[S]=T. Sauer, Numerical Analysis, Second International Edition, Pearson, 2014

## “Teorioppgaver”

1 Oppgave 6.1.3 (b), (e), s. 291, [S]

**Solution:**

**(b):** Separation of variables leads to the equation $\int dy/y = \int t^2\,dt$, or $\log(y) = t^3/3 + C$ where $C$ is an arbitrary integration constant. Thus $y(t) = \exp(t^3/3 + C)$, where $C = 0$ from the equation $y(0) = 1$.

**(e):** Here we get $\int y^2\,dy = \int 1\,dt$, or $y^3/3 = t + C$, $y(t) = (3t + 3C)^{1/3}$. Finally $y(0) = 1$ and therefore $3C = 1$.

2 Oppgave 6.1.4 (a), (b), s. 291, [S]

**Solution:**

**(a):** The general solution to the homogeneous system $y' = y$ is $y_0(t) = C\exp(t)$, where $C$ is an arbitrary constant.

Further, a particular solution to the system $y' = t + y$ can be search in the form $y_1(t) = at + b$, from which it follows that $a = b = -1$.

Thus we can put $y(t) = y_0(t) + y_1(t)$. The constant $C$ is then determined from the equation $y(0) = 0$ and thus $y(t) = -t - 1 + \exp(t)$.

**(b):** Similarly to (a), $y_0(t) = C\exp(-t)$, $y_1(t) = t - 1$, and as a result $y(t) = t - 1 + \exp(-t)$.

3 Oppgave 6.1.9, s. 292, [S]

**Solution:**

**(a):** Here $f(t, y) = t$ - independent from $y$ and thus $f$ is uniformly Lipschitz continuous with respect to $y$ on $[a, b] \times (-\infty, +\infty)$ with the constant $L = 0$. Theorem 6.2 then guarantees the existence and uniqueness of solutions on an arbitrary interval.

**(b), (c):** Here $f(t, y) = \pm y$, which is a linear function with the slope $\pm 1$. In either case $f$ is uniformly Lipschitz continuous with respect to $y$ with constant $L = 1$ on

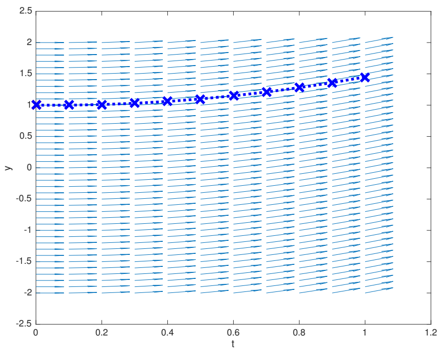
$[a, b] \times (-\infty, +\infty)$, regardless of $a, b$. Theorem 6.2 then guarantees the existence and uniqueness of solutions on an arbitrary interval.

**(d):** Here $f(t, y) = -y^3$. This function is independent from $t$ and is continuously differentiable with respect to $y$. Thus it is uniformly Lipschitz continuous on any finite square $S = [a, b] \times [\alpha, \beta]$ with the Lipschitz constant $L = \max_{y\in[\alpha,\beta]} |df/dy| = \max_{y\in[\alpha,\beta]} |3y^2|$. Note that the Lipschitz constant depends and "grows" with the interval $[\alpha, \beta]$.

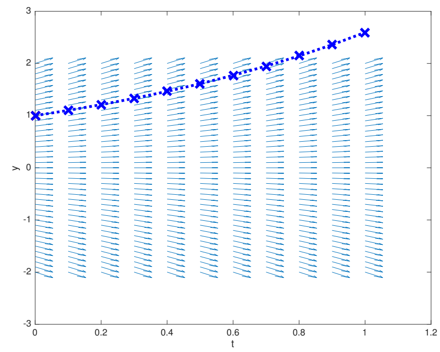
Thus Theorem 6.2 guarantees the existence and uniqueness of solutions on some sub-interval $[a, c]$, $a < c \le b$. As a result we cannot guarantee existence of solutions on the whole interval $[0, 1]$, only on a sub-interval.

4 Oppgave 6.1.10, s. 292, [S]

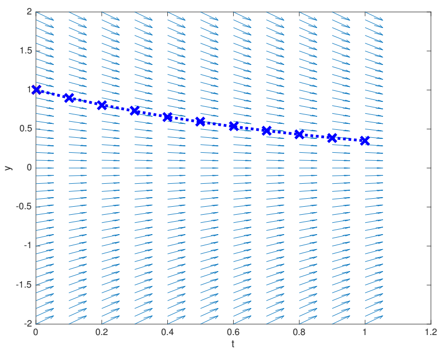
**Solution:**



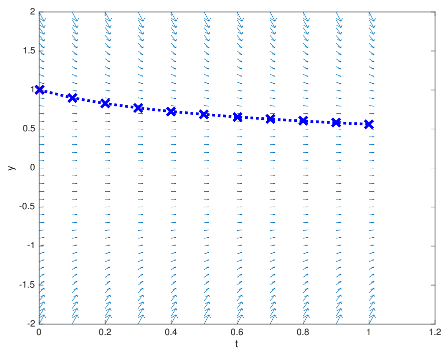

(a)




(b)




(c)




(d)

5 Oppgave 6.1.11, s. 292, [S]

**Solution:**

**(a):** $y(t) = t^2/2 + y(0)$. Theorem 6.3 is clearly verified with $L = 0$.

**(b,c):** $y(t) = y(0)\exp(\pm t)$. Theorem 6.3 is verified with $L = 1$. In fact, in (c) any non-negative $L$ or even $L \ge -1$ is sufficient, but this is of course is not a valid value for the Lipschitz constant.

**(d):** If $y(0) = 0$ we can use the solution $y(t) \equiv 0$. For $y(0) = 1$ we can use separation of variables to find that $y(t) = (2t + C)^{-1/2}$, where $1 = y(0) = C^{-1/2}$. Note that the difference between the two solutions decreases with time, and therefore the estimate of Theorem 6.3 holds with any $L \geq 0$ in this case.

$\boxed{6}$ Oppgave 6.2.2, s. 302, [S]

**Solution:**

**(a):**

```
t_i y_i
0.000000e+00 0.000000e+00
2.500000e-01 3.125000e-02
5.000000e-01 1.416016e-01
7.500000e-01 3.533020e-01
1.000000e+00 6.948557e-01

err =

    0.0234
```

**(b):**

```
t_i y_i
0.000000e+00 0.000000e+00
2.500000e-01 3.125000e-02
5.000000e-01 1.103516e-01
7.500000e-01 2.268372e-01
1.000000e+00 3.725290e-01

err =

    0.0046
```

$\boxed{7}$ Oppgave 6.3.3, s. 302, [S]

**Solution:**

We introduce a new variable $z = y'$ so that $z' = y''$.

**(a):**

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} z \\ ty \end{pmatrix}$$

**(b):**

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} z \\ 2tz - 2y \end{pmatrix}$$

**(c):**

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} z \\ tz + y \end{pmatrix}$$

$\boxed{8}$ Consider the initial value problem

$$y'(t) = \lambda y(t), \qquad t > 0$$
$$y(0) = y_0,$$

where $\lambda \in \mathbb{C}$. Its solution is $y(t) = y_0 \exp(\lambda t)$.

Suppose that we use a numerical method (such as e.g. forward Euler or explicit trapezoid) to solve this problem starting from a point $w_0 = y_0$. *The stability region* for the method is a set of points $z = \lambda h$ in the complex plane, such that the numerical solution $(w_0, w_1, \dots)$ stays *bounded* (i.e., $\exists C > 0 : \forall i, |w_i| \leq C$).

Find the stability region for (a) forward Euler method; (b) explicit Trapezoid method.

**Solution:**

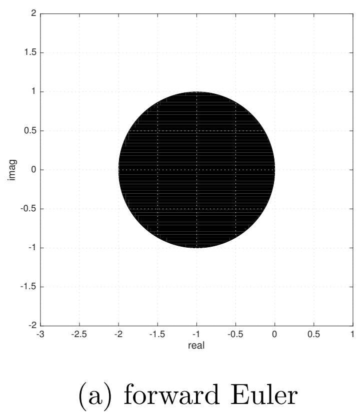
**(a):** In this case $w_k = w_{k-1} + hf(t_{k-1}, w_{k-1}) = (1 + h\lambda)w_{k-1} = (1 + h\lambda)^k w_0$. Thus $w_k$ stays bounded iff $|1 + h\lambda| \leq 1$. That is, the stability region for the forward Euler method is a circle in the complex plane of radius 1 around the point $-1$.

Note: this implies, in particular, that if $\lambda = i\omega$ is purely imaginary then there is no $h > 0$ such that $h\lambda$ is in the stability region. Thus whereas $\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t)$ is oscillatory (bounded) in this case, the Euler's method will result in an unbounded solution (long term behaviour) regardless of how small we chose $h$.

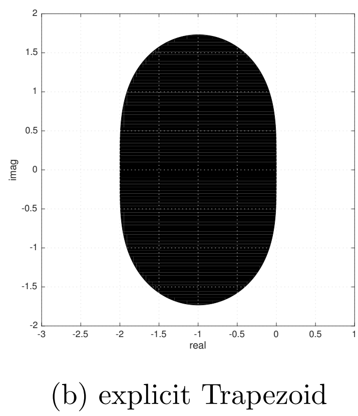
**(b):** Now we have $w_k = w_{k-1} + h/2[f(t_{k-1}, w_{k-1}) + f(t_{k-1} + h, w_{k-1} + hf(t_{k-1}, w_{k-1}))] + w_{k-1} + h/2[\lambda w_{k-1} + \lambda(w_{k-1} + h\lambda w_{k-1})][1 + h\lambda + 0.5(h\lambda)^2]w_{k-1} = [1 + h\lambda + 0.5(h\lambda)^2]^k w_0$. Thus $z = h\lambda$ is in the stability region of the explicit trapezoid method if and only if $|1 + z + 0.5z^2| \leq 1$.

For an arbitrary purely imaginary number $\lambda = i\omega$ and any $h > 0$ we have $|p(i\omega h)| = |1 - 0.5\omega^2 h^2 + i\omega h| = [(1 - 0.5\omega^2 h^2)^2 + \omega^2 h^2]^{1/2} = [1 + \omega^4 h^4/4]^{1/2} > 1$ as in the case of forward Euler, with the same implications. However, for small $h > 0$ we can use a first order Taylor series expansion to get $[1 + \omega^4 h^4/4]^{1/2} \approx 1 + \omega^4 h^4/8$ which is much closer to 1 than $|1 + i\omega h| = (1 + \omega^2 h^2)^{1/2} \approx 1 + \omega^2 h^2/2$ (forward Euler).

Here is the plot of stability regions:




(a) forward Euler



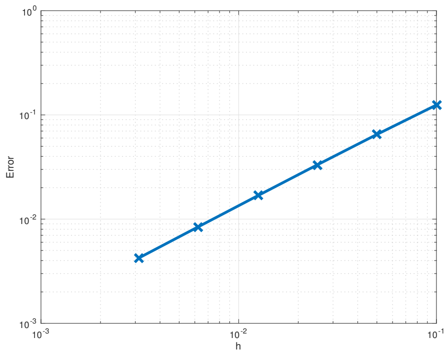

(b) explicit Trapezoid

# "Computeroppgaver"

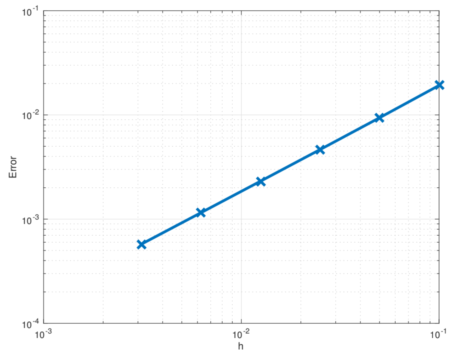
**9** Oppgave 6.1.5, s. 293, [S]

**Solution:**

See `oppgave_6_1_5.m`




(a)

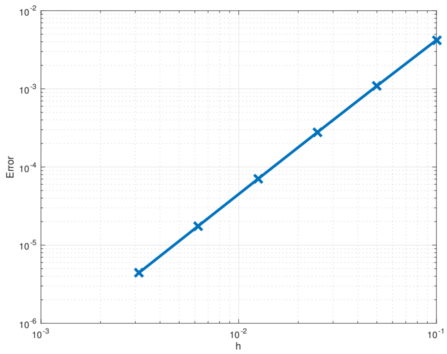



(b)

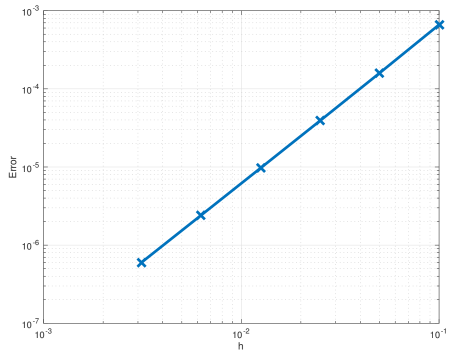
**10** Repeat the previous exercise, but use the explicit trapezoid method instead.

**Solution:**

See `oppgave_10.m`




(a)

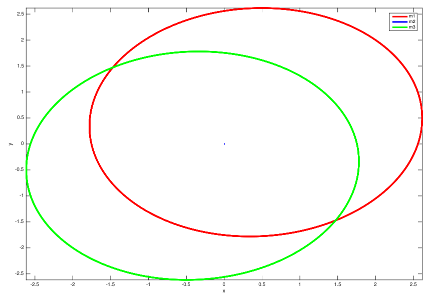



(b)

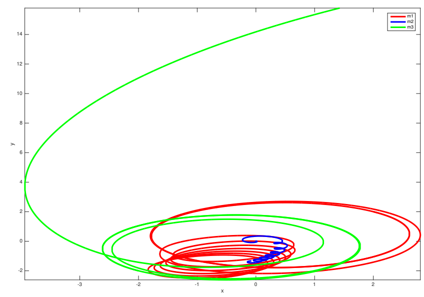
**11** Oppgave 6.3.10, s. 314, [S]. Use the initial conditions specified in the book but different masses: $m_1 = m_3 = 0.03$, $m_2 = 0.3$. Use the explicit Trapezoid method.

**Solution:**

See `three_body_problem.m`



$x_1'(0) = 0.2$

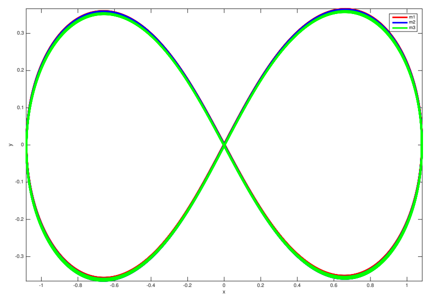


$x_1'(0) = 0.20001$

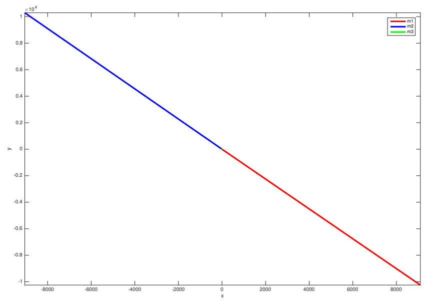
$\boxed{12}$ Oppgave 6.3.11, s. 314, [S]. Use the explicit Trapezoid method.

**Solution:**

See `three_body_problem.m`



(a)



(b)