Methods of checking the properties of relations

- by definition
- by matrix
- by graph

The number of pairs in the relation can be

- infinite
- finite - large
- finite - small

If the number of pairs is infinite or finite but large, we can only use the definition. However, when the number is small, we can also use the matrix and the graph.

Matrix of a relation

Let us consider the set U={0,1,2,3,4} and the relation R such that xRy ⇔ x+y = 4. We can draw the following matrix of the relation:

R	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	1
1	0	0	0	1	0
2	0	0	1	0	0
3	0	1	0	0	0
4	1	0	0	0	0

We put 1 where the element in the row and in the column are in relation, 0 when they are not.

Graph of a relation

When drawing a graph of a relation we draw all the elements of the set as nodes (dots). Then we connect nodes which are in relation with arrows (thus indicating the direction of the relation).

Checking the properties of a relation using matrix or graph

Property	**Matrix**	**Graph**
Reflexiveness	The main diagonal (top-left -> bottom right) contains only 1's	There is a loop in every node
Anti-reflexiveness	The main diagonal contains only 0's	There are no loops in any node
Symmetry	The matrix is symmetric (with respect to the main diagonal)	Every arrow is a double arrow or a loop
Anti-symmetry	*Difficult to observe*	There are no double arrows (however, loops are allowed)
Transitivity	*Difficult to observe*	If there is any path from one node to another there must also be a shortcut between them
Connexity	There is no pair of 0's placed symmetrical with respect to the main diagonal	Every pair of nodes is connected

Example 1. Check the properites of the relation $R_1 = \{(m,n) \in S \times S : m + n \leq 6\}$, $S=\{1,2,3,4\}$

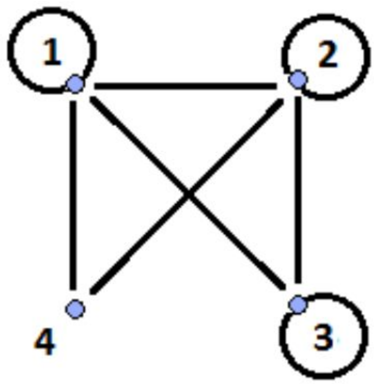
Pairs in the relation:

R_1 = {1,1 , 1,2 , 1,3 , 1,4 , 2,1 , 2,2 , 2,3 , 2,4 , 3,1 , 3,2 , 3,3 , 4,1 , 4,2} - we only have 13 pairs, so we can use the matrix and the graph.

Matrix:

R_1	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	1	1	1
3	1	1	1	0
4	1	1	0	0

Graph:



Solution:

Property	Definition	Matrix	Graph
Reflexiveness **NO**	The pair (4,4) is not in the relation.	There is a 0 on the main diagonal	The node 4 does not have a loop.
Anit-reflexiveness **NO**	The pair (1,1) is in the relation.	There is a 1 on the main diagonal.	The node 1 has a loop.
Symmetry **YES**	Follows from commutativeness of addition	The matrix is symmetric.	All arrows are double.
Anti-symmetry **NO**	Pairs (1,2) and (2,1) are both in the relation, but 1 does not equal 2.	-	-
Transitivity **NO**	4R2 and 2R3 bot not 4R3.	-	There is a path from 4 to 2 and from 2 to 3 but there is no direct shortcut 4 to 3.
Connexity **NO**	Neither 4R3 nor 3R4	There are 0's placed symmetrically with respect to the main diagonal	There is no arrow between 3 and 4.

Example 2. Check the properties of $R_2 = \{(m,n) \in S \times S: m \cdot n = 0\}$, where $S = \{0,1,2,3\}$

Example 3 Check the properties of $R_3 = \{(m,n) \in S \times S: m - n \text{ is even}\}$, where $S = \{0,1,2,3\}$

Example 4 Check the properties of $R_4 = \{(m,n) \in S \times S: m \leq n\}$, where $S = \{0,1,2,3\}$

The equivalence relation

We say that a relation is an equivalence relation if it is reflexive, symmetric and transitive.

For example, previously defined relation R_3 is an equivalence relation.

Let us define the notion of **class of abstraction**. Let there be a set S, an equivalence relation R and an element x belonging to S. A class of abstraction of x, denoted [x], is the set of all the elements which are in relation with x. The element x is called a **representative** of the class of abstraction.

Note that if xRy, then [x] = [y]. This means that you can switch representatives within one class of abstraction. In other words - the definition of the class of abstraction does not depend on the choice of the representative.

The classes of abstraction form a **division** of the set S. This means that:

1. the classes of abstraction do not intersect
2. the classes of abstraction fill the whole set S, i.e. every element in S belongs to some class of abstraction.