Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2017

Exercise set 4

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Show that the sets $U, V \subset \mathcal{P}_4$, the space of polynomials of degree at most 4, defined by

$$
\begin{aligned}
U :=&\{p \in \mathcal{P}_4 : \; p(-1) = p(1) = 0\},\\
V :=&\{p \in \mathcal{P}_4 : \; p(1) = p(2) = p(3) = 0\}
\end{aligned}
$$

are subspaces of $\mathcal{P}_4$ and determine the subspace $U \cap V$.

2 Prove that $(l^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is a normed space, where for any bounded sequence $x = (x_n) \in l^\infty(\mathbb{R})$ we define

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|.$$

Is this norm associated with an inner product?

3 Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries. For $A \in M_n(\mathbb{C})$ we define its *trace* by $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$.

a) Show that for $A, B \in M_3(\mathbb{C})$ we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and try to show this property of the trace for $n \times n$ matrices.

b) Let $\mathcal{D}$ be the set of all diagonal $n \times n$ matrices. Show that $\mathcal{D}$ is a subspace of $M_n(\mathbb{C})$ and that for any $A, B \in D$ we have $AB = BA$ (in contrast to arbitrary matrices in $M_n(\mathbb{C})$).

c) Let $S \subset M_n(\mathbb{C})$ be defined as the matrices with $\mathrm{tr}(A) = 0$. Show that S is a subspace of $M_n(\mathbb{C})$.

4 Suppose $(X, \langle .,. \rangle)$ is an innerproduct space.

a) Let ω be a n^{th} root of unity, i.e. $\omega^n = 1$. Show that

$$\langle x, y \rangle = \frac{1}{n} \sum_{k=1}^{n} \omega^k \|x + \omega^k y\|^2.$$

b) Show that

$$\langle x, y \rangle = \int_0^1 e^{2\pi i \varphi} \|x + e^{2\pi i \varphi} y\|^2 d\varphi.$$

$\boxed{5}$ Let $(\mathbb{R}^n, \|.\|_p)$ be the space of real n-tuples with the p-norms $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$. Show that

$$\sum_{i=1}^{n} |x_i| \leq n^{(p-1)/p} (\sum_{i=1}^{n} |x_i|^p)^{1/p}.$$