MA0301 ELEMENTARY DISCRETE MATHEMATICS
SPRING 2017

1. Homework Set 8 – Solutions

Exercise 1. *Let* $X := \{1, 2, 3, \ldots, 9, 10\}$*. Decide whether the following sentences are statements (and determine its truth value) or propositional functions (and determine its truth set).*

a) $\forall x \in X\ \exists y \in X\ (x + y < 14)$
b) $\forall y \in X\ (x + y < 14)$
c) $\forall x \in X\ \forall y \in X\ (x + y < 14)$
d) $\exists y \in X\ (x + y < 14)$

Solution 1. a) The sentence is a statement and it is true.
b) This is a propositional function with truth set $\{1, 2, 3\}$.
c) This is a statement and it is false, e.g., choose $x = y = 8$.
d) This is an open statement with truth set X.

Exercise 2. *Which of the following sets are equal?*
$A_1 := \{x \mid x^2 = 4x - 3\}$, $A_2 := \{x \mid x^2 = 3x - 2\}$, $A_3 := \{x \mid x \in \mathbb{N}, x < 3\}$, $A_4 := \{x \mid x \in \mathbb{N}, x \text{ odd}, x < 5\}$, $A_5 := \{1, 2\}$, $A_6 := \{1, 2, 1\}$, $A_7 := \{3, 1\}$, $A_8 := \{1, 1, 3\}$.

Solution 2. $A_1 = A_4 = A_7 = A_8$ and $A_2 = A_3 = A_5 = A_6$

Exercise 3. *Grimaldi's book (5. ed., Exercises 7.3, page 364): solve* **Ex. 5** *Note: study in detail pages 360-364 in Grimaldi's book, in particular, topological sorting on page 360.*

Solution 3. One – of several possible – topological sorting is: $\emptyset < \{1\} < \{2\} < \{3\} < \{1, 2\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}$.

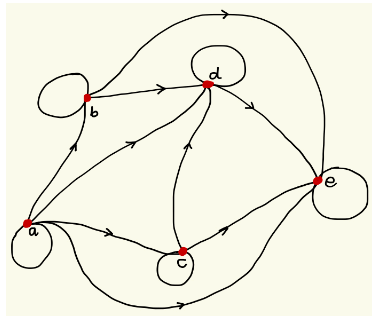
Exercise 4. *Grimaldi's book (5. ed., Exercises 7.3, page 365): solve* **Ex. 6** *Note: study in detail pages 344-347 in Grimaldi's book, in particular, the notion of relation matrix on page 346.*

Solution 4. Relation matrix of R:

$$\begin{pmatrix} & (a) & (b) & (c) & (d) & (e) \\ (a) & 1 & 1 & 1 & 1 & 1 \\ (b) & 0 & 1 & 0 & 1 & 1 \\ (c) & 0 & 0 & 1 & 1 & 1 \\ (d) & 0 & 0 & 0 & 1 & 1 \\ (e) & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Date: March 13, 2017.

Directed graph G:



One possible topological sorting: $a < b < c < d < e$

Exercise 5. *Grimaldi's book (5. ed., Exercises 7.3, page 366): solve* **Ex. 26**

Solution 5. a) 5
b) $n+1$
c) $n+1$
d) 10
e) $1/2(n(n+1))$
f) $1/2(n(n+1))$

Exercise 6. *Grimaldi's book (5. ed., Exercises 1.3, page 24): solve* **Ex. 13**

Solution 6. There are $\frac{7!}{4!2!}$ arrangements of the letters $M\ I\ I\ I\ P\ P\ I$, and there are $\binom{8}{4}$ places where to put the four S – such that that they are non-consecutive.

$$\binom{8}{4}\frac{7!}{4!2!}$$

Exercise 7. *Grimaldi's book (5. ed., Exercises 1.3, page 25): solve* **Ex. 20**

Solution 7. a) $\binom{8}{3} = 56$
b) $\binom{8}{4} = 70$
c)

$$\begin{aligned}\binom{8}{3}+\binom{8}{4}+\binom{8}{5}+\binom{8}{6}+\binom{8}{7}+\binom{8}{8} &= 2^8 - \binom{8}{0} - \binom{8}{1} - \binom{8}{2}\\ &= 256 - 1 - 8 - 28\\ &= 219\end{aligned}$$

Exercise 8. *Grimaldi's book (5. ed., Exercises 1.3, page 25): solve* **Ex. 23**

Solution 8. a) $\binom{12}{9}$
b) $\binom{12}{9}8$
c) $-\binom{12}{9}2^9 3^3$

Exercise 9. *Grimaldi's book (5. ed., Exercises 1.4, page 34): solve* **Ex. 7 a), b), c), f)**

Solution 9. a) $\binom{35}{32}$
b) $\binom{31}{28}$
c) $\binom{11}{8}$
f) $\binom{31}{28}$ - $\binom{6}{3}$

Exercise 10. *Grimaldi's book (5. ed., Exercises 1.4, page 35): solve* **Ex. 12**

Solution 10. a) $\binom{44}{39}$ (Add a variable $x_6 \geq 0$ such that $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 39$.)
b) $\binom{59}{54}$

Exercise 11. *Grimaldi's book (5. ed., Exercises 1.4, page 36): solve* **Ex. 28 a), b)**

Solution 11. a) The strings we want to count are those of the form $1^{x_1}0^{x_2}1^{x_3}0^{x_4}1^{x_5}0^{x_6}$, where

$$\sum_{i=1}^{6} x_i = n, \qquad x_1, x_6 \geq 0,\ x_2, x_3, x_4, x_5 > 0.$$

The number of solutions is $\binom{n+1}{5}$.
b) Find number of integer solutions of

$$\sum_{i=1}^{8} x_i = n, \qquad x_1, x_8 \geq 0,\ x_2, x_3, x_4, x_5, x_6, x_7 > 0,$$

which is $\binom{n+1}{7}$.

2. Classroom Set 8 – Solutions

Exercise 12. *Grimaldi's book (5. ed., Exercises 5.2, page 259): solve* **Ex. 26**

Solution 12. a) $|S_1| = 1$: $f(a) = f(b) = 1,\ f(c) = 2$
b) $|S_2| = 4$
c) $|S_i| = i^2$
d) T_1 is the number of ways of picking two separate elements $f(a) = f(b)$ and $f(c)$ out of S, which has $n + 1$ elements.
e) $f(a) < f(b) < f(c)$: we need three distinct elements from B, and there are $\binom{n+1}{3}$
f) both equal S
g) verify

Exercise 13. *Grimaldi's book (5. ed., Exercises 7.3, page 365): solve* **Ex. 7**

Solution 13. a)

$$\begin{array}{ccc} 4 & & \\ | & & \\ 1 & & 2 \\ | & / & \\ 3 & & \end{array}$$

b) $3 < 2 < 1 < 4$
c) 2

Exercise 14. *Grimaldi's book (5. ed., Exercises 1.3, page 25): solve* **Ex. 25**

Solution 14. a) 12
b) 12
c) -24
d) -216
e) 161280

Exercise 15. *Grimaldi's book (5. ed., Exercises 1.4, page 34): solve* **Ex. 7 d), e)**

Solution 15. d) 1
e) $\binom{43}{40}$

Exercise 16. *Grimaldi's book (5. ed., Exercises 1.4, page 36): solve* **Ex. 28 c)**

Solution 16. c) 2^n is the total number of $0-1$ strings of length n. There are $n+1$ such strings with k 1s followed by $n-k$ 0s, for $k = 0, 1, \ldots, n$. None of these strings has the occurrence of the 01 substring. Therefore there are $2^n - n + 1$ strings that have at least one 01 substring. From the earlier part of this exercise, we know that there are $\binom{n+1}{2k+1}$ strings with k occurrences of exactly k 01 substrings.

Let n be odd: then we can have at most $n/2 - 1/2$ occurrences of 01. The number of strings with exactly $n/2 - 1/2$ occurrences is $\binom{n+1}{n}$. This is the number of solutions of

$$\sum_{i=1}^{n+1} x_i = n, \qquad x_1, x_{n+1} \geq 0,\ x_2, \ldots, x_n > 0.$$

Let n be even: then we can have at most $n/2$ occurrences of 01. The number of strings with exactly $n/2$ occurrences is $\binom{n+1}{n}$. This is the number of solutions of

$$\sum_{i=1}^{n+2} x_i = n, \qquad x_1, x_{n+2} \geq 0,\ x_2, \ldots, x_{n+1} > 0,$$

which is $\binom{n+1}{n+1}$. From this the result follows.

Alternative solution: 2^n is the number of subsets of $\{1, 2, \ldots, n\}$. The right side is the number of subsets of $\{1, 2, \ldots, n+1\}$ with an odd number of elements. A subset $S \subseteq \{1, 2, \ldots, n\}$ corresponds to a subset $S' \subseteq \{1, 2, \ldots, n+1\}$ where $S' = S$ if $|S|$ is odd, and $S' = S \cup \{n+1\}$ if $|S|$ is even, and this correspondence is a bijection between the sets counted on the left side and the sets counted on the right side.