Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4165 Differential Equations and Dynamical Systems Spring 2017

Exercise set 3

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J. S.: 2.1 (i)-(iv), (vi)

These exercises will be presented / discussed in the exercise class:

$\boxed{\text{E7}}$ The velocity v of a relativistic electron in a constant electric field in one dimension satisfies

$$\begin{aligned} \dot{v} &= (1-\varepsilon^2 v^2)^{\frac{3}{2}} \\ v(0) &= v_0, \end{aligned} \tag{1}$$

where $|\varepsilon| < 1$ and $|v_0| < 1$.
Aim: Show that this initial value problem cannot have more than one solution and show that the solution exists for $|t| < 1$.

a) Show that the function $f(v) = (1-\varepsilon^2 v^2)^{\frac{3}{2}}$ is Lipschitz continuous with Lipschitz constant $L \leq 3\varepsilon$ on $\{v \in \mathbb{R} \mid |v| < \frac{1}{\varepsilon}\}$.

b) Show that a) implies that the solution to (1) is unique.

c) Use a comparison argument to show that the solution to (1) cannot leave $\{v \in \mathbb{R} \mid |v| < \frac{1}{\varepsilon}\}$ within finite time.
Hint: Have a look at the initial value problem for $z = (1-\varepsilon^2 v^2)$.

$\boxed{\text{E8}}$ Given the initial value problem

$$\ddot{x} + \varepsilon\dot{x} + \sin(x) = 0, \quad x(0) = x_0, \quad \dot{x}(0) = v, \tag{2}$$

where x_0 and v denote some real constants and $0 < \varepsilon < 1$. Denote by $x^0(t)$ and $x^\varepsilon(t)$ the solutions of (2) with $\varepsilon = 0$ and $\varepsilon \neq 0$, respectively. Aim: Show that there exists a function $K(t) \geq 0$ such that

$$|x^0(t) - x^\varepsilon(t)| \leq K(t)\varepsilon \tag{3}$$

for all $t \geq 0$ and $0 < \varepsilon < 1$.

a) Introduce $y^\varepsilon(t) = \dot{x}^\varepsilon(t)$ and rewrite (2) as a system of differential equations of first order.

b) Show that there exists an increasing function $M(t)$ such that $|y^0(t)| \leq M(T)$ for all $0 \leq t \leq T$.

c) Let $z(t) = (x^0(t) - x^\varepsilon(t))^2 + (y^0(t) - y^\varepsilon(t))^2$ and show that $z(t)$ satisfies the differential inequality

$$\dot{z}(t) \leq 4z(t) + 2\varepsilon|y^0(t)|\sqrt{z(t)} \leq 5z(t) + \epsilon^2(y^0(t))^2. \tag{4}$$

d) Finally show that (3) follows from (4).

$\boxed{\text{E9}}$ Aim: Show that the phase plane for the equation $\ddot{x} - \varepsilon x\dot{x} + x = 0$ $(\varepsilon > 0)$ has a centre at the origin, by finding the equation of the phase paths.

a) Find the equilibrium points.

b) Investigate and sketch the function $g(y) = \ln(1 - \varepsilon y) - (1 - \varepsilon y)$.

c) Sketch the phase portrait in a neighbourhood of the origin with the help of b) to determine wether or not the origin is a center.