Norwegian University of Science
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Department of Mathematics

MA2501: Numerical Methods
Spring 2017

Solutions to exercise set 2

1 Let $\mathbf{y} = \mathbf{A}\mathbf{x}$ in the following.

1. We start by proving that the only matrix with 0 norm is $\mathbf{0}$, the matrix of 0's. If $\mathbf{A} = \mathbf{0}$ it is obvious that $\mathbf{A}\mathbf{x} = \mathbf{0}$ for any vector $\mathbf{x}$, and thus $\|\mathbf{A}\| = 0$ in this case.

   Suppose now $\mathbf{A} \neq \mathbf{0}$. Let row i be a row of $\mathbf{A}$ such that $a_{ij} \neq 0$ for one or more values of j. Choose for $\mathbf{x}$ a normalized vector such that $x_j \neq 0$ and has the same sign as a_{ij} when $a_{ij} \neq 0$. Then $a_{ij}x_j \geq 0$ for all j and $a_{ij}x_j > 0$ for those j such that $a_{ij} \neq 0$. This implies that $y_i > 0$, and since $\|\cdot\|$ is a proper vector norm, $\|\mathbf{A}\mathbf{x}\| > 0$ for this choice of $\mathbf{x}$. From the definition of the matrix norm it is then apparent that $\|\mathbf{A}\| > 0$ for any $\mathbf{A} \neq \mathbf{0}$.

2. Next we show
$$\|\alpha\mathbf{A}\| = |\alpha|\|\mathbf{A}\|,$$
for any real scalar α and matrix $\mathbf{A}$. We know that for any $\mathbf{x}$, $\mathbf{A}$ and α we have $(\alpha\mathbf{A})\mathbf{x} = \alpha(\mathbf{A}\mathbf{x})$. This and
$$\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|,$$
for any vector norm implies that
$$\begin{aligned}\|\alpha\mathbf{A}\| &= \max_{\|\mathbf{x}\|=1} \|(\alpha\mathbf{A})\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|\alpha(\mathbf{A}\mathbf{x})\| = \max_{\|\mathbf{x}\|=1} |\alpha|\|(\mathbf{A}\mathbf{x})\| \\ &= |\alpha| \max_{\|\mathbf{x}\|=1} \|(\mathbf{A}\mathbf{x})\| = |\alpha|\|\mathbf{A}\|.\end{aligned}$$

3. Finally we need to show the triangle inequality
$$\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|,$$
for arbitrary matrices $\mathbf{A}$ and $\mathbf{B}$. This follows readily from the triangle inequality for vector norms and the properties of the max operation.
$$\begin{aligned}\|\mathbf{A} + \mathbf{B}\| &= \max_{\|\mathbf{x}\|=1} \|(\mathbf{A} + \mathbf{B})\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x}\| \leq \max_{\|\mathbf{x}\|=1} (\|\mathbf{A}\mathbf{x}\| + \|\mathbf{B}\mathbf{x}\|) \\ &\leq \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| + \max_{\|\mathbf{x}\|=1} \|\mathbf{B}\mathbf{x}\| = \|\mathbf{A}\| + \|\mathbf{B}\|.\end{aligned}$$

We have thus shown that a subordinate matrix norm is a matrix norm.

2 First step: Work with exercise set 1, task 5 instead.

a) In the first step, equations 1 and 3 are exchanged; in the second step equations 2 and 3. The solution is $[x_1, x_2, x_3] = [43/55, -347/275, -1/11]$.

b) In step 2, equations 2 and 3 are exchanged. The final solution is $[x_1, x_2, x_3] = [19/15, -1/15, 1/5]$.

c) In step 2, equations 2 and 3 are exchanged. The final solution is $[x_1, x_2, x_3] = [109, 27, -66]$.

d) Equations 1 and 3 are exchanged in step ; equations 2 and 3 are exchanged in step 2. The solution is $[x_1, x_2, x_3] = [21, 11, -5]$.

3 **Cf. Cheney and Kincaid, Exercise 2.2.24**

In all the following solutions I have carried out the rounding only after performing all the computations in a line. For instance, I have computed $a_{22} - a_{21}a_{12}/a_{11}$ in double precision and then rounded the result to 4 significant digits. A different possibility would be to round after each single computation, that is, do this calculation as: divide a_{21} by a_{11}, round; multiply the result by a_{12}, round; subtract the result from a_{22}, round. With this approach to rounding, the results of the following equation will be slightly different (the required permutations of rows and columns will stay the same, though). In an exam both approaches would be correct.

a)

- Without pivoting:
  After the first elimination step and rounding to four significant digits we obtain the system

$$\begin{aligned} 0.004000x + \quad 69.13y &= \quad 69.17, \\ -73990y &= -73990. \end{aligned}$$

  Back substitution yields the result

$$x = 10.00, \qquad y = 1.000.$$

- With partial pivoting:
  In the first step, the two equations are exchanged and we obtain

$$\begin{aligned} 4.281x - 5.230y &= 41.91, \\ 0.004000x + 69.13y &= 69.17. \end{aligned}$$

  Elimination and rounding yields

$$\begin{aligned} 4.281x - 5.230y &= 41.91, \\ 69.13y &= 69.13, \end{aligned}$$

  the rounded solution of which is

$$x = 11.01, \qquad y = 1.000.$$

- With scaled partial pivoting:
  For this example it is the same as partial pivoting.

- With complete pivoting:
  In the first pivoting step we obtain

$$\begin{aligned} 69.13y + 0.004000x &= 69.17, \\ -5.230y + \quad 4.281x &= 41.91. \end{aligned}$$

  Elimination and rounding yields

$$\begin{aligned} 69.13y + 0.004000x &= 69.17, \\ 4.281x &= 47.14, \end{aligned}$$

  the rounded solution of which is

$$x = 11.01, \qquad y = 0.9999.$$

- The true solution is

$$x \approx 11.01137, \qquad y \approx 0.9999415.$$

  (Note that I have given 7 significant digits, although it does not really make sense to give more than 4).

b)

- Without pivoting:
  After the first elimination step and rounding to four significant digits we obtain the system

$$\begin{aligned} 30.00x + \quad 591400y &= \quad 591700, \\ -104300y &= -104300. \end{aligned}$$

  Back substitution yields the result

$$x = 10.00, \qquad y = 1.000.$$

- With partial pivoting:
  Here this is the same as without pivoting.
- With scaled partial pivoting:
  In the first step, the two equations are exchanged and we obtain

$$\begin{aligned} 5.291x - \quad 6.130y &= \quad 46.78, \\ 30.00x + 591400y &= 591700. \end{aligned}$$

  Elimination yields

$$\begin{aligned} 5.291x - \quad 6.130y &= \quad 46.78, \\ 591400y &= 591400. \end{aligned}$$

  This system has the rounded solution

$$x = 10.00, \qquad y = 1.000.$$

- With complete pivoting:
  In the first pivoting step we obtain

$$\begin{aligned} 591400y + 30.00x &= 591700, \\ -6.130y + 5.291x &= \quad 46.78. \end{aligned}$$

  Elimination and rounding yields

$$\begin{aligned} 591400y + 30.00x &= 591700, \\ 5.291x &= \quad 52.91, \end{aligned}$$

  the rounded solution of which is

$$x = 10.00, \qquad y = 1.000.$$

- The true solution is (surprisingly the same as all numerical ones, namely)

$$x = 10, \qquad y = 1.$$

4 You can find example programs on the course page (`lusolve.m`, `lusolve_long.m`, and `lusolve_nopivoting.m`). The program `lusolve_long.m` contains an almost verbatim implementation of the pseudocode given in the lecture. The differences are that we also store and return the multipliers, so we can return the LU decomposition, as is done in the textbook We also include a check that the elements of the problem have correct size (not strictly required). The problem with this code is that it uses a triple loop, and loops tend to be slow in MATLAB and should be avoided whenever sensibly possible. The program `lusolve.m` (and also `lusolve_nopivoting.m`) is an example how this can be done for Gaussian elimination using vectorization of most of the operations. On my computer, this code runs about three times faster than the non-vectorized one. Unfortunately, it also becomes more difficult to read. Reading vectorized code becomes easier once you get more familiar with it.

5 **Cf. Cheney and Kincaid, Computer Exercise 2.2.14.**
Implementations of the functions `CompDet` which computes the determinant and `Gauss` which implements the forward elimination part of Gaussian elimination with scaled partial pivoting are given on the course page.