HOMEWORK 5

THE KOLMOGOROV'S FOUNDATIONS OF PROBABILITY THEORY

For the following two problems, begin by describing a discrete probability model for the given experiment.

Problem 1. Two unbiased, six-sided dice are rolled.

(a) Find the probability that the sum of the two outcomes is equal to 5.
(b) Find the probability that the sum of the two outcomes is greater than 8.
(c) Find the expectation of the sum of the two outcomes.

Problem 2. (a) Find the probability of getting exactly 3 heads in 5 flips of a fair coin.
(b) Find the probability of getting at least 3 heads in 5 flips of a fair coin.

For the following two problems, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Problem 3. If X is a random variable, show that the distribution of X, defined as

$$\mu_X(A) := \mathbb{P}(X \in A) \quad \text{for every Borel set } A \subset \mathbb{R},$$

is indeed a probability measure on $\mathbb{R}$.

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function w.r.t. the Lebesgue measure. Define $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(x) := \int_{-\infty}^{x} f(y)\, dy.$$

Show that F is a cumulative distribution function (of some random variable) if and only if $f \geq 0$ a.e. and $\int_{\mathbb{R}} f(y)dy = 1$.

Problem 5. Let $(\Omega, \{p(\omega)\colon \omega \in \Omega\})$ be a *discrete* probability space model. Let $\mathbb{P}$ be the corresponding probability measure on Ω, where for every $E \subset \Omega$, $\mathbb{P}(E) := \sum_{\omega \in E} p(\omega)$.

We defined the expectation of a random variable $X \geq 0$ on Ω as $\mathbb{E}X := \sum_{\omega \in \Omega} X(\omega)\, p(\omega)$. Show that

$$\mathbb{E}X = \sum_{x} x \cdot \mathbb{P}(X = x) = \int_{\Omega} X\, d\mathbb{P}.$$

Problem 6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that for any measurable function $f\colon \Omega \to [0, \infty)$ we have

$$\int_{\Omega} f\, d\mu = \int_{0}^{\infty} \mu\left(\{x \in \Omega\colon f(x) > t\}\right) dt.$$

Hint: Prove this first for an indicator function, then for a simple function and in the end use the monotone convergence theorem.