Supplement to "Averages of Unlabeled Networks: Geometric Characterization and Asymptotic Behavior", by Kolaczyk, Lin, Rosenberg, Walters, and Xu

Supplement B

Proof of Proposition 4.2

The arguments in Lemma 4.1 show that

$$F^o = \{\vec{w}' \in \mathcal{O}_D : d_E(\vec{w}, \vec{w}') < d_E(\vec{w}, \sigma \cdot \vec{w}'),\ \forall \sigma \in \Sigma_d, \sigma \neq \mathrm{Id}\}.$$

We must show that if $\vec{w}'$ is distinct, then there is a unique permutation σ_0 with $\sigma_0 \cdot \vec{w}' \in F$. After relabeling $\sigma_0 \cdot \vec{w}'$ as $\vec{w}'$, we must show

$$\sum_i (w_i - w_i')^2 < \sum_i (w_i - \sigma \cdot w_i')^2 \tag{0.1}$$

for permutation $\sigma \in \Sigma_d$, $\sigma \neq \mathrm{Id}$.

We do this in a number of steps.

Case I: We treat the case $d = 2$. $\vec{w} = (a, b), \vec{w}' = (c, d)$. Let θ be the counterclockwise angle from $\vec{w}$ to $\vec{w}'$. Let ψ be the counterclockwise angle from $\vec{w}$ to $(12) \cdot \vec{w}' = (d, c)$. Then (0.1) fails iff

$$\begin{aligned}
&(a-c)^2 + (b-d)^2 = (a-d)^2 + (b-c)^2 \\
&\quad \Leftrightarrow \quad ac + bd = ad + bc \Leftrightarrow \vec{w} \cdot \vec{w}' = \vec{w} \cdot [(12) \cdot \vec{w}'] \\
&\quad \Leftrightarrow \quad |\vec{w}||\vec{w}'| \cos\theta = |\vec{w}|\ |(12) \cdot \vec{w}'| \cos\psi \Leftrightarrow \theta = \pm\psi.
\end{aligned}$$

This is clearly impossible, as $c \neq d$ rules out $\theta = \psi$ and $a \neq b$ rules out $\theta = -\psi$.

Case II: We now do an induction proof for permutations $\sigma = (a_1 a_2) \ldots (a_{2k-1} a_{2k})$ with all a_i distinct. Assume for ease of notation that $\sigma = (12) \ldots (2k-1\ 2k)$. For $k = 1$, we must show that

$$\sum_{i=1}^N (w_i - w_i')^2 < (w_1 - w_2')^2 + (w_2 - w_1')^2 + \sum_{i=3}^N (w_i - w_i')^2.$$

This reduces to Case I. For the induction step, we assume (0.1) holds for $\sigma' = (12) \ldots (2k-3\ 2k-2)$. Thus

$$\sum_i (w_i - w_i')^2 < \sum_i (w_i - \sigma' \cdot w_i')^2 < \sum_i (w_i - \sigma \cdot w_i')^2.$$

The last inequality follows as above, since the last two sums differ only by a transposition of indices, so we can again use Case I.

This handles all permutations whose cycle decomposition has cycles of weight at most two and with no repeated indices.

Case III: For all other permutations, we will do induction on $e(\sigma) = \#\{i : \sigma(i) \neq i.\}$. The case $e(\sigma) = 0$ is trivial, $e(\sigma) = 1$ is not possible, and we have treated the case $e(\sigma) = 2$. For the induction step, we first pick a cycle $\tau = (a_1 \dots a_k)$ with $k \geq 3$ inside σ; such a cycle must exist. Since $(a_1 \dots a_k) = (a_2 \dots a_k a_1)$ etc., we may assume that

$$(w_{a_1} - w'_{a_2})^2 + (w_{a_k} - w_{a_1})^2 \geq (w_{a_1} - w'_{a_1})^2 + (w_{a_k} - w_{a_2})^2.$$

(If this fails for all $a_{1+i}, a_{2+i}, a_{k+i}$ mod k, then we get a chain of inequalities that end up with $(w_{a_1} - w'_{a_1})^2 + (w_{a_k} - w_{a_2})^2 < (w_{a_1} - w'_{a_1})^2 + (w_{a_k} - w_{a_2})^2$.) For $\tau' = \tau(a_1 a_2) = (a_2 \dots a_k) \dots$, we have $e(\tau') = e(\tau) - 1$. Then

$$\begin{aligned}
&\sum_i (w_i - \tau \cdot w'_i)^2 - \sum_i (w_i - \tau' \cdot w'_i)^2 \\
&= [(w_{a_1} - w'_{a_2})^2 + \dots + (w_{a_{k-1}} - w'_{a_k})^2 + (w_{a_k} - w_{a_1})^2] \\
&\quad -[(w_{a_1} - w'_{a_1})^2 + (w_{a_2} - w'_{a_3})^2 + \dots + (w_{a_{k-1}} - w'_{a_k})^2 + (w_{a_k} - w_{a_2})^2] \\
&= [(w_{a_1} - w'_{a_2})^2 + (w_{a_k} - w_{a_1})^2] - [(w_{a_1} - w'_{a_1})^2 + (w_{a_k} - w_{a_2})^2] \\
&\geq 0.
\end{aligned}$$

Thus $\sum_i (w_i - \tau' \cdot w'_i)^2 \leq \sum_i (w_i - \tau \cdot w'_i)^2$, and

$$\sum_i (w_i - w'_i)^2 < \sum_i (w_i - \tau' \cdot w'_i)^2 \leq \sum_i (w_i - \tau \cdot w'_i)^2,$$

where the first inequality is the induction hypothesis. For a general permutation σ, we write σ as a product of cycles $\tau_1, \dots, \tau_r$ as above, and apply the argument above to each τ_i.