1. Notation and conventions

We now set the following notation, which remains fixed for the remainder of this paper:

- We strongly believe to D. Knuth's words in [citation]

  I realized long ago that "boundary conditions" on indices of summation are often a handicap and a waste of time.

  For example, instead of writing
  $$2^n = \sum_{k=0}^{n} \binom{n}{k}$$
  it is much better to write
  $$2^n = \sum_{k} \binom{n}{k}$$
  the sum now extends over all integers k, but only finitely many terms are nonzero.

- We believe to [citation] that exponential function 0^x should be defined for all x as
  $$0^x = 1.$$

- Iverson's convention $[P(k)]$, where $P(k)$ is logical sentence depending on k
  $$[P(k)] = \begin{cases} 1, & P(k) \textit{ is true}, \\ 0, & \textit{otherwise} \end{cases}$$

- $f^r(n)$ is a power function defined piecewise by means of Iverson's convention
  $$f^r(n) := n^r [n \geq 0], \quad n \in \mathbb{N}.$$

- Discrete convolution transform $(f * f)[n]$ of function $f(n)$
  $$(f * f)[n] = \sum_{k} f[k] f[n-k].$$

- $\mathbf{A}_{m,r}$ is a real coefficient defined recursively as
  $$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m, \\ (2r+1)\binom{2r}{r} \sum\limits_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m, \\ 0, & \text{if } r < 0 \text{ or } r > m. \end{cases}$$
  where B_t are Bernoulli numbers. We assume that $B_1 = \frac{1}{2}$. For $m \geq 11$ the $\mathbf{A}_{m,r}$ takes the fractional values for certain r.

- $\mathbf{Q}^r_{a,b}(n)$ is the power sum defined as
  $$\mathbf{Q}^r_{a,b}(n) := \sum_{a \leq k < b} k^r (n-k)^r, \quad (n,r) \in \mathbb{Z}.$$
  Notation $\mathbf{Q}^r_b(n)$ is an equivalent to $\mathbf{Q}^r_{a,b}(n)$ with set $a = 0$, i.e $\mathbf{Q}^r_b(n) \equiv \mathbf{Q}^r_{0,b}(n)$.

- $S_p(n)$ is a common power sum [possible cite of mathworld]
$$S_p(n) = \sum_{0 \le k < n} k^p.$$
- $\mathbf{L}_m(n,k)$ are polynomials of degree $2m$ in n,k defined involving coefficients $\mathbf{A}_{m,r}$
$$\mathbf{L}_m(n,k) := \sum_r \mathbf{A}_{m,r} k^r (n-k)^r, \quad m \in \mathbb{N}, \quad (n,k) \in \mathbb{Z}.$$
- $\mathbf{P}^m_{a,b}(n)$ are polynomials of degree $2m$ in a,b,n. Polynomials $\mathbf{P}^m_{a,b}(n)$ are defined as a sum of $\mathbf{L}_m(n,k)$ over $k \in [a,..,b]$,
$$\mathbf{P}^m_{a,b}(n) := \sum_{a \le k < b} \mathbf{L}_m(n,k), \quad n \in \mathbb{Z}.$$
Notation $\mathbf{P}^m_b(n)$ is an equivalent to $\mathbf{P}^m_{a,b}(n)$ with set $a = 0$, i.e $\mathbf{P}^m_b(n) \equiv \mathbf{P}^m_{0,b}(n)$.
- $\mathbf{X}^m_t(a,b)$ are polynomials of degree $2m - t$ in a,b defined as
$$\mathbf{X}^m_t(a,b) := (-1)^m \sum_{j \ge t} \mathbf{A}_{m,j} (-1)^j \binom{j}{t} \sum_{a \le k < b} k^{2j-t}, \quad 0 \le t \le m.$$
Notation $\mathbf{X}^m_t(b)$ is an equivalent to $\mathbf{X}^m_t(a,b)$ with set $a = 0$, i.e $\mathbf{X}^m_t(b) \equiv \mathbf{X}^m_t(0,b)$.
- $\mathbf{H}_{m,t}(k)$ are real coefficients defined in terms of Bernoulli numbers B_t, Binomial coefficients $\binom{t}{k}$ and $\mathbf{A}_{m,r}$
$$\mathbf{H}_{m,t}(k) := \sum_{j \ge t} \binom{j}{t} \mathbf{A}_{m,j} \frac{(-1)^j}{2j - t + 1} \binom{2j - t + 1}{k} B_{2j-t+1-k}, \quad 0 \le t \le m,$$
where $B_1 = \frac{1}{2}$.