Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4230 Functional Analysis
Spring 2017

Exercise set 6

1 Show that the following operators are closed.

a) On the space of continuously differentiable functions $C^1[0,1]$ on $[0,1]$ we define the differentiation operator $Tf(x) = f'(x)$.

b) On the space $\{x \in \ell^1 : \sum_{n=1}^{\infty} n|x_n| < \infty\}$ look at the multiplication operator $Tx = (nx_n)$.

2 Let X be a Banach space and T a bounded linear operator on X.

a) Show that the following are equivalent:

- T is invertible.
- The range of T is dense in X and their exists a $C > 0$ such that $\|Tx\| \geq C\|x\|$ for all $x \in X$.

b) Use the result in the first part to formulate a criterion for the non-invertibility of a linear operator on a Banach space.

3 Let T_a be the multiplication operator on ℓ^2 defined by a bounded sequence $a = (a_n)$.

a) If $\inf\{|a_n| : n = 1, 2, ...\}>0$, show that T_a is invertible on ℓ^2 and find its inverse.

b) Show that T_a is not invertible on ℓ^2 for $a = (1/n)$.

c) If λ is not in the closure of $\{a_n : n = 1, 2, ...\}$, then show that $T_a - \lambda I$ is invertible on ℓ^2.

4 Suppose T is an open mapping between Banach spaces X and Y. Show that the preimage of a dense subset in Y is dense in X.

5 Let X be a Banach space and T a (bounded) linear operator on X with domain $D(T)$. Show that if there exists a sequence $(x_n) \in D(T)$ such that $\|x_n\| = 1$ and $Tx_n \to 0$, then T does not have a bounded inverse.