Norwegian University of Science and Technology
Institutt for matematiske fag

Solutions to exercise set 7

$\boxed{1}$ Taking Laplace transforms gives

$$(s^2 - 9)Y(s) - 2s = e^{-s}$$

and hence

$$Y(s) = \frac{1}{3}\left(\frac{3}{s^2 - 9}\right) e^{-s} + 2\left(\frac{s}{s^2 - 9}\right)$$

Inverting using the shifting theorem gives

$$y(t) = \frac{1}{3}u(t-1)\sinh 3(t-1) + 2\cosh 3t$$

$\boxed{2}$ We write

$$y(t) + y(t) * \cosh t = t + e^t$$

Taking Fourier transforms of both sides gives

$$Y(s)(1 + \frac{s}{s^2 - 1}) = \frac{1}{s^2} + \frac{1}{s - 1}$$

Collecting the fractions gives

$$Y(s)\left(\frac{s^2 - 1 + s}{s^2 - 1}\right) = \frac{s - 1 + s^2}{s^2(s - 1)}$$

The terms $s^2 - 1 + s$ cancel from both sides to give

$$Y(s) = \frac{s^2 - 1}{s^2(s - 1)}$$

We can simplify further by noting $s^2 - 1 = (s-1)(s+1)$, to give

$$Y(s) = \frac{s + 1}{s^2} = \frac{1}{s} + \frac{1}{s^2}$$

Inverting then gives the solution $y(t) = 1 + t$

$\boxed{3}$ **a)** Taking Laplace transforms gives

$$(s^2 + 4)Y(s) = R(s)$$

from which we conclude $Y(s) = Q(s)R(s)$, where

$$Q(s) = \frac{1}{s^2 + 4}$$

Since $\mathcal{L}^{-1}(Q) = \frac{1}{2}\sin 2t$, we can invert using the convolution theorem to obtain

$$y(t) = \frac{1}{2}\sin 2t * r(t)$$

b) Now letting $r(t) = 4\cos 2t$ in the above gives

$$y(t) = 2\sin 2t * \cos 2t = 2\int_0^t \sin 2\tau \cos 2(t-\tau)d\tau$$

This can be rewritten using the suggested trigonometric identity; we find

$$y(t) = \int_0^t \sin 2t + \sin(2t - 4\tau)d\tau = t\sin 2t - \frac{1}{4}\left[-\cos(2t-4\tau)\right]_0^t$$

and the terms in the square brackets cancel each other, so we conclude

$$y(t) = t\sin 2t$$

Note that $y \to \infty$ as $t \to \infty$, this is a case of resonance in the system.

$\boxed{4}$ **a)** We take the Laplace transform term-by-term, using the rule that $\mathcal{L}\big(tf(t)\big) = -F'(s)$:

$$\mathcal{L}(ty'') = -s^2\frac{dY}{ds} - 2sY + y(0)$$

$$\mathcal{L}(-2ty') = 2s\frac{dY}{ds} + 2Y$$

We therefore find

$$(2s - s^2)\frac{dY}{ds} + (-2s + 2 + 2)Y + 1 = \frac{2}{s}$$

Rearranging gives

$$s(2-s)\frac{dY}{ds} + 2(2-s)Y = \frac{2-s}{s}$$

and we divide through by $2 - s$ to obtian the desired equation

$$s\frac{dY}{ds} + 2Y = \frac{1}{s}$$

b) We first solve the homogeneous equation

$$s\frac{dY}{ds} + 2Y = 0,$$

trying solutions of the form $Y = s^k$; setting this in gives $(k+2)s^k = 0$, and hence $k = -2$, i.e. s^{-2} solves the homogeneous equation. We look for a particular solution of the form $Y = as^k$, this time we require

$$a(k+2)s^k = s^{-1},$$

which is solved for $k = -1$ and $a = 1$. Adding the two solutions gives

$$Y(s) = \frac{1}{s} + \frac{C}{s^2},$$

for some constant C. Inverting then gives $y(t) = 1 + Ct$; if we wish we can express C in terms of $y'(0)$, i.e.

$$y(t) = 1 + y'(0)t$$

5 **a)** Taking Laplace transforms of the first equation and inserting the initial data gives

$$(s^2+8)Y_1 - 4Y_2 = s + 1 + \frac{11}{s^2+1}$$

Similarly, for the second equation, we find

$$(s^2+8)Y_2 - 4Y_1 = s - 1 - \frac{11}{s^2+1}$$

These two equations can be collected in matrix form to give

$$\begin{pmatrix} s^2+8 & -4 \\ -4 & s^2+8 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} s+1+\frac{11}{s^2+1} \\ s-1-\frac{11}{s^2+1} \end{pmatrix}$$

b) The inversion of the linear system is a long calculation! I used MAPLE to find the expressions

$$Y_1 = \frac{s^3+s^2+s+4}{(s^2+1)(s^2+4)}, \quad Y_2 = \frac{s^3-s^2+s-4}{s^4+5s^2+4}$$

These may be converted using partial fractions to

$$Y_1 = \frac{1}{s^2+1} + \frac{s}{s^2+4}, \quad Y_2 = \frac{-1}{s^2+1} + \frac{s}{s^2+4}.$$

Inverting the above gives the solution

$$y(t) = \begin{pmatrix} \cos 2t + \sin t \\ \cos 2t - \sin t \end{pmatrix}$$