FORMULAS

1. Formula 1 - Coefficients A

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m, \\ (2r+1)\binom{2r}{r} \displaystyle\sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \tbinom{d}{2r+1} \tfrac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \le r < m, \\ 0, & \text{if } r < 0 \text{ or } r > m. \end{cases}$$

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \le r < m \\ 0, & \text{if } r < 0 \text{ or } r > m \end{cases}$$

And it is NOT true that A can be defined as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & r = m, \\ (2r+1)\binom{2r}{r} \displaystyle\sum_{\ell} \mathbf{A}_{m,\ell} \tbinom{\ell}{2r+1} \tfrac{(-1)^{\ell-1}}{\ell-r} B_{2\ell-2r} [0 \le r < \lfloor m/2 \rfloor], & r \ne m \end{cases}$$

2. Formula 2 - Extended form of P

$$\begin{aligned} \mathbf{P}^m_{a,b}(n) &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{Q}^r_{a,b}(n) = \sum_{r=0}^{m} \mathbf{A}_{m,r} (\mathbf{Q}^r_b(n) - \mathbf{Q}^r_{a-1}(n)) \\ &= \sum_{t=0}^{m} \mathbf{X}^m_t(a,b)(-1)^{m-t} n^t = \sum_{t=0}^{m} (\mathbf{X}^m_t(b) - \mathbf{X}^m_t(a-1))(-1)^{m-t} n^t \\ &= \sum_{t=0}^{m} (-1)^{2m-t} \sum_{k=1}^{2m-t+1} \mathbf{H}_{m,t}(k)((b+1)^k - a^k) n^t \end{aligned}$$

3. Formula 3 - Derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{P}^{(m)}_{i,j}(t) &= \lim_{t \to u} \frac{\mathbf{P}^{(m)}_{i,j}(t) - \mathbf{P}^{(m)}_{i,j}(u)}{t-u} \\ &= \lim_{t \to u} \frac{1}{t-u} \sum_{r=0}^{m} \mathbf{A}_{m,r} (\mathbf{Q}^{(r)}_{i,j}(t) - \mathbf{Q}^{(r)}_{i,j}(u)) \end{aligned}$$

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4. Formula 3 - Odd finite differences

$$\begin{aligned}\Delta_h \mathbf{P}_{i,j}^{(m)}(a) &= \mathbf{P}_{i,j}^{(m)}(a) - \mathbf{P}_{i,j}^{(m)}(b) \\ &= \left(\sum_{r=0}^{m} \mathbf{A}_{m,r}\mathbf{Q}_{i,j}^{(r)}(a)\right) - \left(\sum_{r=0}^{m} \mathbf{A}_{m,r}\mathbf{Q}_{i,j}^{(r)}(b)\right) \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r}(\mathbf{Q}_{i,j}^{(r)}(a) - \mathbf{Q}_{i,j}^{(r)}(b)), \quad b \le a\end{aligned}$$

5. Formula 4 - Even finite differences

$$\begin{aligned}\Delta_h(x^{2m}) = (x+h)^{2m} - x^{2m} &= \frac{(x+h)^{2m+1}}{x+h} - \frac{x^{2m+1}}{x} = \frac{x(x+h)^{2m+1} - (x+h)x^{2m+1}}{x(x+h)} \\ &= \frac{x(x+h)^{2m+1} - xx^{2m+1} - hx^{2m+1}}{x(x+h)} = \frac{(x+h)^{2m+1} - x^{2m+1} - hx^{2m}}{x+h} \\ &= \frac{\Delta_h(x^{2m+1}) - hx^{2m}}{x+h}\end{aligned}$$

6. Formula 5 - r-fold power sum of odds

$$\sum_{n=0}^{s} n^{2m+1} = \sum_{n=0}^{s} \mathbf{X}_m(n) = \sum_{n=0}^{s}\sum_{r=0}^{m} \mathbf{A}_{m,r}\mathbf{Q}_r(n) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{n=0}^{s} \mathbf{Q}_r(n)$$

7. Formula 6 - Binomial coefficients as polynomials

$$\binom{t}{k} = \frac{t(t-1)(t-2)\cdots(t-k+1)}{k(k-1)(k-2)\cdots 2\cdot 1} = \frac{1}{k!}\prod_{w=0}^{k-1}(t-w)$$

8. Formula 7 - Binomial expansion and L(m,n,k) identity

$$\mathbf{P}_{0,a+n-1}^{(m)}(a+n) \equiv (a+n)\sum_{k=0}^{2m}\binom{2m}{k} a^{2m-k} n^k$$

9. Formula 8 - Limits of sums

$$\lim_{t\to n} \mathbf{P}_m(n,t) = n^{2m+1}$$

10. Formula 9 - Symmetry of Lm(n,k)

$$\mathbf{L}_m(n,k) = \mathbf{L}_m(n,n-k)$$

11. Formula 10 - Relations between P, A, Q (new notation)

$$\mathbf{P}_{i,j}^{(m)}(n)=\sum_{k=i}^{j}\mathbf{L}_m(n,k)=\sum_{r=0}^{m}\mathbf{A}_{m,r}\mathbf{Q}_{i,j}^{(r)}(n)=\sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=i}^{j}\mathbf{U}^r(n,k)$$

12. Formula 11 - Odd power identities

$$\begin{aligned}
n^{2m+1}&=\sum_{r=0}^{m}\mathbf{A}_{m,r}\mathbf{Q}_{0,n-1}^{(r)}(n)\\
n^{2m+1}&=\sum_{r=0}^{m}\mathbf{A}_{m,r}\mathbf{Q}_{1,n}^{(r)}(n)\\
n^{2m-1}&=\sum_{r=0}^{m-1}\mathbf{A}_{m-1,r}\mathbf{Q}_{1,n}^{(r)}(n)\\
n^{s}&=\frac{1}{n^{\delta_{1,s \bmod 2}}}\sum_{r=0}^{\lfloor s/2\rfloor}\mathbf{A}_{\lfloor s/2\rfloor,r}\mathbf{Q}_{1,n}^{(r)}(n)\\
&=\frac{1}{n^{\delta_{s \bmod 2}}}\sum_{r=0}^{\lfloor s/2\rfloor}\mathbf{A}_{\lfloor s/2\rfloor,r}\mathbf{Q}_{1,n}^{(r)}(n)\\
&=n^{\delta_{s \bmod 2}}\sum_{r=0}^{\lfloor (s-1)/2\rfloor}\mathbf{A}_{\lfloor (s-1)/2\rfloor,r}\mathbf{Q}_{1,n}^{(r)}(n)\\
n^{s}&=\frac{1}{n^{\delta_{1,s \bmod 2}}}\mathbf{P}_{1,n}^{\lfloor s/2\rfloor}(n)\\
&=\frac{1}{n^{\delta_{s \bmod 2}}}\sum_{r=0}^{\lfloor s/2\rfloor}\mathbf{A}_{\lfloor s/2\rfloor,r}\mathbf{Q}_{1,n}^{(r)}(n)\\
&=n^{\delta_{s \bmod 2}}\sum_{r=0}^{\lfloor (s-1)/2\rfloor}\mathbf{A}_{\lfloor (s-1)/2\rfloor,r}\mathbf{Q}_{1,n}^{(r)}(n)
\end{aligned}$$

13. Formula 12 - Other properties of odd power identities

$$(2s+1)^{2m+1}+1=2\sum_{k=0}^{s}\mathbf{L}_m(2s+1,k)$$

$$(2s+2)^{2m+1}+1=\mathbf{L}_m(2s+2,s+1)+2\sum_{k=0}^{s}\mathbf{L}_m(2s+2,k)$$

$$n^{2m+1}=[n \text{ is even}]\mathbf{L}_m(n,n/2)+\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\mathbf{L}_m(n,k)$$

14. Formula 13 - Polynomials P(m,n,i,j) complete form

$$\begin{aligned}\mathbf{P}^{(1)}_{i,j}(n) = 1 &\\ &- 3i^2 + 2i^3\\ &- 3j^2 - 2j^3\\ &+ 3in - 3i^2n\\ &+ 3jn + 3j^2n\end{aligned}$$

$$\begin{aligned}\mathbf{P}^{(2)}_{i,j}(n) = 1 &- 10i^3 + 15i^4 - 6i^5\\ &+ 10j^3 + 15j^4 + 6j^5\\ &+ 15i^2n - 30i^3n + 15i^4n\\ &- 15j^2n - 30j^3n - 15j^4n\\ &- 5in^2 + 15i^2n^2 - 10i^3n^2\\ &+ 5jn^2 + 15j^2n^2 + 10j^3n^2\end{aligned}$$

$$\begin{aligned}\mathbf{P}^{(3)}_{i,j}(n) = 1 &\\ &+ 7i^2 - 28i^3 + 70i^5 - 70i^6 + 20i^7\\ &+ 7j^2 + 28j^3 - 70j^5 - 70j^6 - 20j^7\\ &- 7in + 42i^2n - 175i^4n + 210i^5n - 70i^6n\\ &- 7jn - 42j^2n + 175j^4n + 210j^5n + 70j^6n\\ &- 14in^2 + 140i^3n^2 - 210i^4n^2 + 84i^5n^2\\ &+ 14jn^2 - 140j^3n^2 - 210j^4n^2 - 84j^5n^2\\ &- 35i^2n^3 + 70i^3n^3 - 35i^4n^3\\ &+ 35j^2n^3 + 70j^3n^3 + 35j^4n^3\end{aligned}$$

$$
\begin{aligned}
\mathbf{P}^{(4)}_{i,j}(n) = 1&\\
&+ 60i^2 - 180i^3 + 294i^5 - 420i^7 + 315i^8 - 70i^9\\
&+ 60j^2 + 180j^3 - 294j^5 + 420j^7 + 315j^8 + 70j^9\\
&- 60in + 270i^2n - 735i^4n + 1470i^6n - 1260i^7n + 315i^8n\\
&- 60jn - 270j^2n + 735j^4n - 1470j^6n - 1260j^7n - 315j^8n\\
&- 90in^2 + 630i^3n^2 - 1890i^5n^2 + 1890i^6n^2 - 540i^7n^2\\
&+ 90jn^2 - 630j^3n^2 + 1890j^5n^2 + 1890j^6n^2 + 540j^7n^2\\
&- 210i^2n^3 + 1050i^4n^3 - 1260i^5n^3 + 420i^6n^3\\
&+ 210j^2n^3 - 1050j^4n^3 - 1260j^5n^3 - 420j^6n^3\\
&+ 21in^4 - 210i^3n^4 + 315i^4n^4 - 126i^5n^4\\
&- 21jn^4 + 210j^3n^4 + 315j^4n^4 + 126j^5n^4
\end{aligned}
$$

15. Formula 14 - Another Form

$$T^{2s+1} = \sum_{r=0}^{s} \sum_{\kappa=1}^{2s-r+1} (-1)^{2s-r} \mathscr{L}_{s,r}(\kappa) \cdot T^{\kappa+r}$$

16. Formula 15 - Why polynomials P(m,n,i,j) in i,j,n ?

$$
\begin{aligned}
\sum_{k=a}^{b}\sum_{j=0}^{m} A_{m,j}k^j(n-k)^j &= \sum_{k=a}^{b}\sum_{j=0}^{m} A_{m,j}k^j \sum_{t=0}^{j}\binom{j}{t} n^t(-1)^{j-t}k^{j-t}\\
&= \sum_{t=0}^{m} n^t \sum_{k=a}^{b}\sum_{j=t}^{m}\binom{j}{t} A_{m,j}k^{2j-t}(-1)^{j-t}\\
&= \sum_{t=0}^{m} n^t \sum_{j=t}^{m}(-1)^{j-t}\binom{j}{t} A_{m,j}\sum_{k=a}^{b} k^{2j-t}
\end{aligned}
$$

17. Formulas 16 - Alekseyev's approach on U coefficients

Originally from https://mathoverflow.net/q/304130/113033. (truth)

$$
\begin{aligned}
\mathbf{P}^{(m)}_{0,T}(n) = \sum_{k=0}^{T}\sum_{j=0}^{m} A_{m,j}k^j(n-k)^j &= \sum_{j=0}^{m}(-1)^{m-j}\mathbf{X}^{(m)}_{0,T}(j)\cdot n^j\\
= \sum_{k=0}^{T}\sum_{j=0}^{m} A_{m,j}k^j \sum_{t=0}^{j}\binom{j}{t} n^t(-1)^{t-t}k^{j-t}&\\
= \sum_{t=0}^{m} n^t \sum_{k=0}^{T}\sum_{j=t}^{m}\binom{j}{t} A_{m,j}k^{2j-t}(-1)^{j-t}.&
\end{aligned}
$$

Now, taking the coefficient of n^t in above gives (truth):

$$\mathbf{X}_{0,T}^{(m)}(t) = (-1)^m \sum_{k=0}^{T} \sum_{j=t}^{m} \binom{j}{t} A_{m,j} k^{2j-t} (-1)^j, \quad 0 \le t \le m$$

From this formula it may be not immediately clear why $U_m(T,t)$ represent polynomials in T. However, this can be seen if we change the summation order again and use Faulhaber's formula to obtain:

$$\mathbf{X}_{0,T}^{(m)}(t) = (-1)^m \sum_{j=t}^{m} \binom{j}{t} A_{m,j} \frac{(-1)^j}{2j-t+1} \sum_{\ell=0}^{2j-t} \binom{2j-t+1}{\ell} B_\ell T^{2j-t+1-\ell}.$$

Introducing $k = 2j - t + 1 - \ell$, we further get the formula:

$$\begin{aligned}
\mathbf{X}_{0,T}^{(m)}(t) &= (-1)^m \sum_{k=1}^{2m-t+1} T^k \underbrace{\sum_{j=t}^{m} \binom{j}{t} A_{m,j} \frac{(-1)^j}{2j-t+1} \binom{2j-t+1}{k} B_{2j-t+1-k}}_{\mathbf{H}_{m,t}(k)} \\
&= (-1)^m \sum_{k=1}^{2m-t+1} \mathbf{H}_{m,t}(k)(T+1)^k
\end{aligned}$$

Then polynomial $\mathbf{P}_{0,T}^{(m)}(n)$ can be expressed as

$$\begin{aligned}
\mathbf{P}_{0,T}^{(m)}(n) &= \sum_{j=0}^{m} (-1)^{m-j} (-1)^m \sum_{k=1}^{2m-j+1} \mathbf{H}_{m,j}(k) T^k \cdot n^j \\
&= \sum_{j=0}^{m} \sum_{k=1}^{2m-j+1} (-1)^{2m-j} \mathbf{H}_{m,j}(k) T^k \cdot n^j
\end{aligned}$$

In general, for $\mathbf{P}_{a,b}^{(m)}(n)$ we have

$$\begin{aligned}
\mathbf{P}_{a,b}^{(m)}(n) &= \sum_{k=0}^{b} \mathbf{L}_m(n,k) - \sum_{k=0}^{a-1} \mathbf{L}_m(n,k) \\
&= \sum_{j=0}^{m} \sum_{k=1}^{2m-j+1} (-1)^{2m-j} \mathbf{H}_{m,j}(k) b^k \cdot n^j - \sum_{j=0}^{m} \sum_{k=1}^{2m-j+1} (-1)^{2m-j} \mathbf{H}_{m,j}(k)(a-1)^k \cdot n^j \\
&= \sum_{j=0}^{m} (-1)^{2m-j} \sum_{k=1}^{2m-j+1} \mathbf{H}_{m,j}(k) \cdot n^j (b^k - (a-1)^k) \\
&= \sum_{j=0}^{m} (-1)^{2m-j} \sum_{k=1}^{2m-j+1} \mathbf{H}_{m,j}(k) \cdot n^j \Delta_h(b^k), \quad h = b - a + 1
\end{aligned}$$

$$\begin{aligned}\Delta_h(b^k) = (b+h)^k - b^k &= \left(\sum_{s=0}^{k}\binom{k}{s}b^{k-s}h^s\right) - b^k\\ &= b^k + \left(\sum_{s=1}^{k}\binom{k}{s}b^{k-s}h^s\right) - b^k\\ &= \sum_{s=1}^{k}\binom{k}{s}b^{k-s}h^s\end{aligned}$$

Let be $h = b - a + 1$, thus

$$\begin{aligned}\Delta_h(b^k) = (b+h)^k - b^k &= \sum_{s=1}^{k}\binom{k}{s}b^{k-s}h^s = \sum_{s=1}^{k}\binom{k}{s}b^{k-s}(b-a+1)^s\\ &= \sum_{s=1}^{k}\binom{k}{s}b^{k-s}h^s = \sum_{s=1}^{k}\binom{k}{s}b^{k-s}\sum_{k_1+k^2+k^3=s}\binom{s}{k_1,k_2,k_3}b^{k_1}(-a)^{k_2}\\ &= \sum_{s=1}^{k}\binom{k}{s}b^{k-s}\sum_{k_1+k^2+k^3=s}(-1)^{k_2}\binom{s}{k_1,k_2,k_3}b^{k_1}a^{k_2}\end{aligned}$$

$$\begin{aligned}\mathbf{P}^{(m)}_{a,b}(n) &= \sum_{j=0}^{m}(-1)^{2m-j}\sum_{k=1}^{2m-j+1}\mathbf{H}_{m,j}(k)\cdot n^j(b^k - (a-1)^k)\\ &= \sum_{j=0}^{m}(-1)^{2m-j}\sum_{k=1}^{2m-j+1}\mathbf{H}_{m,j}(k)(n^j b^k - n^j(a-1)^k)\end{aligned}$$

18. Formulas 17 - Odd power and convolution identities

$$\begin{aligned}n^{2m+1}+1 &= \sum_{r\geq 0}\mathbf{A}_{m,r}(f_r * f_r)[n], \quad n>0, \quad n\in\mathbb{N}\\ n^{2m+1}-1 &= \sum_{r\geq 0}\mathbf{A}_{m,r}(f_{r,1} * f_{r,1})[n], \quad n>0, \quad n\in\mathbb{N}\end{aligned}$$

19. Formula 18 - Definition of H

$\mathbf{H}_{m,t}(k)$ is real coefficient defined in terms of Bernoulli numbers, Binomial coefficients and $\mathbf{A}_{m,r}$

$$\mathbf{H}_{m,t}(k) := \sum_{j=t}^{m}\binom{j}{t}\mathbf{A}_{m,j}\frac{(-1)^j}{2j-t+1}\binom{2j-t+1}{k}B_{2j-t+1-k}, \quad 0\leq t\leq m. \tag{19.1}$$

20. Formula 19 - Definition of X

$\mathbf{X}^m_{a,b}(t)$ is $2m-t$ degree polynomial in a,b defined involving $\mathbf{A}_{m,r}$

$$\mathbf{X}^m_{a,b}(t) := (-1)^m\sum_{k=a}^{b}\sum_{j=t}^{m}\mathbf{A}_{m,j}(-1)^j\binom{j}{t}k^{2j-t}, \quad 0\leq t\leq m \tag{20.1}$$

21. Experiment on convolution and Iverson's

$$(f * f)[n] = \sum_k f[k]f[n-k].$$

$$\begin{aligned} n^r_{\geq t} * n^r_{\geq t} &= \sum_k k^r_{\geq t}(n-k)^r_{\geq t} = \sum_k k^r[k \geq t](n-k)^r[n-k \geq t] = \sum_k k^r(n-k)^r[k \geq t][n-k \geq t] \\ &= \sum_k k^r(n-k)^r[t \leq k \leq n-t] \end{aligned}$$

22. Power sum and Iversons notation

$$\begin{aligned} \sum_{k=0}^{b} k^r(n-k)^r &= \sum_k k^r(n-k)^r[0 \leq k \leq b] = \sum_k k^r \sum_{t=0}^{r}(-1)^t \binom{r}{t} n^{r-t}k^t[0 \leq k \leq b] \\ &= \sum_k k^r \sum_t (-1)^t \binom{r}{t} n^{r-t}k^t[0 \leq k \leq b] \\ &= \sum_k \sum_t (-1)^t \binom{r}{t} n^{r-t}k^{2t}[0 \leq k \leq b] \\ &= \sum_t (-1)^t \binom{r}{t} n^{r-t} \sum_k k^{2t}[0 \leq k \leq b] \\ &= \sum_t \binom{r}{t} n^{r-t} \sum_{j=0}^{2t} \frac{(-1)^t}{2t+1}\binom{2t+1}{j} B_j b^{r+t+1-j} \\ &= \sum_t \binom{r}{t} n^{r-t} \sum_j \frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} B_j b^{r+t+1-j}[0 \leq j \leq r+t] \end{aligned}$$

23. P involving Iversons

24. Modification on X

$$\begin{aligned} \mathbf{X}^m_t(a,b) &= (-1)^m \sum_j \mathbf{A}_{m,j}(-1)^j \binom{j}{t} \sum_k k^{2j-t}[a \leq k < b][j \geq t] \\ &= (-1)^m \sum_j \mathbf{A}_{m,j}(-1)^j \binom{j}{t} \left(\sum_{k=0}^{b} k^{2j-t} - \sum_{k=0}^{a-1} k^{2j-t} \right) [j \geq t] \\ &= \sum_{j \geq t} (-1)^{j+m} \mathbf{A}_{m,j} \binom{j}{t} (S_{2j-t}(b) - S_{2j-t}(a)) \end{aligned}$$

25. Binomials and convolution

$$\sum_r \binom{2m+1}{r} a^{2m+1-r} b^r \equiv -1 + \sum_r \mathbf{A}_{m,r} (f_0^r * f_0^r)[a+b]$$
$$\sum_r \binom{2m+1}{r} a^{2m+1-r} b^r \equiv 1 + \sum_r \mathbf{A}_{m,r} (f_1^r * f_1^r)[a+b]$$

26. Convolutional derivative

$$\begin{aligned}(a+b)^{2m+1} = \sum_r \binom{2m+1}{r} a^{2m+1-r} b^r &\equiv -1 + \mathbf{P}^m_{a+b+1}(a+b) \\ &= -1 + \sum_r \mathbf{A}_{m,r} \mathbf{Q}^r_{a+b+1}(a+b) \\ &= -1 + \sum_r \mathbf{A}_{m,r} (f^r * f^r)[a+b].\end{aligned}$$

$$\begin{aligned}\Delta(x^{2m+1}) &= \sum_r \mathbf{A}_{m,r} (f^r * f^r)[x+\Delta x] - \sum_r \mathbf{A}_{m,r} (f^r * f^r)[x] \\ &= \sum_r \mathbf{A}_{m,r} \{(f^r * f^r)[x+\Delta x] - (f^r * f^r)[x]\} \\ &= \sum_r \mathbf{A}_{m,r} \Delta (f^r * f^r)[x] \\ \frac{d}{dx} x^{2m+1} &= \lim_{\Delta x \to 0} \sum_r \mathbf{A}_{m,r} \frac{\Delta (f^r * f^r)[x]}{\Delta x} \\ &= \sum_r \mathbf{A}_{m,r} \lim_{\Delta x \to 0} \frac{\Delta (f^r * f^r)[x]}{\Delta x} \\ &= \sum_r \mathbf{A}_{m,r} \frac{d}{dx} (f^r * f^r)[x] = \sum_r \mathbf{A}_{m,r} (\tfrac{df^r}{dx} * f^r)[x] \\ &= \sum_r \mathbf{A}_{m,r} r (f^{r-1} * f^r)[x]\end{aligned}$$