Supplement to "Averages of Unlabeled Networks: Geometric Characterization and Asymptotic Behavior", by Kolaczyk, Lin, Rosenberg, Walters, and Xu

Supplement A

In this Supplement we work out the space of unlabeled networks $\mathcal{U}_3$ with three vertices in detail (Example 2.2), sketch the proof of Theorem 2.3, and prove results about the topology of $\mathcal{U}_d$.

1. Example 2.2: The quotient space $\mathcal{U}_3$ of unlabeled graphs on three vertices. Denote the equivalence class of a point in $\mathcal{U}_3$ by $[x, y, z]$. Each equivalence class in $\mathcal{O}_3^{\neq 0}/\mathcal{G}_3$ contains six points. In general, the number of elements in an equivalence class $[x, y, z]$ equals $|\Sigma_3|/|\Sigma_3^{[x,y,z]}| = 6/\Sigma_3^{[x,y,z]}|$, where $\Sigma_3^{[x,y,z]}| = \{\sigma \in \Sigma_3 : \sigma \cdot (x, y, z) = (x, y, z)\}$ is the stabilizer subgroup of (x, y, z).

We now analyze the gluings on the boundary of $\mathcal{O}_3$. First, the three coordinate quadrants $\{(0, y, z) : y, z \geq 0\}$, etc. forming part of the boundary of $\mathcal{O}_3^{\neq 0}$ get glued under the action of Σ_3. For example, $(12) \cdot (0, 5, 7) = (5, 0, 7)$ and $(13) \cdot (0, 5, 7) = (7, 5, 0)$. Each coordinate plane gets further glued, e.g., $(0, 5, 7)$ gets glued to $(0, 7, 5)$. This further gluing (in this case by the subgroup $\{\mathrm{Id}, (23)\} \subset \Sigma_3$) acts freely on ($\alpha$ of) the set of graphs $\{(y, z) : y, z \geq 0, y \neq z\}$ and fixes the diagonal line $\{y = z\}$. This is because $(23) \cdot (y, z) = (z, y)$. Thus the quotient space of the three coordinate quadrants is homeomorphic to one quadrant which is then folded along its diagonal to create a closed pie wedge. This wedge is a stratified 2-manifold: it contains an open, dense set which is a 2-manifold, the two edges (minus the origin) which are 1-manifolds, and the origin as a 0-manifold.

This pie wedge is glued onto the 3-manifold $\mathcal{O}_3^{\neq 0}/\mathcal{G}_3$ as follows: given $[x_i, y, z]$ with $x_i \to 0$, we declare the limit point of this sequence to be $[0, y, z]$. We make the similar definition if $y_i \to 0$ or $z_i \to 0$. This is clearly well defined. We make a similar definition if $[x_i, y_i, z]$ has $x_i, y_i \to 0$, etc. and if $[x_i, y_i.z_i]$ has $x_i, y_i, z_i \to 0$.

From now on, for expository reasons, we drop the automatic conditions $x \geq 0, y \geq 0, z \geq 0$ from description of subsets of $\mathcal{O}_3$.

Similarly, the three planes $\{(x, y, z) : x = y\}$, $\{x = z\}, \{y = z\}$ get glued together (e.g., $(123) \cdot (5, 5, 7) = (7, 5, 5)$). Note that e.g. $|[x, x, z]| = 3$ if $x \neq z$ and $|[x, x, x]| = 1$. For example $(5, 5, 7)$ is glued to $(5, 7, 5), (7, 5, 5)$. These three planes intersect at the line $\{(x, x, x)\}$. Thinking of the three planes as troughs with edge $\{x = y = z\}$, the three troughs are glued together. The two sides of a trough are not glued to each other, but are glued to sides of two other troughs. As above, $[x_i, y_i, z]$ has limit point/is glued to $[x, x, z]$ if $x_i, y_i \to x$, etc. In particular, if $x_i, y_i \to 0$, this is consistent with the previous gluing.

The final quotient $\mathcal{O}_D/\Sigma_3$ is a stratified 3-manifold:

- The dense 3-dimensional piece is $\mathcal{O}_3^{\neq 0}/\Sigma_3$, which is topologically a 3-ball.
- With increasingly terse notation, the 2d strata are
  - (i) $\{[x, y, z] : x = 0, y \neq z; y, z \neq 0\} = \{[x, y, z] : y = 0, x \neq z\}$ $= \{[x, y, z] : z = 0, x \neq y\}$;
  - (ii) $\{x = y, z > x\} = \{y = z, x > y\} = \{x = z, y > x\}$;
  - (iii) $\{x = y, 0 < z < x\} = \{y = z, 0 < x < y\}, \{x = z, 0 < y < x\}$.
- The 1d strata are
  - (i) $\{x = 0, y = z \neq 0\} = \{y = 0, x = z \neq 0\} = \{z = 0, x = y \neq 0\}$;
  - (ii) $\{x = y = 0, z > 0\} = \{x = z = 0, y > 0\}$ $= \{y = z = 0, x > 0\}$;
  - (iii) $\{x = y = z > 0\}$.
- The 0d stratum is $\{[0, 0, 0]\}$.

The point $[0, 2, 2]$ in the 1d stratum can be perturbed into a 2d stratum point $[\epsilon, 2, 2]$ or $[0, 2, 2 + \epsilon]$ or into a 3d stratum point $[\epsilon, 2 + \delta, 2 + \mu]$. This agrees with the fact that the 1d stratum $\{x = 0, y = z\}$ glues both to a trough (a 2d stratum) and to an open wedge in a coordinate plane, and that this 1d stratum also glues to the big cell.

2. Proof of Theorem 2.3.

THEOREM 2.3. The space of unlabeled graphs $\mathcal{U}_d = \mathcal{G}/\Sigma_d = \mathcal{O}_D/\Sigma_d$ is a stratified space.

PROOF. We just sketch the proof, since this result is not used in the paper. We don't need the technical definition of a stratified space, just a general understanding that $\mathcal{U}_d$ consists of a sequence of n-dimensional manifolds with boundary, $n = 1, \ldots, D$, with n-dimensional strata glued coherently to $(n + 1)$ (or higher) dimensional strata. The big open cell of dimension D is $\mathcal{O}_D^{\neq 0}/\Sigma_d$. Lower strata are characterized by the number of zero entries and the number of equal nonzero entries. More precisely, say the weight vector $\vec{x} \in \mathcal{O}_D$ has k zeros and $s_1 > 1$ entries equal to $w_1 \neq 0$, $\ldots$, s_t entries equal to $w_t \neq 0$, with all w_i distinct. Then $[\vec{x}] \in \mathcal{U}_d$ belongs to a stratum of dimension

$$D - k - \sum_{i=1}^{t} s_i.$$

A higher dimensional stratum S has a lower dimensional stratum S' as part of ∂S if a sequence of points in S converges to a point in S'. This can occur either by an entry in this sequence going to zero, or by entries in this sequence going to a common positive limit. $\square$

3. The topology of $\mathcal{U}_d$. In this section we show that $\mathcal{U}_d$ is contractible (an easy result), and that a natural slice of $\mathcal{U}_d$ is also contractible.

LEMMA 3.1. *$\mathcal{U}_d$ is contractible.*

This is an easy consequence of the contractibility of a cone, in this case the fundamental domain F.

PROOF. Take $[x] \in \mathcal{U}_d$ and a representative $x \in F$. Take the line segment $x_t := tx + (1-t)\vec{0}, t \in [0,1]$. If $x' = \sigma \cdot x \in F$ is another representative of $[x]$, then

$$(\sigma \cdot x)_t = t\sigma \cdot x + (1-t)\vec{0} = t\sigma \cdot x + (1-t)\sigma \cdot \vec{0} = \sigma \cdot (tx + (1-t)\vec{0}) = \sigma \cdot x_t, \tag{3.1}$$

since σ acts via a linear transformation. (Note the trivial but crucial fact that $\sigma \cdot \vec{0} = \vec{0}$.) Thus the contraction $[x]_t := t[x] + (1-t)[\vec{0}]$ is well defined in $\mathcal{U}_d$. $\square$

Every graph except the graph with no edges can be scaled to a graph whose edge lengths sum to one. This says that $\mathcal{U}'_d := \mathcal{U}_d \setminus \{[\vec{0}]\}$ deformation retracts onto the slice $S = \{[x] : x_1 + \ldots + x_D = 1\}$: there is a continuous map $F : [0,1] \times \mathcal{U}'_d \to \mathcal{U}'_d : F(0,[x]) = [x], F(1,[x]) \in S, F(t,[s]) = [s]$ for all $[x] \in \mathcal{U}'_d, [s] \in S$. Note that the slice condition $\sum x_i = 1$ is independent of the representative of $[x]$, as is the scaling of $[x]$ to an element of the slice.

Thus all the topology of $\mathcal{U}_d$ is contained in $\mathcal{U}'_d$.

LEMMA 3.2. *$\mathcal{U}'_d$ and the slice S are contractible.*

This is not as obvious as the previous lemma, as S is a polygonal cross-section in F with complicated gluings on its boundary.

PROOF. First, it suffices to show that S is contractible. For if S is contractible to a point $[x_0] \in S$ and we pick $[x] \in \mathcal{U}'_d$, then the path that scales $[x]$ to $[x'] \in S$ and then contracts $[x']$ to $[x_0]$ gives a contraction of $\mathcal{U}'_d$ to $[x_0]$.

The contractibility of S: The complete graph with edge weights all equal to $1/D$ has its equivalence class $[x_0]$ in S. For $[x] \in S$, take a representative x and the line segment $x_t := tx + (1-t)x_0$. As in (**??**), this descends to a well-defined path $[x]_t = t[x] + (1-t)[x_0]$ precisely because $\sigma \cdot x_0 = x_0$ for all σ. Thus S contracts to $[x_0]$. $\square$