TMA4295 Statistical inference
Exercise 5 - solution

Problem 5.6
$X \sim f_X(x)$ and $Y \sim f_Y(y)$ independent.

a) Use the transformation $Z = X - Y$ and $W = X$. Then $X = W$ and $Y = W - Z$ and the Jacobian $J = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1 \Rightarrow |J| = 1$. So

$$f_{Z,W}(z,w) = f_X(w)f_Y(z-w) \Rightarrow \int_{-\infty}^{+\infty} f_X(w)f_Y(z-w)dw.$$

b) Use the transformation $Z = XY$ and $W = X$. Then $X = W$ and $Y = Z/W$ and the Jacobian $J = \begin{vmatrix} 1 & 0 \\ -\frac{z}{w^2} & \frac{1}{w} \end{vmatrix} = -\frac{1}{w} \Rightarrow |J| = \left|-\frac{1}{w}\right|$. So

$$f_{Z,W}(z,w) = f_X(w)f_Y(z/w)\left|-\frac{1}{w}\right| \Rightarrow \int_{-\infty}^{+\infty} f_X(w)f_Y(z/w)\left|-\frac{1}{w}\right| dw.$$

c) Use the transformation $Z = X/Y$ and $W = X$. Then $X = W$ and $Y = W/Z$ and the Jacobian $J = \begin{vmatrix} 0 & 1 \\ -\frac{w}{z^2} & \frac{1}{z} \end{vmatrix} = \frac{w}{z^2} \Rightarrow |J| = \left|\frac{w}{z^2}\right|$. So

$$f_{Z,W}(z,w) = f_X(w)f_Y(w/z)\left|\frac{w}{z^2}\right| \Rightarrow \int_{-\infty}^{+\infty} f_X(w)f_Y(w/z)\left|\frac{w}{z^2}\right| dw.$$

Problem 5.17

a) Let $U \sim \chi^2(p)$, $V \sim \chi^2(q)$ and independent, the joint pdf is

$$f(u,v) = \frac{1}{\Gamma(p/2)\Gamma(q/2)2^{\frac{p+q}{2}}} u^{\frac{p}{2}-1} v^{\frac{q}{2}-1} e^{-\frac{u+v}{2}},$$

let $W = \frac{U/p}{V/q}$ and $Z = V$, then $U = \frac{p}{q}ZW$ and $V = Z$ and the Jacobian $J = \begin{vmatrix} \frac{p}{q}Z & \frac{p}{q}W \\ 0 & 1 \end{vmatrix} = \frac{p}{q}Z$. So

$$f(w,z) = k\left(\frac{p}{q}\right)^{\frac{p}{2}} w^{\frac{p}{2}-1} z^{\frac{p+q}{2}-1} e^{-\frac{z}{2}\left(\frac{p}{q}w+1\right)} \quad \text{with } k = \frac{1}{\Gamma(p/2)\Gamma(q/2)2^{\frac{p+q}{2}}},$$

integrating respect to z and using the substitution $y = \frac{z}{2}\left(\frac{p}{q}w + 1\right)$ we have

$$f(w) = k\left(\frac{p}{q}\right)^{\frac{p}{2}} w^{\frac{p}{2}-1} \frac{2^{\frac{p+q}{2}}}{\left(1+\frac{p}{q}w\right)^{\frac{p+q}{2}}} \Gamma\left(\frac{p+q}{2}\right) = \frac{\Gamma\left(\frac{p+q}{2}\right)\left(\frac{p}{q}\right)^{p/2} w^{\frac{p}{2}-1}}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)\left(1+\frac{p}{q}w\right)^{\frac{p+q}{2}}}.$$

b) We first consider $V \sim \chi^2(p)$, and compute

$$E\left(V^{-k}\right) = \frac{1}{\Gamma(q/2)2^{\frac{q}{2}}} \int_0^{\infty} v^{\frac{q}{2}-k-1} e^{-\frac{v}{2}} dv = \frac{\Gamma(\frac{q}{2}-k)}{\Gamma(\frac{q}{2})2^k},$$

then

$$k = 1 \Rightarrow E(V^{-1}) = \frac{1}{q-2}$$

$$k = 2 \Rightarrow E(V^{-2}) = \frac{1}{(q-2)(q-4)}.$$

Since $X = \frac{U}{p}\frac{q}{V}$ with $U \sim \chi^2(p)$, $V \sim \chi^2(q)$ independent, we have

$$E(X) = \frac{E(U)}{p} q E\left(\frac{1}{V}\right) = \frac{q}{q-2},$$

$$Var(X) = \frac{E(U^2)}{p^2} q^2 E\left(\frac{1}{V^2}\right) = \frac{2q^2(p+q-2)}{p(q-2)^2(q-4)}.$$

c) Since $X = \frac{U}{p}\frac{q}{V}$ with $U \sim \chi^2(p)$, $V \sim \chi^2(q)$ independent, then $\frac{1}{X} = \frac{V}{q}\frac{p}{U} \sim F_{q,p}$.

d) $\frac{(p/q)X}{1+(p/q)X} = \frac{1}{1+(q/p)(1/X)} = \frac{1}{1+(q/p)Y}$ with $Y \sim F_{q,p}$. Using the transformation $W = \frac{1}{1+(q/p)Y}$, then $Y = \frac{p}{q}\left(\frac{1}{W} - 1\right)$ with $dy/dw = -\frac{p}{q}\frac{1}{w^2}$. So

$$f_W(w) = \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} w^{\frac{p}{2}-1}(1-w)^{\frac{q}{2}-1}.$$

Problem 5.31

Using the Chebychev's inequality and since $\sigma_{\bar{X}} = 0.09$ we have

$$P(-3t/10 < \bar{X} - \mu < 3t/10) \geq 1 - 1/t^2.$$

So we want $1 - 1/t^2 \geq 0.9 \Rightarrow 1/t^2 \leq 0.1 \Rightarrow t \geq \sqrt{10}$ and $3t/10 = 0.948$. Then

$$P(-0.948 < \bar{X} - \mu < 0.948) \geq 0.9.$$

Using the Central limit theorem we have that $\bar{X} \sim N(\mu, \sigma^2_{\bar{X}})$ and $\frac{\bar{X}-\mu}{0.3} \sim N(0,1)$. So we know that

$$P(-1.645 < \frac{\bar{X}-\mu}{0.3} < 1.645) = 0.9$$

we have

$$P(-0.4935 < \bar{X} - \mu < 0.4935) = 0.9.$$

Problem 5.35

a) $X_i \sim exponential(1)$ with $i = 1, ..., n$.

$$E(X_i) = 1 \quad Var(X_1) = 1 \quad i = 1, ..., n$$

$$\Rightarrow E(\bar{X}_n) = 1 \quad Var(\bar{X}_n) = \frac{1}{n}$$

From the central limit theorem we have

$$\frac{\bar{X}_n - 1}{1/\sqrt{n}} \longrightarrow_{n->\infty} Z \quad \text{with } Z \sim N(0,1).$$

b) We first consider $P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right)$

$$P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) = P\left(\bar{X}_n \leq \frac{x}{\sqrt{n}} + 1\right) = F_{\bar{X}_n}\left(\frac{x}{\sqrt{n}} + 1\right) = F_{\sum X_i}\left(x\sqrt{n} + 1\right),$$

since $X_i \sim gamma(1,1)$ and so $\sum_1^n X_i \sim gamma(n,1)$, taking the derivatives we get

$$\sqrt{n} f_{\sum X_i}\left(x\sqrt{n} + n\right) = \frac{\sqrt{n}}{\Gamma(n)}\left(x\sqrt{n} + n\right)^{n-1} e^{-\left(x\sqrt{n}+n\right)} \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

If we have $x = 0$

$$\frac{\sqrt{n}}{\Gamma(n)} n^{n-1} e^{-n} \approx \frac{1}{\sqrt{2\pi}}.$$