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$\boxed{1}$ Example code can be found on the webpage of the course.

$\boxed{2}$ **a)** Let $\mathbf{L}$, $\mathbf{M} \in \mathbb{R}^{n \times n}$ be two lower triangular matrices. Assume that $1 \le i < j \le n$. Since $\mathbf{L}$ and $\mathbf{M}$ are lower triangular we have $\ell_{ik} = 0$ for $k > i$ and $\ell_{kj} = 0$ for $k < j$. Thus

$$(\mathbf{LM})_{ij} = \sum_{k=1}^{n} \ell_{ik} m_{kj} = \sum_{k=1}^{i} \ell_{ik} \cdot 0 + \sum_{k=i+1}^{j-1} 0 \cdot 0 + \sum_{k=j}^{n} 0 \cdot m_{kj} = 0.$$

Thus $(\mathbf{LM})_{ij} = 0$ whenever $j > i$, proving that $\mathbf{LM}$ is lower diagonal.

b) Let $\mathbf{L}$, $\mathbf{M} \in \mathbb{R}^{n \times n}$ be unit lower triangular. From the previous part we already know that $\mathbf{LM}$ is lower triangular. Now let $1 \le i \le n$. Then

$$(\mathbf{LM})_{ii} = \sum_{k=1}^{n} \ell_{ik} m_{ki} = \sum_{k=1}^{i-1} \ell_{ik} \cdot 0 + \ell_{ii} m_{ii} + \sum_{k=i+1}^{n} 0 \cdot m_{ki} = \ell_{ii} m_{ii} = 1 \cdot 1 = 1.$$

Thus the diagonal entries of $\mathbf{LM}$ are all equal to 1, and therefore $\mathbf{LM}$ is unit lower triangular.

c) Let $\mathbf{L} \in \mathbb{R}^{n \times n}$ be an invertible lower triangular matrix with inverse $\mathbf{M} := L^{-1}$. Assume that $\mathbf{M}$ is not lower triangular. Then there exist indicies $1 \le i < j \le n$ such that $m_{ij} \neq 0$. Moreover, we can choose these indices in such a way that $m_{kj} = 0$ for all $k < i$. Since $\mathbf{LM}$ is the identity matrix, we have $(\mathbf{LM})_{ij} = 0$. Hence

$$0 = (\mathbf{LM})_{ij} = \sum_{k=1}^{n} \ell_{ik} m_{kj} = \sum_{k=1}^{i-1} \ell_{ik} \cdot 0 + \ell_{ii} m_{ij} + \sum_{k=i+1}^{n} 0 \cdot m_{kj} = \ell_{ii} m_{ij}.$$

Since $\mathbf{L}$ is an invertible lower triangular matrix its diagonal elements are different from 0. Thus the equation $0 = \ell_{ii} m_{ij}$ already implies that $m_{ij} = 0$, which is a contradiction to the definition of m_{ij}. Hence $\mathbf{M}$ is lower triangular.

d) Let $\mathbf{L}$ be unit lower triangular. Since $\det \mathbf{L} = 1$, it follows that $\mathbf{L}$ is invertible. Denote its inverse by $\mathbf{M} := \mathbf{L}^{-1}$. We have already shown that the matrix $\mathbf{M}$ is lower triangular. Hence we only have to prove that $\mathbf{M}_{ii} = 1$ for every i. Because $\mathbf{LM}$ is the identity matrix, we have, similar as in the second part of this exercise, the equation

$$1 = (\mathbf{LM})_{ii} = \sum_{k=1}^{n} \ell_{ik} m_{ik} = \ell_{ii} m_{ii}.$$

Since $\ell_{ii} = 1$, this proves that also $m_{ii} = 1$.

e) Assume that $\mathbf{U}$ and $\mathbf{V}$ are upper triangular. Then $\mathbf{U}^T$ and $\mathbf{V}^T$ are lower triangular. Moreover $(\mathbf{U}\mathbf{V})^T = \mathbf{V}^T\mathbf{U}^T$. Thus, using the first part of this exercise we obtain that $(\mathbf{U}\mathbf{V})^T$ is lower triangular and therefore $\mathbf{U}\mathbf{V}$ is upper triangular. The proof for unit upper triangular matrices is similar. Finally, the fact that the inverse of a (unit) upper triangular matrix is (upper) triangular follows from the fact that $(\mathbf{U}^T)^{-1} = (\mathbf{U}^{-1})^T$; the former is lower triangular and therefore the latter as well.

3 a) Assume that $\mathbf{A} = \mathbf{L}_1\mathbf{U}_1$ and $\mathbf{A} = \mathbf{L}_2\mathbf{U}_2$ are two LU factorizations of the matrix $\mathbf{A}$. Then

$$\mathbf{L}_1\mathbf{U}_1 = \mathbf{L}_2\mathbf{U}_2.$$

Since the matrix $\mathbf{A}$ is invertible, so are the matrices $\mathbf{L}_j$ and $\mathbf{U}_j$. Thus we can multiply the equation above with $\mathbf{L}_2^{-1}$ from the left and with $\mathbf{U}_1^{-1}$ from the right. Thus we obtain the equation

$$\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{U}_2\mathbf{U}_1^{-1}.$$

Since $\mathbf{L}_2$ and $\mathbf{L}_1$ are unit lower triangular, so is $\mathbf{L}_2^{-1}\mathbf{L}_1$. Since $\mathbf{U}_2$ and $\mathbf{U}_1$ are upper triangular, so is $\mathbf{U}_2\mathbf{U}_1^{-1}$. Thus $\mathbf{L}_2^{-1}\mathbf{L}_1$ is a unit lower triangular matrix that is at the same time upper triangular. The only possibility is therefore that $\mathbf{L}_2^{-1}\mathbf{L}_1$ is the identity matrix. This shows that we have $\mathbf{L}_2 = \mathbf{L}_1$ and, similarly, $\mathbf{U}_2 = \mathbf{U}_1$. Thus the LU factorization of $\mathbf{A}$ is unique.

b) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Assume that $\mathbf{A} = \mathbf{L}\mathbf{U}$ is an LU factorization of $\mathbf{A}$. Since $\ell_{11} = 1$ and $\ell_{12} = 0$, it follows that $u_{11} = a_{11} = 0$. This implies, however, that the matrix $\mathbf{U}$ is not invertible. As a consequence, also the product $\mathbf{L}\mathbf{U}$ is not invertible. This contradicts, however, the fact that $\det \mathbf{A} = -1$, which would imply that $\mathbf{A} = \mathbf{L}\mathbf{U}$ is an invertible matrix. Thus, $\mathbf{A}$ cannot have an LU factorization.

Consider now the case where $n \geq 2$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the 0-matrix. Let $\mathbf{L}$ be *any* unit lower triangular matrix and let $\mathbf{U} = \mathbf{0} \in \mathbb{R}^{n \times n}$. Then $\mathbf{L}\mathbf{U} = \mathbf{0} = \mathbf{A}$, implying that we have constructed an LU factorization of $\mathbf{A}$. Since there is more than one unit lower triangular matrix in $\mathbb{R}^{n \times n}$ with $n \geq 2$, this proves the non-uniqueness of the factorization.

4 a)

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & 4 \end{bmatrix}.$$

b)

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2.106719 & 1 & 0 \\ 3.067194 & -1.142997 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1.012 & -2.132 & 3.104 \\ 0 & 0.4144743 & -0.4737431 \\ 0 & 0 & -10.04806 \end{bmatrix}.$$

c)

$$
\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & -1.33333 & 2 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

d)

$$
\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.849191 & 1 & 0 & 0 \\ -0.4596433 & -0.2501219 & 1 & 0 \\ 2.768662 & -0.3079436 & -5.352283 & 1 \end{bmatrix},
$$

$$
\mathbf{U} = \begin{bmatrix} 2.1756 & 4.0231 & -2.1732 & 5.1967 \\ 0 & 13.43948 & -4.018662 & 10.80699 \\ 0 & 0 & -0.8929524 & 5.091694 \\ 0 & 0 & 0 & 12.03613 \end{bmatrix}.
$$