Todo list

Boundary condition

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Abstract

Write abstract

1 Introduction

2 Giant viable cluster

Consider a multiplex network with L layers. Let $g_0^{(i)}$ and $g_1^{(i)}$ be the generating functions of respectively the degree and the excess degree in layer i. Moreover define u_i as the probability that a vertex reached after following an edge in layer i is not part of the giant viable cluster. Then if we pick a vertex v at random the probability S that it is part of the giant viable cluster can be written as

$$S = P_0\left(\bigcap_{i=1}^{L} \exists w \in N_i(v)\ w \in GVC\right). \tag{1}$$

By requiring that the layers are independent from one others, we can rewrite S as a product

$$\begin{aligned}
S &= \prod_{i=1}^{L} P_0\left(\exists w \in N_i(v)\ w \in GVC\right) && (2)\\
&= \prod_{i=1}^{L} \left[1 - P\left(w \notin GVC\ \forall w \in N_i(v)\right)\right] && (3)\\
&= \prod_{i=1}^{L} \left[1 - \sum_{k=0}^{\infty} P\left((w \notin GVC\ \forall w \in N_i(v) | deg(v) = k\right) p_k^{(i)}\right] && (4)\\
&= \prod_{i=1}^{L} \left[1 - \sum_{k=0}^{\infty} u_i^k p_k^{(i)}\right] && (5)\\
&= \prod_{i=1}^{L} \left[1 - g_0^{(i)}(u_i)\right]. && (6)
\end{aligned}$$

{Multiplex GCC size final}

We can find u_j by a similar reasoning. First note that $1 - u_j$ is the probability that a vertex reached by following an edge in layer j is in the

giant viable cluster. Which as before can be written in the form

$$1 - u_j = P_1^{(j)} \left(\bigcap_{i=1}^{L} \exists w \in N_i(v)\ w \in GVC \right) \tag{7}$$

$$= \prod_{i=1}^{L} P_1^{(j)} \left(\exists w \in N_i(v)\ w \in GVC \right). \tag{8}$$

Since the layers are independent, the fact that we reached v by following an edge in layer j to reach vertex v is irrelevant in all other layers. However in layer j this means that the degree distribution follows the distribution $q_k^{(j)}$ rather than $p_k^{(j)}$. Putting this together we get

$$1 - u_j = \left[1 - \sum_{k=0}^{\infty} u_j^k q_k^{(j)} \right] \prod_{\substack{i=1 \\ i \neq j}}^{L} \left[1 - \sum_{k=0}^{\infty} u_i^k p_k^{(i)} \right] \tag{9}$$

$$= \left[1 - g_1^{(j)}(u_j) \right] \prod_{\substack{i=1 \\ i \neq j}}^{L} \left[1 - g_0^{(i)}(u_i) \right]. \tag{10}$$

{Multiplex u final}

Is this section too precise and should the results be given with just a citation ?

3 Boundary condition

3.1 General case

Equations (6) and (10) are in principle sufficient to determine the size S of the giant viable cluster in a multiplex network. We see that there always exists a trivial solution

$$u_j = 1, \qquad \forall j, \tag{11}$$

$$S = 0. \tag{12}$$

Moreover this is the only solution for which $S = 0$. Indeed since $g_0^{(j)}$ is a strictly increasing function, we have

$$g_0^{(j)}(z) = 0 \quad \Leftrightarrow \quad z = 1. \tag{13}$$

So if $S = 0$, there exists k such that $u_k = 1$. If we put it back in eq. (10), it forces $1 - u_j = 0$ for all j, and thus all u_j are one.

Up to now, we have considered the multiplex generated to be determined via the degree distributions of each of its layer. However a degree distribution has an infinite number of degrees of freedom, it is therefore more practical to let the degree distributions depend on a finite set of parameters $\lambda_1, \lambda_2, \ldots, \lambda_N$ and express the behaviour of the network in term of them, . Note that the number of parameters N does not need to match the number of layers L.

We now introduce the following notations:

$$\mathbf{u} = (u_1, u_2, \ldots, u_L) \tag{14}$$

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N) \tag{15}$$

$$f_j(\boldsymbol{\lambda}, \mathbf{u}) = 1 - u_j - \left[1 - g_1^{(j)}(u_j) \right] \prod_{\substack{i=1 \\ i \neq j}}^{L} \left[1 - g_0^{(i)}(u_i) \right] \tag{16}$$

{Definition fj}

and the function

$$F : \mathbb{R}^N \times D_L \to \mathbb{R}^L \tag{17}$$

$$(\boldsymbol{\lambda}, \mathbf{u}) \mapsto F(\boldsymbol{\lambda}, \mathbf{u}) = (f_1(\boldsymbol{\lambda}, \mathbf{u}), f_2(\boldsymbol{\lambda}, \mathbf{u}), \ldots, f_L(\boldsymbol{\lambda}, \mathbf{u})), \tag{18}$$

{Definition F}

where $D_L = [0,1]^L$. Since the $g_0^{(i)}$ and $g_1^{(i)}$ are analytic with respect to the u_i, the function

$$F_{\boldsymbol{\lambda}} : D_L \to \mathbb{R}^L \tag{19}$$

$$\mathbf{u} \mapsto F_{\boldsymbol{\lambda}}(\mathbf{u}) = F(\boldsymbol{\lambda}, \mathbf{u}), \tag{20}$$

for a given parameter vector λ is continuously differentiable. Therefore we can define its Jacobi matrix $J(\boldsymbol{\lambda}, \mathbf{u})$ as having coefficients

$$[J(\boldsymbol{\lambda}, \mathbf{u})]_{ij} = \frac{\partial f_i(\boldsymbol{\lambda}, \mathbf{u})}{\partial u_j}. \tag{21}$$

In terms of the introduced notation, solving eq. (10) is equivalent to finding $\mathbf{u}^*$ such that

$$F(\boldsymbol{\lambda}, \mathbf{u}^*) = 0. \tag{22}$$

Assuming that F (and not only $F_{\boldsymbol{\lambda}}$) is continuously differentiable and that we know a solution $\mathbf{u}^*$ for some parameter vector $\boldsymbol{\lambda}^*$, we can use the implicit function theorem which give us the following:

If $\det[J(\boldsymbol{\lambda}^*, \mathbf{u}^*)] \neq 0$ then there is an open neighbourhood $U \subset \mathbb{R}^L$ of $\boldsymbol{\lambda}^*$ such that there is an unique continuously differentiable function $h : U \to D_L$ with

$$h(\boldsymbol{\lambda}^*) = \mathbf{u}^* \tag{23}$$

$$F(\boldsymbol{\lambda}, h(\boldsymbol{\lambda})) = 0, \quad \forall \boldsymbol{\lambda} \in U. \tag{24}$$

{Implicit solution for F}

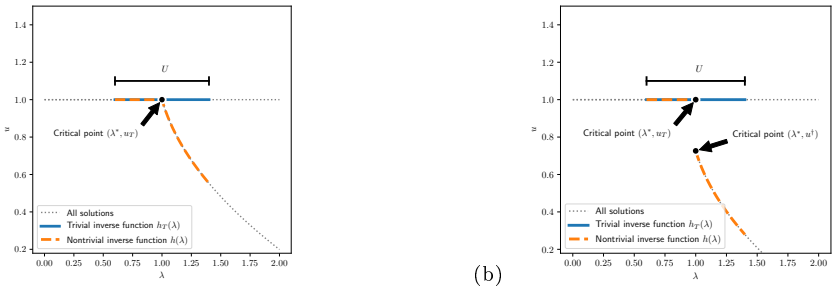
The result in which we are interested here comes from the contrapositive of this statement, namely that if we can not find suitable neighbourhood U or function h, then the determinant of the Jacobi matrix $J(\boldsymbol{\lambda}, \mathbf{u})$ must be zero,

$$\det[J(\boldsymbol{\lambda}^*, \mathbf{u}^*)] = 0. \tag{25}$$

{Boundary condition}

We now prove that such situations arise if $\boldsymbol{\lambda}^*$ is a critical point of a phase transition and therefore that eq. (25) is a sufficient condition to find the boundary critical regions of a phase transition.

First notice that in the context of multiplex network a phase transition appears between the trivial solution $\mathbf{u}_T = (1, 1, \ldots, 1)$ (where $S = 0$) and non trivial solutions ($S > 0$). However, the trivial solution $\mathbf{u}_T$ solves eq. (10) for any generating function and thus for any parameter vector $\boldsymbol{\lambda}$. For a continuous phase transition this immediately gives us $\mathbf{u}^* = \mathbf{u}_T$. Moreover, on one side of the phase transition occurring at $\boldsymbol{\lambda}^*$ one solution exists, while on the other at least two do. Therefore for any U open containing $\boldsymbol{\lambda}^*$ we can define two distinct functions on U that fulfil eq. (24), the trivial $h_T(\boldsymbol{\lambda}) = \mathbf{u}_T$ and another function h corresponding to the non trivial solutions, with $h(\boldsymbol{\lambda}^*) = h_T(\boldsymbol{\lambda}^*) = \mathbf{u}_T$. So the function h of the implicit function theorem is not uniquely defined and thus $\det[J(\boldsymbol{\lambda}^*, \mathbf{u}_T)] = 0$.



(a) (b)

{Figure: Scheme of continuous and di

Figure 1: (a) Scheme of a continuous phase transition. (b) Scheme of a discontinuous phase transition.

On the other hand, let consider a discontinuous phase transition at $\boldsymbol{\lambda}^*$. For any neighbourhood U of $\boldsymbol{\lambda}^*$ there are two sequences $(\boldsymbol{\lambda}_n, \mathbf{u}_n)$ and $(\boldsymbol{\eta}_m, \mathbf{v}_m)$ with $\boldsymbol{\lambda}_n, \boldsymbol{\eta}_m \in U$ such that

$$\lim_{n\to\infty} (\boldsymbol{\lambda}_n, \mathbf{u}_n) = (\boldsymbol{\lambda}^*, \mathbf{u}^\dagger) \quad \text{avec } \mathbf{u}^\dagger \neq \mathbf{u}_T \tag{26}$$

$$\lim_{m\to\infty} (\boldsymbol{\eta}_m, \mathbf{v}_m) = (\boldsymbol{\lambda}^*, \mathbf{u}_T) \tag{27}$$

$$F(\boldsymbol{\lambda}_n, \mathbf{u}_n) = 0 \quad \forall n \tag{28}$$

$$F(\boldsymbol{\eta}_m, \mathbf{v}_m) = 0 \quad \forall m. \tag{29}$$

If we assume that an unique continuous function h solving eq. (24) exists, we would have

$$h(\boldsymbol{\lambda}_n) = \mathbf{u}_n \quad \forall n \tag{30}$$

$$h(\boldsymbol{\eta}_m) = \mathbf{v}_m \quad \forall m. \tag{31}$$

The continuity of h would furthermore imply

$$h(\boldsymbol{\lambda}^*) = \lim_{n\to\infty} h(\boldsymbol{\lambda}_n) = \lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}^\dagger, \tag{32}$$

but also

$$h(\boldsymbol{\lambda}^*) = \lim_{m\to\infty} h(\boldsymbol{\eta}_m) = \lim_{m\to\infty} \mathbf{v}_m = \mathbf{u}_T. \tag{33}$$

Since $\mathbf{u}_T \neq \mathbf{u}^\dagger$, this gives raise to the contradiction $h(\boldsymbol{\lambda}^*) \neq h(\boldsymbol{\lambda}^*)$. Therefore our assumption must be false and no continuous function h can be defined to solve eq. (24). So finally, we have $\det[J(\boldsymbol{\lambda}^*, \mathbf{u}^*)] = 0$, $\mathbf{u}^*$ being either $\mathbf{u}_T$ or $\mathbf{u}^\dagger$.

3.2 One dimensional case

Introduce 1D variables more clearly

If $L = N = 1$, the problem is the classical problem of a one layer network which degree distribution is determined by a single parameter λ. In that case the Jacobi matrix J reduces to the scalar quantity

$$J(\lambda, u) = \frac{\partial}{\partial u}(g_1(u) - u) = \frac{\partial g_1(u)}{\partial u} - 1. \tag{34}$$

Therefore the condition for the boundary $\det J(\lambda, u) = 0$ becomes

$$\frac{\partial g_1(u)}{\partial u} = 1. \tag{35}$$

This condition was already introduced and verified previously by[?]. Missing ref

4 Verification

Define $C \subset \mathcal{R}^L$ as the set of all parameters $\boldsymbol{\lambda}$ corresponding to a critical point. This set correspond to the parameters that solve simultaneously eq. (10) and eq. (25) for some $\mathbf{u} \in D_L$. In this section we present an alternative and independent method to estimate C in order to verify these equations are sufficient to define C.

To do so we use concept borrowed from the field of *interval arithmetic*[?]. An interval I is defined a set of the form Missing ref

$$I = [a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}. \tag{36}$$

The set of all intervals is denoted as $\mathbb{IR}$. The N-dimensional equivalent of an interval is an interval box B, defined as a set of the form

$$B = I_1 \times I_2 \times \cdots \times I_N, \quad I_k \in \mathbb{IR} \quad \forall k = 1, \ldots, N \tag{37}$$

and the set of all N-dimensional interval boxes is denoted $\mathbb{IR}^N$.

Given a function $\phi : \mathbb{R}^M \to \mathbb{R}^N$ the rules of interval arithmetic allow to define a new interval valued function $\Phi : \mathbb{IR}^M \to \mathbb{IR}^N$ such that

$$x \in B \quad \Rightarrow \quad \phi(x) \in \Phi(B). \tag{38}$$

As a consequence of interval arithmetic rules, it is possible, based on eq. (10) and (25), to define a function $H : \mathbb{IR}^L \to \mathbb{I}D_N$ such that

More details

$$F(\boldsymbol{\lambda}, \mathbf{u}) = 0, \quad \forall \mathbf{u} \in U, \forall \boldsymbol{\lambda} \in \Lambda \quad \Rightarrow \quad U \subset H(\Lambda). \tag{39}$$

Then we can separate two cases

1. $H(\Lambda) = \{\mathbf{u}_T\}$, implying that for all $\boldsymbol{\lambda} \in \Lambda$ the system do not display a giant viable component,
2. $H(\Lambda) \neq \{\mathbf{u}_T\}$ and we can not conclude anything.

If H