Norwegian University of Science and Technology
Department of Mathematics

MA2501 Numerical Methods
Spring 2017

Solutions to exercise set 5

1 **a)** First we note that the equation $x^3 = 0$ has the unique solution $x = 0$. Thus, unless one of the iterates becomes 0, we will always have the inequality $a < 0 < b$. Moreover we note that the result of the method given the input $[a, b]$ is

$$c = \frac{ab^3 - ba^3}{b^3 - a^3} = ab\frac{b^2 - a^2}{b^3 - a^3}.$$

Obviously $c = 0$ if and only if $a = -b$.
Assume now that $a \neq -b$. Then either of the inequalities $a^2 < b^2$ or $a^2 > b^2$ holds. Assume first that $a^2 < b^2$. Then, since $a < 0 < b$ and therefore $ab < 0$, we have

$$c = ab\frac{b^2 - a^2}{b^3 - a^3} < 0,$$

implying that in the next step a is replaced by c. Moreover this implies that also in the next step the inequality $a^2 < b^2$ holds, and thus, again, it is the left endpoint of the interval that is updated.
An analogous argumentation shows that in the case $a^2 > b^2$ it is always the point b that gets replaced by c.

b) Assume without loss of generality that at the start of the iteration we have $a^2 < b^2$ (the case where $a^2 > b^2$ can be treated similarly). Then the previous considerations show that the sequence c_k is defined by the iteration $c_0 = a_0$ and

$$c_{k+1} = c_k b\frac{b^2 - c_k^2}{b^3 - c_k^3}.$$

In particular,

$$\frac{c_{k+1}}{c_k} = b\frac{b^2 - c_k^2}{b^3 - c_k^3}.$$

From the assumption we may assume that $c_k \to 0$. Thus

$$\lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|} = \lim_{k\to\infty} b\frac{b^2 - c_k^2}{b^3 - c_k^3} = 1.$$

2 **a)** Since $x \geq \sin(x)$ for $x \geq 0$ and $\cos(x) \geq 0$ on $[0, \pi/2]$, it follows that $\Phi(x) \geq 0$ for $x \in [0, \pi/2]$. Moreover $\sin(x) \geq 0$ on $[0, \pi/2]$, implying that $\Phi(x) \leq x + \cos(x)$ for all $x \in [0, \pi/2]$. Since the maximum of the function $x \mapsto x + \cos(x)$ on $[0, \pi/2]$ is $\pi/2$, this shows that, indeed, Φ maps $[0, \pi/2]$ to $[0, \pi/2]$.
Next note that the function Φ is differentiable with

$$\Phi'(x) = 1 - \sin(x) - \frac{1}{2}\cos(x).$$

Obviously we have $\Phi'(x) \geq 1 - 1 - 1/2 = -1/2$ for every $x \in \mathbb{R}$. In addition, the function Φ' is strictly convex on $[0, \pi/2]$ (since sin and cos are concave on $[0, \pi/2]$, and therefore the function Φ' does not have local maxima in $(0, \pi/2)$. As a consequence,

$$\Phi'(x) \leq \max\{\Phi'(0), \Phi'(\pi/2)\} = \max\{1/2, 0\} = \frac{1}{2}$$

for all $x \in [0, \pi/2]$. Thus we have shown that $|\Phi'(x)| \leq 1/2$ for all $x \in [0, \pi/2]$, showing that Φ is a contraction on $[0, \pi/2]$ with contraction factor $1/2$.

b) The first five iterates are

$$\begin{aligned} x^{(0)} &= 0, \\ x^{(1)} &= 1, \\ x^{(2)} &\approx 1.1196, \\ x^{(3)} &\approx 1.1057, \\ x^{(4)} &\approx 1.1073, \\ x^{(5)} &\approx 1.1071. \end{aligned}$$

c) For estimating the accuracy of the iterates, we use the inequality

$$\left| x^{(k)} - x^* \right| \leq \frac{C}{1 - C} \left| x^{(k)} - x^{(k-1)} \right|$$

with $k = 5$ and $C = 1/2$ (the contraction factor). Thus we obtain

$$\left| x^{(5)} - x^* \right| \leq \left| x^{(5)} - x^{(4)} \right| \approx 2 \cdot 10^{-4}.$$

In order to estimate the number of iterations that are needed for obtaining an error smaller than 10^{-12}, we use the inequality

$$\left| x^{(k)} - x^* \right| \leq \frac{C^{k-4}}{1 - C} \left| x^{(5)} - x^{(4)} \right| = \frac{1}{2^{k-5}} \left| x^{(5)} - x^{(4)} \right| \leq \frac{1}{2^{k-6}} 10^{-4}.$$

Since we want the error to be smaller than 10^{-12}, we obtain from this the condition

$$2^{k-6} \geq 10^8$$

or

$$k \geq 6 + \log_2(10^8) \approx 32.6.$$

Thus we would need 33 iterations.[1]

$\boxed{3}$ **a)** For the bisection method we obtain the intervals given by

$$\begin{aligned} a &= 2, & b &= 3, \\ a &= 2, & b &= 2.5, \\ a &= 2, & b &= 2.25, \\ a &= 2, & b &= 2.125. \end{aligned}$$

[1] All these computations were made with the contraction factor 1/2. It is, however, possible to derive a much smaller one on a small interval around the solution. Thus, in fact, we are vastly overestimating the error and the required number of iterations.

b) For the secant method we obtain

$$\begin{aligned} x^{(0)} &= 3, \\ x^{(1)} &= 3.5, \\ x^{(2)} &\approx 2.4622, \\ x^{(3)} &\approx 2.2615, \\ x^{(4)} &\approx 2.1229. \end{aligned}$$

c) Newton's method yields

$$\begin{aligned} x^{(0)} &= 3, \\ x^{(1)} &= 2.36, \\ x^{(2)} &\approx 2.1271, \\ x^{(3)} &\approx 2.0951. \end{aligned}$$

The actual solution is

$$x^* \approx 2.09455.$$

$\boxed{4}$ For the application of Newton's method we denote

$$F(x,y) := \begin{pmatrix} -5x + 2\sin(x) + \cos(y) \\ 4\cos(x) + 2\sin(y) - 5y \end{pmatrix}.$$

Then the Jacobian of F is

$$\mathbf{J_F}(x,y) = \begin{pmatrix} -5 + 2\cos(x) & -\sin(y) \\ -4\sin(x) & 2\cos(y) - 5 \end{pmatrix}.$$

Its inverse can be calculated analytically as

$$\mathbf{J_F}(x,y)^{-1} = \frac{1}{\det \mathbf{J_F}(x,y)} \begin{pmatrix} 2\cos(y) - 5 & \sin(y) \\ 4\sin(x) & -5 + 2\cos(x) \end{pmatrix}$$

with

$$\det \mathbf{J_F}(x,y) = 25 - 10\cos(x) - 10\cos(y) + 4\cos(x)\cos(y) - 4\sin(x)\sin(y).$$

Newton's method now defines iteratively

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} - \mathbf{J_F}(x^{(k)}, y^{(k)})^{-1} F(x^{(k)}, y^{(k)}).$$

Starting with $(x^{(0)}, y^{(0)}) = (0,0)$, we thus obtain the iterates

$$\begin{aligned} \begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \\ \begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix} &\approx \begin{pmatrix} 0.1302 \\ 1.1838 \end{pmatrix}, \\ \begin{pmatrix} x^{(3)} \\ y^{(3)} \end{pmatrix} &\approx \begin{pmatrix} 0.1330 \\ 1.1597 \end{pmatrix}. \end{aligned}$$

$\boxed{5}$ You find some example code on the webpage of the course.