Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2017

Solutions E27

E27 Given the system

$$\dot{x} = y^2 - x^2,$$
$$\dot{y} = 1 + 2xy.$$

a) Find and classify all equilibrium points of the system.

We see that $\dot{x} = 0$ gives $y = \pm x$. Inserting this into the equation $\dot{y} = 0$ gives $1 \pm 2x^2 = 0$. We need to choose the negative sign and get $x = -y = \pm\frac{1}{\sqrt{2}}$. The matrix of linearization is given by

$$J = \begin{bmatrix} -2x & 2y \\ 2y & 2x \end{bmatrix}.$$

The matrix is symmetric, so it has real eigenvalues. We have $\det J = -4(x^2 + y^2) < 0$ so the eigenvalues are of opposite sign, so $x_0 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is a saddle point for the linearization. Since $\dot{x} = J(x - x_0) + O(|x - x_0|^2)$, it is also a saddle point in the original system.

The system is unchanged if we replace x with $-x$, y with $-y$ and t with $-t$. Hence, the equilibrium point $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is also a saddle point.

b) The index for a saddle point is -1. For a closed path we need the sum of the indices to be equal to one. We can also use Bendixson's negative criterion:

$$\frac{\partial(y^2 - x^2)}{\partial x} + \frac{\partial(1 + 2xy)}{\partial y} = -2x + 2x = 0.$$

c) Show that the given system is Hamiltonian and find a Hamiltonian function for the system. Show that the phase paths through the origin satisfies

$$x = \frac{2y^3}{3\left(1 + \sqrt{1 + \frac{4}{3}y^4}\right)}$$

and sketch the phase diagram.

We write the system of the form

$$\dot{x} = \frac{\partial H}{\partial y},$$
$$\dot{y} = -\frac{\partial H}{\partial x}$$

where $H(x, y)$ satisfies

$$\frac{\partial H}{\partial y} = y^2 - x^2 \quad \text{og} \quad -\frac{\partial H}{\partial x} = 1 + 2xy.$$

The first equation can be integrated to give $H(x, y) = \frac{1}{3}y^3 - x^2y + F(x)$ where $F(x)$ is an arbitrary function. Inserted into the second equation gives $2xy - F'(x) = 1 + 2xy$ which means that $F'(x) = -1$. A Hamiltonian function for the system is

$$H(x, y) = \frac{1}{3}y^3 - x^2y - x.$$

Since $H(x, y)$ is invariant along the phase paths, we have that the path through the origin satisfies $H(x, y) = H(0, 0) = 0$. We solve the equation $yx^2 + x - \frac{1}{3}y^3 = 0$ with respect to x and find

$$x = \frac{-1 \pm \sqrt{1 + \frac{4}{3}y^4}}{2y}.$$

If this path is to pass through the origin we need to choose the positive sign. Hence, we have

$$x = \frac{-1 + \sqrt{1 + \frac{4}{3}y^4}}{2y} \cdot \frac{1 + \sqrt{1 + \frac{4}{3}y^4}}{1 + \sqrt{1 + \frac{4}{3}y^4}} = \frac{2y^3}{3\left(1 + \sqrt{1 + \frac{4}{3}y^4}\right)}.$$

To give a sketch, try to first find the direction of the arrows when $\dot{x} = \dot{y} = 0$. See figure 1 for a sketch of the phase diagram.

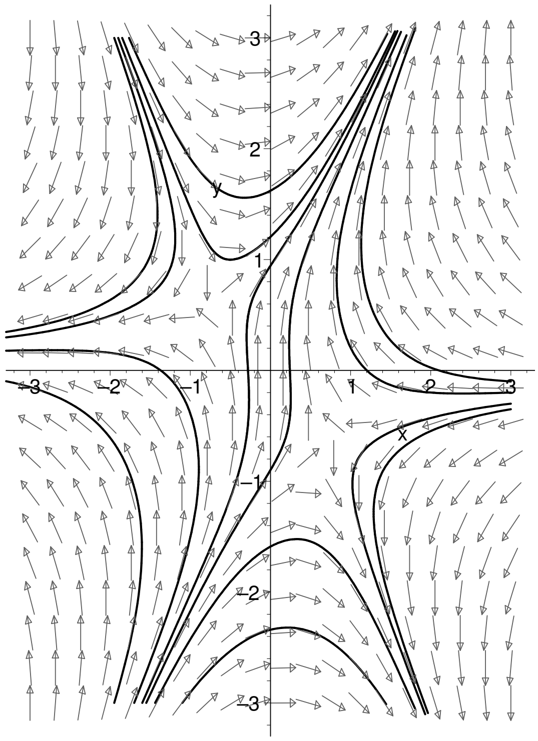

Figure 1: Phase diagram of $\dot{x} = y^2 - x^2$, $\dot{y} = 1 + 2xy$