Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2017

Exercise set 4: Solutions.

1 Show that the sets $U, V \subset \mathcal{P}_4$, the space of polynomials of degree at most 4, defined by

$$U := \{p \in \mathcal{P}_4 : \, p(-1) = p(1) = 0\},$$
$$V := \{p \in \mathcal{P}_4 : \, p(1) = p(2) = p(3) = 0\}$$

are subspaces of $\mathcal{P}_4$ and determine the subspace $U \cap V$.

Solution We show that U is a subspace of $\mathbf{P}_4$.

Let $p_1, \dots, p_n \in U$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Then $p_k(-1) = p_k(1) = 0$ for all indices $k = 1, \dots, n$.

Consider the linear combination $p = \lambda_1 p_1 + \dots + \lambda_n p_n$. Then clearly

$$p(-1) = \lambda_1 p_1(1) + \dots + \lambda_n p_n(1) = \lambda_1 \cdot 0 + \dots \lambda_n \cdot 0 = 0,$$

which shows that $p(-1) = 0$. Similarly, $p(1) = 0$.

Therefore, $p \in U$, so U is a subspace of $\mathbf{P}_4$.

The same kind of argument shows that V is a subspace.

Turning to $U \cap V$, we clearly have

$$U \cap V = \{p \in \mathbf{P}_4 : \, p(-1) = p(1) = p(2) = p(3) = 0\}.$$

This is the set of all real polynomials of degree at most 4 with exactly 4 roots: $-1, 1, 2, 3$.

Let $p_0 := (x+1)(x-1)(x-2)(x-3)$.

Then $U \cap V = \{\lambda \, p_0 \colon \lambda \in \mathbb{R}\}$.

$\boxed{2}$ Prove that $(l^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is a normed space, where for any bounded sequence $x = (x_n) \in l^\infty(\mathbb{R})$ we define

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|.$$

Is this norm associated with an inner product?

Solution. We verify the three axioms of the norm. Let $x = (x_n) \in l^\infty(\mathbb{R})$.

(i) Since $|a| \geq 0$ for all real numbers a,

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| \geq 0.$$

Moreover, if $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| = 0$, then $|x_n| = 0$ for all $n \in \mathbb{N}$, hence $x_n = 0$ for all $n \in \mathbb{N}$. This shows that $x = (x_n) = (0)$, so x is the null vector in $l^\infty(\mathbb{R})$.

(ii) Let $\lambda \in \mathbb{R}$. Then

$$\|\lambda\, x\|_\infty = \sup_{n\in\mathbb{N}} |\lambda\, x_n| = \sup_{n\in\mathbb{N}} |\lambda|\, |x_n| = |\lambda| \sup_{n\in\mathbb{N}} |x_n| = |\lambda|\, \|x\|_\infty.$$

Note that we used the following property of the supremum: if $A \subset \mathbb{R}$ and $c \geq 0$, then

$$\sup(cA) = c \sup(A).$$

(iii) Let $x = (x_n)$ and $y = (y_n) \in l^\infty(\mathbb{R})$. Since for any two real numbers a and b, $|a + b| \leq |a| + |b|$, we have

$$\|x + y\|_\infty = \sup_{n\in\mathbb{N}} |x_n + y_n| \leq \sup_{n\in\mathbb{N}} (|x_n| + |y_n|) \leq \sup_{n\in\mathbb{N}} |x_n| + \sup_{n\in\mathbb{N}} |y_n| = \|x\|_\infty + \|y\|_\infty.$$

Note that we used the following property of the supremum: if $f, g\colon \mathbb{N} \to \mathbb{R}$, then

$$\sup_{n\in\mathbb{N}} (f(n) + g(n)) \leq \sup_{n\in\mathbb{N}} f(n) + \sup_{n\in\mathbb{N}} g(n).$$

Finally, this norm is *not* associated with an inner product because it does not satisfy the parallelogram identity. Indeed, let us consider the sequences

$$\begin{aligned} x &= (x_n) \ \text{ where } x_n = 1 + \frac{1}{n} \ \text{ for all } n \geq 1, \\ y &= (y_n) \ \text{ where } y_n = 1 - \frac{1}{n} \ \text{ for all } n \geq 1, \ \text{so} \end{aligned}$$

$$\begin{aligned} x_n + y_n &= 2 \qquad \text{for all } n \geq 1, \\ x_n - y_n &= \frac{2}{n} \qquad \text{for all } n \geq 1. \end{aligned}$$

Then clearly $\|x\|_\infty = 2$, $\|y\|_\infty = 1$, $\|x+y\|_\infty = 2$ and $\|x-y\|_\infty = 2$, so

$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 8 \neq 10 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2.$$

$\boxed{3}$ Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries. For $A \in M_n(\mathbb{C})$ we define its *trace* by $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$.

a) Show that for $A, B \in M_3(\mathbb{C})$ we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and try to show this property of the trace for $n \times n$ matrices.

b) Let $\mathcal{D}$ be the set of all diagonal $n \times n$ matrices. Show that $\mathcal{D}$ is a subspace of $M_n(\mathbb{C})$ and that for any $A, B \in D$ we have $AB = BA$ (in contrast to arbitrary matrices in $M_n(\mathbb{C})$).

c) Let $S \subset M_n(\mathbb{C})$ be defined as the matrices with $\mathrm{tr}(A) = 0$. Show that S is a subspace of $M_n(\mathbb{C})$.

Solution. a) We will do the general case – the 3×3-case can also be proved by writing A and B as matrices, multiplying them and calculating the traces of AB and BA. Let $A, B \in M_n(\mathbb{C})$ be $n \times n$-matrices with entries a_{ij} and b_ij, respectively. If we let $C = AB$, then we know (or can show) that the entries of C are given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{1}$$

Similarly, if $D = BA$, then the entries of D are given by

$$d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj} = \sum_{k=1}^{n} a_{kj} b_{ik}. \tag{2}$$

The trace is the sum of the diagonal elements. Hence

$$\mathrm{tr}(AB) = \mathrm{tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \tag{3}$$

and

$$\mathrm{tr}(BA) = \mathrm{tr}(D) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ik}. \tag{4}$$

Clearly the sums in equations (3) and (4) are equal – they are the same sum except that the *names* i and k for the variables have been switched.
b) To show that the diagonal matrices form a subspace, we need to show that if A, B are diagonal, then $\lambda A + B$ is diagonal for any $\lambda \in \mathbb{C}$. This is obviously true, since both scalar multiplication λA and addition $A + B$ is performed in each entry of the matrices. The fact that $AB = BA$ similarly follows from the fact that multiplication of diagonal matrices also happens pointwise. To give a formal proof we can use the general expressions for AB and BA in part a). If $C = AB$ we found that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{5}$$

The fact that A is diagonal means that $a_{ij} = 0$ for $i \neq j$. Hence the only value of k such that $a_{ik} \neq 0$ is $k = i$, so in fact

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{ii} b_{ij}. \tag{6}$$

Since B is diagonal, this expression is 0 when $i \neq j$. In conclusion

$$c_{ij} = \begin{cases} a_{ii} b_{ii} & i = j \\ 0 & i \neq j. \end{cases}$$

This just states that the product of two diagonal matrices A and B is the diagonal matrix obtained by multiplying the diagonal elements of A and B, as you hopefully knew. One can then argue that this must mean that $AB = BA$, since both of these matrices are obtained by multiplying the diagonal elements of A and B. If this is not clear, please try to do it for two diagonal 2×2-matrices.
c) We need to show that if $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$ and $\lambda \in \mathbb{C}$, then $\mathrm{tr}(\lambda A + B) = 0$. In fact, we have that the function $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ is a linear transformation, meaning that

$$\mathrm{tr}(\lambda A + B) = \lambda \mathrm{tr}(A) + \mathrm{tr}(B),$$

so if $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$, we must have $\mathrm{tr}(\lambda A + B) = 0$. The fact that tr is linear is rather obvious, but we can show it formally. The trace is the sum of the diagonal elements, so

$$\mathrm{tr}(\lambda A + B) = \sum_{i=1}^{n} \lambda a_{ii} + b_{ii} = \lambda \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \lambda \mathrm{tr}(A) + \mathrm{tr}(B).$$

$\boxed{4}$ Suppose $(X, \langle ., . \rangle)$ is an innerproduct space.

a) Let ω be a n^{th} root of unity, i.e. $\omega^n = 1$. Show that

$$\langle x, y \rangle = \frac{1}{n} \sum_{k=1}^{n} \omega^k \|x + \omega^k y\|^2.$$

b) Show that

$$\langle x, y \rangle = \int_0^1 e^{2\pi i \varphi} \|x + e^{2\pi i \varphi} y\|^2 d\varphi.$$

Solution. a) We will write the right hand side using inner products. We have

$$\begin{aligned}\sum_{k=1}^{n-1}\omega^k\|x+\omega^k y\|^2 &= \sum_{k=1}^{n}\omega^k\langle x+\omega^k y, x+\omega^k y\rangle \\ &= \sum_{k=1}^{n}\omega^k\left(\langle x,x\rangle+\langle\omega^k y,\omega^k y\rangle+\omega^k\langle y,x\rangle+\omega^{-k}\langle x,y\rangle\right) \\ &= \|x\|^2\sum_{k=1}^{n}\omega^k+\|y\|^2\sum_{k=1}^{n}\omega^k+\langle y,x\rangle\sum_{k=1}^{n}\omega^{2k}+\sum_{k=1}^{n}\langle x,y\rangle.\end{aligned}$$

Clearly we need to calculate $\sum_{k=1}^{n}\omega^k$, where ω is an $n'th$ root of unity. This is a geometric sum, and we know that

$$\sum_{k=1}^{n}\omega^k=\frac{1-\omega^{n+1}}{1-\omega}-1=\frac{1-\omega}{1-\omega}-1=0.$$

The -1 appears to compensate for the fact that the usual formula for a geometric sum starts summation at $k=0$. Note that we have used $\omega^{n+1}=\omega$ since ω is an n'th root of unity. The same argument will show that $\sum_{k=1}^{n}\omega^{2k}=0$. If we plug this into our previous calculation, we have

$$\sum_{k=1}^{n-1}\omega^k\|x+\omega^k y\|^2=\sum_{k=1}^{n}\langle x,y\rangle=n\langle x,y\rangle.$$

Divide both sides by n to obtain the desired result.
b) As above we write the norm using inner products, and by using exactly the same kind of simplifications as above we obtain

$$\begin{aligned}\int_0^1 e^{2\pi i\varphi}\|x+e^{2\pi i\varphi}y\|^2 d\varphi &= \int_0^1 e^{2\pi i\varphi}\left(\|x\|^2+e^{2\pi i\varphi}\langle y,x\rangle+e^{-2\pi i\varphi}\langle x,y\rangle+\|y\|^2\right)d\varphi \\ &= \|x\|^2\int_0^1 e^{2\pi i\varphi}d\varphi+\|y\|^2\int_0^1 e^{2\pi i\varphi}d\varphi+\langle y,x\rangle\int_0^1 e^{4\pi i\varphi}d\varphi+\langle x,y\rangle\int_0^1 d\varphi \\ &= \langle x,y\rangle.\end{aligned}$$

The last inequality follows from calculating these integrals, which is straightforward.

$\boxed{5}$ Let $(\mathbb{R}^n, \|.\|_p)$ be the space of real n-tuples with the p-norms $\|x\|_p=(\sum_{i=1}^{n}|x_i|^p)^{1/p}$ for $1\le p<\infty$. Show that

$$\sum_{i=1}^{n}|x_i|\le n^{(p-1)/p}(\sum_{i=1}^{n}|x_i|^p)^{1/p}.$$

Solution. This is an example of Hölder's inequality. Note that $\frac{1}{p}+\frac{p-1}{p}=1$ – in the terminology of the lecture notes we have that $p/(p-1)$ is the conjugate exponent of p. Let x be the n-tuple $(x_1,x_2,...x_n)$ and let $y=(1,1,...,1)$. Hölder's inequality states that

$$
\begin{aligned}
\sum_{i=1}^{n} |x_i||y_i| &\leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} 1^q\right)^{1/q} \\
&= \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} n^{1/q},
\end{aligned}
$$

where q is the conjugate exponent of p. If we now insert that the conjugate exponent of p is $p/(p-1)$, we obtain the desired inequality.