

# MA3203 - PROBLEM SHEET 2

**Problem 1.** Let  $k$  be a field. Find the representations corresponding to the modules  $\Lambda e_i$  for the different possible values of  $i$  and for the different cases of  $\Lambda$  listed below.

(a)  $\Lambda = k\Gamma$ , where  $\Gamma$  is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

(b)  $\Lambda = k\Gamma/\langle\rho\rangle$ , where  $\Gamma$  is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and  $\rho = \{\beta\alpha\}$ .

(c)  $\Lambda = k\Gamma/\langle\rho\rangle$ , where  $\Gamma$  is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightleftharpoons[\gamma]{\beta} 3$$

and  $\rho = \{\beta\alpha\}$ .

(d)  $\Lambda = k\Gamma/\langle\rho\rangle$ , where  $\Gamma$  is the quiver:

$$1 \xrightleftharpoons[\beta]{\alpha} 2 \begin{array}{c} \curvearrowright \gamma \\ \curvearrowleft \end{array}$$

and  $\rho = \{\gamma\alpha, \gamma^3\}$ .

**Problem 2.** Find a composition series for the following representations:

- (a)  $\Lambda e_1$  where  $\Lambda$  is as in (c) above.
- (b)  $\Lambda e_1$  where  $\Lambda$  is as in (d) above.

**Problem 3.**

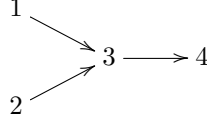
- (a) Given a ring  $\Lambda$ . Show that a  $\Lambda$ -module  $M$  is decomposable if and only if its endomorphism ring  $\text{End}_\Lambda(M) = \{f: M \rightarrow M \mid f \text{ } \Lambda\text{-homomorphism}\}$  contains a nontrivial idempotent (i.e. there is an  $f$  in  $\text{End}_\Lambda(M)$  such that  $f^2 = f$  and  $f \neq 0, 1$ ).
- (b) Use (a) to show that  $\Lambda e_1$  where  $\Lambda$  is as in (b) in Problem 1 is indecomposable. Here we will use without proof that the endomorphism ring of the module  $\Lambda e_1$  is isomorphic to the endomorphism ring of the representation corresponding to  $\Lambda e_1$ .
- (c) Given  $\Lambda = k\Gamma/\langle\rho\rangle$ , where  $\Gamma$  is a quiver with vertices  $\{1, \dots, n\}$  and  $\rho$  is a set of relations. Assume that  $J^t \subset \langle\rho\rangle \subset J^2$  for some  $t$ .  
Show that the endomorphism ring  $\text{End}_\Lambda(\Lambda e_i)^{\text{op}}$  is isomorphic to  $e_i \Lambda e_i$ .  
Conclude (using (a)) that  $\Lambda e_i$  is indecomposable for each  $i$ .
- (d) Given a ring  $\Lambda$  and two simple  $\Lambda$ -modules  $S$  and  $S'$ . Show that if  $f: S \rightarrow S'$  is a nonzero  $\Lambda$ -homomorphism, then  $f$  is an isomorphism.

**Problem 4.** Let  $\Gamma$  be the quiver with relations as in (b) in Problem 1, and let  $V$  be its representation over  $k$  given by:  $V(1) = k$ ,  $V(2) = k^2$ ,  $V(3) = k^2$ ,  $f_\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $f_\beta = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ .

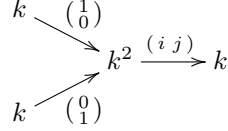
Determine if  $V$  is decomposable, and if it is, find its decomposition into a direct sum of indecomposable representations.

Furthermore, find a composition series for  $V$ .

**Problem 5.** Let  $k$  be a field and let  $\Gamma$  be the quiver



For an ordered pair  $(i, j)$  of elements in  $k$ , let  $M_{ij}$  be the representation given by



- (a) determine for which  $(i, j)$  the representation  $M_{ij}$  is indecomposable and for which  $(i, j)$  it decomposes.
- (b) Prove that if  $M_{ij}$  and  $M_{rs}$  are indecomposable then they are isomorphic.  
Is the same true if  $M_{ij}$  and  $M_{rs}$  decomposes?

**Problem 6.** Let  $\Lambda_c$  be the algebra over  $\mathbb{C}$  with basis  $\{e_0, e_1, e_2, e_3\}$  over  $\mathbb{C}$ , where  $c$  is a given complex number. The multiplication is given by the following multiplication table:

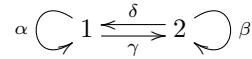
	$e_0$	$e_1$	$e_2$	$e_3$
$e_0$	$e_0$	$e_1$	$e_2$	$e_3$
$e_1$	$e_1$	$e_3$	$e_3$	$0$
$e_2$	$e_2$	$-e_3$	$ce_3$	$0$
$e_3$	$e_3$	$0$	$0$	$0$

*Challenge 1.* For which  $c$  and  $c'$  are the algebras  $\Lambda_c$  and  $\Lambda_{c'}$  isomorphic?

*Challenge 2.* Find a quiver with relations  $\rho_c$  over  $\mathbb{C}$  such that  $\Lambda_c \cong \mathbb{C}\Gamma/\langle\rho_c\rangle$ .

*Challenge 3.* Show that there exists an infinite number of non-isomorphic indecomposable modules over  $\Lambda_c$  for any value of  $c$  in  $\mathbb{C}$ .

**Problem 7.** Let  $k$  be a field and  $\Gamma$  the quiver



with relations  $\rho = \{\delta\gamma - \alpha^2, \alpha^3 - \alpha^2, \gamma\delta - \beta^2, \beta^3 - \beta^2, \alpha\delta - \delta\beta, \gamma\alpha - \beta\gamma\}$ .

- (a) Show that the dimension of  $k\Gamma/\langle\rho\rangle$  over  $k$  is 12.
- (b) Show that the subspace of  $k\Gamma/\langle\rho\rangle$  spanned by  $\alpha^2, \gamma\alpha^2, \alpha^2\delta, \beta^2$  is a ring which is isomorphic to  $M_2(k)$ -ring of  $2 \times 2$ -matrices over  $k$ .

**Problem 8.** We say that a ring  $\Lambda$  is local if the nonunits of  $\Lambda$  (elements in  $\Lambda$  without multiplicative invers) form an ideal in  $\Lambda$ .

Show that if  $\Lambda$  is local, then 0 and 1 are the only idempotents in  $\Lambda$ .