MA3203 - PROBLEM SHEET 2

Problem 1. Let k ba a field. Find the representations corresponding to the modules Λe_i for the different possible values of i and for the different cases of Λ listed below.

(a) $\Lambda = k\Gamma$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

(b) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $\rho = \{\beta\alpha\}$.

(c) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, shift left, "\beta"] \arrow[r, shift right, "\gamma"'] & 3 \end{tikzcd}$$

and $\rho = \{\beta\alpha\}$.

(d) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[loop right, "\gamma"] \end{tikzcd}$$

and $\rho = \{\gamma\alpha, \gamma^3\}$.

Problem 2. Find a composition series for the following representations:

(a) Λe_1 where Λ is as in (c) above.
(b) Λe_1 where Λ is as in (d) above.

Problem 3.

(a) Given a ring Λ. Show that a Λ-module M is decomposable if and only if its endomorphism ring $\mathrm{End}_\Lambda(M) = \{f : M \to M \mid f\ \Lambda\text{-homomorphism}\}$ contains a nontrivial idempotent (i.e. there is an f in $\mathrm{End}_\Lambda(M)$ such that $f^2 = f$ and $f \neq 0, 1$).

(b) Use (a) to show that Λe_1 where Λ is as in (b) in Problem 1 is indecomposable. Here we will use without proof that the endomorphism ring of the module Λe_1 is isomorphic to the endomorhism ring of the representation corresponding to Λe_1.

(c) Given $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is a quiver with vertices $\{1, \ldots, n\}$ and ρ is a set of relations. Assume that $J^t \subset \langle\rho\rangle \subset J^2$ for some t.
Show that the endomorphism ring $\mathrm{End}_\Lambda(\Lambda e_i)^{\mathrm{op}}$ is isomorphic to $e_i\Lambda e_i$. Conclude (using (a)) that Λe_i is indecomposable for each i.

(d) Given a ring Λ and two simple Λ-modules S and S'. Show that if $f : S \to S'$ is a nonzero Λ-homomorphism, then f is an isomorphism.

Problem 4. Let Γ be the quiver with relations as in (b) in Problem 1, and let V be its representation over k given by: $V(1) = k$, $V(2) = k^2$, $V(3) = k^2$, $f_\alpha = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $f_\beta = \left(\begin{smallmatrix}1 & -1\\0 & 0\end{smallmatrix}\right)$.

Determine if V is decomposable, and if it is, find its decomposition into a direct sum of indecomposable representations.

Furthermore, find a composition series for V.

Problem 5. Let k be a field and let Γ be the quiver

$$\begin{tikzcd} 1 \arrow[dr] & & \\ & 3 \arrow[r] & 4 \\ 2 \arrow[ur] & & \end{tikzcd}$$

For an ordered pair (i,j) of elements in k, let M_{ij} be the representation given by

$$\begin{tikzcd} k \arrow[dr, "\binom{1}{0}"] & & \\ & k^2 \arrow[r, "(i\ j)"] & k \\ k \arrow[ur, "\binom{0}{1}"'] & & \end{tikzcd}$$

(a) determine for which (i,j) the representation M_{ij} is indecomposable and for which (i,j) it decomposes.
(b) Prove that if M_{ij} and M_{rs} are indecomposable then they are isomorphic. Is the same true if M_{ij} and M_{rs} decomposes?

Problem 6. Let Λ_c be the algebra over $\mathbb{C}$ with basis $\{e_0, e_1, e_2, e_3\}$ over $\mathbb{C}$, where c is a given complex number. The multiplication is given by the following multiplication table:

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_3	e_3	0
e_2	e_2	$-e_3$	ce_3	0
e_3	e_3	0	0	0

Challenge 1. For which c and c' are the algebras Λ_c and $\Lambda_{c'}$ isomorphic?

Challenge 2. Find a quiver with relations ρ_c over $\mathbb{C}$ such that $\Lambda_c \cong \mathbb{C}\Gamma/\langle\rho_c\rangle$.

Challenge 3. Show that there exists an infinite number of non-isomorphic indecomposable modules over Λ_c for any value of c in $\mathbb{C}$.

Problem 7. Let k be a field and Γ the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift right, "\gamma"'] & 2 \arrow[l, shift right, "\delta"'] \arrow[loop right, "\beta"] \end{tikzcd}$$

with relations $\rho = \{\delta\gamma - \alpha^2, \alpha^3 - \alpha^2, \gamma\delta - \beta^2, \beta^3 - \beta^2, \alpha\delta - \delta\beta, \gamma\alpha - \beta\gamma\}$.

(a) Show that the dimension of $k\Gamma/\langle\rho\rangle$ over k is 12.
(b) Show that the subspace of $k\Gamma/\langle\rho\rangle$ spanned by $\alpha^2, \gamma\alpha^2, \alpha^2\delta, \beta^2$ is a ring which is isomorphic to $M_2(k)$-ring of 2×2-matrices over k.

Problem 8. We say that a ring Λ is local if the nonunits of Λ (elements in Λ without multiplicative invers) form an ideal in Λ.

Show that if Λ is local, then 0 and 1 are the only idempotents in Λ.