Solutions to exercise set 7

Norwegian University of Science and Technology
Department of Mathematics

$\boxed{1}$ **a)** We obtain the following table for the divided differences:

$$
\begin{array}{c||ccccc}
0 & 1 & & & & \\
 & & 8 & & & \\
1 & 9 & & 3 & & \\
 & & 14 & & 1 & \\
2 & 23 & & 7 & & 0 \\
 & & 35 & & 1 & \\
4 & 93 & & 12 & & \\
 & & 83 & & & \\
6 & 259 & & & &
\end{array}
$$

The interpolation polynomial (in nested Newton form) is therefore

$$p(x) = 1 + x(8 + (x-1)(3 + (x-2) + 0 \cdot (x-4))).$$

(Obviously, one can omit the term $0 \cdot (x-4)$ in that expression). Evaluating this polynomial, we see that $p(3) = 49$.

b) For the interpolation polynomial of degree three through the first four interpolation points, we can simply reuse the previously computed table of divided differences and read off the result from there using only the first four columns. In this particular case, we obtain the same interpolation polynomial p as before (therefore, obviously, also its value at the point $x = 3$ is the same as above).

$\boxed{2}$ To start we exploit the fact that divided differences are invariant under permutations of the nodes. Thus, sorting the nodes in increasing order, we have:

$$f[x_3] = 11,\ f[x_2, x_3] = 5,\ f[x_0, x_1, x_2] = -2, \text{ and } f[x_0, x_1, x_2, x_3] = 0.6.$$

We seek $f(-1) = f(x_1) = f[x_1]$. From the recursive definition of divided differences:

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

which rearranged to isolate the unknown $f[x_1, x_2, x_3]$ becomes

$$f[x_1, x_2, x_3] = f[x_0, x_1, x_2] + (x_3 - x_0) f[x_0, x_1, x_2, x_3] = -2 + (4 - (-2))0.6 = 1.6.$$

Similarly we find $f[x_1, x_2]$ as

$$f[x_1, x_2] = f[x_2, x_3] - (x_3 - x_1) f[x_1, x_2, x_3] = 5 - (4 - (-1))1.6 = -3,$$

$f[x_2]$ as

$$f[x_2] = f[x_3] - (x_3 - x_2) f[x_2, x_3] = 11 - (4 - 1)5 = -4,$$

and finally $f[x_1] = f(-1)$ as

$$f[x_1] = f[x_2] - (x_2 - x_1)f[x_1, x_2] = -4 - (1 - (-1))(-3) = 2.$$

$\boxed{3}$ **a)** For $f(x) = x \ln x$ we first calculate $f'(x) = \ln x + 1$, $f''(x) = 1/x$ and $f^{(3)}(x) = -1/x^2$. Since $|f^{(3)}(x)| = 1/x^2$ is a strictly decreasing function of x for positive x we find a bound M on the interval $(8.3, 8.7)$ for $n = 2$ by $M = 1/8.3^2$. Employing the First Interpolation Error Theorem in the textbook with $n = 2$ and $x = 8.4$ on this interval gives the error bound:

$$\begin{aligned}|f(8.4) - p_2(8.4)| &\leq \frac{1}{(2+1)!}\frac{1}{8.3^2}|(8.4 - 8.3)||(8.4 - 8.6)||(8.4 - 8.7)| \\ &= \frac{0.1 \cdot 0.2 \cdot 0.3}{6 \cdot 8.3^2} \leq 1.46 \times 10^{-5}.\end{aligned}$$

The true error is 1.38×10^{-5} correct to the digits given. As expected this is lower than the error bound.

b) For $f(x) = x^4 - x^3 + x^2 - x + 1$ we again calculate $f'(x) = 4x^3 - 3x^2 + 2x - 1$, $f''(x) = 12x^2 - 6x + 2$ and $f^{(3)}(x) = 24x - 6$. Since $|f^{(3)}(x)| = |24x - 6|$ is easily observed to be a strictly decreasing function of x for negative x, we find a bound M on the interval $(-0.5, 0)$ for $n = 2$ by $M = |f^{(3)}(-0.5)| = 18$. Employing the First Interpolation Error Theorem in the textbook with $n = 2$ and $x = -1/3$ on this interval gives the error bound:

$$\begin{aligned}|f(-1/3) - p_2(-1/3)| &\leq \frac{1}{(2+1)!}18|(-1/3 + 1/2)||(-1/3 + 1/4)||(-1/3 - 0)| \\ &= \frac{18}{6 \cdot 6 \cdot 12 \cdot 3} = \frac{1}{72} \leq 1.4 \times 10^{-2}.\end{aligned}$$

The true error is 9.6×10^{-3} correct to the digits given. As expected this is lower than the error bound.

$\boxed{4}$ **a)** It is trivial to see that the formula is valid for $m = 0$. Assume now it is correct for some $m \geq 0$. Then

$$\begin{aligned}f^{(m+1)}(x) &= \frac{d}{dx}f^{(m)}(x) = 2^{m/2}\left[e^x \sin(x + m\pi/4) + e^x \cos(x + m\pi/4)\right] \\ &= 2^{m/2}e^x\left[\sin(x + m\pi/4) + \cos(x + m\pi/4)\right] \\ &= 2^{m/2}e^x\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin(x + m\pi/4) + \frac{1}{\sqrt{2}}\cos(x + m\pi/4)\right] \\ &= 2^{m/2}e^x 2^{1/2}\left[\cos(\pi/4)\sin(x + m\pi/4) + \sin(\pi/4)\cos(x + m\pi/4)\right] \\ &= 2^{(m+1)/2}e^x \sin((x + m\pi/4) + \pi/4) \\ &= 2^{(m+1)/2}e^x \sin(x + (m+1)\pi/4).\end{aligned}$$

This shows that the formula holds for $m + 1$ as well, assuming it holds for m. The proof by induction is then complete, and the stated formula is seen to hold for all $m \geq 0$

b) Our function f has continous derivatives of any order on the entire real line and on the interval $[-4, 2]$ it satisfies

$$|f^{(n+1)}(x)| = 2^{(n+1)/2}e^x|\sin(x + (n+1)\pi/4)| \leq 2^{(n+1)/2}e^2 = M,$$

where we have used that e^x is a strictly increasing positiv function and that $|\sin(x)| \leq 1$ for all x. The Second Interpolation Error Theorem is applicable and we have for $x \in [-4, 2]$

$$\begin{aligned} |f(x) - p(x)| &\leq \frac{1}{4(n+1)}Mh^{n+1} = \frac{1}{4(n+1)}2^{(n+1)/2}e^2\left(\frac{6}{n}\right)^{n+1} \\ &= \frac{e^2}{4(n+1)}\left(\frac{6\sqrt{2}}{n}\right)^{n+1}. \end{aligned}$$

By differentiating, this expression is readliy seen to decrease with increasing positive n. Using trial and error we find that the expression has the value 1.27×10^{-5} for $n = 15$ and 2.26×10^{-6} for $n = 16$. Thus $n = 16$ is required to guarantee an accuracy of 10^{-5}.

c) Using MATLAB we have plotted below $|f(x) - p(x)|$ over the interval, where $p(x)$ is the interpolating polynomial from $b)$ with $n = 16$.

We observe that the error stays several orders of magnitude lower than our requirement of 10^{-5}. This is to be expected. Recall that we were in fact guaranteed an error less than 2.26×10^{-6}, and the theorem only gives fairly loose upper bounds.

Also observe that the errors near the endpoints dominate the errors towards the middle of the interval. This is typical, and illustrates why Chebyshev nodes, which become more clustered as we approach the endpoints, usually gives siginificantly smaller errors for the same number of nodes.