First substitution rule

If **P** is a tautology, then any logical variable in this sentence (e.g. p) can be replaced with any other sentence (e.g. . $q \to r$) and P remains a tautology.

Example

In the law of De Morgan $\sim(\boldsymbol{p} \wedge q) \Leftrightarrow (\sim\boldsymbol{p} \vee \sim q)$ let's substitute p with $\boldsymbol{q} \to \boldsymbol{r}$:

$$\sim[(\boldsymbol{q} \to \boldsymbol{r}) \wedge q] \Leftrightarrow [\sim(\boldsymbol{q} \to \boldsymbol{r}) \vee \sim q].$$

By logical matrix:

p	q	r	$\sim[(q \to r) \wedge q]$	$\Leftrightarrow$	$[\sim(q \to r) \vee \sim q]$
0	0	0	1 1 0	1	0 1 1 1
0	0	1	1 1 0	1	1 1 1 1
0	1	0	1 0 0	1	1 0 1 0
0	1	1	0 1 1	1	0 1 0 0
1	0	0	1 1 0	1	0 1 1 1
1	0	1	1 1 0	1	0 1 1 1
1	1	0	1 0 0	1	1 0 1 0
1	1	1	0 1 1	1	0 1 0 0

we verify that the new sentence is also a tautology.

Second substitution rule

If the sentence P contains Q and if we substitute Q with a sentence equivalent to Q, then the new sentence will be equivalent to P.

Example

In modus tollendo tollens $[(\boldsymbol{p} \to \boldsymbol{q}) \wedge \sim q] \Rightarrow \sim p$ let's substiture $\boldsymbol{p} \to \boldsymbol{q}$ with equivalent $(\sim\boldsymbol{p} \vee \boldsymbol{q})$ (law 10 $(p \to q) \Leftrightarrow (\sim p \vee q)$), to obtain:

$$[(\sim\boldsymbol{p} \vee \boldsymbol{q}) \wedge \sim q] \Rightarrow \sim p,$$

which is equivalent to the original $[(p \to q) \wedge \sim q] \Rightarrow \sim p$. Because the original sentence was a tautology, the modified sentence is also a tautology.

If there were more than one occurences of $p \to q$, in the original sentence, we could substitute all these occurences or only some and the modified sentence would still be equivalent to the original.

Inferring rules

1. $$\begin{array}{l} P \\ \hline \therefore P \lor Q \end{array}$$ introduction of alternative

2. $$\begin{array}{l} P \land Q \\ \hline \therefore P \end{array}$$ ommiting conjunction

3. $$\begin{array}{l} P \\ P \to Q \\ \hline \therefore Q \end{array}$$ modus ponendo ponens

4. $$\begin{array}{l} P \to Q \\ \sim Q \\ \hline \therefore \sim P \end{array}$$ modus tollendo tollens

5. $$\begin{array}{l} P \lor Q \\ \sim P \\ \hline \therefore Q \end{array}$$ modus ponendo tollens

6. $$\begin{array}{l} P \to Q \\ Q \to R \\ \hline \therefore P \to R \end{array}$$ hipothetical syllogism

7. $$\begin{array}{l} P \\ Q \\ \hline \therefore P \land Q \end{array}$$ introduction of conjunction