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Suppose each network can be determined by one parameter. For example, ER network can be determined by the average degree c, SF network can be determined by γ. There are n pieces of networks, suppose their parameters are:$\lambda_1, \lambda_2, \ldots, \lambda_n$. (Actully, we can suppose there are m parameters of n networks, since the prove below does not depend on the number of the parameter. You can think of the number of the parameter as the degree of freedom.)

Now, consider the following n functions: each functions has n variables: $\{u_1, u_2, \ldots, u_n\} \in D \subset \mathbb{R}^n$, and each function also has n parameters: $\lambda_1, \lambda_2, \ldots, \lambda_n$,

$$\begin{cases} f_1(\lambda_1, \lambda_2, \ldots, \lambda_n; u_1, u_2, \ldots, u_n) = 0 \\ f_2(\lambda_1, \lambda_2, \ldots, \lambda_n; u_1, u_2, \ldots, u_n) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \cdots \\ f_n(\lambda_1, \lambda_2, \ldots, \lambda_n; u_1, u_2, \ldots, u_n) = 0 \end{cases} \tag{1}$$

Here, we define $\mathbf{J}f_u$ as the following matrix:

$$\mathbf{J}f_u = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_n} \end{pmatrix}$$

Now, we let:

$$f_i = 1 - u_i - \left[1 - g_1^{(i)}(u_i)\right] \prod_{j=1, j\neq i}^{n} \left[1 - g_0^{(j)}(u_j)\right] \tag{2}$$

the original issue is to find the solution that $0 \leq u_i \leq 1$, that is to let D: $[0,1] \times [0,1] \cdots \times [0,1]$, obviously $(u_1, u_2, \ldots, u_n) = (1, 1 \ldots, 1)$is the trivial solution.

The problem you want to figure out is to find the condition of the parameter that makes the function set has at least one solution and we take one of these(this) solution(s), if we make a little change of the parameters in one direction or more, the function set has no nontrivial solution around this solution, but for the others directions it has nontrivial solution around this solution. If the solution we take is the trivial solution (1,1), that means this might be a continuous phase transition. If the solution we take is a nontrivial solution, the transition must be a discontinuous phase transition. We just consider the second situation.

Consider one simple example, when $n = 2$, we know taht $f_1 = 0$ and $f_2 = 0$ are two curves which have the fixed point (1,1) on the plane $u_1 - u_2$, the situation we talked above means that the two curve should be tangent at one point $(u_1^*, u_2^*) \in [0,1] \times [0,1] \backslash (1,1)$, that is the normal vectors of these two curve are parallel on the point (u_1^0, u_2^0), that means: $\det(\mathbf{J}f_u)|_{(u_1,u_2)=(u_1^0,u_2^0)} = 0$.

Now we consider the general condition, when we fix the parameters, if $\boldsymbol{u} = \boldsymbol{u}_0$ is one nontrivial solution of the function set (1), $f_i(u_1, u_2, \ldots, u_n) = 0$ are n hypersurfaces($(n-1)-$dimentional) of the n-dimentional space, $(\frac{\partial f_i}{\partial u_1}, \frac{\partial f_i}{\partial u_2}, \ldots, \frac{\partial f_i}{\partial u_n})\big|_{\boldsymbol{u}=\boldsymbol{u_0}}$is the normal vector of the hypersurface $f_i = 0$ on the point $\boldsymbol{u}_0$, so we have n vectors(the normal vector of each hypersurface on the point $\boldsymbol{u}_0$), from the enlighten of the situation when $n = 2$, we think that the condition we want to find is that those n vectors are linear correlation on the point $\boldsymbol{u}^*$, so the condition shoud be: $\exists \boldsymbol{u}_0 \in D \backslash (1, 1, \ldots 1)$: $\boldsymbol{f}(\boldsymbol{u}_0) = \boldsymbol{0}$, $\det(\mathbf{J}f_u)|_{\boldsymbol{u}=\boldsymbol{u}_0} = 0$.

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But the statement above is not rigorous. For $n = 2$ we can make gragh and get the conclusion intuitively. But for more than three pieces of networks, for us, cannot imagine a space(or a hypersurface) more than three(or two) dimensions. So we need a more rigorous languege to describe this problem.

Proposition: $\boldsymbol{f} : H \times D \to \mathbb{R}^n$, $\boldsymbol{f} \in C^1(H \times D)$, that means $\boldsymbol{f}$ is differentiable on $H \times D$, now fix the parameter $\boldsymbol{\lambda}_0 \in H$, there is $\boldsymbol{u}_0 \in D \backslash \{\boldsymbol{u}_{tr}\}$($\boldsymbol{u}_{tr}$ is the trivial solution of $\boldsymbol{f} = \boldsymbol{0}$), which satisfies $\boldsymbol{f}(\boldsymbol{\lambda}_0, \boldsymbol{u}_0) = \boldsymbol{0}$.

If $\forall \delta > 0$, $\exists \boldsymbol{\lambda}^*$: $|\boldsymbol{\lambda}^* - \boldsymbol{\lambda}_0| < \delta$, which satisfies either the following two conditions:

(1) for $\forall \boldsymbol{u} \in D \backslash \{\boldsymbol{u}_{tr}\}$, $\boldsymbol{f}(\boldsymbol{\lambda}^*, \boldsymbol{u}) \neq \boldsymbol{0}$,

(2) if $\exists \boldsymbol{u}^* \in D \backslash \{\boldsymbol{u}_{tr}\}$, $\boldsymbol{f}(\boldsymbol{\lambda}^*, \boldsymbol{u}) = \boldsymbol{0}$, but $\lim_{|\boldsymbol{\lambda}^* - \boldsymbol{\lambda}_0| \to 0} \boldsymbol{u}^* \neq \boldsymbol{u}_0$.

then there must be: $\det(\mathbf{J}f_u)\big|_{(\boldsymbol{\lambda},\boldsymbol{u})=(\boldsymbol{\lambda}_0,\boldsymbol{u}_0)} = 0$.

To prove this proposition, it is equivalent to prove the following proposition:

Proposition: $\boldsymbol{f} : H \times D \to \mathbb{R}^n$, $\boldsymbol{f} \in C^1(H \times D)$, now fix the parameter $\boldsymbol{\lambda}_0 \in H$, there is $\boldsymbol{u}_0 \in D \backslash \{\boldsymbol{u}_{tr}\}$, which satisfies $\boldsymbol{f}(\boldsymbol{\lambda}_0, \boldsymbol{u}_0) = \boldsymbol{0}$.

If $|\det(\mathbf{J}f_u)|_{(\boldsymbol{\lambda},\boldsymbol{u})=(\boldsymbol{\lambda}_0,\boldsymbol{u}_0)} \neq 0$, then: $\exists \delta > 0$, $\forall \boldsymbol{\lambda}^* : |\boldsymbol{\lambda}^* - \boldsymbol{\lambda}_0| < \delta$, $\exists \boldsymbol{u}^* \in D \backslash \{\boldsymbol{u}_{tr}\}$, $s.t.$ $\boldsymbol{f}(\boldsymbol{\lambda}^*, \boldsymbol{u}^*) = \boldsymbol{0}$ and $\lim_{|\boldsymbol{\lambda}^* - \boldsymbol{\lambda}_0| \to 0} \boldsymbol{u}^* = \boldsymbol{u}_0$.

Prove: from the Implicit function theorem: there exists a neighbourhood of $(\boldsymbol{\lambda}_0, \boldsymbol{u}_0)$: $H_0 \times D_0 \in H \times D$, $s.t.$ $\forall \boldsymbol{\lambda}^* \in H_0$, the function set $\boldsymbol{f}(\boldsymbol{\lambda}^*, \boldsymbol{u}) = 0$ has a unique solution in D_0: $\boldsymbol{u}^* = \boldsymbol{g}(\boldsymbol{\lambda}^*)$, which satisfies: $\boldsymbol{u}_0 = \boldsymbol{g}(\boldsymbol{\lambda}_0)$, $\boldsymbol{g} \in C^1(H_0)$, thus we can get $\lim_{|\boldsymbol{\lambda}^* - \boldsymbol{\lambda}_0| \to 0} \boldsymbol{u}^* = \lim_{|\boldsymbol{\lambda}^* - \boldsymbol{\lambda}_0| \to 0} \boldsymbol{g}(\boldsymbol{\lambda}^*) = \boldsymbol{g}(\boldsymbol{\lambda}_0) = \boldsymbol{u}_0$, Now we take $\delta = min\{d \big| d = |\boldsymbol{\lambda} - \boldsymbol{\lambda}_0|, \boldsymbol{\lambda} \in \partial H_0\}$, ∂H_0 is the boundary of H_0, then obviously we can obtain the proposition.

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I think the things we also can do is to fix some parameters, change the other(others) to find the condition of this(these) parameter(s). For example, there are n pieces of networks, we fix the first

$n-1$ networks, and make sure there is a giant compoment of these $n-1$ networks, then we change the parameter of the last network, so there is just one parameter. Then there should be a value of this parameter to be the birthpoint of the giant compoment of all the networks.

If $\left|\frac{\partial(f_1,\ldots,f_{n-1})}{\partial(u_1,\ldots,u_{n-1})}\right| \neq 0$, from the Implicit function theorem, there is a neighbourhood we can write as: $u_i = u_i(u_n)$, $(i \in \{1, 2, \ldots, n-1\})$, then the last function can be written as: $f_n(u_1(u_n), \ldots, u_{n-1}(u_n), u_n) = 0$, There are one parameter and one variable in this function, so that we can use the methed that let the curve to be tangent at the solution of the function to find the boundary condition.

Or we can let all the networks be the same, Then from the symmetry of the function set we know that $u_i = u_j$, assume that $u = u_i$, then $1 - u = (1 - g_1(u)) * (1 - g_0(u))^{n-1}$, there are one variable and one parameter. So we can get the boundary condition like before.