Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2017

Exercise set 12

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let

$$z_1 = \sqrt{\frac{2}{3}}\begin{bmatrix}1\\0\end{bmatrix}, \quad z_2 = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\\sqrt{3}/2\end{bmatrix}, \quad z_3 = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\-\sqrt{3}/2\end{bmatrix}.$$

Show that for every $x \in \mathbb{R}^2$ we have

a)

$$\|x\|^2 = \sum_{i=1}^{3} |\langle x, z_i\rangle|^2$$

b)

$$x = \sum_{i=1}^{3} \langle x, z_i\rangle z_i$$

Remark. The vectors z_1, z_2, z_3 span $\mathbb{R}^2$, but they are obviously not an orthonormal basis (they are not even linearly independent). Still, they satisfy a generalization of Parseval's identity and "act like" an orthonormal basis. Such systems appear very naturally in applications (e.g. in signal analysis), and are often called Parseval frames. This concrete system is known as the Mercedes Benz frame (can you think of why?).

2 Let $\mathcal{P}_3$ be the space of polynomials of degree at most 3.

a) Show that $\{B_0^3(x) = (1-x)^3, B_1^3(x) = 3x(1-x)^2, B_2^3(x) = 3x^2(1-x), B_3^3(x) = x^3\}$ is a basis for $\mathcal{P}_3$, known as the Bernstein basis. Try to define a Bernstein basis for the space of polynomials of degree at most n.

b) Since $\{B_i^3(x) : i = 0, \ldots, 3\}$ is a basis of $\mathcal{P}_3$ there exist unique coefficients $\alpha_0, \ldots, \alpha_3$ for any $f \in \mathcal{P}_3$ such that

$$f(x) = \alpha_0 B_0^3(x) + \alpha_1 B_1^3(x) + \alpha_2 B_2^3(x) + \alpha_3 B_3^3(x).$$

On the other hand we have

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Express α_i in terms of a_i for $i = 0, ..., 3$. In other words, how does one convert a polynomial in monomial form to one in the Bernstein basis?

3 Let X be a vector space and $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ be a basis for X. Suppose $T : X \to X$ is a linear mapping. Prove that the following statements are equivalent:

1. The matrix representing T has upper-triangular form.
2. $Tb_j \in \text{span}(b_1, ..., b_j)$ for $j = 1, ..., n$.
3. $\text{span}(b_1, ..., b_j)$ is invariant under T for $j = 1, ..., n$.

4 Let X be a vector space of dimension n. Suppose $\{b_1, b_2, ..., b_k\}$ is a linearly independent subset of X for some $k < n$. Show that there exist vectors $b_{k+1}, ..., b_n$ such that $\{b_1, b_2, ..., b_n\}$ is a basis for X.