1. Introduction

We'd like to begin our discussion from defined above polynomial $\mathbf{P}^m_{a,b}(n)$. Polynomial $\mathbf{P}^m_{a,b}(n)$ is $2m+1$ degree polynomial in a, b, n. Polynomial $\mathbf{P}^m_{a,b}(n)$ is defined as finite sum of $2m$ degree polynomial $\mathbf{L}_m(n,k)$ over k. Being a summation, by means of associativity the polynomials $\mathbf{P}^m_{a,b}(n)$ can be split

$$\mathbf{P}^m_{a,b}(n) = \mathbf{P}^m_b(n) - \mathbf{P}^m_a(n).$$

Polynomials $\mathbf{P}^m_{a,b}(n)$ implicitly involve the polynomials $\mathbf{Q}^r_{a,b}(n)$, $\mathbf{X}^m_t(a,b)$, $\mathbf{H}_{m,t}(k)$ and common power sum $S_p(n)$, see Notation and conventions. In extended form the polynomials $\mathbf{P}^m_{a,b}(n)$ are

$$\begin{aligned}
\mathbf{P}^m_{a,b}(n) &= \sum_{a\le k<b} \mathbf{L}_m(n,k) = \sum_k \mathbf{A}_{m,k}\mathbf{Q}^k_{a,b}(n) = \sum_k \mathbf{A}_{m,k}(\mathbf{Q}^k_b(n) - \mathbf{Q}^k_a(n)) \\
&= \sum_k \mathbf{X}^m_k(a,b)(-1)^{m-k}n^k = \sum_k (\mathbf{X}^m_k(b) - \mathbf{X}^m_k(a))(-1)^{m-k}n^k \\
&= \sum_k \sum_{j\ge k} (-1)^{2m+j-k}\mathbf{A}_{m,j}\binom{j}{k}(S_{2j-k}(b) - S_{2j-k}(a))n^k \\
&= \sum_k (-1)^{2m-k} \sum_{\ell=1}^{2m-k+1} \mathbf{H}_{m,k}(\ell)(b^\ell - a^\ell)n^k.
\end{aligned} \tag{1.1}$$

Last line of the expression (1.1) clearly states why $\mathbf{P}^m_{a,b}(n)$ are polynomials in a, b, n. Let's show a few examples of polynomials $\mathbf{P}^m_{a,b}(n)$

$$\begin{aligned}
\mathbf{P}^0_{a,b}(n) &= -a + b. \\
\mathbf{P}^1_{a,b}(n) &= -3a^2 + 2a^3 \\
&\quad + 3b^2 - 2b^3 \\
&\quad + 3an - 3a^2n \\
&\quad - 3bn + 3b^2n. \\
\mathbf{P}^2_{a,b}(n) &= -10a^3 + 15a^4 - 6a^5 \\
&\quad + 10b^3 - 15b^4 + 6b^5 \\
&\quad + 15a^2n - 30a^3n + 15a^4n \\
&\quad - 15b^2n + 30b^3n - 15b^4n \\
&\quad - 5an^2 + 15a^2n^2 - 10a^3n^2 \\
&\quad + 5bn^2 - 15b^2n^2 + 10b^3n^2.
\end{aligned}$$

$$\begin{aligned}\mathbf{P}^2_{a,b}(n) = \quad & 7a^2 - 28a^3 + 70a^5 - 70a^6 + 20a^7 \\ & - 7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 \\ & - 7an + 42a^2n - 175a^4n + 210a^5n - 70a^6n \\ & + 7bn - 42b^2n + 175b^4n - 210b^5n + 70b^6n \\ & - 14an^2 + 140a^3n^2 - 210a^4n^2 + 84a^5n^2 \\ & + 14bn^2 - 140b^3n^2 + 210b^4n^2 - 84b^5n^2 \\ & - 35a^2n^3 + 70a^3n^3 - 35a^4n^3 \\ & + 35b^2n^3 - 70b^3n^3 + 35b^4n^3.\end{aligned}$$

We consider the polynomials $\mathbf{P}^m_{a,b}(n)$ because of their usefulness in revealing the main topic of the work. By means of partial cases of the polynomial $\mathbf{P}^m_{a,b}(n)$ we establish a relation between the power sum $\mathbf{Q}^r_{a,b}(n)$ and Binomial theorem. For instance, odd powers of n are

$$n^{2m+1} = \mathbf{P}^m_n(n) = \sum_k \mathbf{A}_{m,k}\mathbf{Q}^k_n(n), \quad m \geq 0.$$

Moreover, the Binomial expansion $(a+b)^{2m+1}$ of odd powers can be reached similarly

$$(a+b)^{2m+1} = \sum_k \binom{2m+1}{k} a^{2m+1-k}b^k \equiv \mathbf{P}^m_{a+b}(a+b) \equiv \sum_k \mathbf{A}_{m,k}\mathbf{Q}^k_{a+b}(a+b), \quad m \geq 0.$$

It clearly follows that Multinomial expansion of odd-powered t-fold sum $(a_1+a_2+\cdots+a_t)^{2m+1}$ can be reached by $\mathbf{P}^m_{a,b}(n)$ as well

$$\begin{aligned}(a_1+a_2+\cdots+a_t)^{2m+1} &= \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1,k_2,\ldots,k_t} \prod_{s=1}^{t} a_t^{k_t} \\ &\equiv \mathbf{P}^m_{a_1+a_2+\cdots+a_t}(a_1+a_2+\cdots+a_t) \\ &\equiv \sum_k \mathbf{A}_{m,k}\mathbf{Q}^k_{a_1+a_2+\cdots+a_t}(a_1+a_2+\cdots+a_t), \quad m \geq 0.\end{aligned}$$

Since the $n^s = n^{[s \text{ is even}]} n^{\lfloor (s-1)/2 \rfloor}$, it is easy to generalise previously obtained odd power identity for all exponents $s \in \mathbb{N}$

$$n^s = n^{[s \text{ is even}]}\mathbf{P}^{\lfloor \frac{s-1}{2} \rfloor}_n(n) = n^{[s \text{ is even}]} \sum_k \mathbf{A}_{\lfloor \frac{s-1}{2} \rfloor, k}\mathbf{Q}^k_n(n), \quad s > 0. \tag{1.2}$$

The binomial expansion of $(a+b)^s$ for every integer $s>0$ is

$$(a+b)^s = \sum_k \binom{s}{k} a^{s-k} b^k \equiv (a+b)^{[s \text{ is even}]} \mathbf{P}_{a+b}^{\lfloor \frac{s-1}{2} \rfloor}(a+b)$$

$$\equiv (a+b)^{[s \text{ is even}]} \sum_k \mathbf{A}_{\lfloor \frac{s-1}{2} \rfloor, k} \mathbf{Q}_{a+b}^{k}(a+b).$$

Now we are able to generalise the expression (1.2) even more. For the t-fold s-powered sum $(a_1+a_2+\cdots+a_t)^s, \quad s>0$ we have following Multinomial expansion

$$(a_1+a_2+\cdots+a_t)^s = \sum_{k_1+k_2+\cdots+k_t=s} \binom{s}{k_1,k_2,\ldots,k_t} \prod_{\ell=1}^{t} a_\ell^{k_\ell}$$

$$\equiv (a_1+a_2+\cdots+a_t)^{[s \text{ is even}]} \mathbf{P}_{a_1+a_2+\cdots+a_t}^{\lfloor \frac{s-1}{2} \rfloor}(a_1+a_2+\cdots+a_t)$$

$$\equiv (a_1+a_2+\cdots+a_t)^{[s \text{ is even}]} \sum_k \mathbf{A}_{\lfloor \frac{s-1}{2} \rfloor, k} \mathbf{Q}_{a_1+\cdots+a_t}^{k}(a_1+\cdots+a_t).$$