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Exercise set 7: Solutions

1 Let $\mathcal{H}$ be a Hilbert space and (x_n) a sequence in $\mathcal{H}$ converging weakly to x. Show that the following statements are equivalent:

1. x_n converges strongly to x.
2. $\|x_n\|$ converges to $\|x\|$.

Solution. x_n converges strongly to x if and only if $\|x - x_n\| \to 0$ as $n \to \infty$. Of course, this is equivalent to $\|x - x_n\|^2 = \langle x - x_n, x - x_n \rangle \to 0$ as $n \to \infty$. Let us expand this expression:

$$\langle x - x_n, x - x_n \rangle = \|x\|^2 - \langle x, x_n \rangle - \langle x_n, x \rangle + \|x_n\|^2. \tag{1}$$

Since $x_n \to x$ weakly, we have that $\langle x, x_n \rangle \to \|x\|^2$ and $\langle x_n, x \rangle \to \|x\|^2$ [1] It should now be clear that the expression in equation (1) converges to 0 if and only if $\|x_n\|$ converges to $\|x\|$. Hence $x_n \to x$ if and only if $\|x_n\| \to \|x\|$.

2 Let A be a subset of a Banach space X.

a) Show that if A is relatively sequentially compact, then A is bounded.

Solution. Assume that A is relatively sequentially compact. If A were unbounded, we could find a sequence (x_n) in A with $\|x_n\| > n$ for every $n \in \mathbb{N}$. Since A is relatively sequentially compact, (x_n) has a subsequence converging to some $x \in \overline{A}$, [2] and this convergent subsequence must in particular be bounded. But this is clearly impossible, since we constructed (x_n) such that any subsequence of (x_n) is unbounded.

3 Show the following statement about sets A in a Banach space X.

a) A bounded set A is relatively weakly compact if and only if the weak-* closure of A in X^{**} is in X.

Solution. We will use three facts, all of which may be found in the book by Bowers and Kalton.

[1] Here we use Riesz' representation theorem to identify the dual space of $\mathcal{H}$ with $\mathcal{H}$.
[2] $\overline{A}$ denotes the closure of A in the norm topology.

1. The Banach-Alaoglu theorem says that the closed unit ball in X is weak*-compact. In fact, one can easily show that this implies that any closed ball centered at 0 is weak* compact in X.

2. The weak topology of A agrees with the weak* topology of $\iota(A)$, where $\iota : X \to X^{**}$ is the natural embedding of X into its double dual.

3. The embedding $\iota : X \to X^{**}$ is continuous from the weak topology to the weak* topology.

Assume first that the weak* closure of A in X^{**} is in X. By Banach-Alaoglu, the weak*-closure of A is weak* compact[3]. However, by fact (2), this is the same as saying that the weak closure is weakly compact.

If A is weakly relatively compact, let $\tilde{A}$ denote its weak closure. We wish to show that the weak*-closure of $\iota(A)$ is a subset of $\iota(X)$, where $\iota : X \to X^{**}$ is the natural embedding (This is just a rephrasing of the statement we wish to prove). Since $\tilde{A}$ is assumed to be weakly compact, $\iota(\tilde{A})$ must be weak*-compact by fact (3). But the weak*-closure of $\iota(A)$ must then be a subset of $\iota(\tilde{A})$, since $\iota(\tilde{A})$ is a weak* closed subset containing $\iota(A)$. In particular $\iota(A)$ is a subset of $\iota(X)$, which is what we wanted to prove.

$\boxed{4}$ Give an example of a set Y in a normed space X that is closed but not sequentially weakly closed.

Solution. We will find some inspiration in example 5.28 in the book. This example gives a sequence converging weakly, but not strongly.
Let (e_n) be the standard basis for ℓ^2, given by $e_1 = (1, 0, 0, 0....)$, $e_2 = (0, 1, 0, 0, ...)$ etc . Then $\{e_n : n \in \mathbb{N}\}$ is closed in the norm topology(Why?). However, as example 5.28 in the book shows, $e_n \to 0$ weakly. To see this, let $x^* = (x_n) \in (\ell^2)^* = \ell^2$. Then clearly $x^*(e_n) = x_n \to 0$, hence $e_n \to 0$ weakly. It follows that $\{e_n : n \in \mathbb{N}\}$ is not sequentially weakly closed.

[3]Why? Since A is bounded, it is contained in a closed ball $B_r = \{x \in X : \|x\| \leq r\}$, and B_r is weak* compact by Banach-Alaoglu. The weak* closure of A is closed subset of B_r, and therefore itself weak*-compact.