Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2017

Solutions exercise 5

2.3

(ii) Sketch the phase diagram and characterize the equilibrium points of

$$\dot{x} = ye^y,$$
$$\dot{y} = 1 - x^2.$$

Equilibrium points are found by setting $\dot{x} = 0$ og $\dot{y} = 0$, which gives $x = \pm 1$ and $y = 0$. Hence $(\pm 1, 0)$ are equilibrium points.

We linearize the system about the points $(\pm 1, 0)$.

Near $(1, 0)$ we have $\dot{x} = y$ and $\dot{y} = -2(x-1)$. Hence

$$\frac{dy}{dx} = -2\frac{x-1}{y}.$$

Solving the differential equation gives an ellipse around the point $(1, 0)$

$$2(x-1)^2 + y^2 = C,$$

and hence, a centre. Note that a centre for the linearized system is not necessarily a centre in the original system. In the original system, we might have a spiral (stable or unstable).

The system is Hamiltonian (since $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0$ where $\dot{x} = X(x, y)$ and $\dot{y} = Y(x, y)$) which can be used to decide whether $(1, 0)$ is a centre or not. If the Hamiltonian function has a local maximum or minimum at the point $(1, 0)$, the equilibrium point $(1, 0)$ is a centre, also in original system.

We have $\dot{x} = \frac{\partial H}{\partial y}$ and $\dot{y} = -\frac{\partial H}{\partial x}$, so that

$$\frac{\partial H}{\partial x} = x^2 - 1,$$
$$\frac{\partial H}{\partial y} = ye^y.$$

If we integrate the last equation with respect to y we get

$$H(x, y) = e^y(y - 1) + f(x)$$

which, inserted into the equation $\frac{\partial H}{\partial x} = x^2 - 1$ gives

$$\frac{df}{dx} = x^2 - 1.$$

By integrating this equation with respect to x we get $H(x, y) = e^y(y-1) + \frac{1}{3}x^3 - x$. Then we can use the second derivative test to prove that $H(x, y)$ has a local maximum or minimum at $(1, 0)$. This shows that $(1, 0)$ is a centre in the original system. The direction of the paths are clockwise, since $\dot{y} < 0$ for $y = 0$ and $x > 1$.

Linearization about $(-1, 0)$ gives $\dot{x} = y$ and $\dot{y} = 2(x + 1)$. Hence

$$\frac{dy}{dx} = 2\frac{x+1}{y}.$$

The paths are hyperbolas centered at $(-1, 0)$. We find the asymptotes of the hyperbolas, $y = \pm\sqrt{2}(x + 1)$. The direction along the asymptotes are found by studying the sign of $\dot{y}$ in each quadrant. We may also use the direction of the paths about $(1, 0)$ to sketch the direction along the asymptotes. See figure 1 for a sketch of the phase diagram.

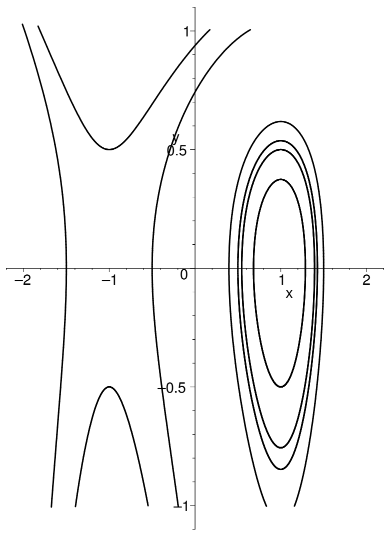

Figure 1: Phase diagram of $\dot{x} = ye^y$, $\dot{y} = 1 - x^2$

(iii) Sketch the phase diagram and characterize the equilibrium points of

$$\dot{x} = 1 - xy,$$
$$\dot{y} = (x - 1)y.$$

We find the equilibrium points of the system by setting $\dot{x} = \dot{y} = 0$, which gives $x = y = 1$ as the equilibrium point.

We linearize the system about the point $(1, 1)$ which gives us the linearized system

$$\begin{aligned} \dot{x} &= -x - y, \\ \dot{y} &= x. \end{aligned}$$

The eigenvalues of the matrix of the system are given as solutions to $(-1-\lambda)(-\lambda)+1 = \lambda^2 + \lambda + 1 = 0$. Hence

$$\lambda = \frac{-1 \pm \sqrt{3}i}{2}.$$

These are complex valued eigenvalues with negative real part. This is a stable spiral locally around $(x, y) = (1, 1)$. The direction is counterclockwise since $\dot{y} > 0$ for $x > 0$ and $y = 0$. Se figure 2 for a sketch of the phase diagram.

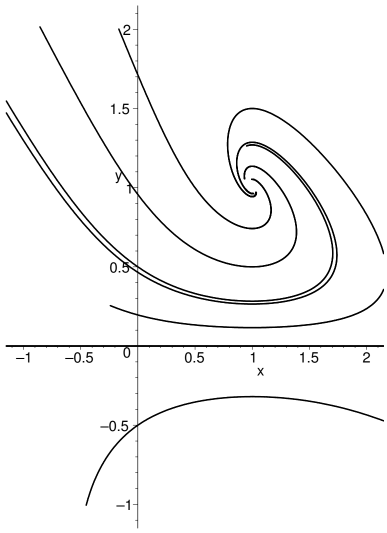

Figure 2: Phase diagram of $\dot{x} = 1 - xy$, $\dot{y} = (x - 1)y$

$\boxed{29}$ Find the equations for the phase paths for the general epidemic described by the system

$$\begin{aligned} \dot{x} &= -\beta xy, \\ \dot{y} &= \beta xy - \gamma y, \\ \dot{z} &= \gamma y. \end{aligned}$$

Sketch the phase diagram in the x, y plane. Confirm that the number of infectives reaches its maximum when $x = \frac{\gamma}{\beta}$.

We find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\beta xy - \gamma y}{-\beta xy} = -1 + \frac{\gamma}{\beta}\frac{1}{x},$$

for $x > 0$. Integration then gives

$$y(x) = -x + \frac{\gamma}{\beta} \ln x + C.$$

We find the maximum of the number of infectives by setting

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1 + \frac{\gamma}{\beta}\frac{1}{x} = 0,$$

hence

$$x = \frac{\gamma}{\beta}.$$

To verify that this is infact a maximum, we see that the sign of the double derivative of y is negative. The solution for z is given by $z = x_0 + y_0 + z_0 - x - y$.

See figure 3 for a sketch of the phase diagram.

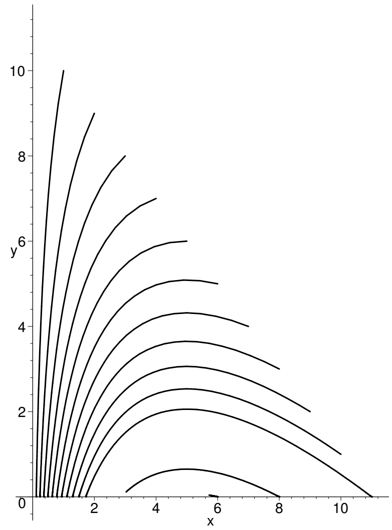

Figure 3: Phase diagram of $\dot{x} = -\beta xy$, $\dot{y} = \beta xy - \gamma y$. Here, $\beta = 2$ and $\gamma = 10$. The total population is set to 10 individuals.

$\boxed{\text{Exam 2011, 5}}$ (a)

We are given the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

with

$$\mathbf{f}(x, y) = \begin{bmatrix} 2y(x^4 - 2x^2 + 2 \\ 4(x - x^3)(y^2 + 1) \end{bmatrix}.$$

The divergence of f is

$$\mathrm{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 2y(4x^3 - 4x) + 4(x - x^3)2y = 0,$$

so the system is Hamiltonian. The Hamilton function satisfies

$$\frac{\partial H}{\partial y} = f_1 \quad \text{and} \quad -\frac{\partial H}{\partial x} = f_2.$$

Hence,

$$H(x,y) = \int_0^y f_1(x,s)s + C(x) = y^2(x^4 - 2x^2 + 2) + C(x),$$

and

$$f_2 = -\frac{\partial H}{\partial x} = -y^2(4x^3 - 4x) - C'(x),$$

so that

$$C'(x) = -4(x - x^3) \implies C(x) = -2x + x^4 + K.$$

By choosing $K = 2$ we get

$$H(x,y) = (y^2 + 1)(x^4 - 2x^2 + 2).$$

(b)

There are three equilibrium points: $(0,0)$, $(1,0)$ og $(-1,0)$. Since the equilibrium points for f are critical points for H, we can classify them by using the second derivative test for H.

$$D^2H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} (12x^2 - 4)(y^2 + 1) & 8y(x^3 - x) \\ 8y(x^3 - x) & 2(x^4 - 2x^2 + 2) \end{bmatrix}.$$

For the point $(0,0)$ we have

$$D^2H(0,0) = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix},$$

$\det D^2H(0,0) = (-4) \cdot 4 < 0$, so $(0,0)$ is a saddle point for H and for the dynamical system. For the points $(\pm 1, 0)$ we find

$$D^2H(\pm 1,0) = \begin{bmatrix} 16 & 0 \\ 0 & 4 \end{bmatrix},$$

$\det D^2H(\pm 1,0) = 16 \cdot 4 > 0$, so $(\pm 1,0)$ are minimium points for H and they are centres for the dynamical system.

$\boxed{\text{Exam 1999, 4}}$ A dynamical system in polar coordinates is given by

$$\dot{\theta} = 1, \quad \dot{r} = \begin{cases} r^2 \sin(\frac{1}{r}) & \text{for } r > 0 \\ 0 & \text{for } r = 0. \end{cases} \tag{1}$$

Determine if the origin is a stable, asymptotically stable or unstable equilibrium point. Sketch the phase diagram nearby the origin.

For $r > 0$, $\dot{r} = 0$ when $r = \frac{1}{n\pi}$ which gives periodic paths. Look at the interval

$$n\pi < \frac{1}{r} < (n+1)\pi.$$

Here, $\dot{r} < 0$ if n is odd and $\dot{r} > 0$ if n is even. Hence, the phase diagram will consist of several limit cycles around the origin, which are closer to each other when r decreases towards zero. Every other cycle is an unstable limit cycle.

After a small perturbation away from the origin, the system will stay in one of the stable limit cycle. The origin is thus a stable equilibrium state, but not asymptotically stable. See figure 4 for a sketch of the phase diagram.

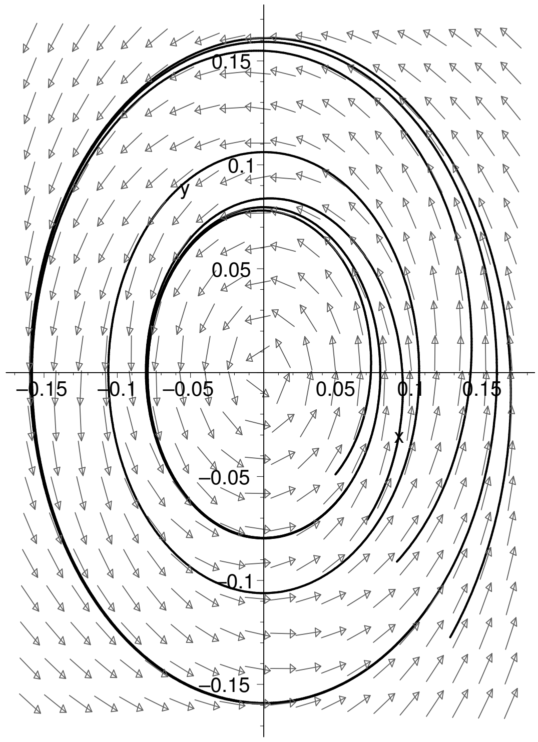

Figure 4: Phase diagram of $\dot{\theta} = 1$, $\dot{r} = r^2 \sin\left(\frac{1}{r}\right)$ for $r > 0$, $\dot{r} = 0$ for $r = 0$