TMA4125 Matematikk 4N
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Solutions to exercise set 8

$\boxed{1}$ **a)** We begin by noting that all of the schemes are valid in the sense that a fixed point of each will be a root of the given polynomial. It remains to check for convergence by computing the derivatives. We first consider

$$x_{n+1} = x_n^3 - 1,$$

i.e. $x_{n+1} = g(x_n)$, where $g(x) = x^3 - 1$. Then $g'(x) = 3x^2$, and hence $|g'(x)| > 1$ on the whole of the interval $[1, 1.5]$. Hence this scheme will not converge. Next we look at

$$x_{n+1} = x_n^{-1} + x_n^{-2}$$

so $g(x) = x^{-1} + x^{-2}$, and $g'(x) = -x^{-2} - 2x^{-3}$. Then g' is increasing on $[1, 1.5]$, and $g(1.5) = -1.1$, i.e. $|g'(x)| > 1$ on $[1, 1.5]$, and this will not converge either. It remains to check

$$x_{n+1} = (x_n + 1)^{\frac{1}{3}},$$

for which we find $g'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}}$. This is decreasing and greater than zero on $[1, 1.5]$. As $g'(x) = 0.2099...$, we have $|g'(x)| \leq 0.21 < 1$ on $[1, 1.5]$. Moreover, since $g(x)$ is increasing each application of g keeps us in the interval $[1, 1.5]$ where $|g'(x)| \leq 0.21$, hence we have convergence.

b) We find the following sequence of values: $x_0 = 1, x_1 = 1.2599, x_2 = 1.3122, x_3 = 1.3223, x_4 = 1.3242$. As the first three digits of x_3 and x_4 are equal, we conclude that $x = 1.32$ to 3 significant digits.

$\boxed{2}$ **a)** As $f(x) = x^m - a$, we have $f'(x) = mx^{m-1}$. Then

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^m - a}{mx^{m-1}} = x(1 - \frac{1}{m}) + \frac{a}{mx^{m-1}}$$

b) Now $\sqrt[3]{2}$ solves $x^3 - 2 = 0$, so we use the above scheme with $m = 3$ and $a = 2$, ie

$$x_{n+1} = \frac{2}{3}(x_n + \frac{1}{x_n^2})$$

As an illustration, we start at $x_0 = 1$, although other initial values are possible. We then find $x_1 = 1.3333, x_2 = 1.2639, x_3 = 1.2599, x_4 = 1.2599$, i.e $x = 1.2599$ to 5 significant figures.

c) For the estimate, we Taylor expand about the solution $s = a^{\frac{1}{m}}$. Then

$$x_{n+1} = g(s - \epsilon_n) = g(s) - g'(s)\epsilon_n + \frac{1}{2}g''(s)\epsilon_n^2 + \ldots$$

Now $g(s) = s$ and $g'(s) = 0$, hence we find

$$\epsilon_{n+1} = s - x_{n+1} \approx \frac{1}{2} g''(s)\epsilon_n^2$$

It remains to compute the second derivative. Then

$$g'(x) = \left(1 - \frac{1}{m}\right) + \frac{a(1-m)}{m} x^{-m}$$

and hence $g''(x) = a(m-1)x^{-m-1}$. Setting in $s = a^{\frac{1}{m}}$ gives $s^{-m-1} = a^{-1}a^{-\frac{1}{m}}$, and hence

$$\frac{1}{2} g''(s) = \frac{m-1}{2} a^{-\frac{1}{m}}$$

i.e. we obtain an estimate

$$\epsilon_{n+1} \approx -\frac{m-1}{2} a^{-\frac{1}{m}} \epsilon_n^2$$

$\boxed{3}$ **a)** Here $f(x) = x^2 - 4x + 1 = 0$, and for the secant method we have

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Now $f(x_n) - f(x_{n-1}) = x_n^2 - x_{n-1}^2 - 4x_n + 4x_{n-1}$, so we can use the suggested factorization to find

$$\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = \frac{1}{(x_n + x_{n-1} - 4)}$$

It follows that

$$x_{n+1} = \frac{x_n(x_n + x_{n-1} - 4) - x_n^2 + 4x_n - 1}{x_n + x_{n-1} - 4} = \frac{x_n x_{n-1} - 1}{x_n + x_{n-1} - 4}$$

b) Implementing the above three times with $x_0 = 5, x_1 = 4$ gives $x_2 = 3.8, x_3 = 3.7368, x_4 = 3.7321, x_5 = 3.7320$ (performed in exact arithmetic, but rounded to 5 significant digits).

$\boxed{4}$ **a)** We begin by computing the Jacobian of the mapping

$$DF = \begin{pmatrix} \frac{\partial}{\partial x}(x - \cos y) & \frac{\partial}{\partial y}(x - \cos y) \\ \frac{\partial}{\partial x}(x - \cos y) & \frac{\partial}{\partial y}(y - \cos x) \end{pmatrix} = \begin{pmatrix} 1 & \sin y \\ \sin x & 1 \end{pmatrix}$$

We then have

$$DF^{-1} = \frac{1}{1 - \sin x \sin y} \begin{pmatrix} 1 & -\sin y \\ -\sin x & 1 \end{pmatrix}$$

Newtons method is then

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \frac{1}{1 - \sin x_n \sin y_n} \begin{pmatrix} 1 & -\sin y_n \\ -\sin x_n & 1 \end{pmatrix} \begin{pmatrix} x_n - \cos y_n \\ y_n - \cos x_n \end{pmatrix}$$

and hence

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{1}{1 - \sin x_n \sin y_n} \begin{pmatrix} x_n(1 - \sin x_n \sin y_n) - (x_n - \cos y_n) + \sin y_n(y_n - \cos x_n) \\ y_n(1 - \sin x_n \sin y_n) + \sin x_n(x_n - \cos y_n) - (y_n - \cos x_n) \end{pmatrix}$$

which on rearranging gives

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{1}{1 - \sin x_n \sin y_n} \begin{pmatrix} \cos y_n + (y_n - x_n \sin x_n - \cos x_n) \sin y_n \\ \cos x_n + (x_n - y_n \sin y_n - \cos y_n) \sin x_n \end{pmatrix}$$

b) Performing three iterations from $x_0 = 0, y_0 = 1$ using the above formula gives $x_1 = 0.5403, y_1 = 1$, then $x_2 = 0.7516, y_2 = 0.7488$, and at last $x_3 = 0.7391, y_3 = 0.7391$ (working to 4 significant figures).