# HOMEWORK 11
## ERGODICITY AND MIXING

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system.

**Problem 1.** Prove that if $f$ is an observable such that for every $t \in \mathbb{R}$ we have

$$\text{either } \; f > t \text{ a.e.} \quad \text{or} \quad f \leq t \text{ a.e.}$$

then $f$ is constant almost everywhere.

**Problem 2.** Prove that for every observable $f$ (which is integrable or non-negative) we have

$$\int_X f \circ T \, d\mu = \int_X f \, d\mu.$$

Deduce from here that if $f \in L^2(d\mu)$ then $\|f \circ T\|_{L^2} = \|f\|_{L^2}$, and more generally, that

$$\|f \circ T^n\|_{L^2} = \|f\|_{L^2} \;\text{ for all }\; n \geq 1.$$

**Problem 3.** Prove that if $f_k \to f$ in $L^2(\mu)$ as $k \to \infty$, then for all $n \geq 1$,

$$f_k \circ T^n \to f \circ T^n \;\text{ in } L^2(\mu) \;\text{ as }\; k \to \infty.$$

**Problem 4.** Use Cauchy-Schwarz to prove that if $f_k \to f$ and $g_k \to g$ in $L^2(\mu)$, then

$$f_k \cdot g_k \to f \cdot g \;\text{ in } L^1(\mu) \;\text{ as }\; k \to \infty.$$

**Problem 5.** Prove that if $f \in L^2(\mu)$ then there is a sequence $\{f_k\}_{k \geq 1}$ of *simple* functions such that

$$f_k \to f \;\text{ in } L^2(\mu) \;\text{ as }\; k \to \infty.$$

**Problem 6.** Prove that if $(X, \mathcal{B}, \mu, T)$ is mixing then for all $f, g \in L^2(\mu)$ we have

$$\int_X f \cdot g \circ T^n \, d\mu \to \int_X f d\mu \int_X g d\mu \;\text{ as }\; n \to \infty.$$

*Hint:* Prove this first for the case when $f = \mathbf{1}_E$ and $g = \mathbf{1}_F$, then for $f, g$ simple functions, then use Problem 5 to approximate by simple functions. You will also need to use Problems 3 and 4.

**Problem 7.** Let $(S, \mathcal{F}, \nu)$ be a probability space and let $\mathcal{X} := S^{\mathbb{N}}$.
Recall the definition of *cylinder sets*: given any $n \in \mathbb{N}$ and given any $\mathcal{F}$-measurable sets $A_0, A_1, \ldots A_{n-1} \subset S$, the corresponding cylinder is the set

$$C[A_0, A_1, \ldots, A_{n-1}] := \{\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}, x_n, \ldots) : x_0 \in A_0, x_1 \in A_1, \ldots, x_{n-1} \in A_{n-1}\}.$$

Let $\mathcal{B}_0$ be the collection of all *finite* unions of cylinder sets. Show that $\mathcal{B}_0$ is a Boolean algebra.