Applied Mathematicans Bake the Best Brownies: The Ultimate Pan Design

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Abstract

When baked in a traditional square pan, the corners of the cake often gets overcooked. However, when a circular pan is used, some of the room inside the oven is wasted during the baking process. We explain the overcooking phenomenon in theoretical models and calculations, and attempt to achieve a compromise between a square pan and a circular pan. We choose rounded rectangles with parameters to achieve shapes in between.

We investigate the heat distribution of the air inside the oven, the pan and the batter. We construct two dimensional heat balance equations with boundary constraints. Stationary solutions are given analytically for the circular and square situation, while for an arbitrary rounded rectangle, we employ the Finite Element Method (FEM) to obtain a numerical solution. PDE solutions show that the temperature distribution is uniform in the pan, while the shape of the pan greatly affects heat distribution among the periphery of a horizontal cross-section of the brownie batter.

We measure the uniformity of heat distribution on the edge by μ, the ratio of the smallest temperature gradient to the biggest temperature gradient throughout the edge. We check typical shapes to validate that it is consistent with our intuition. A function to fit μ is constructed using a two-layer feed-forward neural network.

The maximum number of pans that can fit in the oven can be achieved when the shape is a rectangle with either w or l of the pan equals to the length of one side of the oven. A circular pan undoubtedly attains the most even distribution of heat on the edge. We can get shapes in between by adjusting the parameter p.

We perform sensitivity analysis into a couple of parameters and discuss the strengths and weaknesses of our model.

We strongly recommend our model because of its reasonable assumptions, convincing parameter estimations and clear but robust results.

Contents

1 Introduction

When faced with desserts either creamy or crispy, one can hardly resist the temptation to have a bite. Nowadays ovens, especially toaster ovens which are simple, small and smart, could contribute a lot to happiness of ordinary families, since making desserts using a oven is incredibly simple. Whatever you like, be it brownies, biscotti or bread puddings, you can find and follow a recipe on the Internet and act like a chef.

Nevertheless, when you are ready to invite your family members and friends to enjoy your work of art—fascinating brownies, you will probably find with dismay that the hard edges of brownies are overcooked while the gooey interior are undercooked. This phenomenon is caused by uneven distribution of heat of the pan during the baking process.

In order to get a heat distribution as uniform as possible and secure a good taste of the brownie, Howe[1990] devised and patented a "controlled heating baking pan". The center and the exterior of his plastic baking pan differs in thickness to slow down the baking of the outer edges of a brownie. Another approach is to use a round pan, but the poor space utilization is its inherent downside, for the only kind of ovens available in the market is rectangular.

By modeling what happened inside the oven as a heat transfer process, mathematicians and physicists may guard our brownies against overcooking. We model the baking process of the brownie with two-dimensional heat equations with boundary conditions. We give an analytical solution of two dimensional stationary heat equation, and continue to find a numerical solution by applying Finite Element Method(FEM) to the equation in MATLAB.

For the shape of pans, we choose a spectrum of rounded rectangles which includes circle and rectangle as special circumstances. Given the oven dimensions of W and L, and a fixed pan area of A, we maximize the number of pans that can fit in the oven N, the even distribution of heat H, and a suitable combination of these two where factors p and $(1-p)$ are assigned. We identify the strengths and weaknesses of our model. Finally we gladly advertise for our novel design.

2 Background

2.1 Ingredients of a Brownie

We follow the five-star recipe by Murrin[2003]. The ingredients are listed in Table 1. Pour the mixture into a pan, put in the oven, set the timer for 30 minutes at 350°F $\cong$ 450K(which we take as the operation temperature of the oven afterwards), and what you need is to patiently wait for the sweet smell of yummy brownies. It is indeed a foolproof recipe, but still an inexperienced amateur could spoil the dessert by undercooking or overcooking. To secure the best quatlity of brownies under any circumstances is our propulsion to devise an optimized baking pan.

Unsalted butter	185g
Best dark chocolate	185g
Plain flour	80g
Cocoa powder	40g
White chocolate	50g
Milk chocolate	50g
Eggs	57g×3
Golden caster sugar	275g

Table 1: The ingredients of a brownie(1 kg per serving). Adapted from [Murrin 2003].

2.2 A Glance at Heat Transfer and Ovens

Before we delve into tedious and seemingly formidable heat equations, let's first get ourselves accustomed to some intuitive concepts as to how brownies are baked. Heat, or thermal energy, can be transferred in three fundamental ways: conduction, convection, and radiation.

Conduction happens between objects in physical contact. It is the most direct way to transfer heat when you cook, especially when the substances involved are good thermal conductors.

Convection is distinguished by the flow of thermal energy caused by fluids, like the air when you bake, and the water when you boil. Convection is almost negligible in heat transfer between solids.

Radiation transfers energy by either emission or absorption of electromagnetic waves, so cooking medium does not take part in energy transmission. When we bake a brownie, thermal radiation absorbed by the pan can be neglected because metal surface will perfectly reflect the electromagnetic waves emitted by radiation sources.

Zimmerman [2007] elaborated on comparisons of common cooking methods by identifying modes of heat transfer. His conclusive table reveals that baking is characterized by high levels of conduction and convection.

We turn to a brief introduction of ovens. Descendants of the earliest ovens found in Central Europe dated to 29,000 B.C. [Wikipedia 2013e], modern ovens have grown into a large family, ranging from mini roaster ovens, moderate sized convection ovens to giant double wall ovens.

A traditional oven usually features heat elements in the bottom, which may result in direct heating to the food (a process undesired as opposed to baking). Confronted with the problem, convection ovens are invented to take advantage of forced air to circulate the heat inside the oven cavity. This strategy reduces baking time and temperature, and a more even heating environment is achieved.

The particular oven we will investigate is Siemens HB75GB550B. It is a traditional oven with a cavity width L = 48.2 cm, depth W = 40.5 cm and height D = 32.9 cm [Siemens 2012]. More technical specifications are available on the website.

2.3 Values of Parameters

We list some values of parameters which we will refer to afterwards in Table 2. The sources are [Efunda 2013] and [Urieli n.d.].

Environmental Temperature	T_e	450K
Density of Air	ρ_0	$0.78 \mathrm{kg/m^3}$
Density of Pan	ρ_1	$2700 \mathrm{kg/m^3}$
Density of Brownie	ρ_2	$475 \mathrm{kg/m^3} \sim 660\ \mathrm{kg/m^3}$
Thermal conductivity of Air	k_0	$0.0365 \mathrm{W/m \cdot K}$
Thermal conductivity of Pan	k_1	$238 \mathrm{W/m \cdot K}$
Thermal conductivity of Brownie	k_2	$0.385 \mathrm{W/m \cdot K}$
Heat capacity of Air	c_0	$1020 \mathrm{J/kg \cdot K}$
Heat capacity of Pan	c_1	$975.15 \mathrm{J/kg \cdot K}$
Heat capacity of Brownie	c_2	$700 \mathrm{J/kg \cdot K}$
Convective Heat-transfer Coefficient	h	$15 \mathrm{W/m^2 \cdot K}$
Size of the Pan		$9'' \times 9'' \times 2.25''$
		(or 0.2286m $\times$ 0.2286m $\times$ 0.05715m)
Area of the Pan	A_1	$0.05225 \mathrm{m^2}$

Table 2: Some values of parameters.

3 Assumptions

- The bottom of the baking pans are rounded rectangles in shape, with length l, width w, and radius of the arcs r, illustrated in Figure 1. We consider a two dimensional pan bottom with no thickness.

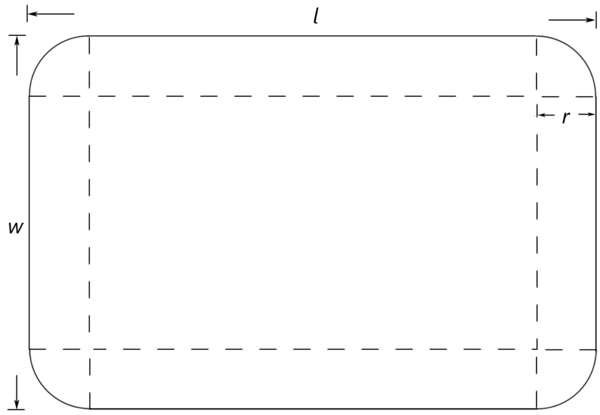


Figure 1: An illustration of the rounded rectangle, with parameters.

The pan is shallow enough that the pan borders can be neglected when we study the pan alone. Still, when we establish the model of the brownie during the baking process, it is supposed that both the bottom and sides of the batter is surrounded by metal. An empty pan is illustrated in Figure 2.

Figure 2: The pan in perspective view.

- All the N pans are identical. In other words, they share the same set of parameters w, l and r.

- The pans are made of metal. More specifically, we use aluminium pans in our model.

- The portion of thermal energy transferred from the wire rack can be neglected. After we confirm the point later in the paper that the temperature is distributed evenly across the pan even without any rack, we can safely neglect the influence of the rack in our model.
- All pans are placed in the same direction, parallel to one side of the rectangular rack. Overlap of the pans is not permitted.
- The outer boundary of the pan shares the same temperature with the surrounding air, as the heat transfer between steel and air is swift, validated by the record in Wikipedia[2013c] of a cooling rate roughly 38°C per minute. Furthermore, we assume that there is always some space between adjacent pans admitting air flow.
- The physical quantities are constant with regard to temperature, an approximation greatly simplifies our model and calculation while still producing reasonable results.

4 Objectives

For a given rectangular oven which has a width to length ratio of W/L and a pan area of A_1 (we leave A_2 for the oven horizontal section area), which we take as 0.05225m^2 in our model. We determine the parameters w, l and r of the pan, so as to

1. Maximize the number of pans (N) that can fit in the oven, which we denote by N_{max}. Notice that there are two evenly spaced racks in the oven.
2. Construct a measure of how even heat is distributed among the outer edges of the brownie batter during the baking process. Denote the relative uniformity of heat distribution of a particular pan shape by μ. We intend to maximize μ by adjusting the shape parameters.
3. Optimize a combination of the above two conditions where weights p and $(1-p)$ are assigned. Define ν as

$$\nu = \frac{N(\text{the particular shape})}{N_0},$$

we seek to maximize

$$\xi = p\nu + (1-p)\mu.$$

4. To determine a suitable pan shape for practical use, we investigate the following scenario:

5 Heat Distribution during the Baking Process

5.1 Air inside the Oven

The oven needs to be preheated to the working temperature according to our recipe, and modern ovens equipped with thermostats perform well in temperature regulation. This corresponds to the experimental results demonstrated in Fig. 2 of [Purlis et al. 2008]. A question naturally arises: is there any temperature variation among various parts of the air?

Luckily, the answer is simple. We assert that the temperature of air in the oven is constant everywhere. In the following pages we will give an estimation of the characteristic time for air to gain thermal equilibrium.

According to the Newton's cooling law [Wikipedia 2013c], which originates from the convection of fluids with solids, we have the following equation:

$$\frac{dQ}{dt} = h_c S(T_e - T(t)) = -h_c A \Delta T(t),$$

where
Q is the thermal energy (in J),
h_c is the heat transfer coefficient (in W·m^{-2}·K),
S is the surface area of the heat being transferred (in m^2),
T is the temperature of the object's surface and interior (since these are the same in this approximation),
T_e is the temperature of the environment; i.e. the temperature suitably far from the surface,
$\Delta T(t) = T(t) - T_e$ is the time-dependent thermal gradient between environment and object.

In thermal dynamics, the change of thermal energy can be represented by the product of special heat capacity, the mass of medium and the change of temperature. That is

$$\Delta Q = cm\Delta T.$$

Therefore, we can rewrite Newton's cooling law as

$$cm\frac{dT}{dt} = h_c S(T_e - T(t)).$$

Assuming that T_e is independent of time, we can get the analytical solution of $T(t)$:

$$T = T_e + (T - T_e)\exp(-\frac{h_c S}{cm}t).$$

If we define parameter τ as the characteristic time for air to gain thermal equilibrium,

$$\tau = \frac{cm}{h_c S} \quad ,$$

then we have

$$\frac{T(t)}{T_e} = 1 + (\frac{T_0}{T_e} - 1)\exp(-\frac{t}{\tau}).$$

Moreover, under the operation temperature (~450K), the density of air is approximately 0.5 kg·m^{-3}. And the dimensions of the oven is $0.35 \times 0.25 \times 0.2$ m^3.

From our data, we estimate the order of magnitudes as follows:

$$\begin{aligned} c &\sim 10^3 \mathrm{J} \cdot \mathrm{kg}^{-1} \cdot \mathrm{K}^{-1}, \\ h_c &\sim 10 \mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-1}, \\ S &\sim 0.1^2 \mathrm{m}^3, \\ V &\sim 0.1^3 \mathrm{m}^3. \end{aligned}$$

Therefore,

$$\tau \sim 1 \mathrm{s},$$

which confirms our assumption that the air in the oven can quickly gain thermal equilibrium with the cavity in approximately one second.

5.2 Pan: Analytical and Numerical Solutions

Analytical Solution For a Rectangle Pan

Combining our assumption that the air temperature is constant inside the oven with Newton's cooling law, we obtain the heat balance equation

$$\begin{cases} \frac{\partial T}{\partial t} - \frac{k_1}{\rho_1 c_1} \nabla^2 T = \frac{Q}{\rho_1 c_1} + \frac{h}{\rho_1 c_1}(T_e - T), \\ T|_{x=0} = T_e, \quad T|_{x=l} = T_e, \\ T|_{y=0} = T_e, \quad T|_{y=w} = T_e. \end{cases}$$

where
k_1 is the thermal conductivity of the pan,
h is the heat transfer coefficient between the pan and the air,
ρ_1 is the density of the pan,
c_1 is the specific heat capacity (in J/(kg·K)),
Q is the power per area unit generated (absorbed) by internal heat source (drain).

Using the residual temperature $u(x,y) = T(x,y) - T_e$, we get

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{k}{\rho c}\nabla^2 u = \frac{Q}{\rho c} + \frac{h}{\rho c}(-u), \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{y=0} = u|_{y=w} = 0. \end{cases}$$

A stationary solution of the problem above satisfies

$$k\nabla^2 u = h(u - \frac{Q}{h}).$$

Then we split $u(x,y)$ into the general solution and a particular solution which only depend on one argument

$$u(x,y) = v(x,y) + \omega(x),$$

where $v(x,y)$ is defined to be the solution of

$$\begin{cases} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})v = \frac{h}{k}v, \\ v|_{x=0} = v|_{x=l} = 0, \\ v|_{y=0} = v|_{y=w} = -\omega(x). \end{cases}$$

Employing the technique of separation of variables, we compute the coefficients

$$C = -\frac{Q}{h},$$

$$D = \frac{-Q/h + Q/h\cosh(\sqrt{h/k}\cdot l)}{\sinh(\sqrt{h/k}\cdot l)},$$

$$Q_n M_n = -\frac{C\cosh(\sqrt{h/k}\cdot x) + D\sinh(\sqrt{h/k}\cdot x) + Q/h}{\sin(\pi n x/l)},$$

$$Q_n N_n = Q_n M_n \cdot \frac{-1 + \cosh(\sqrt{\pi^2 n^2/l^2 + h/k}\cdot w)}{\sinh(\sqrt{\pi^2 n^2/l^2 + h/k}\cdot w)},$$

along with the particular solution

$$\omega(x) = C\cosh(\sqrt{h/k}\cdot x) + D\sinh(\sqrt{h/k}\cdot x) + Q/h,$$

and finally the desired stationary solution is

$$u(x,y) = \sum_{n=0}^{\infty} \sin\left(\frac{\pi n x}{l}\right)\left[Q_n M_n \cosh\left(\sqrt{\frac{\pi^2 n^2}{l^2} + \frac{h}{k}}\cdot y\right) + Q_n N_n \sinh\left(\sqrt{\frac{\pi^2 n^2}{l^2} + \frac{h}{k}}\cdot y\right)\right] + \omega(x).$$

Analytical Solution for a Circular Pan

We continue to investigate the heat balance equation of a circular pan with a radius of R. Again we use the residual temperature $u(r) = T(r) - T_e$. We obtain an ordinary differential equation in polar coordinates:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = -\frac{Q}{k} + \frac{h}{k}u(r),$$

with a general solution as

$$u(r) = C_1 J_0\left(i\sqrt{\frac{h}{k}}r\right) + C_2 Y_0\left(-i\sqrt{\frac{h}{k}}r\right) + \frac{Q}{h},$$

where J_α and Y_α denote the Bessel function of the first and second kinds, respectively. We use the boundary conditions to determine the coefficients C_1, C_2 and obtain the result

$$u(r) = \frac{Q}{h}\left(1 - \frac{J_0(i\sqrt{\frac{h}{k}}r)}{J_0(i\sqrt{\frac{h}{k}}R)}\right).$$

The analytical solution is portrayed in Figure 3. The radius of the circle is R=0.2578m.

Numerical Solutions for an Arbitrary Pan

The heat balance equation for an arbitrary rounded rectangle is

$$\begin{cases} \frac{\partial T}{\partial t} - \frac{k}{\rho c}\nabla^2 T = \frac{Q}{\rho c} + \frac{h}{\rho c}(T_e - T), \\ T|_{x\in\partial G} = T_e, \end{cases}$$

where G represents the rounded rectangular region.

For a complicated boundary condition like this, a concise analytical solution do not exist. In fact, only in quite a few specific conditions like rectangular or circular boundary conditions can we ever possible to get an analytic solution.

Therefore, we employ the powerful Finite Element Method(FEM) to calculate a numerical solution of the heat balance equation. With the aid of the MATLAB Partial Differential Equation Toolbox, we initialize the mesh by partitioning the region into small triangles(A). The mesh can be further refined by dividing small triangles into tiny triangles(B), and another repetition generates even smaller triangles(C). The process can be visualized in Figure 4. In this way, a mesh consisting over 10,000 triangles is generated on the rounded rectangle(in this figure we calculate the square as a special case).

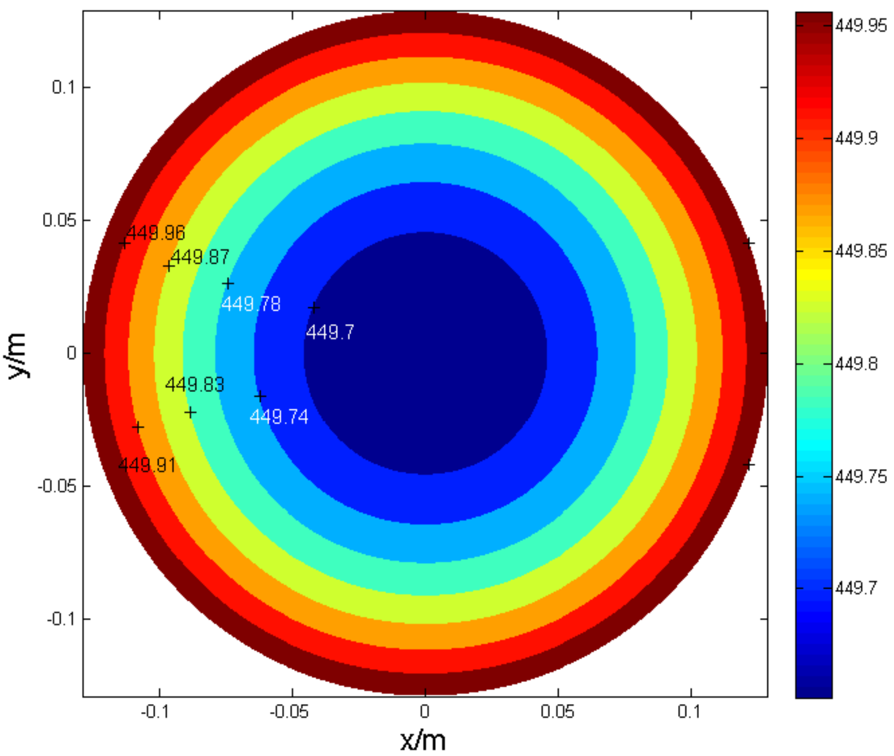
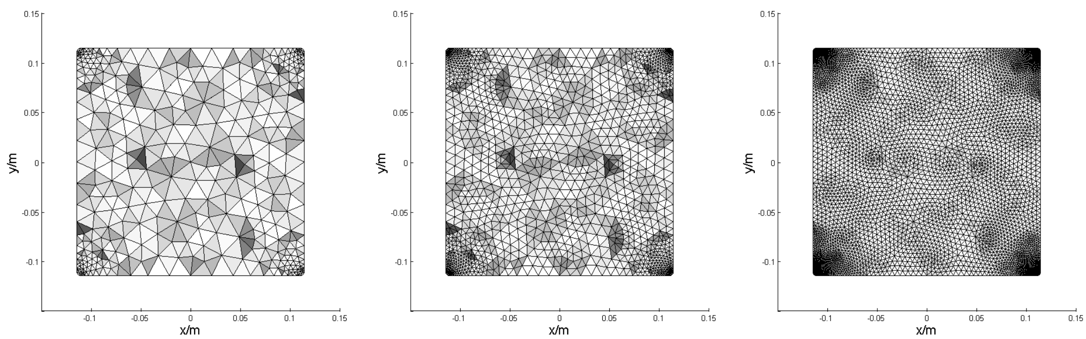


Figure 3: The temperature distribution of a circular pan.



(A) (B) (C)

Figure 4: Triangulation of the region: first step of FEM.

This enables us to approximate the stationary problem with a system of linear equations and obtain numerical solutions. To pave the way for further calculations, we estimate the values on the rectangular grid by interpolating from the vertices of the triangles. For a square pan with $W = L = 0.2286$m, the solution is depicted in Figure 5.

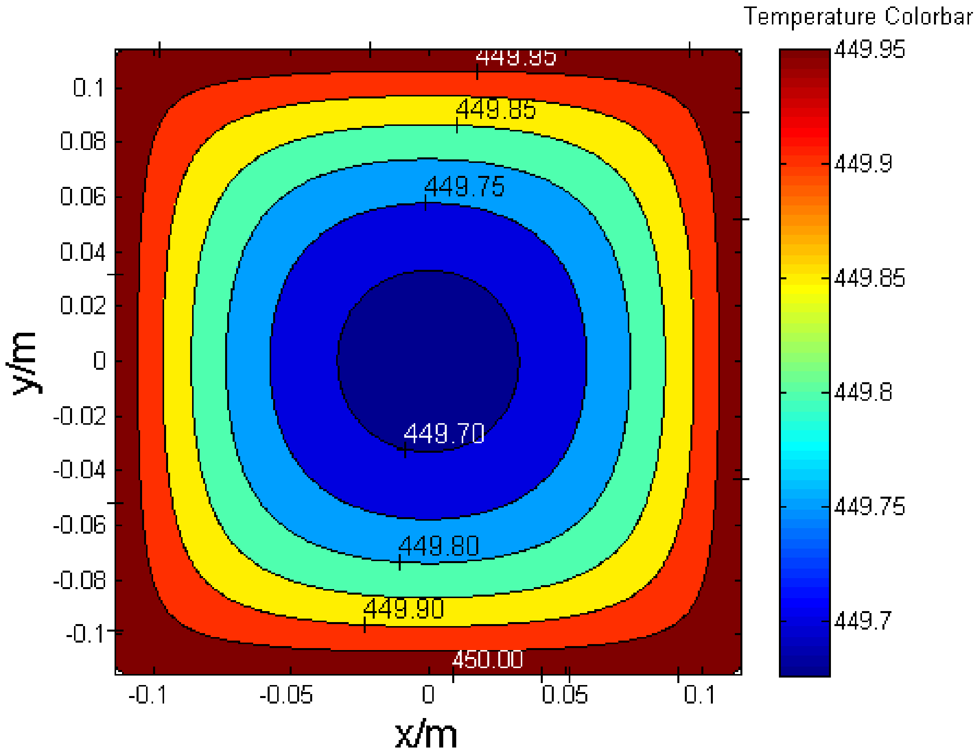


Figure 5: The temperature distribution of a square pan.

Discussions

The temperature colorbar in both Figure 3 and Figure 5 reveals a temperature range of 0.3K. Because the temperature range of the pan is so small, which can be explained by the excellent thermal conductivity of aluminium, we safely conclude that the temperature of every inch of the pan is equal to the air in the cavity.

5.3 Brownies: Vertical and Horizontal Cross-Section

We model the brownie batter as a three dimensional mixed substance, whose shape is identical to the pan, but the thickness H is 4.0cm. Only capable of solving two dimension problems, the MATLAB PDE Toolbox is incompetent in this situation at the first glance. In fact, we need to reduce the dimension of the problem to exploit the full potential of the PDE Toolbox. We tackle the problem by analyzing a cross-section of the batter, thus we obtain a two dimensional problem which we are comfortable with.

A Vertical Cross-Section

We first study the temperature distribution in a vertical cross-section. If our slice-cutting knife is parallel to the front face of the oven, we get a rectangular cross-section of $L \times H$, which is respectively the width and the depth of the oven cavity.

The cross-section satisfies the heat balance equation

$$-k_2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)T = Q(x,z),$$

with boundary conditions

$$\begin{cases} T|_{x=0} = T|_{x=L} = T|_{z=0} = T_e, \\ T|_{z=H} = T_e - \Delta T, \end{cases}$$

where
T is the temperature of the cross-section with coordinates (x,z);
$Q(x,z)$ denotes the heat absorption of batter caused by various thermal effects, including water evaporation and chemical reactions. We fit $Q(x,z)$ as

$$Q = -\frac{50000}{e^{-\frac{0.1}{\sqrt{(\frac{x}{0.2})^2+(\frac{z}{0.02})^2}}}+1}$$

to represent the actual process of baking;
k_2 is the thermal conductivity of the brownie, approximated by the weighted average of the thermal conductivity of its ingredients, shown in Table 3. We compile the table from data in [ASAE 1954], [Clegg 2001], [Mohos 2010], [Řezníček 1988] and [Williams n.d.].

The heat distribution in the vertical cross-section is illustrated in Figure 6. The graph resembles the temperature distribution of the middle cross-section of the bread in literature [Purlis 2008], which strongly supports our model.

unsalted butter	185g	0.21W/m·K
chocolate	285g	0.26W/m·K
plain flour	80g	0.3707W/m·K
cocoa powder	40g	0.1W/m·K
eggs	171g	0.54W/m·K
golden caster sugar	275g	0.58W/m·K
average		0.385W/m·K

Table 3: The thermal conductivity of ingredients of a brownie(1 kg per serving).

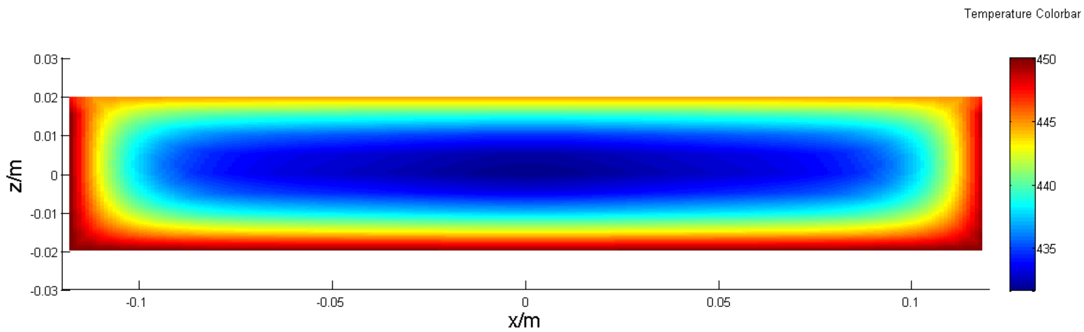


Figure 6: The temperature distribution of a vertical cross-section of the brownie.

The Horizontal Cross-Section

We study a horizontal cross-section of the batter, which bears the same shape as the pan. The layer satisfies the equation

$$-k_2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})T=Q,$$

with the boundary condition

$$T|_{\partial G}=T_e\ ,$$

where G represents the rounded rectangular region.

Unlike the vertical cross-section, the temperature distribution across the outer edges in a horizontal cross-section fluctuates with changes in shape. We consider three typical shapes in Table 4, and plot their temperature distributions and heat flux distributions.

Note that the heat flux distribution of the three shapes clearly reflects how uniform heat distributes across the outer edge. What distinguishes the square from the circle is that heat is concentrated at the corners of the square, while for the circle, heat distributes uniformly across the outer edge.

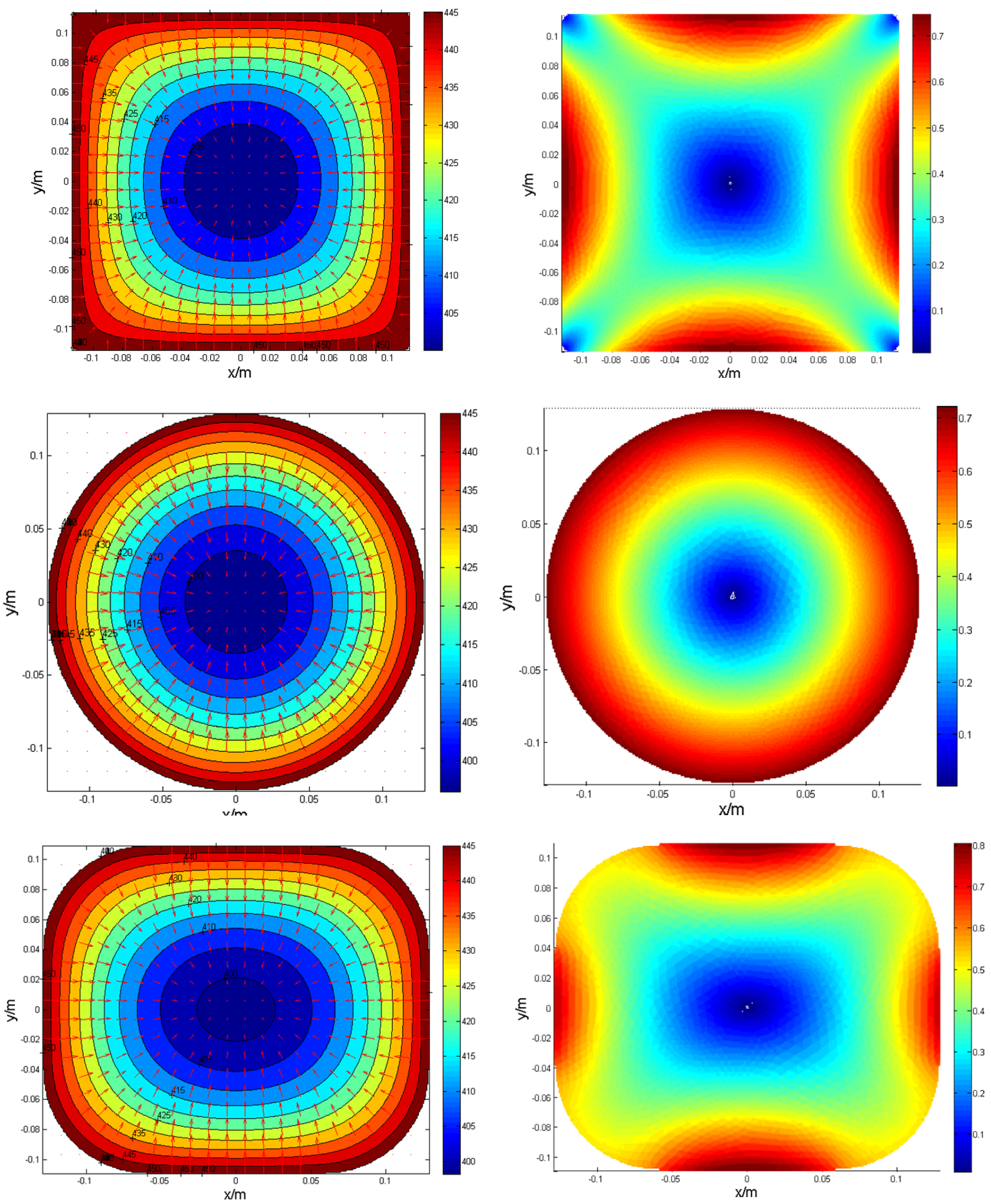


Figure 7: Temperature distribution (left) and heat flux distribution (right) of three typical shapes in Table 4: Square (top), Circle (middle) and Rounded rectangle (bottom). Arrow represents heat flow.

	l	w	r
Square	0.2286m	0.2286m	0.0030m
Circle	0.2578m	0.2578m	0.1286m
Rounded rectangle	0.2580m	0.2186m	0.0695m

Table 4: Parameters of three typical shapes.

6 Optimizing the Pan Shape

6.1 Task 1: Maximize Number of Pans

Given a constant area A, the pan shape parameters w, l and radius of the corner arc r are interrelated, which means we can derive the third if any two parameters are given. We choose $l \geqslant w$ because the length and width are symmetric. If we can establish a rounded rectangle, three geometric restrictions are implied

$$\begin{cases} l \geqslant w, \\ 0 \leqslant 2r \leqslant \min\{w, l\}, \\ wl - r^2(4-\pi) = A. \end{cases}$$

Representing r by w, l, we derive the domain Ω of (w, l):

$$\Omega = \left\{ (w, l) \in \mathbb{R}^2 : wl \geqslant A,\ l \geqslant w,\ 2\sqrt{\frac{wl - A}{4-\pi}} \leqslant \min\{w, l\} \right\}.$$

which is demonstrated in Figure 8. The points on the visible branch of hyperbola represents rectangles, which are special cases of rounded rectangles.

Once we have formulated the condition, we proceed to calculate N, which is the largest number of pans that can fit in the oven, both directions considered. In addition, never forget that we have two racks!

$$N = 2 \max_{(w,l)\in\Omega} \left\{ \left\lfloor \frac{W}{w} \right\rfloor \cdot \left\lfloor \frac{L}{l} \right\rfloor,\ \ \left\lfloor \frac{W}{l} \right\rfloor \cdot \left\lfloor \frac{L}{w} \right\rfloor \right\}.$$

We use MATLAB to find the maximum of N by searching in Ω. Interestingly, without the deduction above, it is easy to recognize that $N = 2\lfloor \frac{WL}{A} \rfloor$, obtained when $r = 0$ and the length of one side of the pan equals W or L.

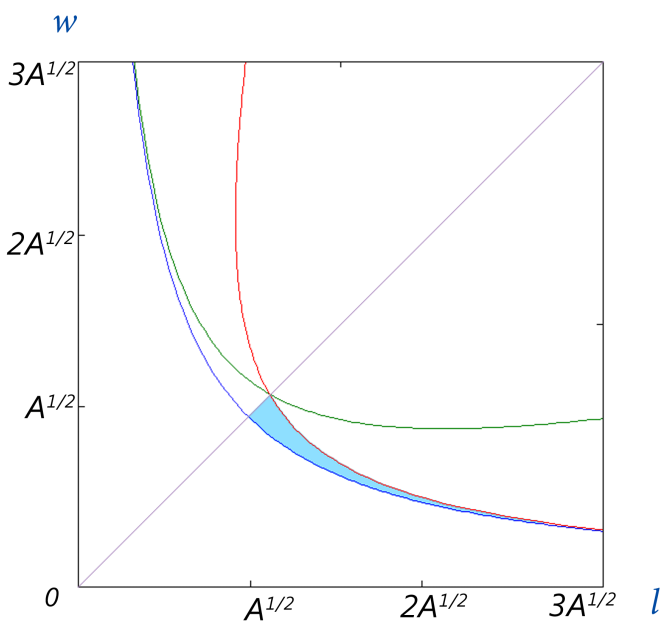


Figure 8: The domain Ω of the parameters w and l, in light cyan.

6.2 Task 2: Maximize Uniformity of Heat Distribution

To begin with, we will define a parameter to measure the distribution of heat for the brownie, which is solely determined by the geometry of the aluminium brownie pan in our model. Considering that brownie tends to be overbaked on the edges and corners, we can reasonably focus our attention on the periphery of the brownie. According to the above analysis, we define the parameter as:

$$\mu = \frac{\min\limits_{(l,w)\in\partial G} \{|\nabla T|\}}{\max\limits_{(l,w)\in\partial G} \{|\nabla T|\}}$$

where ∂G represents the periphery of the brownie.

In practice, we apply the finite difference method to the temperature field that we get from FEM calculation and get the distribution of the gradient of temperature. In fact, according to thermodynamics, the negative gradient of temperature is proportional to the heat flux of the temperature field, and it describes the tendency of temperature change. Since the boundary condition requires the brownie edge to share the same temperature of the oven, the smaller the heat flux of some point on the edge is, the heater the area around that point. Therefore, areas which have a smaller heat flux will be overbaked more quickly. Intuitively, for circle, $\mu = 1$, while for a rectangles with sharp edges, $\mu \sim 0$, and our calculation in Table 5 confirms these speculations.

After we get the parameter μ representing the uniformity of heat distribu-

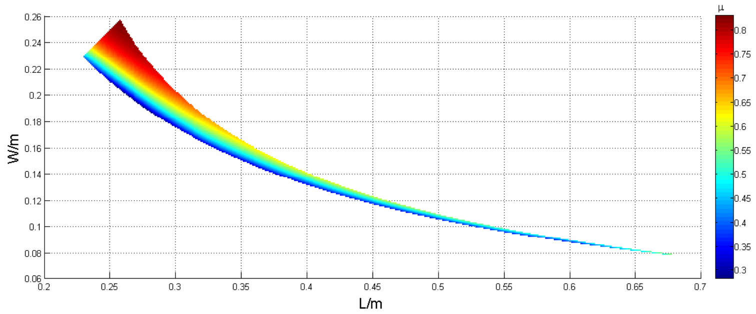


Figure 9: Value of μ in the domain.

	l	w	r	μ
Square	0.2286m	0.2286m	0.0030m	0.0907
Circle	0.2578m	0.2578m	0.1286m	0.9805
Rounded rectangle	0.2580m	0.2186m	0.0695m	0.6350

Table 5: μ of three typical shapes.

tion in a brownie, we will have to fit the function between μ and (w, l), because the size of samples is limited and any later optimization requires high resolution in (w, l) plane. Moreover, since μ is derived from the solution of a partial differential equation with irregular boundary conditions (compared with rectangles or circles), the relation between μ and (w, l) is highly nonlinear. Based on these two considerations, we introduce the MATLAB Neural Network Toolbox, which is capable to tackle nonlinear fitting problems with large number of arguments.

In this context, we construct a two-layer feed-forward neural network with sigmoid hidden neurons(10) and linear output neurons(1), which is illustrated in following Figure 10.

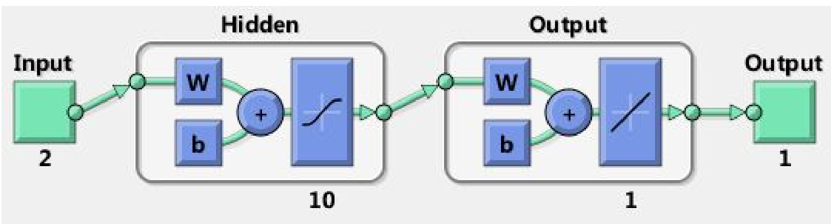


Figure 10: The structure of the neural network.

Moreover, we train the network with Levenberg-Marquardt backpropagation algorithm and the sample is divided into three parts for network training(70%), validation(15%) and testing(15%). As is revealed in Figure 11, the performance of the neural network is measured by the iteration-dependent Mean Squared Error (MSE), Error Histogram (which shows how the error sizes are distributed) and Regression Diagrams (which shows the actual network outputs plotted in terms of the associated target values). All these three diagrams prove that the neural network have successfully fitted the function and is ready to be used to generate high-resolution (w, l) samples. Here we only provide the Error histogram for conviction.

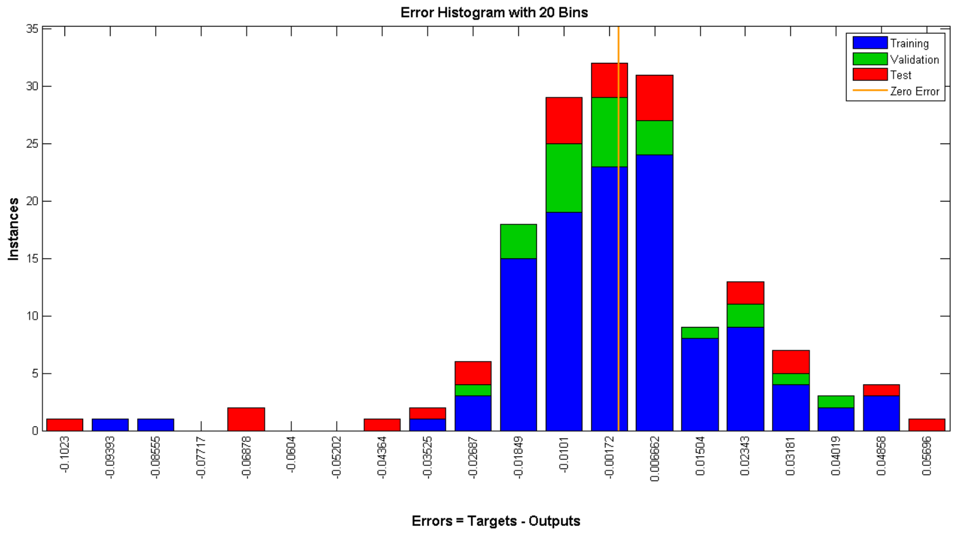


Figure 11: Error Histogram for A=0.05225m^2.

6.3 Task 3: Optimize Combined Conditions

We use the function fitted by the neural network to compute the best pan shape parameters under different levels of p and W/L. Meanwhile, we fix the oven area just like we fix the pan area. The result is illustrated in Figure 12.

From the graph, we can observe that when p increases and we are more concerned with the number of pans, the optimal shape is a long rounded rectangle. When more weight is assigned to the even distribution of heat, we obtain circular pans. So p strikes a balance of the two considerations. With regard to W/L, we find that for the upper half of the square, the ratio w/l after optimization correlate strongly to W/L.

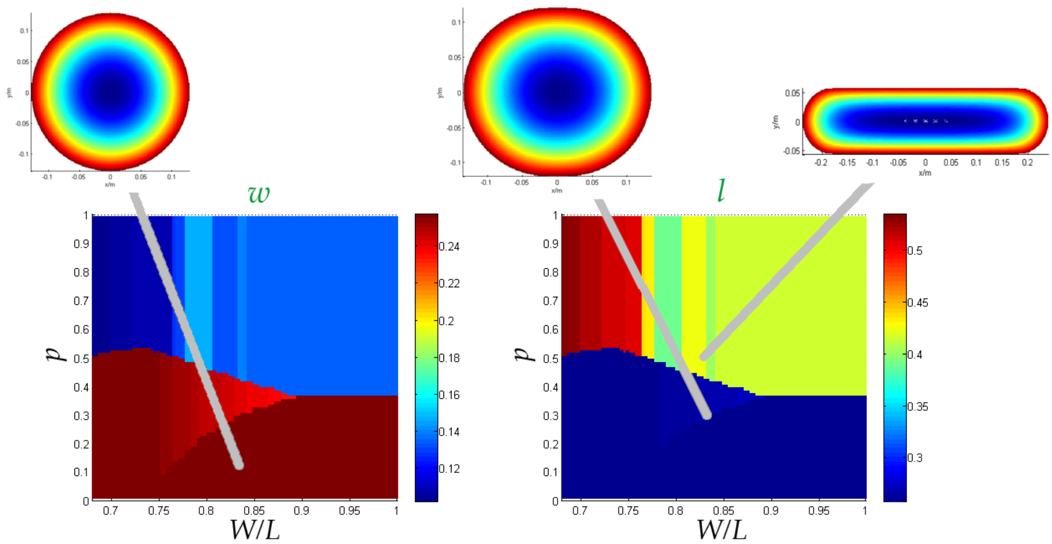


Figure 12: Optimized pan shape parameters w and l, with respect to p and W/L.

7 Analysis of the Results

7.1 Sensitivity Analysis

In our model of baking brownies, we assume that the horizontal section area of oven(A_2) and the pans(A_1) are kept constant,while the heat drain inside the brownie Q ,geometry of pans (w, l) and aspect ratio of oven (W/L) are allowed to change. In this section, we are going to undertake sensitivity analysis by allowing the previously fixed qualities to change while fixing the previous variables.

To begin with, we study the influence of heat drain inside brownie (Q) on temperature distribution indicator (μ) during FEM calculation. Remember that Q is a constant during calculation, and we set Q from $-3000\mathrm{W/m^2}$ to $-8000\mathrm{W/m^2}$ and run the FEM. All the other parameters like A_1, A_2 and thermal parameters are set to standard values. We list the central temperature T_c and μ for each Q.

Surprisingly, we notice that μ will not change with Q while central temperature T_c almost linearly depend on Q. In this way, we can safely employ Q as an adjustable parameter to achieve a reasonable temperature distribution while as the same time securing our definition of μ as the indicator of uniformity of temperature distribution on the periphery of brownie.

Secondly, we keep the aspect ratio of oven while adjusting its horizontal section area from $0.15\mathrm{m}^2$ to $0.25\mathrm{m}^2$. Remember that standard value is around $0.20\mathrm{m}^2$ and therefore we can test the influence of oven size to the geometry op-

Q	μ	T_c
-3000	0.090667	420.0016294
-3500	0.090667	415.0019009
-4000	0.090667	410.0021725
-4500	0.090667	405.0024441
-5000	0.090667	400.0027156
-5500	0.090667	395.0029872
-6000	0.090667	390.0032587
-6500	0.090667	385.0035303
-7000	0.090667	380.0038019
-7500	0.090667	375.0040734
-8000	0.090667	370.004345

Table 6: Sensitivity analysis of Q.

timization of pan. We keep $A_1 = 0.05225\text{m}^2$, which is the standard value, in the calculation and utilize the same neural network to finish our optimization. Results are listed in the following table.

Area of Oven	l_{opti}	w_{opti}	N_{opti}	μ_{opti}
0.15	0.291	0.211	4	0.731959788
0.16	0.436	0.122	6	0.458223606
0.17	0.449	0.123	6	0.658470796
0.18	0.459	0.12	6	0.664181378
0.19	0.466	0.118	6	0.666137368
0.2	0.466	0.118	6	0.666137368
0.21	0.466	0.118	6	0.666137368
0.22	0.508	0.107	8	0.625674381
0.23	0.261	0.219	8	0.663190011
0.24	0.267	0.224	8	0.731394043
0.25	0.272	0.229	8	0.77186925

Table 7: Sensitivity analysis of A_2.

The results are reasonable. When we employ larger oven, the number of optimized pans will increased from 4 to 8. Along with the jump of N_{opti} we can also notice the sudden change in (l_{opti}, w_{opti}) and μ_{opti}. Besides, as the oven become larger than 0.23m^2, the optimized pan become gradually more circular with higher μ_{opti}. while μ_{opti} for A_2 between $[0.170.21]\text{m}^2$ are relatively stabilized (about 0.667), which makes our standard optimization a stable solution. Finally, we change A_1 while fixing all the other parameters to test the influence of pan size on the optimized geometry. We set A_1 from $[0.040.06]\mathrm{m}^2$ and run the FEM. After that, neural network fitting procedures are undertaken to achieve a accurate optimized result. Results are listed in the table.

A_l	p	l_{opti}	w_{opti}	N_{opti}	μ_{opti}
0.03	0.5	0.487	0.063	12	0.8032
0.035	0.5	0.215	0.204	8	0.9168
0.04	0.5	0.241	0.202	8	0.7997
0.05	0.5	0.346	0.16	6	0.6368
0.0511	0.5	0.377	0.148	6	0.6639
0.052	0.5	0.449	0.123	6	0.6618
0.05225	0.5	0.37	0.155	6	0.6528
0.053	0.5	0.392	0.147	6	0.6674
0.0541	0.5	0.372	0.16	6	0.6501
0.055	0.5	0.377	0.16	6	0.5974
0.06	0.5	0.473	0.135	6	0.6271

Table 8: Sensitivity analysis of A_1.

Unfortunately, there seems to be no obvious dependence of optimized geometry $(w_{opti}, v_{opti}, \mu_{opti})$ on A_1. Quite interestingly, we can find a local maximum for μ around standard value, that is , $A_1 = 0.05225\mathrm{m}^2$ and such a result could somehow justify our choice of this standard value. Moreover, we can still confirm the validity of the results because smaller A_1will give larger N_{opti}.

7.2 Strengths and Weaknesses

7.2.1 Strengths

1.In our model, we systematically and correctly estimated the order of magnitudes of various physical quantities,like Q, τ and h_1. Through these reasonable estimations, we are able to get a clear physical picture of the problem.

2.All the unestimated physical quantities in our model can find its origin in literature or reliable websites.

3.In our model, we reduce the 3D time-dependent thermodynamic and fluid dynamics problem inside the oven into a 2D stationary heat equation with carefully checked boundary condition for brownie. Such a treatment greatly facilitate our numerical calculation and physical analysis.

4.We employ various powerful MATLAB toolboxes to facilitate our investigation, and mature and user-friendly methods like Finite Element Method(FEM) and Neural Networks(NN) provided by MATLAB gives us reliable data to analyse the inner physics.

5.Through our paper, we draw many vivid and informative pictures to illustrate our conclusions.

6.We provide a well-organized sensitivity analysis to further verify our models.

7.2.2 Weaknesses

1.Our model is, after all, a simplified one which omits a lot of potentially important parameters like the complex thermal convection of air, vapor emission and mass loss of brownie as well as the contribution of radiation.

2.Although dimensionality reduction greatly facilitate our calculation, it also bring us difficulties to determine the boundary condition and inner thermal conductivity of brownie as revealed in our calculate vertical temperature distribution in brownie.

8 Conclusion

We base our models on thermodynamics and geometric conditions, and we combine them to determine the optimal pan shape in general situations.

- Before investigations into the brownie batter, we use concrete deduction to prove that the air temperature and the pan temperature is uniformly distributed.
- Our estimations of parameters in the equations have been carefully checked to ensure that it reflects the phenomenon in the real world.
- We reduce the dimension of the heat balance problem of the brownie by consider its two dimensional cross-section so as to suit our problem to the MATLAB PDE Toolbox.
- We formulate a measure of how even the temperature is on the edge by observing the heat flux distributions in various typical shapes.
- We exploit the full potential of analysing figures to discover underlying principles.

- We explore how our model respond to small changes and assess the advantages and disadvantages of the model.

References

[1] ASAE Committee on Technical Data. 1954. Thermal Properties of Grain and Grain Products.

[2] Clegg, S. M. 2001. Thermal Conductivity of Chocolate: Summary Report Scientific & Technical Note 203. Leatherhead Food International.

[3] Efunda. 2013. Thermal Conductivity: Aluminum. http://www.efunda.com/materials/elements/TC_Table.cfm?Element_ID=Al. Accessed 3 February 2013.

[4] Howe, David J. U.S. Patent 5,004,121, 1990.

[5] Mohos, Á. Ferenc. 2010 Confectionery and Chocolate Engineering: Principles and Applications.

[6] Murrin, Orlando. 2003. Best-ever brownies recipe - Recipes - BBC Good Food. http://www.bbcgoodfood.com/recipes/1223/bestever-brownies. Accessed 3 February 2013.

[7] Purlis, E., Salvadori, V.O., 2008. Bread baking as a moving boundary problem. Part 1: Mathematical modelling. Journal of Food Engineering 91, 428–433.

[8] Purlis, E., Salvadori, V.O., 2008. Bread baking as a moving boundary problem. Part 2: model validation and numerical simulation. Journal of Food Engineering. 91, 434–442.

[9] Řezníček, Radoš. 1988. Physical properties of agricultural materials and products. Hemisphere Pub. Corp.

[10] Siemens. n.d. iQ 500 Built-in single multi-function activeClean oven stainless steel. http://www.siemens-home.co.uk/our-products/cooking/ovens/HB75GB550B.html. Accessed 3 February 2013.

[11] Urieli, Israel. n.d. Specific Heat Capacities of Air. http://www.ohio.edu/mechanical/thermo/property_tables/air/air_Cp_Cv.html. Accessed 3 February 2013.

[12] Wikipedia. 2013a. Carbon steel. http://en.wikipedia.org/wiki/Carbon_steel

[13] Wikipedia. 2013b. Chocolate brownie. http://en.wikipedia.org/wiki/Chocolate_brownie

[14] Wikipedia. 2013c. Convective heat transfer. http://en.wikipedia.org/wiki/Convective_heat_transfer

[15] Wikipedia. 2013d. Heat transfer. http://en.wikipedia.org/wiki/Heat_transfer

[16] Wikipedia. 2013e. Oven. http://en.wikipedia.org/wiki/Oven

[17] Williams, Charles D. H. n.d. The Science of Boiling an Egg. http://newton.ex.ac.uk/teaching/CDHW/egg/. Accessed 3 February 2013.

[18] Zimmerman, Burr. 2007. Heat Transfer and Cooking - Kitchen Notes - Cooking For Engineers. http://www.cookingforengineers.com/article/224/Heat-Transfer-and-Cooking. Accessed 3 February 2013.

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