TMA4220 Finite Element Method
Fall 2017

Exercise set 3

1 In the lectures, the following bounds were given for the error of linear interpolation:

$$|v - v_K^1|_{H^m(K)} \leq C_{K,m}\frac{h_K^2}{\rho_K^m}|v|_{H^2(K)}, \qquad m = 0, 1$$

where v_K^1 is the linear Lagrange interpolant (using the standard nodal basis) of v on the element K.

a) h_K is the diameter of K and ρ_K is the sphericity. What do these terms mean?

b) Suppose that we use linear finite elements to solve an elliptic equation written in the weak form: find $u \in H^1(\Omega)$ such that $a(u,v) = F(v)$ for all $v \in H^1(\Omega)$, where F is a continuous linear functional and a is a coercive, continuous bilinear form. Suppose we use a refining sequence of triangulations $\mathcal{T}_h$, where h is the maximum diameter of an element, and assume the regularity condition $\frac{h_K}{\rho_K} \leq \delta$ for all K. Under the assumption that the solution $u \in H^2(\Omega)$, use the error bounds given above to prove the convergence result

$$||u - u_h||_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}$$

2 For each of the following problem, *(i)* find a weak formulation of the PDE and *(ii)* choose an appropriate test/trial space V where u and v live. Show that the conditions of the Lax-Milgram theorem are satisfied, thus proving that there exists a unique (weak) solution of the PDE. Assume $\Omega \subset \mathbb{R}^2$ is an open, bounded, connected domain and $f \in L^2(\Omega)$.

a) The biharmonic equation:

$$\begin{aligned} \nabla^4 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

b) The convection-diffusion equation

$$\begin{aligned} -\nabla^2 u + a \cdot \nabla u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where $a = a(\mathbf{x} : \Omega \to \mathbb{R}^2$ is a given differentiable function, the *velocity field*, satisfying $\nabla \cdot a(\mathbf{x}) = 0$ for all $\mathbf{x}$.

3 Suppose we solve the one-dimensional heat equation

$$u_{xx} = u_t, \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = g(x)$$

using linear finite elements on a uniform subdivision of the interval $[0,1]$, with element diameter h. We integrate the resulting ODE forward in time using a theta method with parameter θ (see the lecture notes or Quarteroni, chapter 5).

a) Solve the generalized eigenvalue problem

$$A_h v = \lambda_h M_h v,$$

where A_h and M_h are the stiffness and mass matrices. Use the following: a matrix is Toeplitz if its entries are constant along diagonals, i.e. $a_{i,j} = a_{i+1,j+1}$ for all i, j. Of particular interest are matrices that are tridiagonal, symmetric and Toeplitz (TST):

1. All $n \times n$ TST matrices have the same eigenvectors
2. The eigenvalues of an $n \times n$ TST matrix with diagonal entries a and off-diagonal entries b are given by

$$a + 2b\cos(\frac{k\pi}{n+1}), \quad k = 1, \dots, n$$

What is the largest generalized eigenvalue in the limit $h \to 0$?

b) Use the above to derive a formula for the maximum step-size k in the time direction for which the theta method is stable, as a function of θ.