MA2501 Numerical Methods
Spring 2017

Exercise set 9

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1 Find approximate values of the following integrals using the adaptive Simpson's rule from the textbook implemented in MATLAB with $\epsilon = 5 \times 10^{-6}$ and max recursion level dept equal to 6. Compare the actual error with ϵ.

a)

$$4 \int_0^1 \frac{dx}{1+x^2}$$

b)

$$8 \int_0^{1/\sqrt{2}} (\sqrt{1-x^2} - x)dx$$

2 Compute a numerical approximation of the definite integral

$$\int_0^2 e^{-x^2} \sin(x)\, dx,$$

using the composite Gauss–Legendre rule with $n = 2$ and two subintervals, i.e. $h = 1$. (You may want to use MATLAB for the computations.)

3 Suppose we choose interpolation points in the interval $[0, 1]$:

$$x_0 = \frac{1}{4} \quad x_1 = \frac{1}{2} \quad x_2 = \frac{3}{4}$$

Remember that we obtain a quadrature formula by interpolating a function f at those points, and integrating exactly the resulting interpolation polynomial. The resulting formula has the format

$$\int_0^1 f(x)dx \approx I(f) = \sum_{k=0}^{2} A_k f(x_k)$$

a) Explain why this quadrature formula integrates exactly polynomials of degree up to degree 2. Does it depend on the choice of x_0, x_1, x_2?

b) Using the preceding fact on the polynomials 1, $x - \frac{1}{2}$ and $(x - \frac{1}{2})^2$ to find directly the weights A_k.

c) Write down the Lagrange polynomials ℓ_0, ℓ_1 and ℓ_2 for the interpolation points x_k and compute the integrals $A_k = \int_0^1 \ell_k(x)dx$. Do you find the same values of A_k? If so, why?

d) Show that the quadrature formula integrates exactly polynomials of degree 3, but not 4.
Hint: use the quadrature formula on the polynomials $(x - \frac{1}{2})^3$ and $(x - \frac{1}{2})^4$.

e) Show that the quadrature formula is exact for the polynomial $(x - \frac{1}{2})^5$. Does this mean that the formula is exact for polynomials of degree 5?

f) Write the quadrature formula scaled to an arbitrary interval $[a, b]$, in order to approximate $\int_a^b f(x)dx$.

$\boxed{4}$ Show that the quadrature formula constructed using the $n+1$ nodes $x_0, x_1, \ldots, x_n$ can not possibly have degree of precision $2n+2$, i.e. integrate exactly all polynomials of degree up to and including $2n+2$. This implies that the degree of precision, $2n+1$, obtained by Gaussian quadrature is optimal.

Hint: Consider the polynomial M^2, with

$$M(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

$\boxed{5}$ **Cf. Cheney and Kincaid, Exercise 6.1.9**

Assume that you are interpolating the function $f(x) = \sin(100x)$ on the interval $[0, \pi]$ with a linear spline on a uniformly spaced grid. Estimate how many grid points will be required in order to guarantee that the interpolation error is smaller than 10^{-8}?