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Exercise set 8

1 (Taken from the exam of spring 2015) The polynomial $p(x) = x^3 - x - 1$ has a root in the interval $[1, 1.5]$.

a) Which of the fixed-point iterations schemes

$$x_{n+1} = x_n^3 - 1$$
$$x_{n+1} = x_n^{-1} + x_n^{-2}$$
$$x_{n+1} = (x_n + 1)^{\frac{1}{3}}$$

can be used to find this root? Justify your answer!

b) Use this scheme with initial value $x_0 = 1$ to find the root to three significant digits.

2 In this exercise we use Newton's method (see Kreyszig, 19.2) to find approximate values to mth roots.

a) Find the formula for an iteration of Newton's method, applied to the equation $x^m - a = 0$

b) Use the formula to find $\sqrt[3]{2}$, correct to 5 significant figures.

c) (Optional) By Taylor expanding about $x = a^{\frac{1}{m}}$, find an estimate for the evolution of the error in the form

$$\epsilon_{n+1} \approx C(a, m)\epsilon_n^2,$$

where $C(a, m)$ is a constant you should find, depending on a and m. Consult the section of Kreyszig 19.2 on orders of convergence if you need help.

3 The equation $x^2 - 4x + 1 = 0$ has a solution between 3 and 5.

a) Show that an iteration of the Secant method (see Kreyszig, 19.2) is given by the rule

$$x_{n+1} = \frac{x_n x_{n-1} - 1}{x_n + x_{n-1} - 4}$$

Hint: use the factorization

$$x_n^2 - x_{n-1}^2 - 4x_n + 4x_{n-1} = (x_n - x_{n-1})(x_n + x_{n-1} - 4)$$

b) Compute three steps of the above method, with the initial values $x_0 = 5, x_1 = 4$.

4 We will use Newton's method to solve the system of equations

$$\begin{aligned} x - \cos(y) &= 0 \\ y - \cos(x) &= 0 \end{aligned}$$

(See the notes on the website for information about Newton's method for several dimensions)

a) Show that the Newton iteration is given by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{1}{1 - \sin x_n \sin y_n} \begin{pmatrix} \cos y_n + (y_n - x_n \sin x_n - \cos x_n) \sin y_n \\ \cos x_n + (x_n - y_n \sin y_n - \cos y_n) \sin x_n \end{pmatrix}$$

b) Compute three iterations of the method, starting from $x_0 = 0, y_0 = 1$.

5 (Optional) Write a program that solves equations $f(x) = 0$ using the secant method. Test it on a few problems, such as the equation of exercise 3. Try comparing the results to those of the built-in solvers found in, e.g. Matlab.