HOMEWORK 10
POINCARÉ'S RECURRENCE THEOREM

The goal of problems 1-5 is to prove Liouville's theorem. We first recap some notions.

The states of the system are points $x = (q, p) = (q_1, \ldots, q_m;\, p_1, \ldots, p_m) \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^d$, where $d = 2m$. Let $X \subset \mathbb{R}^d$ be an open and bounded set representing the space of all possible states of the system. The state of the system at time t is $x(t) = (q(t), p(t))$.

We are given a smooth function $H\colon X \to \mathbb{R}$ called a *Hamiltonian*. This determines the Hamiltonian (system of ordinary differential) equations

$$\begin{cases} \dfrac{d}{dt} q_i = \dfrac{\partial H}{\partial p_i} \\ \dfrac{d}{dt} p_i = -\dfrac{\partial H}{\partial q_i}, \end{cases}$$

for $i = 1, \ldots, m$.

Given an initial condition $(q(0), p(0))$, this becomes an IVP (initial value problem).

Let $T_t\colon X \to X$ be the continuous flow generated by this Hamiltonian system.

Now denote

$$\frac{\partial H}{\partial p} := \Big(\frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_m}\Big)$$
$$\frac{\partial H}{\partial q} := \Big(\frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_m}\Big).$$

We can reformulate the above as follows.

Let $y\colon \mathbb{R} \times X \to X$, $y = y(t, x)$ be the (unique) solution of this IVP with initial condition x. In other words, for every x, $y(t, x)$ satisfies

$$\begin{cases} \dfrac{d}{dt}\, y(t, x) = \Big(\dfrac{\partial H}{\partial p}, -\dfrac{\partial H}{\partial q}\Big) \\ \\ y(0, x) = x. \end{cases}$$

Note that $y(t, x) = T_t(x)$.

Problem 1. Show that for all $t, s \in \mathbb{R}$, $x \in X$ we have

$$y(t + s, x) = y(t,\, y(s, x)).$$

Problem 2. Show that the Jacobian matrix of $y = y(x)$ (i.e. the matrix of partial derivatives of y with respect to x) satisfies

$$\frac{\partial y}{\partial x} = I + t\,A + \mathcal{O}(t^2) \quad \text{as } t \to 0,$$

where I is the identity matrix in $\mathrm{Mat}(d, \mathbb{R})$ and A is a matrix in $\mathrm{Mat}(d, \mathbb{R})$ with trace $\mathrm{tr}(A) = 0$.

Problem 3. Show that for every matrix A, the following estimate holds

$$\det\big(I + t\,A + \mathcal{O}(t^2)\big) = 1 + t\,\mathrm{tr}(A) + \mathcal{O}(t^2) \quad \text{as } t \to 0.$$

Problem 4. Prove that the Jacobian of $y = y(x) = T_t(x)$ (i.e. the determinant of the Jacobian matrix) is equal to 1 for all t.

Problem 5. Use the substitution (for multiple variables) formula to derive Liouville's theorem: for every measurable set $E \subset X$ and for every $t \in \mathbb{R}$ we have

$$\lambda(T_t E) = \lambda(E).$$

Problem 6. Prove that the flow T_t has the following properties: $T^n = T_n$ for all $n \geq 1$, the inverse $T^{-1} = T_{-1}$ and more generally, for every integer n (positive or negative),

$$T^n = T_n.$$

Problem 7. Prove the following abstract (measure theoretical) version of Poincaré's recurrence theorem. Let $(X, \mathcal{B}, \mu, T)$ be an MPDS and let $E \in \mathcal{B}$. Then for a.e. $x \in E$ there is an infinite sequence of integers $n_k \to \infty$ such that

$$T^{n_k} x \in E \quad \text{for all } k \geq 1.$$

Problem 8. Let $(X, \mathcal{B}, \mu, T)$ be an MPDS and let $E \in \mathcal{B}$. Define the following set

$$E_0 := \{x \in X \colon T^k x \in E \ \text{ for infinitely many } k \geq 1\}.$$

Show that the set E_0 is $\mathcal{B}$-measurable and T-invariant.