HOMEWORK 3
THE DISTRIBUTION FUNCTION OF A BOREL MEASURE ON $\mathbb{R}$

Before you begin in earnest with the distribution function of a Borel measure, I want that you prove the following general result, which is useful in a variety of contexts. If you get stuck, you may find its proof in Tao's book on measure theory, the PDF file is available here: `https://terrytao.files.wordpress.com/2011/01/measure-book1.pdf`.

Problem 1. (Vitali-type covering lemma) Let $B_1, B_2, \ldots, B_n$ be a finite collection of open balls in $\mathbb{R}^d$ which are *not* necessarily disjoint. Then there exists a sub-collection $B_{m_1}, B_{m_2}, \ldots, B_{m_k}$ of *disjoint* balls in this collection, such that

$$\bigcup_{i=1}^{n} B_i \subset \bigcup_{j=1}^{k} 3\, B_{m_j},$$

where for a ball B, we denote by $3\,B$ the ball with the same center as B but with radius 3 times the radius of B. In particular, by finite sub-additivity, we have:

$$\lambda\big(\bigcup_{i=1}^{n} B_i\big) \le 3^d \sum_{j=1}^{k} \lambda(B_{m_j}).$$

Hint: Use a "greedy" algorithm of selecting balls of maximal radius amongst the ones that are disjoint from the previously selected ones.

The setup for the following problems is the following. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ be the measure space on $\mathbb{R}$ where $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel sets of $\mathbb{R}$ and λ denotes the Lebesgue measure. Let μ be a *finite* Borel measure on $\mathbb{R}$ and denote by F_μ its distribution function.

You will need to use the following theorems.

Theorem. *(FTC part II). If $F \colon \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function, then F is differentiable λ-a.e., $F' \in L^1(\lambda)$ and for any $a \le b$,*

$$\int_{[a,b]} F'(x)\, d\lambda(x) = F(b) - F(a).$$

We did not prove this theorem in the measure theory course (we only proved the FTC part I). You may find its proof in Tao's book or in the textbook.

Theorem. *(Kolmogorov's extension theorem) Let $\mathcal{B}_0$ be a Boolean algebra[1] on a set Ω. Let μ_0 be a pre-measure on $\mathcal{B}_0$. Then μ_0 can be extended to a measure on $\sigma(\mathcal{B}_0)$. Moreover, if μ_0 is σ-finite, then this extension is unique.*

We only proved the existence part in class. Try proving the uniqueness on your own.

[1] A collection of subsets of Ω that contains $\varnothing$ and it is closed under complements and finite unions.

Problem 2. Prove that F_μ is right continuous and that $\lim_{x\to-\infty} F_\mu(x) = 0$.

Problem 3. Prove that F_μ is continuous at a if and only if $\mu(\{a\}) = 0$.

Then conclude that μ is a continuous measure if and only if its distribution function F_μ is continuous.

Hint: Use the monotone convergence theorem for sets.

Problem 4. (a) Prove that $\mu \ll \lambda$ if and only if F_μ is an absolutely continuous function.

(b) Assuming that $\mu \ll \lambda$, and since we know that an absolutely continuous function is differentiable almost everywhere, it is natural to ask what is the derivative of F_μ.

Prove that it must be, almost everywhere, the Radon-Nikodym derivative of μ w.r.t. λ:

$$\frac{dF_\mu}{dx} = \frac{d\mu}{d\lambda}.$$

In other words, if $d\mu = f\,d\lambda$, prove that $F_\mu'(x) = f(x)$ for λ a.e. x.

Hint for part (b): First off, realize that what you actually have to prove is that

$$\mu(E) = \int_E F_\mu'(x)\,d\lambda(x)$$

holds for every Boolean set E.

Verify that this holds: when E is an interval (that is when you need FCT part II); then when E is an elementary set (i.e. a finite union of intervals); and finally when E is the complement of an elementary set. In other words, verify that this holds for the Boolean algebra of all elementary sets and their complements. Conclude by using the uniqueness in the Kolmogorov's extension theorem.

Problem 5. Prove that if $F_\mu'(x) = 0$ for λ a.e. $x \in \mathbb{R}$, then $\mu \perp \lambda$.

Hint: Recall that in class we proved the reverse of this statement, namely that if ν is any measure such that $\nu \perp \lambda$, then $F_\nu' = 0$ a.e.

You will need to use this somewhere in your proof. Also, begin the proof by applying the Lebesgue-Radon-Nicodym theorem to μ.