Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2017

Exercise set 8

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let X and Y be normed spaces.

a) Show that $f : X \to Y$ is continuous if and only if for any closed set $F \subset Y$ its preimage $f^{(-1)}(F)$ is closed in X.
Hint: A useful starting point may be to show that
$$f^{(-1)}(Y \setminus A) = X \setminus f^{(-1)}(A)$$
for $A \subset Y$.

b) Show that the zero set $\{x \in X : f(x) = 0\}$ of a continuous function $f : X \to Y$ is closed. Use the preceding statement.

c) Use the preceding statement to prove that the kernel of a bounded linear transformation $T : X \to Y$ is a closed subspace.

2 Let X and Y be normed spaces. Show that a linear map $T : X \to Y$ is not continuous if and only if there exists a sequence of unit vectors (x_n) in X such that $\|Tx_n\| \geq n$ for $n \in \mathbb{N}$.

3 Let T be a linear mapping $T : (\mathbb{R}^n, \|.\|_\infty) \to (\mathbb{R}^n, \|.\|_\infty)$ given by a $n \times n$ matrix A. Show that the operator norm of T in terms of A is given by $\|T\| = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$.

4 Let T be the integral operator $Tf(x) = \int_0^1 k(x,y)f(y)dy$ defined by a kernel $k \in C([0,1] \times [0,1])$ such that $k(x,y) \geq 0$ for any $(x,y) \in [0,1] \times [0,1]$. Show that the operator norm of T as a mapping on $C[0,1]$ with respect to $\|.\|_\infty$-norm is $\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy$.

5 Let T be a linear operator from $(\ell^\infty, \|.\|_\infty)$ to $(\ell^\infty, \|.\|_\infty)$ defined by an infinite matrix $(a_{ij})_{i,j=1}^\infty$ satisfying $\sum_{j=1}^\infty |a_{ij}| < \infty$. Show that the operator norm of T is given by $\sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}|$.

6 Let T be a linear operator between the normed spaces X and Y. We say that T is an isometry if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$.

a) Show that if T is an isometry, then T is injective.

b) For $1 \leq p \leq \infty$ define the shift operator $T : \ell^p \to \ell^p$ by $Tx = (0, x_1, x_2, ...)$. Show that T is an isometry and determine its range and kernel.