Norwegian University of Science and Technology Institutt for matematiske fag	TMA4125 Matematikk 4N Spring 2017 **Solutions to exercise set 13**

1 **a)** Let $u_m^n \approx u(mh, nk)$ be the approximate solution. We use the finite difference approximations

$$u_{xx}(mh, nk) \approx \frac{1}{h^2}(u_{m-1}^n - 2u_m^n + u_{m+1}^n), \quad u_t(mh, nk) \approx \frac{u_m^{n+1} - u_m^n}{k}$$

Inserting the above into the equation $u_t = u_{xx}$, we obtain

$$u_m^{n+1} = \frac{k}{h^2}(u_{m-1}^n + u_{m+1}^n) + (1 - \frac{2k}{h^2})u_m^n,$$

which using the given values of h, k simplifies to

$$u_m^{n+1} = 4(u_{m-1}^n + u_{m+1}^n) - 7u_m^n$$

For the first step, we use the initial value $u(x, 0) = x(1 - x)$ to obtain the values of u_m^0, e.g. at $u_1^0 \approx u(h, 0) = u(\frac{1}{4}, 0) = \frac{3}{16}$. Writing u^n as a vector with components $(u_0^n, u_1^n, u_2^n, \ldots)$, we have

$$u^0 = (0, \frac{3}{16}, \frac{1}{4}, \frac{3}{16}, 0)$$

For the next step, we find

$$u_1^1 = 4(\frac{1}{4}) - 7(\frac{3}{16}) = -\frac{5}{16}$$

$$u_2^1 = 4(\frac{3}{16} + \frac{3}{16}) - 7(\frac{1}{4}) = -\frac{1}{4}$$

and the calculation for u_3^1 is identical to that of u_1^1. We use the boundary conditions to obtain $u_0^1 = u_4^1 = 0$, so after one timestep we have

$$u^1 = (0, -\frac{5}{16}, -\frac{1}{4}, -\frac{5}{16}, 0)$$

We now use these values to compute the next time step, e.g.

$$u_1^2 = 4(\frac{-1}{4}) - 7(\frac{-5}{16}) = \frac{19}{16},$$

$$u_2^2 = 4(-\frac{5}{16} - \frac{5}{16}) - 7(-\frac{1}{4}) = -\frac{3}{4}$$

as before, $u_1^2 = u_3^2$ (the calculation is the same), and $u_0^2 = u_4^2 = 0$. Combining this, we have

$$u^2 = (0, \frac{19}{16}, -\frac{3}{4}, \frac{19}{16}, 0)$$

b) The Crank-Nicolson scheme discretizes first in space using the usual formula, and then in time using the trapezium rule. Precisely, we have for $u_m(t) \approx u(mh,t)$,

$$\frac{\partial}{\partial t}u_m = \frac{1}{h^2}(u_{m-1} - 2u_m + u_{m+1}) = f(u)$$

Applying the trapezium rule in time,

$$u^{n+1} = u^n + \frac{k}{2}\big(f(u^n) + f(u^{n+1})\big)$$

we obtain the following

$$u_m^{n+1} = u_m^n + \frac{k}{2h^2}\big((u_{m-1}^n - 2u_m^n + u_{m+1}^n) + (u_{m-1}^{n+1} - 2u_m^{n+1} + 2u_{m-1}^{n+1})\big)$$

Inserting in the values of k and h, we rearrange the above to

$$5u_m^{n+1} - 2u_{m-1}^{n+1} - 2u_{m+1}^{n+1} = 2u_{m-1}^n - 3u_m^n + 2u_{m+1}^n$$

Starting from $n = 0$ and inserting the known values on the right hand side, we obtain the following equations

$$5u_1^1 - 2u_2^1 = -3(\frac{3}{16}) + 2(-\frac{1}{4}) = \frac{-1}{16}$$

$$-2u_1^1 + 5u_2^1 - 2u_3^1 = 2(\frac{3}{16} + \frac{3}{16}) - 3(\frac{1}{4}) = 0$$

$$-2u_2^1 + 5u_3^1 = -3(\frac{3}{16}) + 2(-\frac{1}{4}) = \frac{-1}{16}$$

We write this as a matrix equation

$$\begin{pmatrix} 5 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{16} \\ 0 \\ -\frac{1}{16} \end{pmatrix}$$

the solution of which gives (we omit the steps here) in combination with the boundary values $u_0^1 = u_4^1 = 0$,

$$u^1 = (0, -\frac{5}{272}, -\frac{1}{68}, -\frac{5}{272}, 0)$$

Repeating the procedure, we find the equation for the next step

$$\begin{pmatrix} 5 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{pmatrix} = \begin{pmatrix} \frac{7}{272} \\ \frac{-1}{34} \\ \frac{7}{272} \end{pmatrix}$$

which may be solved to get

$$u^2 = (0, \frac{19}{4624}, \frac{-3}{1156}, \frac{19}{4624}, 0)$$

c) In general, we expect the second solution to be more realistic because the parameter $\frac{k}{h^2} = 4 > 0.5$ is such that the first solution is unstable. The results obtained reflect this instability - in the first case the solution is growing in magnitude at each time step, whereas with Crank-Nicolson the solution is decaying towards zero as we expect (recall the behaviour of the heat equation)

$\boxed{2}$ **a)** We set up the discretization as before but include an additional term coming from the approximation

$$u_x(mh, nk) \approx \frac{1}{2h}(u_{m+1}^n - u_{m-1}^n)$$

We accordingly use

$$u_m^{n+1} = u_m^n + \frac{k}{h^2}(u_{m-1}^n - 2u_m^n + u_{m+1}^n) + \frac{k}{2h}(u_{m+1}^n - u_{m-1}^n)$$

and inserting the values of k and h gives

$$u_m^{n+1} = \frac{7}{8}u_{m-1}^n - u_m^n + \frac{9}{8}u_{m+1}^n$$

From the initial conditions, we have $u^0 = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$. We find

$$u_1^1 = -\frac{1}{4} + \frac{9}{8}(\frac{1}{2}) = \frac{5}{16}$$

$$u_2^1 = \frac{7}{8}(\frac{1}{4}) - \frac{1}{2} + \frac{9}{8}(\frac{3}{4}) = \frac{9}{16}$$

$$u_3^1 = \frac{7}{8}(\frac{1}{2}) - \frac{3}{4} + \frac{9}{8}(1) = \frac{13}{16}$$

Combining these with the boundary conditions $u_0^1 = 0$ and $u_4^1 = 1$, we have

$$u^1 = (0, \frac{5}{16}, \frac{9}{16}, \frac{13}{16}, 1)$$

Repeating the procedure, we find

$$u^2 = (0, \frac{41}{128}, \frac{5}{8}, \frac{103}{128}, 1)$$

b) In the same manner, we modify the Crank-Nicolson scheme to include the term

$$u_x(mh, nk) \approx \frac{1}{2h}(u_{m+1}^n - u_{m-1}^n)$$

Our spatially discretized equation becomes

$$\frac{\partial}{\partial t}u_m = \frac{1}{h^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{1}{2h}(u_{m+1} - u_{m-1}) = f(u)$$

Applying the trapezium rule gives

$$u_m^{n+1} = u_m^n + \frac{k}{2h^2}\big((u_{m-1}^n - 2u_m^n + u_{m+1}^n) + (u_{m-1}^{n+1} - 2u_m^{n+1} + 2u_{m-1}^{n+1})\big)$$
$$+\frac{k}{4h}(u_{m+1}^n - u_{m-1}^n + u_{m+1}^{n+1} - u_{m-1}^{n+1})$$

Setting in the values of k and h and collecting all the terms in u^{n+1} on the left, we have

$$5u_m^{n+1} - \frac{7}{4}u_{m-1} - \frac{9}{4}u_{m+1}^{n+1} = \frac{7}{4}u_{m-1}^n + \frac{9}{4}u_{m+1}^n - 3u_m^n$$

For the first time step ($n = 0$), we insert the values of u^0 from the initial condition on the left hand side, and the values $u_0^1 = 0$ and $u_4^1 = 1$ from the boundary conditions on the left. This gives the equation

$$\begin{pmatrix} 5 & -\frac{9}{4} & 0 \\ -\frac{7}{4} & 5 & -\frac{9}{4} \\ 0 & -\frac{7}{4} & 5 \end{pmatrix} \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{1}{4} \\ \frac{25}{8} \end{pmatrix}$$