MA0301 ELEMENTARY DISCRETE MATHEMATICS
SPRING 2017

1. Homework Set 2 – Solutions

Exercise 1. *Grimaldi's book (5. ed., Exercises 2.4, page 101): solve* **Exercise 8**

Solution 1. The universe are the integers $\mathbb{Z}$.

a) T. If x is even, $x^2 - 8x + 15$ is odd and thus $\neq 0$.
b) F. $x = 1 : q(1) : T \ , \ p(1) : F$
c) T. Follows from a).
d) T. Pick any even x, or $x = 3$, $x = 5$.
e) T. Pick x odd or $x \leq 0$.
f) T. Equivalent to a).
g) T. Any x works.
h) F. $(x = -1 : p(-1) \vee q(-1) : T \ , \ r(-1) : F)$

Exercise 2. *Grimaldi's book (5. ed., Exercises 2.4, page 103): solve* **Exercise 19**

Solution 2. Universe U, $\forall_{x\in U}[p(x) \rightarrow q(x)]$
Converse: $\forall_{x\in U}[q(x) \rightarrow p(x)]$
Inverse: $\forall_{x\in U}[\neg p(x) \rightarrow \neg q(x)]$
Contrapositive: $\forall_{x\in U}[\neg q(x) \rightarrow \neg p(x)]$

a) $U = \mathbb{Z}^+$: statement is T
Converse: $\forall_{m,n\in U}[m^2 > n^2]$ then $m > n$ – T
Inverse : $\forall_{m,n\in U}[m \leq n]$ then $m^2 \leq n^2$ – T
Contrapositive: $\forall_{m,n\in U}[m^2 \leq n^2]$ then $m \leq n$ – T

b) $U = \mathbb{Z}$: statement is F $(a = 1, b = -2)$
Converse: $\forall_{a,b\in U}[a^2 > b^2]$ then $a > b$ – F $(a = -5, b = 3)$
Inverse : $\forall_{a,b\in U}[a \leq b]$ then $a^2 \leq b^2$ – F $(a = -5, b = 3)$
Contrapositive: $\forall_{a,b\in U}[a^2 \leq b^2]$ then $a \leq b$ – F $(a = 1, b = -2)$

c) $U = \mathbb{Z}$: statement is T;
Recall that m divides p is equivalent to the existence of $a \in \mathbb{Z}$ such that $p = am$.
Converse: $\forall_{m,p\in U}([p = am] \rightarrow \forall_{n\in U} n = bm \wedge p = cn)$ – F $(m = 1, n = 2, p = 3)$
Inverse : $\forall_{m,n,p\in U}([n \neq am \vee p \neq bn] \rightarrow [p \neq cm]])$ – F $(m = 1, n = 2, p = 3)$
Contrapositive: $\forall_{m,p\in U}([p \neq am] \rightarrow \forall_{n\in U}[n \neq bm \vee p \neq cn])$ – T

d) $U = \mathbb{R}$: statement is T
Converse: $\forall_{x\in U}[x^2 > 9 \rightarrow x > 3]$ – F $(x = -5)$

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Inverse : $\forall_{x\in U}[x \leq 3 \to x^2 \leq 9]$ – F $(x=-5)$
Contrapositive: $\forall_{x\in U}[x^2 \leq 9 \to x \leq 3]$ – T

e) $U=\mathbb{R}$: statement is T
Converse: $\forall_{x\in U}[(x>3 \vee x<-7) \to (x^2+4x-21>0)]$ – T
Inverse : $\forall_{x\in U}[(x^2+4x-21 \leq 0) \to (x \leq 3 \wedge x \geq -7)]$ – T
Contrapositive: $\forall_{x\in U}[(x \leq 3 \wedge x \geq -7) \to (x^2+4x-21 \leq 0)]$ – T

Exercise 3. *Grimaldi's book (5. ed., Exercises 2.5, page 116): solve* **Exercise 8**

Solution 3. a) Universe U, and assume that $\forall_{x\in U}p(x) \vee \forall_{x\in U}q(x)$ is true. Without loss of generality, let $\forall_{x\in U}p(x)$ be true. Then for any $c \in U$, $p(c)$ is true, and also $p(c) \vee q(c)$ is true. Therefore, since $c \in U$ was arbitrary, we find that $\forall_{x\in U}[p(x) \vee q(x)]$ is true, which proves the implication.

b) Let Universe $U = \mathbb{Z}^*$. The open statements are $p(x) : x > 0$ and $q(x) : x < 0$. Then both $\forall_{x\in U}p(x)$, $\forall_{x\in U}q(x)$ are false for $U = \mathbb{Z}^*$. However, for any non-zero integer $a \in U$ it is true that either $a > 0$ or $a < 0$. Hence $\forall_{x\in U}[p(x) \vee q(x)]$ is true.

Exercise 4. *Grimaldi's book (5. ed., Exercises 3.1, page 135): solve* **Exercise 22**

Solution 4. a) There is a 2-to-1 correspondence between the subsets of B and the subsets of A. Namely, we can match a subset $S \subseteq A$ with the subsets S and $S \cup \{x\}$ of B. This way, each subset of B is matched with exactly two subsets of A, so B has twice as many subsets as A, and the answer is $2n$. (This can also be used to show that a finite set X has $2^{|X|}$ subsets by induction.)

b) $4n$: Apply a) twice.

c) $2^k n$: Apply a) k times.

Exercise 5. *Grimaldi's book (5. ed., Exercises 3.2, page 147): solve* **Exercise 16**

Solution 5.

(1) $(A \cap B) \cup [B \cap ([C \cap D] \cup [C \cap \bar{D}])]$

(2) $= (A \cap B) \cup [B \cap [C \cap (D \cup \bar{D})]]$ (distributivity)

(3) $= (A \cap B) \cup [B \cap (C \cap U)]$ $(D \cup \bar{D} = U)$

(4) $= (A \cap B) \cup [B \cap C]$ (identity law)

(5) $= B \cap (A \cup C)$ (commutativity + distributivity)

Exercise 6. *Let*

$$C := \{n \in \mathbb{N} \mid n \text{ is a multiple of } 12\}$$

and

$$D := \{n \in \mathbb{N} \mid n \text{ is a multiple of } 2 \text{ and } n \text{ is a multiple of } 6\}.$$

Which of the statements is true: $C \subseteq D$, $D \subseteq C$, $C = D$.

Solution 6. Let $c \in C$, then $c = 12m$ for some $m \in \mathbb{N}$. We know that $12 = 2 \times 6$, and therefore $c = 2 \times 6m = 6 \times 2m$, which implies that $c \in D$ and $C \subseteq D$. However, $6 \in D$ but 6 is not in C. Therefore $D \subseteq C$ and $D = C$ are false.

2. Classroom Set 2 – Solutions

Exercise 1. *Grimaldi's book (5. ed., Exercises 2.2, page 66): solve* **Exercise 13** *and describe in detail in your own words the relation with the proof of Theorem 3.4 on page 138 in Grimaldi's book (Thm. II. 15. of lecture 4).*

Solution 1. Provide a detailed argument.

Exercise 2. *By using rules of inference, show that the following arguments are true:*

$$i)\ \neg(a \wedge b) \wedge (\neg c \to b) \to (a \to c)$$

$$ii)\ \neg(\neg p \vee q) \wedge (\neg z \to \neg s) \wedge ((p \wedge \neg q) \to s) \wedge (\neg z \vee r) \to r$$

Solution 2. i)

$\neg(a \wedge b)$	premise
$\neg c \to b$	premise
$\neg a \vee \neg b$	(1) DeMorgan
$a \to \neg b$	(3) $p \to q \Leftrightarrow \neg p \vee q$
$\neg b \to c$	(2) and contrapositive
$a \to c$	(4,5) and syllogism

ii)

$\neg(\neg p \vee q)$	premise
$\neg z \to \neg s$	premise
$(p \wedge \neg q) \to s$	premise
$\neg z \vee r$	premise
$p \wedge \neg q$	(1), DeMorgan and double negation
s	(3,5) and modus ponens
$s \to z$	(2) and contrapositive
z	(6,7) and modus ponens
$z \to r$	(4) and $p \to q \Leftrightarrow \neg p \vee q$
r	(8,9) and modus ponens

Exercise 3. *Grimaldi's book (5. ed., Exercises 2.4, page 100): solve* **Exercise 1**

Solution 3. $U = \mathbb{Z}$.

$$a)\ F \quad b)\ F \quad c)\ F \quad d)\ T \quad e)\ F \quad f)\ F.$$

Exercise 4. *Grimaldi's book (5. ed., Exercises 2.4, page 100): solve* **Exercise 2**

Solution 4. $U = \mathbb{Z}$. a) $i)\ T \quad ii)\ T \quad iii)\ T \quad iv)\ T$
b) $x = 2$

Exercise 5. *Let*

$$C := \{n \in \mathbb{N} \mid n \text{ is a multiple of } 6\}$$

and

$$D := \{n \in \mathbb{N} \mid n \text{ is a multiple of } 2 \text{ and } n \text{ is a multiple of } 3\}.$$

Show that $C = D$.

Solution 5. We have to show that $C \subseteq D$ and $D \subseteq C$ for $C = D$ to be true. $C \subseteq D$: for $c \in C$ we have that $c = 6m$, $m \in \mathbb{N}$. But $c = 6m = 2 \times 3m = 3 \times 2m$. Therefore $c \in D$.

$D \subseteq C$: for $d \in D$ we have that $d = 2m$ and $d = 3n$, $m, n \in \mathbb{N}$. Therefore $2m = 3n$ and n must be even. Let $n = 2a$, $a \in \mathbb{N}$, which implies that $d = 6a$, and this yields $d \in C$.

Exercise 6. *Grimaldi's book (5. ed., Exercises 3.1, page 135): solve* **Exercise 17**

Solution 6. (Only c) and d) are necessary.)

a) Start with $x \in A$. From $A \subseteq B$ it follows that $x \in B$, and from $B \subseteq C$ we deduce that $x \in C$. Hence, $x \in A$ implies $x \in C$, and therefore $A \subseteq C$.

b) $A \subset B$ implies that $A \subseteq B$, and therefore $A \subseteq C$ (part a). However, $A \subset B$ implies that there exists an element $b \in B$ which is not in A. Since $B \subseteq C$ it follows that $b \in C$. Therefore, there exists an element $b \in C$ which is not in A, and this implies that $A \subset C$.

c) $B \subset C$ implies that $B \subseteq C$, and therefore $A \subseteq C$ (part a). $B \subset C$ implies that there exists an element $c \in C$ which is not in B. As $A \subseteq B$ the element c is not in A, and therefore $A \subset C$.

d) $A \subset B$ implies that $A \subseteq B$, and the result follows from (part c).