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Solutions to exercise set 12

$\boxed{1}$ We will find an approximate solution to the differential equation

$$y'' - y' + \sin y = 0, \quad y(0) = 0,\ y'(0) = 1$$

a) Let $z = y'$. Then the differential equation becomes

$$z' = z - \sin y = 0$$

Combining this with the equation $y' = z$ gives a first order system.

b) The initial conditions give $y_0 = 0, z_0 = 1$. The first order system takes the form

$$y' = f_1(y, z) = z, \quad z' = f_2(y, z) = z - \sin y$$

At the first time step we compute the predictors:

$$y_1^* = y_0 + hf_1(y_0, z_0) = 0 + hz_0 = 0.2 \times 1 = 0.2$$

$$z_1^* = z_0 + hf_2(y_0, z_0) = 1 + h(z_0 - \sin y_0) = 1.2$$

We can now finish the first time step:

$$y_1 = y_0 + \frac{h}{2}\big(f_1(y_0, z_0) + f_1(y_1^*, z_1^*)\big) = 0 + 0.1(1 + 1.2) = 0.22$$

$$z_1 = z_0 + \frac{h}{2}\big(f_2(y_0, z_0) + f_2(y_1^*, z_1^*)\big) = 1 + 0.1(1 + 1.2 - \sin 0 - \sin 0.2) = 1.2001$$

We repeat the procedure starting from the new values $y_1 = 0.22, z_1 = 1.2001$:

$$y_2^* = y_1 + hz_1 = 0.22 + 0.2(1.2001) = 0.46003$$

$$z_2^* = z_1 + h(z_1 - \sin(y_1)) = 1.2001 + 0.2(1.2001 - \sin(0.22)) = 1.3965$$

We then conclude

$$y_2 = y_1 + \frac{h}{2}(z_1 + z_2^*) = 0.22 + 0.2(1.2001 + 1.3965) = 0.47966$$

$$\begin{aligned} z_2 &= z_1 + \frac{h}{2}(z_1 - \sin y_1 + z_2^* - \sin y_2^*) \\ &= 1.2001 + 0.1(1.2001 - \sin 0.22 + 1.3965 - \sin 0.46003) = 1.3936 \end{aligned}$$

In particular, we find the approximation $y(0.4) \approx y_2 = 0.47966$ (it is y_2 we require, as we have progressed two timesteps of size 0.2 from $t = 0$)

$\boxed{2}$ Consider the third order differential equation

$$y'''(x) = xy(x), \quad y(1) = -1,\ y'(1) = 2,\ y''(1) = 1$$

a) Here we require two new variables, one for y' and one for y''. We rename y as y_1 and then write $y' = y_1' = y_2$, and $y'' = y_2' = y_3$. The original differential equation is then $y_3' = xy_1$, i.e. we have the system

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2, y_3) = y_2 \\ y_2' &= f_2(x, y_1, y_2, y_3) = y_3 \\ y_3' &= f_3(x, y_1, y_2, y_3) = xy_1 \end{aligned}$$

b) We write $y_{n,1}$ for the value of y_1 at the nth timestep, etc. The backward Euler scheme is then

$$y_{n+1,i} = y_{n,i} + hf_i(x_{n+1}, y_{n+1,1}, y_{n+1,2}, y_{n+1,3}), \quad i = 1, 2, 3$$

We first note that as the initial conditions are given at $x = 1$ (i.e. we have $y(1) = \dots, y'(1) = \dots$ etc), we have $x_0 = 1$. Now $h = 1$, so we have $x_{n+1} = x_n + 1$, and hence $x_n = 1 + n$. The initial conditions are $y_{0,1} = -1, y_{0,2} = 2, y_{0,3} = 1$. Combining this, we compute the first time step:

$$\begin{aligned} y_{1,1} &= -1 + y_{1,2} \\ y_{1,2} &= 2 + y_{1,3} \\ y_{1,3} &= 1 + x_1 y_{1,1} = 1 + 2y_{1,1} \end{aligned}$$

This may be written in the following matrix equation form:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

By performing Gauss elimination on the resulting system it may be shown that $y_{1,1} = -2, y_{1,2} = -1, y_{1,3} = -3$. Note that $y_{1,1}$ approximates $y(x)$ at $x = x_1 = 2$.

$\boxed{3}$ Suppose $u(x, y)$ satisfies the Poisson equation

$$u_{xx} + u_{yy} = 9(x + y)$$

on the interior of the square $\mathcal{R}$ with corners at $(0, 0), (0, 1), (0, 1)$ and $(1, 1)$. Moreover, u is given on the boundary of $\mathcal{R}$ by

$$u(x, y) = 9x(1 - x)$$

a) Constructing the rectangular grid, we find points $u_{1,1} \approx u(\frac{1}{3}, \frac{1}{3})$, $u_{1,2} \approx u(\frac{1}{3}, \frac{2}{3})$, $u_{2,1} \approx u(\frac{2}{3}, \frac{1}{3})$ and $u_{2,2} \approx u(\frac{2}{3}, \frac{2}{3})$. The difference equations resulting from the 5-point molecule are

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(x_i, y_j) = \frac{1}{3}(i + j),$$

where for the last we have used that $h^2 = \frac{1}{9}$, and that $x_i = \frac{i}{3}, y_j = \frac{j}{3}$. We now compute the values of u on the boundary, ie at $u_{0,i}, u_{3,i}, u_{j,0}, u_{j,3}$. For this the boundary conditions give

$$u(0,i) = u(3,i) = 0, \quad u(1,0) = u(1,3) = u(2,0) = u(2,3) = 9(\frac{1}{3})(\frac{2}{3}) = 2,$$

The four equations are then

$$\begin{aligned} u_{2,1} + u_{1,2} + u_{0,1} + u_{1,0} - 4u_{1,1} &= \frac{2}{3} \\ u_{2,2} + u_{1,3} + u_{0,2} + u_{1,1} - 4u_{1,2} &= 1 \\ u_{3,1} + u_{2,2} + u_{1,1} + u_{2,0} - 4u_{2,1} &= 1 \\ u_{3,2} + u_{2,3} + u_{1,2} + u_{2,1} - 4u_{2,2} &= \frac{4}{3} \end{aligned}$$

Inserting the values of the terms on the boundary and rewriting as a matrix equation gives

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{-4}{3} \\ -1 \\ -1 \\ \frac{-2}{3} \end{pmatrix}$$

b) (Optional) The values will vary depending on the initial vector chosen. For instance, starting from $u_{i,j} = 0$ for all i gives

$$u_{1,1} = 0.5, \quad u_{1,2} = u_{2,1} = 0.4583, \quad u_{2,2} = 0.3958$$

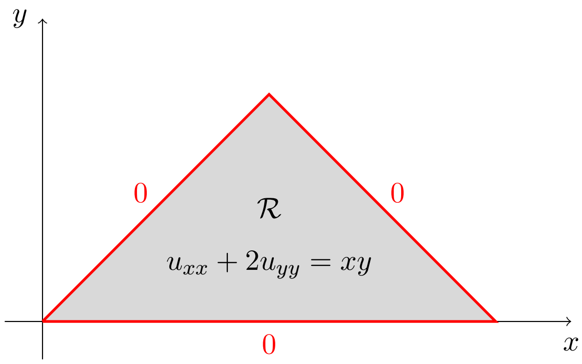
$\boxed{4}$ The steady state heat distribution in a non-uniform triangular plate is modeled by the following PDE

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} = xy, \quad (x,y) \in \mathcal{R}$$

where $\mathcal{R}$ is the triangle defined by the inequalities

$$y \geq 0, \quad y \leq x, \quad y \leq 6 - x.$$

We have in addition $u(x,y) = 0$ on the boundary of $\mathcal{R}$. The picture is as follows:



a) We use the approximations

$$u_{xx}(x,y) \approx \frac{u(x+h,y) + u(x-h,y) - 2u(x,y)}{h^2}$$

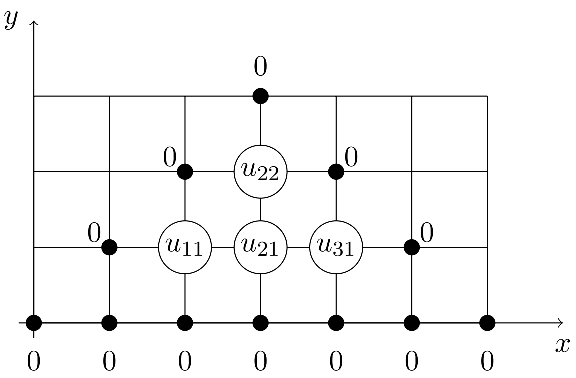
$$u_{yy}(x,y) \approx \frac{u(x,y+h)+u(x,y-h)-2u(x,y)}{h^2}$$

Multiplying the second by two and adding the first gives the desired approximation. (This calculation can be visualized as adding to 3-point molecules, one with a horizontal alignment and one with a vertical alignment)

b) In this case, as $h=1$, we have $u(x_i,y_j)\approx u_{i,j}$, and $f(x_i,x_j)=(i+1)j$, hence the above approximation gives

$$u_{i+1,j}+2u_{i,j+1}+u_{i-1,j}+2u_{i,j-1}-6u_{i,j}=(i+1)j$$

We now apply the approximation to the grid:



The equations are

$$\begin{aligned} u_{2,1}-6u_{1,1}&=2\\ u_{1,1}+2u_{2,2}+u_{3,1}-6u_{2,1}&=3\\ u_{2,1}-6u_{3,1}&=4\\ 2u_{2,1}-6u_{2,2}&=6 \end{aligned}$$

In matrix form, this is

$$\begin{pmatrix} -6 & 1 & 0 & 0\\ 1 & -6 & 1 & 2\\ 0 & 1 & -6 & 0\\ 0 & 2 & 0 & -6 \end{pmatrix} \begin{pmatrix} u_{1,1}\\ u_{2,1}\\ u_{3,1}\\ u_{2,2} \end{pmatrix} = \begin{pmatrix} 2\\ 3\\ 4\\ 6 \end{pmatrix}$$

c) (Optional) Again, the result in general depends on the initial guess. Taking all the values to be zero gives

$$u_{1,1}=-0.4167,\quad u_{1,2}=-1,\quad u_{1,3}=-0.75,\quad u_{2,2}=-1.1667$$

$\boxed{5}$ (Optional) We note that

$$u(x\pm h,y)=u(x,y)\pm hu_x(x,y)+\frac{h^2}{2}u_{xx}(x,y)\pm\frac{h^3}{6}u_{xxx}(x,y)+\dots$$

$$u(x\pm 2h,y)=u(x,y)\pm 2hu_x(x,y)+2h^2u_{xx}(x,y)\pm\frac{4h^3}{3}u_{xxx}(x,y)+\dots$$

If we add together the combination $u(x+2h,y)-2u(x+h,y)+2u(x-h,y)-u(x-2h,y)$ given in the exercise, we see that the coefficients of the $u(x \pm h, y)$ terms are equal and opposite, and hence the terms u, $\frac{h^2}{2}u_x x$ (and all higher order terms with even powers of h) cancel. A similar situation occurs for $u(x \pm 2h, y)$. We then consider the coefficient of u_x in the big sum; it is

$$2h - 2h - 2h + 2h = 0$$

It remains to check the coefficient of u_{xxx}, this is

$$\frac{4h^3}{3} - 2\frac{h^3}{6} - 2\frac{h^3}{6} + \frac{4h^3}{3} = \frac{h^3}{3}(4 - 1 - 1 + 4) = 2h^3$$