HOMEWORK 2
THE LEBESGUE-RADON-NIKODYM THEOREM AND ITS CONSEQUENCES

Let $(\Omega, \mathcal{F}, \mathrm{m})$ be a σ-finite measure space.

Problem 1. Given $f \in L^1(\mathrm{m})$, consider the associated signed measure m_f and prove that its (total variation) norm is simply the norm of f in $L^1(\mathrm{m})$, that is,

$$\|\mathrm{m}_f\| = \int_\Omega |f| \, d\mathrm{m}.$$

Problem 2. Given $f \in L^1(\mathrm{m})$, prove that the associated signed measure m_f is in fact unsigned if and only if $f(x) \geq 0$ for m-a.e. $x \in \Omega$.

Problem 3. Assume that Ω is *countable* and that $\mathcal{F} = \mathcal{P}(\Omega)$, meaning that $\mathcal{F}$ is the collection of all subsets of Ω. Show that every measure on $(\Omega, \mathcal{F})$ is absolutely continuous w.r.t. the counting measure $\#$.

Hint: Recall that the counting measure $\#(E)$ is defined as the number of elements of E if E is finite, and $+\infty$ otherwise. It helps to know (verify this first), that if $f \colon \Omega \to [0, \infty)$, then

$$\int_\Omega f \, d\# = \sum_{k \in \Omega} f(k).$$

Problem 4. Let δ_a be the Dirac measure centered at some point $a \in \Omega$. Prove that if μ is absolutely continuous w.r.t. δ_a, then there is a constant $c \in \mathbb{R}$ such that $\mu = c \, \delta_a$.

Problem 5. Prove that if μ_1 and μ_2 are finite, signed measures on $(\Omega, \mathcal{F})$ such that $\mu_1 \perp \mathrm{m}$ and $\mu_2 \perp \mathrm{m}$, then $(\mu_1 - \mu_2) \perp \mathrm{m}$. In fact, you may prove that the set $\mathrm{m}^\perp$ of all finite signed measures that are singular w.r.t. m is a vector subspace of $\mathcal{M}(\Omega)$.

Problem 6. Prove that if $\{\mu_n\}_{n \geq 1}$ is a sequence of finite, signed measures on $(\Omega, \mathcal{F})$ such that $\mu_n \perp \mathrm{m}$ for all $n \geq 1$, and if, as $n \to \infty$, $\mu_n \to \mu$ strongly, then the limit measure μ also satisfies $\mu \perp \mathrm{m}$.

Problem 7. Let μ be a finite, positive measure on $(\Omega, \mathcal{F})$ and let $\{f_n\}$ be a sequence of non-negative $\mathcal{F}$-measurable functions such that $\mathrm{m}_{f_n} \leq \mu$ for all $n \geq 1$.

Define $f := \sup_{n \geq 1} f_n$ and prove that $\mathrm{m}_f \leq \mu$.

Hint: It would be nice if $\{f_n\}$ was *increasing*, because then we could apply the monotone convergence theorem (MCT).

If it is not increasing, just replace it with $g_n := \max\{f_1, f_2, \ldots, f_n\}$. Then the sequence $\{g_n\}$ is increasing and it satisfies all the properties that $\{f_n\}$ satisfies, meaning: it is non-negative, measurable, $\mathrm{m}_{g_n} \leq \mu$ and $\sup_{n\geq 1} g_n = f$.

Of course, you have to verify these claims.