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Solutions to exercise set 6

1 Let $\mathbf{u} \in \mathbb{R}^n \setminus \{0\}$ be an eigenvector of $\mathbf{I} - \mathbf{Q}^{-1}$ with respect to some eigenvalue λ with $|\lambda| \geq 1$. Define moreover

$$\mathbf{x}^{(0)} = \mathbf{u} + \mathbf{A}^{-1}\mathbf{b}.$$

We show in the following that

$$\mathbf{x}^{(k)} = \lambda^k \mathbf{u} + \mathbf{A}^{-1}\mathbf{b}. \tag{1}$$

For $k = 0$ this is trivial. Assume now that (1) holds for some $k \geq 0$. Then

$$\begin{aligned}
\mathbf{x}^{(k+1)} &= (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A})\mathbf{x}^{(k)} + \mathbf{Q}^{-1}\mathbf{b} \\
&= (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A})(\lambda^k \mathbf{u} + \mathbf{A}^{-1}b) + \mathbf{Q}^{-1}\mathbf{b} \\
&= \lambda^k (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A})\mathbf{u} + \mathbf{A}^{-1}\mathbf{b} - \mathbf{Q}^{-1}\mathbf{A}\mathbf{A}^{-1}\mathbf{b} + \mathbf{Q}^{-1}\mathbf{b} \\
&= \lambda^{k+1}\mathbf{u} + \mathbf{A}^{-1}\mathbf{b},
\end{aligned}$$

which shows that (1) holds for all k. In particular,

$$\mathbf{x}^{(k)} - \mathbf{A}^{-1}\mathbf{b} = \lambda^k \mathbf{u},$$

which does not tend to 0.

2 **a)** Damped Richardson iteration converges if (and only if) all eigenvalues of the defining matrix $\mathbf{I} - \frac{1}{\lambda}\mathbf{A}$ are in absolute value smaller than 1. Now denote by

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0$$

the eigenvalues of $\mathbf{A}$ (these are all real positive, because $\mathbf{A}$ is assumed to be symmetric and positive definite). Then the eigenvalues of $\mathbf{I} - \frac{1}{\lambda}\mathbf{A}$ are

$$1 - \frac{\mu_1}{\lambda} \leq 1 - \frac{\mu_2}{\lambda} \leq \ldots \leq 1 - \frac{\mu_n}{\lambda} < 1.$$

Thus the eigenvalues of $\mathbf{I} - \frac{1}{\lambda}\mathbf{A}$ are *all* in absolute value smaller than 1, if and only if

$$1 - \frac{\mu_1}{\lambda} > -1,$$

which means that

$$\lambda > \frac{\mu_1}{2}. \tag{2}$$

Every parameter $\lambda > 0$ satisfying (2) yields a convergent method.

b) According to part **a)** of this exercise, we have to choose the damping parameter larger than $\mu_1/2$, where μ_1 is the largest eigenvalue of the matrix of the system. Using, the power method we find that $\mu_1 \approx 5.4142$. Thus we can for instance use $\lambda = 3$, which yields the iteration

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \frac{1}{3}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}\mathbf{x}^{(k)} + \frac{1}{3}\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}.$$

Starting with $\mathbf{x}^{(0)} = [0,0,0]^T$, we obtain the iterates

$$\mathbf{x}^{(1)} = \frac{1}{3}\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \frac{1}{9}\begin{bmatrix} 7 \\ -17 \\ 8 \end{bmatrix}, \qquad \mathbf{x}^{(3)} = \frac{1}{27}\begin{bmatrix} 27 \\ -51 \\ 29 \end{bmatrix}.$$

(The solution is $[1,-2,1]$.)

3 **a)** In the Jacobi method, we have the iteration

$$\begin{aligned}\mathbf{x}^{(k+1)} &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x}^{(k)} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 6 \\ -1 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1/3 & -1/3 \\ 0 & 0 & -2/5 \\ 1/3 & 1/3 & 0 \end{bmatrix}\mathbf{x}^{(k)} + \begin{bmatrix} 2 \\ -1/5 \\ 2 \end{bmatrix}.\end{aligned}$$

This yields the iterates

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1/5 \\ 2 \end{bmatrix}, \qquad \mathbf{x}^{(2)} \approx \begin{bmatrix} 1.2667 \\ -1 \\ 2.6 \end{bmatrix}, \qquad \mathbf{x}^{(3)} \approx \begin{bmatrix} 0.8 \\ -1.24 \\ 2.0889 \end{bmatrix}.$$

b) The Gauss–Seidel method reads as

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{3}\big(x_2^{(k)} - x_3^{(k)} + 6\big), \\ x_2^{(k+1)} &= \frac{1}{5}\big(-2x_3^{(k)} - 1\big), \\ x_3^{(k+1)} &= \frac{1}{3}\big(x_1^{(k+1)} + x_2^{(k+1)} + 6\big). \end{aligned}$$

This yields the iterates

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -0.2 \\ 2.6 \end{bmatrix}, \qquad \mathbf{x}^{(2)} \approx \begin{bmatrix} 1.0667 \\ -1.24 \\ 1.9422 \end{bmatrix}, \qquad \mathbf{x}^{(3)} \approx \begin{bmatrix} 0.9393 \\ -0.9769 \\ 1.9875 \end{bmatrix}.$$

c) For the SOR method with $\omega = 3/2$ we obtain the iteration

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{2}\big(x_2^{(k)} - x_3^{(k)} + 6\big) - \frac{1}{2}x_1^{(k)}, \\ x_2^{(k+1)} &= \frac{3}{10}\big(-2x_3^{(k)} - 1\big) - \frac{1}{2}x_2^{(k)}, \\ x_3^{(k+1)} &= \frac{1}{2}\big(x_1^{(k+1)} + x_2^{(k+1)} + 6\big) - \frac{1}{2}x_3^{(k)}. \end{aligned}$$

This yields the iterates

$$\mathbf{x}^{(1)} = \begin{bmatrix} 3 \\ -0.3 \\ 4.35 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} -0.825 \\ -2.76 \\ -0.9675 \end{bmatrix}, \qquad \mathbf{x}^{(3)} \approx \begin{bmatrix} 2.5162 \\ 1.6605 \\ 5.5721 \end{bmatrix}.$$

Apparently, the SOR method does not converge.

4 You can find a possible implementation of the method on the webpage of the course.

5 For the Jacobi iteration, the matrix $\mathbf{B} := \mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$ defining the iteration has the entries

$$b_{ij} = \begin{cases} \frac{a_{ij}}{a_{ii}} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Now recall that the ∞-norm of a matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ is defined as

$$\|\mathbf{B}\|_\infty := \max_{1\le i\le n} \sum_{j=1}^{n} |b_{ij}|.$$

For the Jacobi iteration we have

$$\|\mathbf{B}\|_\infty = \max_{1\le i\le n} \sum_{j\neq i} \frac{|a_{ij}|}{|a_{ii}|}.$$

Since the matrix $\mathbf{A}$ is diagonally dominant, we have

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|$$

for all i. Thus

$$\sum_{j\neq i} \frac{|a_{ij}|}{|a_{ii}|} < 1,$$

and therefore also $\|\mathbf{B}\|_\infty < 1$. Thus the Jacobi iteration converges.

6 **a)** To be accurate, we choose the nodes closest to the x-value where we wish to approximate f. Consequently for the linear case we choose $x_0 = 8.3$ and $x_1 = 8.6$. The polynomial is

$$P_1(x) = f(8.3)\frac{x - 8.6}{-0.3} + f(8.6)\frac{x - 8.3}{0.3}.$$

From this we find

$$P_1(8.4) = 8.3\ln(8.3)\frac{-0.2}{-0.3} + 8.6\ln(8.6)\frac{0.1}{0.3} \approx 17.878332.$$

In the quadratic case we add $x_2 = 8.7$ to keep the interval small. The interpolating polynomial is

$$P_2(x) = f(8.3)\frac{(x-8.6)(x-8.7)}{0.12} + f(8.6)\frac{(x-8.3)(x-8.7)}{-0.03} + \\ + f(8.7)\frac{(x-8.3)(x-8.6)}{0.04},$$

and we get the approximation

$$P_2(8.4) = 8.3\ln(8.3)\frac{0.06}{0.12} + 8.6\ln(8.6)\frac{-0.03}{-0.03} + 8.7\ln(8.7)\frac{-0.02}{0.04} \approx 17.877160$$

In the cubic case we must use all the points so $x_3 = 8.1$. The interpolating polynomial is

$$P_3(x) = f(8.3)\frac{(x-8.6)(x-8.7)(x-8.1)}{0.024} + f(8.6)\frac{(x-8.3)(x-8.7)(x-8.1)}{-0.015} + \\ + f(8.7)\frac{(x-8.3)(x-8.6)(x-8.1)}{0.024} + f(8.1)\frac{(x-8.3)(x-8.6)(x-8.7)}{-0.06},$$

and finally

$$P_3(8.4) = 8.3\ln(8.3)\frac{0.018}{0.024} + 8.6\ln(8.6)\frac{-0.009}{-0.015} + 8.7\ln(8.7)\frac{-0.006}{0.024} + \\ + 8.1\ln(8.1)\frac{0.006}{-0.06} \approx 17.877146.$$

The true value to the digits given is $f(8.4) = 17.877146$.

b) The procedure is analogue to the previous task. The results are summarized in the table below.

n	$x_0, x_1, \ldots, x_n$	$P_n(-1/3)$
1	$-0.5, -0.25$	1.5338542
2	$-0.5, -0.25, 0.0$	1.5034722
3	$-0.5, -0.25, 0.0, -0.75$	1.4918981

The true value is $f(-1/3) = 1.4938272$ to the digits given.

$\boxed{7}$ **a)** We number the nodes in ascending order $x_0 = 8.1$, $x_1 = 8.3$, $x_2 = 8.6$ and $x_3 = 8.7$ for $f(x) = x \ln x$, and compute the table of approximations $S_{ij}(x)$ using the Neville 2nd recurrence relation:

$$S_{ij}(x) = \frac{x - x_{i-j}}{x_i - x_{i-j}} S_{i,j-1}(x) + \frac{x_i - x}{x_i - x_{i-j}} S_{i-1,j-1}(x)$$

at $x = 8.4$.

8.1	16.944099			
8.3	17.564921	17.875332		
8.6	18.505155	17.878332	17.877132	
8.7	18.820910	17.873644	17.877160	17.877146

Here $P_1(8.4)$, $P_2(8.4)$, $P_3(8.4)$ from the previous task corresponds to $S_{21}(8.4)$, $S_{32}(8.4)$, $S_{33}(8.4)$ respectively, and the computed values are seen to be the same.

b) We again number the nodes in ascending order $x_0 = -0.75$, $x_1 = -0.5$, $x_2 = -0.25$ and $x_3 = 0$ with $f(x) = x^4 - x^3 + x^2 - x + 1$, and compute the table of approximations $S_{ij}(x)$ at $x = -1/3$

-0.75	3.0507812			
-0.5	1.9375	1.1953125		
-0.25	1.3320312	1.5338542	1.4774306	
0	1	1.4427083	1.5034722	1.4918981

Again $P_1(-1/3)$, $P_2(-1/3)$, $P_3(-1/3)$ from the previous task corresponds to $S_{21}(-1/3)$, $S_{32}(-1/3)$, $S_{33}(-1/3)$ respectively, and the computed values are seen to match.