Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4125 Matematikk 4N
Spring 2017

Exercise set 6

1 **a)** The temperature in a long, thin bar is modeled by the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

with initial condition $u(x,0) = f(x)$. By taking Fourier transforms of both sides of the above equation with respect to x, find a differential equation satisfied by the Fourier transform $\hat{u}(w,t)$. Solve this equation to obtain an expression for $\hat{u}$ in terms of $\hat{f}(w)$, the Fourier transform of the initial condition. It is not required to use the result to find u. (You may find it helpful to consult Example 2 of Kreyszig, Chapter 12.7)

b) Now consider the PDE

$$\frac{\partial v}{\partial t} = -v + c^2 \frac{\partial^2 v}{\partial x^2},$$

with initial condition $v(x,0) = f(x)$. Take Fourier transforms of both sides to find an equation for $\hat{v}(w,t)$, which you should solve in terms of $\hat{f}(w)$. Assume that $u(x,t)$ solves the heat equation in the previous question for some initial condition $f(x)$. By comparing the expressions for $\hat{u}$ and $\hat{v}$, find a solution $v(x,t)$ to the equation for v with the same initial conditions, in terms of $u(x,t)$. What is the physical effect of the extra term in the second equation?

2 Take Fourier transforms with respect to x on both sides of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Suppose we wish to solve the wave equation with respect to the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = 0$. By solving the equation for $\hat{u}$, show that the Fourier transform of u is given by

$$\hat{u}(w,t) = \hat{f}(w) \cos cwt$$

Using the identity, $\cos cwt = \frac{1}{2}(e^{icwt} + e^{-icwt})$, write down a formula for the inverse Fourier transform of $\hat{u}$ as a integral of $\hat{f}(w)$ and exponentials only. Use this formula to obtain the familiar solution

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big)$$

3 Let $u(t)$ be the Heaviside step function. Compute the Laplace transform of the function $u(t-2) - u(t-3)$ directly from the definition (i.e. without using any formula tables)

$\boxed{4}$ Find the Laplace transforms of the following functions:

a) $f(t) = t^2 + 2t - 3$

b) $f(t) = \sinh t \sin 2t$

$\boxed{5}$ Find the inverse Laplace transforms of the following functions (you will require the s-shifting theorem from Chapter 6.1 of Kreyszig):

a) $F(s) = 2s^{-3} + 5(s-1)^{-1}$

b) $F(s) = \frac{3s-7}{s^2+2s+5}$ (Hint: find a number b such that $s^2 + 2s + 5 = (s+1)^2 + b$)

$\boxed{6}$ Consider the initial value problem (for time $t > 0$),

$$y'' + y = r(t), \quad y(0) = 1, y'(0) = 0$$

where $r(t)$ is the function defined by

$$r(t) = \begin{cases} 1 - t, & 0 < t < 1 \\ 0 & t \geq 1 \end{cases}$$

By writing $r(t)$ as a single expression involving a Heaviside function, and then taking Laplace transforms of both sides of the above equation, show that the Laplace transform $Y(s)$ of $y(t)$ solves the equation

$$Y(s) = \frac{s}{s^2+1} + \frac{s - 1 + e^{-s}}{s^2(s^2+1)}$$

Find an expression for $y(t)$ by taking inverse Laplace transforms of the above (you will require the t-shifting theorem from Chapter 6.3 of Kreyszig). You may find the following expression useful:

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{(s^2+1)}$$

$\boxed{7*}$ The formula for the Laplace transform of $f'(t)$ requires that $f(t)$ be continuous, with piecewise continuous derivative $f'(t)$. Suppose that $f(t)$ is continuous except for a single jump discontinuity at $t = a$ (i.e. $\lim_{s\to a;\, s<a} f(s) = f(a_-)$ and $\lim_{s\to a;\, s>a} f(s) = f(a_+)$ both exist). By first splitting up the domain of integration at a and then integrating by parts, show that

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) - e^{-as}\big(f(a_+) - f(a_-)\big)$$

(The * indicates this exercise is optional)