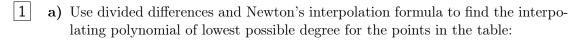


## TMA4215

## Numerical Mathematics Autumn 2017

Norwegian University of Science and Technology Department of Mathematical Sciences

Exercise set 5



- **b)** Add the point (1,0) to the table in **a)**. What is the new interpolating polynomial?
- 2 In this task, you are to approximate  $\sin(x)$  on the interval  $[0, \pi]$  using polynomial interpolation.
  - a) Choose 4 equidistant nodes on the interval. Find the polynomial and an upper bound for the error  $|\sin(x) p_3(x)|$ .
  - b) Repeat a), but use Chebyshev nodes instead (see hint). (You do not have to calculate the polynomial, but show the interpolation points.)
  - c) In the two cases (equidistant and Chebyshev nodes), find an expression for an upper error bound, expressed by n (the polynomial degree). Plot the error bound as a function of n and compare the two cases.

**Hint:** The Chebyshev nodes are defined on the interval [-1, 1]. To move them over to another interval [a, b], the change of variables

$$x = \frac{a+b}{2} + \frac{b-a}{2}t, \qquad t \in [-1,1]$$

is used. The error formula is adjusted in the same way.

Write two Python functions, one that calculates the table of divided differences based on a given dataset, and one that calculates the value of the interpolating polynomial in given points, based on this table.

Iterated multiplication may be useful in this task. Let

$$p(x) = f_0 + f_1(x - x_0) + f_2(x - x_0)(x - x_1) + \dots + f_n(x - x_0)(x - x_0) + \dots + f_n(x - x_0)(x - x_0)(x - x_0) + \dots + f_n(x - x_0)(x - x_0)(x - x_0) + \dots + f_n(x - x_0)(x - x_0)(x - x_0)(x - x_0) + \dots + f_n(x - x_0)(x - x_0)(x - x_0)(x - x_0) + \dots + f_n(x - x_0)(x - x_0)$$

and define the polynomials  $b_n, b_{n-1}, \ldots, b_0$  by

$$b_n(x) = f_n$$

$$b_{n-1}(x) = f_{n-1} + (x - x_{n-1})b_n(x)$$

$$\vdots$$

$$b_0(x) = f_0 + (x - x_0)b_1(x)$$

Then  $b_0(x) = p(x)$ . This equality is easily seen by substituting for  $b_1(x), b_2(x), \ldots, b_n(x)$  in the expression for  $b_0$  and expand the expression. Test the functions on the dataset in Task 1.

4 Let the distance h between nodes be such that  $x_i = a + ih$ , i = 0, 1, 2, ... Let  $f_i = f(x_i), i = 0, 1, 2, ...$ 

On such a sequence  $\{f_i\}_{i=0}^{\infty}$ , we can define a forward difference recursively by

$$\Delta^0 f_0 = f_0, \quad \Delta f_0 = f_1 - f_0, \qquad \Delta^k f_0 = \Delta(\Delta^{k-1} f_0) = \Delta^{k-1} f_1 - \Delta^{k-1} f_0, \qquad k = 1, 2, \dots$$

We get

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0,$$
  $\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0,$  and so on.

Let  $x = x_0 + sh$ , where  $s \in \mathbb{R}$ . The task is about showing that the polynomial interpolating f in the nodes  $x_i$ , i = 0, 1, ..., n can be written

$$p_n(x) = p_n(x_0 + sh) = f_0 + \sum_{k=1}^n \binom{s}{k} \Delta^k f_0$$
 (1)

where

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}.$$

a) Show by induction:

$$f[x_0, x_1, \dots, x_k] = \frac{1}{k!h^k} \Delta^k f_0.$$

**b)** Show that

$$\prod_{i=0}^{k-1} (x - x_i) = k! h^k \binom{s}{k}, \qquad k \ge 1.$$

- c) Use the results from a) and b) to prove Newton's forward difference formula (1).
- d) Apply the formula to the dataset in Task 1b).

**Comment:** Equivalently, it is possible to show Newton's backward difference formula. Backward differences on the sequence  $\{f_n\}_{n=0}^{\infty}$  are defined by

$$\nabla^0 f_n = f_n, \qquad \nabla f_n = f_n - f_{n-1}, \qquad \nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}, \qquad k = 1, 2, \dots$$

Newton's backward difference formula is given by

$$p_n(x) = p_n(x_n + sh) = f_n + \sum_{k=1}^n (-1)^k {-s \choose k} \nabla^n f_n.$$

- 5 Exercise 7.3, chapter 7 from S&M book
- 6 Exercise 7.6, chapter 7 from S&M book

## Note: S&M,

Süli, Endre, and David F. Mayers. An introduction to numerical analysis. Cambridge university press, 2003.