Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions exercise E20

E20

(c) Since $A - A = 0$ and $A(-A) = -A^2 = (-A)A$ we have

$$e^A e^{-A} = e^{A-A} = e^0 = I = e^{-A}e^A$$

by using (i). This shows that e^A has an inverse matrix e^{-A}.

(d) Using the previous properties we find

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \lim_{h\to 0}\frac{e^{A(t+h)} - e^{At}}{h} = \lim_{h\to 0}\frac{e^{At}\left(e^{Ah} - I\right)}{h} \\
&= e^{At}\lim_{h\to 0}\frac{I + Ah + \sum_{k=2}^{\infty}\left(\frac{A^k h^k}{k!}\right) - I}{h} \\
&= e^{At}A\lim_{h\to 0}\left(I + \sum_{k=1}^{\infty}\frac{A^k h^k}{(k+1)!}\right) = e^{At}A.
\end{aligned}$$

The last equality follows since

$$\begin{aligned}
\left|\sum_{k=1}^{\infty}\frac{A^k h^k}{(k+1)!}\right| &\le \sum_{k=1}^{\infty}\frac{\|A^k\||h|^k}{(k+1)!} \le \sum_{k=1}^{\infty}\frac{\|A^k\||h|^k}{k!} \\
&\le \|A\||h|\sum_{k=0}^{\infty}\frac{\|A\|^k|h|^k}{k!} = e^{\|A\||h|}\|A\||h| \to 0
\end{aligned}$$

as $h \to 0$. Similarly, we can write

$$\frac{d}{dt}e^{At} = \lim_{h\to 0}\frac{e^{A(t+h)} - e^{At}}{h} = \lim_{h\to 0}\frac{\left(e^{Ah} - I\right)e^{At}}{h}$$

to get

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

(e)

$$\begin{aligned}
e^{A^T t} &= \sum_{k=0}^{\infty}\frac{(A^T)^k t^k}{k!} = \sum_{k=0}^{\infty}\frac{(A^T t)^k}{k!} = \sum_{k=0}^{\infty}\left(\frac{(At)^k}{k!}\right)^T \\
&\overset{1)}{=} \left(\sum_{k=0}^{\infty}\frac{(At)^k}{k!}\right)^T = (e^{At})^T.
\end{aligned}$$

1) To show this equality, show it first for finite sums, use $(A+B)^T = A^T + B^T$.