Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2017

Solutions exercise 4

2.3

(i) Sketch the phase diagram and characterize the equilibrium points of

$$\dot{x} = x - y,$$
$$\dot{y} = x + y - 2xy.$$

Equilibrium points are found by setting $\dot{x} = \dot{y} = 0$, which gives $x = y$ and $x + y - 2xy = 0$. This gives $x = y = 0$, or $x = y = 1$. Hence, $(0,0)$ and $(1,1)$ are equilibrium points.

The matrix for linearization is

$$J = \begin{bmatrix} 1 & -1 \\ 1-2y & 1-2x \end{bmatrix}.$$

Linearization about the origin gives us the equations $\dot{x} = x - y$ and $\dot{y} = x + y$. This system can be solved in the usual way by looking at eigenvectors and eigenvalues. We find $\lambda = 1 \pm i$. Hence, we have an unstable spiral at the origin. The orientation can be found by setting $y = 0$ and $x > 0$. This gives $\dot{y} > 0$, so the spiral has a motion counterclockwise.

Linearization about $(1,1)$ gives $\dot{x} = x - y$ and $\dot{y} = -x - y$. We find the eigenvalues as the roots of $(1-\lambda)(-1-\lambda) - 1 = 0$, so $\lambda = \pm\sqrt{2}$. Two real eigenvalues with opposite signs gives us a saddle point. We find the asymptotes of the family of phase paths by looking at the eigenvectors

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \pm \sqrt{2} \end{bmatrix}.$$

Now, we are able to give a sketch of the phase diagram by giving a sketch of a spiral which spreads out counterclockwise at the origin, and at $(1,1)$ a saddle point. See figure 1 for a sketch of the phase diagram.

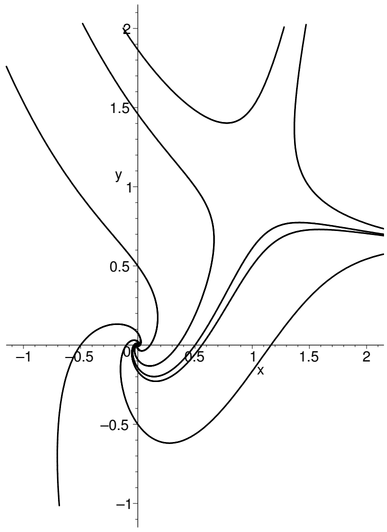

Figure 1: Phase diagram for $\dot{x} = x - y$, $\dot{y} = x + y - 2xy$

(vi) Sketch the phase diagram and characterize the equilibrium points of

$$\dot{x} = -6y + 2xy - 8,$$
$$\dot{y} = y^2 - x^2.$$

At equilibrium we have $\dot{y} = 0$ so that $x = \pm y$. By inserting this into the first equations, we get two possible second order equations for y. Only one of them has real solutions, and the equilibrium points are $(-1, -1)$ and $(4, 4)$.

The matrix for linearization is

$$J = \begin{bmatrix} 2y & -6 + 2x \\ -2x & 2y \end{bmatrix}.$$

We look at the equilibrium point $(-1, -1)$ first. Here,

$$J(-1, -1) = \begin{bmatrix} -2 & -8 \\ 2 & -2 \end{bmatrix}.$$

The eigenvalues of $J(-1, -1)$ are given by $\lambda = -2 \pm 4i$. Since these are complex valued with negative real part, the equilibrium point is a stable spiral.

At the point $(4, 4)$ we find

$$J(4, 4) = \begin{bmatrix} 8 & 2 \\ -8 & 8 \end{bmatrix}.$$

The eigenvalues of $J(4, 4)$ are given by $\lambda = 8 \pm 4i$. Since these are complex valued with positive real part, the equilibrium point is an unstable spiral.

See figure 2 for a sketch of the phase diagram.

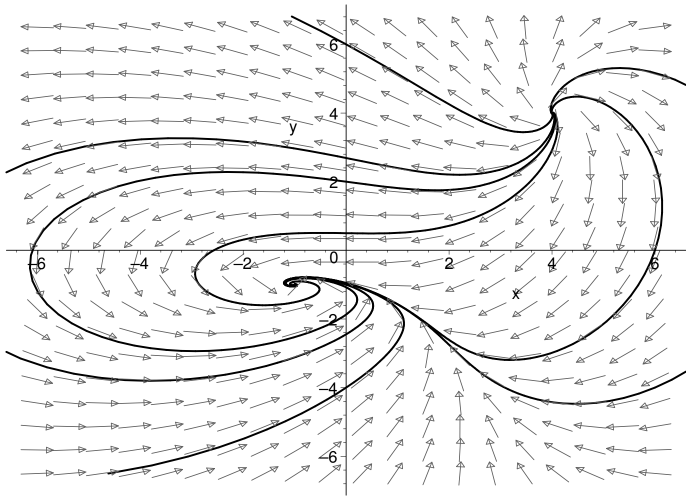

Figure 2: Phase diagram of $\dot{x} = -6y + 2xy - 8$, $\dot{y} = y^2 - x^2$

(ix) Sketch the phase diagram and characterize the equilibrium points of

$$\dot{x} = \sin y,$$
$$\dot{y} = -\sin x.$$

Equilibrium points are found by solving $\dot{x} = \dot{y} = 0$, which gives $x = m\pi$ and $y = n\pi$ for $(m, n) \in \mathbf{Z} \times \mathbf{Z}$. We can linearize $\dot{x}$ and $\dot{y}$ about such a point, and we have to distinguish between the cases where m and n are even and odd numbers.

For n even, $\sin x \approx x$, and for n odd, $\sin x \approx -x$. Hence, we will have four different types of equilibrium points: (1) m even and n even, (2) m even and n odd, (3) m odd and n even and (4) m odd and n odd. When performing the linearization we see that there are only two kinds here. One kind when one of them is even and the other one is odd, and the second kind when either both are even or both are odd.

Linearization about $(m\pi, n\pi)$ when they are both even or odd gives $\dot{x} = \pm y$ and $\dot{y} = \mp x$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

We find that the solution will be closed circles,

$$x^2 + y^2 = C,$$

where C is a constant, and the equilibrium point for the linearized system is a centre. In the nonlinear system we can integrate to find

$$\cos y + \cos x = C$$

where C is a constant. This equation is symmetric about both the x and y axis, which means that we cannot have spirals in the phase diagram, i.e it is a centre.

Linearization about $(m\pi, n\pi)$ where one is even and the other one odd gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}.$$

This has solution $y = \pm x$ der

$$x^2 - y^2 = C$$

where C is a constant. Hence, the equilibrium point is a saddle point.

The phase diagram is given as a network of these types of equilibrium points. See figure 3 for a sketch of the phase diagram.

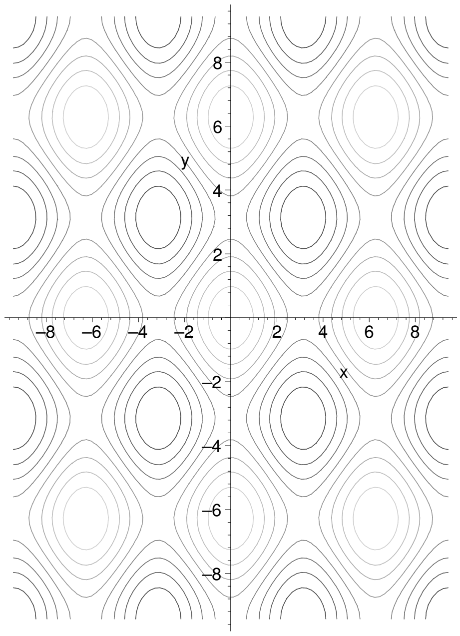

Figure 3: Phase diagram of $\dot{x} = \sin y$, $\dot{y} = -\sin x$

$\boxed{\text{Ex 2013.2}}$ Given H, one can find a dynamical system with H as its Hamiltonian as follows:

$$f(x, y) := \begin{cases} \dot{x} &= H_y = -\cos x \sin y \\ \dot{y} &= -H_x = \sin x \cos y \end{cases}.$$

To show that $(0, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$ are equilibrium points, we simply plug those values into f:

$$f(0, 0) = (-\cos 0 \sin 0, \sin 0 \cos 0) = (0, 0),$$

and

$$f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \left(-\cos\frac{\pi}{2}\sin\frac{\pi}{2}, \sin\frac{\pi}{2}\cos\frac{\pi}{2}\right) = (0, 0),$$

so both points are equilibrium points.

To classify these, we use the second derivative test.

$$D^2H = \begin{bmatrix} -\cos x \cos y & \sin x \sin y \\ \sin x \sin y & -\cos x \cos y \end{bmatrix}$$

Evaluating at each of the points gives

$$D^2H(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

with determinant $\det D^2H(0,0) = 1$, so $(0,0)$ is a center; and

$$D^2H\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with determinant $\det D^2H\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -1$, so $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a saddle.