Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2017

Solutions exercise 2

$\boxed{4}$ Sketch the phase diagram for $\ddot{x} + e^x = a$ for $a < 0$, $a = 0$ and $a > 0$.

Let $y = \dot{x}$ to obtain the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= a - e^x. \end{aligned}$$

The equilibrium points are found by $\dot{x} = y = 0$ and $\dot{y} = a - e^x = 0$. It follows that there is an equilibrium point for $x = \ln a$ when $a > 0$. Otherwise, there are no equilibrium points.

Since the system is of the form $\ddot{x} = f(x)$ we may find the phase diagram by the aid of the potential function. We can rewrite $\ddot{x} = f(x)$ into

$$\begin{aligned} \ddot{x} - f(x) &= 0 \\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\dot{x})^2 - f(x) &= 0 \\ \frac{1}{2}(\dot{x})^2 - \int f(x) dx &= C. \end{aligned}$$

Let $\mathcal{V}(x) = -\int f(x)\mathrm{d}x$. Then $\frac{1}{2}y^2 + \mathcal{V}(x)$ is constant on the paths of the system. Notice the similarity with the expression for total mechanical energy, where $\frac{1}{2}(\dot{x})^2$ is the kinetic energy and $\mathcal{V}(x)$ is the potential energy.

An equilibrium point is a stable center if it is a minimum point for $\mathcal{V}(x)$, and an unstable saddle point for a maximum point of $\mathcal{V}(x)$. Here, $\mathcal{V}(x) = -ax + e^x$. We plot this function, and find that the equilibrium point $x = \ln a$ for $a > 0$ is stable.

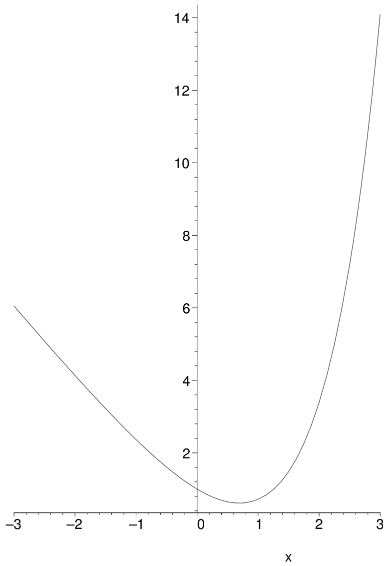

Figure 1: A plot of $\mathcal{V}(x) = -ax + e^x$ for $a = 2$. The equilibrium point $x = \ln a$ is stable.

A recipe for using the potential function to sketch the phase diagram is given in the following: If we plot the potential function and draw a horizontal line, the intersections with the potential function represents points with equal energy levels. If two points of intersection are located on each side of a minimum point of the potential function, these points will belong to the same path in the phase diagram. However, if they are located on each side of a maximum point of the potential function, they will belong to different paths. Therefore, we can decide whether a point on the x-axis belong to a closed path or not.

In this case, we see that, for each point on the x-axis except the equilibrium point, we can find a point on the other side of the minimum point by drawing a horizontal line. All paths are therefore closed for $a > 0$.

Returning to the equations, for $y \neq 0$ we may divide by y on both sides of the equations which gives $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a-e^x}{y}$. Rewriting gives

$$y\mathrm{d}y = (a - e^x)\mathrm{d}x.$$

We can integrate this equation to obtain

$$y^2(x) = 2ax - 2e^x + C$$

where C is a constant.

To give a sketch, note that when $x \ll 0$, $y^2(x) \approx 2ax + C$, and for $x \gg 0$, $y^2(x) \approx -2e^x + C$. We can sketch the phase diagram by noting that the transition between the two cases has to be smooth. Notice the connection between the shape of the function $\mathcal{V}(x)$ and the phase diagrams. See figure 2, 3 and 4 for a phase diagram of $\ddot{x} + e^x = a$ when $a = 2$, $a = 0$ and $a = -2$.

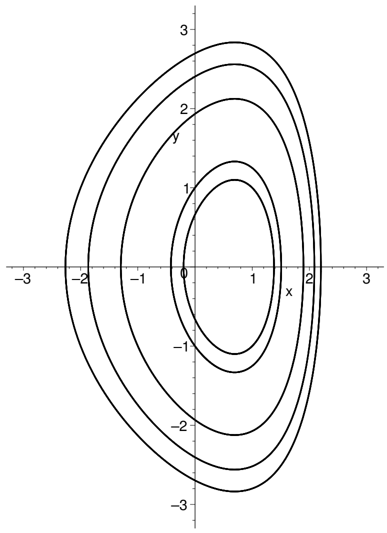

Figure 2: A pahse diagram of $\ddot{x} + e^x = a$ when $a = 2$

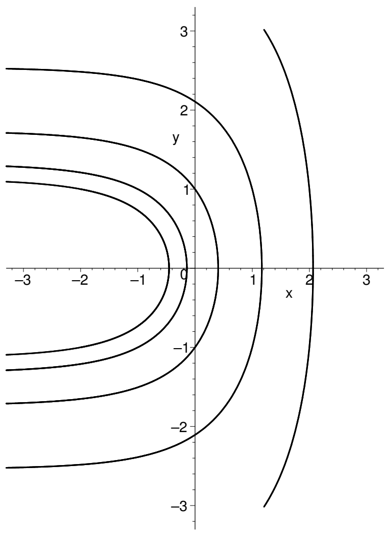

Figure 3: A phase diagram of $\ddot{x} + e^x = a$ when $a = 0$

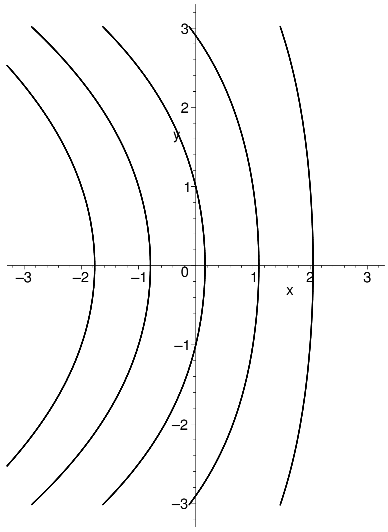

Figure 4: A phase diagram of $\ddot{x} + e^x = a$ when $a = -2$

$\boxed{6}$ The potential energy $\mathcal{V}(x)$ of a conservative system is continuous, and is strictly increasing for $x < -1$, zero for $|x| \leq 1$, and stricly decreasing for $x > 1$. Locate the equilibrium points and sketch the phase diagram for the system.

The equilibrium points are given by $y = 0$ and $\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}x} = 0$. Hence, there are equilibrium points for $|x| < 1$. If we assume that the potential function is differentiable when $|x| = 1$, these will also be equilibrium points. The equilibrium points are unstable, since they are maximum points for the potential function. For the sketch, it will look like we have an equilibrium point in the origin stretched out such that the paths become horizontal lines between -1 and 1. See figure 5 for a sketch of the phase diagram.

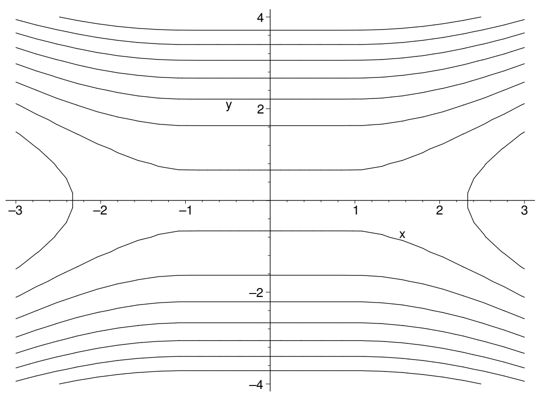

Figure 5: A possible phase diagram of a conservative system where the potential energy is strictly increasing for $x < -1$, zero for $|x| \leq 1$ and strictly declining for $x > 1$.