# HOMEWORK 6
## EXPECTATION AND DISTRIBUTIONS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

**Problem 1.** Prove the Cauchy-Schwarz inequality: if $X$, $Y$ are non-negative random variables, then

$$\mathbb{E}\, XY \le (\mathbb{E}X^2)^{1/2}\, (\mathbb{E}Y^2)^{1/2}.$$

Moreover, if $X$, $Y$ are absolutely integrable random variables, then

$$\big|\mathbb{E}\, XY\big| \le (\mathbb{E}X^2)^{1/2}\, (\mathbb{E}Y^2)^{1/2}.$$

Deduce that for any random variable $X$ we have

$$\mathbb{E}\big|X\big| \le (\mathbb{E}\big|X\big|^2)^{1/2}.$$

*Hint:* Begin with the case when both $X$ and $Y$ are simple functions. It is convenient to represent them as

$$X = \sum_{i=1}^{k} a_i\, \mathbf{1}_{E_i} \quad \text{and} \quad Y = \sum_{i=1}^{k} b_i\, \mathbf{1}_{E_i},$$

that is, using the *same* sets $E_1, \dots, E_k$. Why is this possible?

**Problem 2.** Prove that if $X$ is a random variable such that $0 < \mathbb{E}\big|X\big|^2 < \infty$ then

$$\mathbb{P}(X \neq 0) \ge \frac{(\mathbb{E}\big|X\big|)^2}{\mathbb{E}\big|X\big|^2}.$$

**Problem 3.** Define the $L^\infty$ norm of a random variable $X$ as

$$\|X\|_\infty := \inf\{M \colon \big|X\big| \le M \text{ a.s.}\}.$$

Show that if $X, Y$ are non-negative random variables, then

$$\mathbb{E}\, XY \le \mathbb{E}\big|X\big|\, \|Y\|_\infty.$$

**Problem 4.** Let $X$ be a random variable and let $F_X(x) = \mathbb{P}(X \le x)$ be its CDF.

Show that if $F_X$ is continuous then the random variable $Y := F_X(X)$ has a uniform distribution on $(0, 1)$. In other words, if $y \in [0, 1]$, then $\mathbb{P}(Y \le y) = y$.

**Problem 5.** Suppose the random variable $X$ has probability density function $f$.
Compute the distribution function of $X^2$ and then differentiate to find its density function.

**Problem 6.** Compute the expectation, the variance and the moments of a random variable drawn uniformly from $(0, 1)$.
Compute the expectation of a random variable with (negative) exponential distribution.

**Problem 7.** Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two probability measure spaces and let $\mathcal{F}_1 \times \mathcal{F}_2$ be the $\sigma$-algebra on $\Omega_1 \times \Omega_2$ generated by the family of all Cartesian products $E_1 \times E_2$ where $E_1 \in \mathcal{F}_1$ and $E_2 \in \mathcal{F}_2$.

(a) Show that for every set $E \in \mathcal{F}_1 \times \mathcal{F}_2$, and for every $\omega_1 \in \Omega_1$, the section
$$E_{\omega_1} := \{\omega_2 \in \Omega_2 \colon (\omega_1, \omega_2) \in E\}$$
is $\mathcal{F}_2$-measurable.

(b) Show that the function $\phi \colon \Omega_1 \to [0, 1]$,
$$\phi(\omega_1) := \mu_2(E_{\omega_1})$$
is $\mathcal{F}_1$-measurable.

(c) Finally show that if we define for all $E \in \mathcal{F}_1 \times \mathcal{F}_2$
$$\mu_1 \times \mu_2(E) := \int_{\Omega_1} \mu_2(E_{\omega_1})\, d\mu_1(\omega_1),$$
then $\mu_1 \times \mu_2$ is a probability measure on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times F_2)$.

*Hint:* Use the monotone class lemma.