Norwegian University of Science and Technology Institute of mathematics	TMA4220 Finite Element Method Fall 2016 **Exercise set 2**

1 Consider the homogeneous Dirichlet problem

$$\begin{cases} -\nabla^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where f is a given continuous function. The weak formulation for this problem is: find $u \in H_0^1(\Omega)$:

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega fv \, d\Omega \quad \forall v \in H_0^1(\Omega)$$

(make sure you understand how this is derived!)

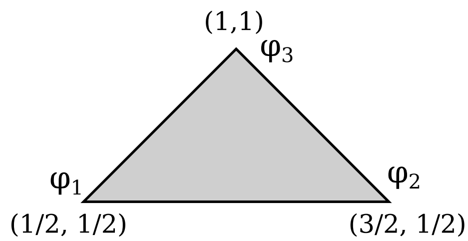
a) Prove that the bilinear form $a(u, v)$ given by the integral on the left-hand side above is symmetric, continuous and coercive. Conclude that there exists a unique solution to the above problem.

b) Let V_h be a closed, finite-dimensional subspace of $H_0^1(\Omega)$; the resulting Galerkin problem gives rise to a linear system

$$Au = f$$

Show that the matrix A is symmetric and positive definite, and hence conclude that the above problem has a unique solution (Hint: how do the properties of $a(u, v)$ proved in the previous part translate to the Galerkin problem? If you are stuck, consult chapter 4.1 of Quarteroni)

2 Consider the triangle with corners $x_1 = (\frac{1}{2}, \frac{1}{2})$, $x_2 = (1, 1)$ and $x_3 = (\frac{3}{2}, \frac{1}{2})$. Let



the reference element $\hat{K}$ (i.e. the triangle with vertices at $(1, 0), (0, 1)$ and $(0, 0)$) be mapped to the physical element above by an affine mapping

$$\mathcal{F} : \mathbf{x}(\boldsymbol{\lambda}) = \mathbf{x}_1\lambda_1 + \mathbf{x}_2\lambda_2 + \mathbf{x}_3\lambda_3$$

a) The linear basis functions on this element can be written as

$$\varphi_i(x, y) = a_i x + b_i y + c$$

Find the expression for these three basis functions in physical coordinates (x, y).

b) Compute the Jacobian of the forward mapping $\mathcal{F}$:

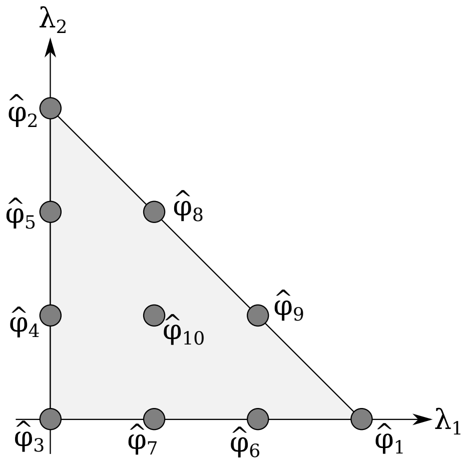
$$J = \begin{bmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{y}} \\ \frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{y}} \end{bmatrix}$$

c) By changing variables to barycentric coordinates on the reference element $\hat{K}$, we can compute derivatives of any basis function $\varphi_i(x)$ in terms of the corresponding function $\varphi(\lambda)$ on $\hat{K}$. Indeed, show that

$$\nabla \varphi_i = \begin{bmatrix} \frac{\partial \varphi_i}{\partial x} \\ \frac{\partial \varphi_i}{\partial y} \end{bmatrix} = G \begin{bmatrix} \frac{\partial \hat{\varphi}_i}{\partial \lambda_1} \\ \frac{\partial \hat{\varphi}_i}{\partial \lambda_2} \\ \frac{\partial \hat{\varphi}_i}{\partial \lambda_3} \end{bmatrix}$$

for some constant matrix $G \in \mathbb{R}^{2\times 3}$. Use the expression for the inverse of the Jacobian matrix J to find this matrix.

$\boxed{3}$ Consider the 10-node reference triangle of unit length: X_h^3. The *cubic* functions can



be written as

$$\hat{\varphi}_i(\lambda_1, \lambda_2, \lambda_3) = \sum_{0\le i+j+k\le 3} a_{ijk}\lambda_1^i \lambda_2^j \lambda_3^k$$

a) Find the expression for the ten basis functions on this element in barycentric (area) coordinates $(\lambda_1, \lambda_2, \lambda_3)$.

b) Suppose the reference element is mapped to the physical element from the previous exercise. Determine the value of the physical derivatives $\nabla\varphi_i$ of the first and fourth basis functions φ_1, φ_4, when evaluated at the element barycentre $(\lambda_1, \lambda_2, \lambda_3) = (1/3, 1/3, 1/3)$. (Hint: use the results of 2c)