MA8502 fall 2016
Assignment 1

1 Problem 1: Spectral approximation in $\mathbf{R}^1$

In the set of notes entitled "The Poisson Problem: Spectral Discretization in $\mathbf{R}^1$," we discussed the numerical solution of the one-dimensional Poisson problem using a spectral method based on high-order polynomials. For example, for the case with homogeneous Dirichlet boundary conditions, we arrived at a system of algebraic equations $\underline{A}\underline{u} = \underline{B}\underline{f}$ for the unknown basis coefficients $\underline{u}$ in the interior of the one-dimensional domain $\Omega = (-1, 1)$; see page 29.

a) Implement this spectral solver in the case of homogeneous Dirichlet boundary conditions at $x = \pm 1$, and where the given right hand side $f = x^3$. You can use Matlab for the implementation. Note that the library routines provided on the web site for the course will be very useful. For example, you will find routines for computing the GLL weights and points, as well as a routine for computing the GLL derivative matrix.

   Provide plots of the error of the numerical approximation (the solution u is easy to find in this and the following examples) in an appropriate norm, show how the error decreases as a function of the number of points N and on the smoothness of u.

b) What is the difference between the exact solution u and the numerical solution u_N when $N = 4, 5$ and 10? Explain your answer.

c) Will you obtain the exact solution when $f(x) = \sin(\pi\, x)$ for any N? What is the behaviour of the error in this case?

d) Modify the program from a) to solve the Poisson problem with the boundary conditions $u(-1) = 0$ and $u_x(1) = 0$. Explain the necessary modifications. Provide plots of the error to give numerical evidence that your code is correct.

e) Modify your program from c) to solve the Poisson problem with the boundary conditions $u(-1) = 0$ and $u_x(1) = 2$. Explain the necessary modifications. Provide plots of the error to give numerical evidence that your code is correct.

2 Problem 2:

Consider the Poisson problem $-u_{xx} = f$ and for the Helmholtz problem $-u_{xx} + ru = f$ (r continuous and $r(x) \geq 0$) with homogeneous Dirichlet boundary conditions. Show that $\exists \gamma > 0$ such that

$$\forall w, v \in H_0^1, \quad |a(w, v)| \leq \gamma \|w\|_{H^1} \|v\|_{H^1}.$$

3 Problem 3: Finite elements error estimates

Let $\Omega = (-1, 1)$. Subdivide Ω in elements of equal size $h = 2/N$ and consider the piecewise linear interpolant I_h associating to any function on Ω the piecewise linear approximation $I_h f$ such that $I_h f(x_i) = f(x_i)$ and $x_i = -1 + i\,h$, $i = 0, \dots, N$. Denote with u_h the linear finite elements approximation of the Poisson problem with homogeneous Dirichlet boundary conditions, and with u the solution of the Poisson problem. In this case $X_h \subset H_0^1$ is generated by the *hat functions*.

a) Give a mathematical description of the hat functions and of X_h. What is the dimension of X_h?

b) By using the best approximation property in the energy norm and the following error estimate for the piecewise linear interpolant $I_h u$ ($u \in H^2(\Omega)$)

$$\|u - I_h u\|_{H^1} \leq C\, h\, \|u\|_{H^2} \tag{1}$$

show that

$$\|u - u_h\|_{H^1} \leq C_1 h \|u\|_{H^2}.$$

c) Find C_1.

d) Review briefly a strategy to derive the estimate (1).

e) Implement a linear finite element discretization for the Poisson problem with homogeneous Dirichlet boundary conditions and compare numerically the errors of the finite element and spectral approximations.