Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2017

Solutions to E9

$\boxed{\text{E9}}$ Show that the phase diagram of

$$\ddot{x} - \epsilon x\dot{x} + x = 0$$

has a center in the origin.

Set $y = \dot{x}$, to see that the equation can be written as $\dot{y} = -x(1 - \epsilon y)$. Substitute $z = 1 - \epsilon y$ to get

$$\dot{x} = y = \frac{1}{\epsilon}(1 - z),$$
$$\dot{z} = -\epsilon\dot{y} = \epsilon xz.$$

From the system of equations we get

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \epsilon^2 \frac{xz}{1-z}.$$

This is a separable differential equation. Rewriting, to get the z and x dependency on different sides of the equation, yields

$$\left(\frac{1}{z} - 1\right)\mathrm{d}z = \epsilon^2 x \mathrm{d}x.$$

Integration then gives

$$\ln|z| - z = \frac{1}{2}\epsilon^2 x^2 + C$$

where C is some constant. Substituting back y gives

$$\ln|1 - \epsilon y| + \epsilon y = \frac{1}{2}(\epsilon x)^2 + C.$$

For simplicity, scale the variables by letting $\epsilon x \to x$ and $\epsilon y \to y$. This gives

$$x(y) = \pm\sqrt{2}\sqrt{\ln|1-y| + y + C}.$$

Since $x(y) = \pm\sqrt{2(\ln|1-y| + y + C)}$, the solution is symmetric about the y-axis. The function $x(y)$ is plotted in figure 1.

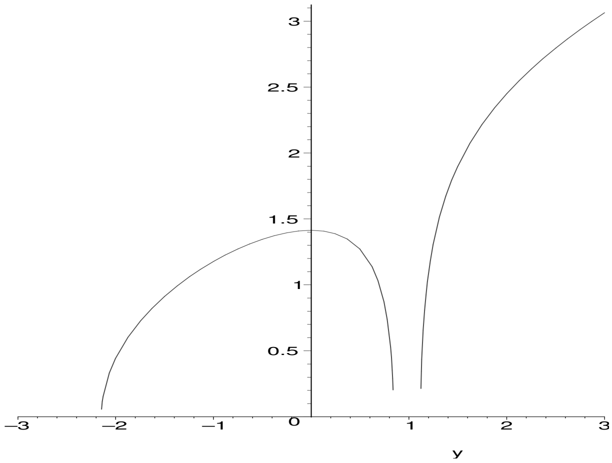

Figure 1: A plot of $x(y) = \sqrt{2}\sqrt{\ln|1-y| + y + C}$ for $C = 1$

Notice that the constant C only shifts the function up and down in the x-direction. By symmetry, if we can show that $y = 0$ is a local maximum, there are some values of C for which the paths are closed. By differentiating the positive solution for $y < 1$ we get

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{x}\left(1 - \frac{1}{|1-y|}\right),$$

which is positive for $y < 0$, zero for $y = 0$ and negative for $y > 0$, as desired. See figure 2 for a sketch of the phase diagram.

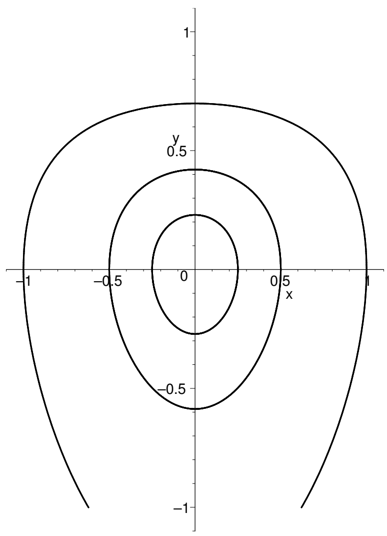

Figure 2: Phase diagram of $\ddot{x} - \epsilon x\dot{x} + x = 0$