Sol1

onsdag 13. september 2017 09.57

1. a) Here the Galerkin problem is: find $u \in V_h$

$$: \int_0^1 u'v' = \underbrace{u'(1)\,v(1)}_{=u'(1)} + \int_0^1 fv$$

setting $u = \sum u_i \varphi_i$, $v = \sum v_j \varphi_j$

we find the equation

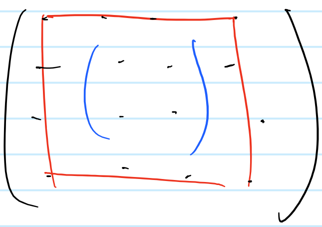
$$Au = f + (0,0,\dots,0,1)^T$$

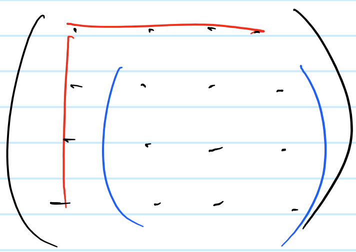
i.e. in comparison with previous, must add a line

$\boxed{b(n) = b(n) + 1}$ after element loop.

and then only remove first row/columns

Dirichlet: Mixed:





e.g. the remove boundary conditions lines should be updated to:

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A(1,:) = [];
A(:,1) = [];
b(1) = [];
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b) We can integrate: $u'(x) = -x + A$.

$$u(x) = -\frac{x^2}{2} + Ax + B$$

then $u(0) = 0 \Rightarrow B = 0$

$u'(1) = 1 \Rightarrow A = 2$

i.e. $u(x) = -\frac{x^2}{2} + 2x = \frac{1}{2}x(4-x)$.

2. a) Multiply by v and integrate

$$\Rightarrow \underbrace{\int -u_{xx} v}_{\int u'v' \text{ as before.}} + \sigma \int uv = \int fv.$$

then substituting $u = \sum u_i \varphi_i$

$v = \sum v_j \varphi_j$

gives

$$\sum_{i,j} u_i v_j \left(\int \varphi_i' \varphi_j' + \sigma \overbrace{\int \varphi_i \varphi_j}^{M} \right) = v_j \int f \varphi_j$$

ie $v^T (A + \sigma M) u = v^T f$

hence the equation $(A + \sigma M) u = f$.

b) deriving mass matrix: ~~X~~

1. linear case

$\varphi_1 = x$

$\varphi_2 = 1 - x$

$$\int_0^1 x^2 \, dx = \frac{1}{3}$$

$$\int_0^1 (1-x)^2 \, dx = \left[-\frac{(1-x)^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\int_0^1 x(1-x) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$x = h\hat{x} + x_0 \quad \Rightarrow \quad \hat{\varphi}_1 = \hat{x}, \quad \hat{\varphi}_2 = 1 - \hat{x}$$

and $\frac{dx}{d\hat{x}} = h$, i.e. $dx = h\,d\hat{x}$

so $\int \varphi_i \varphi_j \, dx = \int \hat{\varphi}_i \hat{\varphi}_j \, h\, d\hat{x}$.

and mass matrix $M = \frac{h}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

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2. quadratic case: a tedious calculation shows

$$M = \frac{h}{30}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}$$

c) See sample code.

3. a) Continuous:

$$a(u,v) = \int_0^1 \underbrace{(1+x)}_{\leq 2 \text{ on } [0,1]} u'(x)\, v'(x)\, dx \leq 2 \int u' v'$$

$$\leq 2\, |u|_{H^1} |v|_{H^1} \leq 2\, \|u\|_{H^1} \|v\|_{H^1}$$

Coercive:

$$a(v,v) = \int_0^1 \underbrace{(1+x)}_{\geq 1 \text{ on } [0,1]} v' v' \geq \int v' v' = |v|_{H^1}^2 \geq \alpha \|v\|_{H^1}^2$$

Poincaré

as $v \mapsto \int f v$ is a continuous functional, we apply Lax-Milgram and are done.

b) Here $u(x) = x^2 - x \Rightarrow u' = 2x - 1$

hence $a(u,v) = \int_0^1 (2x^2 + x - 1)\, v'\, dx$

integrating by parts $= -\int_0^1 (2x^2 + x - 1)'\, v\, dx$

$= \int (-4x-1)\, v\, dx = F(v)$

4. a) (i) integrate: $\int_{-1}^1 |x|^{2\alpha} = 2\int_0^1 |x|^{2\alpha} = \frac{1}{2\alpha+1}\left[x^{2\alpha+1}\right]_0^1$

blows up at 0 if $2\alpha > 1$

if $2\alpha = 1$, $\int \ldots = \log x$ which also blows up at 0

i.e. $|x|^\alpha \in L^2(-1,1)$

$\Leftrightarrow \alpha > -1/2$.

(ii) Similarly $\int_0^\infty |x|^{2\alpha}\, dx = \lim_{x\to\infty} \frac{1}{2\alpha+1}\left[x^{2\alpha+1}\right]_1^x$

and need $\alpha < -1/2$

(again, not $\alpha = -\frac{1}{2}$, as $\log x$ blows up at ∞)

(iii) transform to polar coords

$\|x^\alpha\| = \int_0^{2\pi}\int_0^1 r^{2\alpha}\, dr\, d\theta$, and $\alpha > -1/2$ as per (i)

integral from

integral from (i)

b) as $D \subset \mathbb{R}$ is closed and bounded,
f is continuous, f takes its max on D

and $$\int_D f^2 \, dx \leq M \underbrace{\mu(D)}_{\text{size of } D}$$

c) Uniqueness: if u_1, u_2 are weak derivatives

$$\int_\Omega (v_1 - v_2)\, \phi(x)\, dx = \int (u-u)\, \phi'(x) = 0 \quad \forall \phi$$

hence $v_1 = v_2$ almost everywhere (i.e. equal in L^2).

if $u \in C^1$, then

$$-\int u' \phi \, dx = \int u \phi' \, dx$$ by integration by parts

(boundary values vanish as $\phi \in C_c^\infty(\Omega)$)

d) compute $$\int_0^2 f_1^2 \, dx = \int_0^1 x^2 \, dx + \int_1^2 dx$$

$$= 4/3 < \infty$$

$$\int_0^2 f_2^2 \, dx = \int_0^1 x^2 \, dx + \int_1^2 2 \, dx = \frac{7}{3} < \infty.$$

ie $f_1, f_2 \in L^2$.

Now

$$\int_0^2 f_1 \phi' \, dx = \int_0^1 x \, \phi'(x) \, dx + \int_1^2 \phi'(x) \, dx$$

$$= \left[x \phi(x) \right]_0^1 - \int_0^1 \phi \, dx + \left(\phi(x) \right)_1^2$$

$$= \phi(1) + \cancel{\phi(2)} - \phi(1) - \int_0^1 \phi(x) - \int_1^2 0 \, \phi(x)$$

0 as ϕ vanishes

gives

$$v = \begin{cases} 1 & \text{on } 0 \leq x < 1 \\ 0 & 1 \leq x \leq 2. \end{cases}$$

f_2 is not continuous, so has no weak derivative, by Sobolev embedding theorem.

(ie. $H^1 \subset C^0$)

or check that none could exist, as $\int u \phi'$ will always depend on value of $\phi(1)$.