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MA2501 Numerical Methods
Spring 2017

Solutions to exercise set 8

$\boxed{1}$ The trapezoid rule yields the following approximations of the integral:

$$T(f,0,1) = \Big(\frac{1}{2}\sin(0) + \frac{1}{2}\sin(1)\Big) \approx 0.4207,$$

$$T(f,0,1,1/2) = \frac{1}{2}\Big(\frac{1}{2}\sin(0) + \sin(1/4) + \frac{1}{2}\sin(1)\Big) \approx 0.3341,$$

$$T(f,0,1,1/4) = \frac{1}{4}\Big(\frac{1}{2}\sin(0) + \sin(1/16) + \sin(1/4) + \sin(9/16) + \frac{1}{2}\sin(1)\Big) \approx 0.3160.$$

For the approximation error we use the estimate

$$|T(f,a,b,h) - \int_0^1 f(x)\,dx| \le \frac{b-a}{12}h^2 \sup_{x\in[a,b]} |f''(x)|.$$

In our case,

$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2),$$

and thus (since cos and sin are both bounded by 1)

$$\sup_{x\in[0,1]} |f''(x)| \le 6.$$

Hence we obtain

$$|T(f,a,b,h) - \int_0^1 f(x)\,dx| \le \frac{h^2}{2}.$$

For $h = 1$, $h = 1/2$, and $h = 1/4$ the respective error bounds are therefore $1/2$, $1/8$, and $1/32$.

Note: In this case better estimates can be obtained quite easily: The function f'' is decreasing on the interval $[0,1]$ because both $\cos(x^2)$ and $-x^2\sin(x^2)$ are. Thus we can estimate

$$\sup_{x\in[0,1]} |f''(x)| \le \max\{|f''(0)|, |f''(1)|\} \approx 2.285.$$

From this we would obtain the estimate

$$|T(f,a,b,h) - \int_0^1 f(x)\,dx| \le \frac{2.285}{12}h^2$$

and hence the error estimates 0.1904, 0.0476, and 0.0119.

$\boxed{2}$ **a)** For the trapezoid rule we obtain

$$T(f,0,2,1) = \Big(\frac{1}{2}f(0) + f(1) + \frac{1}{2}f(2)\Big) \approx 0.3179.$$

and

$$T(f,0,2,1/2) = \Big(\frac{1}{2}f(0) + f(1/2) + f(1) + f(3/2) + \frac{1}{2}f(2)\Big) \approx 0.3982.$$

b) The Simpson rule yields

$$S(f,0,2,1) = \frac{1}{6}\big(f(0) + 4f(1/2) + 2f(1) + 4f(3/2) + f(2)\big) \approx 0.4250$$

and

$$\begin{aligned} S(f,0,2,1/2) = \frac{1}{12}\big(&f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + 2f(1) \\ &+ 4f(5/4) + 2f(3/2) + 4f(7/4) + f(2)\big) \approx 0.4213. \end{aligned}$$

$\boxed{3}$ The results of the numerical approximations with the trapezoid and the Simpson method are listed in the folowing table (all rounded to 7 significant digits):

$f(x) = x^{7/2}$:

h	Trapezoid	Simpson
1	0.5	0.2255922
0.5	0.2941942	0.2224537
0.25	0.2403888	0.2222377

$f(x) = x^{5/2}$:

h	Trapezoid	Simpson
1	0.5	0.2845178
0.5	0.3383883	0.2855925
0.25	0.2987915	0.2857024

a) The exact values of the integrals are:

$$\int_0^1 x^{7/2}\,dx = \frac{2}{9} \approx 0.2222222,$$
$$\int_0^1 x^{5/2}\,dx = \frac{2}{7} \approx 0.2857143.$$

Thus we obtain the following approximate errors err_h:

$f(x) = x^{7/2}$:

	Trapezoid	Simpson
err_1	0.2777778	0.0033701
$\mathrm{err}_{0.5}$	0.0719720	0.0002315
$\mathrm{err}_{0.25}$	0.0181666	0.0000154

$f(x) = x^{5/2}$:

	Trapezoid	Simpson
err_1	0.2142857	0.0011965
$\mathrm{err}_{0.5}$	0.0526741	0.0001217
$\mathrm{err}_{0.25}$	0.0130772	0.0000118

In order to estimate the convergence rates, we compute next the ratios of consecutive approximations. For these we get

$f(x) = x^{7/2}$:

	Trapezoid	Simpson
$\frac{\mathrm{err}_1}{\mathrm{err}_{0.5}}$	3.860	14.558
$\frac{\mathrm{err}_{0.5}}{\mathrm{err}_{0.25}}$	3.962	15.003

$f(x) = x^{5/2}$:

	Trapezoid	Simpson
$\frac{\mathrm{err}_1}{\mathrm{err}_{0.5}}$	4.068	9.828
$\frac{\mathrm{err}_{0.5}}{\mathrm{err}_{0.25}}$	4.028	10.312

In case a method has approximation order $\mathcal{O}(h^p)$ for some $p > 0$, we would expect that

$$\frac{\mathrm{err}_h}{\mathrm{err}_{h/2}} \approx \frac{Ch^p}{C(h/2)^p} = 2^p.$$

Consequently, the number

$$\hat{p}_h := \log_2\Big(\frac{\mathrm{err}_h}{\mathrm{err}_{h/2}}\Big)$$

should give us an estimate of the convergence order. We obtain

$f(x) = x^{7/2}$:

	Trapezoid	Simpson
$\hat{p}_1$	1.949	3.864
$\hat{p}_{0.5}$	1.986	3.907

$f(x) = x^{5/2}$:

	Trapezoid	Simpson
$\hat{p}_1$	2.024	3.297
$\hat{p}_{0.5}$	2.010	3.366

These results indicate that, most probably, the trapezoid rule has for both functions a convergence order of $\mathcal{O}(h^2)$. The convergence rates for the Simpson rule are not that easy to estimate. In case of the function $f(x) = x^{7/2}$ one can reasonably assume that the convergence order is $\mathcal{O}(h^4)$, although it would be also possible that it is slightly smaller; for the function $f(x) = x^{5/2}$, this convergence rate seems unlikely, and one would rather estimate a rate of $\mathcal{O}(h^{3.3})$ or $\mathcal{O}(h^{3.4})$.

b) Usually, one would expect for the trapezoid method a convergence rate of $\mathcal{O}(h^2)$, and the computations confirm this expectation. The convergence rate for the Simpson method is usually $\mathcal{O}(h^4)$, but the numerical computations indicate that the rate is maybe not reached for the function $f(x) = x^{7/2}$ and almost surely not reached for the function $f(x) = x^{5/2}$. The problem is that the convergence rates result for the Simpson rule requires the function f to be four times differentiable. Both functions in this example, however, are not, and the second function is in fact only twice differentiable. (This is also the reasons why the trapezoid rule works just as expected.)

4 The weights for this Newton–Cotes rule have to satisfy the equations

$$\begin{aligned}
2 &= \int_{-1}^{1} 1\,dx = 2\bigl(c_0 + c_1 + c_2 + c_3 + c_4\bigr),\\
0 &= \int_{-1}^{1} x\,dx = 2\bigl(-c_0 - \frac{1}{2}c_1 + \frac{1}{2}c_3 + c_4\bigr),\\
\frac{2}{3} &= \int_{-1}^{1} x^2\,dx = 2\bigl(c_0 + \frac{1}{4}c_1 + \frac{1}{4}c_3 + c_4\bigr),\\
0 &= \int_{-1}^{1} x^3\,dx = 2\bigl(-c_0 - \frac{1}{8}c_1 + \frac{1}{8}c_3 + c_4\bigr),\\
\frac{2}{5} &= \int_{-1}^{1} x^4\,dx = 2\bigl(c_0 + \frac{1}{16}c_1 + \frac{1}{16}c_3 + c_4\bigr).
\end{aligned}$$

From the second and the fourth equation (or the symmetry of the nodes) we obtain that $c_0 = c_4$ and $c_1 = c_3$. Thus the third and the fifth equation simplify to

$$\begin{aligned}
\frac{1}{3} &= 2c_0 + \frac{1}{2}c_1,\\
\frac{1}{5} &= 2c_0 + \frac{1}{8}c_1.
\end{aligned}$$

From this we immediately obtain that

$$c_1 = c_3 = \frac{16}{45}$$

and thus

$$c_0 = c_4 = \frac{1}{6} - \frac{1}{4}c_1 = \frac{7}{90}.$$

Finally,

$$c_2 = 1 - c_0 - c_1 - c_3 - c_4 = \frac{2}{15}.$$

Thus the closed Newton–Cotes rule with $n = 4$

$$Q(f,a,b) = (b-a)\Big(\frac{7}{90}f(y_0) + \frac{16}{45}f(y_1) + \frac{2}{15}f(y_2) + \frac{16}{45}f(y_3) + \frac{7}{90}f(y_4)\Big).$$

$\boxed{5}$ The rule has to satisfy the equations

$$\begin{aligned} 2 &= \int_{-1}^{1} 1\,dx = 2(c_0 + c_1 + c_2), \\ 0 &= \int_{-1}^{1} x\,dx = 2(-\alpha c_0 + \alpha c_2), \\ \frac{2}{3} &= \int_{-1}^{1} x^2\,dx = 2(\alpha^2 c_0 + \alpha^2 c_2). \end{aligned}$$

The second equation implies that $c_0 = c_2$; the third equation implies

$$c_0 = c_2 = \frac{1}{6\alpha^2}.$$

The first equation now implies that

$$c_1 = 1 - \frac{1}{3\alpha^2} = \frac{3\alpha^2 - 1}{3\alpha^2}.$$

For finding the actual degree of precision, we check whether also polynomials of higher degree are integrated exactly. We note first that

$$Q(x^3,-1,1) = 2\Big(-\frac{1}{6\alpha^2}\alpha^3 + 0 + \frac{1}{6\alpha^2}\alpha^3\Big) = 0 \int_{-1}^{1} x^3\,dx,$$

which shows that the (composite) rules have degree of precision at least 3. Next we note that

$$\int_{-1}^{1} x^4\,dx = \frac{2}{5}$$

and

$$Q(x^4,-1,1) = 2\Big(\frac{1}{6\alpha^2}\alpha^4 + \frac{1}{6\alpha^2}\alpha^4\Big) = \frac{2}{3}\alpha^2.$$

Thus the function x^4 is integrated exactly, if and only if $\alpha^2 = 3/5$ or

$$\alpha = \sqrt{\frac{3}{5}}.$$

Thus the degree of precision for $\alpha^2 \neq 3/5$ is 3. For the case $\alpha^2 = 3/5$

$$Q(x^5,-1,1) = 2\Big(-\frac{1}{6\alpha^2}\alpha^5 + \frac{1}{6\alpha^2}\alpha^5\Big) = 0 = \int_{-1}^{1} x^5\,dx,$$

but

$$Q(x^6, -1, 1) = 2\Big(\frac{1}{6\alpha^2}\alpha^6 + \frac{1}{6\alpha^2}\alpha^6\Big) = \frac{2}{3}\alpha^4 = \frac{2}{3}\frac{9}{25} = \frac{6}{25}$$

and

$$\int_{-1}^{1} x^6\, dx = \frac{2}{7}.$$

This shows that the degree of precision for $\alpha^2 = 3/5$ is 5.

To summarize, we have a degree of precision 3 for $\alpha^2 \neq 3/5$, and 5 for $\alpha^2 = 3/5$.

6 A function that computes the Romberg Array in MATLAB is implemented in the function `romint`.

For the integral $\int_0^2 e^{-x^2}\sin(x)\,dx$ the algorithm converges for $n = 6$ to the approximation $R(6, 6) = 0.4211642$. This matches the true solution to the digits given. The actual error is 1.67×10^{-12}, well below 10^{-6}.

7 We have $b - a = 1$ and can thus compute $R(0, 0)$ since

$$R(0, 0) = \frac{1}{2}(f(2) + f(3)) = 0.44065.$$

In addition

$$R(1, 0) = \frac{1}{2}(R(0, 0) + f(2.5)).$$

From the Richardson extrapolation formula and the above relation

$$\begin{aligned}
R(1,1) &= \frac{4}{3}R(1,0) - \frac{1}{3}R(0,0) = \frac{1}{3}R(0,0) + \frac{2}{3}f(2.5)\\
R(2,1) &= \frac{4}{3}R(2,0) - \frac{1}{3}R(1,0) = \frac{4}{3}R(2,0) - \frac{1}{6}R(0,0) - \frac{1}{6}f(2.5)\\
R(2,2) &= \frac{16}{15}R(2,1) - \frac{1}{15}R(1,1)\\
&= \frac{16}{15}\left(\frac{4}{3}R(2,0) - \frac{1}{6}R(0,0) - \frac{1}{6}f(2.5)\right) - \frac{1}{15}\left(\frac{1}{3}R(0,0) + \frac{2}{3}f(2.5)\right)\\
&= \frac{64}{45}R(2,0) - \frac{1}{5}R(0,0) - \frac{2}{9}f(2.5)
\end{aligned}$$

Isolating $f(2.5)$ in the last equation yields

$$f(2.5) = \frac{64R(2,0) - 9R(0,0) - 45R(2,2)}{10} \approx 0.43459$$