1 Let X and Y be Banach spaces and $(T_n)_{n\in\mathbb{N}}$ a sequence of bounded linear operators between X and Y.
Then $(T_n x)_{n\in\mathbb{N}}$ converges for all $x \in X$ if and only if the following two conditions hold:

a) $(T_n x)_{n\in\mathbb{N}}$ converges for every $x \in S$, where S is a dense subset of X.

b) $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.

Solution. Assume first that $(T_n x)_{n\in\mathbb{N}}$ converges for all $x \in X$. Then **a** holds trivially, and **b** holds by the uniform boundedness theorem. To be more precise, $(T_n x)_{n\in\mathbb{N}}$ must be bounded for all $x \in X$, since any convergent sequence is bounded. Hence there must exist $C_x > 0$ for every $x \in X$ such that $\|T_n x\|_Y \leq C_x$ for all $n \in \mathbb{N}$. Now the uniform boundedness theorem gives a $C > 0$ such that $\|T_n\|_Y \leq C$, hence $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.

Now assume that **a** and **b** hold, and let $x \in X$. We want to prove that $T_n x$ converges, or equivalently that it is Cauchy. Let $\epsilon > 0$, and pick $s \in S$ such that $\|x - s\|_X < \epsilon$. Now consider $\|T_n x - T_m x\|_Y$, which we want to be able to make small by picking n and m large. By the triangle inequality,

$$\|T_n x - T_m x\|_Y \leq \|T_n x - T_n s\|_Y + \|T_n s - T_m s\|_Y + \|T_m s - T_m x\|_Y.$$

Let $\sup_{n\in\mathbb{N}} \|T_n\| = C < \infty$. Since $(T_n s)_{n\in\mathbb{N}}$ is assumed to converge, it is in particular Cauchy. Hence we can find $N \in \mathbb{N}$ such that $\|T_n x - T_n s\|_Y < \epsilon$ for $m, n \geq N$. If we consider $m, n \geq N$, we find that

$$\begin{aligned}
\|T_n x - T_m x\|_Y &\leq \|T_n x - T_n s\|_Y + \|T_n s - T_m s\|_Y + \|T_m s - T_m x\|_Y \\
&\leq C\|x - s\|_X + \epsilon + C\|x - s\|_X \\
&< \epsilon(2C + 1),
\end{aligned}$$

which proves that the sequence is Cauchy.

2 Show that the limit operator in the theorem of Banach-Steinhaus is not necessarily bounded for a sequence of bounded linear mappings $(S_n)_{n\in\mathbb{N}}$ on a normed space X. Take X to be the space of real-valued sequences of finite support with the supremums norm. Consider the partial sum operator $S_n x = \sum_{k=1}^{n} x_k$ where $x = (x_k)_{k\in\mathbb{N}}$ on this space.

Solution. The limit operator is defined by $Sx = \lim_{n\to\infty} S_n x$. To show that this is not bounded, it is sufficient to consider the elements $x_N \in X$ for $N \in \mathbb{N}$, such that the first N components of x_N are 1 and the rest are zero. In other words, $x_N = (1, 1, ...1, 0, 0, 0....)$ where the last 1 appears in the N'th position. Clearly $\|x_N\|_X = 1$ for every N, but

$$\begin{aligned} Sx_N &= \lim_{n\to\infty} S_n x_N \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} (x_N)k \\ &= N. \end{aligned}$$

Hence we have elements x_N of norm 1 such that $|Sx_N|_X = N$, and therefore S cannot be bounded.

One can also check that the operators $(S_n)_{n\in\mathbb{N}}$ satisfy the condition for Banach Steinhaus (except that X is not complete, of course!), namely that $\lim_{n\to\infty} S_n x$ exists for any $x \in X$. This is trivially true since x is assumed to have finite support.

$\boxed{3}$ Let $\mathcal{H}$ be a real Hilbert space and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form on $\mathcal{H}$:

$B(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 B(x_1, y) + \alpha_2 B(x_2, y)$ and $B(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 B(x, y_1 y) + \beta_2 B(x, y_2)$

for all $x_1, x_2, x, y_1, y_2, y \in \mathcal{H}$ and for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.

Show that if $B(\cdot, y)$ is continuous for every $y \in \mathcal{H}$ and $B(x, \cdot)$ is continuous for every $x \in \mathcal{H}$. Then B is bounded.

Hint: Banach-Steinhaus

Solution. Define T_y by $T_y(x) = \frac{B(x,y)}{\|y\|}$ for $y \neq 0$. We want to show that $(T_y)_{y\in\mathcal{H}}$ is pointwise bounded, and thus uniformly bounded by the uniform boundedness theorem. For $x \in \mathcal{H}$,

$$|T_y(x)| = \frac{1}{\|y\|} |B(x, y)|.$$

Since we assume that $B(x, \cdot)$ is bounded for any $x \in \mathcal{H}$, there must exist a constant $C_x < \infty$ such that $|B(x, y)| \leq C_x \|y\|$. Inserting this into our calculation, we get

$$|T_y(x)| \leq C_x,$$

hence the family $(T_y)_{y\in\mathcal{H}}$ is pointwise bounded. By the uniform boundedness theorem, there must exist a $C_1 < \infty$ such $\|T_y\| < C_1$ for any $y \in \mathcal{H}$, i.e. $|B(x, y)| \leq C_1 \|y\|$ for any $x, y \in \mathcal{H}$.

By exactly the same argument, we find a constant C_2 such that $|B(x, y)| \leq C_2 \|x\|$ for any $x, y \in \mathcal{H}$. If $C = \max\{C_1, C_2\}$, then we have that

$$|B(x, y)| \leq C\|x\| \qquad\qquad |B(x, y)| \leq C\|y\|.$$

By adding these two equations, we find that

$$|B(x, y)| \leq \frac{C}{2}(\|x\| + \|y\|),$$

so B is bounded.