Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems
Spring 2017

Solution to E6

E6

(a) When $z_1(t) = x(-t)$ we can write $z_1(t) = x(f(t))$ where $f(t) = -t$. By the chain rule $z_1' = x'(f(t))f'(t)$. In this case, $f(t) = -t$ so we get

$$\dot{z_1} = -(-4x(-t) + 2y(-t)) = +4z_1 - 2z_2$$

A similar calculation for the time derivative of z_2 yields

$$\begin{aligned} \dot{z_1} &= 4z_1 - 2z_2, \\ \dot{z_2} &= 3z_1 - z_2. \end{aligned}$$

(b) Sketch the phase diagram for z_1 and z_2.

The eigenvalues of the system obtained in (a) are solutions to the equation

$$(4 - \lambda)(-1 - \lambda) + 6 = 0,$$

with solutions $\lambda_1 = 1, \lambda_2 = 2$. This is an unstable node, since both eigenvalues are real and positive. The eigenvectors

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

give us the asymptotes of the family of phase paths. See figure 1 for a sketch of the phase diagram.

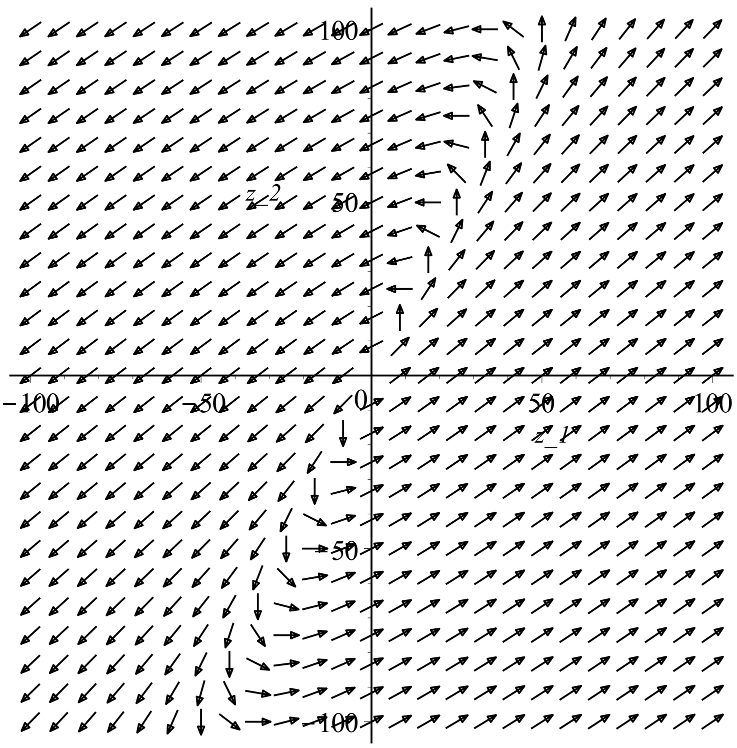

Figure 1: Phase diagram of the system found in (a).

(c) Sketch the phase diagram for

$$\begin{aligned} \dot{x} &= -4x + 2y, \\ \dot{y} &= -3x + y. \end{aligned}$$

The eigenvalues of the matrix associated with the system of equations are given as solutions to

$$(-4-\lambda)(1-\lambda) + 6 = 0,$$

with solution $\lambda_1 = -1, \lambda_2 = -2$. Since both eigenvalues are real and negative, we get a stable node. Again, the eigenvectors

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

give us the asymptotes of the family of phase paths. A faster way to calculate the eigenvectors and eigenvalues, is to use the previous exercise. If $\dot{z} = Az$ and $\dot{x} = (-A)x$, we can use the defining equation $Av = \lambda v$ to find the relationship between the eigenvectors and eigenvalues of the two systems. See figure 1 for a sketch of the phase diagram.

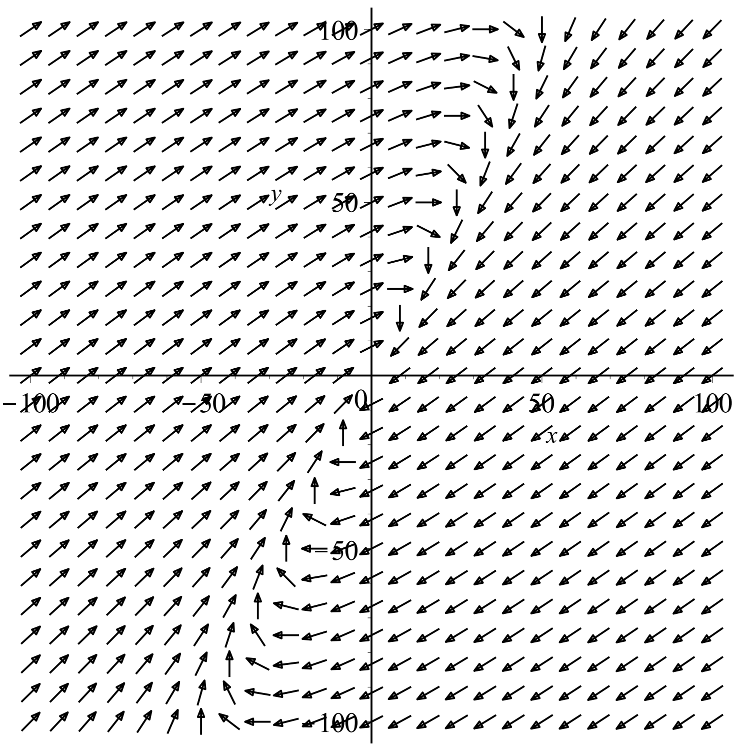

Figure 2: Phase diagram for the system $\dot{x} = -4x + 2y, \dot{y} = -3x + y$.

(d) The figures 1 and 1 look exactly the same, except the direction of the arrows are in the opposite direction. Note that the original system given in the text has a stable equilibrium point, while the transformed one has an unstable equilibrium point at the origin.