TMA4205 Numerical Linear Algebra Fall 2017

Exercise set 6

$\boxed{1}$ Perform two steps of the the Rayleigh quotient iteration with starting vector $v^{(0)} = (1, 0, 0)^T$ for approximating an eigenvalue and eigenvector of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Possible solution:

We start with computing

$$\lambda_0 = (v^{(0)})^T A v^{(0)} = 2.$$

Now we need to find the solution w of $(A - 2\,\mathrm{Id})w = v^{(0)}$, that is, of

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix} w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

A short computation yields

$$w = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now we compute $v^{(1)}$ by normalising w and obtain

$$v^{(1)} = \frac{1}{\sqrt{2}} w = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

In the second step, we first compute

$$\lambda_1 = (v^{(1)})^T A v^{(1)} = \frac{5}{2}.$$

In the next step we therefore need to solve the equation $(A - \frac{5}{2}\,\mathrm{Id})w = v^{(1)}$, or

$$\begin{pmatrix} -1/2 & 1 & 1 \\ 1 & -3/2 & -2 \\ 0 & 0 & -3/2 \end{pmatrix} w = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We obtain

$$w = \frac{1}{\sqrt{2}} \begin{pmatrix} 10 \\ 6 \\ 0 \end{pmatrix}.$$

Normalising this vector yields

$$v^{(2)} = \frac{1}{\sqrt{34}} \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}.$$

(Note that $\lambda_2 = (v^{(2)})^T A v^{(2)} = 89/34$, and the matrix A has an eigenvalue $(3+\sqrt{5})/2$, which is closer than 0.0004 to λ_2.)

2 Perform one step of the QR-iteration with shift $\mu = 2$ in order to find the eigenvalues of the matrix

$$A = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}.$$

Possible solution:

The shifted matrix is

$$A - 2\,\mathrm{Id} = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}.$$

A possible QR-decomposition of this matrix is

$$\begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix} = QR = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 12/5 \\ 0 & -9/5 \end{pmatrix}.$$

Multiplying R with Q yields

$$RQ = \begin{pmatrix} 5 & 12/5 \\ 0 & -9/5 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 136 & -27 \\ -27 & -36 \end{pmatrix}.$$

Adding back the shift 2 yields as a result

$$A^{(1)} = \frac{1}{25} \begin{pmatrix} 186 & -27 \\ -27 & 14 \end{pmatrix}.$$

3 Assume that you apply one step of the QR-iteration with shift μ in order to find the eigenvalues of a matrix A, and that this shift is actually equal to one of the eigenvalues of A. How can you easily detect this situation based on the QR-decomposition of the shifted matrix?

Possible solution:

If the shift μ is actually equal to one of the eigenvalues, then the matrix $A - \mu\,\mathrm{Id}$ does not have full rank. Since the matrix Q in the QR-decomposition of $A - \mu\,\mathrm{Id}$ is orthogonal, it always has full rank, which implies that the matrix R cannot have full rank. This, however, is only possible, if one of the diagonal entries of R is equal to 0.