HOMEWORK 9
THE CENTRAL LIMIT THEOREM

Problem 1. Let $X_1, X_2, \ldots$ be a sequence of scalar random variables, and let X be another scalar random variable. The goal is to complete the proof of the following fact discussed in class.

If $X_n \to X$ in distribution, then $F_{X_n}(t) \to F_X(t)$ for any point t where F_X is continuous. (Recall that F_Y denotes the cumulative distribution function of the random variable Y).

The remaining part for you to prove is that for any such point t and given any $\epsilon > 0$,

$$F_{X_n}(t) \leq F_X(t) + \mathcal{O}(\epsilon),$$

if n is large enough.

Problem 2. Let $X_1, X_2, \ldots$ be a sequence of scalar random variables, and let X be another scalar random variable. Prove that if $X_n \to X$ in *probability*, then $F_{X_n}(t) \to F_X(t)$ for any point t where F_X is continuous.

Note: Of course you could just say: if $X_n \to X$ in probability then it converges in distributions as well, so you could just apply the previous problem. But I want that you prove this result directly, without using the previous problems, since it is much easier.

Let X be a scalar random variable and let μ_X be its probability distribution. Compute:
(a) Then mean μ;
(b) The standard deviation σ;
(c) The characteristic function $\varphi_X(t) := \mathbb{E}\, e^{itX}$
for the following examples.

Problem 3. The Bernoulli r.v. with values 1 and -1 with equal probabilities $\frac{1}{2}$.

Problem 4. The standard normal distribution $N(0, 1)$.
If you do the calculation correctly, you should get $\varphi(t) = e^{-t^2/2}$.

Problem 5. The uniform distribution of the interval $(0, 1)$.

The next problem will tell us that under appropriate assumptions, derivates and integrals may be interchanged (just like limits and integrals may be interchanged). We will need it for the proof of the central limit theorem.

Problem 6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $I \subset \mathbb{R}$ be an interval. Given any absolutely integrable function $f \colon I \times \Omega \to \mathbb{C}$, define

$$\varphi(t) := \int_\Omega f(t,\omega) d\mu(\omega).$$

We assume that for every $\omega \in \Omega$, the function $I \ni t \mapsto f(t,\omega) \in \mathbb{C}$ is differentiable at every point $t \in I$ and that its derivative satisfies

$$\left| \frac{d}{dt} f(t,\omega) \right| \leq g(\omega) \quad \text{for all } t \in I,$$

where $g \in L^1(\Omega, \mu)$.

Prove that φ is differentiable at all points $t \in I$ and

$$\frac{d}{dt}\varphi(t) = \int_\Omega \frac{d}{dt} f(t,\omega)\, d\mu(\omega).$$

Hint: Interpret the derivative as a limit and use dominated convergence.

Problem 7. Prove that if X is a scalar random variable with $\mathbb{E}\,|X| < \infty$, then its characteristic function $\varphi_X(t)$ is differentiable everywhere and

$$\varphi_X'(t) = i\,\mathbb{E}\,(Xe^{itX}).$$

Hint: The random variable X is modeled by some measurable function $X \colon \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then simply apply the previous problem to the function $f \colon \mathbb{R} \times \Omega \to \mathbb{C}$, defined by $f(t,\omega) := e^{i\,t\,X(\omega)}$.