



a) You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:

J. S.: 1.1 (i)–(iii).

b) These exercises will be presented / discussed in the exercise class:

E1 Aim: Find the equilibrium points and sketch the phase diagram for $\ddot{x} + 3\dot{x} + 2x = 0$.
Idea: Use methods from linear algebra to understand the dynamics of the above equation.

Background: A (real) 2×2 matrix A is diagonalizable, if there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If A has 2 real eigenvalues λ_1 and λ_2 such that either

- $\lambda_1 \neq \lambda_2$ or
- $\lambda_1 = \lambda_2$ and there exist 2 linearly independent eigenvectors to λ_1 ,

this is always possible. Given a system of ODEs

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad (1)$$

where \mathbf{x} denotes a vector valued function from \mathbb{R} to \mathbb{R}^2 and A a diagonalizable matrix, it can be written as $\dot{\mathbf{x}} = PDP^{-1}\mathbf{x}$. Multiplying with P^{-1} from the left and introducing $\mathbf{z} = P^{-1}\mathbf{x}$, we obtain that \mathbf{z} solves

$$\dot{\mathbf{z}} = D\mathbf{z}. \quad (2)$$

- Rewrite $\ddot{x} + 3\dot{x} + 2x = 0$ as a system of differential equations of first order, i.e. as in (1).
- Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.
- Find the equilibrium points and sketch the phase diagram for $\dot{\mathbf{z}} = D\mathbf{z}$.
- Find the equilibrium points and sketch the phase diagram for $\dot{\mathbf{x}} = A\mathbf{x}$, with the help of c). Find the connections between the two phase diagrams.

E2 Aim: Sketch the phase diagram for $\ddot{x} = -x - \alpha x^3$ with $\alpha \in \mathbb{R}$.
Idea: Use the connection between computing phase paths and drawing phase diagrams to check that your result is correct.
Background: Consider the differential equation

$$\dot{x} = \alpha x, \quad x(0) = C, \quad \alpha \in \mathbb{R}. \quad (3)$$

Then the corresponding solutions is given by

$$x(t) = Ce^{\alpha t},$$

and in particular, one observes that the behavior of the solution for $t \rightarrow \pm\infty$ depends heavily on the value of α .

- a) Investigate and sketch the function $f(x) = -x - \alpha x^3$ for different values of α .
- b) Rewrite $\ddot{x} = -x - \alpha x^3$ as a system of differential equations of first order.
- c) Find the equilibrium points and sketch the phase diagram.
- d) Classify all the equilibrium points (with the help of phase paths). Find the connection between the phase diagram close to the equilibrium points and the classification.

E3 Aim: Determine if $2y^2 = 1 - 2x$ is a separatrix for the equation $\ddot{x} + \dot{x}^2 + x = 0$.
Idea: Sketch the phase diagram and identify the curve $2y^2 = 1 - 2x$.

- a) Rewrite $\ddot{x} + \dot{x}^2 + x = 0$ as a system of differential equations of first order.
- b) Compute the corresponding phase paths.
- c) Sketch the phase diagram.
- d) Is $2y^2 = 1 - 2x$ a separatrix or not?