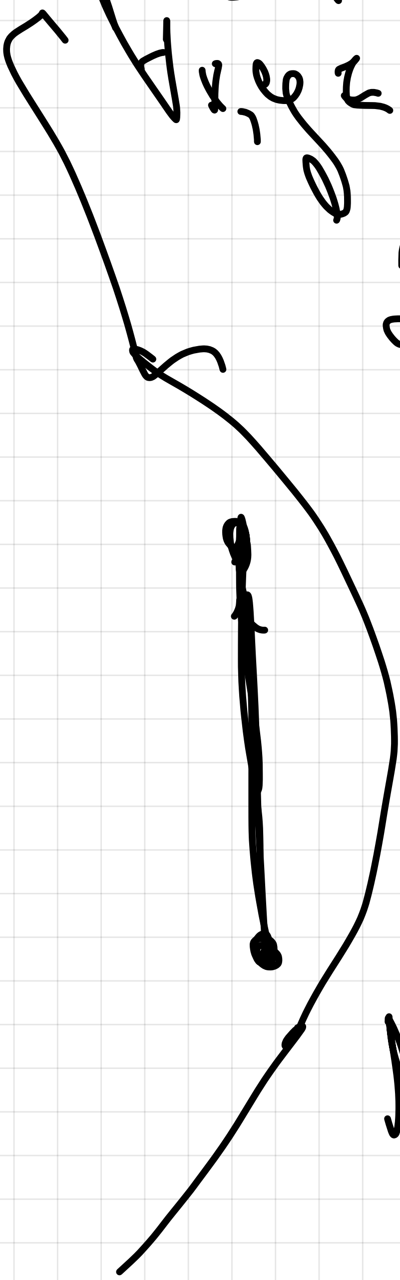
1) Выпуклые множества

Опр.

$\forall x, y \in C \Rightarrow \forall \lambda \in (0,1)$

$\lambda x + (1-\lambda) y \in C$!!



$Ax - b = 0$ — [illegible] ≤ 0

$A(\lambda x + (1-\lambda) y) - b =$

$= \lambda A x + (1-\lambda) A y - \lambda b - (1-\lambda) b =$

$= 0$

$\lambda(Ax - b) + (1-\lambda)(Ay - b)$

$x^{(T)} V x \leq 1 \qquad V \succ 0$

$\forall x : x^{(T)} V x > 0$

$$(\alpha x + (1-\alpha) y)^{(T)} V (\alpha x + (1-\alpha) y) - 1 =$$

$$= \alpha^2 x^{(T)} V x + (1-\alpha)^2 y^{(T)} V y + \quad ?$$

$$+ 2\alpha x^{(T)} V y (1-\alpha) - 1 \leq 0 \quad ?$$

$$\pm 2 x^{(T)} A x \qquad (1-\alpha) y^{(T)} A y$$

$$(\alpha x + (1-\alpha) y)^{(T)} V (\alpha x + (1-\alpha) y) =$$

$$= \lambda(\lambda-1)\left(x^T V x - 2x^T N y +\right.$$

$$\left. + y^T V y\right) + \lambda x^T V x + (1-\lambda) y^T V y$$

$$\lambda^2 - \lambda \quad x^T V x$$

$$\lambda^2 - \lambda \to (1-\lambda)^2 = 1 - 2\lambda + \lambda^2 =$$

$$= 1 - \lambda + \lambda^2 - \lambda =$$

$$= (1-\lambda) + \lambda(\lambda - 1)$$

$$\leq 0$$

$$(\lambda x + (1-\lambda) y)^T V(\ldots) = \lambda(\lambda-1)(x-y)^T V \cdot$$

$$\cdot (x-y) + \lambda x^T A x + (1-\lambda) y^T A y \leq 1$$

$$\leq \lambda \qquad \leq 1-\lambda$$

$$\left[(x-y)^T V(x-y) = x^T V x - \underline{\underline{2x^T V y}} + y^T V y\right]$$

$$\underbrace{\lambda(\alpha-1)}_{<0}\overbrace{(x-y)^T V(x-y)}^{>0} < 0$$

! C выпукло $\Leftrightarrow$ $\forall$ прямой

λC – выпукло

$$\max f(x)$$

$$\text{s.t. } \underline{\underline{x \in C}}$$

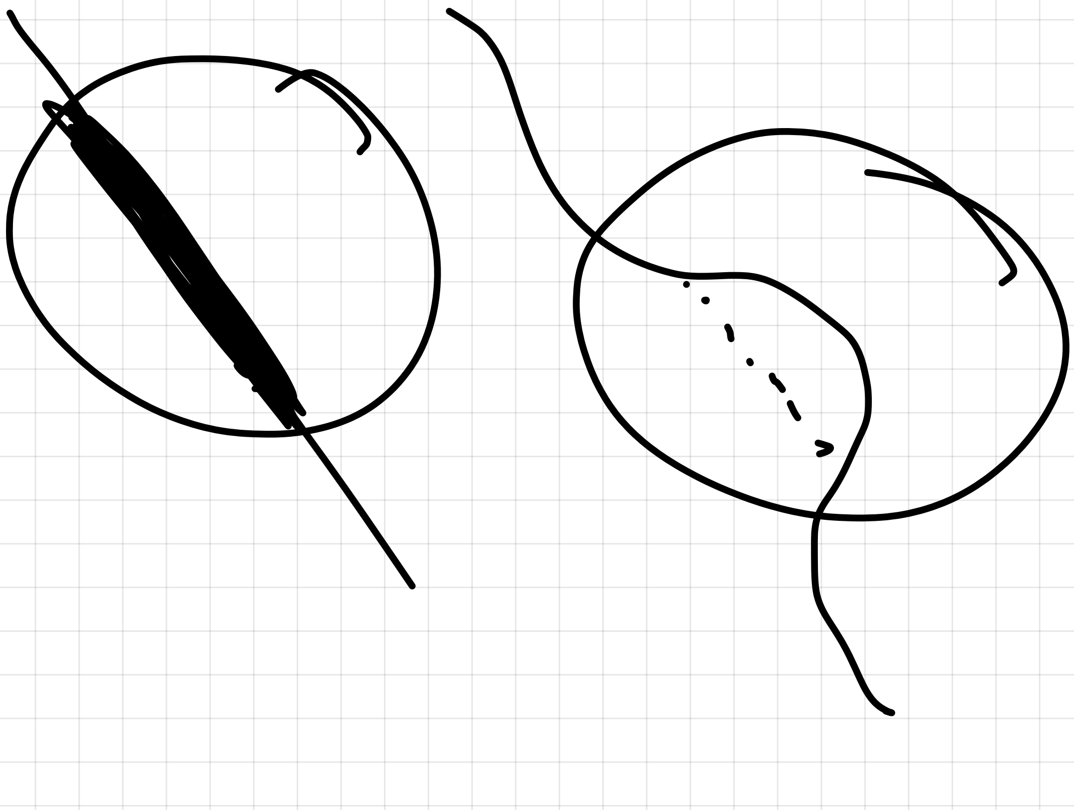
$g(x) \le c$ — $g(x)$ – выпуклая функция

$>$ str $\quad \ge$ concave

$$g(\lambda x + (1-\lambda) y) \le \lambda g(x) + (1-\lambda) g(y)$$

$<$ str $\quad$ convex

$g(x) = c$ — $g(x)$ – только мн



Convex

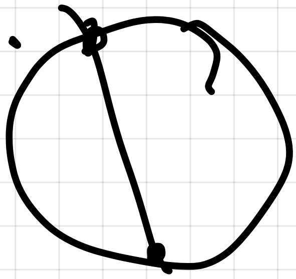
$$\min f(x) \text{ - convex}$$
$$\text{s.t. } x \in C$$

Concave

$$\max f(x) \text{ - concave}$$
$$\text{s.t. } x \in C$$

$$\begin{cases} \min \underline{x^{T} V x} \\ \text{s.t. } x^{T} e = \underline{1} \end{cases} \qquad V > 0$$

Yes!



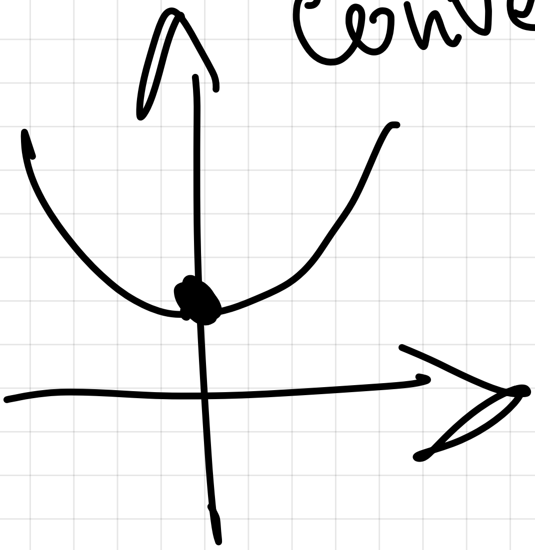
$$\begin{cases} \max x^{T} c \\ \text{s.t. } x^{T} V x = \underline{1} \end{cases}$$

No!

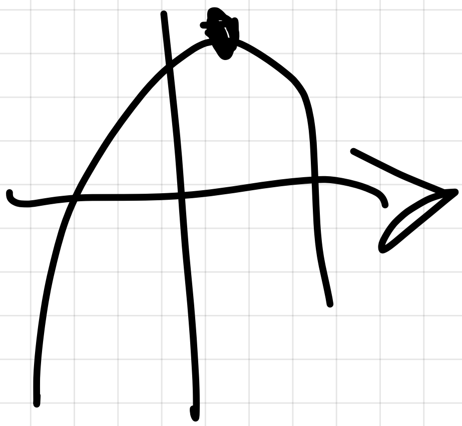
$$\begin{cases} \max x^T c \\ \text{s.t. } x^T A x \le 1 \end{cases}$$

Yes!

convex



& concave



Markowitz model

x - shares $[-1, 1]$ long

short

$$\sum_i x_i = x^T e = 1$$

1,2

r - return   2018 $r = 0,3$

$$\frac{P_t - P_{t-1}}{P_{t-1}} = r \quad (!)$$

$\sum_i x_i r_i = x^{[T]} r = r_p$

assets $\to$ V - covariance

$\underline{x^{[T]} V x}$ - variance of portfolio

$x \cdot \tilde{r}$

$Var(x \cdot \tilde{r}) =$
$= x^{[T]} V x$

$\min x^{[T]} V x$

s.t. $\begin{cases} x^{[T]} e = 1 \\ x^{[T]} r = r_p \end{cases}$

$\max - x^{[T]} V x$

s.t. ...

$$\mathcal{L} = -\frac{1}{2} x^{T} V x + \mu \left(x^{T} e - 1 \right) + \lambda \left(x^{T} r - r_p \right)$$

Ограничение строгое

$\lambda - \forall$ $\quad \mu - \forall$

$g(x) \geq c \qquad \underset{\geq 0}{\lambda} \left(g(x) - c \right) = 0$

≥ 0

$$\begin{cases} \lambda, \mu - \forall \\ \frac{\partial \mathcal{L}}{\partial x} = -Vx + \mu e + \lambda r = 0 \\ \lambda \left(x^{T} e - 1 \right) = 0 \qquad \lambda \left(g(x) - c \right) = 0 \\ \mu \left(x^{T} r - r_p \right) = 0 \end{cases}$$

$$x = \lambda V^{-1} a + \mu V^{-1} e$$

$x^T r = r_p$

$$\begin{cases} \lambda a^T V^{-1} a + \mu e^T V^{-1} a = r_p \\ \lambda a^T V^{-1} e + \mu e^T V^{-1} e = 1 \end{cases}$$

$x^T e = 1$

V^{-1} – симм. матрица

$$V_p^{-1} = \begin{pmatrix} a^T V^{-1} a & a^T V^{-1} e \\ a^T V^{-1} e & e^T V^{-1} e \end{pmatrix}$$

$$V_p^{-1} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} r_p \\ 1 \end{pmatrix}$$

$$\begin{pmatrix}\lambda \\ \mu\end{pmatrix} = V_p \begin{pmatrix} r_p \\ 1\end{pmatrix}$$

$$x = \left(V^{-1} r, V^{-1} e\right)\begin{pmatrix}\lambda \\ \mu\end{pmatrix} =$$

$$= V^{-1} r \cdot \lambda + V^{-1} e \cdot \mu$$

$$x = \left(V^{-1} r, V^{-1} e\right) V_p \cdot \begin{pmatrix} r_p \\ 1\end{pmatrix}$$

$$x^T V x = \left(r_p \; 1\right) V_p \begin{pmatrix} r^T V^{-1} \\ e^T V^{-1}\end{pmatrix} \cdot V$$

$$\cdot \left(V^{-1} r, V^{-1} e\right) \cdot V_p \cdot \begin{pmatrix} r_p \\ 1\end{pmatrix}$$

$$x y^{[T]} = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$$

$$\begin{pmatrix} r^{[T]} \\ e^{[T]} \end{pmatrix} \left(V^{-1} r, V^{-1} e \right) = V_p^{-1}$$

$$\begin{pmatrix} r^{[T]} V^{-1} r, & r^{T} V^{-1} e \\ \dots & \dots \end{pmatrix}$$

$e = $ np.ones $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$$x^{[T]} V x = (r_p \; 1) V_p V_p^{-1} V_p \begin{pmatrix} r_p \\ 1 \end{pmatrix} =$$

$$= \underline{(r_p \; 1) V_p \begin{pmatrix} r_p \\ 1 \end{pmatrix}} = \sigma_p^2$$

$$V_p^{-1} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \qquad A = r^T V^{-1} r$$

$$B = r^T V^{-1} e$$

$$C = e^T V^{-1} e$$

$$V_p = \frac{1}{D} \begin{pmatrix} C & -B \\ -B & A \end{pmatrix}$$

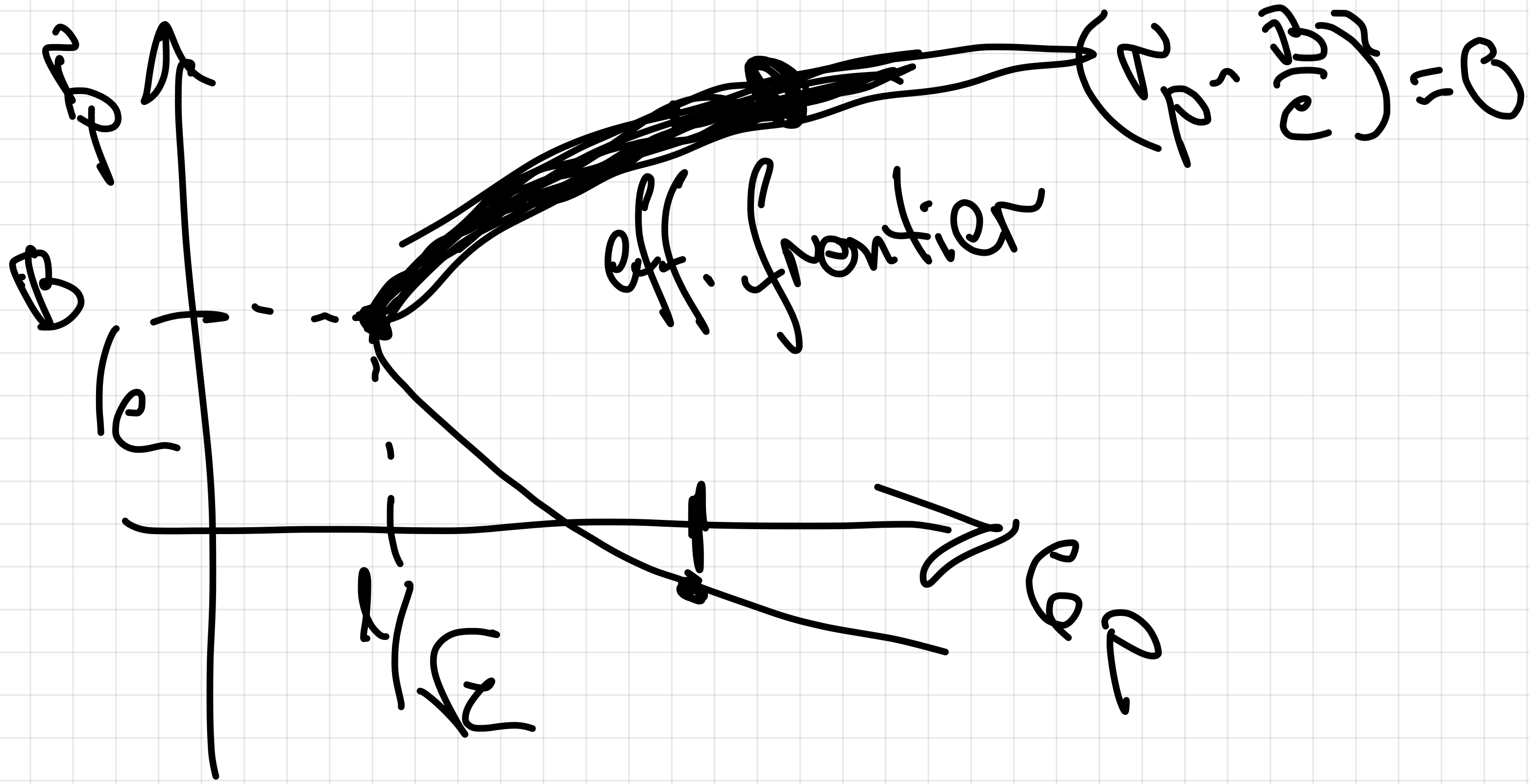
$$\sigma_p^2 = \begin{pmatrix} r_p & 1 \end{pmatrix} \frac{1}{D} \cdot \begin{pmatrix} C & -B \\ -B & A \end{pmatrix} \begin{pmatrix} r_p \\ 1 \end{pmatrix} =$$

$$= \frac{C}{D}\left(r_p - \frac{B}{C}\right)^2 + \frac{1}{C}$$

$C>0 \quad A>0$

$D>0$

$$C\sigma_p^2 - \frac{C^2}{D}\left(r_p - \frac{B}{C}\right)^2 = 1$$



$$\left[\begin{array}{l} \min x^T V x \\ \left\{\begin{array}{l} x^T r = r_p \\ x^T e = 1 \\ x \geq 0 \end{array}\right. \end{array}\right] \geq \begin{array}{l} \min x^T V x \\ \sim x \geq 0 \end{array}$$

$$\mathcal{L} = -\frac{1}{2}x^T V x + \lambda(x^T r - r_p) +$$

$$+ \mu(x^T e - 1) + \xi^T x$$

$\min\{x_1, x_2\}$

$\underline{\text{K-T}}$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = -Vx + \lambda r + \mu e + \xi = 0 \\ (x^T r - r_p) = 0 \\ (x^T e - 1) = 0 \\ \lambda, \mu - \forall \\ \xi \cdot x = 0 \\ \underline{\xi \geq 0} \end{cases}$$

$>0 \quad <0$

$x \geq 0$

!

$$x = 0 \rightarrow \underset{>0}{\lambda r} + \underset{<0}{\mu} = -\xi \leq 0$$

Семинар 2

1) $u(x) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho}$

$\rho \in (-\infty, 1)$, $\rho \neq 0$

$\alpha_1, \alpha_2 \geq 0$

CES - constant elasticity of substitution function

$$\begin{cases} \max u(x) \\ \text{s.t. } p_1 x_1 + p_2 x_2 \leq w \end{cases}$$

1) $u(x)$ - concave

CES - concave?

$f''(x) < 0$ $\quad \forall \xi \quad \xi^T f''(x) \xi < 0$

strictly concave!

$g(x)$ - concave $\quad g(x) > 0$

$s(x) = g(x)^\alpha$ - concave?

$s'(x) = \alpha g^{\alpha-1}(x) g'(x)$ $\quad > 0$

≥ 0 $\quad > 0$ $\quad g$

$s''(x) = \alpha(\alpha-1) g^{\alpha-2}(x) (g'(x))^2 +$

< 0 $\quad ?$

$+ \alpha g^{\alpha-1}(x) g''(x) < 0$

≥ 0 $\quad > 0$ $\quad < 0$ $\quad \alpha > 0$

$\alpha(\alpha - 1) < 0$ $\quad \alpha - 1 < 0$ $\quad \alpha \in (0,1)$

$\alpha < 1$

$$u(x) = (\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho})^{1/\rho}$$

$$\boxed{\rho \in (-\infty, 1)} \qquad \uparrow \rho$$

$$\frac{1}{\rho} - 1 = \frac{1-\rho}{\rho}$$

$$\frac{\partial u}{\partial x_1} = \alpha_1 x_1^{\rho-1} (\ldots)^{\frac{1-\rho}{\rho}}$$

$$\frac{\partial u}{\partial x_2} = \alpha_2 x_2^{\rho-1} (\ldots)^{\frac{1-\rho}{\rho}}$$

!

$$\rho(\rho-1)\alpha_1 x_1^{\rho-2}$$

$$0$$

$$\rho(\rho-1)\alpha_2 x_2^{\rho-1}$$

$$\rho < 1$$

$$\rho(\rho-1) < 0$$

$$\rho(\rho-1) > 0$$

concave

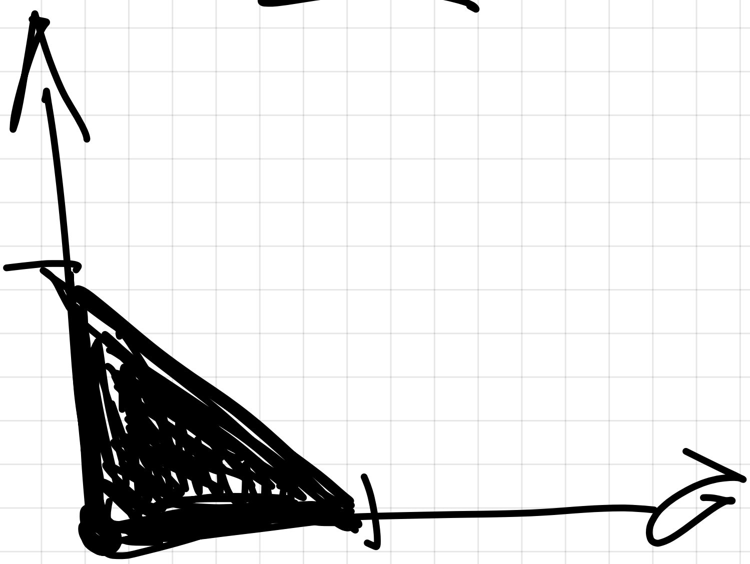
$$\rho \in (0,1)$$

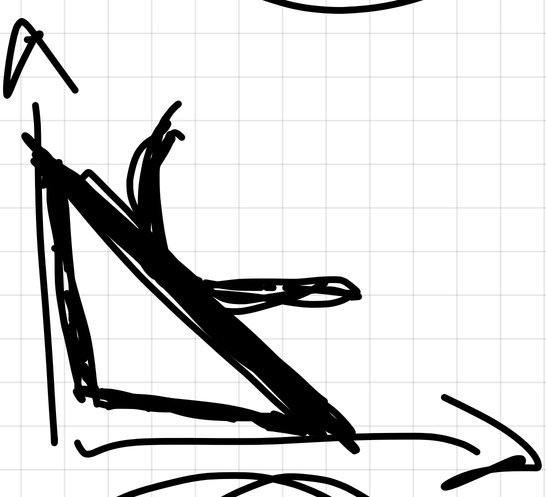
$$\mathcal{L} = (a_1 x_1^{\rho} + a_2 x_2^{\rho})^{1/\rho} + \lambda(w - p^T x)$$

$w \geq p^T x$ $\quad (\theta_1 - \rho)$ $\qquad x_1, x_2 \geq 0$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_1} = a_1 x_1^{\rho-1} (\dots)^{\frac{1-\rho}{\rho}} - \lambda p_1 = 0 \\[2ex] \dfrac{\partial \mathcal{L}}{\partial x_2} = a_2 x_2^{\rho-1} (\dots)^{\frac{1-\rho}{\rho}} - \lambda p_2 = 0 \\[2ex] w = p^T x \\[1ex] \lambda \geq 0 \\[1ex] \lambda(w - p^T x) = 0 \end{cases}$$

K.T. condition





$w = p^T x$

$$\frac{\alpha_1}{\alpha_2}\cdot\left(\frac{x_1}{x_2}\right)^{\rho-1} = \frac{p_1}{p_2}$$

$$\ln\frac{\alpha_1}{\alpha_2} + \overset{<0}{(\rho-1)}\ln\frac{x_1}{x_2} = \ln\frac{p_1}{p_2}$$

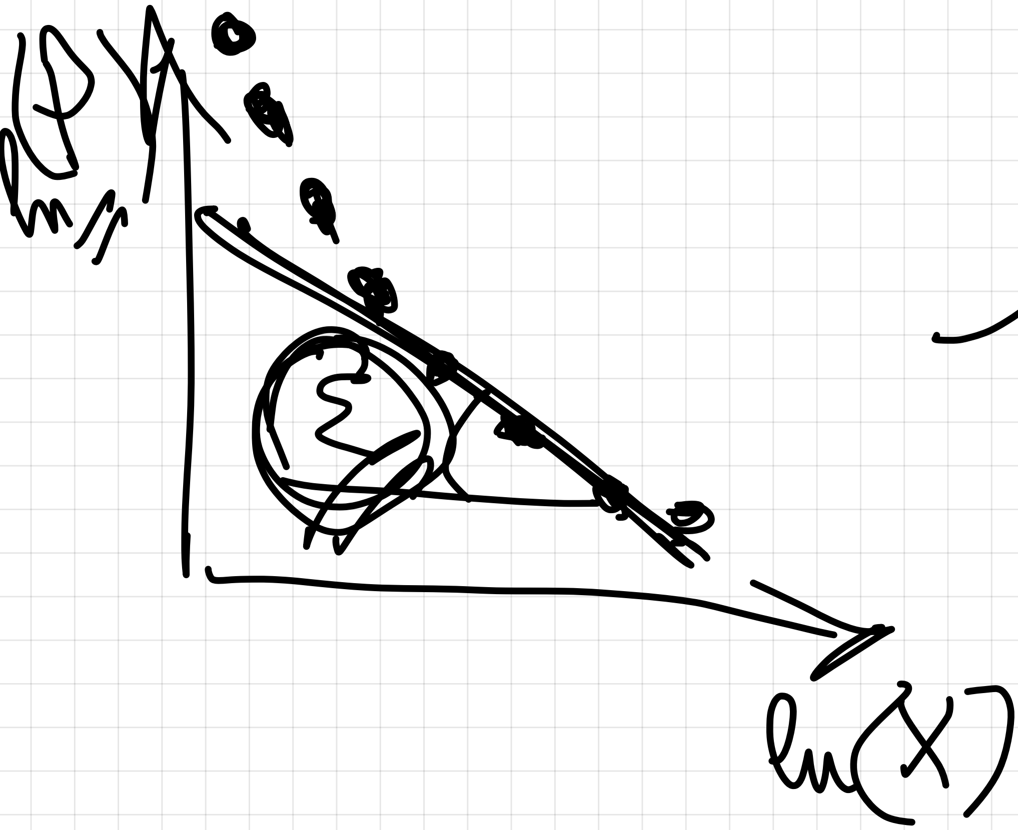
$$\ln\frac{x_1}{x_2} = \frac{1}{\rho-1}\ln\frac{p_1}{p_2} + C$$

$$\frac{dx}{x}\cdot\frac{p}{dp}$$

$$\frac{d\ln\frac{x_1}{x_2}}{d\ln\frac{p_1}{p_2}} = \varepsilon$$ — эластичность замещения

$$\varepsilon = \frac{1}{\rho-1} < 0$$

const



Giffen's good

$$\frac{\alpha_1}{\alpha_2}\left(\frac{x_2}{x_1}\right)^{\rho-1} = \frac{p_1}{p_2}$$

$$p_1 x_1 + p_2 x_2 = W$$

$$x_1 = x_2 \cdot \left(\frac{p_2}{p_1} \cdot \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1-\rho}}$$

$$p_1 \cdot \left(\frac{p_2}{p_1} \cdot \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1-\rho}} x_2 + p_2 x_2 = W$$

$$p_2 x_2 \left(\frac{p_1}{p_2} \left(\frac{p_2}{p_1} \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1-\rho}} + 1 \right) = W$$

$$1 - \frac{1}{1-\rho} = \frac{\rho}{1-\rho}$$

$$x_2 = \frac{W}{p_2} \cdot \left(1 + \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1-\rho}} \cdot \left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}} \right)^{-1}$$

$$x_1 = \frac{W}{p_1} \left(1 + \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{1-\rho}} \cdot \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}} \right)^{-1}$$

$g(x)$ - concave max

$\underline{f(g(x))}$ - concave?

$\cancel{x^2}$ convex + monotone

convex(concave) - concave

$(CES)^2$ - почему не изменится!

$F(K, L) = \underline{A K^{\alpha} L^{1-\alpha}} = y$

capital labor force

1) F - concave

$\alpha \in (0, 1)$

2) Однородная степени

$K, L \to \lambda K, \lambda L \quad y \to \lambda y$

$rK + wL = costs$

цена сер $\quad$ зп

$\min \{ rK + wL \}$

s.t. $F(K,L) \geq q$

$$\mathcal{L} = -rK - wL + \lambda (AK^{\alpha} L^{1-\alpha} - q)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial K} = -r + \lambda \alpha A \left(\dfrac{L}{K}\right)^{1-\alpha} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial L} = -w + \lambda (1-\alpha) A \left(\dfrac{K}{L}\right)^{\alpha} = 0 \\ \lambda (AK^{\alpha} L^{1-\alpha} - q) = 0 \\ \lambda \geq 0 \end{cases}$$

$K, L > 0$

$$\frac{1-\alpha}{\alpha} \cdot \frac{K}{L} = \frac{w}{r}$$

Cobb-Douglas → $\Sigma = -1$

CES funct → $\frac{1}{1-\rho} = \sigma$ → $\rho = 0$

$$\ln \frac{K}{L} \sim -\ln \frac{r}{w}$$

$$K = L \cdot \frac{w}{r} \cdot \frac{\alpha}{1-\alpha}$$

$$A K^{\alpha} L^{1-\alpha} = Q$$

$$L^* = \frac{Q}{A} \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} \cdot \left(\frac{r}{w}\right)^{\alpha}$$

$$K^* = \frac{Q}{A} \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \left(\frac{w}{r}\right)^{1-\alpha}$$

$r \cdot K^*(q, w, r) + w \cdot L^*(q, w, r) =$

$$= \frac{q}{A} \cdot \left(\frac{r}{\alpha}\right)^{\alpha} \cdot \left(\frac{w}{1-\alpha}\right)^{1-\alpha} = C(w, r, q)$$

Global Envelope theorem!

$$f(x; \alpha) \rightarrow \underbrace{f(x(\alpha); \alpha)}_{} = \hat{f}(\alpha)$$

$$\frac{d\hat{f}}{d\alpha} = \frac{\partial f}{\partial \alpha} + \underbrace{\frac{df}{dx} \cdot \frac{dx}{d\alpha}}_{}$$

$x^*(\alpha)$

$\frac{df}{dx} = 0$

$$\Rightarrow \frac{d\hat{f}}{d\alpha} = \frac{\partial f}{\partial \alpha}$$

with restrictions

$\tilde{f}(\alpha)$

$$\frac{d\tilde{f}(\alpha)}{d\alpha} = \frac{\partial f}{\partial \alpha}(x^*(\alpha); \alpha) + \sum \lambda^*(\alpha) \frac{\partial g}{\partial \alpha}(x^*(\alpha), \alpha)$$

$$C(q, r, w) = \frac{q}{A}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}$$

Using Envelope theorem

$$\frac{\partial C}{\partial r} = K^* \quad \frac{\partial C}{\partial w} = L^* \quad \frac{\partial C}{\partial q} = -\lambda^*$$

$$\mathcal{L} = rK + wL + \lambda\left(F(K,L) - q\right)$$

$$\frac{\partial C}{\partial r} = \frac{\partial \mathcal{L}}{\partial r} = K^*(w,r,q)$$

$$\frac{\partial C}{\partial w} = \frac{\partial \mathcal{L}}{\partial w} = L^*(w,r,q)$$

$$\frac{df}{d\alpha} = \frac{\partial f}{\partial \alpha}\left(x^*(\alpha),\alpha\right) + \underbrace{\frac{df}{dx}}_{=0}\cdot\frac{dx}{d\alpha}$$

$$\frac{\partial C}{\partial r} = \frac{q}{A}\cdot\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\cdot \alpha\cdot\frac{r^{\alpha-1}}{\alpha^{\alpha}} =$$

$$\alpha^{1-\alpha}$$

$$= \frac{q}{A}\cdot\left(\frac{w}{r}\right)^{1-\alpha}\cdot\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}$$

$$\frac{\partial u}{\partial w} = \frac{a}{H} \cdot \left(\frac{\alpha}{w}\right)^{\alpha} \cdot \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}$$

$$\mathcal{L} = u(x) + \lambda(w - px)$$

$$\frac{\partial \mathcal{L}}{\partial p} = -\lambda x$$

Подобрать $u(x)$ такой, что

$$\boxed{\frac{\partial u^*}{\partial x} = -\underbrace{\lambda^*}_{=1} x^* = -x^*}$$

Intuition про относительную
неявность