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TMA4230 Functional Analysis
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Exercise set 4 – Solutions

$\boxed{1}$ Use Zorn's lemma to show the following statements:

a) Any vector space has a Hamel basis.

b) Any Hilbert space has an orthonormal basis.

Solution. When we apply Zorn's lemma we show that there exists a maximal element in some sense. In these problems we are asked to show that there exists a Hamel/orthonormal basis in a space X, so to use Zorn's lemma we need to think of a way in which these bases are maximal.

a) In the first case, i.e. showing that there is a Hamel basis, we intuitively think of a Hamel basis as a maximal linearly independent set. Our partially ordered set will therefore be the set of linearly independent subsets of X. These subsets are ordered by inclusion; if U, V are two linearly independent subsets of X we say that $U \leq V$ if $U \subset V$. To apply Zorn's lemma we need that any chain has an upper bound: if $\mathcal{C}$ is a chain of linearly independent subsets of X, an upper bound is given by the union $\cup_{U \in \mathcal{C}} U$ (check that this set is linearly independent!). Zorn's lemma then gives us a maximal linearly independent subset U of X, which means that if V is a linearly independent set with $U \subset V$, then $U = V$. We need to show that U is in fact a Hamel basis. Proceeding by contradiction, we assume that there is some vector $x \in X$ that cannot be written as a finite linear combination of elements of U. In this case the set $\{x\} \cup U$ will also be linearly independent, but this is impossible since U is a *maximal* linearly independent subset.

b)In the second case we think of an orthonormal basis as a largest possible orthonormal subset. By using Zorn's lemma as above, now with the partially ordered set given by orthonormal subsets of X, we obtain a maximal orthonormal subset U of X, and we need to show that it is an orthonormal basis. We know (from linear methods) that U is an orthonormal basis if and only if $U^{\perp} = \{0\}$. Therefore we assume that there is some $x \neq 0$ such that $x \in U^{\perp}$, we are done if we can show that this is impossible. But now we conclude just as before: in this case $\{x\} \cup U$ would be a bigger orthonormal subset, contradicting the maximality of U.

A couple of comments:

1. The maximal element produced by Zorn's lemma is in no way unique. We know that a vector space may have many different Hamel bases.
2. The bases that we have produced need not be countable. In fact, many spaces do not have countable bases.

$\boxed{2}$ Let X be a vector space and p, q sublinear functionals on X. If a linear functional φ on X satisfies

$$|\varphi(x)| \le p(x) + q(x) \quad \text{for all} \quad x \in X.$$

Then there exist linear functionals φ_1 and φ_2 on X such that $\varphi = \varphi_1 + \varphi_2$ satisfying

$$|\varphi_1(x)| \le p(x) \quad \text{and} \quad |\varphi_2(x)| \le q(x)$$

for all $x \in X$.

Hint: Relate the sublinear functional $p+q$ and the assumption on φ with the diagonal $\Delta = \{(x, x) : x \in X\} \subset X \times X$.

Solution. It should not come as a surprise that the proof will use the Hahn-Banach theorem, since we want to show the existence of linear functionals. Note that the diagonal Δ is a subspace of $X \times X$, and we can define a linear functional $\tilde{\varphi}$ on Δ by

$$\tilde{\varphi}(x, x) = \varphi(x).$$

We would like to use Hahn-Banach to extend $\tilde{\varphi}(x, x)$ to all of $X \times X$, but the Hahn-Banach theorem requires a sublinear functional defined on all of $X \times X$. The next step is therefore to introduce the sublinear functional $r(x, y) = p(x) + q(y)$ (check that it is sublinear), which is defined on $X \times X$.

Note that $|\tilde{\varphi}(x, x)| \le r(x, x)$ by assumption, so $|\tilde{\varphi}| \le r$ on Δ. The Hahn-Banach theorem allows us to extend $\tilde{\varphi}$ to a linear functional on $X \times X$ such that $|\tilde{\varphi}(x, y)| \le r(x, y) = p(x) + q(y)$. We may pick $\varphi_1(x) = \tilde{\varphi}(x, 0)$ and $\varphi_2(x) = \tilde{\varphi}_1(0, x)$ to conclude the proof.

$\boxed{3}$ Let X be $(\mathbb{R}^2, \|.\|_p)$ and $Y = \{x \in \mathbb{R}^2 : x_1 - 2x_2 = 0\}$ a subspace of $\mathbb{R}^2$. Define the linear functional φ on Y by $\varphi(x_1, x_2) = x_1$.

a) Compute the norm of φ.

b) Determine the norm-preserving linear functionals that extend to $(\mathbb{R}^2, \|.\|_p)$ for $p = 1, 2, \infty$

Solution. **a)** The space Y consists of all points of the form $(2x, x)$ for $x \in \mathbb{R}$. To find the norm of φ for $p = 1$, we need to calculate

$$\|\varphi\| = \sup_{x \ne 0} \frac{|\varphi(2x, x)|}{\|(2x, x)\|_1} = \sup_{|x| \ne 0} \frac{|2x|}{|3x|} = \frac{2}{3}.$$

Similarly, for $p = \infty$, we find

$$\|\varphi\| = \sup_{x \ne 0} \frac{|\varphi(2x, x)|}{\|(2x, x)\|_\infty} = \sup_{|x| \ne 0} \frac{|2x|}{|2x|} = 1.$$

And exactly the same calculation for $p = 2$ yields $\|\varphi\| = \frac{2}{\sqrt{5}}$.

b) One can certainly solve this problem by straightforward computations, but it is quicker to use that the dual space of $\mathbb{R}$ with the p-norm is $\mathbb{R}$ with the q-norm, where $\frac{1}{p} + \frac{1}{q} = 1$. This means that the extension $\tilde{\varphi}$ of φ is given by $\tilde{\varphi}(x, y) = ax + by$ for some $a, b \in \mathbb{R}$. Since $\tilde{\varphi}$ should extend φ, we get that

$$\tilde{\varphi}(2x, x) = 2ax + bx = 2x,$$

hence $2a + b = 2$. Now let $p = 1$. In this case $q = \infty$, and the norm of $\tilde{\varphi}$ is given by $\|\tilde{\varphi}\| = \max\{|a|, |b|\}$, and we want this to be $\frac{2}{3}$. Using the condition $2a + b = 2$, we can write this requirement as

$$\max\{|a|, 2|1 - a|\} = \frac{2}{3},$$

and we see that this holds if and only $a = \frac{2}{3}$. Hence the norm-preserving extension for $p = 1$ is unique and given by $\tilde{\varphi}(x, y) = \frac{2}{3}(x + y)$.

The calculations for $p = 2$ are similar, but a bit messier.