Calculus: #1. Basic Properties of Numbers

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Alexandre Leibler

Problem 1

Prove the following:

(i). If $ax = a$ for some number $a \neq 0$, then $x = 1$.

(ii). $(x^2 - y^2) = (x - y)(x + y)$.

(iii). If $x^2 = y^2$, then $x = y$ or $x = -y$.

(iv). $(x^3 - y^3) = (x - y)(x^2 + xy + y^2)$.

(v). $(x^n - y^n) = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

(vi). $(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$.

Solutions Problem 1

(i). *Proof.*

$$\begin{aligned}
&\text{Since } a \neq 0,\ a^{-1}\text{exists.} && \text{(P7)}\\
&\text{hence, } a^{-1} \cdot (a \cdot x) = a^{-1} \cdot a; && \\
&\text{hence, } (a^{-1} \cdot a) \cdot x = a^{-1} \cdot a; && \text{(P5)}\\
&\text{hence, } 1 \cdot x = 1; && \text{(P7)}\\
&\text{consequently, } x = 1. && \text{(P6)}
\end{aligned}$$

$\square$

(ii). *Proof.*

$$\begin{aligned}
(x-y)(x+y) &= x(x+y) + (-y)(x+y); && \text{(P9)}\\
&= x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y; && \text{(P9)}\\
&= x^2 + xy - yx - y^2; && \\
&= x^2 + xy - xy - y^2; && \text{(P8)}\\
&= x^2 + 0 - y^2; && \text{(P3)}\\
&= 0 + x^2 - y^2; && \text{(P4)}\\
&= 0 + (x^2 - y^2); && \text{(P1)}\\
&= x^2 - y^2. && \text{(P2)}
\end{aligned}$$

$\square$

(iii). First we should prove that if $a \cdot b = 0$, then either $a = 0$, or $b = 0$.

Proof. To do this we must also prove that $a \cdot 0 = 0$:

$$\begin{aligned}
0 &= 0; && \\
0 + 0 &= 0; && \text{(P2)} \\
a \cdot (0 + 0) &= a \cdot 0; && \\
(a \cdot 0) + (a \cdot 0) &= a \cdot 0; && \text{(P9)} \\
((a \cdot 0) + (a \cdot 0)) - (a \cdot 0) &= (a \cdot 0) - (a \cdot 0); && \\
(a \cdot 0) + ((a \cdot 0) - (a \cdot 0)) &= (a - a) \cdot 0; && \text{(P5)(P9)} \\
(a \cdot 0) + ((a - a) \cdot 0) &= (0 \cdot 0); && \text{(P9)(P3)} \\
(a \cdot 0) + (0 \cdot 0) &= (0 \cdot 0); && \text{(P3)} \\
(a \cdot 0) + ((0 \cdot 0) - (0 \cdot 0)) &= (0 \cdot 0) - (0 \cdot 0); && \\
(a \cdot 0) + 0 &= 0; && \text{(P3)} \\
\therefore a \cdot 0 &= 0. && \text{(P2)}
\end{aligned}$$

□

Proof. Show that $(a \cdot b = 0) \implies (a = 0 \text{ or } b = 0)$

Given $a \cdot b = 0$,

Suppose that $a \neq 0$,
then $\exists a^{-1} : a \cdot a^{-1} = 1$; (P7)
hence $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$;
hence $(a^{-1} \cdot a) \cdot b = 0$; (P5)(above proof)
hence $1 \cdot b = 0$; (P7)
consequently $b = 0$. (P6)

Suppose that $b \neq 0$,
then $\exists b^{-1} : b \cdot b^{-1} = 1$; (P7)
hence $(a \cdot b) \cdot b^{-1} = 0 \cdot b^{-1}$;
hence $a \cdot (b \cdot b^{-1}) = 0$; (P5)(above proof)
hence $a \cdot 1 = 0$; (P7)
consequently $a = 0$. (P6)

□

Proof.

If $x^2 = y^2$,
then $x^2 - y^2 = y^2 - y^2$;
hence $x^2 - y^2 = 0$; (P3)
hence $(x - y) \cdot (x + y) = 0$; (see p1.ii.)
thus either $(x - y) = 0$ or $(x + y) = 0$; (see above proof)

Case 1:

$$\begin{aligned}
&\text{If } x - y = 0, \\
&\text{then } (x-y) + y = 0 + y; \\
&\text{hence } x + (-y + y) = y; && \text{(P1)(P2)} \\
&\text{hence } x + 0 = y; && \text{(P3)} \\
&\text{consequently } x = y. && \text{(P2)}
\end{aligned}$$

Case 2:

$$\begin{aligned}
&\text{If } x + y = 0, \\
&\text{then } (x+y) - y = 0 - y; \\
&\text{hence } x + (y - y) = -y; && \text{(P1)(P2)} \\
&\text{hence } x + 0 = -y; && \text{(P3)} \\
&\text{consequently } x = -y. && \text{(P2)}
\end{aligned}$$

□

(iv). *Proof.*

$$\begin{aligned}
(x-y)(x^2+xy+y^2) &= x \cdot x^2 + x \cdot xy + x \cdot y^2 - y \cdot x^2 - y \cdot xy - y \cdot y^2 && \text{(P9)} \\
&= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 && \text{(P8)} \\
&= x^3 - y^3 + (x^2y - x^2y) + (xy^2 - xy^2) && \text{(P4)(P1)} \\
&= x^3 - y^3 + 0 + 0 && \text{(P3)} \\
&= x^3 - y^3 && \text{(P2)}
\end{aligned}$$

□

Problem 2

What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned}
x^2 &= xy, \\
x^2 - y^2 &= xy - y^2, \\
(x+y)(x-y) &= y(x-y), \\
x + y &= y, \\
2y &= y, \\
2 &= 1.
\end{aligned}$$

Solution Problem 2 Since $x = y$, $x - y = 0$. Therefore, the multiplicative inverse of $(x-y)$ does not exist. This means that $(x+y)(x-y) = y(x-y)$ cannot be simplified to $x + y = y$ (division by 0 is not defined).

Problem 3

(i). $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

Problem 3 continued on next page...

(ii). $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, if $b, d \neq 0$.

(iii). $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

(iv). $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$, if $b, d \neq 0$.

(v). $\frac{a}{b}\Big/\frac{c}{d}$, if $b, c, d \neq 0$.

(vi). If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

Problem 4

Find all numbers x for which

(i). $4 - x < 3 - 2x$.

(ii). $5 - x^2 < 8$.

(iii). $5 - x^2 < -2$.

(iv). $(x-1)(x-3) > 0$.

(v). $x^2 - 2x + 2 > 0$.

(vi). $x^2 + x + 1 > 2$.

(vii). $x^2 - x + 10 > 16$.

(viii). $x^2 + x + 1 > 0$.

(ix). $(x-\pi)(x+5)(x-3) > 0$.

(x). $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.

(xi). $2^x < 8$.

(xii). $x + 3^x < 4$.

(xiii). $\frac{1}{x} + \frac{1}{1-x} > 0$.

(xiv). $\frac{x-1}{x+1} > 0$.