Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2017

Solutions exercise 8

$\boxed{10.1}$ Find a simple function V to establish the stability or instability of the zero solution of the following equations:

(i) For the system

$$\dot{x} = -x + y - xy^2,$$
$$\dot{y} = -2x - y - x^2y,$$

try

$$V(x,y) = x^2 + by^2$$

where $b > 0$. Then $V(x,y) > 0$ for all $x, y \in \mathbf{R} \setminus \{0\}$. Further,

$$\begin{aligned}\dot{V}(x,y) &= 2x\dot{x} + 2by\dot{y}\\ &= -2x^2 + 2xy - 2x^2y^2 - 4bxy - 2by^2 - 2bx^2y^2\\ &= -2x^2 - 2by^2 - 2x^2y^2(1+b) + 2xy(2b-1).\end{aligned}$$

Put $b = \frac{1}{2}$ so that

$$\dot{V}(x,y) = -2x^2 - y^2 - 3x^2y^2 < 0$$

for all $x, y \in \mathbf{R} \setminus \{0\}$. The function V satisfies the conditions in theorem 10.5, so the zero solution is asymptotically stable.

(ii) For the system

$$\dot{x} = y^3 + x^2y,$$
$$\dot{y} = x^3 - xy^2,$$

try $V(x,y) = xy$. $V(x,y)$ satisfies the requirements for theorem 10.13: $V(x,y)$ and its partial derivatives are continuous and $V(0,0) = 0$. We also have

$$\begin{aligned}\dot{V}(x,y) &= y\dot{x} + x\dot{y}\\ &= y^4 + x^2y^2 + x^4 - x^2y^2\\ &= y^4 + x^4 > 0.\end{aligned}$$

For an arbitrary close neighborhood of the origin we can find a point (ϵ, ϵ) for small enough ϵ. Here, $V(\epsilon, \epsilon) = \epsilon^2$ and we can conclude that the origin is unstable by theorem 10.13.

(iii) For the system

$$\begin{aligned}\dot{x} &= 2x + y + xy,\\ \dot{y} &= x - 2y + x^2 + y^2,\end{aligned}$$

try $V(x,y) = x^2 - y^2$. We find

$$\begin{aligned}\dot{V}(x,y) &= 2x\dot{x} - 2y\dot{y}\\ &= 4(x^2 + y^2) - 2y^3.\end{aligned}$$

Hence, $\dot{V}(x,y) > 0$ for all $|\mathbf{x}| < 1$, so the origin is unstable.

(iv) For the system

$$\begin{aligned}\dot{x} &= -x^3 + y^4,\\ \dot{y} &= -y^3 + y^4,\end{aligned}$$

try

$$V(x,y) = \frac{1}{2}(x^2 + y^2).$$

Then $V(x,y) > 0$ for all $x, y \in \mathbf{R} \setminus \{0\}$. Further,

$$\begin{aligned}\dot{V}(x,y) &= x\dot{x} + y\dot{y}\\ &= -x^4 + xy^4 - y^4 + y^5\\ &= -x^4 - y^4(1 - x - y) < 0\end{aligned}$$

when $|\mathbf{x}| \leq \frac{1}{2}$. Hence, the function $V(x,y)$ is a strong Liapunov function so the origin is asymptotically stable.

$\boxed{10.2}$ Show that α can be chosen so that $V(x,y) = x^2 + \alpha y^2$ is a strong Liapunov function for the system

$$\begin{aligned}\dot{x} &= y - \sin^3 x,\\ \dot{y} &= -4x + \sin^3 y.\end{aligned}$$

We note first that $V(x,y)$ satisfies definition 10.1. Further,

$$\begin{aligned}\frac{1}{2}\dot{V}(x,y) &= x\dot{x} + \alpha y\dot{y}\\ &= x(y - \sin^3 x) + \alpha y(-4x - \sin^3 y)\\ &= (1 - 4\alpha)xy - (x\sin^3 x + \alpha y \sin^3 y).\end{aligned}$$

Put $\alpha = \frac{1}{4}$ to get

$$\dot{V}(x,y) = -2\left(x\sin^3 x + \frac{1}{4}y\sin^3 y\right).$$

Since $x\sin^3 x$ and $y \sin^3 y$ are positive for $|x|, |y| \leq \frac{\pi}{2}$ we can look at the ball centered at $(0,0)$ with radius $\frac{\pi}{2}$. Here, $\dot{V}(x,y) < 0$ so V is a strong Liapunov function.

$\boxed{10.7}$ Discuss the stability at the origin for the system

$$\dot{x} = x^2 - y^2,$$
$$\dot{y} = -2xy$$

by using $V(x, y) = \alpha xy^2 + \beta x^3$ where α and β are constants.

We find

$$\begin{aligned}\dot{V}(x, y) &= \alpha(\dot{x}y^2 + 2xy\dot{y}) + 3\beta x^2\dot{x} \\ &= \alpha[(x^2 - y^2)y^2 + 2xy(-2xy)] + 3\beta x^2(x^2 - y^2) \\ &= \alpha(-3x^2y^2 - y^4) + \beta(3x^4 - 3x^2y^2) \\ &= -3x^2y^2(\alpha + \beta) - \alpha y^4 + 3\beta x^4.\end{aligned}$$

Choosing $\alpha = -1$ and $\beta = 1$, we get

$$V(x, y) = -xy^2 + x^3 \quad \text{where} \quad \dot{V}(x, y) = 3x^4 + y^4$$

Then $\dot{V}(x, y) > 0$ for all $x, y \in \mathbf{R} \setminus \{0\}$. We conclude that the origin is unstable.

$\boxed{1996,4}$ Given $V \in C^1(\mathbb{R}^n, \mathbb{R})$.

a) Show that, if x_0 is a strict minimum for $V(x)$, then x_0 is an asymptotically stable equilibrium point for the system

$$\dot{x} = -\nabla V(x).$$

b) Let

$$V(x, y) = x^2(x - 1)^2 + y^2.$$

Sketch the phase diagram of the system

$$\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} = -\nabla V(x, y).$$

a) Since x_0 is a minimum for V we have $\nabla V(x_0) = 0$, showing that x_0 is an equilibrium point.

We want to find a function $L(x)$ so that

$$\dot{L} = \frac{\partial L}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial L}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t} = -\frac{\partial L}{\partial x_1}\frac{\partial V}{\partial x_1} - \cdots - \frac{\partial L}{\partial x_n}\frac{\partial V}{\partial x_n} = -\nabla L \cdot \nabla V < 0.$$

Hence, we try $L(x) = V(x) + C$ for some constant C. Since we need $L(x_0) = 0$, we take $C = -V(x_0)$. Now we have

$$\dot{L} = -\left(\left(\frac{\partial V}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial V}{\partial x_n}\right)^2\right) < 0$$

and x_0 is an asymptotically stable equilibrium point.

b) We calculate the gradient of V and find

$$\begin{aligned}\dot{x} &= -2x(x-1)(2x-1)\\ \dot{y} &= -2y.\end{aligned}$$

The equilibrium points for the system are $(0,0),(1,0)$ and $(\frac{1}{2},0)$. The matrix of linearization at the point $(0,0)$ and $(1,0)$ are given by

$$J = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

which give us a stable nodes. The matrix of linearization at the point $(\frac{1}{2},0)$ is given by

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

which is a saddle point. See figure 1 for a sketch of the phase diagram.

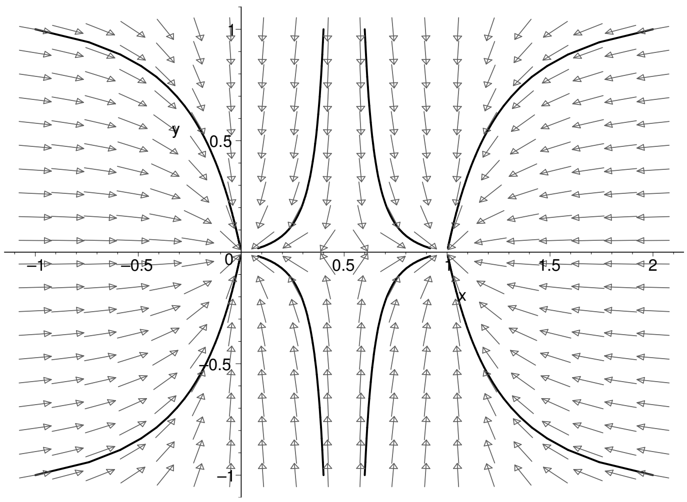

Figure 1: Phase diagram of $\dot{\mathbf{x}} = -\nabla V(x,y)$