Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 5

1 Let X and Y be Banach spaces and $(T_n)_{n\in\mathbb{N}}$ a sequence of bounded linear operators between X and Y.
Then $(T_n x)_{n\in\mathbb{N}}$ converges for all $x \in X$ if and only if the following two conditions hold:

a) $(T_n x)_{n\in\mathbb{N}}$ converges for every $x \in S$, where S is a dense subset of X.

b) $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.

2 Show that the limit operator in the theorem of Banach-Steinhaus is not necessarily bounded for a sequence of bounded linear mappings $(S_n)_{n\in\mathbb{N}}$ on a normed space X. Take X to be the space of real-valued sequences of finite support with the supremums norm. Consider the partial sum operator $S_n x = \sum_{k=1}^n x_k$ where $x = (x_k)_{k\in\mathbb{N}}$ on this space.

3 Let $\mathcal{H}$ be a real Hilbert space and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form on $\mathcal{H}$:

$$B(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 B(x_1, y) + \alpha_2 B(x_2, y) \text{ and } B(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 B(x, y_1 y) + \beta_2 B(x, y_2)$$

for all $x_1, x_2, x, y_1, y_2, y \in \mathcal{H}$ and for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.

Show that if $B(\cdot, y)$ is continuous for every $y \in \mathcal{H}$ and $B(x, \cdot)$ is continuous for every $x \in \mathcal{H}$. Then B is bounded.

Hint: Banach-Steinhaus