1. Homework 3

In mathematics, an equivalence relation is a binary relation that is reflexive, symmetric and transitive. For instance,

- Reflexiveness condition of relation R over set U

$$\forall_{x \in U}\ xRx \ \to\ \forall_{x \in U} :\ (x, x) \in \mathbf{R},$$

  where $\mathbf{R}$ is subset of Cartesian product $U \times U$. We can denote $\mathbf{R}$ as well as $xRy\colon U \mapsto \mathbf{R}$
- Symmetry of relation R is given by

$$\forall_{x,y \in U}\ xRy \ \to\ yRx$$

  It implies that

$$\forall_{x,y \in U}\ \{(x, y), (y, x)\} \subset \mathbf{R}$$

- Transitivity of relation R over given set U is given by

$$\forall_{x,y,z \in U}\ (xRy \wedge yRz) \ \to\ xRz$$

2. Task 1.

Let's define relations R_1 and R_2 in the set of all straight lines on a plane $\mathbb{R}$:

- Two lines a, b are in relation R_1 if they are parallel.
- Two lines a, b are in relation R_2 if they are perpendicular.

Case 2.1. *Two lines a, b are in relation R_1 if they are parallel.*

We examining if the relation R_1 is equivalence.

- Relation R_1 is reflexive, since $\forall_{x \in \mathbb{R}}\ (x \parallel x)$.
- Relation R_1 is symmetric, since $\forall_{x,y \in \mathbb{R}}\ (x \parallel y) \to (y \parallel x)$.
- Relation R_1 is transitive, since $\forall_{x,y,z \in \mathbb{R}}\ (x \parallel y) \wedge (y \parallel z) \to x \parallel z$.

Thus, relation R_1 is equivalence relation.

Case 2.2. *Two lines a, b are in relation R_2 if they are perpendicular.*

- Relation R_2 is not reflexive, since $\forall_{x \in \mathbb{R}}\ (x \perp x)$ is false.
- Relation R_2 is symmetric, since $\forall_{x,y \in \mathbb{R}}\ (x \perp y) \to (y \perp x)$.
- Relation R_2 is not transitive, since $\forall_{x,y,z \in \mathbb{R}}\ (x \perp y) \wedge (y \perp z) \to x \parallel z$.

3. Task 2.

Let be a set $U = \{0, 1, 2, 3, 4\}$. Define the following relations within U:

- $(x, y) \in R_1$ if $2x = y$.
- $(x, y) \in R_2$ if $x - y = 0$.

Case 3.1. $(x, y) \in R_1$ *if* $2x = y$.

The following Matrix shows the relation R_1 over set U

xR_1y	0	1	2	3	4
0	1	0	0	0	0
1	0	0	1	0	0
2	0	0	0	0	1
3	0	0	0	0	0
4	0	0	0	0	0

- Relation R_1 is not reflexive since main diagonal is not totally consists of 1's.

- Relation R_1 is not symmetric, since Matrix is not symmetric with respect to main diagonal.
- Relation R_1 is not transitive, since $\forall_{x,y,z\in U}\ (xRy \wedge yRz) \not\to xRz$.

Thus, relation R_1 is not equivalence.

Case 3.2. $(x, y) \in R_2$ *if* $x - y = 0$.

The following Matrix shows the relation R_1 over set U

xR_2y	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	0	1	0	0
3	0	0	0	1	0
4	0	0	0	0	1

- Relation R_2 is reflexive since main diagonal is totally consists of 1's.
- Relation R_2 is symmetric since the matrix is symmetric over main diagonal.
- Relation R_2 is not transitive, since iff $x = y \to x-y = 0$, thus $\forall_{x,y,z\in U}(xR_2y\wedge yR_2z) \not\to xR_2z$. Unless $U \neq \{const\}$.

Thus, relation R_2 is not equivalence.

4. Task 3.

Give an examples of equivalence relations in sets of:

- Computer Science Students, students with same average score is equivalence relation.
- Cars, cars with the same color is equivalence relation.
- Citizens of Europe, people with passport of EU with same age is equivalence relation.
- Natural Numbers, two parallel vectors (a, b) on the plane $\mathbb{N}$.

5. Task 4.

Describe all equivalence classes in the modulo 4 over the set of natural numbers.

$$[0] = \{0, 4, 8, 12, ..., \} = \{0 + 4k|k \in \mathbb{N}\}$$
$$[1] = \{1, 5, 9, 13, ..., \} = \{1 + 4k|k \in \mathbb{N}\}$$
$$[2] = \{2, 6, 10, 14, ..., \} = \{2 + 4k|k \in \mathbb{N}\}$$
$$[3] = \{3, 7, 11, 15, ..., \} = \{3 + 4k|k \in \mathbb{N}\}$$