TMA 4275 Lifetime analysis
Exercise 3 - solution

Problem 1

Basic statics: `Stats > Basic Statistics > Display Descriptive Statistics`

```
Variable    N  N*    Mean  SE Mean  StDev  Minimum     Q1  Median     Q3  Maximum
C1         12   0  40.16     9.22  31.95     0.80  13.48   31.00  68.40    96.00
```

Histogram: `Graph > Histogram`

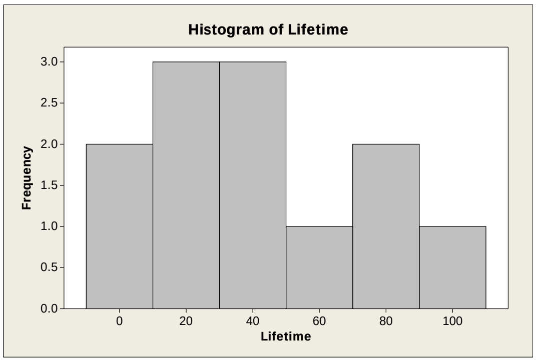

Figure 1: Histogram of lifetime

Empirical cumulative distribution function: `Graph > Empirical CDF`
Kaplan-Meier estimator: `Stats > Reliability/Survival > Distribution Analysis (Right censoring) > Nonparametric Distribution Analysis`

```
Distribution Analysis: Lifetime

Variable: Lifetime

Censoring Information  Count
Uncensored value          12


Nonparametric Estimates


Characteristics of Variable

             Standard   95.0% Normal CI
Mean(MTTF)      Error    Lower    Upper
   40.1583   9.22398  22.0797  58.2370

Median = 30.4
```

```
IQR = 43.8  Q1 = 10.2  Q3 = 54


Kaplan-Meier Estimates

        Number  Number     Survival  Standard   95.0% Normal CI
Time  at Risk  Failed  Probability     Error     Lower     Upper
 0.8       12       1     0.916667  0.079786  0.760290  1.00000
 3.6       11       1     0.833333  0.107583  0.622475  1.00000
10.2       10       1     0.750000  0.125000  0.505005  0.99500
23.3        9       1     0.666667  0.136083  0.399949  0.93338
28.0        8       1     0.583333  0.142319  0.304394  0.86227
30.4        7       1     0.500000  0.144338  0.217104  0.78290
31.6        6       1     0.416667  0.142319  0.137727  0.69561
41.2        5       1     0.333333  0.136083  0.066616  0.60005
54.0        4       1     0.250000  0.125000  0.005005  0.49500
73.2        3       1     0.166667  0.107583  0.000000  0.37753
89.6        2       1     0.083333  0.079786  0.000000  0.23971
96.0        1       1     0.000000  0.000000  0.000000  0.00000
```

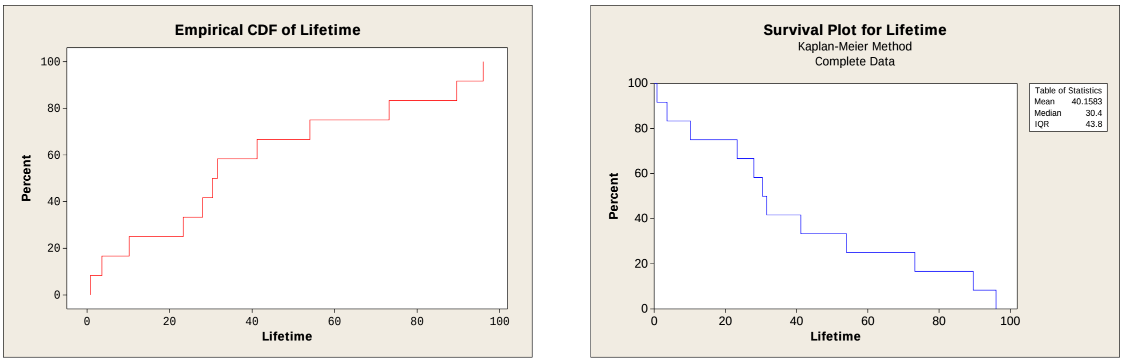

Figure 2: Empirical cumulative distribution function and Kaplan Meier estimator

Parametric fit overview plot: `Stats > Reliability/Survival > Distribution Analysis (Right censoring) > Distribution Overview plot`

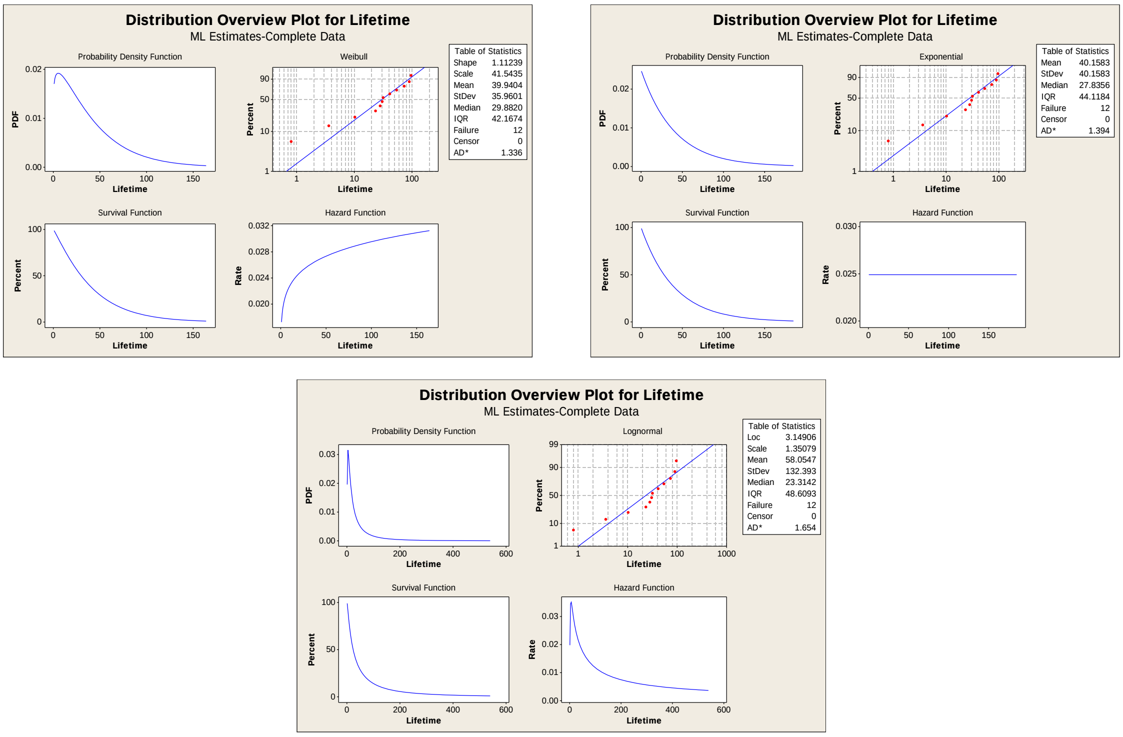

Figure 3: Distribution overview plots for Weibull, exponential and log-normal distribution. Based on the probability plots, the Weibull distribution and the exponential distribution seems to be more appropriate than the lognormal distribution

Problem 2

For the non-censored data, the empirical survival plot obtained by the Kaplan-Meier method is 1-(empirical cumulative distribution function).

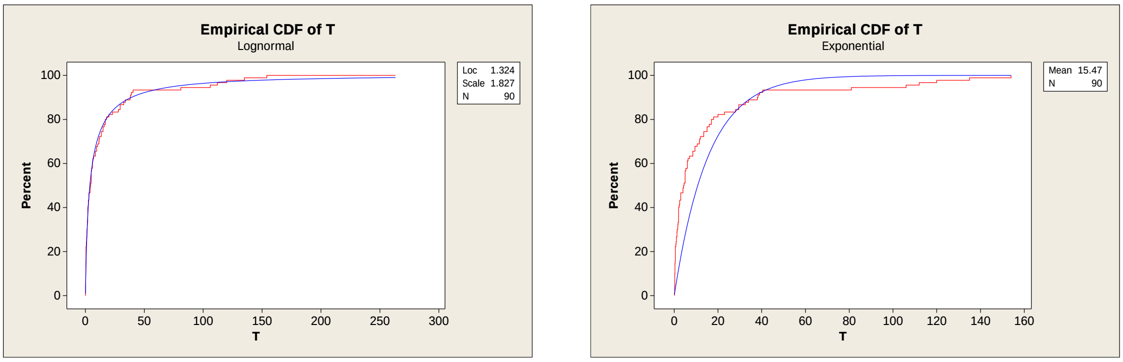

Figure 4: Empirical cumulative distribution function with the estimated curves for the log-normal distribution and exponential distribution. These plots indicates that log-normal distribution is better fit than exponential distribution.

Problem 3

a)

$$R_Z(z) = P(Z > z) = P(T > z \cap V > z) = P(T > z)P(V > z)$$
$$= R_T(z)R_V(z) = e^{-\lambda z}e^{-\mu z} = e^{-(\lambda+\mu)z}$$

which is the reliability function of the exponential distribution with failure rate $\lambda + \mu$.

b) A serial system fails when there is a failure of any of the components of the system. That is, the lifetime is $Z = \min(T, V)$ with distribution as in a).

The probability that component A is the failure cause can be computed as

$$P(\text{failure due to the } A) = P(V > T) = \int_0^\infty P(V > T|T = z)f_T(z)dz$$
$$= \int_0^\infty P(V > z)f_T(z)dz = \int_0^\infty e^{-\mu z}\lambda e^{-\lambda z}dz$$
$$= \frac{\lambda}{\lambda + \mu}$$

c)

$$R(t) = 1 - P(T \le t \cap V \le t) = 1 - P(T \le t)P(V \le t) = 1 - (1 - e^{-\lambda t})(1 - e^{-\mu t})$$

d) Let T_{AB} be the time until failure of the system. Then

$$T_{AB} = T + V$$

The MTTF is given by

$$E(T_{AB}) = E(T) + E(V) = \frac{1}{\lambda} + \frac{1}{\mu}$$

and the survival function of the system

$$\begin{aligned}
R(t) = P(T_{AB} > t) &= \int_0^t P(T_{AB} > t | T = z) f_T(z) dz \\
&= \int_0^t P(T_{AB} > t) f_T(z) dz + \int_t^\infty f_T(z) dz \\
&= \int_0^t P(T + V > t | T = z) f_T(z) dz + R_T(t) = \int_0^t P(V > t - z) f_T(z) dz + R_T(t) \\
&= \int_0^t exp(-\mu(t - z)) \lambda exp(-\lambda z) dz + exp(-\lambda t) \\
&= \lambda exp(-\mu t) \int_0^t exp(z(\mu - \lambda)) dz + exp(-\lambda t) \\
&= \lambda exp(-\mu t) \frac{exp(t(\mu - \lambda)) - 1}{\mu - \lambda} + exp(-\lambda t) = \frac{\lambda}{\mu - \lambda}(exp(-\lambda t) - exp(-\mu t)) + exp(-\lambda t) \\
&= (\frac{\lambda}{\mu - \lambda} - 1) exp(-\lambda t) - \frac{\lambda}{\mu - \lambda} exp(-\mu t) = \frac{\mu}{\mu - \lambda} exp(-\lambda t) - \frac{\lambda}{\mu - \lambda} exp(-\mu t)
\end{aligned}$$

Because $P(T_{AB} > t | T = z) = 1$ for $z > t$.
If $\mu = \lambda$ we can go back to the 5-th line of the above equation and replace μ with λ (or we can take the limit of the last part of the 7-th line and use L'Hospital's rule). The result is then:

$$\begin{aligned}
&\lambda exp(-\lambda t) \int_0^t exp(z(\lambda - \lambda)) dz + exp(-\lambda t) \\
&= \lambda exp(-\lambda t) \int_0^t 1 dz + exp(-\lambda t) \\
&= \lambda t exp(-\lambda t) + exp(-\lambda t) = exp(-\lambda t)(1 + \lambda t)
\end{aligned}$$

Which is the Gamma distribution.