TMA4295 Statistical inference
Exercise 4 - solution

Problem 4.15
$X \sim Poisson(\theta)$ and $Y \sim Poisson(\lambda)$
$Z = X + Y \sim Poisson(\theta + \lambda)$

$$P(X = x|Z = z) = \frac{P(X = x, Y = y)}{P(Z = z)} = \frac{\frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^{z-x} e^{-\lambda}}{(z-x)!}}{\frac{(\theta+\lambda)^z e^{-(\theta+\lambda)}}{z!}} = \binom{z}{x} \left(\frac{\theta}{\theta + \lambda}\right)^x \left(1 - \frac{\theta}{\theta + \lambda}\right)^{z-x}$$

which is a binomial with $p = \left(\frac{\theta}{\theta+\lambda}\right)$.

Problem 4.30
$Y|X = x \sim N(x, x^2)$ and $X \sim Uniform(0, 1)$.

a)

$$E(Y) = E(E(Y|X)) = E(X) = \frac{1}{2},$$
$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)) = \frac{1}{12} + \frac{1}{3} = \frac{5}{12},$$
$$Cov(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

b)

$$f(x, y) = f(y|x)f(x) = \frac{1}{2\sqrt{\pi}x} e^{\frac{-(y-x)^2}{2x^2}} I_{(0,1)}(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{y}{x}} \frac{1}{x} e^{-\frac{1}{2}} I_{(0,1)}(x) = g\left(\frac{y}{x}\right) h(x)$$

$$\Rightarrow \frac{Y}{X} \text{ and } X \text{ independent.}$$

Problem 4.31
$Y|X = x \sim B(n, x)$ and $X \sim Uniform(0, 1)$.

a)

$$E(Y) = E(E(Y|X)) = E(nX) = \frac{n}{2},$$
$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)) = E(nX(1 - X)) + Var(nX) = \frac{n}{6} + \frac{n^2}{12}.$$

b)

$$f(x, y) = f(y|x)f(x) = \binom{n}{y} x^y (1 - x)^{n-y} I_{(0,1)}.$$

c)

$$f_Y(y) = \int_0^1 f(x, y)dx = \int_0^1 \binom{n}{y} x^y (1 - x)^{n-y} dx = \binom{n}{y} \frac{\Gamma(y + 1)\Gamma(n - y + 1)}{\Gamma(n + 2)}.$$

Problem 4.32
$Y|\Lambda \sim Poisson(\Lambda)$ and $\Lambda \sim Gamma(\alpha, \beta)$.

a)

$$f_Y(y) = \int_0^\infty f(y|\lambda)f(\lambda)d\lambda = \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!}\frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}e^{-\frac{\lambda}{\beta}}d\lambda$$

$$= \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \lambda^{y+\alpha-1}exp\left(\frac{-\lambda}{\frac{\beta}{1+\beta}}\right)d\lambda = \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\Gamma(y+\alpha)\left(\frac{\beta}{1+\beta}\right)^{y+\alpha}.$$

If α is an integer

$$f_Y(y) = \binom{y+\alpha-1}{y}\left(\frac{\beta}{1+\beta}\right)^y\left(\frac{1}{1+\beta}\right)^\alpha$$

that is a negative binomial.

$$E(Y) = E(E(Y|\Lambda)) = E(\Lambda) = \alpha\beta,$$

$$Var(Y) = Var(E(Y|\Lambda)) + E(Var(y|\Lambda)) = E(\Lambda) + Var(\Lambda) = \alpha\beta + \alpha\beta^2 = \alpha\beta(\beta+1).$$

b) $Y|N \sim Binomial(N,p)$, $N|\Lambda \sim Poisson(\Lambda)$ and $\Lambda \sim gamma(\alpha,\beta)$ from example 4.4.2 we see that

$$P(Y=y|\lambda) = \sum_{n=y}^\infty P(Y=y|N=n,\lambda)P(N=n|\lambda) = \frac{(p\lambda)^y e^{-p\lambda}}{y!},$$

so we have $Y|\Lambda \sim Poisson(p\Lambda)$, calculations similar to those in a) prove that Y is negative binomial distributed (if α is a positive integer).

Problem 4.35
$X|P \sim Binomial(n,P)$ and $Y \sim Beta(\alpha,\beta)$

a)

$$Var(X) = E(Var(X|P)) + Var(E(X|P)) = E(nP(1-P)) + Var(nP)$$
$$= n(E(P) - E(P^2)) + n^2Var(P) = nE(P) - nVar(P) - n(E(P)^2) + n^2Var(P)$$
$$= nE(P)(1-E(P)) + n(n-1)Var(P).$$

b)

$$Var(Y) = \alpha\beta + \alpha\beta^2 = \mu + \frac{\mu^2}{\alpha}.$$

Problem 4.36
$X_i|P_i \sim Bernoulli(P_i) \quad i = 1,...,k$
$P_i \sim Beta(\alpha,\beta)$.

We first compute

$$E(X_i) = E(E(X_i|P_i)) = E(P_i) = \frac{\alpha}{\alpha+\beta},$$

$$Var(X_i) = E(Var(X_i|P_i)) + Var(E(X_i|P_i)) = \frac{\alpha\beta}{(\alpha+\beta)^2}.$$

a) Since $Y = \sum_{i=1}^k X_i$

$$E(X) = \sum_{i=1}^k E(X_i) = \frac{n\alpha}{\alpha+\beta}.$$

b)

$$Var(X) = \sum_{i=1}^k Var(X_i) = \frac{n\alpha\beta}{(\alpha+\beta)^2}.$$

c) $X_i|P_i \sim Binomial(n_i, P_i)$ and $P_i \sim Beta(\alpha, \beta)$ for $i = 1, ..., k$.

$$E(X_i) = E(E(X_i|P_i)) = E(n_i P_i) = \frac{n_i \alpha}{\alpha + \beta},$$

$$Var(X_i) = E(Var(X_i|P_i)) + Var(E(X_i|P_i)) = \frac{n_i \alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

$$\Rightarrow E(Y) = \sum_{i=1}^{k} E(X_i) = \frac{\alpha}{\alpha+\beta}\sum_{i=1}^{k} n_i$$

$$Var(Y) = \sum_{i=1}^{k} Var(X_i).$$

Problem 4.58

a)

$$\begin{aligned} Cov(X,Y) &= \int\int (x-\mu_x)(y-\mu_y)f(x,y)dxdy \\ &= \int (x-\mu_x)\int (y-\mu_y)f(y|x)dyf(x)dx \\ &= \int (x-\mu_x)(E(Y|X)-\mu_y)f(x)dx \\ &= Cov(X, X(Y|X)). \end{aligned}$$

b)

$$\begin{aligned} Cov(X,(Y-E(Y|X))) &= E((X-\mu_x)(Y-E(Y|X))) \\ &= \int\int (x-\mu_x)(y-E(Y|x))f(y|x)dyf(x)dy \\ &= \int (x-\mu_x)\int (y-E(Y|x))f(y|x)dyf(x)dx \\ &= \int (x-\mu_x)(E(Y|x)-E(Y|x))f(x)dx = 0. \end{aligned}$$

c)

$$\begin{aligned} Var(Y-E(Y|X)) &= \int\int (y-E(Y|x))^2 f(x,y)dxdy \\ &= \int\int (y-E(Y|x))^2 f(y|x)dyf(x)dx \\ &= \int Var(Y|x)f(x)dx = E(Var(Y|X)). \end{aligned}$$