TMA4295 Statistical inference
Exercise 3 - solution

4.1
Since the distribution is uniform, the probability corresponds to the ratio of the area representing the events and the total area.

a) It is a circle with the center at $(0,0)$ and radius 1, i.e. $\mathrm{P}\left(X^2+Y^2<1\right)=\frac{\pi}{4}$

b) The area equals to the half of the square, i.e. $P(2X-Y>0)=0.5$

c) $P(|X+Y|<2)=1$

4.4

a) C is the normalization constant, i.e. $\int_0^1\int_0^2 f(x,y)dxdy=1$, which gives $C=\frac{1}{4}$

b)

$$f(x)=\begin{cases}\int_0^1 f(x,y)dy=\frac{x+1}{4} & 0<x<2\\ 0 & \text{otherwise}\end{cases}$$

c)

$$F_{X,Y}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)dxdy=\begin{cases}0 & x\le 0 \text{ or } y\le 0\\ \frac{x^2y}{8}+\frac{y^2x}{4} & 0<x<2 \text{ and } 0<y<1\\ \frac{y}{2}+\frac{y^2}{2} & 2\le x \text{ and } 0<y<1\\ \frac{x^2}{8}+\frac{x}{4} & 0<x<2 \text{ and } 1\le y\\ 1 & 2\le x \text{ and } 1\le y\end{cases}$$

d) Theorem 2.1.5 gives $f_Z(z)=\frac{9}{8z^2}$, $1<z<9$.

4.10

a) $\mathrm{P}(X=x)\,\mathrm{P}(Y=y)\neq\mathrm{P}(X=x,Y=y)$

b)

		U		
		1	2	3
	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
V	3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	4	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

4.34

a) Use the definition of the beta function, $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$, and the relation between the beta and gamma function $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

b) Using the same approach as in a) gives

$$\mathrm{P}(X=x)=\binom{r+x+1}{x}\frac{\Gamma(\alpha+\beta)\Gamma(r+\alpha)\Gamma(x+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(r+x+\alpha+\beta)},\quad x=0,1,2,\ldots$$

The mean and variance can be computed with use of the theorems 4.4.3 and 4.4.7, which gives

$$\mathrm{E}(X)=\frac{r\beta}{\alpha-1}$$
$$\mathrm{Var}(X)=\frac{r\beta(\alpha+\beta-1)}{(\alpha-1)(\alpha-2)}+\frac{r^2\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)}$$

In the computation of the mean and variance it is necessary to compute $\mathrm{E}\left(\frac{1-P}{P}\right)$, $\mathrm{E}\left(\frac{1-P}{P^2}\right)$ and $\mathrm{Var}\left(\frac{1-P}{P}\right)$, which can be done with the use of the definition of the beta function and the relation between the beta and gamma function listed in a).

4.56

a)

$$\begin{aligned}\mathrm{P}\left(\text{test is positive for group of } k \text{ people}\right) &= 1 - \mathrm{P}\left(0 \text{ positive tests in group of } k \text{ people}\right)\\ &= 1 - (1-p)^k\end{aligned}$$

b) It is necessary to perform $m + ik$ tests, where $i = 1, 2, \ldots, m$ denotes the number of positive groups. Groups are independent and each group is positive with the probability computed in a) which leads to the binomial distribution. The expected number of blood tests necessary under the plan (ii) is then equal to $m + km(1 - (1-p)^k)$.

c) The limit of the expected number of blood tests necessary under the plan (ii) for $p \to 0$ is $m < N$, therefore the plan (ii) is preferred.

5.11 Use Jensen's inequality.

5.29 The weight of ith booklet represents a random variable with mean $\mathrm{E}\left(X_i\right) = 1$ and variance $\mathrm{Var}(X_i) = 0.05^2$ and the booklets are independent. The CLT gives

$$\mathrm{P}\left(\sum_{i=1}^{100} X_i > 100.4\right) = \mathrm{P}\left(\bar{X} > 1.004\right) \approx \mathrm{P}\left(Z > 0.8\right) = 0.2119$$