1. Derivation of A(m,r) coeffs

Assuming that

$$n^{2m+1} = \mathbf{P}_n^m(n) = \sum_k \mathbf{A}_{m,k}\mathbf{Q}_n^k(n), \quad m \geq 0.$$

The coefficients $A_{m,r}$ could be evaluated expanding $\sum_{k=0}^{n-1} k^r(n-k)^r$ and using Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$, we get

$$\begin{aligned}
&\sum_{k=0}^{n-1} k^r(n-k)^r \\
&= \sum_{k=0}^{n-1} k^r \sum_j (-1)^j \binom{r}{j} n^{r-j} k^j = \sum_j (-1)^j \binom{r}{j} n^{r-j} \left(\sum_{k=0}^{n-1} k^{r+j}\right) \\
&= \sum_j \binom{r}{j} n^{r-j} \frac{(-1)^j}{r+j+1}\left[\sum_s \binom{r+j+1}{s} B_s n^{r+j+1-s} - B_{r+j+1}\right] \\
&= \sum_{j,s} \binom{r}{j} \frac{(-1)^j}{r+j+1}\binom{r+j+1}{s} B_s n^{2r+1-s} - \sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j} \\
&= \sum_s \underbrace{\sum_j \binom{r}{j} \frac{(-1)^j}{r+j+1}\binom{r+j+1}{s}}_{S(r)} B_s n^{2r+1-s} - \sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}
\end{aligned} \tag{1.1}$$

where B_s are Bernoulli numbers and $B_1 = \frac{1}{2}$. Now, we notice that

$$S(r) = \sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1}\binom{r+j+1}{s} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } s = 0; \\ \frac{(-1)^r}{s}\binom{r}{2r-s+1}, & \text{if } s > 0. \end{cases}$$

In particular, the last sum is zero for $0 < s \leq r$. Therefore, expression (3.1) takes the form

$$\sum_{k=0}^{n-1} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\sum_{s\geq 1} \frac{(-1)^r}{s}\binom{r}{2r-s+1} B_s n^{2r+1-s}}_{(\star)}$$

$$- \underbrace{\sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}}_{(\diamond)}$$

Hence, introducing $\ell = 2r+1-s$ to $(\star)$ and $\ell = r-j$ to $(\diamond)$, we get

$$\begin{aligned}\sum_{k=0}^{n-1} k^r(n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{\ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \\ &\quad - \sum_{\ell}\binom{r}{\ell}\frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\end{aligned}$$

Using the definition of $A_{m,r}$ coefficients, we obtain the following identity for polynomials in n

$$\sum_r A_{m,r}\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{r, \text{ odd } \ell} A_{m,r}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \tag{1.2}$$

Taking the coefficient of n^{2m+1} in (3.2) we get $A_{m,m} = (2m+1)\binom{2m}{m}$ and taking the coefficient of n^{2d+1} for an integer d in the range $m/2 \le d < m$, we get $A_{m,d} = 0$. Taking the coefficient of n^{2d+1} for d in the range $m/4 \le d < m/2$, we get

$$A_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d} B_{2m-2d} = 0,$$

i.e,

$$A_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d} B_{2m-2d}.$$

Continue similarly, we can express $A_{m,d}$ for each integer d in range $m/2^{s+1} \le d < m/2^s$ (iterating consecutively $s = 1, 2...$) via previously determined values of $A_{m,j}$ as follows

$$A_{m,d} = (2d+1)\binom{2d}{d}\sum_{j \ge 2d+1} A_{m,j}\binom{j}{2d+1}\frac{(-1)^{j-1}}{j-d} B_{2j-2d}.$$

Thus, for every $(n, m) \in \mathbb{N}$ holds

$$n^{2m+1} = \sum_{r=0}^{m} A_{m,r}\sum_{k=0}^{n-1} k^r(n-k)^r.$$