Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2017

Exercise set 1

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

$\boxed{1}$ Based on your exposure to mathematics in school and at university answer the following questions.

a) State your favorite mathematical theorem, explain all the notions of the statement and explain in a few words your choice.

b) Give three applications of mathematics to real-world problems.

The answers should be given such that your fellow students in the course are able to understand them.

$\boxed{2}$ Let X, Y and Z be sets.

a) Show that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

b) Show that $X \backslash (Y \cup Z) = (X \backslash Y) \cap (X \backslash Z)$.

Solution. a) We want to show that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$, and it is enough to show that $x \in X \cap (Y \cup Z) \iff x \in (X \cap Y) \cup (X \cap Z)$. We show this by the following chain of equivalences:

$$\begin{aligned}
x \in X \cap (Y \cup Z) &\iff x \in X \text{ and } x \in Y \cup Z && \text{by the definition of } \cap \\
&\iff [x \in X \text{ and } x \in Y] \textbf{ or } [x \in X \text{ and } x \in Z] && \text{by definition of } \cup \\
&\iff x \in (X \cap Y) \cup (X \cap Z) && \text{by definition of } \cup.
\end{aligned}$$

By following these equivalences, we have shown that $x \in X \cap (Y \cup Z) \iff x \in (X \cap Y) \cup (X \cap Z)$.

b) We now want to show that $X \backslash (Y \cup Z) = (X \backslash Y) \cap (X \backslash Z)$, and we will do so by showing that $x \in X \backslash (Y \cup Z) \iff x \in (X \backslash Y) \cap (X \backslash Z)$.

$$
\begin{aligned}
x \in X\backslash(Y \cup Z) &\iff x \in X \text{ and } x \notin Y \cup Z && \text{definition of } \backslash \\
&\iff x \in X \text{ and } x \in Y^C \cap Z^C && \text{de Morgan's law} \\
&\iff x \in X \text{ and } x \notin Y \text{ and } x \notin Z && \text{definition of } \cap \text{ and complement} \\
&\iff [x \in X \text{ and } x \notin Y] \textbf{ and } [x \in X \text{ and } x \notin Z] \\
&\iff x \in X\backslash Y \cap X\backslash Z && \text{definition of } \cap \text{ and } \backslash
\end{aligned}
\qquad \blacksquare
$$

$\boxed{3}$ Show that the sets $\mathbb{Z}$ of integers and $\mathbb{Q}$ of rational numbers are countable.

Solution. Let us start by showing that $\mathbb{Z}$ is countable. The quick way of solving this is to use proposition 2.4.4 in the lecture notes: countable unions of countable subsets are themselves countable. In this case $\mathbb{Z}$ is the union of three countable sets: the positive integers (countable by definition), the negative integers (obviously countable - make sure that you would know how to prove it!) and $\{0\}$ – hence $\mathbb{Z}$ is countable.

For those interested, we also solve the problem using the definition in a way that hopefully makes the result obvious. We need to find a bijection φ from $\mathbb{Z}$ to $\mathbb{N}$. To construct φ, we need to assign to each integer a natural number. There is an obvious way of doing this:

Integer n	Natural number $\varphi(n)$
...	...
-3	7
-2	5
-1	3
0	1
1	2
2	4
3	6
...	...

It is not difficult to find the general formula for φ:

$$\varphi(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2|n| + 1 & \text{if } n < 0 \\ 1 & \text{if } n = 0. \end{cases}$$

We leave it to the reader to check that φ is bijective - it is not difficult.

Not let us turn to $\mathbb{Q}$. Any number in $\mathbb{Q}$ can be written in a unique way as $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ have no common divisor (this last statement means, for instance, that we would write $\frac{1}{5}$ and not $\frac{10}{50}$). By sending $\frac{p}{q}$ to $(p, q) \in \mathbb{Z} \times \mathbb{N}$, we have actually defined an injection [1] from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{N}$. By proposition 2.4.4, $\mathbb{Z} \times \mathbb{N}$ is countable. Thus $\mathbb{Q}$ is at most countable, since we have an injection from $\mathbb{Q}$ into a countable set. Since $\mathbb{Q}$ is certainly not finite, it must be countable by proposition 1.4.3.

[1] check that you see why the map $\frac{p}{q} \mapsto (p, q)$ is injective

Another way to prove that $\mathbb{Q}$ is countable is to find a surjective map from $\mathbb{N}$ to $\mathbb{Q}$. By the lecture notes, this would imply that $\mathbb{Q}$ is countable. Since we have seen that $\mathbb{Z}$ and $\mathbb{N}$ are countable, we know by the lecture notes that $\mathbb{Z} \times \mathbb{N}$ is countable. Hence there exists a bijection $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$. Furthermore, we define a surjection $g : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ by $g(m, n) = \frac{m}{n}$. Since both f and g are surjective, it follows that $g \circ f : \mathbb{N} \to \mathbb{Q}$ is surjective, which is what we needed.

$\boxed{4}$ Define functions on $\mathbb{R}$ with values in $\mathbb{R}$. (i) A function that is not left invertible; (ii) A function that is not right invertible. Show that the given functions have their respective properties.

Solution. i) This is, by the lecture notes, the same as finding a function that is not injective. The function f defined by $f(x) = x^2$ is such a function. It is not injective, since $f(-1) = f(1) = 1$. **ii)** We need to find a function that is not surjective. The same function as before will actually work, since its image contains no negative values. A slightly more interesting example is the function $x \mapsto e^x$, which is injective yet not surjective.

$\boxed{5}$ Given the linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T = Ax$ with

$$A = \begin{pmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{pmatrix}.$$

a) Show that the matrix

$$A_l^{-1} = \frac{1}{9}\begin{pmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{pmatrix}$$

induces a left inverse T_l^{-1} of T.
This left inverse is not unique. Show that

$$\frac{1}{2}\begin{pmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{pmatrix}$$

gives another left inverse.

b) Turn this example into one for right inverses. Concretely, find a mapping $S : \mathbb{R}^3 \to \mathbb{R}^2$ that is based on the mapping T and give a right inverse for this mapping.

Solution.
a) A_l^{-1} "induces a left inverse T_l^{-1} of T" if we define $T_l^{-1}y = A_l^{-1}y$ for $y \in \mathbb{R}^3$. To check that this is indeed a left inverse, we need to check that $T_l^{-1}Tx = x$ for any $x \in \mathbb{R}^2$. By the definitions of the mappings, we need to check that $A_l^{-1}Ay = y$ for any $y \in \mathbb{R}^3$, or, equivalently, that $A_l^{-1}A$ is the identity matrix:

$$A_l^{-1}A = \frac{1}{9}\begin{pmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{pmatrix}\begin{pmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly we can show that the other matrix gives a left inverse, since

$$\frac{1}{2}\begin{pmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) The simplest way of finding such an operator S and a right inverse S_r^{-1} is to exploit some properties of the transpose of matrices. We know from linear algebra that if X and Y are matrices such that the matrix product XY is defined, then $(XY)^T = Y^T X^T$. In the previous problem we found that $A_l^{-1}A = I$, where I denotes the identity matrix. Taking the transpose we find that $I = I^T = (A_l^{-1}A)^T = A^T(A_l^{-1})^T$. Hence, if we define S to be the mapping induced by A^T, we see that the mapping induced by $(A_l^{-1})^T$ is a right inverse of this S.