Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2017
Exercise set 6

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Show that $(\mathbb{C}^n, \|.\|_2))$ is complete.

Hint: Recall from problem (5) of problem set 4 that for $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we have that
$$\sum_{i=1}^{n} |x_i| \leq n^{1/2}(\sum_{i=1}^{n} |x_i|^2)^{1/2}.$$
This might be useful at some point in your proof.

2 Show that $(\ell^2, \|.\|_2)$ is a Banach space.

3 Let $(X, \|.\|)$ be a normed space.

a) Show that a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in X.

b) Suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and $(y_n)_{n\in\mathbb{N}}$ another sequence in X. Show that if
$$\|y_n - y_m\| \leq \|x_n - x_m\|$$
for all $m, n \in \mathbb{N}$, then $(y_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in X.

4 Prove the following two statements for a normed space $(X, \|.\|)$.

a) Any ball $B_r(x) = \{y \in X : \|x - y\| < r\}$ in $(X, \|.\|)$ is bounded and $\mathrm{diam}(B_r(x)) \leq 2r$.

b) If A is a bounded subset of $(X, \|.\|)$, then for any $a \in A$ we have $A \subseteq \bar{B}_{\mathrm{diam}(A)}(a)$. (Recall that the a closed ball $\bar{B}_r(x)$ is the set $\{y \in X : \|y - x\| \leq r\}$.)

$\boxed{5}$ **a)** Let $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(t) = \begin{cases} 0 & \text{for } a \leq t \leq \frac{a+b}{2}, \\ n(t - \frac{a+b}{2}) & \text{for } \frac{a+b}{2} < t \leq \frac{a+b}{2} + \frac{1}{n}, \\ 1 & \text{for } \frac{a+b}{2} + \frac{1}{n} \leq t \leq b. \end{cases}$$

in $C[a,b]$. Determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$.

b) Let $(f_n)_{n\in\mathbb{N}}$ be the sequence on $[0,1]$ defined by $f_n(x) = \frac{1}{1+nx}$. Determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,1]$.

$\boxed{6}$ Let f be a Lipschitz function $f : (X, \|.\|_X) \to (Y, \|.\|_Y)$. Show that f is continuous.