Norwegian University of Science
and Technology
Department of Mathematical
Sciences

TMA4205 Numerical Linear Algebra Fall 2017

Exercise set 2

1 We return to solving the one dimensional Poisson problem that was discussed in the first exercise set.

$$\begin{aligned} -\frac{\partial^2 u}{\partial x^2} &= 4\pi^2 \sin(2\pi x), \quad && x \in [0,1], \\ u &= 0, && x \in \{0,1\}. \end{aligned}$$

Let us again use the finite difference method on a uniform grid with step-size $h = 1/n$ and grid points $x_j = jh$. The discretized equations can be expressed as $Au = b$ where A represents the discrete Laplacian. We now want to solve this system of linear equations by three different iterative methods: the Jacobi iteration, steepest descent (SD), and minimal residual (MR) iteration.

a) Suppose that we want to reduce the initial error by 5 orders of magnitude. Estimate the number of iterations required in the Jacobi method and with SD.

b) Suppose that we want to reduce the initial residual by 5 orders of magnitude. Estimate the number of iterations required in MR.

c) Discuss the computational cost (complexity) of the three iterative methods.

d) What are the relative advantages (if they exist) of the various methods, both in terms of solving the Poisson problem, and in the more general context of solving linear systems?

2 **Saad, Exercise 5.3** In Section 5.3.3, it is shown that using a one-dimensional projection method with $\mathcal{K} = \text{span}\{A^T r\}$ and $\mathcal{L} = \text{span}\{AA^T r\}$ is equivalent to using the steepest descent method on the normal equations $A^T Ax = A^T b$ for a matrix $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$.

Show that an orthogonal projection method with solution space $\mathcal{K} = \mathcal{L}$ for solving the equation $A^T Ax = A^T b$ is equivalent to applying a projection method onto $\mathcal{K}$ orthogonally to $\mathcal{L} = A\mathcal{K}$ for the problem $Ax = b$.

3 Assume that a real matrix A is anti-symmetric, that is, $A^T = -A$. Describe the structure of the Hessenberg matrix H_m resulting from Arnoldi process in this case. Explain how this structure can be utilized for an efficient implementation of the Arnoldi algorithm.

4 Show that both CG and GMRES are scaling independent. That is, when applied to a system $(\lambda A)x = \lambda b$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ ($\lambda > 0$ in the case of CG in order to

preserve positive definiteness) they produce exactly the same sequence of iterates in exact arithmetics. (Of course, since the residual vectors will be scaled by λ, this may affect stopping criteria for the methods.)