Discrete Mathematics
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Basic mathematical logic

Propositional calculus

Logic is the systematic study of the form of valid inference, and the most general laws of truth. (Gottlob Frege)

Inference is a process of obtaining a conclusion from a set of premises. Logic is concerned with inferring true statements from other statements.

Propositional calculus is a formal system that deals with propositions in a mathematical way. **Proposition** is expressed by a declarative logical sentence and can either be true or false. For every proposition It must be possible to determine whether it is true or false regardless of any circumstances or external conditions. That is to say that a proposition must either *always* be true or *always* be false.

Examples of propositions:

- ✓ Warsaw was the capital of Poland in 2018.
- ✓ Earth orbits around the sun.
- ✓ Every odd number is divisable by 2. *(this proposition is always false).*
- ✓ It is not true that $x \geq 3$ for every natural x.

Examples of sentences which do not express a proposition

- ✓ Come to the blackboard!
- ✓ Do you own a dog?
- ✓ Lawyers are rich.
- ✓ $x - y = y - x$

Propositions in logic are typically denoted with lowercase letters p,q,r. We will be joining them together with standard operators:

$\sim$ negation

$\vee$ alternative (or)

$\wedge$ conjunction (and)

$\rightarrow$ implication (if.. then…)

$\leftrightarrow$ equivalence „…if and only if…”

$\oplus$ exclusive alternative XOR

| Sheffer stroke NAND

$\downarrow$ Pierce’s arrow NOR

Tables for the operators

negation	
p	$\sim p$
0	1
1	0

0	1	0
1	0	0
1	1	1

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

alternative

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

implication

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

exclusive or

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

conjunction

p	q	$p \wedge q$
0	0	0

equivalence

Sheffer stroke

p	q	$p \vert q$
0	0	1
0	1	1
1	0	1
1	1	0

Pierce's arrow

p	q	$p \downarrow q$
0	0	1
0	1	0
1	0	0
1	1	0

An **inverse** proposition for $p \to q$ is $q \to p$. A **contraposition** of $p \to q$ is $(\sim q) \to (\sim p)$.

Logical matrix of a complex sentence built from simple propositions p,q,r, .. is a table of all possible logical values of the sentences calculated from the values of the propositions.

<u>Example 1</u>

Sentence $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$\wedge$	$q \to p$
0	0	1	1	1
0	1	1	0	0
1	0	0	0	1
1	1	1	1	1

is equivalent to $p \leftrightarrow q$.

<u>Example 2</u>

Logical matrix of $(p \to q) \wedge [(q \to \sim r) \to (p \vee r)]$

p	q	r	$p \to q$	$\wedge$	$[(q \to \sim r) \to (p \vee r)]$
0	0	0	1	0	1 1 0 0

0 0 1	1	1	1 0 1 1
0 1 0	1	0	1 1 0 0
0 1 1	1	1	0 0 1 1
1 0 0	0	0	1 1 1 1
1 0 1	0	0	1 0 1 1
1 1 0	1	1	1 1 1 1
1 1 1	1	1	0 0 1 1

Tautology is a sentence which is always true for every value of the p,q, r, .. variables.

Conversely, a sentence which is always false is a **contradiction**. Negation of any tautology is a contradiction.

Equivalent sentences (laws of logic)

1. double negation

$(\sim\sim p) \Leftrightarrow p$

2. laws of commutation

$$(p \vee q) \Leftrightarrow (q \vee p)$$

$$(p \wedge q) \Leftrightarrow (q \wedge p)$$

$(p \leftrightarrow q) \Leftrightarrow (q \leftrightarrow p)$

3. laws of association

$$[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$$

$[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$

4. laws of distribution

$$[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$$

$[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$

5. laws of idempotence

$$(p \vee p) \Leftrightarrow p$$

$(p \wedge p) \Leftrightarrow p$

6. laws of identity

$(p \vee 0) \Leftrightarrow p$
$(p \vee 1) \Leftrightarrow 1$
$(p \wedge 0) \Leftrightarrow 0$

7. law of excluded middle

$$(p \vee \sim p) \Leftrightarrow 1$$

$$(p \wedge \sim p) \Leftrightarrow 0$$

8. De Morgan laws for negation

$$\sim(p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$$

$\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$
$(p \vee q) \Leftrightarrow \sim(\sim p \wedge \sim q)$
$(p \wedge q) \Leftrightarrow \sim(\sim p \vee \sim q)$

9. law of contraposition (transposition)

$(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$

10. reformulating implication with alternative or conjunction

$(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$
$(p \rightarrow q) \Leftrightarrow \sim(p \wedge \sim q)$

11. $(p \vee q) \Leftrightarrow (\sim p \rightarrow q)$

$$(p \wedge q) \Leftrightarrow \sim(p \rightarrow \sim q)$$

12. $[(p \rightarrow r) \wedge (q \rightarrow r)] \Leftrightarrow [(p \vee q) \rightarrow r]$

$$[(p \rightarrow q) \wedge (p \rightarrow r)] \Leftrightarrow [p \rightarrow (q \wedge r)]$$

13. definition of equivalence

$(p \leftrightarrow q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

14. exportation law

$[(p \wedge q) \rightarrow r] \Leftrightarrow [p \rightarrow (q \rightarrow r)]$

15. reductio ad absurdum

$(p \rightarrow q) \Leftrightarrow [(p \wedge \sim q) \rightarrow 0]$

Two logical sentences P and Q are **logically equivalent** if they assume the same logical values for every set of values of their propositions p,q,r...

We denote two logically equivalent sentences with $P \Leftrightarrow Q$.

$$P \Leftrightarrow Q \textit{ if and only if } P \leftrightarrow Q \textit{ is a tautology.}$$

Sentence P **implies** Q when Q always has the value 1 when P has the value 1. We denote $P \Rightarrow Q$.

$$P \Rightarrow Q \textit{ if and only if } P \rightarrow Q \textit{ is a tautology.}$$

Logical implications

16. $p \Rightarrow (p \vee q)$ introduction of alternative

17. $(p \wedge q) \Rightarrow p$ omitting conjunction

18. $(p \to 0) \Rightarrow \sim p$ reductio ad absurdum

19. $[p \wedge (p \to q)] \Rightarrow q$ modus ponendo ponens

20. $[(p \to q) \wedge \sim q] \Rightarrow \sim p$ modus tollendo tollens

21. $[(p \vee q) \wedge \sim p] \Rightarrow q$ modus ponendo tollens

22. $p \Rightarrow [q \to (p \wedge q)]$

23. $[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \Rightarrow (p \leftrightarrow r)$ transitivity of$\leftrightarrow$

24. $[(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$ transitivity of $\to$