

# SUPPLEMENTARY TO ARXIV:1603.02468 - ARXIV160302468 MATHEMATICA PACKAGE DOCUMENTATION

PETRO KOLOSOV

ABSTRACT. This PDF presents a brief description of Mathematica programs, related to preprint arXiv:1603.02468.

## CONTENTS

1. Introduction	1
2. CoeffA	1
3. TriangleCoeffA	2
4. ColumnTriangleA	2
5. DiagonalTriangleA	2
6. CenteredColumnTriangleA	2
7. L	3
8. TriangleL	3
9. RowTriangleL	3
10. ColumnTriangleL	4
11. ColumnSumTriangleL	4
12. CenteredColumnTriangleL	4
13. GeneralizedTriangleL	4
14. RowGeneralizedTriangleL	5
15. RowSumGeneralizedTriangleL	5
16. OddPowerIdentity	5
17. ClosedFormOddPowerIdentity	5
18. ClosedFormOddPowerIdentityList	6
19. ClosedFormCoefficient	6
20. ClosedFormCoefficientList	6
21. ClosedFormCoefficientColumn	7
22. BinomialTriangle	7
23. Numerical	7
24. CoeffLamda	8
25. LamdaOddPowerIdentity	8
26. LamdaOddPowerIdentityClosedForm	8
27. Program 4 - Convolution tables and their application	8
28. Program 5 - Closed forms of identity of $n^{2m+1}$ , polynomials	8
29. Program 6 - Evaluating coefficients of Closed forms	8
30. Program 7 - Graphs of closed forms of identity of $n^{2m+1}$	8
31. Not Present in OEIS sequences	8
References	8

## 1. INTRODUCTION

### 2. COEFFA

**Description.** CoeffA[n,k] produces the coefficient  $A[n, k]$ .

**Definition.** CoeffA[n,k] is defined regarding to formula (number) in arxiv1603version

```
In[1] := CoeffA[3, 1]
Out[1] = -14
```

### 3. TRIANGLECOEFFA

**Description.** `TriangleCoeffA[j]` prints a triangular array of coefficients `CoeffA[n,k]` consisting of  $j$ -rows.

**Definition.** `TriangleCoeffA[j_]` := `Column[Table[CoeffA[n, k], {n, 0, j}, {k, 0, n}], Left];`

```
In[1] := TriangleCoeffA[5]
Out[1] =
  {1},
  {1, 6},
  {1, 0, 30},
  {1, -14, 0, 140},
  {1, -120, 0, 0, 630},
  {1, -1386, 660, 0, 0, 2772}
```

### 4. COLUMNTRIANGLEA

**Description.** `ColumnTriangleA[j, t]` prints  $t$  terms of  $j$ -th column of `TriangleCoeffA[h]`.

**Definition.** `ColumnTriangleA[j_, t_]` := `Column[Table[CoeffA[n, j], {n, j, t}], Left];`

```
In[1] := ColumnTriangleA[3, 10]
Out[1] = {140, 0, 0, 0, -60060, -3712800, -196409840, -10863652800}
```

### 5. DIAGONALTRIANGLEA

**Description.** `DiagonalTriangleA[r,t]` prints  $t$  items of  $r$ -th diagonal of `TriangleCoeffA[h]`.

**Definition.** `DiagonalTriangleA[r_, t_]` := `Table[CoeffA[j + r, j], {j, 0, t}];`

**Inputs.**

```
In[1] := DiagonalTriangleA[3, 10]
Out[1] = {1, -120, 660, 0, 0, 0, 0, 0, 0, 0, 0}
```

### 6. CENTEREDCOLUMNTRIANGLEA

**Description.** `CenteredColumnTriangleA[r,t]` prints  $t$  items of  $r$ -th centered column of `TriangleCoeffA[h]`.

**Definition.** `CenteredColumnTriangleA[r_, t_]` := `Table[CoeffA[2 n + r, n], {n, 0, t}];`

```

In[1] := CenteredColumnTriangleA[2, 10]
Out[1] = {1, -120, 18018, -3712800, 1031151660, -374796021600, 173441819530980,
          -(698546939374627200/7), 69970095348694681140,
          -58779399490200841452000, 58311848381539710311691480}

```

## 7. L

**Description.**  $L[m, n, k]$  gives an integer value of polynomial  $L$  with respect to integers  $m, n, k$ .

**Definition.**  $L[m_-, n_-, k_-] := \text{Sum}[\text{CoeffA}[m, r] * k^r * (n - k)^r, \{r, 0, m\}];$

```

In[1] := L[1, 10, 5]
Out[1] = 151

```

## 8. TRIANGLEL

**Description.**  $\text{TriangleL}[m, t]$  generates numerical triangle of  $t$ -rows filled by  $L[m, n, k]$ .

**Definition.**  $\text{TriangleL}[m_-, t_-] := \text{Column}[\text{Table}[L[m, n, k], \{n, 0, t\}, \{k, 0, n\}], \text{Left}];$

```

In[1] := TriangleL[1, 7]
Out[1] =
  {1},
  {1, 1},
  {1, 7, 1},
  {1, 13, 13, 1},
  {1, 19, 25, 19, 1},
  {1, 25, 37, 37, 25, 1},
  {1, 31, 49, 55, 49, 31, 1},
  {1, 37, 61, 73, 73, 61, 37, 1}

```

## 9. ROWTRIANGLEL

**Description.**  $\text{RowTriangleL}[m, n]$  prints  $n$ -th row of  $\text{TriangleL}[m, t], t \geq n$ .

**Definition.**  $\text{RowTriangleL}[m_-, n_-] := \text{Table}[L[m, n, k], \{k, 0, n\}];$

```

In[1] := RowTriangleL[1, 5]
Out[1] = {1, 25, 37, 37, 25, 1}

```

## 10. COLUMNTRIANGLEL

**Description.** `ColumnTriangleL[m, k, t]` prints the  $t$  terms of  $k$ -th column of `TriangleL[m, n]`,  $n \geq k$ .

**Definition.** `ColumnTriangleL[m_, k_, t_] := Table[L[m, n, k], {n, k, t+k}];`

```
In[1] := ColumnTriangleL[1, 2, 10]
Out[1] = {1, 13, 25, 37, 49, 61, 73, 85, 97, 109, 121}
```

## 11. COLUMNSUMTRIANGLEL

**Description.** `ColumnSumTriangleL[m, r, s]` gives the partial sums of `ColumnTriangleL[m, k, t]` over  $k$  from 0 to  $s$ .

**Definition.** `ColumnSumTriangleL[m_, r_, s_] := Table[Sum[L[m, k, r], {k, r, t}], {t, r, s+r}];`

```
In[1] := ColumnSumTriangleL[1, 2, 10]
Out[1] = {1, 14, 39, 76, 125, 186, 259, 344, 441, 550, 671}
```

## 12. CENTEREDCOLUMNTRIANGLEL

**Description.** `CenteredColumnTriangleL[m, r, t]` gives  $t$  terms of  $r$ -th centered column of `TriangleL[m, t]`.

**Definition.** `CenteredColumnTriangleL[m_, r_, t_] := Table[L[m, 2 n + r, n], {n, 0, t}];`

```
In[1] := CenteredColumnTriangleL[1, 2, 10]
Out[1] = {1, 19, 49, 91, 145, 211, 289, 379, 481, 595, 721}
```

## 13. GENERALIZEDTRIANGLEL

**Description.** `GeneralizedTriangleL[m, t, radius]` gives a generalized `TriangleL[m, t]` with rows from  $-radius$  to  $+radius$ .

**Definition.**

`GeneralizedTriangleL[m_, t_, radius_] := Column[Table[L[m, n, k], {n, 0, t}, {k, -radius, n+radius}], Left];`

```
In[1] := GeneralizedTriangleL[1, 7, 1]
Out[1] =
  {-5, 1, -5},
  {-11, 1, 1, -11},
  {-17, 1, 7, 1, -17},
  {-23, 1, 13, 13, 1, -23},
  {-29, 1, 19, 25, 19, 1, -29},
  {-35, 1, 25, 37, 37, 25, 1, -35},
  {-41, 1, 31, 49, 55, 49, 31, 1, -41},
  {-47, 1, 37, 61, 73, 73, 61, 37, 1, -47}
```

## 14. ROWGENERALIZEDTRIANGLEL

**Description.** RowGeneralizedTriangleL[m, t, radius] gives a t-th row of GeneralizedTriangleL[m, t, radius] from -radius to +rad.

**Definition.** RowGeneralizedTriangleL[m\_, n\_, radius\_] := Table[L[m, n, s], {s, -radius, n+radius}];

```
In[1] := RowGeneralizedTriangleL[1, 2, 1]
Out[1] = {-17, 1, 7, 1, -17}
```

## 15. ROWSUMGENERALIZEDTRIANGLEL

**Description.** RowSumGeneralizedTriangleL[m, n, radius] gives the sum of n-th row of GeneralizedTriangleL[m, t, radius] with radius.

**Definition.** RowSumGeneralizedTriangleL[m\_, n\_, radius\_] := Sum[L[m, n, k], {k, -radius, n+radius}];

```
In[1] := RowSumGeneralizedTriangleL[1, 3, 0]
Out[1] = 28
```

## 16. ODDPOWERIDENTITY

**Description.** OddPowerIdentity[n, m] gives integer  $n^{(2m+1)}$ .

**Definition.** OddPowerIdentity[n\_, m\_] := Sum[CoeffA[m, r]\*Sum[k^r(n-k)^r, {k, 0, n}], {r, 0, m}];

```
In[1] := OddPowerIdentity[X, 2]
Out[1] = 1 + X + X (1 + X) (-1 + X - X^2 + X^3) = X^5 + 1
```

$$\frac{1}{X+1} \sum_{r=0}^m A_{m,r} \text{Conv}_{r,N}[X] = \frac{X^5+1}{X+1} = \sum_{r=0}^{2m-2} (-1)^r X^r$$

## 17. CLOSEDFORMODDPOWERIDENTITY

**Description.** ClosedFormOddPowerIdentity[m, T] gives closed form of identity for any particular natural T.

**Definition.** ClosedFormOddPowerIdentity[m\_, n\_, T\_] := Expand[Sum[L[m, n, k], {k, 0, T}]];

```
In[1] := ClosedFormOddPowerIdentity[1, n, T]
Out[1] = 1 + 3 n T - 3 T^2 + 3 n T^2 - 2 T^3
In[2] := ClosedFormOddPowerIdentity[1, n + 1, n]
Out[2] = 1 + 3 n + 3 n^2 + n^3
```

$$\text{ClosedFormOddPowerIdentity}[1, a+b, a+b-1] = \sum_j \binom{2m+1}{j} a^{2m+1-k} b^k = (a+b)^{2m+1}$$

## 18. CLOSEDFORMODDPOWERIDENTITYLIST

**Description.** `ClosedFormOddPowerIdentityList[m, n, t]` generates a list of `ClosedFormOddPowerIdentity[m, T]` containing `t` terms.

**Definition.**

`ClosedFormOddPowerIdentityList[m_, n_, t_]`  
`:= Column[Table[ClosedFormOddPowerIdentity[m, n, f], {f, 0, t}], Left];`

**In[1]** := `ClosedFormOddPowerIdentityList[2, n, 7]`

**Out[1]** =  
 $\{1\},$   
 $\{32 - 60n + 30n^2\},$   
 $\{513 - 540n + 150n^2\},$   
 $\{2944 - 2160n + 420n^2\},$   
 $\{10625 - 6000n + 900n^2\},$   
 $\{29376 - 13500n + 1650n^2\},$   
 $\{68257 - 26460n + 2730n^2\},$   
 $\{140288 - 47040n + 4200n^2\}$

## 19. CLOSEDFORMCOEFFICIENT

**Description.** `ClosedFormCoefficient[m, l, t]` prints the coefficient of `l`-th power in `ClosedFormOddPowerIdentity[m, T]` for any given `T`.

**Definition.**

`ClosedFormCoefficient[m_, l_, t_]` :=  
 $(-1)^m \text{Sum}[\text{Sum}[\text{Binomial}[j, t] \text{CoeffA}[m, j] k^{(2j - t)} (-1)^j, \{j, t, m\}], \{k, 0, 1\}];$

**In[1]** := `ClosedFormCoefficient[3, 2, 1]`

**Out[1]** = 13818

## 20. CLOSEDFORMCOEFFICIENTLIST

**Description.** `ClosedFormCoefficientList[m, r]` prints `r` lines of `ClosedFormCoefficient[m, l, t]`.

**Definition.**

`ClosedFormCoefficientList[m_, r_]`  
`:= Column[Table[ClosedFormCoefficient[m, l, t], {l, 0, r}, {t, 0, m}], Left];`

**In[1]** := `ClosedFormCoefficientList[2, 10]`

**Out[1]** =  
 $\{1, 0, 0\},$   
 $\{32, 60, 30\},$   
 $\{513, 540, 150\},$   
 $\{2944, 2160, 420\},$   
 $\{10625, 6000, 900\},$   
 $\{29376, 13500, 1650\},$

```
{68257, 26460, 2730},
{140288, 47040, 4200},
{263169, 77760, 6120},
{460000, 121500, 8550},
{760001, 181500, 11550}
```

## 21. CLOSEDFORMCOEFFICIENTCOLUMN

**Description.** `ClosedFormCoefficientColumn[m, t, r]` gives column of `ClosedFormCoefficientList[m, r]` of power  $t \leq m$ .

**Definition.** `ClosedFormCoefficientColumn[m_, t_, r_] := Table[ClosedFormCoefficient[m, l, t], {l, 0, r}];`

```
In[1] := ClosedFormCoefficientColumn[2, 0, 10]
Out[1] = {1, 32, 513, 2944, 10625, 29376, 68257, 140288, 263169, 460000, 760001}
```

## 22. BINOMIALTRIANGLE

**Description.** `BinomialTriangle[n, s]` prints binomial triangle of  $s$  rows for any integer  $n$ .

**Definition.** `BinomialTriangle[s_, t_] := Column[Table[Binomial[n, k] * s^k, {n, 0, t}, {k, 0, n}], Center];`

```
In[1] := BinomialTriangle[2, 5]
Out[1] =
{1},
{1, 2},
{1, 4, 4},
{1, 6, 12, 8},
{1, 8, 24, 32, 16},
{1, 10, 40, 80, 80, 32}
```

Generalization

$$n^m = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{n-1}} \binom{m}{k_1} \binom{k_1}{k_2} \cdots \binom{k_{n-2}}{k_{n-1}}$$

## 23. NUMERICAL

**Description.** `Numerical[n, k]` gives terms of numerical expansion of monomials.”

**Definition.**

```
Numerical[n_, k_] := 0
Numerical[n_, k_] := 1 /; k == 0 || k == n
Numerical[n_, k_] := Sum[n^s, {s, 0, n-1}] /; 0 < k < n
```

```
In[1] := Numerical[4, 2]
Out[1] = 85
```

## 24. COEFFLAMDA

**Description.** `CoeffLamda[m, r, k]` gives Lamda coefficients from definition 1.6 at [https://kolosovpetro.github.io/pdf/faulhabers/formula\\_binomial\\_theorem\\_identity.pdf](https://kolosovpetro.github.io/pdf/faulhabers/formula_binomial_theorem_identity.pdf).

**Definition.** `CoeffLamda[m_, r_, k_] := Sum[Binomial[j, r] * CoeffA[m, j] * (-1)^j / (2j - r + 1) * Binomial[2j - r + 1, k] * BernoulliB[2j - r + 1 - k], {j, r, m}];`

```
In[1] := CoeffLamda[2, 2, 1]
Out[1] = 5
```

## 25. LAMDAODDPOWERIDENTITY

**Description.** `LamdaOddPowerIdentity[T, 1, s]` gives a closed form of odd power identity  $T^{2s+1}$  for two variables  $T, l$  from [https://kolosovpetro.github.io/pdf/faulhabers\\_formula\\_binomial\\_theorem\\_identity.pdf](https://kolosovpetro.github.io/pdf/faulhabers_formula_binomial_theorem_identity.pdf).

**Definition.** `LamdaOddPowerIdentity[T_, s_] := Sum[Sum[(-1)^(2s-r) * CoeffLamda[s, r, k] * T^(k+r), {k, 1, 2s+1-r}], {r, 0, s}];`

```
In[1] := LamdaOddPowerIdentity[T, 1]
Out[1] = T^3
```

## 26. LAMDAODDPOWERIDENTITYCLOSEDFORM

**Description.** `LamdaOddPowerIdentity[T, l, s]` gives a closed form of odd power identity  $T^{2s+1}$  for two variables  $T, l$  from [https://kolosovpetro.github.io/pdf/faulhabers\\_formula\\_binomial\\_theorem\\_identity.pdf](https://kolosovpetro.github.io/pdf/faulhabers_formula_binomial_theorem_identity.pdf).

**Definition.** `LamdaOddPowerIdentityClosedForm[T_, l_, s_] := Sum[Sum[(-1)^(2s-r) * CoeffLamda[s, r, k] * T^r * l^k, {k, 1, 2s+1-r}], {r, 0, s}];`

```
In[1] := LamdaOddPowerIdentityClosedForm[T, S, 1]
Out[1] = 3 S^2 - 2 S^3 - 3 S T + 3 S^2 T
```

## 27. PROGRAM 4 - CONVOLUTION TABLES AND THEIR APPLICATION

28. PROGRAM 5 - CLOSED FORMS OF IDENTITY OF  $n^{2m+1}$ , POLYNOMIALS

## 29. PROGRAM 6 - EVALUATING COEFFICIENTS OF CLOSED FORMS

30. PROGRAM 7 - GRAPHS OF CLOSED FORMS OF IDENTITY OF  $n^{2m+1}$ 

## 31. NOT PRESENT IN OEIS SEQUENCES

## REFERENCES

[1] arXiv:1603.02468v18 [math.NT]

[2] arXiv:1603.02468v15 [math.NT]