TMA4125 Matematikk 4N
Spring 2017

Exercise set 11

1 In this problem we consider numerical solution of the differential equation

$$y'(t) = -2y(t) - \exp(y(t)) + t^2, \quad y(0) = 1$$

a) Perform two steps of both the Euler method and the improved Euler method with a step size of $h = 0.5$ to solve the equation.

b) Show that applying the backward Euler method with $h = 1$ gives the following equation for y_{n+1} which we must solve:

$$3y_{n+1} + \exp(y_{n+1}) - y_n - t_{n+1}^2 = 0$$

c) Perform one step of the backward Euler method, using two iterations of Newton's method to solve the equation for y_1 (see Kreyszig 19.2 for revision of Newton's method)

2 (Adapted from an exam question, autumn 2014) Consider the differential equation

$$y'(t) = -1000y(t), \quad y(0) = 1,$$

which has the exact solution $y = \exp(-1000t)$.

a) Suppose we find approximate solutions $y_n \approx y(t_n)$ using the improved Euler method (see Kreyszig 21.1). By first writing y_{n+1} in terms of y_n, show that

$$y_n = \left(1 - 1000h + \frac{10^6}{2}h^2\right)^n$$

Explain why $y_n \to \infty$ as $n \to \infty$ if $h = 0.1$. Note that this in stark contrast to the behaviour of the exact solution, which tends to zero as $t \to \infty$.

b) (optional) Show that $y_n \to 0$ as $n \to \infty$ if and only if the step size $h < \frac{1}{500}$.

c) Now find a formula for y_n using the backward Euler scheme, and hence show that in this case $y_n \to 0$ as $n \to \infty$ for all values of $h > 0$.

3 Here, we study Runge-Kutta methods.

a) Use the classical RK4 scheme (see Kreyszig, Table 21.3) to approximate the value of $y(1)$ if $y(t)$ satisfies the differential equation

$$\dot{y} - ty^2 = 0, \quad y(0) = 1$$

Use a step size $h = 1$.

b) (Optional) Look at the notes on Butcher tableaux on the course website (under fremdriftsplan). A 3-stage Runge-Kutta method of order 3 is encoded in the following tableau:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

Is this method explicit or implicit? Write down formulae for k_1, k_2, k_3 and y_{n+1} in terms of f, y_n and h (and when necessary, other k_i).

$\boxed{4}$ (Optional) Write a program that implements the RK4 scheme. Try applying this to the equation

$$\dot{y} = (y - t - 1)^2 + 2, \quad y(0) = 1,$$

and compare your answers to the exact solution given in the textbook on page 904.