UNIVERSITY OF FRIBOURG

MASTER PROJECT

Giant connected component in networks

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Abstract

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Master Project

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by Benoît RICHARD

The abstract. Write it

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P	Probability

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Chapter 1

Introduction

1.1 Definition of a network

Many systems in real world can be conceptually represented as some kind of objects being or not connected with others and we call such representation a network. For example, a road network can be represented as a set of crossings that are connected by direct roads. However, the concept of network does not require the object or the link between them to be physical. We can represent friendship relations as a network: two people are connected if they consider to be friends.

Mathematically, networks are represented as *graphs*. A graph is an object composed of a set V of nodes (also referred to as vertices) and a set of edges E. An edge is characterized by the fact that it connect two nodes together, which in mathematical terms translate to the fact that an edge can be written as a pair of nodes or equivalently $E \subset \{(v_1, v_2)|v_1, v_2 \in V\}$. To fit the numerous systems many extension of this simple model can be considered, for example edges may have a direction (*directed graph*), meaning that $(v_1, v_2) \neq (v_2, v_1)$, edges or vertices can also have carry a value (*weighted graph*) or multiple edges between two vertices may be allowed.

Chapter 2

Single layer networks

Single layer networks correspond to the classic picture of a network

2.1 Giant connected component

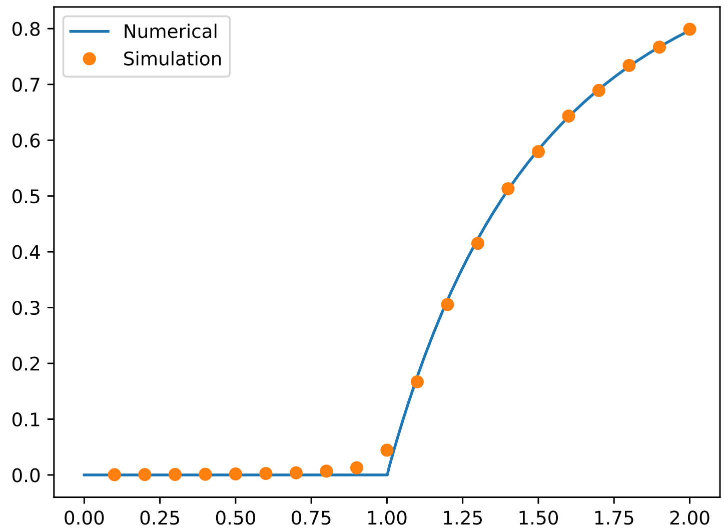
$$S = 1 - g_0(u) \tag{2.1}$$

{Single layer GCC final}

$$u = 1 - g_1(u) \tag{2.2}$$

{Single layer u final}

(A)



(B)

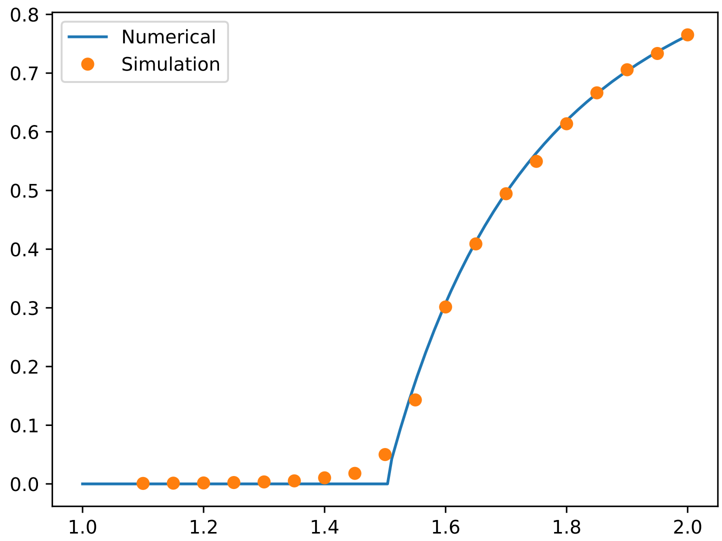


FIGURE 2.1: Numerical solution of eqs. (2.1) and (2.2), together with results on simulated networks. Results of simulations are average over 10 runs and the network size was set to 10^4 nodes. Simulations ran for several seconds in total, indicating that much larger network should be easily usable. (A) Poisson degree distribution. (B) Geometric degree distribution.

2.2 Degree distribution in the GCC

Per Bayes theorem we have for two event A and B

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(B|A)\frac{P(A)}{P(B)}. \tag{2.3}$$

{Bayes theorem}

This allows us to compute the degree distribution of the vertices in the giant connected component

$$P(deg(v) = k|k \in GCC) = P(k \in GCC|deg(v) = k)\frac{P(deg(v) = k)}{P(k \in GCC)} \tag{2.4}$$

$$= (1 - P(k \notin GCC|deg(v) = k))\frac{p_k}{S} \tag{2.5}$$

$$= \frac{p_k}{S}(1 - u^k). \tag{2.6}$$

By multiplying this expression by z^k for each k and summing, we find the generating function $G_0(z)$ of the degree distribution in the giant connected component

$$G_0(z) = \frac{1}{S}(g_0(z) - g_0(uz)). \tag{2.7}$$

Chapter 3

Multiplex network

3.1 Giant viable cluster

Consider a multiplex network with L layers. Let $g_0^{(i)}$ and $g_1^{(i)}$ be the generating functions of respectively the degree and the excess degree in layer i. Moreover define u_i as the probability that a vertex reached after following an edge in layer i is not part of the giant viable cluster. Then if we pick a vertex v at random the probability S that it is part of the giant viable cluster can be written as

$$S = P\left(\bigcap_{j=1}^{L} v \text{ has at least 1 neighbor in the GVC in layer } j\right). \tag{3.1}$$

By requiring that the layers are independent from one others, we can rewrite S as a product

$$\begin{aligned}
S &= \prod_{i=1}^{L} P\left(v \text{ has at least 1 neighbor in the GVC in layer } i\right) && (3.2)\\
&= \prod_{i=1}^{L} \left[1 - P\left(v \text{ has no neighbors in the GVC in layer } i\right)\right] && (3.3)\\
&= \prod_{i=1}^{L} \left[1 - \sum_{k=0}^{\infty} P\left(\text{no neig. of } v \text{ in GVC in layer } i | deg(v) = k\right) p_k^{(i)}\right] && (3.4)\\
&= \prod_{i=1}^{L} \left[1 - \sum_{k=0}^{\infty} u_i^k p_k^{(i)}\right] && (3.5)\\
&= \prod_{i=1}^{L} \left[1 - g_0^{(i)}(u_i)\right]. && (3.6)
\end{aligned}$$

{Multiplex GCC size final}

We can find u_j by a similar reasoning. First note that $1 - u_j$ is the probability that a vertex reached by following an edge in layer j is in the giant viable cluster. Which as before can be written in the form

$$\begin{aligned}
1 - u_j &= P\left(\bigcap_{j=1}^{L} v \text{ has at least 1 neighbor in the GVC in layer } j\right) && (3.7)\\
&= \prod_{i=1}^{L} P\left(v \text{ has at least 1 neighbor in the GVC in layer } i\right). && (3.8)
\end{aligned}$$

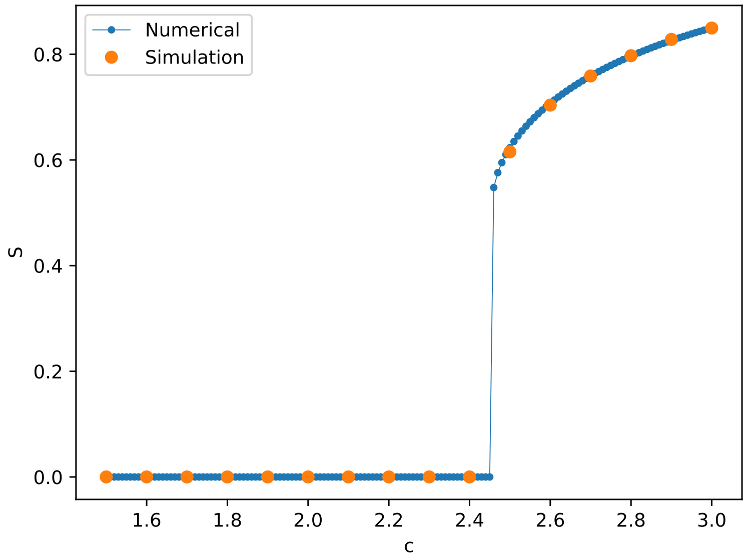
Since the layers are independent, the fact that we reached v by following an edge in layer j to reach vertex v is irrelevant in all other layers. However in layer j this means that the degree distribution follows the distribution $q_k^{(j)}$ rather than $p_k^{(j)}$. Putting this together we get

$$1 - u_j = \left[1 - \sum_{k=0}^{\infty} u_j^k q_k^{(j)}\right] \prod_{\substack{i=1\\ i\neq j}}^{L} \left[1 - \sum_{k=0}^{\infty} u_i^k p_k^{(i)}\right] \tag{3.9}$$

{Multiplex u final}

$$= \left[1 - g_1^{(j)}(u_j)\right] \prod_{\substack{i=1\\ i\neq j}}^{L} \left[1 - g_0^{(i)}(u_i)\right]. \tag{3.10}$$

(A)



(B)

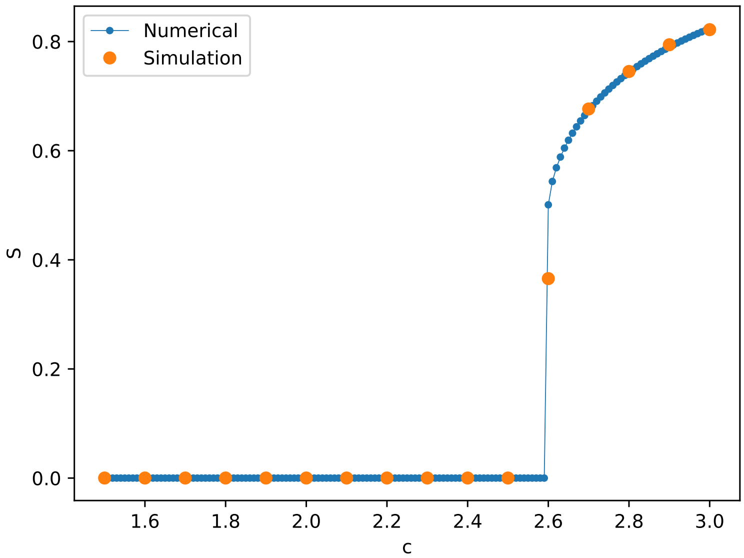


FIGURE 3.1: Numerical solution of eqs. (3.6) and (3.10), together with results on simulated networks for multiplex networks composed of two layers with the same distribution and mean number of edge c. Results of simulations are average over 10 runs and the network size was set to 10^4 nodes. Simulations ran for several seconds in total, indicating that much larger network should be easily usable. (A) Poisson degree distribution. (B) Geometric degree distribution.

3.2 Boundary condition

Equations (3.6) and (3.10) are in principle sufficient to determine the size S of the giant viable cluster in a multiplex network. We see that there always exists a trivial solution

$$u_j = 1, \qquad \forall j, \tag{3.11}$$
$$S = 0. \tag{3.12}$$

First notice that S can be zero if and only if all u_j are one. Indeed since $g_0^{(j)}$ is a strictly increasing function, we have

$$g_0^{(j)}(z) = 0 \quad \Leftrightarrow \quad z = 1. \tag{3.13}$$

So if $S = 0$, there exists k such that $u_k = 1$. If we put it back in eq. (3.10), it forces $1 - u_j = 0$ for all j, and thus all u_j are one.

Despite the fact that there is always an unique solution such that there is no giant viable cluster, the condition to have more than this solution is not immediately clear. To find it, observe that a new solution for eq. (**??**) appears when the curves

$$1 - u_j \tag{3.14}$$

$$\text{and} \qquad \left[1 - g_1^{(j)}(u_j)\right] \prod_{\substack{i=1\\ i\neq j}}^{L} \left[1 - g_0^{(i)}(u_i)\right] \tag{3.15}$$

are tangent. So the derivative of both with respect to u_j must be equals, which means

$$-1 = \frac{\mathrm{d}}{\mathrm{d}u_j}(1 - u_j) = \frac{\mathrm{d}}{\mathrm{d}u_j}\left[1 - g_1^{(j)}(u_j)\right] \prod_{\substack{i=1\\ i\neq j}}^{L} \left[1 - g_0^{(i)}(u_i)\right] \tag{3.16}$$

$$= -\left(\frac{\mathrm{d}}{\mathrm{d}u_j} g_1^{(j)}(u_j)\right) \prod_{\substack{i=1\\ i\neq j}}^{L} \left[1 - g_0^{(i)}(u_i)\right] \tag{3.17}$$

$$-\sum_{k=1}^{L} \left[1 - g_1^{(j)}(u_j)\right] \frac{\mathrm{d}u_k}{\mathrm{d}u_j} \left(\frac{\mathrm{d}}{\mathrm{d}u_k} g_0^{(k)}(u_k)\right) \prod_{\substack{i=1\\ i\neq j\\ i\neq k}}^{L} \left[1 - g_0^{(i)}(u_i)\right]. \tag{3.18}$$

This can be rewritten as

$$0 = \sum_{k=1}^{L} Q_{jk} \frac{\mathrm{d}u_k}{\mathrm{d}u_j} \tag{3.19}$$

$$= Q_{jj} + \sum_{\substack{k=1\\ k\neq j}}^{L} Q_{jk} \frac{\mathrm{d}u_k}{\mathrm{d}\lambda} \frac{\mathrm{d}\lambda}{\mathrm{d}u_j}, \tag{3.20}$$

where

$$Q_{jj} = \left(\frac{\mathrm{d}}{\mathrm{d}u_j} g_1^{(j)}(u_j) \right) \prod_{\substack{i=1 \\ i \neq j}}^{L} \left[1 - g_0^{(i)}(u_i) \right] - 1, \tag{3.21}$$

$$Q_{jk} = \left[1 - g_1^{(j)}(u_j) \right] \left(\frac{\mathrm{d}}{\mathrm{d}u_k} g_0^{(k)}(u_k) \right) \prod_{\substack{i=1 \\ i \neq j \\ i \neq k}}^{L} \left[1 - g_0^{(i)}(u_i) \right], \qquad k \neq j, \tag{3.22}$$

and where λ is some parameter, from which the u_j depend. For example it could be a composition of the parameters of the degree distribution for each layer.

If u_j is a smooth enough function of λ and both are not independent we have

$$\frac{\mathrm{d}u_j}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}u_j} \right)^{-1} \neq 0 \tag{3.23}$$

and we can rewrite the tangency condition for each j as

$$\sum_{k=1}^{L} Q_{jk} \frac{\mathrm{d}u_k}{\mathrm{d}\lambda} = 0, \tag{3.24}$$

which can be translated in matrix form as

$$Q \cdot v = 0, \tag{3.25}$$

where Q is the matrix with elements Q_{kj} and v the vector with elements

$$v_k = \frac{\mathrm{d}u_k}{\mathrm{d}\lambda}. \tag{3.26}$$

The system has an obvious solution $v_k = 0$ for all k, but this implies that λ has been chosen independent of the u_k, a possibility which is not very interesting. Otherwise, the system has a solution if Q is singular, which implies

$$\det Q = 0. \tag{3.27}$$

{Tangency condition as determinant

In this case, the value of v is of not relevant to the problem, nor is the value of parameter λ.

This gives us an additional equation which allows in principle to find the boundary of the region (in the space of parameters of the degree distributions) where S is not zero. Indeed the set of equations (3.10) give us implicitly the u_j as a function of the degree distribution parameters, and eq. (3.27) restrict the parameters to where they first encounter a non trivial solution.