TMA4165 Differential Equations and Dynamical Systems Spring 2017

Exercise set 10

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
Ex 1992.2, Ex 1993.1, Ex 1995.5, Ex 2013.5

These exercises will be presented / discussed in the exercise class:
E19, E20, E21

Exam 1992, 2 Given the system

$$\begin{aligned}\dot{x} &= -x - xy^2 - x^3\\ \dot{y} &= -7y + 3x^2y - 2yz^2 - y^3\\ \dot{z} &= -5z + y^2z - z^3.\end{aligned}$$

a) Show that the origin is an asymptotically stable equilibrium point.

b) Show that every solution $(x(t), y(t), z(t))$ tends to the origin as $t \to \infty$.

Exam 1993, 1 Given the system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}. \tag{1}$$

Find, with the help of the matrix exponential, the solution which satisfies $x(0) = y(0) = z(0) = 1$.

Exam 1995, 5 Given the linear system

$$\begin{aligned}\dot{x} &= 3x - 2y\\ \dot{y} &= 2x + 3y.\end{aligned}$$

a) Consider the solution $(x(t), y(t))$ which satisfies $(x(0), y(0)) = (x_0, 0)$ where $x_0 > 0$. Denote by $t_1 > 0$ the first time the solution crosses the positive x-axis. Show that $(x(t_1), y(t_1)) = (x_0e^{3\pi}, 0)$.

b) Classify the equilibrium point $(0, 0)$ of the given system.
Classify the equilibrium point $(0, 0)$ of the following system

$$\dot{x} = 3x - 2y + (x^2 + y^2)^2,$$
$$\dot{y} = 2x + 3y + x^3 + y^4.$$

$\boxed{\text{E19}}$ Aim: Prove the following generalization of the Gronwall inequality presented in the lecture. If, for $t \geq 0$

- $u(t)$ and $g(t)$ are continuous and $g(t) \geq 0$ and $u(t) \geq 0$,
- $f(t)$ is continuous, non-decreasing and $f(t) > 0$
- $$u(t) \leq f(t) + \int_0^t g(s)u(s)ds, \tag{2}$$

then
$$u(t) \leq f(t) \exp\left(\int_0^t g(s)ds\right). \tag{3}$$

Background: We proved in the lecture the following lemma: If, for $t \geq 0$

- $w(t)$ and $v(t)$ are continuous and $w(t) \geq 0$ and $v(t) \geq 0$,
- $$w(t) \leq K + \int_0^t w(s)v(s)ds, \quad K > 0, \tag{4}$$

then
$$w(t) \leq K \exp\left(\int_0^t v(s)ds\right). \tag{5}$$

a) Rewrite equation (2) in such a way that it is of the form (4).

b) Use the Gronwall inequality stated in the background to prove (3).

$\boxed{\text{E20}}$ Aim: Use the series definition of e^{At} to prove some properties of the exponential function of a matrix A.

a) Show that $e^{A+B} = e^A e^B$ if $AB{=}BA$.

b) Find 2×2 matrices A and B such that $e^{A+B} \neq e^A e^B$.

c) Show that e^A is nonsingular and $(e^A)^{-1} = e^{-A}$.

d) Show that $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$.

e) Show that $(e^{At})^T = e^{A^T t}$.

$\boxed{\text{E21}}$ Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $x : \mathbb{R} \mapsto \mathbb{R}^n$ be a solution of

$$\dot{x}(t) = Ax(t) + b.$$

a) Define what it means for $x(t)$ to be (Liapunov) stable.

b) Show that all solutions of the above equation are (Liapunov) stable if there is a fundamental matrix $\Phi : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ for $\dot{x} = Ax$ such that

$$\|\Phi(t)\| \leq C < \infty \quad \text{for all} \quad t \geq 0.$$