1. Homework 3

1. Let $A = \{a, b, c\},\ B = \{a, b, c, d\}$. Write down all the elements of the set:
   - Cartesian product $A \times A$:
   $$A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, c), (c, c)\}$$
   - Cartesian product $A \times B$:
   $$A \times B = \{a, b, c\} \times \{a, b, c, d\} =$$
   $$\{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d)\}$$
   - Cartesian product $B \times A$:
   $$A \times B = \{a, b, c, d\} \times \{a, b, c\} =$$
   $$= \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, a), (d, b), (d, c)\}$$
   - Find the set $\{(x, y) \in A \times B :\ x = y\}$
   $$\{(x, y) \in A \times B :\ x = y\} = \{(a, a), (b, b), (c, c)\}$$
   - Find the set $\{(x, y) \in B \times A :\ x = y\}$
   $$\{(x, y) \in B \times A :\ x = y\} = \{(a, a), (b, b), (c, c)\}$$
2. Check the properties of the following relations in U:
   - $(x, y) \in R,$ if $x + y = 4;\ U = \{0, 1, 2, 3, 4\}$. Denote the relation $x + y$ as xRy. Relation R is the following subset $\mathbf{R}$ of Cartesian product $U \times U$
   $$\mathbf{R} = \{(1, 3), (2, 2), (0, 4), (3, 1), (2, 2), (4, 0)\}$$
     - Is R Reflexive ? To be Reflexive, R must meet the conditions $\forall_{x \in U}\ xRx$ i.e for every $x \in U : (x, x) \in \mathbf{R}$. Thus, relation R is not reflexive, since at least $\{(0, 0), (1, 1), (3, 3), (4, 4)\} \notin \mathbf{R}$.
     - Is R anti-reflexive ? Anti-reflexive condition is $\forall_{x \in U}\ (x, x) \notin \mathbf{R}$. Relation R is not anti-reflexive since $2 \in U,\ (2, 2) \in \mathbf{R}$.
     - Is R Symmetric ? Symmetry condition $\forall_{x \in U,\ y \in U}\ xRy \to yRx$. It could be said as "For each $x, y \in U$ if x is in relation R with y then y is in relation R with x. So, to be hold, for every $x, y \in U$ such that xRy the set $\mathbf{R}$ contains (y, x). In other words, $\forall_{x,y \in U}\ xRy :\ (y, x) \in \mathbf{R}$. Reviewing set $\mathbf{R}$ we can conclude that relation R is reflexive. Commutativity of summation also proves it.
     - Is R anti-symmetrical ? To be anti-symmetrical, R has to meet the condition $\forall_{x,y \in U}\ (xRy \wedge yRx) \to x = y$. By the commutativity of summation, the R is not anti-reflexive. Example of anti-reflexive relation $xRy = x^y$.
     - Is the R transitive? No, since there no one triple of x, y, z, such that $\forall_{x,y,z \in U}\ (xRy \vee yRz) \to xRz$
     - Connexity of relation the R is given by the condition $\forall_{x,y \in U}\ (xRy \vee yRx)$, and it is true.
   - $(x, y) \in R,$ if $x + y = 6;\ U = \{1, 2, 3, 4, 5\}$. Denote the relation $x + y$ as xRy. Relation R is the following subset $\mathbf{R}$ of Cartesian product $U \times U$
   $$\mathbf{R} = \{(1, 5), (2, 4), (3, 3), (5, 1), (4, 2), (3, 3)\}$$
     - Reflexive - not
     - Anti-reflexive - not
     - Symmetric - yes
     - Anti-Symmetric - not
     - Transitive - not
     - Connex - not

- $(x, y) \in R,$ if $x + y \leq 4;$ $U = \{0, 1, 2, 3\}$. Term x in relation R with y iff $x + y \leq 4$

  $$U \times U = \{\underline{(0,0)}, \underline{(0,1)}, \underline{(0,2)}, \underline{(0,3)}, \underline{(1,0)}, \underline{(1,1)}, \underline{(1,2)}, \underline{(1,3)},$$
  $$\underline{(2,0)}, \underline{(2,1)}, \underline{(2,2)}, (2,3), \underline{(3,0)}, \underline{(3,1)}, (3,2), (3,3)\}$$

  Therefore, relation R represents the following set $\mathbf{R}$ such that $\mathbf{R} \in U \times U$

  $$\mathbf{R} = \{\underline{(0,0)}, \underline{(0,1)}, \underline{(0,2)}, \underline{(0,3)}, \underline{(1,0)}, \underline{(1,1)}, \underline{(1,2)}, \underline{(1,3)}, \underline{(2,0)}, \underline{(2,1)}, \underline{(2,2)}, \underline{(3,0)}, \underline{(3,1)}\}$$

  - Relation R is not reflexive since $(3, 3) \notin \mathbf{R}$.
  - Relation R is not anti-reflexive since $0, 1, 2 \in U,$ $\{(0, 0), (1, 1), (2, 2)\} \subset \mathbf{R}$.
  - Relation R is symmetric since $\forall_{x,y \in U}\ xRy :\ (y, x) \in \mathbf{R}$.
  - Relation R is not anti-symmetrical since summation operator is commutative.
  - Relation R is transitive since $\forall_{x,y,z \in U}\ (xRy \wedge yRz) \rightarrow xRz$. For example, $x = 0,$ $y = 1,$ $z = 2$ then $(0, 1), (1, 2) \rightarrow (2, 0)$.
  - Relation R is connex.
- $(x, y) \in R$ if $x - y$ is even. $U = \{2, 4, 6\}$. Let write Cartesian product of U

  $$U \times U = \{\underline{(2,2)}, (2,4), (2,6), \underline{(4,2)}, \underline{(4,4)}, (4,6), \underline{(6,2)}, \underline{(6,4)}, \underline{(6,6)}\}.$$

  Relation R gives the following set $\mathbf{R} \subset U \times U$

  $$\mathbf{R} = \{\underline{(2,2)}, \underline{(4,2)}, \underline{(4,4)}, \underline{(6,2)}, \underline{(6,4)}, \underline{(6,6)}\}$$

  - Relation R is reflexive since $\forall_{x \in U}\ xRx :\ (x, x) \in \mathbf{R}$.
  - Relation R is not anti-reflexive.
  - Relation R is not symmetric since $\urcorner(\exists_{x,y}\ xRy) :\ (y, x) \in \mathbf{R}$.
  - Relation R is anti-symmetric since $\forall_{x,y \in U}\ (xRy \wedge yRx) \rightarrow x = y$.
  - Relation R is transitive since $\forall_{x,y,z \in U}\ (xRy \wedge yRz) \rightarrow zRy =$ false.