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Solutions to exercise set 5

1 **a)** Writing $u(x,t) = F(x)G(t)$ we have $u_t = F\dot{G}$, $u_{xx} = F''G$, and hence we obtain the equation

$$F\dot{G} - c^2F''G = 0$$

Separating variables by dividing by c^2FG, we have

$$\frac{\dot{G}}{c^2G} - \frac{F''}{F} = 0,$$

and as both terms must be constant we obtain the system of equations

$$F'' - kF = 0, \quad \dot{G} - c^2kG = 0$$

for unknown constant k. The boundary conditions stated in terms of F and G give

$$F'(0)G(t) = F(1)G(t) = 0$$

for all $t > 0$, and hence to obtain nontrivial solutions we require $F'(0) = F(1) = 0$. We use this to solve the equation for F. As usual, there are three possible solutions depending on the sign of k.

1. $k = n^2$ (i.e. $k > 0$) Here we have

$$F_n = A_ne^{nx} + B_ne^{-nx}$$

$$F_n' = nA_ne^{nx} - nB_ne^{-nx}$$

and setting $F'(0) = 0$ gives $A_n - B_n = 0$, i.e. $F_n = A_n \cosh nx$. Then $F_n(1) = 0$ implies that $A_n = 0$, as cosh is always strictly greater than zero, and hence $F_n = 0$. So we can ignore this case.

2. $k = 0$. Then

$$F_0 = Ax + B,$$

and $F_0' = B$. So $F'(0) = 0$ gives $B = 0$, but then $F(1) = A = 0$, so here we have again $F_0 = 0$.

3. $k = -n^2$ (i.e. $k < 0$) Now

$$F_n = A_n \cos nx + B_n \sin nx$$

$$F_n' = -nA_n \sin nx + nB_n \cos nx$$

Setting $F'(1) = 0$ gives $B_n = 0$, so we have $F_n = A_n \cos nx$. To satisfy $F(1) = 0$ with a non-trivial solution, we require $\cos nx = 0$ which is satisfied if and only if $n = \frac{2m+1}{2}\pi$, for some integer m, which we can take to be greater than or equal to zero without loss of generality as cos is even.

We therefore obtain $k = -\left(\frac{2m+1}{2}\pi\right)^2$, for $m = 0, 1, 2, \ldots$, and

$$F_m = \cos\left(\frac{2m+1}{2}\pi x\right)$$

We now solve the equation for G,

$$\dot{G}_m + \left(\frac{c\pi(2m+1)}{2}\right)^2 G_m = 0,$$

to obtain

$$G_m = A_m e^{-\left(\frac{c\pi(2m+1)}{2}\right)^2 t}$$

The general solution is therefore

$$u(x,t) = \sum_{m=0}^{\infty} A_m \cos\left(\frac{2m+1}{2}\pi x\right) e^{-\left(\frac{c\pi(2m+1)}{2}\right)^2 t}$$

b) We now wish to solve the above equation given the initial condition

$$u(x,0) = 100\cos(\frac{\pi x}{2})$$

Setting $t = 0$ in the general solution gives

$$u(x,0) = \sum_{m=0}^{\infty} A_m \cos\left(\frac{2m+1}{2}\pi x\right).$$

We then recognize that $\cos(\frac{\pi x}{2})$ is of the form appearing in the sum on the right, where $m = 0$. So we have $A_0 = 100$ and all other A_n equal to zero. The solution is then

$$u(x,t) = 100\cos(\frac{\pi x}{2})e^{\frac{c^2\pi^2}{4}t}$$

For the temperature to cool to 20°C we require that

$$e^{-\frac{c^2\pi^2}{4}t} = \frac{1}{5}$$

and taking logarithms on both sides gives

$$t = -\frac{4}{c^2\pi^2}\ln\left(\frac{1}{5}\right) \approx 5136(\text{seconds})$$

$\boxed{2}$ Writing $u = F(x)G(y)$ and separating variables results in the equations

$$\frac{d^2F}{dx^2} - kF = 0, \quad \frac{d^2G}{dy^2} + kG = 0$$

The boundary conditions give (for non trivial solutions)

$$F'(0) = F'(2\pi) = 0$$

and the equation for F is of a type encountered before for the 1d heat equation in a bar with insulating ends, see Kreyszig Chapter 12.6. In particular, solving as before we obtain $k = -\left(\frac{n}{2}\right)^2$ for $n = 0, 1, 2, \dots$ (note that the sum starts from 0 to include a constant solution corresponding to the case $k = 0$), with solution

$$F_n(x) = \cos\frac{nx}{2}$$

We then solve for G:

$$\frac{d^2 G_n}{dy^2} - \left(\frac{n}{2}\right)^2 G_n = 0, \quad n = 0, 1, 2, \dots$$

The first boundary condition for G gives $G'(0) = 0$, and solving the equation subject to this condition gives $G_0 = A_0$ constant, and for $n > 0$ we have

$$G_n = A_n \cosh\frac{ny}{2}$$

The solution is then

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{nx}{2} \cosh\frac{ny}{2}$$

There is one final boundary condition to satisfy, namely

$$u(x, \pi) = 50(1 + \cos x)$$

Setting $y = \pi$ in the expression for u gives

$$u(x, \pi) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{nx}{2} \cosh\frac{n\pi}{2},$$

and by comparing the two expressions for $u(x, \pi)$ (all the A_n are zero except A_0 and A_2) we get the final form of the solution

$$u(x, t) = 50\left(1 + \frac{\cos x \cosh y}{\cosh \pi}\right) \text{ (°C)}$$

3 **a)** This is an exercise in a (slightly) unfamiliar use of separation of variables. As usual, we write $u = F(x)G(t)$ and compute derivatives: $u_t = F\dot{G}$, $u_{tt} = F\ddot{G}$, $u_{xx} = F''G$. Substituting into the equation we obtain

$$F\ddot{G} - 2F\dot{G} + FG - F''G = 0$$

We separate variables in the normal manner by dividing by FG; this gives

$$\frac{\ddot{G} - 2\dot{G}}{G} + 1 - \frac{F''}{F} = 0$$

The easiest way of dealing with the constant term is to include it in the term containing the G's, i.e. by rewriting the above as

$$\frac{\ddot{G} - 2\dot{G} + G}{G} - \frac{F''}{F} = 0$$

(We could instead have included this term in a similar manner with the F's without difficulty, but this way the equation for F will be one we have seen many times) The usual argument that the first term is a function of t alone and the second a function of x alone, and they are both equal and hence constant, applies here. This gives the system of equations

$$F'' - kF = 0, \quad \ddot{G} - 2\dot{G} + (1-k)G$$

The boundary conditions $u(0,t) = u(\pi,t) = 0$ give $F(0) = F(\pi)$, and the equation for F is one encountered many times, first in the Chapter 12.3 of Kreyszig on the wave equation. It requires that $k = -n^2$, $n = 1, 2, \ldots$, and

$$F_n = \sin nx$$

The equation for G is then

$$\ddot{G} - 2\dot{G} + (1+n^2)G = 0$$

To solve this, we must solve the characteristic equation $m^2 - 2m + (1+n^2)m = 0$. The solution obtained using the quadratic formula is

$$m = \frac{2 \pm \sqrt{4 - 4(1+n^2)}}{2} = 1 \pm in$$

We therefore have

$$G_n = e^t \left(A_n \cos nt + B_n \sin nt\right)$$

Hence the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} e^t \left(A_n \cos nt + B_n \sin nt\right) \sin nx$$

b) We now require the solution satisfying the initial conditions

$$u(x,0) = \sin x + \sin 2x$$

$$\frac{\partial u(x,0)}{\partial t} = -\sin 2x$$

We note that

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx$$

hence the first condition gives that $A_1 = A_2 = 1$, whilst all other A_n are zero. Differentiating $u(x,t)$ (we assume this can be done term-by-term), we obtain

$$u_t(x,t) = \sum_{n=1}^{\infty} e^t \big((A_n + nB_n)\cos nt + (B_n - nA_n)\sin nt\big) \sin nx$$

It follows that

$$u_t(x,0) = \sum_{n=1}^{\infty} (A_n + nB_n) \sin nx$$

We require that $A_n + nB_n = -1$ for $n = 2$ and 0 for all other n. If $n > 2$ this clearly requires that $B_n = 0$. It remains to consider the cases $n = 1, 2$. First (setting $A_1 = 1$)the equation for $n = 1$ reads $1 + B_1 = 0$, hence $B_1 = -1$. Then the equation for $n = 2$ is $1 + 2B_2 = -1$, and we have $B_2 = -1$. Collecting this together, we find the solution

$$u(x,t) = e^t \sin x \big(\cos t - \sin t\big) + e^t \sin 2x \big(\cos 2t - \sin 2t\big)$$