3.1 Given a dynamical system

$$\dot{x} = X(x, y),$$
$$\dot{y} = Y(x, y)$$

let p and q be the number of times

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{Y(x, y)}{X(x, y)}$$

changes from, respectively, ∞ to $-\infty$, and from $-\infty$ to ∞. Then the index of the critical point P inside a counterclockwise oriented curve C is given by

$$I(P) = \frac{1}{2}(p - q). \tag{1}$$

Alternatively, we can use the Bendixson's index formula(see the note [H] chapter 5) given by,

$$I(P) = 1 + \frac{e - h}{2} \tag{2}$$

where e is the number of elliptical sectors and h is the number of hyperbolic sectors.

(i) We find $p = q = 1$. Then we get by equation (1), $I(P) = 0$. We can also use Bendixson's index formula, $e = 0$ and $h = 2$ to get $I = 0$. See figure 1 for an illustration.

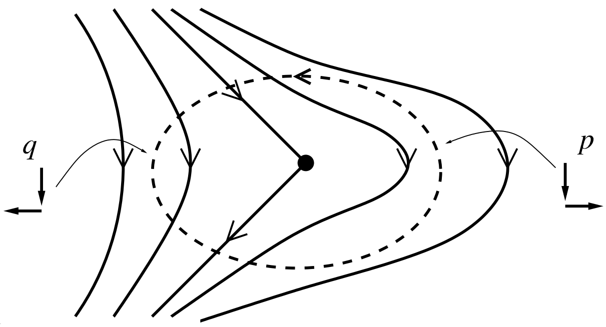

Figure 1: Phase diagram of a dynamical system where $I(P) = 0$.

(ii) We find $p = q = 1$ so that $I(P) = 0$. Using the notation in equation (2) we find $e = 0$ and $h = 2$ which gives $I(P) = 0$.

(iii) Here, $p = 3$ and $q = 1$ which gives $I(P) = 1$. Using Bendixsons's index formula gives the same result, with $h = e = 2$.

(iv) We find $p = 2$ and $q = 0$. Equation (1) gives $I(P) = 1$. Using Bendixsons's index formula gives the same result, with $h = e = 1$.

(v) We find $p = 0$ and $q = 4$. Equation (1) gives $I(P) = -2$. Using Bendixsons's index formula gives the same result, with $h = 6$ and $e = 0$.

$\boxed{3.3}$ Find the index of the equilibrium points of the following systems

(i) $\dot{x} = 2xy,$ $\dot{y} = 3x^2 - y^2.$ (ii) $\dot{x} = y^2 - x^4,$ $\dot{y} = x^3 y.$ (iii) $\dot{x} = x - y,$ $\dot{y} = x - y^2.$

(i) The only equilibrium point of this system is at the origin. We find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - y^2}{2xy} = \frac{3x}{2y} - \frac{y}{2x}. \tag{3}$$

We put a test square C around the origin passing through the points $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$.

Along the line from $(1, 1)$ to $(-1, 1)$, equation (3) changes sign from $-\infty$ to ∞ one time. Similarly, equation (3) changes sign from $-\infty$ to ∞ one time along each of the four lines. This means that $p = 0$ and $q = 4$ and so the index is $I(0, 0) = -2$.

See figure 2 for a sketch of the phase diagram.

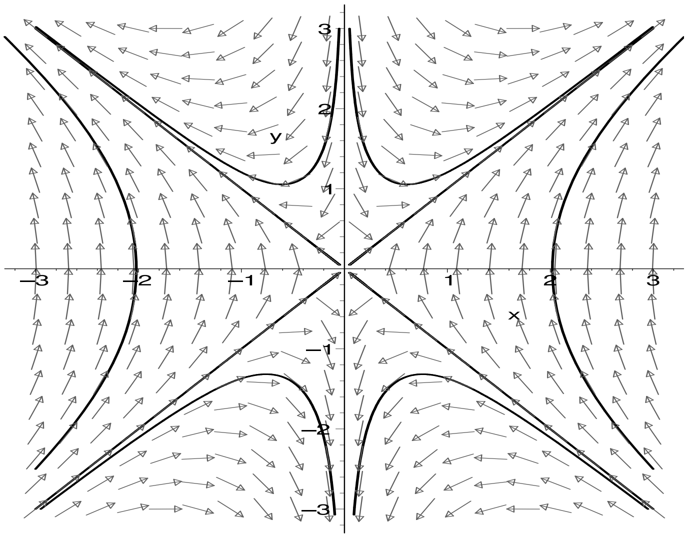

Figure 2: Phase diagram of $\dot{x} = 2xy$, $\dot{y} = 3x^2 - y^2$

(ii) Again, the origin is the only equilibrium point. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 y}{y^2 - x^4}. \tag{4}$$

We put a test square C about the origin passing through the points. $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$.

We find that equation (4) changes sign from $-\infty$ to ∞ one time on each of the four lines of the square. This means that $p = 0$ and $q = 4$, so the index is $I(0,0) = -2$.

See figure 3 for a sketch of the phase diagram.

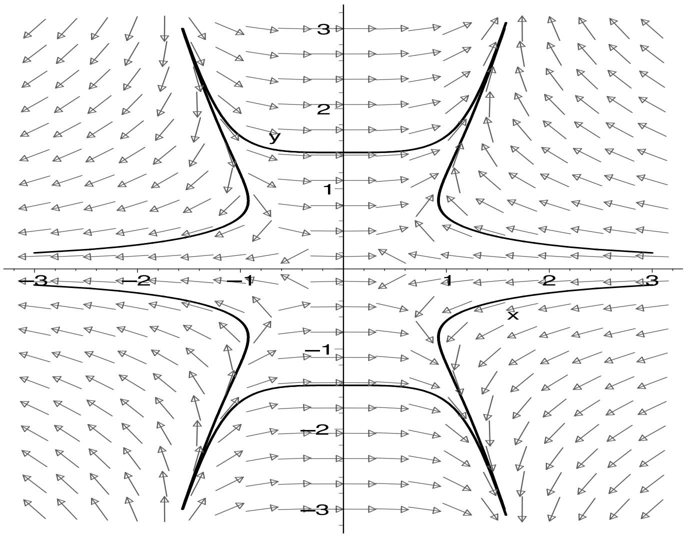

Figure 3: Phase diagram of $\dot{x} = y^2 - x^4$, $\dot{y} = x^3 y$

(iii) Here, there are two equilibrium points, namely $(0,0)$ and $(1,1)$. The matrix of linearization is given by

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -2y \end{bmatrix}$$

at the point (x, y).

The eigenvalues of A at the origin can be found by solving the system $-\lambda(1-\lambda) + 1 = 0$ which gives $\lambda = \frac{1}{2} \pm +\frac{\sqrt{3}}{2}i$. This is an unstable spiral and so the index at the origin is $I(0,0) = 1$.

The eigenvalues of A at $(1,1)$ can be found by solving $(1-\lambda)(-2-\lambda)+1 = 0$. This gives $\lambda = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. This is a saddle point, so $I(1,1) = -1$.

See figure 4 for a sketch of the phase diagram.

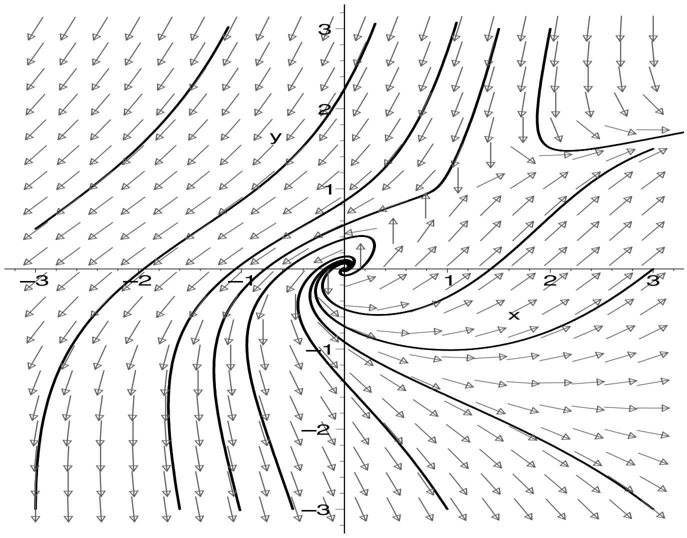

Figure 4: Phase diagram of $\dot{x} = x - y$, $\dot{y} = x - y^2$

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(a) Determine if the following system is stable or unstable at the origin.

$$\dot{x} = e^{-x-3y} - 1,$$
$$\dot{y} = x(1 - y^2).$$

The matrix of linearization is given by

$$J = \begin{bmatrix} -1 & -3 \\ 1 & 0 \end{bmatrix}$$

at the origin. We find the eigenvalues of J as a solution to the equation $\lambda^2 + \lambda + 3 = 0$. Hence

$$\lambda = \frac{-1 \pm \sqrt{-11}}{2} = -\frac{1}{2} \pm \frac{\sqrt{11}}{2} i.$$

This gives a stable spiral, both in the linear and the original system, so the origin is stable.

b) Given the system

$$\dot{x} = x - y,$$
$$\dot{y} = 1 - xy.$$

Find and characterize the equilibrium points. Sketch the phase diagram with orientation.

Setting $\dot{x} = \dot{y} = 0$ gives $x = y$ from the first equation. Inserting this into the second equation gives us the equilibrium points $(-1, -1)$ and $(1, 1)$. The matrix of linearization is given by

$$J = \begin{bmatrix} 1 & -1 \\ -y & -x \end{bmatrix}.$$

at the point (x, y). The eigenvalues of J are given as solutions to the equation $\lambda^2 + (x-1)\lambda - (x+y) = 0$. This gives

$$\lambda = \frac{1 - x \pm \sqrt{(1-x)^2 + 4(x+y)}}{2}.$$

At the point $(-1,-1)$, $\lambda = 1 \pm i$. Hence, both in the original and the linear system, we have an unstable spiral. The direction of the spiral is counterclockwise, which we find from studying the sign of $\dot{x}$ when $y < 0$.

At the point $(1,1)$, $\lambda_\pm = \pm\sqrt{2}$. Hence we have an unstable saddle in the linear system, and also in the nonlinear system. The eigenvectors are given by

$$\mathbf{x}_\pm = \begin{bmatrix} 1 \\ 1 - \lambda_\pm \end{bmatrix}$$

giving us the asymptotes of the phase paths. See figure 5 for a sketch of the phase diagram.

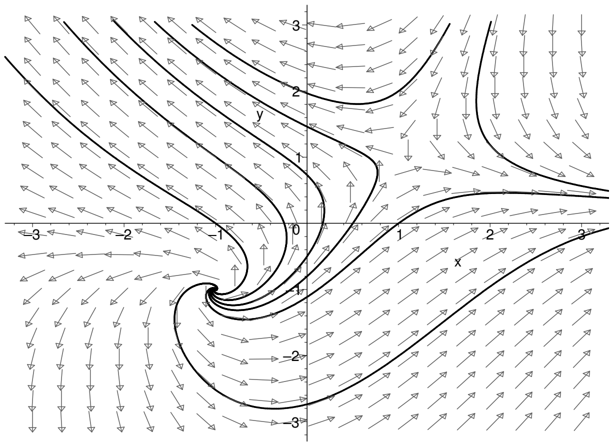

Figure 5: Phase diagram of $\dot{x} = x - y$,$\dot{y} = 1 - xy$

$\boxed{1.16}$ The system $\ddot{x} + x = -F_0 \operatorname{sgn}(\dot{x})$, $F_0 > 0$, has initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ with $x_0 > 0$. Show that the phase path will spiral n times before entering equilibrium if

$$(4n-1)F_0 < x_0 < (4n+1)F_0.$$

The phase paths are described by the equations $y^2 + (x + F_0)^2 = C$ for $y > 0$ and $y^2 + (x - F_0)^2 = C$ for $y < 0$. (See page 33 in the book.)

We start at $(x_0, 0)$. Since $\dot{y} = -x$ we move downwards from this point, so that y is negative. We follow the path $y^2 + (x - F_0)^2 = (x_0 - F_0)^2$ until we hit the x-axis in $(\tilde{x}_1, 0)$ where $(\tilde{x}_1 - F_0)^2 = (x_0 - F_0)^2$. From this we get $\tilde{x}_1 = 2F_0 - x_0$. Similarly, we now move through the upper half plane described by $y^2 + (x + F_0)^2 = (\tilde{x}_1 + F_0)^2 =$

$(3F_0 - x_0)^2$, until we hit $(x_1, 0)$ after one round. Here, $(x_1 + F_0)^2 = (3F_0 - x_0)^2$, so that $x_1 = x_0 - 4F_0$. By induction,

$$x_n = x_0 - 4nF_0.$$

On the x-axis, between $-F_0$ and F_0, there will be equilibrium - see figure 6. After n spirals we enter equilibrium provided

$$-F_0 < x_0 - 4nF_0 < F_0,$$

which can be rearranged to give the desired inequality.

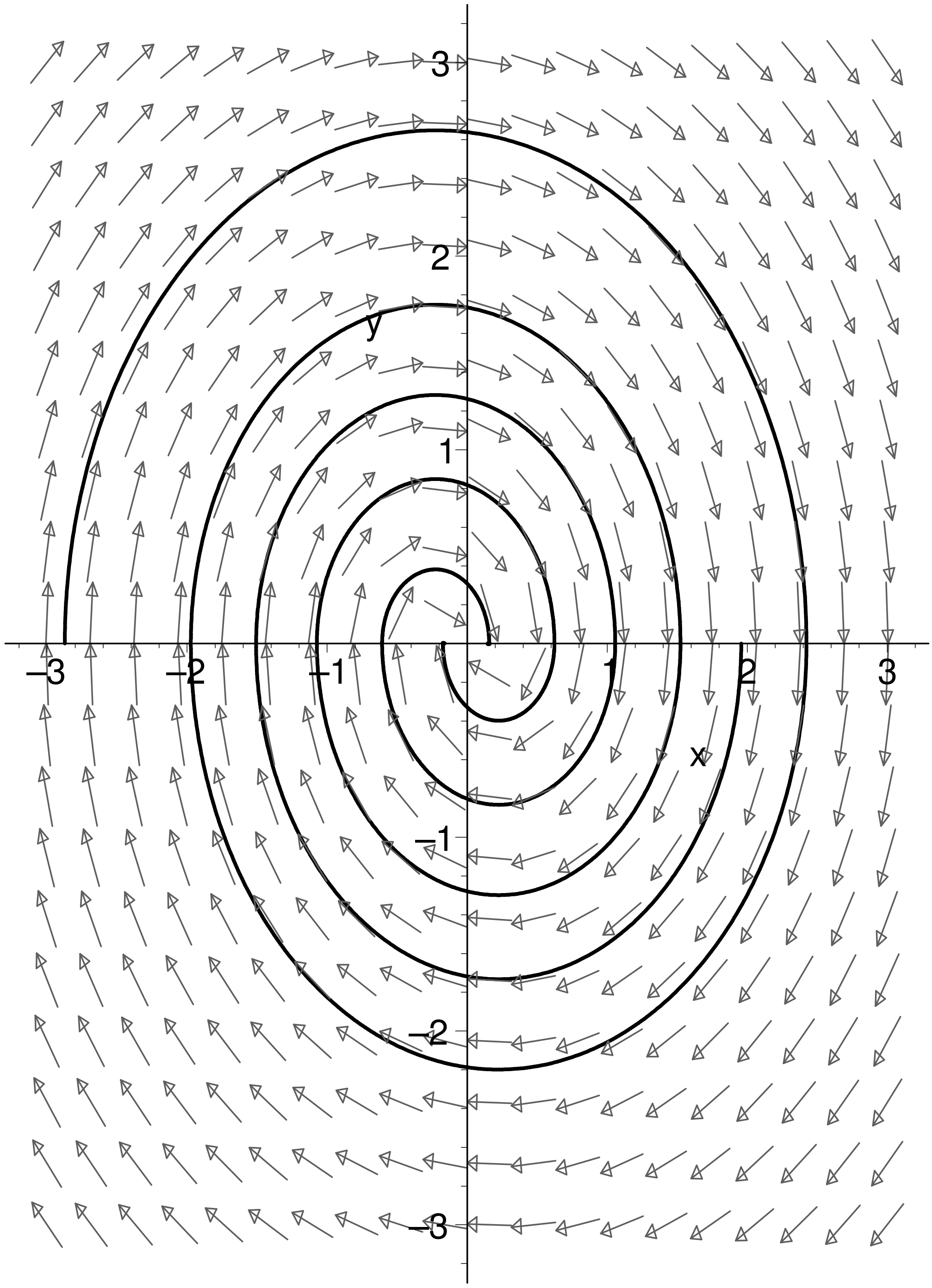

Figure 6: Phase diagram of $\ddot{x} + x = -F_0 \operatorname{sgn}(\dot{x})$ for $F_0 = \frac{9}{40}$