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Solutions to exercise set 11

1 **a)** We begin with the Euler method. We have

$$y_1 = y_0 + h\big(-2y_0 - e^{y_0} + t_0^2\big) = 1 + 0.5(-2 - e^1 + 0^2) = -1.359$$
$$y_2 = -1.359 + 0.5 * (-2 \times -1.359 - e^{-1.359} + 0.5^2) = -0.003441$$

Next we use the Improved Euler method. This time $y_1^* = -1.359$ as the predictor step of the method coincides with the Euler method. Then

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2}(f(y_0, t_0) + f(y_1^*, t_1)) \\ &= 1 + 0.25(-2 - e^1 + 0^2 - 2 \times -1.359 - e^{-1.359} + 0.5^2) = 0.4983 \end{aligned}$$

For the next step we must first calculate y_2^*:

$$y_2^* = 0.4983 + 0.5(-2 \times 0.4983 - e^{-0.4983} + 0.5^2) = -0.6979$$

We then calculate

$$\begin{aligned} y_2 &= 0.4983 + 0.25(-2 \times -0.4983 - e^{-0.4983} + 0.5^2 - 2 \times -0.6979 - e^{-0.6979} + 1^2) \\ &= 0.3747 \end{aligned}$$

b) The backward Euler method is

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) = y_n - 2y_{n+1} - \exp y_{n+1} + t_{n+1}^2$$

Rearranging gives

$$3y_{n+1} + \exp(y_{n+1}) - y_n - t_{n+1}^2 = 0$$

c) A step of backward Euler results in

$$3y_1 + \exp(y_1) - 1 - 1 = 0,$$

i.e. y_1 solves the equation

$$f(x) = 3x + e^x - 2 = 0$$

(We have renamed the variable x so as not to confuse the nth Newton iterate x_n with the nth timestep of backward Euler y_n). A Newton iteration consists of

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We first calculate $f'(x) = 3 + e^x$. The Newton iterations therefore proceed as follows. We take $x_0 = 0$ as our initial guess, note that other valid answers may be obtained for different initial guesses.

$$x_1 = -\frac{1-2}{4} = 0.25$$

$$x_2 = 0.25 - \frac{3 \times 30.25 + e^{0.25} - 2}{3 + e^{0.25}} = 0.2421$$

$\boxed{2}$ **a)** The predictor step of the improved Euler scheme is

$$y_{n+1}^* = y_n + hf(t_n, y_n) = (1 - 1000h)y_n$$

The improved Euler scheme is

$$y_{n+1} = y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_n, y_{n+1}^*)\big)$$

substituting in the formula for y_{n+1}^* gives

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\big(-1000y_n - 1000(1 - 1000h)y_n\big) \\ &= (1 - 1000h + \frac{10^6}{2}h^2)y_n \end{aligned}$$

By applying the above repeatedly we find

$$y_n = (1 - 1000h + \frac{10^6}{2}h^2)^n y_0,$$

and setting $y_0 = 1$ gives the requested result. Now suppose we set $h = 0.1$ into the above formula, we find

$$y_n = 4901^n,$$

which clearly goes to infinity as $n \to \infty$.

b) Examining the formula more closely, we recall that $x^n \to 0$ as $n \to \infty$ if and only if $|x| < 1$. We therefore require $|1 - 1000h + \frac{10^6}{2}h^2| < 1$. This is a quadratic expression in h which is always positive. We therefore solve the equation

$$1 - 1000h + \frac{10^6}{2}h^2 = 1$$

which has roots at $h = 0$ and $h = \frac{1}{500}$. We therefore have $y_n \to 0$ if and only if h lies between these values.

c) Applying backward Euler we find

$$y_{n+1} = y_n - 1000hy_{n+1},$$

which is rearranged to give

$$(1 + 1000h)y_{n+1} = y_n$$

Dividing both sides by $1 + 1000h$ and iterating as before gives

$$y_{n+1} = \left(\frac{1}{1 + 1000h}\right)^n$$

In this case we have $y_n \to 0$ as $n \to \infty$ if and only if

$$\left|\frac{1}{1 + 1000h}\right| < 1$$

This holds for all $h > 0$.

$\boxed{3}$ **a)** We first note that $y_1 \approx y(1)$, so we require one step only. We then calculate k_1, k_2, k_3, k_4 in succession, followed by y_1. We have

$$k_1 = f(t_0, y_0) = f(0, 1) = 0 \times 1^2 = 0$$

$$k_2 = f(t_0 + \frac{h}{2}, y_n + \frac{h}{2}k_1) = \frac{1}{2}(1 + 0)^2 = 0.5$$

$$k_3 = f(t_0 + \frac{h}{2}, y_n + \frac{h}{2}k_2) = \frac{1}{2}(1 + 0.5^2)^2 = 0.78125$$

$$k_4 = f(t_0 + h, y_n + hk_3) = (1 + 0.78125)^2 = 3.173$$

We conclude that

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.9559$$

b) The Butcher tableau is interpreted as follows:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \frac{1}{2}h, y_n + \frac{h}{2}k_1)$$

$$k_3 = f(t_n + h, y_n - hk_1 + 2hk_2)$$

which are then combined to give

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3)$$