LØSNINGSSKISSER ØVING 12

Vanlige forbehold!
KH

Oppgave 1

$$C: r(t) = (\underset{x}{a\cos^3 t},\ \underset{y}{b\sin^3 t}) \quad t \in [a,b]\ ;\ a,b>0$$

Arealet er

$$\frac{1}{2}\int_0^{2\pi} -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_0^{2\pi} b\sin^4 t \cdot 3a\cos^2 t + a\cos^3 t \cdot 3b\sin^2 t \cdot \cos t\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} 3ab\ \sin^2 t\ \cos^2 t\ \overbrace{(\sin^2 t + \cos^2 t)}^{1}\,dt$$

$$= \frac{3}{4\cdot 2}\,ab\underbrace{\int_0^{2\pi}\sin^2(2t)\,dt}_{2\pi/2} = \underline{\underline{\frac{3}{8}\pi ab}}$$

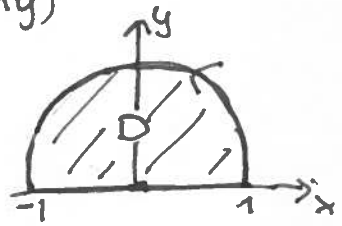
Oppgave 2

Beregner kurveintegralet ved Green

$$(F_1, F_2) = (xy + \ln(x^2+1),\ 4x + e^{y^2} + 3\arctan y)$$

$$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,d(x,y) = \iint_D 4 - x\,d(x,y)$$

$$= 4\cdot\overbrace{\frac{\pi\cdot 1^2}{2}}^{\text{areal}} - \overset{\text{symmetri}}{0} = \underline{\underline{2\pi}}$$



Oppgave 3

Ellipseligning $9x^2 + 4y^2 - 18x + 16y = 11$

a) Ligningen kan skrives

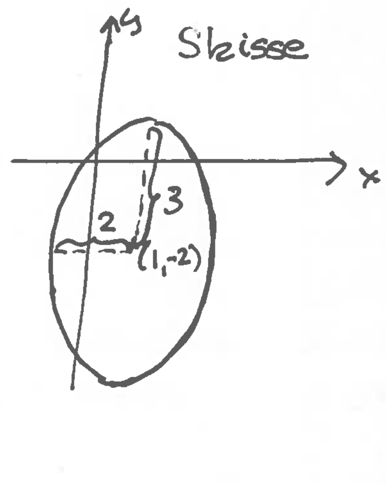
$$\frac{x^2 - 2x + 1}{\frac{36}{9}} + \frac{y^2 + 4y + 2^2}{\frac{36}{4}} = 1$$

eller

$$\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} = 1$$

$\therefore$ $\underline{\underline{\text{sentrum i } (1,-2),\ \text{halvakser } 2 \text{ og } 3}}$

Skisse



b) $\frac{x-1}{2} = \cos t$, $\frac{y+2}{3} = \sin t$ en parametrisering ; $t \in [0, 2\pi)$

mao.

$$r(t) = (1 + 2\cos t, -2 + 3\sin t)$$

$$r'(t) = (-2\sin t, 3\cos t)$$

$$\underline{F(x,y) = (y^2, x)} = ((-2 + 3\sin t)^2, 1 + 2\cos t)$$

Altså

$$\int_C F \cdot dr = \int_0^{2\pi} ((-2+3\sin t)^2, 1+2\cos t) \cdot (-2\sin t, 3\cos t)\,dt$$

$$-2(4 - 12\sin t + 9\sin^2 t)\sin t + 3\cos t + 6\cos^2 t$$

$$= 0 + 24\pi + 0 + 0 + 6\pi = \underline{\underline{30\pi}}$$

c) $\underline{\underline{30\pi}} = \int_C F \cdot dr \underset{\text{Green}}{=} \underline{\iint_R 1 - 2y \, d(x,y)}$

Oppgave 4

$$F = (\phi_x, \phi_y) \underset{\phi \text{ glatt}}{\Longrightarrow} \phi_{xy} = \phi_{yx}$$

$$\int_C F \cdot dr \underset{\text{Green}}{=} \iint_R \underbrace{\phi_{xy} - \phi_{yx}}_{0} \, d(x,y) = \underline{\underline{0}}$$

Oppgave 5

i) $I = \iiint_A xyz \, d(x,y,z) = ?$ når $A = [0,1] \times [0,1] \times [0,1]$

$$I = \int_0^1 z\,dz \int_0^1 y\,dy \int_0^1 x\,dx = \left(\int_0^1 x\,dx\right)^3 = \left(\left[\frac{x^2}{2}\right]_0^1\right)^3 = \underline{\underline{\frac{1}{8}}}$$

ii)

$$I = \iiint_A (xy+z)\, d(x,y,z) = ? \text{ når } A = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le x^2 y\}$$

$$I = \iint_{\underset{R}{[0,1]\times[0,2]}} \int_0^{x^2 y} (xy + z)\, dz \; d(x,y)$$

$$= \iint_{[0,1]\times[0,2]} xy\, x^2 y + (x^2 y)^2/2 \; d(x,y)$$

$$= \left(\int_0^1 x^3\, dx\right)\left(\int_0^2 y^2 dy\right) + \tfrac{1}{2}\left(\int_0^1 x^4 dx\right)\left(\int_0^2 y^2 dy\right)$$

$$= \frac{1}{4} + \frac{2^3}{3} + \frac{1}{2}\cdot\frac{1}{5}\cdot\frac{2^3}{3} = \frac{1}{3}\left(2 + \frac{4}{5}\right) = \underline{\underline{\frac{14}{15}}}$$

Oppgave 6

$$I = \iiint_R \sqrt{x^2+y^2}\; d(x,y,z) = ? \quad R = \{(x,y,z) \mid x^2+y^2+z^2 \le 4\}$$

Bruker kulekoordinater

$$\left.\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \end{aligned}\right\} \; x^2+y^2 = \rho^2 \sin^2\phi$$

$$d(x,y,z) = \rho^2 \sin\phi \; d(\rho,\phi,\theta)$$

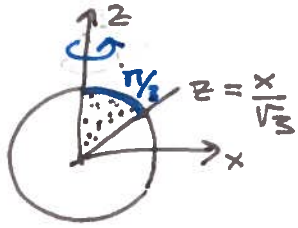
Altså

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho \sin\phi \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= 2\pi \left(\int_0^2 \rho^3 d\rho\right)\left(\int_0^{\pi} \sin^2\phi \; d\phi\right)$$

$$= 2\pi \cdot \frac{2^4}{4} \cdot \frac{\pi}{2} = \underline{\underline{4\pi^2}}$$

Oppgave 7



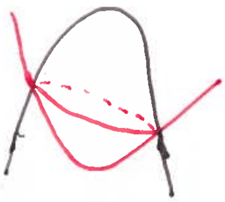
$$V = \int_0^{2\pi}\int_0^{R}\int_0^{\pi/3} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$

$$= 2\pi \cdot \frac{R^3}{3} \left[-\cos\phi\right]_0^{\pi/3} = 2\pi R^3/3 \left(1-\tfrac{1}{2}\right) = \underline{\underline{\frac{\pi R^3}{3}}}$$

Oppgave 8

R er området avgrenset av paraboloiden

$z = 6 - x^2 - y^2$ (topp opp) og $z = x^2 - 4x + y^2 = -4 + (x-2)^2 + y^2$



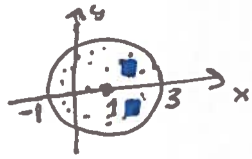
a) Projeksjonen av skjæringskurven mellom paraboloidene blir

$$6 - x^2 - y^2 = x^2 - 4x + y^2 \Leftrightarrow (x-1)^2 + y^2 = 4 \quad (*)$$

Altså: $$I = \iint_S \int_{x^2-4x+y^2}^{6-x^2-y^2} y \, dz \, d(x,y)$$

$$= \iint_S y\left(6 - x^2 - y^2 - x^2 + 4x - y^2\right) d(x,y)$$

der S er sirkelskiva $(x-1)^2 + y^2 \le 4$



$$I = \iint_S \left(6y - 2x^2y - 2y^3 + 4xy\right) d(x,y)$$

b) $I = 0$ da området S er symmetrisk om x-aksen, og vi har odde potenser av y slik at alle ledd i summen blir 0, „de positive y-verdiene balanserer de negative"; kfr TL øverst s. 685. Alternativt polar-koordinater med pol i (1,0).

c) $(*)$: $x - 1 = 2\cos t$, $y = 2\sin t \Rightarrow z = 6 - x^2 - y^2 = 1 - 4\cos t$

$$\oint_C z\,dx + \underbrace{y\,dy}_{\text{gir } 0} + x\,dz = \int_0^{2\pi} \underbrace{(1-4\cos t)(-2\sin t)}_{0+0} + \underbrace{(1+2\cos t)4\sin t}_{0+0}\, dt = \underline{\underline{0}}$$