1. Area mean value theorem:

Prove that if u is harmonic into the disk $\{z : |z| < R\}$ and continuous in the closed disk $\{z : |z| \leq R\}$ then

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\,d\varphi =$$

$$= \frac{1}{\pi R^2}\iint_{|z|<R} u(re^{i\varphi})\,r\,dr\,d\varphi$$

2. Poisson and Schwarz formula for arbitrary ~~radii~~ disk.

Let u is harmonic into the disk $\{z : |z| < R\}$ and continuous into the closed disk $\{z : |z| \leq R\}$. Let also $f(z) = u(z) + iv(z)$ be the corresponding analytic function. Prove that

a. (Poisson)

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\varphi - \theta)}\, u(Re^{i\theta})\,d\theta$$

b. (Schwarz)

$$f(z) = \frac{1}{2i\pi}\int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z}\,\frac{d\zeta}{\zeta} + iC$$

for some $C \in \mathbb{R}$.

c) (Conjugate harmonic function)

Find representation for the conjugate harmonic function $v(z)$ through the boundary values $u(Re^{i\theta})$, $-\pi \le \theta \le \pi$.

3. Obtain the Schwarz and Poisson representations (for the unit disk, say) as convolution theorems.

4. (Poisson representation in the half-plane.) Let $u(z)$ be harmonic in $\mathbb{C}_+ = \{z;\ \operatorname{Im} z > 0\}$, ~~bounded~~ continuous in $\overline{\mathbb{C}}_+ = \{z : \operatorname{Im} z \ge 0\}$ and bounded in $\overline{\mathbb{C}}_+$. Prove that

$$u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} u(t) \frac{dt}{(x-t)^2 + y^2}, \qquad z = x+iy \in \mathbb{C}_+$$

(Hint: Conformal mapping).

~~5. Find function $u(z)$ harmonic in $\mathbb{C}_+$ continuous in $\overline{\mathbb{C}}_+ \setminus \{\pm 1\}$ and such that~~

~~$$u(x) = \begin{cases} 0, & |x| > 1 \\ 1, & |x| < 1 \end{cases}$$~~

~~What is~~

5. Observe that

$$\frac{y}{(x-t)^2+y^2} = \operatorname{Im} \frac{1}{z-t}, \quad z = x+iy$$

and obtain the version of the Schwarz formula for the upper half-plane.

6. Find function $u(z)$ harmonic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+ \setminus \{\pm 1\}$ and such that

$$u(x) = \begin{cases} 0 & |x|>1 \\ 1 & |x|<1 \end{cases}.$$

What is the corresponding function $f(z)$?

7. If $u(z)$ is harmonic and bounded in $0<|z|<\rho$ show that the origin is a removable singularity in the sense that u becomes harmonic in $|z|<\rho$ if $u(0)$ is properly defined.

8. Let $u(z)$ be harmonic in $\mathbb{C}_+$ and $0 \le u(x+iy) \le Ky$ for $y>0$.
Prove that $u(z) = ky$ for some $k \in [0, K]$.