Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4230 Functional Analysis
Spring 2017

Exercise set 4

$\boxed{1}$ Use Zorn's lemma to show the following statements:

a) Any vector space has a Hamel basis.

b) Any Hilbert space has an orthonormal basis.

$\boxed{2}$ Let X be a vector space and p, q sublinear functionals on X. If a linear functional φ on X satisfies

$$\varphi(x) \leq p(x) + q(x) \quad \text{for all} \quad x \in X.$$

Then there exist linear functionals φ_1 and φ_2 on X such that $\varphi = \varphi_1 + \varphi_2$ satisfying

$$\varphi_1(x) \leq p(x) \quad \text{and} \quad \varphi_2(x) \leq q(x)$$

for all $x \in X$.

Hint: Relate the sublinear functional $p + q$ and the assumption on φ with the diagonal $\Delta = \{(x, x) : x \in X\}$ in $X \times X$.

$\boxed{3}$ Let X be $(\mathbb{R}^2, \|.\|_p)$ and $Y = \{x \in \mathbb{R}^2 : x_1 - 2x_2 = 0\}$ a subspace of $\mathbb{R}^2$. Define the linear functional φ on Y by $\varphi(x_1, x_2) = x_1$.

a) Compute the norm of φ.

b) Determine the norm-preserving linear functionals that extend to $(\mathbb{R}^2, \|.\|_p)$ for $p = 1, 2, \infty$