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Exercise set 2 – Solutions

$\boxed{1}$ Suppose $(x_n)_n$ is a Schauder basis for a Banach space X. Show the following assertions.

a) X is separable.

b) The set $(x_n)_n$ is linearly independent.

Solution. **a)**To avoid making the calculations more involved than necessary, we will assume that X is a real vector space and that $\|x_n\| = 1$ for any $n \in \mathbb{N}$. We will end the proof by mentioning how to extend the proof to the general case.

To prove that X is separable we need to find a countable, dense subset of X. I claim that the set $Y = \{\sum_{n=1}^{N} c_n x_n : c_n \in \mathbb{Q}, N \in \mathbb{N}\}$ satisfies these criteria, i.e. the set of finite linear combinations of basis elements with rational coefficients.

Let us first ask whether this set is countable. For this we will need three different facts on countability:

1. $\mathbb{Q}$ is countable
2. A finite cartesian product of countable sets is countable
3. A countable union of countable sets is countable.

If any of these facts are unfamiliar I recommend that you google them; the usual proofs are clever and interesting.

First note that we can write $Y = \cup_{N\in\mathbb{N}} Y_N$, where $Y_N = \{\sum_{n=1}^{N} c_n x_n : c_n \in \mathbb{Q}\}$. Since this is a countable union, point (3) shows that we need only show that Y_N is countable for every $N \in \mathbb{N}$ to show that Y is countable. I claim that we can find a bijection from Y_N to $\mathbb{Q}^N$, and the latter space is countable by points (1) and (2). This bijection is given by sending $\sum_{n=1}^{N} c_n x_n$ in Y_N to $(c_1, c_2, ..., c_N)$ in $\mathbb{Q}^N$. This is clearly both injective (since $(x_n)_n$ is a Schauder basis) and surjective, so Y_N is countable. As we have shown, this implies that Y is countable.

Now we need to prove that Y is dense. Pick an $x \in X$ and an $\epsilon > 0$; we need to find $y \in Y$ such that $\|x-y\|_X < \epsilon$. Since $(x_n)_n$ is a Schauder basis, there is a finite linear combination $\sum\limits_{n=1}^{N} c_n x_n$ such that $\|x - \sum\limits_{n=1}^{N} c_n x_n\|_X < \frac{\epsilon}{2}$. It is well known that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, so for every c_n we pick a $q_n \in \mathbb{Q}$ such that $|c_n - q_n| < \frac{\epsilon}{2N}$. Now let $y = \sum_{n=1}^{N} q_n x_n$. We use the triangle inequality to calculate that

$$\begin{aligned}
\|x - y\| &= \|x - \sum_{n=1}^{N} q_n x_n\| \\
&= \| \left(x - \sum_{n=1}^{N} c_n x_n \right) + \left(\sum_{n=1}^{N} c_n x_n - \sum_{n=1}^{N} q_n x_n \right) \| \\
&\leq \|x - \sum_{n=1}^{N} c_n x_n\| + \|\sum_{n=1}^{N} c_n x_n - \sum_{n=1}^{N} q_n x_n\| \\
&< \frac{\epsilon}{2} + \sum_{n=1}^{N} |c_n - q_n| \|x_n\| \\
&\leq \frac{\epsilon}{2} + \sum_{n=1}^{N} \frac{\epsilon}{2N} = \epsilon.
\end{aligned}$$

Note that we used our assumption that $\|x_n\| = 1$ during this calculation. If this were not true, one would simply need to pick q_n such that $|q_n - c_n| < \frac{\epsilon}{2N\|x_n\|}$. To sum this up we have shown that an arbitrary $x \in X$ may be approximated within a distance of any $\epsilon > 0$ by points in Y, so Y is dense.

If X is a complex vector space, one would modify the density proof slightly. We approximate x with $\sum_{n=1}^{N} c_n x_n$ as before, now with $c_n \in \mathbb{C}$. Then we need to approximate both the real and imaginary parts of c_n with rational numbers.

b) We will prove the statement by contradiction. Assume that some x in the Schauder basis can be written as finite linear combination $x = a_1 x_{n_1} + a_2 x_{n_2} + ... + a_m x_{n_m}$ where $a_i \in \mathbb{C}$ for $i = 1, 2...m$, not all a_i are zero and x is not among the x_{n_i}. This is impossible by the *uniqueness* part of the definition of a Schauder basis: any element in X can be written in a unique way as a possibly infinite linear combination of elements of the Schauder basis. If x is an element of the Schauder basis, one way of writing x in this way is simply $x = x$, and so we cannot find an expansion like $x = a_1 x_{n_1} + a_2 x_{n_2} + ... + a_m x_{n_m}$.

$\boxed{2}$ Let X be a Banach space with a Schauder basis $(x_n)_n$. Define a norm on X by $\|x\|_a = \sup_{N \in \mathbb{N}} \|\sum_{n=1}^{N} a_n x_n\|$ where (a_n) are the coefficients of x with respect to $(x_n)_n$.

Show that $\|\cdot\|_a$ is a norm on X.

Solution. We will check the axioms for being a norm one by one. In each case we will use that $\|\cdot\|$ is a norm, and some simple properties of the supremum.

1. $\|x\|_a = 0 \iff x = 0$: If $x = 0$, then the coefficients of x with respect to $(x_n)_n$ are all zero, and clearly $\|x\|_a = \sup\{0\} = 0$. Now assume that $\|x\|_a = 0$ for some

x, and let $(a_n)_n$ be the coefficients of x with respect to $(x_n)_n$. Since $0 = \|x\|_a = \sup_{N\in\mathbb{N}} \|\sum_{n=1}^N a_n x_n\|$, we must have that $\|\sum_{n=1}^N a_n x_n\| = 0$ for every $N \in \mathbb{N}$ (why?). Furthermore, $\|\cdot\|$ *is* a norm, so this implies that $\sum_{n=1}^N a_n x_n = 0$ for every $N \in \mathbb{N}$. Now, since $(x_n)_n$ is a Schauder basis, we get that $a_n = 0$ for every $n \in \mathbb{N}$ (The expansion of an element in a Schauder basis is unique, so since we have that $\sum_{n=1}^N a_n x_n = 0$, we know that $\sum_{n=1}^N a_n x_n = 0$ must be the expansion of 0 where all coefficients are zero), hence $x = 0$.

2. $\|\lambda x\|_a = |\lambda|\|x\|_a$ for $\lambda \in \mathbb{C}$: If $x = \sum_{n=1}^\infty a_n x_n$, then $\lambda x = \sum_{n=1}^\infty \lambda a_n x_n$. In other words, the coefficients of λx with respect to $(x_n)_n$ are just the coefficients of x multiplied by λ. Therefore we can use known properties of the supremum to calculate that

$$\begin{aligned}
\|\lambda x\|_a &= \sup_{N\in\mathbb{N}} \|\sum_{n=1}^N \lambda a_n x_n\| \\
&= \sup_{N\in\mathbb{N}} |\lambda| \|\sum_{n=1}^N a_n x_n\| \\
&= |\lambda| \sup_{N\in\mathbb{N}} \|\sum_{n=1}^N a_n x_n\| = |\lambda|\|x\|_a.
\end{aligned}$$

3. $\|x+y\|_a \le \|x\|_a + \|y\|_a$: If $x = \sum_{n=1}^\infty a_n x_n$ and $y = \sum_{n=1}^\infty b_n x_n$, then $x + y = \sum_{n=1}^\infty (a_n + b_n) x_n$. Therefore the coefficients of $x + y$ with respect to $(x_n)_n$ are the sum of the coefficients of x and of y: $a_n + b_n$. Using the triangle inequality for the norm $\|\cdot\|$ and a known property of the supremum we find that

$$\begin{aligned}
\|x+y\|_a &= \sup_{N\in\mathbb{N}} \|\sum_{n=1}^N (a_n + b_n) x_n\| \\
&\le \sup_{N\in\mathbb{N}} \left(\|\sum_{n=1}^N a_n x_n\| + \|\sum_{n=1}^N b_n x_n\| \right) \\
&= \sup_{N\in\mathbb{N}} \|\sum_{n=1}^N a_n x_n\| + \sup_{N\in\mathbb{N}} \|\sum_{n=1}^N b_n x_n\| \\
&= \|x\|_a + \|y\|_a.
\end{aligned}$$

$\boxed{3}$ Show that ℓ^∞ is not separable.

Solution. To show that ℓ^∞ is not separable we need to show that it does not have a dense, countable subset. Assume that X is a dense subset of ℓ^∞, we will show that it is uncountable. In order to do this we will need some facts. First recall that the set $\{0,1\}^{\mathbb{N}}$ is uncountable. This set consists of infinite sequences of zeros and ones, and one approach to prove that it is uncountable is to use that the interval $(0,1)$ is uncountable (google this if you didn't know it!). Every point in $(0,1)$ has a (possibly two) binary representation of the form $0.a_1a_2a_3...$ with $a_i \in \{0,1\}$, and therefore the cardinality of $\{0,1\}^{\mathbb{N}}$ is at least that of $(0,1)$, and hence uncountable.

We will now establish an injective map from the uncountable set $\{0,1\}^{\mathbb{N}}$ to the dense set X, and thus show that X is uncountable. Given a sequence a in $\{0,1\}^{\mathbb{N}} \subset \ell^\infty$, pick x_a to be an element in X such that $\|a - x_a\|_\infty < \frac{1}{2}$. This is possible since X is dense. We need to show that $a \mapsto x_a$ is injective, so assume that a, b are two different elements in $\{0,1\}^{\mathbb{N}}$. It is trivial to show that $\|a - b\|_\infty = 1$ in this case (think about this!), and therefore the triangle inequality gives

$$\begin{aligned} 1 = \|a - b\|_\infty &= \|a - x_a + b - x_b + x_b - x_a\|_\infty \\ &\leq \|a - x_a\|_\infty + \|b - x_b\|_\infty + \|x_a - x_b\|_\infty, \end{aligned}$$

and since $\|a - x_a\|_\infty < \frac{1}{2}$ and $\|b - x_b\|_\infty < \frac{1}{2}$, this implies that $\|x_a - x_b\|_\infty > 0$, so $x_a \neq x_b$. Therefore $a \mapsto x_a$ is injective, and X is uncountable.