Quantifiers

Sentence function P(x) is every expression containing variables. A sentence function becomes a sentence when we substitute its variables with values.

The operators we were using on sentences can also be used on sentence functions.

Sentence functions can also become sentences when we introduce quantifiers. We have:

- **universal** quantifier $\forall x \in X \ P(x)$ „**for all** x ∈ X"
- **existential** quantifier $\exists x \in X \ P(x)$ „**there exists** such x ∈ X"

Example 1.

By using arithmetic operations and logical symbols write the sentence:

a) The arccos function assumes values from the set (0, π).
b) Every number x ∈ (0,1) is larger than its square.

Answer a)

$\forall x \in R \ 0 < arccos(x) < \pi$

Answer b)

$\forall x \in (0,1) \ x > x^2$

There are logic laws for quantifiers, including **De Morgan laws:**

$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$

$\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$

It is possible to have more than one quantifier in a sentence, for example:

For every country in the world there exists a city which is the capital of that country.

or:

There exists a natural number n that for every real x $\sqrt[n]{x}$ *is not negative.*

The latter sentence can be written $\exists n \in N \ \forall x \in R \ \sqrt[n]{x} \geq 0$

De Morgan laws for two quantifiers look similar to the one quantifier version. We only need to remember to reverse both quantifiers:
$\neg \forall x \in X \ \exists y \in Y \ P(x,y) \Leftrightarrow \exists x \in X \ \forall y \in Y \ \neg P(x,y)$
$\neg \exists x \in X \ \forall y \in Y \ P(x,y) \Leftrightarrow \forall x \in X \ \exists y \in Y \ \neg P(x,y)$

It is possible to inverse the order of two quantifiers of the same type but not of different types.

De Morgan laws can be useful when trying to check the logical value of a sentence and it is not easy to do it straight away. It is sometimes easier to check the value of a negation than the value of the original sentence.

Example. Write the negation of the sentence and check its logical value:
$\forall x \in X \ \forall y \in Y \ x^2 + y^2 \geq 0$

Negation:
$\neg \forall x \in X \ \forall y \in Y \ x^2 + y^2 \geq 0 \Leftrightarrow \exists x \in X \ \exists y \in Y \ x^2 + y^2 < 0$

The negated sentence is false, so the original was true.

Exercise Write the negation of the sentence and check its logical value:
$\forall x \in N \ \exists y \in R \ y^2 = x$