Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4165 Differential Equations and Dynamical Systems Spring 2017

Exercise set 2

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J. S.: 1.4, 1.6.
Some explanation to 1.6: The potential energy $\mathcal{V}(x)$ of a conservative system ..., means that you consider the problem $\ddot{x} = f(x)$ where $\mathcal{V}(x) = -\int f dx$ and $\mathcal{V}(x)$ as given in the text.

These exercises will be presented / discussed in the exercise class:

$\boxed{\text{E4}}$ Aim: Sketch the phase diagram for the equation $\ddot{x} - e^x = a$, $a \in \mathbb{R}$.

a) Investigate and sketch the function $f(x) = e^x + a$ for different values of a

b) Rewrite $\ddot{x} - e^x = a$ as a system of differential equations of first order.

c) Find the equilibrium points and sketch the phase diagram.

d) Mark the separatrix in the phase diagram.

$\boxed{\text{E5}}$ Aim: Show that the initial value problem

$$\dot{x} = \frac{1}{2}x^2 + \frac{1}{2}|x|^3, \quad x(0) = 1 \tag{1}$$

has exactly one local solution, that $\lim_{t\to-\infty} x(t) = 0$ and that there exists $a \in (\frac{1}{2}, 1)$ such that $\lim_{t\to a-} x(t) = \infty$.
Idea: Estimate the maximal interval of existence for the above initial value problem by using a comparison argument.
Background: Given two continuously differentiable functions $y : \mathbb{R} \mapsto \mathbb{R}$ and $z : \mathbb{R} \mapsto \mathbb{R}$ such that $y(0) = z(0)$. Then one has that $\dot{y}(t) \leq \dot{z}(t)$ for all $t \in \mathbb{R}$ implies that

$$y(t) - y(0) = \int_0^t \dot{y}(s)ds \leq \int_0^t \dot{z}(s)ds = z(t) - z(0) \quad \text{for all } t \geq 0,$$

which is equivalent to $y(t) \leq z(t)$ for all $t \geq 0$. Similarly it follows that $z(t) \leq y(t)$ for all $t \leq 0$.

a) Show that the functions $f(x) = x^2$ and $g(x) = |x|^3$ are locally Lipschitz continuous and conclude that (1) has a unique local solution.

b) Find the unique solution to the initial value problem $\dot{x} = f(x)$, $x(0) = x_0$ for $x_0 < 0$, $x_0 = 0$, $x_0 > 0$.

c) Find the unique solution to the initial value problem $\dot{x} = g(x)$, $x(0) = x_0$ for $x_0 < 0$, $x_0 = 0$, $x_0 > 0$.

d) Use a comparison argument based on the solutions found in b) and c) to show that $x(t)$, the solution of (1), satisfies that $\lim_{t\to-\infty} x(t) = 0$ and that there exists $a \in (\frac{1}{2}, 1)$ such that $\lim_{t\to a-} x(t) = \infty$. Note, this means in particular that the maximal interval of existence of $x(t)$ is given by $(-\infty, a)$.

$\boxed{\text{E6}}$ Aim: sketch the phase diagram for the linear system:

$$\begin{aligned} \dot{x} &= -4x + 2y \\ \dot{y} &= -3x + y. \end{aligned} \tag{2}$$

Background: Given some initial value problem $\dot{x}(t) = f(x(t))$, $x(0) = a$, then $y(t) = x(-t)$ solves the initial value problem $\dot{y}(t) = -f(y(t))$, $y(0) = a$. That means in particular, if $x(t) \to \infty$ as $t \to \infty$ then $y(t) \to \infty$ as $t \to -\infty$ and if $x(t) \to 0$ as $t \to -\infty$ then $y(t) \to 0$ as $t \to \infty$.

a) Define $z_1(t) = x(-t)$ and $z_2(t) = y(-t)$. Find the linear system for $z_1(t)$ and $z_2(t)$.

b) Sketch the phase diagram for the system for $z_1(t)$ and $z_2(t)$.

c) Sketch the phase diagram for (2).

d) Compare the two diagrams obtained in b) and c).