HOMEWORK 1
SIGNED MEASURES, HAHN AND JORDAN DECOMPOSITION THEOREMS

Let $(\Omega, \mathcal{F})$ be a measurable space and let μ be a *signed* measure on it.

Problem 1. Let $E_1, E_2, \dots$ be $\mathcal{F}$-measurable disjoint sets. Prove that if $\mu(\cup_{k=1}^{\infty} E_k) < \infty$ then $\sum_{k=1}^{\infty} \mu(E_k)$ is absolutely summable, meaning that:

$$\sum_{k=1}^{\infty} \big|\mu(E_k)\big| < \infty.$$

Hint: Use the Riemann rearrangement theorem, which you could find online here: `https://en.wikipedia.org/wiki/Riemann_series_theorem`.

Problem 2. Verify that the difference $\mu := \mu_1 - \mu_2$ of two *positive* measures at least one of which is finite, is a signed measure.

Problem 3. Let $(\Omega, \mathcal{F}, \mathrm{m})$ be an honest measure space and let $f \in L^1(\mathrm{m})$. Define the map $\mathrm{m}_f \colon \mathcal{F} \to [-\infty, \infty]$ by

$$\mathrm{m}_f(E) := \int_E f \, d\mathrm{m} = \int_\Omega f \, \mathbf{1}_E \, d\mathrm{m} \quad \text{for all } E \in \mathcal{F}.$$

(a) Verify that m_f is a finite signed measure on $(\Omega, \mathcal{F})$.
(b) Prove that for any $\mathcal{F}$-measurable function $g \colon \Omega \to \mathbb{R}$ we have

$$\int_\Omega g \, d\mathrm{m}_f = \int_\Omega g \, f \, d\mathrm{m}.$$

Problem 4. Let E, F be two $\mathcal{F}$-measurable sets. Prove that if $E \subset F$ and $\big|\mu(F)\big| < \infty$ then $\big|\mu(E)\big| < \infty$.

Problem 5. (monotone convergence theorem for sets) Let $E_1, E_2, \dots$ be a sequence of $\mathcal{F}$-measurable sets.

(a) If $E_1 \subset E_2 \subset \dots$ then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} \mu(E_n)$.

(b) If $E_1 \supset E_2 \supset \dots$ and $\big|\mu(E_1)\big| < \infty$ then $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n \to \infty} \mu(E_n)$.

Problem 6. Let $\mu := \lambda - \delta_a$, where λ is the Lebesgue measure on $\mathbb{R}$ and δ_a is the Dirac measure on $\mathbb{R}$ centered at the point $a \in \mathbb{R}$.

Find two sets P and N representing a Hahn decomposition of μ.

Problem 7. Find two sets P and N representing a Hahn decomposition of the signed measure m_f in Problem 3.

Problem 8. Prove the *uniqueness* of the Jordan decomposition $\mu = \mu^+ - \mu^-$ of a signed measure μ.

Problem 9. For every $\mathcal{F}$-measurable set E show that:

(a) $\big|\mu(E)\big| \le \big|\mu\big|(E)$. This of course implies that $\mu(E)$ and $-\mu(E)$ are both $\le \big|\mu\big|(E)$.

(b) If $\Omega = P \sqcup N$ is a Hahn decomposition of μ, then $\big|\mu\big|(E) = \mu(E \cap P) - \mu(E \cap N)$.
In particular, $\big|\mu\big|(\Omega) = \mu(P) - \mu(N)$.

(c) $\big|\mu\big|(E) = \max\left\{ \sum_{k=1}^{\infty} \big|\mu(E_k)\big| \ : E_1, E_2, \dots \text{ are disjoint and } E = \bigcup_{k=1}^{\infty} E_k \right\}$.

Problem 10. Let $\mathcal{M}(\Omega)$ be the space of all finite signed measures on $(\Omega, \mathcal{F})$. For every $\mu \in \mathcal{M}$ let

$$\|\mu\| := \big|\mu\big|(\Omega).$$

Prove that $(\mathcal{M}, \|\ \|)$ is a normed space.
In fact, this is a Banach space (you may try proving this as well).

Hint: The trickier part is verifying the triangle inequality $\|\mu_1 + \mu_2\| \le \|\mu_1\| + \|\mu_2\|$.

Begin by considering a Hahn decomposition $\Omega = P \sqcup N$ for the signed measure $\mu := \mu_1 + \mu_2$. Use Problem 9 part (b) for μ and then part (a) for μ_1 and μ_2.