Discrete mathematics - Relations, partial ordering based on materials by Joanna Pomianowska

Relations

A relation **R** in sets X and Y is any subset of the Cartesian product of X and Y. We say that x ∈ X and y ∈ Y are in R if the pair (x,y) ∈ R. We denote it xRy.

Basic relation types:

- Reflexive: $\forall x \in X\ xRx$
- Anti-reflexive: $\forall x \in X\ (x,x) \notin R$
- Symmetric: $\forall x \in X, y \in X\ xRy \rightarrow yRx$
- Anti-symmetrical: $\forall x \in X, y \in X(xRy \wedge yRx) \rightarrow x = y$
- Transitive: $\forall x \in X, y \in X, z \in X(xRy \wedge yRz) \rightarrow xRz$
- Connex: $\forall x \in X, y \in X\ xRy \vee yRx$

A relation which is reflexive, anti-symmetric and transitive is a relation of **partial ordering.**
A relation of partial ordering which is also connex is a relation of **linear ordering.**

In mathematics relations are described with symbols:
$\leq < \geq > \subseteq \subset \supseteq \supset$

Example 1: check if the relation $\leq$ is a relation of linear ordering in the set of real numbers R.

- Reflexiveness: $\forall x \in R\ x \leq x$ OK
- Anti-symmetry: $\forall x \in R, y \in R\ (x \leq y \wedge y \leq x) \rightarrow x = y$ OK
- Transitivity: $\forall x \in R, y \in R, z \in R(x \leq y \wedge y \leq z) \rightarrow xRz$ OK
- Connexity: $\forall x \in R, y \in R\ x \leq y \vee y \leq x$ OK

Example 2: check if the relation $\subseteq$ is a relation of linear ordering in the set of sets.

- Reflexiveness: $\forall A \in U\ A \subseteq A$ OK
- Anti-symmetry: $\forall A \in U, B \in U\ (A \subseteq B \wedge B \subseteq A) \rightarrow A = B$ OK
- Transitivity: $\forall A \in U, B \in U, C \in U(A \subseteq B \wedge B \subseteq C) \rightarrow A \subseteq C$ OK
- Connexity: $\forall A \in U, B \in U\ A \subseteq B \vee B \subseteq A$ **not ok!**

For example, there are sets which are disjoint (contain completely different elements). Neither of them contains the other. Set inclusion is therefore a relation of partial ordering.

Example 3.
Is the divisibility relation a linear ordering in the set: {1,2,3,4,5,6}? Is it in the set {3,6,12} ?