SOLUTION TRIAL EXAM 2017

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Problem 2

a) Cubic spline is given by:

- K knots $\xi_1, \xi_2, \dots, \xi_K$
- Between knots given by 3rd order polynomials
- Continuous at knots
  Continuous first and second derivatives at knots.

[Natural Cubic spline: linear outside boundary knots.

Dimension of cubic splines:

$K+1$ regions with 4 coefficients $\quad 4K+4$

$-$ K knots $\cdot$ 3 restrictions $\quad -3K$

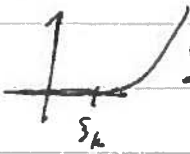
$\Rightarrow$ $K+4$ basis functions needed.

These can be given as in exercise

(Must check that they are

- linearly independent
- requirements at knots:

Consider $g(x) = (x-\xi_k)^3_+$



So $g(\xi_k+) = g(\xi_k-) = 0$

$g'(x) = 3(x-\xi_k)^2_+ \qquad g'(\xi_k+) = g'(\xi_k-) = 0$

$g''(x) = 6(x-\xi_k)_+ \qquad g''(\xi_k+) = g''(\xi_k-) = 0.$

b) <u>Natural</u> requires
for $x < \xi_1$ is $f(x)$ linear
now the given $f(x)$ is for $x < \xi_1$

$$f(x) = \sum_{j=0}^{3} \beta_j x^j$$

so linear $\Rightarrow$ $\beta_2 = \beta_3 = 0$.

For $x > \xi_k$ is

$$f(x) = \beta_0 + \beta_1 x + \sum_{k=1}^{K} \theta_k (x - \xi_k)^3 \qquad \text{(without + 's)}$$

So

$$f'(x) = \beta_1 + \sum_{k=1}^{K} \theta_k \, 3(x - \xi_k)^2$$

$$f''(x) = \sum_{k=1}^{K} 6\theta_k (x - \xi_k)$$

$$f'''(x) = \sum_{k=1}^{K} 6\theta_k$$

Hence: $f'''(x) = 0 \Rightarrow \sum_{k=1}^{K} \theta_k = 0$

$$f''(x) = 0 \Rightarrow \underbrace{\sum_{k=1}^{K} \theta_k x}_{=0} = \sum_{k=1}^{K} \theta_k \xi_k$$

$$\Rightarrow \sum_{k=1}^{K} \theta_k \xi_k = 0$$

c) For natural splines we need

$$\underbrace{N+4}_{\text{cubic spline}} - 4 = N \text{ basis function}$$

coeff. for x^2 and x^3 term are 0 outside both ξ_1 and ξ_K.

Now note that

$$\sum_{k=1}^{K} \theta_k = 0 \Rightarrow \boxed{\theta_K = -\sum_{k=1}^{K-1} \theta_k \quad (*)}$$

$$\sum_{k=1}^{K} \xi_k \theta_k = 0 \Rightarrow \sum_{k=1}^{K-1} \xi_k \theta_k + \xi_K \theta_K = 0$$

$\Rightarrow$ using $(*)$

$$\boxed{\begin{gathered}\sum_{k=1}^{K-1} (\xi_k - \xi_K)\,\theta_k = 0 \quad (**) \\ \text{or} \quad \sum_{k=1}^{K-2} (\xi_k - \xi_K)\theta_k = -(\xi_{K-1} - \xi_K)\theta_{K-1}\end{gathered}}$$

Now under constraints in b) is

$$f(x) = \beta_0 + \beta_1 x + \sum_{k=1}^{K} \theta_k (x-\xi_k)_+^3 = \beta_0 + \beta_1 x + \sum_{k=1}^{K-1} \theta_k (x-\xi_k)_+^3 + \theta_K (x-\xi_K)_+^3$$

$$\overset{(*)}{=} \beta_0 + \beta_1 x + \sum_{k=1}^{K-1} \theta_k (x-\xi_k)_+^3 - \sum_{k=1}^{K-1} \theta_k (x-\xi_K)_+^3$$

$$= \beta_0 + \beta_1 x + \sum_{k=1}^{K-1} \theta_k \left[(x-\xi_k)_+^3 - (x-\xi_K)_+^3 \right]$$

$$= \beta_0 + \beta_1 x + \sum_{k=1}^{K-1} \theta_k (\xi_K - \xi_k)\, d_k(X)$$

~~But using (**) this is~~

$$= \beta_0 + \beta_1 x + \sum_{k=1}^{K-2} \theta_k (\xi_K - \xi_k)\, d_k(x)$$

$$+ \theta_{K-1} (\xi_K - \xi_{K-1})\, d_{K-1}(x)$$

$$\cancel{= \sum_{k=1}^{K-2} (\xi_k - \xi_K)\, \theta_{K-1} d_{K-1} \text{ by } (**)}$$

by (**) is this $= \sum_{k=1}^{K-2} (\xi_k - \xi_K)\, \theta_k\, d_{K-1}(x)$

$$= \beta_0 + \beta_1 x + \sum_{k=1}^{K-2} \theta_k (\xi_K - \xi_k)\left(d_k(x) - d_{K-1}(x)\right)$$

$$= \beta_0 N_1(x) + \beta_1 N_2(x) + \sum_{k=1}^{K-2} \theta_k (\xi_K - \xi_k)\, N_{k+2}(x)$$

which gives a linear combination of the $N_1(\cdot), \ldots, N_K(\cdot)$ } Q.E.D.

Alternatively:

$$\beta_0 + \beta_1 x + \sum_{k=1}^{K-2} \alpha_k N_{k+2}(x)$$

$$= \beta_0 + \beta_1 x + \sum_{k=1}^{K-2} \alpha_k \left[d_k(x) - d_{K-1}(x) \right]$$

$$= \beta_0 + \beta_1 x + \sum_{k=1}^{K-2} \alpha_k d_k(x) - d_{K-1}(x) \sum_{k=1}^{K-2} \alpha_k$$

$$= \beta_0 + \beta_1 x + \sum_{k=1}^{K-2} \frac{\alpha_k}{\xi_K - \xi_k} (x - \xi_k)_+^3 - \sum_{k=1}^{K-2} \frac{\alpha_k}{\xi_K - \xi_k} (x - \xi_K)_+^3$$

$$- \frac{(x - \xi_{K-1})_+^3}{\xi_K - \xi_{K-1}} \sum_{k=1}^{K-2} \alpha_k + \frac{(x - \xi_K)_+^3}{\xi_K - \xi_{K-1}} \sum_{k=1}^{K-2} \alpha_k$$

~~Coeff to~~ θ_k

~~For $k \le K-2$:~~ $\theta_k = \frac{\alpha_k}{\xi_K - \xi_k}$

$$\theta_{K-1} = - \left(\sum_{k=1}^{K-2} \alpha_k \right) \cdot \frac{1}{\xi_K - \xi_{K-1}}$$

$$\theta_K = - \sum_{k=1}^{K-2} \frac{\alpha_k}{\xi_K - \xi_k} + \frac{\sum_{k=1}^{K-2} \alpha_k}{\xi_K - \xi_{K-1}}$$

Now

$$\sum_{k=1}^{K}\theta_k = \sum_{k=1}^{K-2}\frac{\alpha_k}{\xi_K-\xi_k} - \frac{\sum_{k=1}^{K-2}\alpha_k}{\xi_K-\xi_{K-1}} - \underset{i\,\theta_K}{\text{def same}} = \underline{\underline{0}}$$

$$\sum_{k=1}^{K}\xi_k\theta_k = \sum_{k=1}^{K-2}\frac{\xi_k\alpha_k}{\xi_K-\xi_k} - \frac{\left(\sum_{k=1}^{K-2}\alpha_k\right)\xi_{K-1}}{\xi_K-\xi_{K-1}}$$

$$-\left(\sum_{k=1}^{K-2}\frac{\alpha_k}{\xi_K-\xi_k}\right)\xi_K + \frac{\left(\sum_{k=1}^{K-2}\alpha_k\right)\xi_K}{\xi_K-\xi_{K-1}}$$

$$= \sum_{k=1}^{K-2}\frac{\alpha_k}{\xi_K-\xi_k}\left(\xi_k-\xi_K\right) \qquad \text{"} \left(\sum_{k=1}^{K-2}\alpha_k\right)$$

$$= -\sum_{k=1}^{K-2}\alpha_k$$

$$= \underline{\underline{0}}$$