Norwegian University of Science and Technology
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TMA4230 Functional Analysis
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Exercise set 3 – Solutions

$\boxed{1}$ Show that the space of converging sequences c is isomorphic to the space of sequences converging to zero, c_0.

Hint: Let T be the mapping $T : c \to c_0$ defined by

$$T(a_1, a_2, ...) = (-a, a_1 - a, a_2 - a, ...),$$

where $a = \lim_n a_n$. Show that T is bijective, determine its inverse, and prove that these maps are linear and contiuous.

Solution. Let us start by showing that T is injective. Assume that $T(a_1, a_2, ...) = (-a, a_1 - a, a_2 - a, ...) = (0, 0, ...)$. By the equality in the first coordinate, we get that $a = 0$. Then we easily get that every $a_i = 0$ by looking at the other coordinates. Thus $(a_1, a_2, ...) = (0, 0, ...)$, and hence T is injective.

We now wish to prove that T is surjective, so let $(a_n)_n$ be a sequence such that $\lim_n a_n = 0$. We need to construct a preimage of $(a_n)_n$, and I claim that the sequence $(a_2 - a_1, a_3 - a_1, a_4 - a_1, ...)$ works. I leave to you to check that this is true, i.e. that $T(a_2 - a_1, a_3 - a_1, a_4 - a_1, ...) = (a_1, a_2, ...)$.

The inverse of T is given by $T^{-1}(a_1, a_2, a_3, ...) = (a_2 - a_1, a_3 - a_1, a_4 - a_1, ...)$, which I also leave for you to check (Note that you have already checked one part, namely that TT^{-1} is the identity. This was actually what I asked you to check when we proved surjectivity!).

Linearity is also a simple exercise that I leave to you – just check it by computation.

Let us now check that T is continuous. First recall that both of these spaces are Banach spaces under the supremum norm, i.e. $\|(a_n)_n\|_\infty = \sup_n |a_n|$ for a sequence $(a_n)_n$ in c, so continuity is equivalent to being bounded. We therefore look at $\|T(a_1, a_2, ...)\|_\infty$. We write $a = \lim_n a_n$, and use the triangle inequality to find that

$$\begin{aligned}\|T(a_1, a_2, ...)\|_\infty &= \sup\{|a|, |a_1 - a|, |a_2 - a|, ...\} \\ &\leq \sup\{|a|, |a_1| + |a|, |a_2| + |a|, ...\} \\ &= |a| + \sup_n |a_n|.\end{aligned}$$

Now, since $\lim_n a_n = a$, we do in fact have that $|a| \leq \sup_n |a_n|$ (think about this!). Hence $\|T(a_1, a_2, ...)\|_\infty \leq 2 \sup_n |a_n| = 2\|(a_1, a_2, ...\|_\infty$, thus T is bounded.

Checking that T^{-1} is a bit simpler. Let $(a_n)_n$ be a sequence in c_0. As above we calculate:

$$\begin{aligned}\|T^{-1}(a_1, a_2, ...)\|_\infty &= \sup_n |a_n - a_1| \\ &\leq |a_1| + \sup_n |a_n|\end{aligned}$$

by the triangle inequality $|a_n - a_1| \leq |a_1| + |a_n|$. Clearly $|a_1| \leq \sup_n |a_n|$ by the definition of supremum, so the inequality really says that $\|T^{-1}(a_1, a_2, ...)\|_\infty \leq 2\|(a_1, a_2, ...)\|_\infty$, and so T^{-1} is bounded.[1]

$\boxed{2}$ Show that the space of continuous linear functionals on ℓ^1 is isometrically isomorphic to ℓ^∞.

Solution. We need to show the following:

1. Any sequence $a = (a_n)_n$ in ℓ^∞ determines a linear functional on ℓ^1.
2. The norm of $(a_n)_n$ as an element of the dual space of ℓ^1 equals $\|(a_n)_n\|_\infty$[2].
3. Any linear functional on ℓ^1 is given by an element of ℓ^∞

We start with the first point. If $x = (x_n)_n \in \ell^1$, how does a act on x? In fact, the action is given in the same way as for the other ℓ^p spaces that you have seen in the lectures:

$$a(x) = \sum_n x_n a_n. \tag{1}$$

It is easy to check that this is linear. To see that it is continuous, note that

$$\begin{aligned}|a(x)| &\leq \sum_n |x_n a_n| \\ &\leq \left(\sup_n |a_n|\right) \sum_n |x_n| \\ &= \|a\|_\infty \|x\|_1.\end{aligned}$$

In words, a defines a continuous linear functional on ℓ^1 with operator norm less than or equal to $\|a\|_\infty$.

Recall that the operator norm of a is given by $\sup\limits_{\|x\|_1=1} |a(x)|$. We have shown that $\|a\|_\infty$ is an upper bound of $\{|a(x)| : \|x\|_1 = 1\}$, to prove (2) we need to show that it is the *least* upper bound. We will check this in detail, so let $\epsilon > 0$. We need to find $x \in \ell^1$ with $\|x\|_1 = 1$ such that $|a(x)| > \|a\|_\infty - \epsilon$ (This would show that $\|a\|_\infty - \epsilon$ is never an upper bound, so $\|a\|_\infty$ must be the least upper bound).

[1] Checking that both the mapping itself and its inverse is bounded can often be tedious. One of the most important results in this course is the bounded inverse theorem. This theorem states that a linear, bounded bijection between Banach spaces always has a bounded inverse. In particular, we could have skipped the proof that T^{-1} was bounded, and used the bounded inverse theorem.

[2] This means that our correspondence is injective, since isometry implies injectivity.

Since $\|a\|_\infty = \sup_n |a_n|$, there is some $n \in \mathbb{N}$ such that $|a_n| > \|a\|_\infty - \epsilon$. Therefore pick $x \in \ell^1$ to be the sequence with zeros in every position except for the n'th position, where we will have $\frac{a_n^*}{|a_n|}$: $x = (0, 0, 0, ..., \frac{a_n^*}{|a_n|}, 0, ...)$.

Then $a(x) = \frac{a_n^*}{|a_n|} a_n = |a_n|$, so $|a(x)| > \|a\|_\infty - \epsilon$, which is what we wanted to show.

Now we turn to part (3), so let φ be a linear functional on ℓ^1. Let e_n be the standard basis of ℓ^1. We define the sequence $(a_n)_n$ by $a_n = \varphi(e_n)$, and want to show that $(a_n)_n \in \ell^\infty$ and that φ is the linear functional determined by $(a_n)_n$ by equation **??**.

To show that $(a_n)_n \in \ell^\infty$ we need only note that $|a_n| = |\varphi(e_n)| \leq \|\varphi\| \|e_n\|_1 = \|\varphi\|$, so $(a_n)_n$ is bounded.

Finally, let $x \in \ell^\infty$. We will check that $a(x) = \varphi(x)$, where $a(x)$ is determined by equation **??**. From an earlier exercise we know that we can write $x = \sum_n x_n e_n$. Then

$$\begin{aligned} \varphi(x) &= \sum_n x_n \varphi(e_n) \\ &= \sum_n x_n a_n = a(x). \end{aligned}$$

This concludes the proof.

$\boxed{3}$ Let X be a normed space. Show the following assertions:

a) For $\varphi \in X^*$ the kernel of φ is a closed hyperplane, i.e. a closed subspace of codimension 1. (Note even more is true, which is not part of the problem: A linear functional on X is continuous if and only if $\ker\varphi$ is closed. Hence $\ker\varphi$ is either closed or dense in X.)

b) Suppose $\varphi, \eta \in X^*$ satisfy $\ker\varphi = \ker\eta$. Then $\varphi = c\eta$ for some constant c.

c) Given a hyperplane M in X. Then there exists a $\varphi \in X^*$ such that $M = \ker\varphi$.

Solution. We will assume that the vector space is complex.

a) The kernel of a continuous function is always a closed subspace, which you probably know from linear methods (it is not difficult to prove, either). That the codimension is 1 follows from the first isomorphism theorem from linear algebra: If $\phi : Y \to Z$ is a linear mapping between vector spaces Y and Z, then there is an isomorphism

$$\frac{Y}{\ker\phi} \cong \mathrm{im}\phi.$$

Now, if $\varphi : X \to \mathbb{C}$ is a non-zero linear functional, then the image of φ is has dimension 1 (why?). By the isomorphism theorem $\frac{X}{\ker\varphi}$ has dimension 1, which is exactly what it means for $\ker\varphi$ to have codimension 1.

b) First assume that $\varphi, \eta \neq 0$ – the proof is obvious if one of them is zero (pick $c = 0$). Since $\ker\varphi = \ker\eta$ is a hyperplane, we know that $X = \ker\varphi \oplus A$ for some one-dimensional subspace $A \subset X$. Let $e \in A$ be a generator of A, in other words e is such that $A = \{ae : a \in \mathbb{C}\}$.

Now let $c = \frac{\varphi(e)}{\eta(e)}$, and let x be an arbitrary element of X. By the above, x can be written in a unique way as $x = y + ae$, where $y \in \ker \varphi = \ker \eta$ and $a \in \mathbb{C}$. Then $\varphi(x) = a\varphi(e)$ and $\eta(x) = a\eta(e)$. Clearly $\varphi(x) = c\eta(x)$ (just look at the definition of c!), and since x was arbitrary this means that $\varphi = c\eta$.

c) By assumption $\frac{X}{M}$ has dimension 1. From linear algebra we know that there must exist an isomorphism $\phi : \frac{X}{M} \to \mathbb{C}$. We can also define a linear map $\psi : X \to \frac{X}{M}$ by $\psi(x) = x + M$, where $x + M$ denotes the equivalence class of x. Clearly $\phi \circ \psi : X \to \mathbb{C}$ is a linear functional on X, with kernel M.