MA0301 ELEMENTARY DISCRETE MATHEMATICS
SPRING 2017

1. Homework Set 7 – Solutions

Exercise 1. *Grimaldi's book (5. ed., Exercises 3.1, page 134): solve* **Ex. 6**

Solution 1. a) T, b) T, c) T, d) F, e) T, f) F

Exercise 2. *Grimaldi's book (5. ed., Exercises 3.2, page 147): solve* **Ex. 13**

Solution 2. Statement a) is not true. Counterexample: $X := \{x_1, x_2, x_3\}$, $X_1 = \{x_1\}$, $X_2 = \{x_2\}$. Then it follows that $\mathcal{P}(X_1) = \{\emptyset, X_1\}$, $\mathcal{P}(X_2) = \{\emptyset, X_2\}$, and $\mathcal{P}(X_1 \cup X_2) = \{\emptyset, X_1, X_2, \{x_1, x_2\}\}$, but $\mathcal{P}(X_1) \cup \mathcal{P}(X_2) = \{\emptyset, X_1, X_2\}$.

Statement b) is true. Start with $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$ which is equivalent to $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, which is equivalent to $X \subset A$ and $X \subset B$, which is equivalent to $X \subset A \cap B$, which is equivalent to $X \in \mathcal{P}(A \cap B)$. Therefore $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

Exercise 3. *Grimaldi's book (5. ed., Exercises 5.1, page 252): solve* **Ex. 2**

Solution 3. a) clear, b) clear

Exercise 4. *Grimaldi's book (5. ed., Exercises 5.1, page 252): solve* **Ex. 3**

Solution 4. a) $|A| = |B| = 3$ and $|A \times B| = |A||B| = 9$.

b) Recall that relations from A to B are subsets of the cartesian product $A \times B$. There are $|\mathcal{P}(A \times B)|$ many subsets of $A \times B$. Hence, there are 2^9 possible relations from A to B.

c) $|A| = 3$ and $|A \times A| = |A||A| = 9$. Hence, there are $|\mathcal{P}(A \times A)| = 2^9$ possible relations on A.

d) The pairs $(1, 2)$ and $(1, 5)$ are two out of 9 pairs in $A \times B$. Relations from A to B are subsets of $A \times B$. Hence there are seven pairs left in $A \times B$ that can be added or not to the two pairs $(1, 2)$ and $(1, 5)$. Therefore there are 2^7 possible relations that contain $(1, 2)$ and $(1, 5)$.

e) Choose five out of nine: $\binom{9}{5}$.

f) Choose either seven, eight or nine out of nine: $\binom{9}{7} + \binom{9}{8} + \binom{9}{9}$.

Exercise 5. *Grimaldi's book (5. ed., Exercises 5.1, page 252): solve* **Ex. 7**

Solution 5. a) $|A| = 5$ and $|B| = 4$ so that $|A \times B| = |A||B| = 20$ and $|\mathcal{P}(A \times B)| = 2^{20}$.

b) $|A| = a$ and $|B| = b$ so that $|A \times B| = |A||B| = ab$ and $|\mathcal{P}(A \times B)| = 2^{ab}$.

Exercise 6. *Grimaldi's book (5. ed., Exercises 5.2, page 258): solve* **Ex. 1**

Solution 6. a) Defines a function with range $\{7, 8, 11, 16, 23 \dots\}$.

b) Is not a function since $(4, 2)$ and $(4, -2)$ are in the relation.

c) Defines a function with range $\mathbb{R}$.

d) Is merely a relation since $(0, 1)$ and $(0, -1)$ are included.

e) Is merely a relation since $|R| > 5$, i.e., it must contain two pairs with the same initial element.

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Exercise 7. *Grimaldi's book (5. ed., Exercises 5.2, page 258): solve* **Ex. 5**

Solution 7. a) $A \cap B = \{(x, y) \in \mathbb{R}^2 \mid y - 2x - 1 = 0 \wedge y - 3x = 0\}$. It follows that $x = 1$ and therefore $A \cap B = \{(1, 3)\}$.

b) $B \cap C = \{(x, y) \in \mathbb{R}^2 \mid y - 3x = 0 \wedge y - x + 7 = 0\}$. It follows that $x = -7/2$ and therefore $B \cap C = \{(-7/2, -21/2)\}$.

c) $A \cap C = \{(x, y) \in \mathbb{R}^2 \mid y - 2x - 1 = 0 \wedge y - x + 7 = 0\}$. It follows that $x = -8$ and therefore $A \cap C = \{(-8, -15)\}$.

d) $B' \cup C' = (B \cap C)' = \mathbb{R}^2 - \{(-7/2, -21/2)\}$.

Exercise 8. *Grimaldi's book (5. ed., Exercises 5.2, page 259): solve* **Ex. 15 c), d), f)**

Solution 8. c) $f(0) = f(1)$ implies that it is not injective. The range is $\{0, \pm 6, \pm 24, \pm 60, \ldots\}$.

d) Is injective with range $\mathbb{R}^+$.

f) Among others, $f(\pi/4) = f(3\pi/4)$, and this implies that it is not injective. The range is $[0, 1]$.

Exercise 9. *Grimaldi's book (5. ed., Exercises 5.3, page 265): solve* **Ex. 2 b), d), f)**

Solution 9. b) It is injective and not surjective. The range are the odd integers.

d) It is not injective ($f(1) = f(-1)$) and not surjective. The range is $\{0, 1, 4, 9, 16, \ldots\}$.

f) It is injective and not surjective. The range is $\{\ldots, -64, -27, -8, -1, 0, 1, 8, \ldots\}$.

Exercise 10. *Grimaldi's book (5. ed., Exercises 5.3, page 265): solve* **Ex. 3 b), d), f)**

Solution 10. b) It is injective and surjective.

d) It is not injective and not surjective. The range is $[0, \infty[$.

f) It is injective and surjective.

Exercise 11. *Grimaldi's book (5. ed., Exercises 5.6, page 288): solve* **Ex. 5**

Solution 11. We have by definition $g(A) := T \cap (S \cup A)$. Then $g(g(A)) = T \cap (S \cup g(A))$. Now, we see that $T \cap (S \cup g(A)) = T \cap (S \cup (T \cap (S \cup A))) = T \cap (S \cup T) \cap (S \cup A) = (T \cap (S \cup T)) \cap (S \cup A))) = T \cap (S \cup A) = g(A)$.

Exercise 12. *Grimaldi's book (5. ed., Exercises 5.6, page 288): solve* **Ex. 10 b), d)**

Solution 12. b) Invertible for $a \neq 0$, $f^{-1} = \{(x, y) \mid ay = c - bx,\ b \neq 0,\ a \neq 0\}$

d) f is not injective and therefore not bijective.

Exercise 13. *Grimaldi's book (5. ed., Exercises 5.6, page 288): solve* **Ex. 14 b), c), f)**

Solution 13. b) $f^{-1}(B) = \{-1, 0, 1\}$

c) $f^{-1}(B) = [-1, 1]$

f) $f^{-1}(B) =]-3, -2[\cup [-1, 0[\cup]0, 1] \cup]2, 3[$.

2. Classroom Set 7 – Solutions

Exercise 14. *Grimaldi's book (5. ed., Exercises 5.2, page 258): solve* **Ex. 12**

Solution 14. If $n = ka$, a positive integer, i.e., k divides n, then $\lceil n/k \rceil = a$. On the other hand, $(n-1)/k = a - (1/k)$ and $a - 1 \leq a - (1/k) \leq a$, which implies that $\lfloor (n-1)/k \rfloor + 1 = a - 1 + 1 = a$. If k does not divide n, then $n = ak + b$, with a, b positive integers and $b < k$. This implies that

$n/k = a + (b/k)$ with $0 < b/k < 1$. $n - 1 = ak + b - 1$ and $(n-1)/k = a + (b-1)/k$ and $0 \le (b-1)/k < 1$. From this it follows that

$$\lceil n/k \rceil = a + 1 = \lfloor (n-1)/k \rfloor + 1.$$

Exercise 15. *Grimaldi's book (5. ed., Exercises 5.2, page 258): solve* **Ex. 14**

Solution 15. a) $a_2 = 2, a_3 = 2, a_4 = 4, a_5 = 4, a_6 = 4, a_7 = 4, a_8 = 8$

b) Proof by induction (alternative form): $a_1 = 1 \le 1$, which provides the basis step. Assume that the statement holds for $n = 1$ up to $n = k$. For $n = k+1$ it follows from the ind.-hyp. that $a_{k+1} = 2a_{\lfloor (k+1)/2 \rfloor}$ is smaller or equal to $2\lfloor (k+1)/2 \rfloor$. For odd k it follows that $2\lfloor (k+1)/2 \rfloor = (k+1)$, which implies that $a_{k+1} = 2a_{\lfloor (k+1)/2 \rfloor} \le k+1$. For even k we find that $2\lfloor (k/2) + (1/2) \rfloor = k$, which implies that $a_{k+1} = 2a_{\lfloor (k+1)/2 \rfloor} = k \le k+1$. Therefore it follows that for all positive integers n that $a_n \le n$.

Exercise 16. *Grimaldi's book (5. ed., Exercises 5.2, page 259): solve* **Ex. 15 a), b), e)**

Solution 16. a) This function is injective. The range consists of the odd integers.

b) This function is injective and the range is $\mathbb{Q}$.

e) This function is injective and the range is $[-1, 1]$.

Exercise 17. *Grimaldi's book (5. ed., Exercises 5.2, page 259): solve* **Ex. 22**

Solution 17. a) $\binom{11}{7}$

b) $\binom{14}{6}$

c) $\binom{m+n-1}{m}$

d) We must have that $f(\{1,2,3\}) \subset \{1,2,3,4\}$, $f(4) = 4$ and $f(\{5,6,7,8,9,10\}) \subset \{4,5,6,7,8\}$. Therefore there are $\binom{6}{3}\binom{10}{6}$ many such functions.

e) $\binom{12}{4}\binom{5}{2}$

f) Let $f : X_m \to X_n$, $1 \le k \le m$ and $1 \le l \le n$, and $f(k) = l$. Then $f(\{1, \dots, k-1\}) \subset \{1, \dots, l\}$ and $f(\{k+1, \dots, m\}) \subset \{l, l+1, \dots, n\}$, and there are $\binom{l+k-2}{k-1}\binom{n+m-l-k}{m-k}$ such functions.

Exercise 18. *Grimaldi's book (5. ed., Exercises 5.3, page 265): solve* **Ex. 2 a), c), e)**

Solution 18. a) It is injective and surjective.

c) It is injective and surjective.

e) $f(0) = f(-1)$ implies that it is not injective. It is not surjective and the range is $\{0, 2, 6, 12, 20, \dots\}$

Exercise 19. *Grimaldi's book (5. ed., Exercises 5.3, page 265): solve* **Ex. 3 a), c), e)**

Solution 19. a) It is injective and surjective.

c) It is injective and surjective.

e) It is not injective and not surjective. The range is $[-1/4, \infty[$.

Exercise 20. *Grimaldi's book (5. ed., Exercises 5.6, page 288): solve* **Ex. 10 a), c)**

Solution 20. a) f is invertible with $f^{-1} = \{(x,y) \mid 2y = 7 - 3x\}$.

c) f is invertible with $f^{-1} = \{(x,y) \mid y^3 = x\}$.

Exercise 21. *Grimaldi's book (5. ed., Exercises 5.6, page 288): solve* **Ex. 14 a), d), e)**

Solution 21. a) $f^{-1}(B) = \{-1, 0, 1\}$
d) $f^{-1}(B) =]-1, 1[$
e) $f^{-1}(B) = [-2, 2]$

Exercise 22. *Grimaldi's book (5. ed., Exercises 5.6, page 288): solve* **Ex. 15**

Solution 22. First observe that $3, 4, 5 \in f^{-1}(\{9, 10, 11, 12\})$, i.e., there are four choices for $f(3)$, $f(4)$, $f(5)$, and for each of the images of $f(1)$ and $f(2)$ we have three choices. Hence, rule of product implies that there are $3^2 4^3$ such functions.