HOMEWORK 7
INDEPENDENCE

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Problem 1. Show that if E and F are independent events, so are: $\overline{E}$ and F; $\overline{E}$ and $\overline{F}$.

Problem 2. Prove the following basic properties:

(a) If a random variable X is independent of itself, then it is deterministic, meaning that $X = c$ a.s. for some constant c.

(b) A deterministic random variable is independent of any other random variable.

(c) Show that the scalar random variables $X_1, \dots, X_n$ are jointly independent if and only if
$$\mathbb{P}\big(\wedge_{i=1}^n (X_i \le t_i)\big) = \prod_{i=1}^n \mathbb{P}(X_i \le t_i)$$
for all $t_1, \dots, t_n \in \mathbb{R}$.

(d) Show that the discrete random variables $X_1, \dots, X_n$ are jointly independent if and only if
$$\mathbb{P}\big(\wedge_{i=1}^n (X_i = x_i)\big) = \prod_{i=1}^n \mathbb{P}(X_i = x_i)$$
for all $x_1, \dots, x_n$ in their corresponding ranges of values.

Problem 3. Show that if X and Y are independent random variables with values in R and S, and if $F\colon R \to \mathbb{R}$ and $G\colon S \to \mathbb{R}$ are measurable functions (either both nonnegative or both absolutely integrable), then
$$\mathbb{E}\, F(X)\, G(Y) = \big(\mathbb{E} F(X)\big)\, \big(\mathbb{E} G(Y)\big).$$

We did not get the chance to define independent events and σ-algebras in class (in a more conceptual manner). So here are these concepts.

Definition. *A family of events $\{E_i\}_{i \in N}$ is jointly independent if the corresponding family of indicator random variables $\{\mathbf{1}_{E_i}\}_{i \in N}$ is jointly independent.*

A family of σ-algebras $\{\mathcal{F}_i\}_{i \in N}$ is jointly independent if for every $E_i \in \mathcal{F}_i$, the family of events $\{E_i\}_{i \in N}$ is jointly independent.

Problem 4. Show that the family of events $\{E_i\}_{i \in N}$ is jointly independent if and only if for any disjoint finite sets of indices $J_1, J_2 \subset N$,

$$\mathbb{P}\big(\wedge_{i \in J_1} E_i \wedge \wedge_{i \in J_2} \overline{E_i} \big) = \prod_{i \in J_1} \mathbb{P}(E_i) \prod_{i \in J_2} \mathbb{P}(\overline{E_i}).$$

In particular this says that if the events E_1, E_2, E_3 are jointly independent, then

$$\mathbb{P}(E_1 \wedge E_2 \wedge E_3) = \mathbb{P}(E_1)\,\mathbb{P}(E_2)\,\mathbb{P}(E_3).$$

Give an example showing that the reverse is not necessarily true, meaning that this last equality may hold without E_1, E_2, E_3 being jointly independent.

Problem 5. Show that a family of σ-algebras $\{\mathcal{F}_i\}_{i \in N}$ is jointly independent if and only if any family $\{X_i\}_{i \in N}$ of random variables such that X_i is $\mathcal{F}_i$-measurable for all $i \in N$, is jointly independent.

Problem 6. Show that a family $\{X_i\}_{i \in N}$ of random variables is jointly independent if and only if the family $\{\sigma(X_i)\}_{i \in N}$ of σ-algebras is independent.