Todo list

Boundary condition

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June 22, 2018

Abstract

Write abstract

1 Introduction

2 Giant viable cluster

Consider a multiplex network with L layers. Let $g_0^{(i)}$ and $g_1^{(i)}$ be the generating functions of respectively the degree and the excess degree in layer i. Moreover define u_i as the probability that a vertex reached after following an edge in layer i is not part of the giant viable cluster. Then if we pick a vertex v at random the probability S that it is part of the giant viable cluster can be written as

$$S = P_0 \left(\bigcap_{i=1}^{L} \exists w \in N_i(v) \; w \in GVC \right). \tag{1}$$

By requiring that the layers are independent from one others, we can rewrite S as a product

$$\begin{aligned} S &= \prod_{i=1}^{L} P_0 \left(\exists w \in N_i(v) \; w \in GVC \right) && (2) \\ &= \prod_{i=1}^{L} \left[1 - P \left(w \notin GVC \; \forall w \in N_i(v) \right) \right] && (3) \\ &= \prod_{i=1}^{L} \left[1 - \sum_{k=0}^{\infty} P \left((w \notin GVC \; \forall w \in N_i(v) | deg(v) = k \right) p_k^{(i)} \right] && (4) \\ &= \prod_{i=1}^{L} \left[1 - \sum_{k=0}^{\infty} u_i^k p_k^{(i)} \right] && (5) \\ &= \prod_{i=1}^{L} \left[1 - g_0^{(i)}(u_i) \right]. && (6) \end{aligned}$$

{Multiplex GCC size final}

We can find u_j by a similar reasoning. First note that $1 - u_j$ is the probability that a vertex reached by following an edge in layer j is in the giant viable cluster. Which as before can be written in the form

$$1 - u_j = P_1^{(j)}\left(\bigcap_{i=1}^{L} \exists w \in N_i(v)\ w \in GVC\right) \tag{7}$$

$$= \prod_{i=1}^{L} P_1^{(j)}\left(\exists w \in N_i(v)\ w \in GVC\right). \tag{8}$$

Since the layers are independent, the fact that we reached v by following an edge in layer j to reach vertex v is irrelevant in all other layers. However in layer j this means that the degree distribution follows the distribution $q_k^{(j)}$ rather than $p_k^{(j)}$. Putting this together we get

$$1 - u_j = \left[1 - \sum_{k=0}^{\infty} u_j^k q_k^{(j)}\right] \prod_{\substack{i=1 \\ i\neq j}}^{L} \left[1 - \sum_{k=0}^{\infty} u_i^k p_k^{(i)}\right] \tag{9}$$

$$= \left[1 - g_1^{(j)}(u_j)\right] \prod_{\substack{i=1 \\ i\neq j}}^{L} \left[1 - g_0^{(i)}(u_i)\right]. \tag{10}$$

{Multiplex u final}

Is this section too precise and should the results be given with just a citation ?

3 Boundary condition

Equations (6) and (10) are in principle sufficient to determine the size S of the giant viable cluster in a multiplex network. We see that there always exists a trivial solution

$$u_j = 1, \qquad \forall j, \tag{11}$$

$$S = 0. \tag{12}$$

Moreover this is the only solution for which $S = 0$. Indeed since $g_0^{(j)}$ is a strictly increasing function, we have

$$g_0^{(j)}(z) = 0 \quad \Leftrightarrow \quad z = 1. \tag{13}$$

So if $S = 0$, there exists k such that $u_k = 1$. If we put it back in eq. (10), it forces $1 - u_j = 0$ for all j, and thus all u_j are one.

Something about the fact that the degree distributions are determined by some set of parameters λ_j

We now introduce the following notations:

$$\mathbf{u} = (u_1, u_2, \ldots, u_L) \tag{14}$$

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N) \tag{15}$$

$$f_j(\boldsymbol{\lambda}, \mathbf{u}) = 1 - u_j - \left[1 - g_1^{(j)}(u_j)\right] \prod_{\substack{i=1 \\ i\neq j}}^{L} \left[1 - g_0^{(i)}(u_i)\right] \tag{16}$$

{Definition fj}

and the function

$$F : \mathbb{R}^N \times D_L \to \mathbb{R}^L \tag{17}$$

$$(\boldsymbol{\lambda}, \mathbf{u}) \mapsto F(\boldsymbol{\lambda}, \mathbf{u}) = (f_1(\boldsymbol{\lambda}, \mathbf{u}), f_2(\boldsymbol{\lambda}, \mathbf{u}), \ldots, f_L(\boldsymbol{\lambda}, \mathbf{u})), \tag{18}$$

{Definition F}

where $D_L = [0,1]^L$. Since the $g_0^{(i)}$ and $g_1^{(i)}$ are analytic with respect to the u_i, the function

$$F_{\boldsymbol{\lambda}} : D_L \to \mathbb{R}^L \tag{19}$$

$$\mathbf{u} \mapsto F_{\boldsymbol{\lambda}}(\mathbf{u}) = F(\boldsymbol{\lambda}, \mathbf{u}), \tag{20}$$

for a given parameter vector λ is continuously differentiable as well. Therefore we can define its Jacobi matrix $J(\boldsymbol{\lambda}, \mathbf{u})$ as having coefficients

$$[J(\boldsymbol{\lambda}, \mathbf{u})]_{ij} = \frac{\partial f_i(\boldsymbol{\lambda}, \mathbf{u})}{\partial u_j}. \tag{21}$$

In terms of the introduced notation, solving eq. (10) is equivalent to finding $\mathbf{u}^*$ such that

$$F(\boldsymbol{\lambda}, \mathbf{u}^*) = 0. \tag{22}$$

Assuming that F (and not only $F_{\boldsymbol{\lambda}}$) is continuously differentiable and that we know a solution $\mathbf{u}^*$ for some parameter vector $\boldsymbol{\lambda}^*$, we can use the implicit function theorem which give us the following:

If $\det[J(\boldsymbol{\lambda}, \mathbf{u}^*)] \neq 0$ then there is an open neighbourhood $U \subset \mathbb{R}^L$ of $\boldsymbol{\lambda}^*$ such that there is an unique continuously differentiable function $h : U \to D_L$ with

$$h(\boldsymbol{\lambda}^*) = \mathbf{u}^* \tag{23}$$

$$F(\boldsymbol{\lambda}, h(\boldsymbol{\lambda})) = 0, \quad \forall \boldsymbol{\lambda} \in U. \tag{24}$$

{Implicit solution for F}

The result in which we are interested here however comes from the contrapositive statement, namely that if we can not find suitable neighbourhood U and function h, then the determinant of the Jacobi matrix $J(\boldsymbol{\lambda}, \mathbf{u})$ must be zero. We now prove that it arises if $\boldsymbol{\lambda}^*$ is a critical point of a phase transition.

First notice that in the context of multiplex network a phase transition appears between the trivial solution $\mathbf{u}_T = (1, 1, \ldots, 1)$ (where $S = 0$) and non trivial solutions ($S > 0$). However, the trivial solution $\mathbf{u}_T$ solves eq. (10) for any generating function and thus for any parameter vector $\boldsymbol{\lambda}$. This implies that at one side of the phase transition only one solution exists, while on the other at least two does. Therefore if there is a continuous phase transition at $\boldsymbol{\lambda}^*$, for any U open containing $\boldsymbol{\lambda}$ we can define two distinct functions on U that fulfil eq. (24), the trivial $h_T(\boldsymbol{\lambda}) = \mathbf{u}_T$ and another function h corresponding to the non trivial solutions. Therefore the function h of the implicit function theorem is not uniquely defined and thus $\det[J(\boldsymbol{\lambda}^*, \mathbf{u}^*)] = 0$.

On the other hand, for a discontinuous phase transition at $\boldsymbol{\lambda}^*$, there are two sequences $\boldsymbol{\lambda}_n$ and $\mathbf{u}_n$ such that

$$\lim_{n\to\infty} \boldsymbol{\lambda}_n = \boldsymbol{\lambda}^* \tag{25}$$

$$\lim_{n\to\infty} \mathbf{u}_n \neq \mathbf{u}_T \tag{26}$$

$$F(\boldsymbol{\lambda}_n, \mathbf{u}_n) = 0. \tag{27}$$

This implies that no continuous function h can be defined to solve eq. (24). Therefore, once again, we have $\det\left[J(\boldsymbol{\lambda}^*, \mathbf{u}^*)\right] = 0$.

More explanation ?