TMA4125 Matematikk 4N
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Solutions to exercise set 6

$\boxed{1}$ **a)** We have

$$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u e^{-iwx} dx = \hat{u}_t$$

Moreover, using the rule for Fourier transforms of (second) derivatives gives $\mathcal{F}(u_{xx}) = -w^2 \hat{u}$. We thus obtain the equation

$$\frac{\partial \hat{u}}{\partial t} = -c^2 w^2 \hat{u}$$

This is readily solved to obtain

$$\hat{u} = \hat{u}(x,0) e^{-c^2 w^2 t} = \hat{f}(w) e^{-c^2 w^2 t}$$

b) Proceeding as above, we obtain the equation

$$\frac{\partial \hat{v}}{\partial t} = (1 - c^2 w^2) \hat{v}$$

This is solved by

$$\hat{v} = \hat{v}(x,0) e^{(1-c^2 w^2)t} = \hat{f}(w) e^{-t} e^{-c^2 w^2 t}$$

We then note the relation $\hat{v}(w,t) = e^{-t} \hat{u}(w,t)$. As e^{-t} is independent of w, the inverse Fourier transform immediately gives $v(x,t) = e^{-t} u(x,t)$. The extra term in the equation has therefore caused the solution to be modified by a factor of e^{-t}, which is a rapidly decaying term. Physically, this would correspond to some dissipation of heat.

$\boxed{2}$ By similar arguments to the previous question, taking Fourier transforms on both sides gives the equation

$$\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 w^2 \hat{u}$$

This has solution

$$\hat{u} = A(w) \cos cwt + B(w) \sin cwt$$

The initial condition $u(x,0) = f(x)$ implies that $A(w) = \hat{f}(w)$, whilst differentiating the expression for $\hat{u}$ and setting $u_t(x,0) = 0$ shows that $B(w) = 0$, i.e. we have $\hat{u} = \hat{f}(w) \cos cwt$. The integral for the inverse Fourier transform is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) \cos cwt e^{iwx} dw$$

Rewriting the cos in terms of exponentials as suggested gives

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2}(\hat{f}(w)e^{icwt}e^{iwx} + \hat{f}(w)e^{-icwt}e^{iwx})dw \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2}(\hat{f}(w)e^{iw(x+ct)} + \hat{f}(w)e^{iw(x-ct)})dw \\
&= \frac{1}{2}\big(f(x+ct) - f(x-ct)\big),
\end{aligned}$$

where in the final step we recognized the expressions for the inverse Fourier transform of $\hat{f}$.

3 We compute

$$\begin{aligned}
\mathcal{L}\big(u(t-2) - u(t-3)\big) &= \int_0^{\infty} \big(u(t-2) - u(t-3)\big)e^{-st}dt \\
&= \int_2^3 e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_2^3 = \frac{e^{-2s} - e^{-3s}}{s}
\end{aligned}$$

4 **a)** We have

$$\mathcal{L}(t^2 + 2t - 3) = \mathcal{L}(t^2) + 2\mathcal{L}(t) - 3\mathcal{L}(1) = \frac{2}{s^3} + \frac{2}{s^2} - \frac{3}{s}$$

b) We write in terms of exponentials:

$$\begin{aligned}
\sinh(t)\sin(2t) &= \left(\frac{e^t - e^{-t}}{2}\right)\left(\frac{e^{2it} - e^{-2it}}{2i}\right) \\
&= \frac{1}{4i}(e^{(2i+1)t} - e^{(1-2i)t} - e^{(2i-1)t} + e^{-(2i+1)t})
\end{aligned}$$

Taking Laplace transforms of each of the exponentials gives

$$\begin{aligned}
\mathcal{L}\big(f(t)\big) &= \frac{1}{4i}\left(\frac{1}{s-1-2i} - \frac{1}{s-1+2i} - \frac{1}{s+1-2i} + \frac{1}{s+1+2i}\right) \\
&= \frac{1}{4i}\left(\frac{s-1+2i-(s-1-2i)}{(s-1-2i)(s-1+2i)} + \frac{s+1-2i-(s+1+2i)}{(s+1+2i)(s+1-2i)}\right) \\
&= \frac{1}{(s-1)^2+4} - \frac{1}{(s+1)^2+4}
\end{aligned}$$

Note that the grouping into quadratic terms in s was done such that the each pair of factors in the denominator was a conjugate pair, leading to the resulting simplification. There are other equally valid ways to reach the same answer, e.g. by writing

$$\sinh(t)\sin(2t) = \frac{1}{2}(e^t\sin(2t) - e^{-t}\sin(2t))$$

and using the s-shifting formula together with the Laplace transform for $\sin(2t)$.

5 **a)** From the tables, we have that $\mathcal{L}(t^2) = \frac{2}{s^3}$, and $\mathcal{L}(e^t) = \frac{1}{s-1}$. It follows immediately that $f(t) = t^2 + 5e^t$

b) Following the hint, we note that

$$\frac{3s-7}{s^2+2s+5} = \frac{3s-7}{(s+1)^2+4}$$

We recall that the Laplace transforms of $\cos at$ and $\sin at$ are $\frac{s}{s^2+a^2}$ and $\frac{a}{s^2+a^2}$, respectively. By the s-shifting formula, the Laplace transforms of $e^{-t}\cos 2t$ and $e^{-t}\sin 2t$ are therefore

$$\frac{s+1}{(s+1)^2+4}, \quad \frac{2}{(s+1)^2+4}$$

We bring our function into this form by writing

$$F(s) = \frac{3s-7}{(s+1)^2+4} = \frac{3(s+1)}{(s+1)^2+4} - \frac{5*2}{(s+1)^2+4},$$

from which it follows that

$$f(t) = e^{-t}(3\cos 2t - 5\sin 2t)$$

6 We begin by writing, for $t > 0$, $r(t) = \big(1-t\big)\big(1-u(t-1)\big)$. To compute the Laplace transform of r we will use the t-shifting theorem, so we write

$$r(t) = 1 - t + (t-1)u(t-1),$$

from which it follows that

$$\mathcal{L}(r) = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2} = \frac{s-1+e^{-s}}{s^2}$$

Taking Laplace transforms on both sides of the equation and using the formula $\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy'(0) - y(0)$ gives

$$(s^2+1)Y(s) - s = \frac{s-1+e^{-s}}{s^2}$$

which we rearrange to the form given in the exercise:

$$Y(s) = \frac{s}{s^2+1} + \frac{s-1+e^{-s}}{s^2(s^2+1)}.$$

We recognize immediately that the first term in the above expression for Y is the Laplace transform of $\cos t$. For the second, we use the suggested partial fraction expansion

$$\frac{s-1+e^{-s}}{s^2(s^2+1)} = (s-1+e^{-s})\left(\frac{1}{s^2} - \frac{1}{(s^2+1)}\right)$$

Expanding out the brackets, we compute the inverse Laplace transform term-by-term using t-shifting where appropriate, to give

$$1 - t + (t-1)u(t-1) - \cos t + \sin t - u(t-1)\sin(t-1)$$

Summing the inverse Laplace transforms for the two terms of $Y(s)$ then gives

$$y(t) = 1 - t + \sin t + u(t-1)\big(t - 1 - \sin(t-1)\big)$$

$\boxed{7}$ We begin by writing

$$\int_0^\infty f'(t)e^{-st}ds = \int_0^{a_-} f'(t)e^{-st}ds + \int_{a_+}^\infty f'(t)e^{-st}ds$$

Integrating each term on the right by parts gives

$$\begin{aligned} F(s) &= \left[f(t)e^{-st}\right]_0^{a_-} + \int_0^{a_-} sf(t)e^{-st}ds + \left[f(t)e^{-st}\right]_{a_+}^\infty + \int_{a_+}^\infty sf(t)e^{-st}ds \\ &= f(a_-)e^{-st} - f(0) - f(a_+)e^{-st} + s\mathcal{L}(f), \end{aligned}$$

where we recombined the two integrals above to get the $s\mathcal{L}(f)$ term.