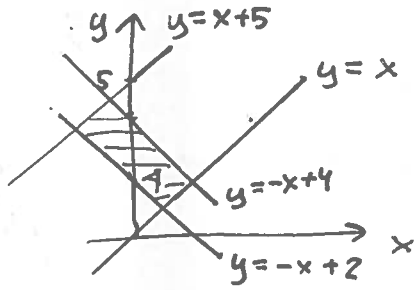
Vanlige forbehold! KH

Løsningsskisser Øving 11

Oppgave 1

a) $\iint_A \frac{e^{x-y}}{x+y}\, d(x,y) = ?$



A: $0 \le \underbrace{y-x}_{u} \le 5$, $2 \le \underbrace{y+x}_{v} \le 4$

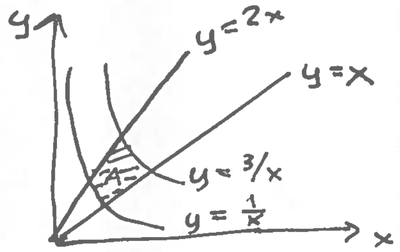
$$\left.\begin{aligned} u &= y-x \\ v &= y+x \end{aligned}\right\} \Rightarrow y = \frac{u+v}{2},\ x = \frac{v-u}{2} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \underline{-\frac{1}{2}}$$

$$\iint_A \frac{e^{x-y}}{x+y}\, d(x,y) = \iint_D \frac{e^{-u}}{2v}\, d(u,v) = \int_2^4 \left[\int_0^5 \frac{1}{2v} e^{-u}\, du\right] dv$$

$$= \int_2^4 \frac{1}{2v} \left[-e^{-u}\right]_0^5 dv = \underline{\underline{\frac{(1-e^{-5})\ln 2}{2}}}$$

b) Se Eksempel 6.7.3

$$I = \iint_A xy\, dx\, dy$$



$1 \le \underbrace{xy}_{u} \le 3 \qquad 1 \le \underbrace{\frac{y}{x}}_{v} \le 2$

$$\left.\begin{aligned} y &= xv \\ u &= x^2 v \end{aligned}\right\} \quad x = \sqrt{\frac{u}{v}},\ y = \sqrt{uv}\ ; \quad \frac{\partial(x,y)}{\partial(u,v)} = \underline{\frac{1}{2v}} \quad (s.630)$$

(vi er i 1. kvadrant)

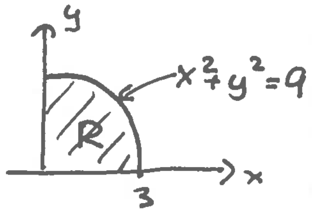
$$I = \int_1^2 \left[\int_1^3 u \frac{1}{2v}\, du\right] dv = \left[\frac{u^2}{2}\right]_2^3 \left[\frac{1}{2}\ln v\right]_1^2 = \left(9/2 - 1/2\right)\frac{\ln 2}{2}$$

$$= \underline{\underline{2\ln 2}}$$

Kan også bruke Metode 2 (s630)

Oppgave 2

i)

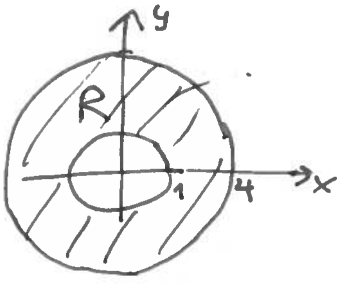


$$\iint_R x y^2 \, d(x,y)$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^3 r\cos\theta \; r^2 \sin^2\theta \; r \, dr \right] d\theta$$

$$= \left[\frac{\sin^3\theta}{3} \right]_0^{\pi/2} \cdot \left[\frac{r^5}{5} \right]_0^3 = \underline{\underline{\frac{81}{5}}}$$

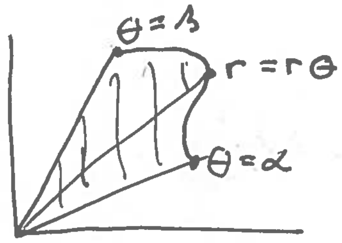
ii)



$$\iint_R e^{r^2} r \, d(r,\theta)$$

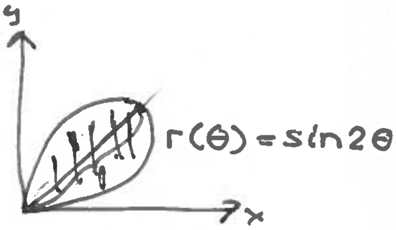
$$= 2\pi \; \tfrac{1}{2} \left[e^{r^2} \right]_1^4 = \underline{\underline{\pi(e^{16}-e)}}$$

Oppgave 3



$$\underline{|A|} = \iint_A d(x,y) = \int_\alpha^\beta \left[\int_0^{r(\theta)} r \, dr \right] d\theta$$

$$= \underline{\frac{1}{2} \int_\alpha^\beta r(\theta)^2 \, d\theta}$$



$$|A| = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2(2\theta) \, d\theta$$

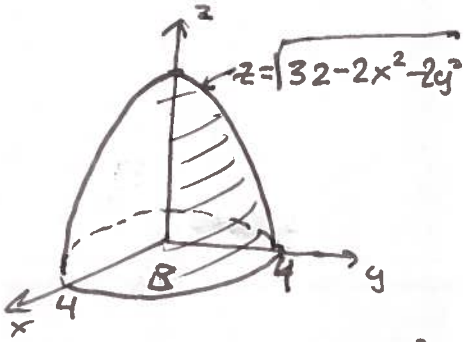
$$\underset{t=2\theta}{=} \frac{1}{2} \underline{\int_0^{\pi}} \sin^2 t \cdot \frac{1}{2} \underline{dt}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \; \underline{\frac{\pi}{2}} \; \underset{*)}{=} \; \underline{\underline{\frac{\pi}{8}}} \qquad \left(\int_0^{\pi} \sin^2 t \, dt = \frac{1}{2} \int_0^{\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{1} = \underline{\frac{1}{2}\pi} \right)$$

*) „Ser" altså integralverdien, alternativt $\sin^2 t = \dfrac{1-\cos 2t}{2}$

evt. delvis int.

Oppgave 4



i)

$$E:\quad 0 \le z \le \sqrt{32-2x^2-2y^2}$$

$$V(E) = \iint_B \sqrt{32-2(x^2+y^2)}\; d(x,y)$$

$$= 2\pi \int_0^4 \sqrt{32-2r^2}\; r\,dr \,]\, dr$$

$B:\ 0 \le x^2+y^2 \le 4^2$

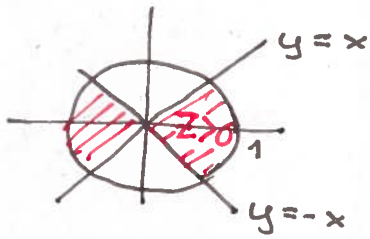
$\supset:\ 0 \le r < 4$

$(0 \le \theta < 2\pi)$

$$\underset{\uparrow}{=} + \frac{2\pi}{4} \int_0^{32} \sqrt{u}\, du = \frac{2\pi}{4}\left[\frac{2}{3} u^{3/2}\right]_0^{32} = \frac{128\sqrt{2}}{3}\pi$$

$$\boxed{\begin{array}{l} u = 32-2r^2 \\ du = -4r\,dr \end{array}}$$

ii)



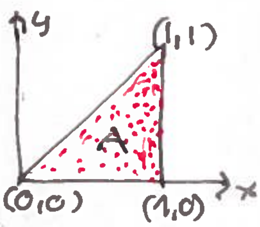
$$\frac{V}{4} = \iint_{\substack{x^2+y^2<1 \\ x^2 \ge y^2,\, x>0}} (x^2-y^2)\, d(x,y)$$

$$\underset{\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}}}{=} \int_0^{\frac{\pi}{4}} \left[\int_0^1 r^2(\cos^2\theta - \sin^2\theta)\, r\, dr\right] d\theta$$

$$= \left[\frac{r^4}{4}\right]_0^1 \int_0^{\pi/4} \cos 2\theta\, d\theta = \frac{1}{8}\left[\sin 2\theta\right]_0^{\pi/4} = \frac{1}{8}$$

$V = \frac{1}{2}$ Fasit er vel gal?

Oppgave 5



Massen er $\iint_A x\, d(x,y) = \int_0^1 \left[\int_0^x x\, dy\right] dx = \int_0^1 x^2 dx = \frac{1}{3}$

$$\bar{x} = \int_0^1 \left[\int_0^x x^2 dy\right] dx \Big/ \frac{1}{3} = \frac{1/4}{1/3} = \frac{3}{4}\ ;\quad \bar{y} = \int_0^1 \left[\int_0^x xy\, dy\right] \Big/ 1/3 = \frac{\int_0^1 x^3/2\, dx}{1/3} = \frac{3}{8}$$

Oppgave 6

$$z = x^2 - y^2 \ ; \quad z_x = 2x \ , \quad z_y = -2y$$

$$A = \iint\limits_{x^2+y^2 \le 4} \sqrt{1+4(x^2+y^2)}\, d(x,y) = 2\pi \int_0^2 \sqrt{1+4r^2}\, r\, dr$$

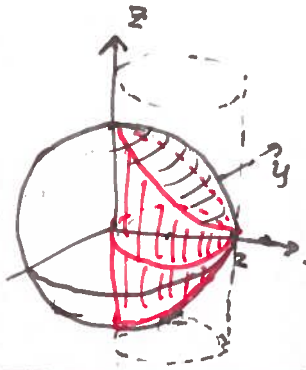
$$= 2\pi \tfrac{2}{3} \cdot \tfrac{1}{8} \left(1+4r^2\right)^{3/2} \Big]_0^2 = \underline{\underline{\frac{\pi}{6}\left(17^{3/2}-1\right)}}$$

Oppgave 7

$x^2 - 2x + y^2 = 0 \Leftrightarrow \underline{(x-1)^2 + y^2 = 1}$

Altså

$T: \ x^2+y^2+z^2 \le 2^2$ og $(x-1)^2+y^2 \le 1$



a)

La $0 \le z \le \sqrt{4-(x^2+y^2)}$; $(x,y) \in D$

Har da

$x^2 - 2x + y^2 = 0$ i polark.

$r^2 - 2r\cos\theta = 0 \ ; \ \underline{r = 2\cos\theta}$

$$\underline{\frac{V}{4}} = \iint\limits_D \sqrt{4-(x^2+y^2)}\, d(x,y)$$

$$\underset{\text{polark.}}{=} \int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{4-r^2}\; r\, dr\, d\theta$$

$$= -\frac{1}{3} \int_0^{\pi/2} \left[\left(4-r^2\right)^{3/2}\right]_0^{2\cos\theta} d\theta = -\frac{1}{3} \int_0^{\frac{\pi}{2}} \Big[\underbrace{\left(4-4\cos^2\theta\right)^{3/2}}_{8\sin^3\theta} - 8\Big] d\theta$$

$$= -\frac{8}{3}\left[\int_0^{\frac{\pi}{2}} \sin\theta\left(1-\cos^2\theta\right) d\theta - \frac{\pi}{2}\right]$$

$$= -\frac{8}{3}\left[\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi/2} - \frac{\pi}{2}\right]$$

$$= -\frac{8}{3}\left(\frac{2}{3} - \frac{\pi}{2}\right) = \underline{\frac{8\pi}{6} - \frac{16}{9}} \ ,$$

og $V = \underline{\underline{\dfrac{16\pi}{3} - \dfrac{64}{9}}}$

b)

$$z = \sqrt{4-(x^2+y^2)} \quad ; \quad z_x = \frac{x}{\sqrt{4-(x^2+y^2)}} \, , \quad z_y = \frac{y}{\sqrt{4-(x^2+y^2)}}$$

$$\frac{A}{4} = \iint_D \sqrt{1 + \frac{x^2+y^2}{4-(x^2+y^2)}} \; d(x,y)$$

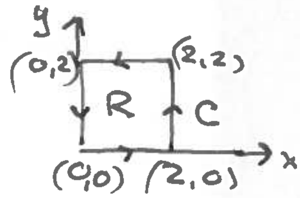
$$\underset{\text{polar}}{=} \int_0^{\frac{\pi}{2}} \left[\int_0^{2\cos\theta} \sqrt{1+\frac{r^2}{4-r^2}} \; r \, dr \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{2r}{\sqrt{4-r^2}} \, dr = -2 \int_0^{\frac{\pi}{2}} \left[(4-r^2)^{1/2} \right]_{r=0}^{r=2\cos\theta}$$

$$= -4 \int_0^{\frac{\pi}{2}} \sin\theta - 1 \, d\theta = 4\left[\cos\theta\right]_0^{\frac{\pi}{2}} + 2\pi = \underline{2\pi - 4}$$

$$A = \underline{\underline{8\pi - 16}}$$

<u>Oppgave 8</u>



$$F(x,y) = (\overset{F_1}{x^2+y}, \; \overset{F_2}{x^2 y})$$

$$\int_C F \cdot dr = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \; d(x,y)$$

$$= \iint_R 2xy - 1 \; d(x,y)$$

$$= 2 \int_0^2 x\,dx \int_0^2 y\,dy \quad - \text{Areal} R$$

$$= 2 \cdot 2 \cdot 2 - 4 = \underline{4}$$