

# Calculus: #1. Basic Properties of Numbers

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## Problem 1

Prove the following:

- (i). If  $ax = a$  for some number  $a \neq 0$ , then  $x = 1$ .
- (ii).  $(x^2 - y^2) = (x - y)(x + y)$ .
- (iii). If  $x^2 = y^2$ , then  $x = y$  or  $x = -y$ .
- (iv).  $(x^3 - y^3) = (x - y)(x^2 + xy + y^2)$ .
- (v).  $(x^n - y^n) = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ .
- (vi).  $(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$ .

### Solution

(i). *Proof.*

Since  $a \neq 0$ ,  $a^{-1}$  exists. (P7)

hence,  $a^{-1} \cdot (a \cdot x) = a^{-1} \cdot a$ ;

hence,  $(a^{-1} \cdot a) \cdot x = a^{-1} \cdot a$ ; (P5)

hence,  $1 \cdot x = 1$ ; (P7)

consequently,  $x = 1$ . (P6) □

(ii). *Proof.*

$$(x - y)(x + y) = x(x + y) + (-y)(x + y); \quad (\text{P9})$$

$$= x \cdot x + x \cdot y + (-y) \cdot x + (-y) \cdot y; \quad (\text{P9})$$

$$= x^2 + xy - yx - y^2;$$

$$= x^2 + xy - xy - y^2; \quad (\text{P8})$$

$$= x^2 + 0 - y^2; \quad (\text{P3})$$

$$= 0 + x^2 - y^2; \quad (\text{P4})$$

$$= 0 + (x^2 - y^2); \quad (\text{P1})$$

$$= x^2 - y^2. \quad (\text{P2}) \quad \square$$

## Problem 2

What is wrong with the following “proof”? Let  $x = y$ . Then

$$x^2 = xy,$$

$$x^2 - y^2 = xy - y^2,$$

$$(x + y)(x - y) = y(x - y),$$

$$x + y = y,$$

$$2y = y,$$

$$2 = 1.$$

### Problem 3

- (i).  $\frac{a}{b} = \frac{ac}{bc}$ , if  $b, c \neq 0$ .
- (ii).  $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ , if  $b, d \neq 0$ .
- (iii).  $(ab)^{-1} = a^{-1}b^{-1}$ , if  $a, b \neq 0$ .
- (iv).  $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$ , if  $b, d \neq 0$ .
- (v).  $\frac{a}{b} \bigg/ \frac{c}{d}$ , if  $b, c, d \neq 0$ .
- (vi). If  $b, d \neq 0$ , then  $\frac{a}{b} = \frac{c}{d}$  if and only if  $ad = bc$ . Also determine when  $\frac{a}{b} = \frac{b}{a}$ .

### Problem 4

Find all numbers  $x$  for which

- (i).  $4 - x < 3 - 2x$ .
- (ii).  $5 - x^2 < 8$ .
- (iii).  $5 - x^2 < -2$ .
- (iv).  $(x - 1)(x - 3) > 0$ .
- (v).  $x^2 - 2x + 2 > 0$ .
- (vi).  $x^2 + x + 1 > 2$ .
- (vii).  $x^2 - x + 10 > 16$ .
- (viii).  $x^2 + x + 1 > 0$ .
- (ix).  $(x - \pi)(x + 5)(x - 3) > 0$ .
- (x).  $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$ .
- (xi).  $2^x < 8$ .
- (xii).  $x + 3^x < 4$ .
- (xiii).  $\frac{1}{x} + \frac{1}{1 - x} > 0$ .
- (xiv).  $\frac{x - 1}{x + 1} > 0$ .