Norwegian University of Science and Technology
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TMA4220 Finite Element Method Autumn 2017

Exercise set 1

1 We consider the homogeneous Dirichlet problem

$$\begin{aligned} -u''(x) &= 1, \quad x \in [0,1] \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

Let $\mathcal{T}_h$ be a triangulation of $[0,1]$, i.e. a set of points $0 = x_0 < x_1 < \ldots < x_N = 1$. Let X_h^1 be the space of continuous, piecewise linear polynomials with respect to $\mathcal{T}_h$, i.e.

$$X_h^1 = \{v \in C^0([0,1]) : v|_{[x_i,x_{i+1}]} \in \mathbb{P}_1 \;\; \forall x_i, x_{i+1} \in \mathcal{T}_h\}$$

The corresponding Galerkin problem is then: find $u_h \in X_h^1$ such that

$$\begin{aligned} a(u_h, v_h) &= F(v), \quad \forall v_h \in X_h^1 \\ a(u,v) &= \int_0^1 u'(x)v'(x)\mathrm{d}x \\ F(v) &= \int_0^1 v(x)\mathrm{d}x \end{aligned}$$

This problem is solved by the following `Matlab` code, where $\mathcal{T}_h$ is set to be 20 equidistant points on $[0,1]$:

```
n = 20;                % number of nodal points
x = linspace(0,1,n); % nodal points
A = zeros(n);         % system matrix
b = zeros(n,1);       % right-hand side
h = diff(x);          % element size
for el=1:n-1          % element loop
  k = el:el+1;
  A(k,k) = A(k,k) + [1,-1;-1,1]/h(el);
  b(k)   = b(k)   + h(el)/2;
end
A([1,n],:) = [];     % remove boundary conditions
A(:,[1,n]) = [];
b([1,n])   = [];
u = A \ b;           % solve system
```

a) How would you modify the code to instead solve the following mixed boundary value problem

$$
\begin{aligned}
-u''(x) &= 1, \quad x \in [0,1] \\
u(0) &= 0 \\
u'(1) &= 1
\end{aligned}
$$

b) What is the exact solution to this problem? Plot your finite element solution and the exact solution in the same plot.

c) Modify your code to solve the problem using quadratic elements, i.e. where $V_h = X_h^2$, the space of piecewise quadratic polynomials on $\mathcal{T}_h$. Recall that you will require to specify an internal node x_{i+1} on each element $[x_i, x_{i+2}]$. The stiffness matrix A should be constructed from 3×3 sub-blocks of the form

$$
\frac{1}{3h}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & 1 \\ 1 & -8 & 7 \end{pmatrix}
$$

where h is the width of the element. Explain how this form is derived.

$\boxed{2}$ We now consider the Helmholtz equation

$$
\begin{gathered}
-u_{xx} + \sigma u = f(x) \text{ on } (0,1)\,, \\
u(0) = u(1) = 0.
\end{gathered}
$$

where $\sigma > 0$ is a constant.

a) Set up the weak form for this problem. Show that, when this problem is solved by a Galerkin method, using $V_h = \text{span}\,\{\varphi_i\}_{i=1}^{N}$, the discrete problem can be written as

$$
(A + \sigma M)\mathbf{u} = \mathbf{f},
$$

where the matrix M is the 'mass matrix'

$$
M_{ij} = \int_0^1 \varphi_i \varphi_j dx
$$

b) Set up the matrix M for $V_h = X_h^1$ (i.e. linear piecewise polynomials) on a uniform grid. What about the quadratic elements? (Hint: construct M in a manner analogous to the construction of A, by finding 2×2 or 3×3 block submatrices)

c) Modify your code from question 1 to solve this problem, using linear and quadratic elements. To test your code, let $\sigma = 1$, $f = \sin(\pi x)$ in which case $u(x) = \sin(\pi x)/(1 + \pi^2)$. Plot the solutions you obtain against the exact solution.

This time, to compute the load vector $\mathbf{f}$ you will need code that computes integrals. Try writing a function that uses the following quadrature formula:

$$
\int_0^1 g(x)dx \approx \frac{1}{2}(g(c_1) + g(c_2)), \qquad c_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}.
$$

Alternatively, at this stage you may use the built-in `Matlab` function `integral`.

3 Let $V = H_0^1(0,1)$, and take $a : V \times V \to \mathbb{R}$ and $F : V \to \mathbb{R}$ defined in the following way:

$$F(v) = \int_0^1 (-1-4x)v(x)\mathrm{d}x, \quad a(u,v) = \int_0^1 (1+x)u'(x)v'(x)\mathrm{d}x$$

a) Show that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive and that the problem "find $u_h \in V$ such that $a(u,v) = F(v)$" has a unique solution by the Lax Milgram theorem.

b) Verify that this solution is $u(x) = x^2 - x$.

4 **a)** For which $\alpha \in \mathbb{R}$ does the function $f(x) := |x|^\alpha$ lie in $L^2([-1,1])$? What about $L^2([1,\infty))$? What about $L^2(B_1(0))$, where $B_1(0) = \{x \in \mathbb{R}^2 : |x| < 1\}$ is the unit ball in $\mathbb{R}^2$?

b) If $D \subset \mathbb{R}$ is a closed, bounded subset of $\mathbb{R}$ and $f \in C^0(D)$, show that $f \in L^2(D)$.

c) Let $\Omega \subset \mathbb{R}$ be some open interval. A *weak derivative* of a function $u : \Omega \to \mathbb{R}$ is a function $v : \Omega \to \mathbb{R}$ such that

$$\int_\Omega u(x)\phi'(x)\ dx = -\int_\Omega v(x)\phi(x)\ dx$$

for every $\phi \in C_c^\infty(\Omega)$, the set of infinitely differentiable functions with compact support in Ω. Show that the weak derivative (if it exists) is unique. Show that if u is continuously differentiable (i.e. $u \in C^1(\Omega)$), then $\frac{du}{dx}$ is its weak derivative.

d) Let

$$f_1(x) := \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leq x < 2, \end{cases} \qquad f_2(x) := \begin{cases} x & \text{if } 0 < x < 1 \\ 2 & \text{if } 1 \leq x < 2 \end{cases}$$

for $x \in \Omega := (0,2)$. Show that $f_1, f_2 \in L^2(\Omega)$. Show that $f_1 \in H^1(\Omega)$ by finding its weak derivative. Show that $f_2 \notin H^1(\Omega)$.