MA0301 ELEMENTARY DISCRETE MATHEMATICS
SPRING 2017

1. Homework Set 5 – Solutions

Exercise 1. *Find the least number n_0 for which it is true that $n! \geq 2^n$. Taking the case $n = n_0$ as the induction basis, show that the statement is true for all $n \geq n_0$.*

Solution 1. Induction:

Check: $n = 4$: $2^4 = 16 < 4! = 24$

Assume that the statement holds for $n = k > 3$: $2^k < k!$. $2^k 2 < k!(k+1) = (k+1)!$, since $k+1 > 2$ for $k > 3$. Therefore the statement holds for all $n > 3$.

Exercise 2. *Grimaldi's book (5. ed., Exercises 4.2, page 219): solve* **Exercise 1 a,c,e**

Solution 2. a) $c_1 = 7$; for $n > 0$ $c_{n+1} = c_n + 7$

c) $c_1 = 10$; for $n > 0$ $c_{n+1} = c_n + 3$

e) $c_1 = 1$; for $n > 0$ $c_{n+1} = c_n + 2n + 1$

Exercise 3. *Grimaldi's book (5. ed., Exercises 4.2, page 220): solve* **Exercise 14**

Solution 3. Induction:

Check: $L_1 L_1 = 1^2 = 1 = 3 - 2 = L_1 L_2 - 2$.

Assume that the statement holds for $n = k$, i.e., $\sum_{i=1}^{k} L_i^2 = L_k L_{k+1} - 2$. Let us consider $\sum_{i=1}^{k+1} L_i^2 = \sum_{i=1}^{k} L_i^2 + L_{k+1}^2 = L_k L_{k+1} - 2 + L_{k+1}^2 = L_{k+1}(L_k + L_{k+1}) - 2 = L_{k+1} L_{k+2} - 2$. Hence, the statement holds for all n.

Exercise 4. *Grimaldi's book (5. ed., Exercises 4.1, page 209): solve* **Exercise 15**

Solution 4. Induction

Check the case $n = 5$: $2^5 = 32 > 5^2 = 25$. Assume that for $n = k > 4$: $2^k > k^2$ holds. For $k > 2$ we have $k^2 > 2k + 1$. With $2^k > k^2$ we have that $2^{k+1} > 2k^2 > k^2 + 2k + 1 = (k+1)^2$. Therefore the result holds for $n > 4$.

Exercise 5. *Grimaldi's book (5. ed., Exercises 4.2, page 220): solve* **Exercise 20**

Solution 5. a)

$$
\begin{aligned}
(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_n \to p_{n+1}) &\implies (p_n \to p_{n+1}) \\
&\implies \neg p_n \vee p_{n+1} \\
&\implies (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n) \vee p_{n+1} \\
&\implies \neg (p_1 \wedge p_2 \wedge \cdots \wedge p_n) \vee p_{n+1} \\
&\implies (p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to p_{n+1}
\end{aligned}
$$

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b) Assume the hypotheses in Theorem 1, that is $S(1)$ true and $S(k) \implies S(k+1)$ for all $k \geq 1$. We want to prove $S(n)$ for all $n \geq 1$ using Theorem 2. Let $n_0 = n_1 = 1$ in Theorem 2. Then condition a) in Theorem 2 is the same as condition a) in Theorem 1, and condition 2 in Theorem 2 follows from $S(k) \implies S(k+1)$. Thus the conditions in Theorem 2 are fulfilled, and we get that $S(n)$ is true for all $n \geq n_0 = 1$, which was what we wanted to prove.

c) Assume that $S \neq \emptyset$ does not have a least element, and let $P(k)$ be the statement 'S does not contain any numbers $\leq k$'. $P(1)$ is true, because if not, 1 would have been the least element in S. $P(k) \implies P(k+1)$ is also true for $k \geq 1$, because if not, $k+1$ would have been the least element in S. By Theorem 1/induction, we get that $P(n)$ is true for all n, implying that S is empty, so we have a contradiction.

d) Let $P(k)$ be the statement '$S(n_0), S(n_0+1), \ldots, S(n_1+k-1)$ are true'. By the first condition in Theorem 2, $P(1)$ is true. By the second condition, $P(k) \implies P(k+1)$ holds for $k \geq 1$. Theorem 1 then says that $P(n)$ is true for all $n \geq 1$, which implies that $S(n)$ is true for all $n \geq n_0$.

Exercise 6. *Use the alternative principle of induction to show that if u_n is defined recursively by the rules $u_1 = 3$, $u_2 = 5$ and for all $n > 1$*

$$u_{n+1} = 3u_n - 2u_{n-1}$$

then $u_n = 2^n + 1$ for all $n \in \mathbb{N}$.

Solution 6. Check that $u_1 = 3 = 2+1$ and $u_2 = 5 = 2^2+1$. Calculate $u_{k+1} = 3(2^k+1) - 2(2^{k-1}+1) = 2^k 3 - 2^k + 3 - 2 = 2^k 2 + 1$. Therefore the statement that $u_n = 2^n + 1$ holds for all n.

2. Classroom Set 5 – Solutions

Exercise 1. *1) Find the appropriate values of n_0 such that $n^2 - 6n + 8 \geq 0$. Then show that the statement is true for all $n \geq n_0$.*

2) Find the appropriate values of n_0 such that $n^3 \geq 6n^2$. Then show that the statement is true for all $n \geq n_0$.

Solution 1. 1) Define $f(n) := n^2 - 6n + 8$ and check that: $f(1) = 3$, $f(2) = 0$, $f(3) = -1$, $f(4) = 0$, $f(5) = 3$, $f(6) = 8$. Hence we may want to assume that $n_0 = 4$. Induction: check that: $f(4) \geq 0$ and suppose that $f(k) \geq 0$ for $k > 3$. We want to show that $f(k+1) \geq 0$: $f(k+1) = f(k) + 2k - 5$. Since $k > 3$ we have that $2k - 5 > 0$, and therefore $f(k+1) > f(k) \geq 0$. Hence the statement follows.

2) Define $f(n) := n^3 - 6n^2 = n^2(n-6)$. We want to show that $f(n) \geq 0$ for $n \geq n_0$. Suppose $n_0 = 6$. Induction: check that $f(6) = 0 \geq 0$, and suppose that $f(k) \geq 0$ for $k > 5$. Then $f(k+1) = f(k) + k^2 + (2k+1)(k-5) \geq f(k)$ since $k - 5 > 0$ for $k > 5$. Hence the statement follows.

Exercise 2. *Use the alternative principle of induction to show that if u_n is defined recursively by the rules $u_1 = 1$, $u_2 = 5$ and for all $n > 1$*

$$u_{n+1} = 5u_n - 6u_{n-1}$$

then $u_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.

Solution 2. Verify for $n = 1$ and $n = 2$: $3 - 2 = 1 = u_1$ and $9 - 4 = 5 = u_2$.

Assume that the formula holds for u_k and u_{k-1}. Deduce that it is correct for u_{k+1}:

$$u_{k+1} = 5u_k - 6u_{k-1} = 9(3^{k-1}) - 4(2^{k-1}) = 3^{k+1} + 2^{k+1}.$$

This shows that the statement holds for $n > 0$.

Exercise 3. *Grimaldi's book (5. ed., Exercises 4.2, page 219): solve* **Exercise 1 b,d,f**

Solution 3. b) $c_1 = 7$; for $n > 0$ $c_{n+1} = 7c_n$

d) $c_1 = 7$; for $n > 0$ $c_{n+1} = 7$ or $c_{n+1} = c_n$

f) $c_1 = 2,\ c_2 = 3$; for $n > 0$ $c_{n+2} = c_n$

Exercise 4. *Grimaldi's book (5. ed., Exercises 4.2, page 219): solve* **Exercise 12**

Solution 4. Induction: $n = 0$: $F_0 = 0 = 1 - 1 = F_2 - 1$. Assume that the statement holds for $n = k \geq 0$, i.e., $\sum_{i=0}^{k} F_i = F_{k+2} - 1$. Consider $n = k + 1$ and calculate

$$\sum_{i=0}^{k+1} F_i = F_{k+2} - 1 + F_{k+1} = F_{k+1} + F_{k+2} - 1 = F_{k+3} - 1.$$

This shows that the statement holds for n.

Exercise 5. *Grimaldi's book (5. ed., Exercises 4.2, page 219): solve* **Exercise 13**

Solution 5. Induction: $n = 1$: $F_0/2 = 0 = 1 - 2/2 = 1 - F_3/2$. Assume that the statement holds for $n = k > 0$, i.e., $\sum_{i=1}^{k} F_{i-1}/2^i = 1 - F_{k+2}/2^k$. Consider $n = k + 1$ and calculate

$$\sum_{i=1}^{k+1} F_{i-1}/2^i = 1 - F_{k+2}/2^k + F_k/2^{k+1} = 1 - F_{k+3}/2^k 2.$$

This shows that the statement holds for $n > 0$.

Exercise 6. *List 5 examples of objects that are counted by the Catalan numbers, e.g., the number of complete parenthesizations of words in $n+1$ letters. For the four letter word $w = abcd$ you'll find $C_3 = 5$ parenthesizations*

$$(ab)(cd),\ ((ab)c)d,\ (a(bc))d,\ a(b(cd)),\ a((bc)d)$$

Solution 6. Check out:

– The On-Line Encyclopedia of Integer Sequences (OEIS) `https://oeis.org/`

– See entry: Catalan numbers `https://oeis.org/A000108`

– Richard Stanley's website at MIT