Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4165 Differential Equations and Dynamical Systems
Spring 2017

Exercise set 11

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J.S.: 11.5
Exam 1996.1; Exam 1996.6; Exam 2002.3

These exercises will be presented / discussed in the exercise class:

E31, E32, E33

$\boxed{\text{Exam 1996, 1}}$ Given the system

$$\begin{aligned}\dot{x} &= x - y\\ \dot{y} &= x^2 - 1.\end{aligned}$$

a) Find and classify all equilibrium points of the system. Sketch the phase diagram.

b) Does there exist a closed phase path surrounding all equilibrium points?

$\boxed{\text{Exam 1996, 6}}$ Compute the index of the origin for the following systems

a)

$$\begin{aligned}\dot{x} &= x\\ \dot{y} &= -y.\end{aligned}$$

b)

$$\begin{aligned}\dot{x} &= x + x^4 + y^5\\ \dot{y} &= -y + xy^3.\end{aligned}$$

$\boxed{\text{Exam 2002, 3}}$ **a)** State Bendixson's negative criterion.

b) Determine whether or not the following system has non-constant periodic solutions.

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x - y(1 + x^2 + x^4).\end{aligned}$$

c) Given the population model

$$\dot{x} = xF(x,y)$$
$$\dot{y} = yG(x,y),$$

where F and G are C^1 functions. Assume that $\frac{\partial F}{\partial x} < 0$ and $\frac{\partial G}{\partial y} < 0$. Show that there are no closed phase paths in the first quadrant.

E31 **a)** Show that the system

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3$$

has a closed phase path inside the region

$$A_{a,b} = \{(x,y) \mid a \leq x^2 + y^2 \leq b, 0 < a < 1, b > 2\}.$$

b) Consider the system in a) for the region $A_{\frac{3}{4},3}$. Explain why the result in a) does not contradict Bendixson's negative criterion.

E32 Given the system

$$\dot{x} = x + y - x\sqrt{x^2 + y^2} \tag{1}$$
$$\dot{y} = -x + y - y\sqrt{x^2 + y^2}.$$

a) Classify the equilibrium point $(0,0)$ for both (1) and its linearisation.

b) Show that the system has exactly one closed phase path.

c) Define what it means to be a Poincaré map with Poincaré section Σ.

d) Determine the Poincaré map with Poincaré section $\Sigma = \{(x,0) \mid x > 0\}$.

E33 **a)** Given the autonomous two-dimensional system $\dot{x} = f(x)$, where $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a Lipschitz function.
Explain which ω-limit sets a phase path Γ can have if Γ lies inside a closed, bounded subset K of $\mathbb{R}^2$.

b) Given the following systems in polar coordinates

$$\dot{r} = (1 - r^2)^2 r \tag{2a}$$
$$\dot{\theta} = 1 \tag{2b}$$

and

$$\dot{r} = (1 - r^2)^2 r \tag{3a}$$
$$\dot{\theta} = 1 - r^2 \tag{3b}$$

Find and classifiy all possible ω-limit sets and determine whether they are stable or unstable.