Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4125 Matematikk 4N
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Solutions to exercise set 9

$\boxed{1}$ **a)** We compute by the Lagrange method. First we compute the linear functions L_i that are one at x_i and zero at the other point. These are:

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 0.5}{0 - 0.5} = 1 - 2x, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x}{0.5} = 2x$$

We then find $p_1(x) = f_0L_0(x) + f_1L_1(x) = (1 - 2x) + 0.60653 \times 2x$, and hence

$$p_1(x) = 1 - 0.78694x$$

(Solutions given to 5 significant figures). We find the following approximation of $e^{-0.25}$:

$$p_1(0.25) = 0.80327$$

b) We again use the Lagrange method. The L_i are in this case given by

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0.5)(x - 1)}{-0.5 \times -1} = 2(x - 0.5)(x - 1)$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x - 1)}{0.5 \times -0.5} = -4x(x - 1)$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x - 0.5)}{0.5} = 2x(x - 0.5)$$

Writing $p_2(x) = f_0L_0(x) + f_1L_1(x) + f_2L_2(x)$ then gives

$$p_2(x) = 1 - 0.94175x + 0.30963x^2,$$

which in turn gives the approximation

$$p_2(0.25) = 0.78391$$

c) As a crude estimate we have $\epsilon_1(x) = f(x) - p_1(x) \approx p_2(x) - p_1(x)$. In this case we obtain

$$\epsilon_1(x) \approx 0.78391 - 0.80327 = -0.019352$$

(Answers displayed to 5 significant figures but computed in exact arithmetic)

d) We use the result that the error of linear interpolation is given by

$$\epsilon_1(x) = (x - x_0)(x - x_1)\frac{f''(t)}{2} = \frac{1}{2}x(x - 0.5)e^{-t},$$

for some $t \in (0, 1)$. As e^{-t} is decreasing on $[0, 1]$, it is bounded from above by $e^0 = 1$ and from below by $e^{-1} = 0.36789$. We therefore find

$$0.5x(x - 0.5) \leq \epsilon_1(x) \leq 0.18394x(x - 0.5)$$

Substituting in $x = 0.25$ then gives the error bound

$$-0.03125 \le \epsilon_1(0.25) \le -0.011496$$

The true error is $e^{-0.25} - p_1(x) = -0.024465$, which lies within the above range (as does the estimate we found previously).

$\boxed{2}$ **a)** We construct the table of divided differences:

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, \ldots, x_{i+3}]$
-2	-1	$\frac{1-(-1)}{0-(-2)} = 1$	$\frac{-2-1}{1-(-2)} = -1$	$\frac{3-(-1)}{2-(-2)} = 1$
0	1	$\frac{-1-1}{1-0} = -2$	$\frac{4-(-2)}{2-0} = 3$	
1	-1	$\frac{3-(-1)}{2-1} = 4$		
2	3			

We then use the Newton interpolation formula

$$p(x) = f_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, \ldots, x_3](x - x_0)(x - x_1)(x - x_2)$$

from which we obtain

$$p(x) = -1 + (x + 2) - x(x + 2) + x(x + 2)(x - 1),$$

which can be rearranged to give

$$p(x) = 1 - 3x + x^3$$

b) We keep the same divided difference table, but add an additional point. The computations are indicated below:

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, \ldots, x_{i+3}]$	$f[x_i, \ldots, x_{i+4}]$
-2	-1	1	-1	1	$\frac{1-1}{2.5-(-2)} = 0$
0	1	-2	3	$\frac{5.5-3}{2.5-0} = 1$	
1	-1	4	$\frac{12.25-4}{2.5-1} = 5.5$		
2	3	$\frac{9.125-3}{2.5-2} = 12.25$			
2.5	9.125				

As we have

$$p_4(x) = p_3(x) + (x - x_0) \cdots (x - x_3) f[x_0, \ldots, x_4],$$

and $f[x_0, \ldots, x_4] = 0$ from the above table, the polynomials p_3 and p_4 are the same.

$\boxed{3}$ **a)** We begin with the rectangular rule. First we must find the position of the integration nodes $x_1^*, \ldots, x_4^*$. These lie in the middle of the intervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$ and $[0.75, 1]$ respectively. We thus have $x_1^* = 0.125, x_2^* = 0.375, x_3^* = 0.625, x_4^* = 0.875$. Now $h = 1/4$, so the approximation is then

$$J_r = \frac{1}{4}\big(\tan(0.125) + \tan(0.375) + \tan(0.625) + \tan(0.875)\big) = 0.609545$$

For the trapezoidal rule, $x_1 = 0.25, x_2 = 0.5, x_3 = 0.75$. As we still have $h = \frac{1}{4}$, the formula gives

$$J_t = \frac{1}{4}\big(\frac{1}{2}\tan(0) + \tan(0.25) + \tan(0.5) + \tan(0.75) + \frac{1}{2}\tan(1)\big) = 0.627986$$

b) We use the formula for the error of the trapezoid rule approximation:

$$\epsilon = -\frac{b-a}{12}h^2 f''(t),$$

for some $t \in (a, b)$. In our case, we find

$$\epsilon = -\frac{1}{96}\tan(t)\sec^2(t)$$

As this function is increasing, it is bounded from above by the value at $t = 1$. We find

$$\epsilon \leq 0.055572$$

$\boxed{4}$ **a)** First we note that $h = \frac{1}{4}$ and that the integration nodes are at $a = 1, x_1 = 1.25, x_2 = 1.5, x_3 = 1.75, b = 2$. Simpson's rule then gives

$$J_{\frac{1}{4}} = \frac{1}{12}\big(\cos(e^{1-1}) + 4\cos(e^{1-1.25^2}) + 2\cos(e^{1-1.5^2}) + 4\cos(e^{1-1.75^2}) + \cos(e^{1-2^2})\big)$$
$$= 0.89944394$$

b) We now recompute with $h = \frac{1}{2}$ and nodes at $a = 1, x_1 = 1.5, b = 2$. Simpson's rule is

$$J_{\frac{1}{2}} = \frac{1}{6}\big(\cos(e^{1-1}) + 4\cos(e^{1-1.5^2}) + \cos(e^{1-2^2})\big) = 0.896002$$

We then use the estimate

$$\epsilon_{\frac{1}{4}} = \frac{1}{15}(J_{\frac{1}{4}} - J_{\frac{1}{2}}) = 0.00022945$$