$ og $\phi(b) = c$ har vi

$$I_1 = -\int_d^c f(r_2(\phi(u)) v_2(u)\,du \underset{\text{pr def } \int_d^c = -\int_c^d}{=} \int_c^d f(r_2(\phi(u)) v_2(u)\,du \qquad \square$$

Merk Setning 9.2.7 i „Kalkulus" essensiell enten ϕ er strengt voksende eller strengt avtagende.

Oppgave 4

a) $$\int_C F\cdot dr = \int_1^3 (-3t, 2t)\cdot(2,-3)\,dt = \int_1^3 (-12t)\,dt$$
$$= -6\left[t^2\right]_1^3 = -54+6 = \underline{\underline{-48}}$$

b) $$\int_C F\cdot dr = \int_0^2 (t^3\cdot t^2, t^2, t^4)(1, 2t, 3t^2) = \left[\frac{t^6}{6}+\frac{t^4}{2}+\frac{3t^7}{7}\right]_0^2$$
$$= \frac{32}{3}+8+\frac{384}{7} = \underline{\underline{\frac{1544}{21}}}$$

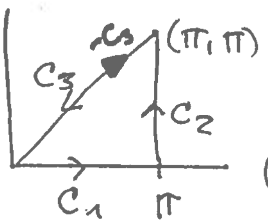
Oppgave 5

C : $x = 5\cos t$, $y = 5\sin t$; $t\in[0, 2\pi]$ (Kjent!)

$F = (5\cos t, 5\sin t)$

$$\int_C F\cdot dr = \int_0^{2\pi} 5^2(\cos t, \sin t)\cdot(-\sin t, \cos t)\,dt = \int_0^{2\pi} 0\,dt = \underline{\underline{0}}$$

Oppgave 6



$F(x,y) = (\cos x \sin y, x)$

$$\int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr + \int_{C_3} F\cdot dr$$

C_1: $(x,y) = (t, 0)$, $t\in[0,\pi]$ $\quad C_2$: $(x,y) = (\pi, t)$, $t\in[0,\pi]$

$-C_3$: $(x,y) = (t, t)$, $t\in[0,\pi]$

$$\int_{C_1} F \cdot dr = \int_0^{\pi} (0,t)\cdot(1,0)\,dt = \underline{0}$$

$$\int_{C_2} \ldots = \int_0^{\pi} (\cos\pi \sin t, \pi)\cdot(0,1)\,dt = \int_0^{\pi} \pi\,dt = \underline{\pi^2}$$

$$\int_{-C_3} \ldots = \int_0^{\pi} (\cos t \sin t, t)(1,1)\,dt = \left[\frac{\sin^2 t}{2} + \frac{t^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2}$$

$$\int_C F\cdot dr = \pi^2 - \frac{\pi^2}{2} = \underline{\underline{\frac{\pi^2}{2}}}$$

Oppgave 7

Alle vektorfeltene F er definert i $\mathbb{R}^2$ og har kont. part. deriverte. Da er F konservativt hvis og bare hvis $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ for alle $(x,y) \in \mathbb{R}^2$ (T 3.5.7).
At F er konservativt betyr at $F = \nabla\phi$ (D 3.5.3).

a) $F(x,y) = (x^2+y^2, 2xy) \Rightarrow \frac{\partial F_1}{\partial y} = 2y = \frac{\partial F_2}{\partial x}$; konservativt.
Potensialfunksjonen ϕ må tilfredsstille

$$\frac{\partial\phi}{\partial x} = x^2 + y^2 \Rightarrow \phi(x,y) = \underline{\tfrac{1}{3}x^3} + \underline{\underline{y^2x}} + C_1(y)$$

$$\frac{\partial\phi}{\partial y} = 2xy \Rightarrow \phi(x,y) = \underline{\underline{xy^2}} + \underline{C_2(x)}$$

Samler vi kravene ser vi at $\underline{\underline{\phi(x,y) = \tfrac{1}{3}x^3 + xy^2}}$ passer.
(Det gjør også $\phi(x,y) = \frac{1}{3}x^3 + xy^2 + C$)

b) $F(x,y) = (xy, xy) \Rightarrow \frac{\partial F_1}{\partial y} = x$, $\frac{\partial F_2}{\partial x} = y$ ⊃: ikke konservativt.
(Har ikke $x = y$ for alle $(x,y) \in \mathbb{R}^2$)

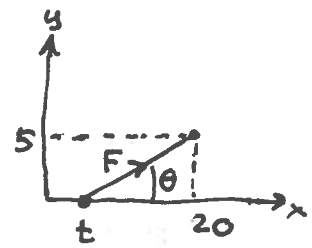
c) $F(x,y) = (2x\cos y + \cos y, x^2 \sin y + x\sin y)$

$\Rightarrow \frac{\partial F_1}{\partial y} = -2x\sin y - \sin y$, $\frac{\partial F_2}{\partial x} = 2x\sin y + \sin y$

F er ikke konservativt. ($\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$ for $(x,y) = (0, \frac{\pi}{2})$ f.eks)

Oppgave 8

Trekkraften i tauet er konstant lik K (rettet langs tauet)



a) $K\cos\theta = K \dfrac{20-t}{\sqrt{5^2+(20-t)^2}}$

$$(*)\ W = \int_C F \cdot dr\ ;\ F = (K\cos\theta, K\sin\theta),\ r(t) = (t,0);\ 0 \le t \le 20$$
$$r'(t) = (1,0)$$

$$W = \int_0^{20} K\cos\theta\, dt = K \int_0^{20} \frac{20-t}{\sqrt{25+(20-t)^2}}\, dt$$

Bmk Kanskje litt kunstig å bruke (*) her? Tangensial-komp. er jo $K\cos\theta$. – Oppgaven er nesten identisk med 8.6.14 i „Kalkulus."

$$\int_0^{20} \frac{20-t}{\sqrt{25+(20-t)^2}}\, dt \underset{\boxed{\substack{s=20-t\\ ds=-dt}}}{=} -\int_{20}^{0} \frac{s}{\sqrt{25+s^2}}\, ds$$

$$= \int_0^{20} \frac{s}{\sqrt{25+s^2}}\, ds = \left[\sqrt{25+s^2}\right]_0^{20} = \sqrt{425} - 5$$

Dermed

$$W = K(\sqrt{425} - 5) = 5K(\sqrt{17} - 1)$$

Oppgave 9

$F(x,y,z) = (2xy, x^2, z)$; $\int_C F \cdot dr = ?$

Spm ii) indikerer at F må være et gradientfelt; $\phi = ?$

$$\frac{\partial\phi}{\partial x} = 2xy \Rightarrow \phi = x^2y + C_1(y,z)$$
$$\frac{\partial\phi}{\partial y} = x^2 \Rightarrow \phi = x^2y + C_2(x,z)$$
$$\frac{\partial\phi}{\partial z} = z \Rightarrow \phi = \tfrac{1}{2}z^2 + C_3(x,y)$$

$\phi = x^2y + \frac{1}{2}z^2$ passer

(Sjekk at $\nabla\phi = F$!)

i) $r(0) = (1,0,0) = r(2\pi)$. Lukket kurve, så $\int_C F \cdot dr = 0$

ii) $\int_C F \cdot dr = \phi(1,-2,\sqrt{2}) - \phi(0,0,0) = 1(-2) + \frac{1}{2}(\sqrt{2})^2 = -1$