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Solutions to exercise set 2

$\boxed{1}$ **a)** Here we calculate

$$\begin{aligned}
b_n &= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} x\sin nx\,dx + \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi}\sin nx\,dx\right)\\
&= \frac{2}{\pi}\left(\left[\frac{-x\cos nx}{n}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{\cos nx\,dx}{n} - \frac{\pi}{2}\left[\frac{\cos nx}{n}\right]_{\frac{\pi}{2}}^{\pi}\right)\\
&= \frac{2}{\pi}\left(-\frac{\pi\cos\frac{n\pi}{2}}{2n} + \left[\frac{\sin nx}{n^2}\right]_0^{\frac{\pi}{2}} - \frac{\pi\cos n\pi}{2n} + \frac{\pi\cos\frac{n\pi}{2}}{2n}\right)\\
&= \frac{(-1)^{n+1}}{n} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}
\end{aligned}$$

We can rewrite the above by noting that $\sin\frac{n\pi}{2}$ evaluates to zero whenever n is even, and $(-1)^{k+1}$ if $n = 2k-1$, thus

$$f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin(2k-1)x$$

b) First we compute

$$a_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} x\,dx + \frac{1}{2\pi}(\pi.\frac{\pi}{2}) = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + \frac{\pi}{4} = \frac{3\pi}{8}$$

We also have

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} x\cos nx\,dx + \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx\right)\\
&= \frac{2}{\pi}\left(\left[\frac{x\sin nx}{n}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\frac{\sin nx\,dx}{n} + \frac{\pi}{2}\left[\frac{\sin nx}{n}\right]_{\frac{\pi}{2}}^{\pi}\right)\\
&= \frac{2}{\pi}\left(\frac{\pi\sin\frac{n\pi}{2}}{2n} + \left[\frac{\cos nx}{n^2}\right]_0^{\frac{\pi}{2}} - \frac{\pi\sin\frac{n\pi}{2}}{2n}\right)\\
&= \frac{2}{n^2\pi}\left(\cos(\frac{n\pi}{2}) - 1\right)
\end{aligned}$$

If we wish we can note that the above evaluates to $\frac{-2}{n^2\pi}$ whenever n is odd, to 0 if n is divisible by 4, and to $\frac{-4}{n^2\pi}$ if n is even but not divisible by 4, i.e. we have

$$g(x) = \frac{3\pi}{8} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2} + \frac{2\cos(4k-2)x}{(4k-2)^2}$$

$\boxed{2}$ **a)** Here we use the given Fourier series

$$r(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)t$$

To express the steady state solution of $y'' + 0.02y' + 25y = r(t)$ in the form $y = \sum_{n \text{ odd}} y_n$, where y_n solves

$$y'' + 0.02y' + 25y = \frac{4}{n^2\pi} \cos nt$$

We expect solutions of the form $y_n = A_n \cos nt + B_n \sin nt$; by differentiation we obtain

$$\begin{aligned} y_n &= A_n \cos nt + B_n \sin nt \\ y_n' &= -nA_n \sin nt + nB_n \cos nt \\ y_n'' &= -n^2 A_n \cos nt - n^2 B_n \sin nt \end{aligned}$$

By setting this into the differential equation and equating the coefficients of $\cos nt$ and $\sin nt$, we obtain the system of equations

$$\begin{pmatrix} 25 - n^2 & 0.02n \\ -0.02n & 25 - n^2 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \frac{4}{n^2\pi} \\ 0 \end{pmatrix}$$

which we invert

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \frac{1}{D_n} \begin{pmatrix} 25 - n^2 & -0.02n \\ 0.02n & 25 - n^2 \end{pmatrix} \begin{pmatrix} \frac{4}{n^2\pi} \\ 0 \end{pmatrix}$$

where $D_n = (25 - n^2)^2 + (0.02n)^2$. We therefore obtain

$$y = \frac{1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4(25 - n^2)}{n^2 D_n} \cos nt + \frac{0.08}{nD_n} \sin nt$$

b) By the theorem on best approximation by trigonometric polynomials we know that we require the first 5 terms of the Fourier series for y. As the previous expression is this Fourier series, we simply take the first 5 terms, ie

$$y = 0.053 \cos t + 0.000044 \sin t + 0.0088 \cos 3t + 0.000033 \sin 3t + 0.51 \sin 5t$$

Note that the magnitude of the $\sin 5t$ term is nearly 10 times that of the largest of the other terms!

$\boxed{3}$ **a)** By Parseval's theorem we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \big(f(x)\big)^2 dx = \sum_{n=1}^{\infty} \left(\frac{2(-1)^n}{n} \right)^2 = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

b) Alternatively, we compute the above integral in the usual manner:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \big(f(x)\big)^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

c) Comparing the above, we have

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{3}$$

from which we simplify to obtain the famous result $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

d) The best such approximation of f comprises the first 3 terms of the Fourier series $f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)$, i.e.

$$\hat{f}(x) = -2\sin x + \sin 2x - \frac{2}{3}\sin(3x)$$

We then calculate the error as follows

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}(f-\hat{f})^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2 - \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}^2 \\ &= \frac{\pi^2}{3} - 2\sum_{n=1}^{3}\frac{1}{n^2} \\ &= \frac{\pi^2}{3} - 2(1+\frac{1}{4}+\frac{1}{9}) = \frac{\pi^2}{3} - \frac{49}{18}\end{aligned}$$

e) The argument involving integration by parts presented there requires that f be continuous, and that its derivatives f' and f'' are also continuous. This is not true in this case as there are discontinuities at, for instance $f = \pi$ (this is because $f(\pi) \neq f(-\pi)$).

$\boxed{4}$ We calculate

$$\begin{aligned}c_n &= \frac{1}{2\pi}\left(-\int_{-\pi}^{0}e^{-inx}\,dx + \int_{0}^{\pi}e^{-inx}\,dx\right) \\ &= \frac{1}{2\pi}\left(\left[\frac{e^{-inx}}{in}\right]_{-\pi}^{0} - \left[\frac{e^{-inx}}{in}\right]_{0}^{\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{in} - \frac{e^{in\pi}}{in} - \frac{e^{-in\pi}}{in} + \frac{1}{in}\right) \\ &= -\frac{i}{n\pi} + \frac{i}{n\pi}\left(\frac{e^{in\pi}+e^{-in\pi}}{2}\right) \\ &= \frac{i}{n\pi}\left(-1+\cos n\pi\right) \\ &= \begin{cases} 0 & n \text{ even} \\ -\frac{2i}{n\pi} & n \text{ odd}\end{cases}\end{aligned}$$

Accordingly, we have

$$f(x) = \sum_{k=-\infty}^{\infty} -\frac{2i}{(2k-1)\pi}e^{(2k-1)ix}$$