Norwegian University of Science and Technology Institutt for matematiske fag	TMA4125 Matematikk 4N Spring 2017 **Exercise set 12**

1 We will find an approximate solution to the differential equation

$$y'' - y' + \sin y = 0, \quad y(0) = 0, \ y'(0) = 1$$

a) Rewrite this equation as a system of first order equations.

b) Find an approximation to $y(0.4)$ by using two steps of the improved Euler scheme with $h = 0.2$

2 Consider the third order differential equation

$$y'''(x) = xy(x), \quad y(1) = -1, \ y'(1) = 2, \ y''(1) = 1$$

a) Rewrite the equation as a system of first order equations.

b) Compute one iteration of the backward Euler method with stepsize $h = 1$. You will need to solve a 3×3 system of linear equations. For what value of x does the value of y_1 you obtain approximate $y(x)$?

3 Suppose $u(x, y)$ satisfies the Poisson equation

$$u_{xx} + u_{yy} = 9(x + y)$$

on the interior of the square $\mathcal{R}$ with corners at $(0, 0)$, $(0, 1)$, $(0, 1)$ and $(1, 1)$. Moreover, u is given on the boundary of $\mathcal{R}$ by

$$u(x, y) = 9x(1 - x)$$

a) Use the difference equation for Poisson equations (outlined in Kreyszig 21.4) with $h = \frac{1}{3}$ to construct a linear system for the approximations to $u(x, y)$ at four interior points of $\mathcal{R}$ (if you are stuck see Example 1 from Kreyszig 21.4, which shows a similar problem).

b) (Optional) Perform two Gauss-Seidel iterations on the resulting linear system.

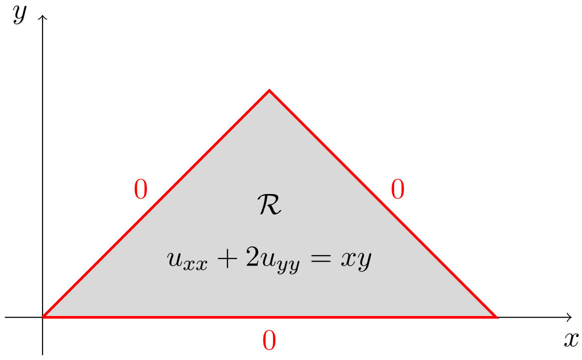
4 The steady state heat distribution in a non-uniform triangular plate is modeled by the following PDE

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} = xy, \quad (x, y) \in \mathcal{R}$$

where $\mathcal{R}$ is the triangle defined by the inequalities

$$y \geq 0, \quad y \leq x, \quad y \leq 6 - x.$$

We have in addition $u(x, y) = 0$ on the boundary of $\mathcal{R}$. The picture is as follows:



a) Derive the approximation

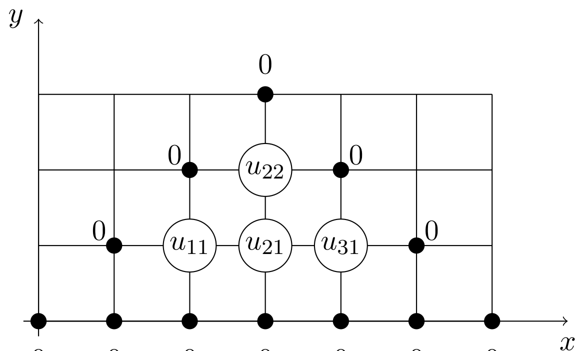
$$u_{xx}(x, y)+2u_{yy}(x, y) \approx \frac{u(x+h, y) + 2u(x, y+h) + u(x-h, y) + 2u(x, y-h) - 6u(x, y)}{h^2}$$

(the derivation is similar to that of the approximation to Laplace and Poisson equations given in Kreyszig, 21.4)

b) Show that applying the above finite difference approximation to the PDE with $h = 1$ gives rise to the difference equation

$$u_{i+1,j} + 2u_{i,j+1} + u_{i-1,j} + 2u_{i,j-1} - 6u_{i,j} = ij$$

Use this result to approximate u on the interior of $\mathcal{R}$, i.e. find a linear system of equations satisfied by the approximate values of u on the grid:



c) (Optional) Compute two steps of the Jacobi iteration on the resulting linear system.

$\boxed{5}$ (Optional) By summing the Taylor expansions of the various terms on the right hand side below, derive the approximation

$$2h^3 u_{xxx}(x, y) \approx u(x + 2h, y) - 2u(x + h, y) + 2u(x - h, y) - u(x - 2h, y)$$

Give a similar approximation to $u_{yyy}(x, y)$.