TMA 4275 Lifetime analysis
Exercise 5 - solution

Problem 1

a) MINITAB output:

Time	Cum Time	Tot Time	i/n	TTT
271	271	2710	0.1	0.25116
320	591	3151	0.2	0.29203
629	1220	5623	0.3	0.52113
706	1926	6162	0.4	0.57108
777	2703	6588	0.5	0.61057
1182	3885	8613	0.6	0.79824
1463	5348	9737	0.7	0.90241
1603	6951	10157	0.8	0.94133
1484	8435	9919	0.9	0.91928
2355	10790	10790	1.0	1.00000

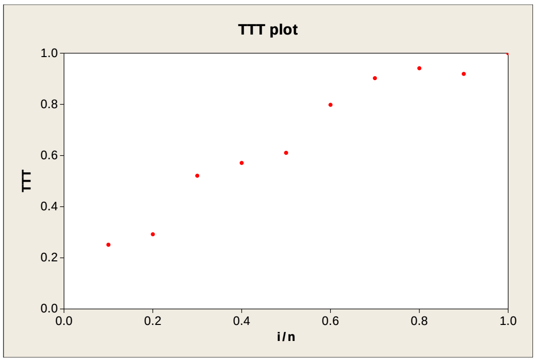

Figure 1: TTT-plot for Problem 1

It is not very clear from the Figure 1 whether the underlying life distribution is exponential or with increasing failure rate. The estimated scaled TTT transform shows some concavity but it also looks like an estimate of a straight line. Therefore, Barlow-Proschan's statistical testing is needed.

b)

$$H_0 : T \text{ is exponentially distributed} \qquad vs. \qquad H_1 : T \text{ has IFR distribution}$$

The value of the test statistic is $W = 5.80723$, which has under the null hypothesis approximately normal distribution with mean $\frac{10-1}{2} = 4.5$ and variance $\frac{10-1}{12} = 0.75$, i.e. the value of the transformed statistic is $Z = 1.50946$. Since $1.50946 < 1.645 = z_\alpha$, the null hypotheses is not rejected.

c) Using the formula derived in the materials from lectures

$$\hat{\theta} = \frac{\sum_{i=1}^{10} T_i}{10} = 1079$$

The 95% confidence intervals are given by $\hat{\theta} \pm 1.96\widehat{SD(\hat{\theta})}$ where $\widehat{SD(\hat{\theta})} = \frac{\hat{\theta}}{\sqrt{n}}$, i.e. (580.561, 2005.37).

d) The maximum value of the log likelihood function is -79.838 (with use of formulas from the materials from lectures $l(\hat{\theta}) = -n\log(\hat{\theta}) - n$).

e) MINITAB output:

```
                        Standard    95.0% Normal CI
Parameter  Estimate        Error     Lower     Upper
Shape       1.80913     0.455330   1.10467   2.96281
Scale       1216.90      224.323   847.900   1746.49

Log-Likelihood = -77.702

                                   Standard    95.0% Normal CI
                         Estimate     Error     Lower     Upper
Mean(MTTF)                1081.91   195.765   758.880   1542.45
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The estimated MTTF is similar to the one from the exponential model.

f) A so-called goodness-of-fit test for the exponential distribution can be defined as the test of hypotheses (α denotes the shape parameter of the Weibull distribution)

$$H_0 : \alpha = 1 \text{ versus } H_1 : \alpha \neq 1$$

The idea behind this is that exponential distribution is the special case of the Weibull distribution, and that we in fact are testing exponential vs. Weibull. If we accept H_0, we may assume that the data follow an exponential distribution, while a rejection result indicates that the exponential distribution is not appropriate.

From the likelihood theory (see course materials) we have that asymptotically

$$W(1) = 2[l(\hat{\theta}) - l(\hat{\theta}, \alpha = 1)] \sim \chi_1^2$$

if H_0 holds, i.e. if $\alpha = 1$ ($l(\cdot)$ denotes the loglikelihood function). The decision rule for significance level 1% is therefore taken to be

$$\text{Reject } H_0 \text{ if } W > \chi^2_{1,\alpha}.$$

In our case, $W(1) = 4.272$ therefore the exponential model cannot be rejected for $\alpha = 0.025$ but is rejected for $\alpha = 0.05$.

Problem 2

a) Since $\lambda = \frac{1}{\theta}$, using the properties of the MLE gives $\hat{\lambda} = \frac{1}{\hat{\theta}}$ and it is possible to use directly the formulas from the lecture materials. That is, $\hat{\lambda} = 0.025$.

b) The standard 95% confidence intervals can be computed with use of the formulas and approach as in the lecture materials as $\hat{\lambda} \pm 1.96\frac{\hat{\lambda}}{\sqrt{n}} = (0.0109, 0.0391)$. Note, that you will get the same formula for the standard deviation as for the other parametrization.

Following the recipe given by the materials from lectures (or direct transformation of the formulas in these materials) gives $P\left(\frac{\hat{\lambda}}{e^{z\frac{\alpha}{2}/\sqrt{n}}} \leq \lambda \leq \frac{\hat{\lambda}}{e^{-z\frac{\alpha}{2}/\sqrt{n}}}\right) = 1-\alpha$, i.e. (0.0142,0.0440).

Following the recipe given by the materials from lectures leads to the solving of nonlinear equation $2(l(\hat{\lambda}) - n\log(\lambda) + \lambda\sum_{i=1}^{12} T_i) = \chi^2_{1,0.05}$. This equation can be solved numerically, which gives (0.0133,0.0417).

Problem 3

Note, that this problem cannot be solved by MINITAB, since MINITAB needs at least one failure.

Also note, that this problem is easier to solve in the other parametrization, but the solution is made in agreement with the parametrization specified by problem.

a) If the assumed lifetime is exponential with the density $f(t) = \frac{1}{\theta}\exp^{-\frac{t}{\theta}}$, then probability that failure occurs between times and is

$$P(t_1 < T \leq t_2) = \int_{t_1}^{t_2} \frac{1}{\theta}\exp^{-\frac{t}{\theta}} = \exp^{-\frac{t_1}{\theta}} - \exp^{-\frac{t_2}{\theta}}$$

and the likelihood is the (because of iid) $L(\theta) = \left(\exp^{-\frac{t_1}{\theta}} - \exp^{-\frac{t_2}{\theta}}\right)^n$.

By implanting the standard procedures (logarithm, first derivative, putting equal to zero) it is possible to obtain the maximum likelihood estimator, which in this case has the form $\hat{\theta} = \frac{t_1 - t_2}{\ln(t_1) - ln(t_2)}$, i.e. $\hat{\theta} = 2.164$

b) Taking the second derivative of the loglikelihood leads to the expression

$$n\left(-2\,t_1 \mathrm{e}^{-\frac{t_1}{\theta}}\theta^{-3} + {t_1}^2\mathrm{e}^{-\frac{t_1}{\theta}}\theta^{-4} + 2\,t_2\mathrm{e}^{-\frac{t_2}{\theta}}\theta^{-3} - {t_2}^2\mathrm{e}^{-\frac{t_2}{\theta}}\theta^{-4}\right)\left(\mathrm{e}^{-\frac{t_1}{\theta}} - \mathrm{e}^{-\frac{t_2}{\theta}}\right)^{-1} - n\left(t_1\mathrm{e}^{-\frac{t_1}{\theta}}\theta^{-2} - t_2\mathrm{e}^{-\frac{t_2}{\theta}}\theta^{-2}\right)^2\left(\mathrm{e}^{-\frac{t_1}{\theta}} - \mathrm{e}^{-\frac{t_2}{\theta}}\right)^{-2}$$

Taking the negative inversion of this, using the derived estimator and taking the square root gives you the standard deviation $\widehat{SD(\hat{\theta})} = 1.047$.

c) The suitable method is the likelihood method (because of the low number of observations). You can directly follow the recipe given by the course materials. Note, that the resulting expression can be solved only numerically.

Problem 3

a) The loglikelihood is given by

$$\begin{aligned} l(\lambda) &= \log\left(\prod_{i=1}^{n} \lambda^2 T_i e^{-\lambda T_i}\right) \\ &= 2n\log(\lambda) + \sum_{i=1}^{n}\log(T_i) - \lambda\sum_{i=1}^{n} T_i \end{aligned}$$

Solving the score function gives

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} T_i}$$

The important properties of $\hat{\lambda}$ are:

- $\hat{\lambda}$ is unbiased, i.e. $E(\hat{\lambda}) = \lambda$
- $\hat{\lambda}$ is asymptotically normal $\hat{\lambda} \sim N(\lambda, (-\frac{d^2 l}{d\lambda^2})^{-1})$

b) The estimated value is $\hat{\lambda} = 0.111$.

The variance can be computed as the negative inversion of the second derivative of the loglikelihood with use of the derived estimator, which gives

$$\widehat{Var(\hat{\lambda})} = \frac{\hat{\lambda}^2}{2n} = 0.0006.$$

c) Theory shows that $W(\lambda_0) = 2(l(\hat{\lambda}) - l(\lambda_0))$ is χ^2_1 distributed under H_0. H_0 is rejected if $W(0.25) > \chi^2_{1,0.05} = 3.841$ (at the 5% significance level). Since $W(0.25) = 17.56$, the null hypotheses is rejected.

d) Follow the methodology from the Problem 1c). The resulting equation can be solved numerically.