Inclusion-Exclusion

Selected Exercises

Powerpoint Presentation taken from Peter Cappello's webpage

www.cs.ucsb.edu/~capello

The Principle of Inclusion-Exclusion

$$| A_1 \cup A_2 | = | A_1 | + | A_2 | - | A_1 \cap A_2 |$$

$$| A_1 \cup A_2 \cup A_3 | = | A_1 | + | A_2 | + | A_3 |$$

$$- | A_1 \cap A_2 | - | A_1 \cap A_3 | - | A_2 \cap A_3 |$$

$$+ | A_1 \cap A_2 \cap A_3 |$$

The Principle of Inclusion-Exclusion

$$| A_1 \cup A_2 \cup . . . \cup A_n | = \Sigma | A_i |$$

$$- \Sigma | A_i \cap A_j |$$

$$+ \Sigma | A_i \cap A_j \cap A_k |$$

$$- \quad . . .$$

$$+ (-1)^{n-1} \Sigma | A_1 \cap A_2 \cap ... \cap A_n |$$

Exercise 10

Find the number of positive integers not exceeding 100

that are *not* divisible by 5 or 7.

(Numbers ≤ 100 that are not divisible by 5 *and* are

not divisible by 7.)

See Venn diagram.

Exercise 10 Solution

Let the "bad" numbers be those that are divisible by 5 or 7.

Subtract the number of "bad" numbers from the *size of the universe*: 100.

- Let A_5 denote the numbers $\leq$ 100 that *are* divisible by 5.
- Let A_7 denote the numbers $\leq$ 100 that *are* divisible by 7.
- $| A_5 \cup A_7 | = | A_5 | + | A_7 | - | A_5 \cap A_7 |$

$$= \lfloor 100/5 \rfloor + \lfloor 100/7 \rfloor - \lfloor 100/(5 \cdot 7) \rfloor$$

$$= 20 + 14 - 2$$

$$= 32.$$

The answer is $100 - 32 = 68$.

Exercise 14

How many permutations of the 26 letters of the English alphabet *do not* contain any of the strings *fish*, *rat*, or *bird*?

See Venn diagram.

Exercise 14 Solution

1. Start with the universe: 26!
2. Subtract the permutations that contain:
   - *fish*.

     To count the number of permutations with a fixed substring:

     Treat the fixed substring as 1 letter.

     There are (26 – 4 + 1)! such permutations.
   - *rat*: (26 – 3 + 1)!
   - *bird*: : (26 – 4 + 1)!
3. Add the permutations that contain:
   - *fish* & *rat*: (26 – 4 – 3 + 2)!
   - *fish* & *bird*: 0
   - *rat* & *bird*: 0
4. Subtract the permutations that contain all 3 strings

   There are 0 such permutations.

Answer: 26! – 23! – 24! – 23! + 21!

Proof

1. An element in exactly 0 of the sets is counted by the RHS 0 times.
2. An element in exactly 1 of the sets is counted by the RHS 1 time.
3. An element in exactly 2 of the sets is counted by the RHS 2 – 1 = 1 time.
4. An element in exactly 3 of the sets is counted by the RHS 3 – 3 + 1 = 1 time.

m. An element in exactly m of the sets is counted by the RHS

$C(m, 1) – C(m, 2) + C(m, 3) - \ldots + (-1)^{m-1}C(m, m)$ times.

Proof

$(x + y)^n = C(n, 0)x^n y^0 + C(n, 1)x^{n-1}y^1 + \ldots C(n, n)x^0 y^n$

Evaluate the Binomial Theorem, at $x = 1$, $y = -1$, we obtain

$(1 – 1)^n = C(n, 0) – C(n, 1) + C(n, 2) - \ldots + (-1)^{n-1}C(n, n)$

$\Leftrightarrow C(n, 1) – C(n, 2) + \ldots + (-1)^n C(n, n) = C(n, 0)$

$= 1.$

Exercise 18

How many terms are there in the formula for the # of elements in the union of 10 sets given by the inclusion-exclusion principle?

Exercise 18 Solution

It is the sum of the:

- Terms with 1 set: C(10, 1)
- Terms with 2-way intersections: C(10, 2)
- Terms with 3-way intersections: C(10, 3)
- Terms with 4-way intersections: C(10, 4)
- Terms with 5-way intersections: C(10, 5)
- Terms with 6-way intersections: C(10, 6)
- Terms with 7-way intersections: C(10, 7)
- Terms with 8-way intersections: C(10, 8)
- Terms with 9-way intersections: C(10, 9)
- Terms with 10-way intersections: C(10, 10)

The total is $2^{10} - 1 = 1{,}023$.

Exercise 20

How many elements are in the *union* of 5 sets if:

- The sets contain 10,000 elements
- Each pair of sets has 1,000 common elements
- Each triple of sets has 100 common elements
- Each quadruple of sets has 10 common elements
- All 5 sets have 1 common element.

Exercise 20 Solution

To count the *size* of the *union* of the 5 sets:

- Add the sizes of each set: C(5, 1)10,000
- Subtract the sizes of the 2-way intersections: C(5, 2)1,000
- Add the sizes of the 3-way intersections: C(5, 3)100
- Subtract the sizes of the 4-way intersections: C(5, 4)10
- Add the size of the 5-way intersection: C(5, 5)1

C(5, 1)10,000 - C(5, 2)1,000 + C(5, 3)100 - C(5, 4)10 + C(5, 5)1