Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems
Spring 2017

Solutions exercise 3

$\boxed{2.1}$

(i) Sketch the phase diagram and find the equilibrium points of

$$\dot{x} = x - 5y,$$
$$\dot{y} = x - y.$$

Rewriting into the form $\dot{\mathbf{x}} = A\mathbf{x}$ yields

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Equilibrium points are points where $A\mathbf{x} = 0$. Since A is non-singular, we must have $\mathbf{x} = 0$ so the only equilibrium point is the origin.

The eigenvalues satisfy the equation $(1-\lambda)(-1-\lambda)+5 = 0$, which implies $\lambda = \pm 2i$. These are purely imaginary, so the origin is a center. The direction of the paths can be found by setting $y = 0$ and $x > 0$, which tells us that the direction is counterclockwise. See figure 1 for a sketch of the phasediagram.

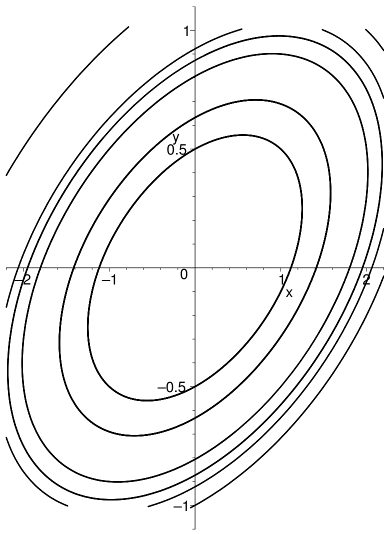

Figure 1: Phase diagram of $\dot{x} = x - 5y$, $\dot{y} = x - y$

(ii) Sketch the phase diagram and find the equilibrium points of

$$\dot{x} = x + y,$$
$$\dot{y} = x - 2y.$$

We set $\dot{x} = \dot{y} = 0$ to find the only equilibrium point in the origin.

We find the eigenvalues of the system by solving $(1-\lambda)(-2-\lambda)-1 = 0$, which gives

$$\lambda = \frac{-1 \pm \sqrt{5}}{2}.$$

The real eigenvalues of opposite sign represents a saddle point in the phase diagram.

The eigenvectors give us the axes of the saddle point. The eigenvector $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ associated with the eigenvalue λ can be found by noting that it satisfies the equation $\left[\begin{smallmatrix} 1-\lambda & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] = 0$. We get $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ \lambda-1 \end{smallmatrix}\right]$. The direction is outwards from the origin along the eigenvector with positive eigenvalue and inwards to the origin for the eigenvector corresponding to the negative eigenvalue. See figure 2 for a sketch of the phase diagram.

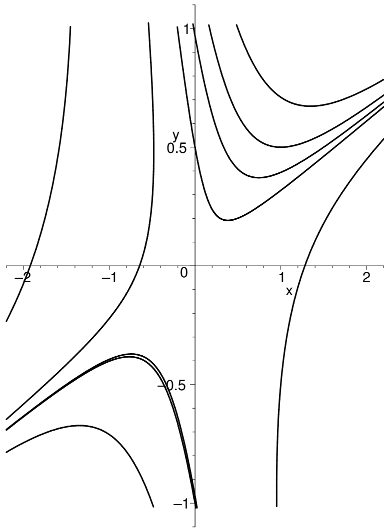

Figure 2: Phase diagram of $\dot{x} = x + y$, $\dot{y} = x - 2y$

(iii) Sketch the phase diagram and find the equilibrium points of

$$\dot{x} = -4x + 2y,$$
$$\dot{y} = 3x - 2y.$$

Again, the origin is the only equilibrium point.

We find the eigenvalues as a solution to the equation $(-4-\lambda)(-2-\lambda)-6=0$, yielding $\lambda = -3 \pm \sqrt{7}$. The real eigenvalues with negative sign gives a stable node. See figure 3 for a sketch of the phase diagram.

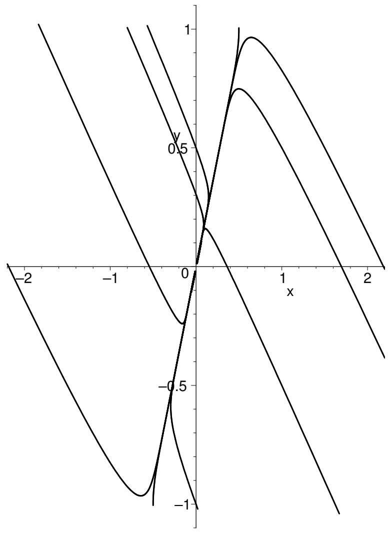

Figure 3: Phase diagram of $\dot{x} = -4x + 2y$, $\dot{y} = 3x - 2y$

(iv) Sketch the phase diagram and find the equilibrium points of

$$\dot{x} = x + 2y,$$
$$\dot{y} = 2x + 2y.$$

The origin is the only equilibrium point.

The eigenvalues are given as a solution to $(1-\lambda)(2-\lambda)-4=0$. Hence,

$$\lambda = \frac{3 \pm \sqrt{17}}{2}.$$

This gives us the eigenvectors

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1 \pm \sqrt{17}}{4} \end{bmatrix}.$$

Here, the eigenvalues are real with opposite sign, which yields a saddle point in the phase diagram. The eigenvectors describe the axes for the saddle. See figure 4 for a sketch of the phase diagram.

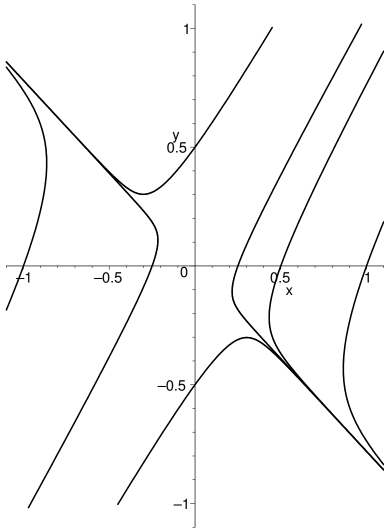

Figure 4: Phase diagram of $\dot{x} = x + 2y$, $\dot{y} = 2x + 2y$

(vi) Sketch the phase diagram and find the equilibrium points of

$$\dot{x} = 2x + y,$$
$$\dot{y} = -x + y.$$

The origin is the only equilibrium point.

The eigenvalues are given as solutions to the equation $(2 - \lambda)(1 - \lambda) + 1 = 0$. Hence,

$$\lambda = \frac{3 \pm \sqrt{3}i}{2}.$$

We get complex eigenvalues with positive real part, so we get an unstable spiral. Set $y = 0$ and $x > 0$ to find the direction of the paths. This gives $\dot{y} < 0$, so the direction is clockwise. See figure 5 for a sketch of the phase diagram.

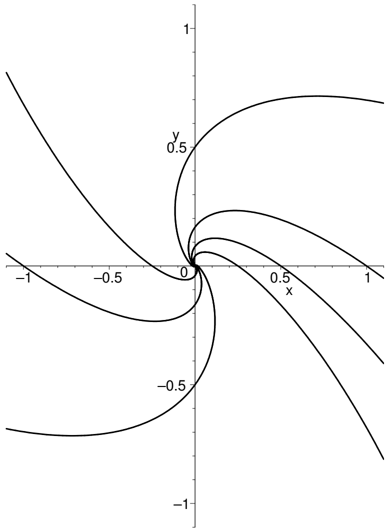

Figure 5: Phase digram of $\dot{x} = 2x + y$, $\dot{y} = -x + y$