HOMEWORK 4
THE RIESZ-MARKOV-KAKUTANI REPRESENTATION THEOREM

Problem 1. Let (X, d) be a metric space, let $A \subset X$ and define the function $d_A \colon X \to \mathbb{R}$,

$$d_A(x) := \operatorname{dist}(x, A) = \inf \{ d(x, a) \colon a \in A \}.$$

(a) Let $L \subset X$ be a closed set. Show that

$$d_L(x) = 0 \iff x \in L.$$

(b) For any $A \subset X$, prove that the function d_A is continuous.

Problem 2. Let K be a compact subset of $\mathbb{R}$ and let U be an open subset of $\mathbb{R}$ such that $K \subset U$. Prove the following refinement of Urysohn's lemma: there is *continuous* function $f \colon \mathbb{R} \to \mathbb{R}$, with compact support, such that

$$f(x) = 1 \text{ if } x \in K \quad \text{and} \quad \operatorname{supp}(f) \subset U,$$

where $\operatorname{supp}(f)$ is the *topological support* of f.

Hint: It would *not* be quite enough to apply Urysohn's lemma to K and $L := U^{\complement}$. You should instead apply it to K and to a slightly increased version of L.

Problem 3. Prove that if $f \colon K \to \mathbb{R}$ is a lower semi-continuous function, where K is a compact set, then there is $m \in \mathbb{R}$ such that

$$f(x) \geq m \quad \text{for all } x \in [0, 1].$$

We denote by $\mathrm{BLSC}_+[0, 1]$ the family of all bounded, *lower semi continuous* functions $f \colon [0, 1] \to \mathbb{R}$ with $f \geq 0$.

Let $I \colon C[0, 1] \to \mathbb{R}$ be a positive linear functional. For every $f \in \mathrm{BLSC}_+[0, 1]$ we define the extension

$$\widetilde{I}(f) := \sup \{ I(g) \colon g \in C[0, 1] \text{ and } 0 \leq g \leq f \}.$$

Since for every open set $U \subset [0, 1]$, the function $\mathbf{1}_U \in \mathrm{BLSC}_+[0, 1]$, we may then define

$$\mu_0(U) := \widetilde{I}(\mathbf{1}_U).$$

Problem 4. Use Problem 2 to show that

$$\mu_0(U) = \sup \{ I(g) \colon g \in C[0, 1],\ 0 \leq g \leq 1 \text{ and } \operatorname{supp}(g) \subset U \}.$$

We would like to define $\mu_0(F)$ also for a closed set F. To do that, we create an even larger space and further extend $\widetilde{I}$ to that space.

Let $\mathcal{L}[0,1]$ be the family of all functions $f\colon [0,1] \to \mathbb{R}$ that can be written as $f = f_1 - f_2$, where $f_1, f_2 \in \mathrm{BLSC}_+[0,1]$. For such a function f define

$$\widetilde{I}(f) := \widetilde{I}(f_1) - \widetilde{I}(f_2).$$

Problem 5. (a) Prove that $\mathcal{L}[0,1]$ is a vector space.

Then it clearly becomes a normed space with the uniform norm $\|f\| := \sup_{x\in[0,1]} \big|f(x)\big|$.

(b) Prove that $\widetilde{I}$ is a positive linear functional on $\mathcal{L}[0,1]$.

Then in particular $\widetilde{I}$ is continuous on $\mathcal{L}[0,1]$.