MA8502 fall 2016
Assignment 3

1 Steady state Stokes on a rectangular domain

In this assignment we consider the model problems described in the note "The steady Stokes problem", p. 5 Example 1.3. The exact solution is given in the note.

We want to use a spectral Galerkin method approximation with $\mathbb{P}_N$ polynomials for the two velocity components and $\mathbb{P}_{N-2}$ polynomials for the pressure, on Gauss-Lobatto Legendre nodes.

- What error do you expect to obtain for this problem, assuming the polynomial degree N is high enough? Explain your answer.

Consider now the Stokes problem on on $\Omega = [-1,1]^2$, with solution

$$\begin{cases} u_1 &= -\cos(\pi x_1)\sin(\pi x_2), \\ u_2 &= \sin(\pi x_1)\cos(\pi x_2), \\ p &= -\frac{1}{4}[\cos(2\pi x_1) + \cos(2\pi x_2)]. \end{cases}$$

This Stokes problem can be formulated with non-homogeneous Dirichlet boundary conditions. Discretize the equations and find the appropriate discrete Laplacian A, discrete gradient D^T and discrete divergence D operators.

Use the Usawa algorithm combined with a conjugate gradient algorithm to solve the linear system (you can either implement the Conjugate-Gradient yourself, or try to use `pcg` in MATLAB).

1. Show how the error for the pressure and the velocity in the appropriate (approximated) norms decreases as a function of the polynomial degree.

2. Report numerical results on the convergence of the conjugate gradient algorithm: how does the number of iterations increase with the polynomial degree?

3. Compare the results of your numerical experiments with the estimates of convergence of the conjugate gradient algorithm. Use the estimates of the condition number for the Uzawa operator $U = DAD^T$ in terms of the polynomial degree N.

2 Navier-Stokes equations

We consider the numerical solution of the Navier-Stokes equations on $\Omega = [-1,1]^2$, with Dirichlet boundary conditions. We consider the Taylor vortex problem [2], [1] with solution given by

$$\begin{cases} u_1 &= -\cos(\pi x_1)\sin(\pi x_2)\exp(-2\pi^2 t/Re), \\ u_2 &= \sin(\pi x_1)\cos(\pi x_2)\exp(-2\pi^2 t/Re), \\ p &= -\frac{1}{4}[\cos(2\pi x_1) + \cos(2\pi x_2)]\exp(-4\pi^2 t/Re). \end{cases}$$

We use the same space discretization as for the Stokes problem. For the time integration we use an implicit-explicit scheme, treating the linear diffusion part implicitly and the nonlinear convection part explicitly. With this approach we need to solve only one linear system per time step, and no Newton-iteration is necessary. Start first with a method of order 1 combining forward Euler and backward Euler. Then use a method of order 2 and 3 in time.

1. Show a plot of the spectral convergence in space, i.e. show the decrease of the error in space as a function of the polynomial degree. Compared to the similar experiment in the previous exercise, this experiment will give evidence that you have discretized correctly the convection operator.

2. Show also a plot of the convergence in time for both the velocity and the pressure. Fix a space discretization and show how the error decreases when you decrease the time step.

See [1] and [2] for comparable experiments and suggestions of how to choose N, the size of the time interval and the range of step-sizes.

References

[1] E. Celledoni, B.K. Kometa, Semi-Lagrangian multistep exponential integrators for index 2 differential-algebraic systems, *J. Comp. Phys.*, **230**, pp. 3413-3429, 2011.

[2] Y. Maday, A.T. Patera, E.M. Rønquist, An operator-integration-factor splitting method for time-dependent problems: application to incompressible fluid flow *J. Sci. Comput.*, 5 (4) (1990), pp. 263-292