MA0301 ELEMENTARY DISCRETE MATHEMATICS
SPRING 2017

1. Homework Set 4 – Solutions

Exercise 1. *Grimaldi's book (5. ed., Exercises 4.1): solve* **Exercise 2**

Solution 1. a) 1) Ind. basis: $S(1): 2^0 = 1 = 2^1 - 1$.

2) Ind. step: ind. hyp.: $S(k): \sum_{i=1}^{k} 2^{i-1} = 2^k + 1$ holds; then

$$\sum_{i=1}^{k+1} 2^{i-1} = \sum_{i=1}^{k} 2^{i-1} + 2^k = 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1.$$

Hence, $S(k) \Rightarrow S(k+1)$, and therefore by induction, $S(n)$ is true for all positive integers.

b) 1) Ind. basis: $S(1): 2 = 2 + 0$.

2) Ind. step: ind. hyp.: $S(k): \sum_{i=1}^{k} i2^i = 2 + (k-1)2^{k+1}$ holds; then

$$\begin{aligned}\sum_{i=1}^{k+1} i2^i &= \sum_{i=1}^{k} i2^i + (k+1)2^{k+1} \\ &= 2 + (k-1)2^{k+1} + (k+1)2^{k+1} \\ &= 2 + 2(k+1-1)2^{k+1} \\ &= 2 + k2^{k+2}.\end{aligned}$$

Hence, $S(k) \Rightarrow S(k+1)$, and therefore by induction, $S(n)$ is true for all positive integers.

c) 1) Ind. basis: $S(1): 1(1!) = 1 = 2! - 1$.

2) Ind. step: ind. hyp.: $S(k): \sum_{i=1}^{k} i(i!) = (k+1)! - 1$ holds; then

$$\begin{aligned}\sum_{i=1}^{k+1} i(i!) &= \sum_{i=1}^{k} i(i!) + (k+1)((k+1)!) \\ &= (k+1)! - 1 + (k+1)((k+1)!) \\ &= (k+1)!(k+1+1) - 1 \\ &= (k+2)! - 1.\end{aligned}$$

Hence, $S(k) \Rightarrow S(k+1)$, and therefore by induction, $S(n)$ is true for all positive integers.

Exercise 2. *Use the principle of induction to show that for all* $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Solution 2. 1) Ind. basis: $S(1): \sum_{i=1}^{1} i^3 = 1^3 = 1 = 1^2$.

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2) Ind. step: ind. hyp.: $S(k): \sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$; then

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= (k+1)^2\left(\frac{k^2}{4} + k + 1\right) \\ &= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) \\ &= (k+1)^2\left(\frac{(k+2)^2}{4}\right) \\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2.\end{aligned}$$

Hence, $S(k) \Rightarrow S(k+1)$, and therefore by induction, $S(n)$ is true for all positive integers.

Exercise 3. *Use the principle of induction to show that for all $n \in \mathbb{Z}^+$, $n^3 + 3n^2 + 2n$ is a multiple of 6.*

Solution 3. 1) Ind. basis: $S(1): 1 + 3 + 2 = 6$.
2) Ind. step: ind. hyp.: $S(k): k^3 + 3k^2 + 2k = 6l$, $l \in \mathbb{Z}^+$; then

$$\begin{aligned}(k+1)^3 + 3(k+1)^2 + 2(k+1) &= k^3 + 3k^2 + 2k + 3k^2 + 3k + 1 + 6k + 3 + 2 \\ &= 6l + 3k^2 + 9k + 6 \\ &= 6l + 3(k+1)(k+2).\end{aligned}$$

Since either $(k+1)$ or $(k+2)$ is even, it follows that $3(k+1)(k+2) = 6m$. Therefore, if follows that $(k+1)^3 + 3(k+1)^2 + 2(k+1) = 6(l+m)$. Hence, $S(k) \Rightarrow S(k+1)$, and therefore by induction, $S(n)$ is true for all positive integers.

Exercise 4. *Grimaldi's book (5. ed., Exercises 4.1): solve* **Exercise 4**

Solution 4. We have the numbers $1, 2, 3, 4, \dots, 24, 25$ in random order on the wheel. Assume that $n_1, n_2, n_3, n_4, \dots, n_{24}, n_{25}$ is such a random arrangement. Now, assume the opposite, i.e., that $m_i := \sum_{j=0}^{2} n_{i+j} < 39$, for all $1 \leq i \leq 23$, and as well that $n_{24} + n_{25} + n_1 < 39$ and $n_{25} + n_1 + n_2 < 39$. Then we have that $\sum_{k=1}^{25} 3n_k < 25 \cdot 39$. From the formula for $\sum_{i=1}^{n} i = n(n+1)/2$ is follows that $\sum_{k=1}^{25} 3n_k = 3\sum_{k=1}^{25} k = 3 \cdot 25 \cdot 13 = 25 \cdot 39$, which is a contradiction.

Exercise 5. *Grimaldi's book (5. ed., Exercises 4.1): solve* **Exercise 17**

Solution 5. Recall that for a positive integer m, the mth Harmonic number is $H_m := \sum_{i=1}^{m} \frac{1}{i}$.
a) 1) Ind. basis: $S(0)$: $1 = 1 + 0/2 \leq H_{2^0}$.
2) Ind. step: ind. hyp. $S(k)$: $1 + k/2 \leq H_{2^k}$. For $n = k+1$ we have

$$H_{2^n} = \sum_{i=1}^{2^n} \frac{1}{i} = \sum_{i=1}^{2^k} \frac{1}{i} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \frac{1}{2^k+3} + \dots + \frac{1}{2^k 2}.$$

This is larger than or equal to

$$\begin{aligned} H_{2^k} + \frac{2^k}{2^k 2} &= H_{2^k} + \frac{1}{2} \\ &\geq 1 + k/2 + 1/2 \\ &= 1 + (k+1)/2. \end{aligned}$$

Hence, $S(k) \Rightarrow S(k+1)$, and therefore by induction, $S(n)$ is true for all non-negative integers.

b) 1) Ind. basis: $S(1)$: $\sum_{i=1}^{1} iH_i = 1 = H_2 - 1/2$.
2) Ind. step: ind. hyp. $S(k)$: $\sum_{i=1}^{k} iH_i = \frac{k(k+1)}{2} H_{k+1} - \frac{k(k+1)}{4}$. For $n = k+1$ we have

$$\begin{aligned} \sum_{i=1}^{k+1} iH_i &= \sum_{i=1}^{k} iH_i + (k+1)H_{k+1} \\ &= \frac{k(k+1)}{2} H_{k+1} - \frac{k(k+1)}{4} + (k+1)H_{k+1} \\ &= (k+1)\left(1 + \frac{k}{2}\right) H_{k+1} - \frac{k(k+1)}{4} \\ &= (k+1)\left(1 + \frac{k}{2}\right)\left(H_{k+2} - \frac{1}{k+2}\right) - \frac{k(k+1)}{4} \\ &= \frac{(k+1)(k+2)}{2} H_{k+2} - \frac{(k+1)(k+2)}{4}. \end{aligned}$$

Hence, $S(k) \Rightarrow S(k+1)$, and therefore by induction, $S(n)$ is true for all positive integers.

Exercise 6. *Grimaldi's book (5. ed., Exercises 4.1): solve* **Exercise 11 a,b,c**

Solution 6. Rmk.: I called these numbers "triangle numbers".
a) We want to find a formula for $\sum_{i=1}^{n} t_{2i}$.

$$\begin{aligned} \sum_{i=1}^{n} t_{2i} &= 2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \\ &= n(n+1)(2n+1)/3 + n(n+1)/2 \\ &= n(n+1)(4n+5)/6 \end{aligned}$$

b) 681750
c) Check...

2. Classroom Set 4 – Solutions

Exercise 1. *Let $Y := \{1, 2, 3, 4, \ldots, 600\}$. Use the inclusion-exclusion principle to find the numbers of positive integers in Y that are not divisibile by 3 or 5 or 7.*

Solution 1. The subset $A \subset Y$ of numbers divisible 3 has 200 elements. The subset $B \subset Y$ of numbers divisible 5 has 120 elements, and the subset $C \subset Y$ of numbers divisible 7 has $\lfloor 600/7 \rfloor = 85$ elements (explain the floor function $\lfloor x \rfloor$); $|A| + |B| + |C| = 405$. The set of numbers that are divisible by 15 has $|A \cap B| = \lfloor 600/15 \rfloor = 40$ elements; the set of numbers that are divisible by 21 has $|A \cap B| = \lfloor 600/21 \rfloor = 28$ elements; and the set of numbers that are divisible by 35 has $|A \cap B| = \lfloor 600/35 \rfloor = 17$. Now, the number of elements $|A \cap B \cap C| = \lfloor 600/105 \rfloor = 5$. Therefore $|\bar{A} \cap \bar{B} \cap \bar{C}| = 275$, i.e., there 275 elements in Y that are neither divisible by 3 or 5 or 7.

Exercise 2. *Grimaldi's book (5. ed., Exercises 4.1): solve* **Exercise 1 a,b,c**

Solution 2. a), b), c) should follow by straightforward induction.

Exercise 3. *Grimaldi's book (5. ed., Exercises 4.1, page 208): solve* **Exercise 27**

Solution 3. Define the set of positive integers $T := \{n \mid n \geq n_0,\ S(n) \text{ false }\} \subset \mathbb{Z}^+$. The alternative induction principle implies that $n_0, n_0+1, \ldots, n_1 \notin T$. Assuming that the set T is non-empty implies by the well-ordering that T has a minimal element m. As $S(n_0)$ up to $S(m-1)$ are true, it follows that $S(m)$ is true, too. Therefore T must be empty, which shows that $S(n)$ is true for all $n \geq n_0$.

Exercise 4. *Use the principle of induction to show that for all natural numbers n, $4\sum_{i=1}^n i(i+2)(i+4) = n(n+1)(n+4)(n+5)$.*

Solution 4. Should follow by straightforward induction.

Exercise 5. *1) Guess a formula for $\sum_{i=1}^n (bi+c)$, where b, c are given numbers, and prove it using the principle of induction.*

2) Use the well-known result $6\sum_{i=1}^n i^2 = n(n+1)(2n+1)$ and the result of 1) to write down a formula for $\sum_{i=1}^n ai^2 + bi + c$, where a, b, c are given numbers.

Solution 5. 1) $\sum_{i=1}^n (bi+c) = b\sum_{i=1}^n i + \sum_{i=1}^n c = bn(n+1)/2 + nc$. Rest follows by an easy induction.

2) Follows from $\sum_{i=1}^n ai^2 + bi + c = a\sum_{i=1}^n i^2 + \sum_{i=1}^n bi + c$, which implies that

$$\sum_{i=1}^n ai^2 + bi + c = an(n+1)(2n+1)/6 + bn(n+1)/2 + nc.$$