TMA4125 Matematikk 4N
Spring 2017

Norwegian University of Science and Technology
Institutt for matematiske fag

Exercise set 7

1 Use the Laplace transform to solve the Initial Value Problem

$$y'' - 9y = \delta(t-1),$$

with initial conditions $y(0) = 2, y'(0) = 0$. (See Kreyszig, Chapter 6.4 for details on how to handle the delta function $\delta(t-1)$)

2 In this exercise we will solve the integral equation

$$y(t) + \int_0^t y(\tau)\cosh(t-\tau)d\tau = t + e^t$$

a) Rewrite the integral appearing in the equation as a convolution (see Kreyszig, Chapter 6.5)

b) By taking Laplace transforms of both sides and using the convolution theorem, show that the Laplace transform of $y(t)$ obeys the equation

$$Y(s) = \frac{s^2-1}{s^2(s-1)}$$

c) Solve the integral equation for $y(t)$ by simplifying the above expression and then taking the inverse Laplace transform.

3 The oscillations of a spring under a forcing term $r(t)$ are described by the equation

$$y'' + 4y = r(t),$$

where the system is initially at rest, i.e. $y(0) = y'(0) = 0$.

a) Find an expression for the Laplace transform of $y(t)$ in the form $Y(s) = Q(s)R(s)$, where $R(s)$ is the Laplace transform of $r(t)$, and $Q(s)$ is a function you should find (this $Q(s)$ is the transfer function of the system, see Kreyszig 6.5). Conclude that

$$y(t) = \frac{1}{2}\sin 2t * r(t),$$

b) By computing the above convolution, solve for y in the case that $r(t) = 4\cos 2t$. You may use the identity

$$\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\big)$$

What is happening as t becomes large? What physical phenomenon does this correspond to?

4 We investigate the variable coefficient ODE,

$$ty'' - 2ty' + 2y = 2$$

with initial condition $y(0) = 1$.

a) Show that the Laplace transform of $y(t)$ obeys the first order linear ODE

$$s\frac{dY}{ds} + 2Y = \frac{1}{s}$$

You will require the formula for $\mathcal{L}\big(tf(t)\big)$ given in Chapter 6.6 of Kreyszig.

b) (Optional) Solve the equation for Y and hence find $y(t)$. (Hint: consider solutions of the form $Y = s^k$. It may be helpful to solve the homogeneous equation $sY' + 2Y = 0$ separately. Alternatively, use an integrating factor)

5 A system of two connected springs subject to an external force is described by the equations

$$y_1'' = -8y_1 + 4y_2 + 11\sin t$$
$$y_2'' = 4y_1 - 8y_2 - 11\sin t$$

with initial conditions $y_1(0) = 1, y_1'(0) = 1, y_2(0) = 1, y_2'(0) = -1$. (See Kreyszig, Chapter 6.7 for a discussion of such systems)

a) Show that the Laplace transforms of y_1 and y_2 satisfy the linear system

$$\begin{pmatrix} s^2+8 & -4 \\ -4 & s^2+8 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} s+1+\frac{11}{s^2+1} \\ s-1-\frac{11}{s^2+1} \end{pmatrix}$$

b) (Optional, for those who wish to practise their algebra, or can solve linear systems by computer!) By solving for Y and then inverting, show that the solution is given by

$$y(t) = \begin{pmatrix} \cos 2t + \sin t \\ \cos 2t - \sin t \end{pmatrix}$$