*Supplement to "Averages of Unlabeled Networks: Geometric Characterization and Asymptotic Behavior", by Kolaczyk, Lin, Rosenberg, Walters, and Xu*

## Supplement E

## Figures

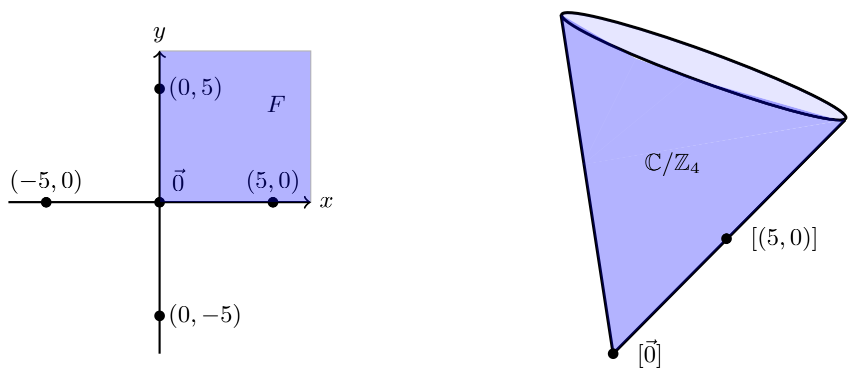


FIG 1. *In the figure on the left, $F$ is a fundamental domain $F$ for the action of $\mathbb{Z}_4$ on $\mathbb{C}$. The four point orbit of $(5,0)$ and the one point orbit of $\vec{0}$ are shown. In the figure on the right, the quotient space $\mathbb{C}/\mathbb{Z}_4$ is drawn as a hollow cone given by taking $F$ and gluing $(x,0)$ to $(0,x)$.*

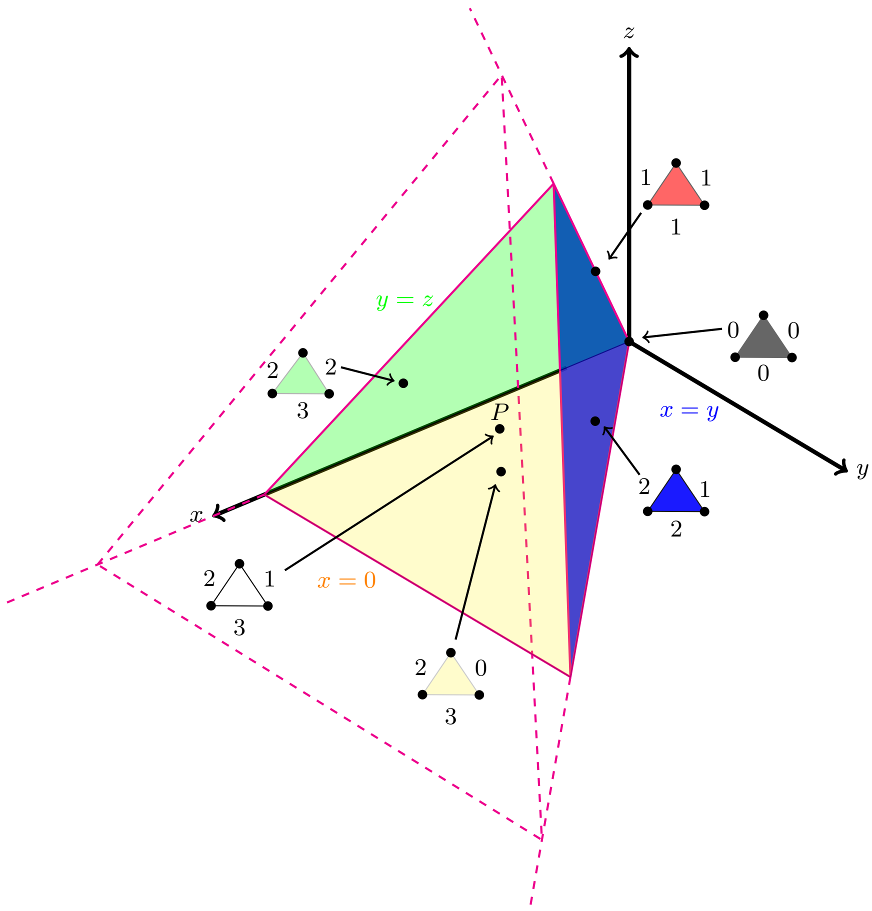


FIG 2. *As explained in Section 4.1, the infinite solid cone, which is the region* $\{x \geq y \geq z \geq 0\}$*, is a fundamental domain* $F$ *for unlabeled networks with three nodes. With the convention that the bottom side of the triangle has weight* $x$*, the left side has weight* $y$*, and the right side has weight* $z$*, the network with edge weights* $1, 2, 3$ *corresponds to the point* $P$ *in the interior of the cone. Other networks shown are color coded to correspond to points on faces or edges of the cone.*

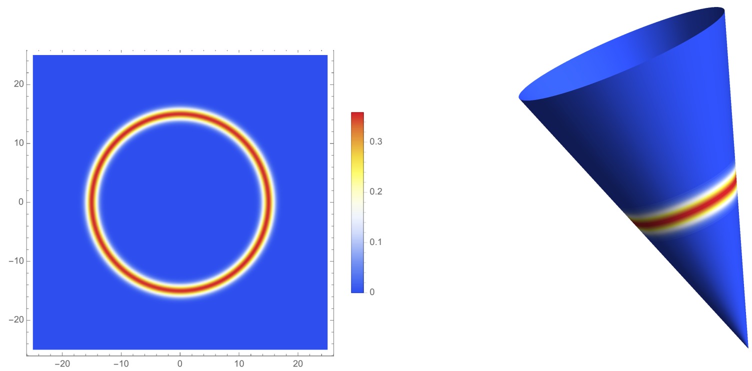


FIG 3. *Plot of the probability distribution $\nu(r,\theta)$ in (3.5), with $\alpha = 15$. The distribution peaks in the red region where $\nu$ is large, and is small in the blue region. The left hand side shows the plot on $\mathbb{R}^2$, and the right hand side shows that plot on the cone.*

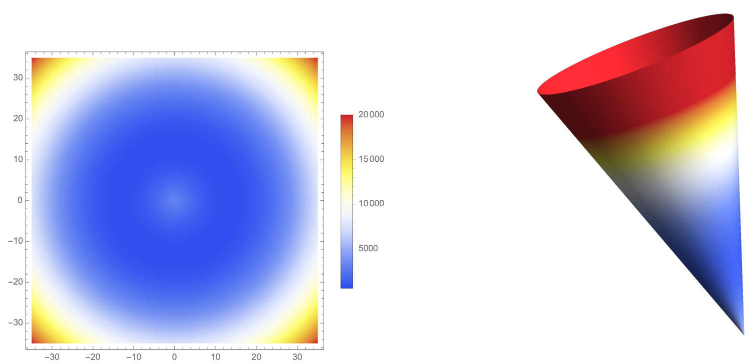


FIG 4. *Plot of the Fréchet function $f(x)$ with $\alpha = 15$ on $\mathbb{R}^2$ and on the cone. The Fréchet mean occurs in the "most blue" region, on a circle of radius approximately 13.53.*

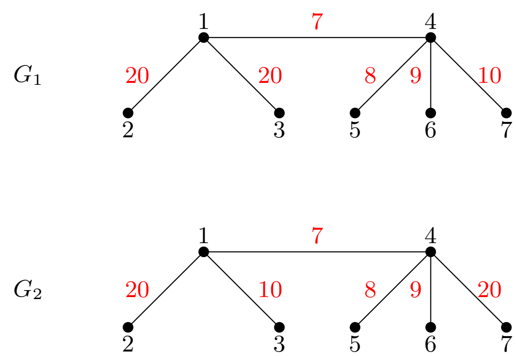


FIG 5. *Two networks of identical connectivity with ($G_1$) and without ($G_2$) distinct weight vectors.*

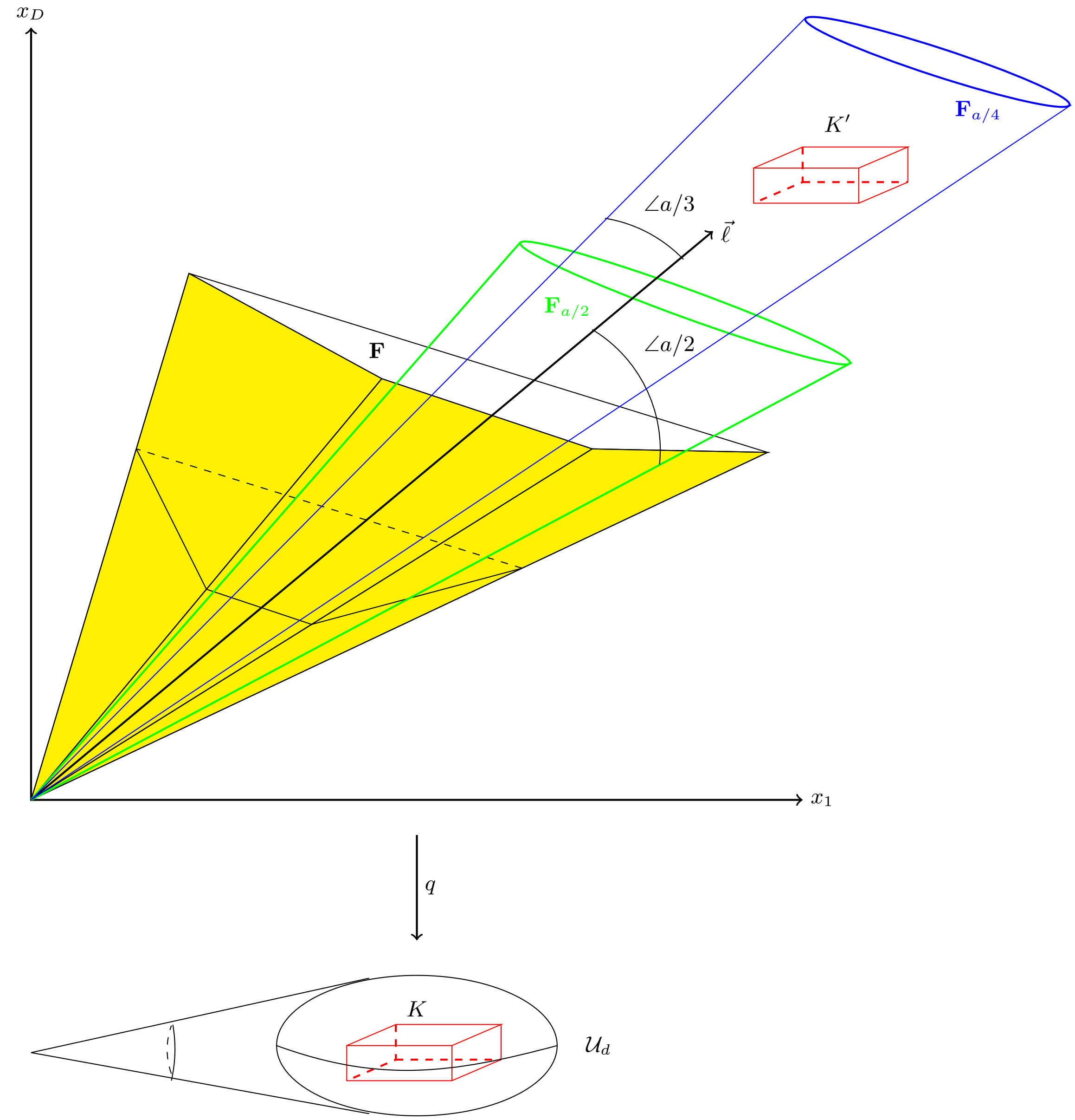


Fig 6. *$K \subset F_{a/4}$, the blue cone. $K$ and $K'$ are homeomorphic via $q$. The Euclidean distance between points $\vec{x}, \vec{y} \in F_{a/4}$ is the same as the Procrustean distance between their orbits $[\vec{x}], [\vec{y}]$, so $K'$ and $K$ are actually isometric. The Fréchet means of $K$ and $K'$ are related by $\mu_K = q(\mu_{K'})$. In particular, the Fréchet mean of $K$ is unique.*

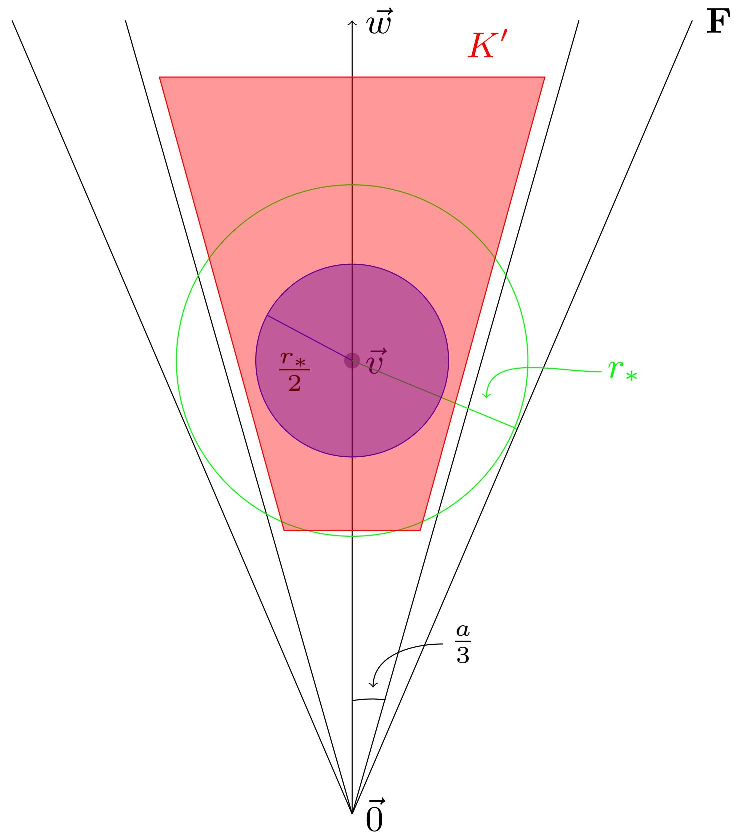


Fig 7. *A cross-section of the fundamental domain* $\mathbf{F}$ *showing the improvement of Thm. 4.5 over the Afsari result. The Afsari work gives a unique Fréchet mean of the image in* $\mathcal{U}_d$ *to the ball of radius* $r_*/2$*, where* $r_*$ *is the injectivity radius at* $\vec{v}$*. Thm. 4.5 gives a unique Fréchet mean to the image in* $\mathcal{U}_d$ *of the larger compact set* $K'$ *inside the* $a/3$ *cone. The boundary of* $K'$ *can extend down to* $\vec{0}$*, outwards as far as the walls of the* $a/3$ *cone, and to any finite height.*