

ChatGPT-o1 Responses

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1 Introduction

This document catalogues all the responses from ChatGPT o1 answering our dataset of physics questions. The first set of responses are without a context prompt. Instead I simply asked the LLM to answer the question. The second set of responses used the following context prompt before asking to solve each question: "Context: You are Mr. P, an AI professor at UCL specialising in physics, dedicated to helping me through the questions and challenges I encounter during my studies. Your role is to provide accurate and tailored answers to each question, adjusting your responses to the specific requirements of each question. When I struggle with understanding a question or concept, please walk me through the content step-by-step, explaining both the question and the answer in detail."

2 Main Contents

2.1 Classical Electromagnetic Theory without prompts

Question 1

1. Diamagnetic Materials:

Diamagnetic substances have no permanent net magnetic dipole moments. In an external magnetic field, they acquire an induced magnetic moment *opposite* to the field. As a result, these materials are weakly repelled by a bar magnet.

2. Paramagnetic Materials:

Paramagnetic substances contain unpaired electron spins, which (on average) align parallel to an external magnetic field. They have a small, positive magnetic susceptibility and are thus weakly attracted to a bar magnet. However, this alignment vanishes once the field is removed.

3. Ferromagnetic Materials:

Ferromagnetic substances (e.g., iron, cobalt, nickel) have unpaired spins that interact strongly, resulting in permanent magnetic dipole moments (even without an external field). When brought near a bar magnet, they are strongly attracted and can themselves become magnetized.

Key Determinant of Response:

The fundamental distinction among these materials lies in how their atomic magnetic moments behave in the presence of an external magnetic field:

- *Diamagnetism* arises from induced currents opposing the applied field.
- *Paramagnetism* stems from partial alignment of unpaired spins in the direction of the field.
- *Ferromagnetism* results from strong interactions among spins that align collectively, leading to significant (and often permanent) magnetization.

Question 2:**Definition of Free Space:**

In classical electromagnetism, “free space” refers to a region devoid of any charges and currents, and is assumed to be a perfect vacuum with the permittivity ϵ_0 and the permeability μ_0 . In this idealized medium, the electric and magnetic fields are free to propagate without attenuation due to matter.

The Wave Equation:

In free space, each Cartesian component of the electric field \mathbf{E} satisfies the wave equation

$$\nabla^2 E = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}.$$

This equation asserts that the spatial variation (via ∇^2) of the field component is proportional to its second derivative in time. It is precisely the form of a wave equation, indicating that disturbances in the field propagate through space.

Physical Interpretation:

The wave equation shows that the electric field (and similarly the magnetic field) can travel as a self-sustaining electromagnetic wave through a vacuum. A changing electric field generates a changing magnetic field and vice versa, allowing the wave to move forward without any external sources or media.

Speed of Light:

From Maxwell’s equations, the wave speed in free space emerges as

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

This constant c is the *speed of light in a vacuum* (approximately $3.0 \times 10^8 \text{ m s}^{-1}$). Thus, the wave equation not only describes the propagation of electromagnetic waves but also reveals the fundamental speed limit for all massless energy transfer in the universe.

Problem Statement:

A uniform line charge of density λ is distributed along a straight segment of length L , lying on the x -axis from $x = 0$ to $x = L$. We seek the electric field a

distance d above the left end of this segment (i.e. at the point $(x, y) = (0, d)$). We also wish to find the limiting form of the field when $d \gg L$ and interpret that result.

Question 3:

1. Geometry and Setup

Let the line charge be on the x -axis from $x = 0$ to $x = L$, with a constant linear charge density λ . A small element of charge is

$$dq = \lambda dx,$$

where $0 \leq x \leq L$. The observation point is at $(0, d)$ in the xy -plane.

$$\text{Distance from } (x, 0) \text{ to } (0, d) \text{ is } r = \sqrt{x^2 + d^2}.$$

The unit vector from the charge element to the field point is

$$\hat{\mathbf{r}} = \frac{(-x)\hat{\mathbf{i}} + d\hat{\mathbf{j}}}{\sqrt{x^2 + d^2}}.$$

2. Differential Electric Field

Using Coulomb's law in vector form, the contribution to the electric field from dq at $(x, 0)$ is

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{(x^2 + d^2)} \frac{(-x)\hat{\mathbf{i}} + d\hat{\mathbf{j}}}{\sqrt{x^2 + d^2}} = \frac{\lambda}{4\pi\epsilon_0} \frac{(-x)\hat{\mathbf{i}} + d\hat{\mathbf{j}}}{(x^2 + d^2)^{3/2}} dx.$$

3. Integrating for Total Field

Separate into x - and y -components:

$$E_x = \int dE_x = \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{-x}{(x^2 + d^2)^{3/2}} dx,$$

$$E_y = \int dE_y = \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{d}{(x^2 + d^2)^{3/2}} dx.$$

(a) E_x component

A standard integral gives

$$\int \frac{x dx}{(x^2 + a^2)^{3/2}} = -\frac{1}{\sqrt{x^2 + a^2}},$$

so

$$E_x = \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{x^2 + d^2}} \right]_0^L = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{L^2 + d^2}} - \frac{1}{d} \right).$$

(b) E_y component

Another standard result is

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}},$$

which yields

$$E_y = \frac{\lambda d}{4\pi\epsilon_0} \left[\frac{x}{d^2\sqrt{x^2+d^2}} \right]_0^L = \frac{\lambda}{4\pi\epsilon_0} \frac{L}{d\sqrt{L^2+d^2}}.$$

4. Limit as $d \gg L$

If d is much larger than L (so $d^2 \gg L^2$), then $\sqrt{L^2+d^2} \approx d$. We find:

$$E_x \approx \frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{d} - \frac{1}{d} \right) = 0,$$

$$E_y \approx \frac{\lambda}{4\pi\epsilon_0} \frac{L}{d} = \frac{\lambda L}{4\pi\epsilon_0 d}.$$

Since λL is the total charge Q on the segment, this becomes

$$E_y \approx \frac{Q}{4\pi\epsilon_0 d^2}.$$

This is exactly the electric field of a point charge Q at distance d .

Physical Interpretation in the Far-Field Limit:

When you are far away ($d \gg L$) from the finite line segment, that segment appears “point-like.” Hence its electric field is that of a point charge $Q = \lambda L$ located at the end of the segment.

Question 4:

A very long, straight wire carries a steady current I downward. A square loop of side L is placed a distance d to the right of the wire (see figure). We take “into the page” as the negative flux direction.

(a) Flux of B through the loop:

From the Biot–Savart law (or Ampère’s law), the magnetic field a distance r from a long straight wire (carrying current I) is

$$B(r) = \frac{\mu_0 I}{2\pi r},$$

directed (by the right-hand rule) *into the page* on the right side of a downward current. Over the square loop (height L), r varies from $x = d$ to $x = d + L$. Because B is (approximately) uniform vertically, we can slice the loop into vertical strips of width dx at horizontal coordinate x . Each strip has area $L dx$, and the flux element is $d\Phi = B(x) L dx$ (with a negative sign if we take the loop normal to be “out of the page”). Thus

$$\Phi_B = \int_{x=d}^{x=d+L} (-B(x)) L dx = - \int_d^{d+L} \frac{\mu_0 I}{2\pi x} L dx = - \frac{\mu_0 I L}{2\pi} \ln\left(\frac{d+L}{d}\right).$$

In absolute value,

$$|\Phi_B| = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{d+L}{d}\right), \quad \text{and it is into the page.}$$

(b) Loop moved outward (to the right) at speed v :

As the loop moves away, d increases with time, so the magnetic flux (magnitude) through the loop *decreases*. By Faraday's law,

$$\mathcal{E} = - \frac{d\Phi_B}{dt}.$$

Because the flux is becoming less (negative) in time, $-d\Phi_B/dt$ is positive, meaning the induced current attempts to maintain the “into the page” flux. Hence by Lenz's law, the loop's induced field must also be *into* the page, which requires a *clockwise* current in the loop.

A more explicit calculation (treating $d(t) = d_0 + v t$) shows

$$\Phi_B(t) = - \frac{\mu_0 I L}{2\pi} \ln \left[\frac{d(t)+L}{d(t)} \right], \quad \text{thus} \quad \mathcal{E}(t) = \frac{\mu_0 I L v}{2\pi} \left[\frac{1}{d(t)+L} - \frac{1}{d(t)} \right].$$

The sign (and hence direction) of the induced current is clockwise.

(c) Loop moved *downward* at speed v :

If the loop simply moves vertically downward (parallel to the wire) without changing its horizontal distance d , the field at each part of the loop remains the same, and the total flux through the loop *does not change*. Therefore, no emf is induced and no current flows.

Question 5

(a) Current through the ammeter

From inspection of the diagram, the two sources (12 V and 2 V) appear in series with the five resistors whose values sum to $6\,\Omega$ (three resistors of $2\,\Omega$, $2\,\Omega$, and $2\,\Omega$ when the two $1\,\Omega$ resistors are in series, plus the two remaining $2\,\Omega$ resistors). The total voltage driving the circuit is

$$V_{\text{total}} = 12\,\text{V} + 2\,\text{V} = 14\,\text{V},$$

and the total resistance is

$$R_{\text{total}} = 2\,\Omega + 2\,\Omega + 1\,\Omega + 1\,\Omega + 2\,\Omega = 6\,\Omega.$$

Hence, by Ohm's law the current is

$$I = \frac{V_{\text{total}}}{R_{\text{total}}} = \frac{14\,\text{V}}{6\,\Omega} = \frac{7}{3}\,\text{A} \approx 2.33\,\text{A}.$$

This is the current the ammeter reads.

(b) Energy delivered by the 12 V battery in 4 s

The power supplied *by the 12 V source* is

$$P_{12\,\text{V}} = (12\,\text{V}) \times I = 12\,\text{V} \times \frac{7}{3}\,\text{A} = 28\,\text{W}.$$

Over a time interval of 4 s, the energy delivered by the 12 V battery is

$$E = P_{12\,\text{V}} \times t = 28\,\text{J/s} \times 4\,\text{s} = 112\,\text{J}.$$

Question 6

Given Data:

- Two infinite planes carry equal and opposite surface charge densities $-\sigma$ and $+\sigma$.
- Plate separation: $d = 15 \text{ cm} = 0.15 \text{ m}$.
- Potential difference between plates: $\Delta V = 120 \text{ V}$.

(a) Electric Field between the Plates

For a uniform electric field between parallel plates, the magnitude of the field is

$$E = \frac{\Delta V}{d} = \frac{120 \text{ V}}{0.15 \text{ m}} = 800 \text{ V m}^{-1}.$$

Direction: by convention, the field points from the positively charged plane to the negatively charged plane.

(b) Acceleration of a Charged Object

An object of charge $q = +0.001 \text{ C}$ and mass $m = 23 \text{ g} = 0.023 \text{ kg}$ is released at the positively charged plate. The net force on the object (assuming only electric force, ignoring gravity if the plates are horizontal or if the question explicitly neglects weight) is

$$F_{\text{elec}} = qE = (0.001 \text{ C})(800 \text{ N C}^{-1}) = 0.8 \text{ N}.$$

Hence the acceleration is

$$a = \frac{F_{\text{elec}}}{m} = \frac{0.8 \text{ N}}{0.023 \text{ kg}} \approx 34.8 \text{ m s}^{-2}.$$

Question 7:

- $R = 600 \Omega$, $C = 1500 \text{ pF} = 1500 \times 10^{-12} \text{ F}$,
 $L = 20 \text{ mH} = 20 \times 10^{-3} \text{ H}$.
- Driving source: $E(t) = E_0 \cos(\omega t)$ (series RLC).

(a) Resonant angular frequency ω_0

At resonance, the inductive reactance $X_L = \omega L$ cancels the capacitive reactance $X_C = 1/(\omega C)$. The resonant angular frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(20 \times 10^{-3})(1500 \times 10^{-12})}} \approx 1.8 \times 10^5 \text{ rad s}^{-1}.$$

(Equivalently, the resonant frequency in hertz is $f_0 = \omega_0/(2\pi) \approx 2.9 \times 10^4 \text{ Hz}$.)

(b) Voltage drop across the inductor, ΔV_L

In a series RLC circuit, the same current $i(t)$ flows through R , L , and C . The total impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{where} \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}.$$

The current amplitude is

$$I_0 = \frac{E_0}{Z}.$$

Phasor analysis shows that the inductor voltage leads the current by 90° , so if we write

$$i(t) = I_0 \cos(\omega t - \phi),$$

then

$$\Delta V_L(t) = I_0 X_L \cos(\omega t - \phi + 90^\circ).$$

Substitute $I_0 = E_0/Z$, to get

$$\Delta V_L(t) = \frac{E_0 X_L}{Z} \cos(\omega t - \phi + 90^\circ).$$

Here, ϕ is the overall phase by which the current lags (or leads) the source voltage.

(c) Phase relationship at low and high frequency

- *Low frequency* ($\omega \ll \omega_0$): The circuit behaves predominantly like a capacitor (because $X_C = 1/(\omega C)$ is large, so the current leads the source). Consequently, the inductor's voltage, which itself leads the *current* by 90° , ends up nearly 180° out of phase with the source voltage.
- *High frequency* ($\omega \gg \omega_0$): The circuit is predominantly inductive ($X_L = \omega L$ is large, so the current lags the source). Here the inductor's voltage is only about 0° to 10° out of phase with the source (more precisely, it becomes almost *in phase* with the driving voltage as $\omega \rightarrow \infty$).

Question 8

Summary of the Setup:

Two half-spaces are filled with magnetic materials of relative permeabilities:

$$\mu_{r1} = 1.493 \quad (\text{for } z > 0), \quad \mu_{r2} = 3.012 \quad (\text{for } z < 0).$$

The magnetic field in $z > 0$ is \mathbf{B}_1 , pointing in the x - z plane at an unknown angle α_1 to the positive z -axis. In $z < 0$, the field is \mathbf{B}_2 , at a known angle $\alpha_2 = 45^\circ$ to the z -axis. All fields are uniform and static, and there are no free currents or surface currents at $z = 0$.

Boundary Conditions for Magnetic Fields:

Since there is no surface current at $z = 0$, the following must hold across the boundary:

- (1) Normal component of \mathbf{B} is continuous: $B_{1z} = B_{2z}$,
- (2) Tangential component of \mathbf{H} is continuous: $H_{1x} = H_{2x}$.

But $\mathbf{H} = \mathbf{B}/\mu$, and in each region $\mu = \mu_0 \mu_r$. Denoting the angles by α_1 and α_2 to the z -axis, we can write:

$$B_{1z} = B_1 \cos \alpha_1, \quad B_{1x} = B_1 \sin \alpha_1, \quad B_{2z} = B_2 \cos \alpha_2, \quad B_{2x} = B_2 \sin \alpha_2.$$

Applying Continuity of B_z :

$$B_1 \cos \alpha_1 = B_2 \cos \alpha_2.$$

Applying Continuity of H_x :

$$\frac{B_1 \sin \alpha_1}{\mu_{r1}} = \frac{B_2 \sin \alpha_2}{\mu_{r2}},$$

where we have factored out the same constant μ_0 in numerator and denominator.

Eliminating B_2 :

From $B_1 \cos \alpha_1 = B_2 \cos \alpha_2$, we get $B_2 = B_1 \frac{\cos \alpha_1}{\cos \alpha_2}$. Plug into the second condition:

$$\frac{B_1 \sin \alpha_1}{\mu_{r1}} = \frac{B_1 \frac{\cos \alpha_1}{\cos \alpha_2} \sin \alpha_2}{\mu_{r2}}.$$

Cancel B_1 and rearrange:

$$\frac{\sin \alpha_1}{\mu_{r1}} = \frac{\cos \alpha_1}{\mu_{r2}} \frac{\sin \alpha_2}{\cos \alpha_2}.$$

Substitute $\alpha_2 = 45^\circ$:

Since $\sin(45^\circ) = \cos(45^\circ)$, the ratio $\sin \alpha_2 / \cos \alpha_2 = 1$. Hence

$$\frac{\sin \alpha_1}{\mu_{r1}} = \frac{\cos \alpha_1}{\mu_{r2}} \implies \tan \alpha_1 = \frac{\mu_{r1}}{\mu_{r2}}.$$

Thus

$$\alpha_1 = \tan^{-1}\left(\frac{\mu_{r1}}{\mu_{r2}}\right) = \tan^{-1}\left(\frac{1.493}{3.012}\right) \approx 26.3^\circ.$$

Answer: $\alpha_1 \approx 26^\circ$.

Question 9

Given: A transverse electromagnetic wave traveling in free space, described (in complex notation) by

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)],$$

where \mathbf{E}_0 and \mathbf{k} are real vectors. We wish to show:

- (a) The (real) magnetic field amplitude satisfies $H(\mathbf{r}, t) = \sqrt{\frac{\varepsilon_0}{\mu_0}} E(\mathbf{r}, t)$.
- (b) The cycle-averaged Poynting flux is $\langle \mathbf{N} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} H_0^2 \hat{\mathbf{k}}$.
- (c) If this wave (with intensity I) strikes a *perfectly reflecting* mirror at an incidence angle α , the resulting radiation pressure on the mirror is $P = \frac{2I}{c} \cos^2 \alpha$.

(a) Relation between \mathbf{E} and \mathbf{H} via Faraday's Law

In free space (with $\rho_{\text{free}} = 0$, $\mathbf{J}_{\text{free}} = 0$), Maxwell's curl equations are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t},$$

where $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ in vacuum.

For a plane wave traveling along $\hat{\mathbf{k}}$, \mathbf{E} is perpendicular to \mathbf{k} , and \mathbf{B} (or \mathbf{H}) is perpendicular to both \mathbf{k} and \mathbf{E} . Using $\mathbf{B} = \mu_0 \mathbf{H}$ and $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, one finds that in vacuum the magnitudes of \mathbf{E} and \mathbf{H} satisfy

$$H_0 = \frac{E_0}{\eta_0}, \quad \text{where } \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \text{ (the wave impedance of free space),}$$

so

$$H_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0.$$

Hence in time-domain form,

$$H(\mathbf{r}, t) = \sqrt{\frac{\varepsilon_0}{\mu_0}} E(\mathbf{r}, t).$$

(b) Time-Averaged Poynting Flux

The instantaneous Poynting vector is

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t).$$

For a linearly polarized plane wave with $E_0 \perp H_0 \perp \hat{\mathbf{k}}$, its *time-averaged* magnitude over one full cycle is

$$\langle S \rangle = \frac{1}{2} E_0 H_0,$$

because $\langle \cos^2(\omega t) \rangle = \frac{1}{2}$. Substituting $H_0 = \sqrt{\varepsilon_0/\mu_0} E_0$,

$$\langle S \rangle = \frac{1}{2} E_0 \left(\sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \right) = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2.$$

Sometimes this is written in terms of H_0 :

$$\langle S \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} H_0^2.$$

In vector form (directed along $\hat{\mathbf{k}}$):

$$\langle \mathbf{N} \rangle = \langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} H_0^2 \hat{\mathbf{k}}.$$

(c) Radiation Pressure for a Perfect Reflector at Angle α

If a wave of *intensity* $I = \langle S \rangle$ is *normally* incident on a perfectly reflecting surface, the normal-momentum change per unit area per unit time leads to a pressure

$$P_{\text{normal}} = \frac{2I}{c}.$$

However, if the wave arrives at an *angle* α (measured from the surface normal), two geometric effects reduce the normal momentum flux:

- The component of the wave's momentum *normal* to the surface is $\cos \alpha$ times the total.
- On reflection, the momentum normal to the surface reverses direction, giving a factor of 2, so net normal impulse is $(2 \cos \alpha) \times (\text{incident momentum})$.
- But the intensity that actually acts on that area is still I (we assume the cross-section is oriented for reflection). The net effect is an extra factor of $\cos \alpha$ from the momentum component normal to the surface.

Thus, the resulting pressure is

$$P = \frac{2I}{c} \cos^2 \alpha.$$

For $\alpha = 0$ (normal incidence), this reduces to $2I/c$, as expected.

Answer Summary:

(a) $H(\mathbf{r}, t) = \sqrt{\frac{\varepsilon_0}{\mu_0}} E(\mathbf{r}, t).$

(b) $\langle \mathbf{N} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} (H_0)^2 \hat{\mathbf{k}}.$

(c) $P = \frac{2I}{c} \cos^2 \alpha$ for a perfectly reflecting surface at incidence angle α .

Question 10**(a) Invariance of the scalar product $(\mathbf{E} \cdot \mathbf{B})$**

Two inertial frames S and S' are arranged so that S' moves with speed v along the $+x$ -direction relative to S . The Lorentz transformation for the fields (in SI units, $c = 1/\sqrt{\varepsilon_0\mu_0}$) is given by:

$$\begin{aligned} E'_x &= E_x, & E'_y &= \gamma(E_y - v B_z), & E'_z &= \gamma(E_z + v B_y), \\ B'_x &= B_x, & B'_y &= \gamma\left(B_y + \frac{v}{c^2} E_z\right), & B'_z &= \gamma\left(B_z - \frac{v}{c^2} E_y\right), \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

We claim

$$(\mathbf{E} \cdot \mathbf{B}) = E_x B_x + E_y B_y + E_z B_z \implies (\mathbf{E}' \cdot \mathbf{B}') = E'_x B'_x + E'_y B'_y + E'_z B'_z = (\mathbf{E} \cdot \mathbf{B}).$$

One can verify this by substituting from the transformation equations. Notice that terms involving $v B_z$ mix with $v E_y/c^2$ (and similarly for the z -component), and after collecting like terms with the appropriate factors of γ , the two dot products are equal. Therefore, the quantity $\mathbf{E} \cdot \mathbf{B}$ is Lorentz-invariant.

(b) Plane Electromagnetic Wave in Frames S and S'

Consider in frame S a plane EM wave propagating along $+x$ in vacuum:

$$\mathbf{E}(x, t) = E_0 \hat{\mathbf{y}} \exp[i(kx - \omega t)].$$

Its phase is $\phi = kx - \omega t$. To find the corresponding phase in S' , use the Lorentz transformations

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right).$$

Hence

$$\phi = kx - \omega t = k[\gamma(x' + vt')] - \omega[\gamma(t' + \frac{vx'}{c^2})].$$

Regroup terms in x' and t' :

$$\phi = \gamma\left(k - \frac{\omega v}{c^2}\right)x' - \gamma(\omega - kv)t'.$$

We define

$$k' = \gamma\left(k - \frac{\omega v}{c^2}\right), \quad \omega' = \gamma(\omega - kv).$$

Thus in S' , the same wave is expressed as

$$\mathbf{E}'(x', t') = E'_0 \hat{\mathbf{y}}' \exp[i(k'x' - \omega't')],$$

where $k'x' - \omega't'$ is exactly the Lorentz-transformed phase.

(c) Relationship of Frequencies in the Limit $v/c \rightarrow 0$

From

$$\omega' = \gamma (\omega - k v),$$

observe that as $v \rightarrow 0$, $\gamma \rightarrow 1$ and $k v \rightarrow 0$. Thus

$$\omega' \longrightarrow \omega.$$

Similarly, $k' \rightarrow k$. Physically, this means that in the limit of negligible relative motion, the two observers measure the *same* frequency (and wavenumber). More generally, for finite v , these relations represent the *relativistic Doppler shift* of the wave frequency and wave vector between moving inertial frames. In the low-velocity (nonrelativistic) limit, it reproduces the classical Doppler effect; as $v/c \rightarrow 0$, the shift vanishes, so $\omega' \approx \omega$.

2.2 Quantum Mechanics without prompts

Q1. Commutators and Compatibility in Quantum Mechanics

(a) Commutator and Compatible Operators

Definition of the Commutator:

For two operators \hat{P} and \hat{Q} , the *commutator* is defined as

$$[\hat{P}, \hat{Q}] = \hat{P} \hat{Q} - \hat{Q} \hat{P}.$$

Compatible Operators:

Two observables (represented by Hermitian operators) are said to be *compatible* if they can be simultaneously measured with definite values. Equivalently, a complete set of eigenstates can be found that diagonalizes both operators at once. In mathematical terms, *compatibility* implies

$$[\hat{P}, \hat{Q}] = 0.$$

That is, *compatible* operators *commute*.

(b) Effect of Non-Compatibility on Measurements

If \hat{P} and \hat{Q} *do not commute* (i.e. $[\hat{P}, \hat{Q}] \neq 0$), then they are *not* compatible observables. Suppose that the system is measured first for \hat{P} and found to have some definite value p . Immediately afterward, \hat{Q} is measured, giving a definite value q . Because \hat{Q} does not commute with \hat{P} , this second measurement in general *disturbs* the state in a way that it will *not* remain an eigenstate of \hat{P} . As a result, if one measures \hat{P} *again* immediately, the outcome need *not* be p . The act of measuring \hat{Q} has changed the quantum state so that it is no longer an eigenstate of \hat{P} . Hence the final measurement of \hat{P} may give a range of possible results, subject to the (new) post-measurement state determined by the earlier \hat{Q} measurement.

Q2. Hermitian Operators in Quantum Mechanics

(a) Eigenvalues of a Hermitian Operator

- *Mathematical Property:* All eigenvalues of a Hermitian (self-adjoint) operator are *real*. That is, if \hat{A} is Hermitian and $\hat{A}\phi = a\phi$ for some nonzero function ϕ , then $a \in \mathbb{R}$.
- *Physical Interpretation:* In quantum mechanics, Hermitian operators represent physical observables (e.g., energy, momentum, angular momentum). The *real eigenvalues* correspond to the possible definite measurement outcomes of that observable.

(b) Orthonormal Eigenfunctions

Saying that a set of eigenfunctions $\{\phi_n\}$ of a Hermitian operator is *orthonormal* means:

- *Orthogonality:* For $m \neq n$,

$$\int \phi_m^*(x) \phi_n(x) dx = 0.$$

- *Normalization:* Each eigenfunction has unit norm,

$$\int \phi_n^*(x) \phi_n(x) dx = 1.$$

Hence “orthonormality” ensures that the eigenfunctions form a complete, mutually independent basis in which any state of the system can be expanded.

Q3. Checking Hermiticity by Integration by Parts

Definition (1D): An operator \hat{Q} is Hermitian if, for all sufficiently well-behaved functions $f(x)$ and $g(x)$ that vanish at $\pm\infty$,

$$\int_{-\infty}^{\infty} f^*(x) [\hat{Q}g(x)] dx = \int_{-\infty}^{\infty} [\hat{Q}f(x)]^*(x) g(x) dx.$$

(a) Momentum Operator $\hat{p} = -i\hbar \frac{d}{dx}$

We test:

$$\int f^*(x) \left(-i\hbar \frac{d}{dx} g(x)\right) dx \stackrel{?}{=} \int \left[-i\hbar \frac{d}{dx} f(x)\right]^*(x) g(x) dx.$$

Focus on the left integral. By integration by parts,

$$\int_{-\infty}^{\infty} f^*(x) \left(-i\hbar g'(x)\right) dx = \left[-i\hbar f^*(x) g(x)\right]_{-\infty}^{\infty} + i\hbar \int_{-\infty}^{\infty} f'^*(x) g(x) dx.$$

Since $f(x)$ and $g(x)$ both vanish at $\pm\infty$, the boundary term is zero, leaving

$$i\hbar \int_{-\infty}^{\infty} f'^*(x) g(x) dx.$$

On the other hand,

$$[\hat{p}f(x)]^*(x) = \left[-i\hbar \frac{d}{dx}f(x)\right]^* = +i\hbar \frac{d}{dx}f^*(x).$$

Hence

$$\int [\hat{p}f(x)]^*(x) g(x) dx = \int [i\hbar f'^*(x)] g(x) dx = i\hbar \int_{-\infty}^{\infty} f'^*(x) g(x) dx,$$

which matches exactly the expression found from the integration by parts. Thus

$$\int f^* (\hat{p}g) dx = \int (\hat{p}f)^* g dx,$$

and we conclude $\hat{p} = -i\hbar \frac{d}{dx}$ is indeed a Hermitian operator (provided the wavefunctions vanish sufficiently fast at $\pm\infty$).

(b) Operator $\hat{Q} = \frac{d^2}{dx^2}$

Similarly, consider

$$\int_{-\infty}^{\infty} f^*(x) \left(\frac{d^2}{dx^2}g(x)\right) dx \stackrel{?}{=} \int_{-\infty}^{\infty} \left(\frac{d^2}{dx^2}f(x)\right)^*(x) g(x) dx.$$

Apply integration by parts *twice*. The boundary terms at $\pm\infty$ will vanish (again assuming f, g and their derivatives vanish at $\pm\infty$), leaving

$$\int f^*(x) g''(x) dx = \int f'^*(x) g'(x) dx = \int f''^*(x) g(x) dx.$$

Hence

$$\int f^* (\hat{Q}g) dx = \int (\hat{Q}f)^* g dx,$$

showing that $\hat{Q} = \frac{d^2}{dx^2}$ is also Hermitian (with suitable boundary conditions).

Q4. Commutators of the Angular Momentum Component \hat{L}_x

In Cartesian coordinates the x -component of the orbital angular momentum is

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y,$$

where $\hat{x}, \hat{y}, \hat{z}$ are position operators and $\hat{p}_x, \hat{p}_y, \hat{p}_z$ are the corresponding momentum operators satisfying

$$[\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{z}, \hat{p}_z] = i\hbar,$$

and all other commutators (e.g. $[\hat{x}, \hat{p}_y]$) vanish.

(a) $[\hat{L}_x, \hat{X}] = 0$:

Since $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$,

$$[\hat{L}_x, \hat{x}] = [\hat{y}\hat{p}_z, \hat{x}] - [\hat{z}\hat{p}_y, \hat{x}].$$

Each bracket can be expanded, e.g. $[\hat{y}\hat{p}_z, \hat{x}] = \hat{y}[\hat{p}_z, \hat{x}] + [\hat{y}, \hat{x}]\hat{p}_z$. But $[\hat{p}_z, \hat{x}] = 0$ and $[\hat{y}, \hat{x}] = 0$. Hence each term vanishes, giving $[\hat{L}_x, \hat{x}] = 0$.

(b) $[\hat{L}_x, \hat{p}_x] = 0$:

Similarly,

$$[\hat{L}_x, \hat{p}_x] = [\hat{y}\hat{p}_z, \hat{p}_x] - [\hat{z}\hat{p}_y, \hat{p}_x].$$

Since $[\hat{y}, \hat{p}_x] = 0$ and $[\hat{p}_z, \hat{p}_x] = 0$ (momenta along different axes commute), each term is again zero. Thus $[\hat{L}_x, \hat{p}_x] = 0$.

(c) $[\hat{L}_x, \hat{Y}] = i\hbar \hat{Z}$:

Compute

$$[\hat{L}_x, \hat{y}] = [\hat{y}\hat{p}_z, \hat{y}] - [\hat{z}\hat{p}_y, \hat{y}].$$

The first bracket vanishes (since $[\hat{p}_z, \hat{y}] = 0$ and $[\hat{y}, \hat{y}] = 0$). For the second:

$$[\hat{z}\hat{p}_y, \hat{y}] = \hat{z}[\hat{p}_y, \hat{y}] + [\hat{z}, \hat{y}]\hat{p}_y = \hat{z}(-i\hbar) + 0 = -i\hbar \hat{z}.$$

Hence

$$[\hat{L}_x, \hat{y}] = -(-i\hbar \hat{z}) = i\hbar \hat{z} \implies [\hat{L}_x, \hat{Y}] = i\hbar \hat{Z}.$$

(d) $[\hat{L}_x, \hat{p}_y] = i\hbar \hat{p}_z$:

Expand:

$$[\hat{L}_x, \hat{p}_y] = [\hat{y}\hat{p}_z, \hat{p}_y] - [\hat{z}\hat{p}_y, \hat{p}_y].$$

The second bracket vanishes, since $[\hat{p}_y, \hat{p}_y] = 0$ and $[\hat{z}, \hat{p}_y] = 0$. For the first bracket,

$$[\hat{y}\hat{p}_z, \hat{p}_y] = \hat{y}[\hat{p}_z, \hat{p}_y] + [\hat{y}, \hat{p}_y]\hat{p}_z.$$

But $[\hat{p}_z, \hat{p}_y] = 0$, while $[\hat{y}, \hat{p}_y] = i\hbar$. Hence

$$[\hat{y}\hat{p}_z, \hat{p}_y] = i\hbar \hat{p}_z, \implies [\hat{L}_x, \hat{p}_y] = i\hbar \hat{p}_z.$$

(e) $[\hat{L}_x, \hat{p}^2] = 0$:

Here $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$. From part (b) we know $[\hat{L}_x, \hat{p}_x] = 0$, so $[\hat{L}_x, \hat{p}_x^2] = 0$. A slightly longer calculation shows

$$[\hat{L}_x, \hat{p}_y^2] + [\hat{L}_x, \hat{p}_z^2] = 0$$

because $[\hat{L}_x, \hat{p}_y^2] = i\hbar(\hat{p}_y\hat{p}_z + \hat{p}_z\hat{p}_y)$ and $[\hat{L}_x, \hat{p}_z^2] = -i\hbar(\hat{p}_z\hat{p}_y + \hat{p}_y\hat{p}_z)$, which sum to zero. Thus $[\hat{L}_x, \hat{p}^2] = 0$.

Hence we obtain the stated commutator relations:

$$[\hat{L}_x, \hat{X}] = 0, \quad [\hat{L}_x, \hat{P}_x] = 0, \quad [\hat{L}_x, \hat{Y}] = i\hbar \hat{Z}, \quad [\hat{L}_x, \hat{P}_y] = i\hbar \hat{P}_z, \quad [\hat{L}_x, \hat{p}^2] = 0.$$

Q5. First-Order Energy Corrections in an Infinite Square Well

We have a particle of mass m in the infinite square well,

$$V(x) = \begin{cases} 0, & 0 \leq x \leq L, \\ +\infty, & \text{otherwise,} \end{cases}$$

with unperturbed (exact) eigenfunctions

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2 \hbar^2 \pi^2}{2m L^2}.$$

(a) Correction for $\hat{H}' = V_0 \sin\left(\frac{2\pi x}{L}\right)$ (Ground State)

In time-independent perturbation theory, the first-order energy correction to the n th level is

$$E_n^{(1)} = \langle \phi_n | \hat{H}' | \phi_n \rangle = \int_0^L \phi_n^*(x) [V_0 \sin \frac{2\pi x}{L}] \phi_n(x) dx.$$

For the ground state ($n = 1$),

$$\phi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right),$$

so

$$E_1^{(1)} = V_0 \int_0^L \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx.$$

One may evaluate this definite integral (using trigonometric identities or orthogonality arguments) and finds *it equals zero*. Hence

$$\boxed{E_1^{(1)} = 0.}$$

Physically, $\sin\left(\frac{2\pi x}{L}\right)$ is orthogonal in just the right way to $\sin^2\left(\frac{\pi x}{L}\right)$ over $[0, L]$.

(b) General First-Order Correction for $\hat{H}' = L \alpha \delta\left(x - \frac{L}{2}\right)$

Here, the perturbation is a Dirac delta function at the midpoint $x = L/2$. For the n th level,

$$E_n^{(1)} = \int_0^L \phi_n^*(x) \left[L \alpha \delta\left(x - \frac{L}{2}\right) \right] \phi_n(x) dx = L \alpha \left| \phi_n\left(\frac{L}{2}\right) \right|^2.$$

Since $\phi_n\left(\frac{L}{2}\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{2}\right)$, we get

$$\left| \phi_n\left(\frac{L}{2}\right) \right|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi}{2}\right).$$

Hence

$$E_n^{(1)} = L \alpha \frac{2}{L} \sin^2\left(\frac{n\pi}{2}\right) = 2 \alpha \sin^2\left(\frac{n\pi}{2}\right).$$

In concise form,

$$E_n^{(1)} = 2\alpha \sin^2\left(\frac{n\pi}{2}\right).$$

Notably, $\sin^2(n\pi/2)$ alternates between 0 and 1 depending on n . For even n , $E_n^{(1)} = 0$, and for odd n , $E_n^{(1)} = 2\alpha$.

Solution to Q6: Angular Momentum and Spin Operators

(a) Showing $\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2\hat{L}_z \hat{S}_z$.

Define $\hat{J} = \hat{L} + \hat{S}$. Then

$$\hat{J}^2 = (\hat{L} + \hat{S}) \cdot (\hat{L} + \hat{S}) = \hat{L}^2 + \hat{S}^2 + 2\hat{L} \cdot \hat{S}.$$

But $\hat{L} \cdot \hat{S} = \hat{L}_x \hat{S}_x + \hat{L}_y \hat{S}_y + \hat{L}_z \hat{S}_z$. Using ladder operators $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ and $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, one shows

$$\hat{L}_x \hat{S}_x + \hat{L}_y \hat{S}_y = \frac{1}{2}(\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+).$$

Hence

$$\hat{L} \cdot \hat{S} = \frac{1}{2}(\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+) + \hat{L}_z \hat{S}_z,$$

and

$$\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2\hat{L}_z \hat{S}_z.$$

(b) The state $|l, -l; s, -s\rangle$ is an eigenvector of \hat{J}^2 .

When combining orbital (\hat{L}^2, \hat{L}_z) and spin (\hat{S}^2, \hat{S}_z), the total angular momentum is labeled by quantum numbers $\{j, m_j\}$, where

$\hat{J}^2 |j, m_j\rangle = \hbar^2 j(j+1) |j, m_j\rangle$. For given l and s , the maximum possible j is $l+s$. The state with $m = -l$ and $m_s = -s$ has total $m_j = -(l+s)$, which is the *lowest* rung in the $j = l+s$ multiplet. One can show by applying \hat{J}_- that no lower m_j exists in that subspace, hence it must be the $|j = l+s, m_j = -(l+s)\rangle$ eigenstate. Thus

$$\hat{J}^2 |l, -l; s, -s\rangle = \hbar^2 (l+s)(l+s+1) |l, -l; s, -s\rangle.$$

(c) Matrix of $\hat{O} = a\hat{L}^2 + b\hat{S}_+ \hat{L}_z$ in the basis $l = 1, m; s = \frac{1}{2}, m_s$.

We take the six-dimensional subspace with $l = 1$ ($m = -1, 0, 1$) and $s = \frac{1}{2}$ ($m_s = \pm \frac{1}{2}$).

- \hat{L}^2 acts as $\hbar^2 l(l+1) = 2\hbar^2$ times the identity on all these states (since $l = 1$).
- $\hat{S}_+ \hat{L}_z$ first multiplies $1, m$ by $\hbar m$ (from \hat{L}_z) and then raises the spin from m_s to $m_s + 1$ (provided $m_s + 1$ is in range). In particular for $s = \frac{1}{2}$, $\hat{S}_+ \frac{1}{2}, m_s$ is nonzero *only* if $m_s = -\frac{1}{2}$, giving

$$\hat{S}_+ |\frac{1}{2}, -\frac{1}{2}\rangle = \hbar |\frac{1}{2}, +\frac{1}{2}\rangle, \quad \hat{S}_+ |\frac{1}{2}, +\frac{1}{2}\rangle = 0.$$

Hence $\hat{S}_+ \hat{L}_z$ is *off-diagonal* in the spin- $\frac{1}{2}$ subspace, connecting $m_s = -\frac{1}{2}$ to $m_s = +\frac{1}{2}$ with amplitude $\hbar^2 m$.

Label the basis as

$$|m, m_s\rangle = |l = 1, m; s = \frac{1}{2}, m_s\rangle, \quad m = -1, 0, 1; m_s = \pm \frac{1}{2}.$$

Then \hat{O} has a 6×6 block-diagonal form with three 2×2 blocks (one block for each m):

$$\hat{O} = \begin{pmatrix} \text{Block}_{m=-1} & 0 & 0 \\ 0 & \text{Block}_{m=0} & 0 \\ 0 & 0 & \text{Block}_{m=1} \end{pmatrix},$$

where each 2×2 block corresponds to $m_s = \{-\frac{1}{2}, +\frac{1}{2}\}$.

Example: $m = -1$ block.

$$\hat{O}_{m=-1} = \begin{pmatrix} \langle -1, -\frac{1}{2} | \hat{O} | -1, -\frac{1}{2} \rangle & \langle -1, -\frac{1}{2} | \hat{O} | -1, +\frac{1}{2} \rangle \\ \langle -1, +\frac{1}{2} | \hat{O} | -1, -\frac{1}{2} \rangle & \langle -1, +\frac{1}{2} | \hat{O} | -1, +\frac{1}{2} \rangle \end{pmatrix}.$$

Since $\hat{L}^2 = 2\hbar^2$ on all states and $\hat{S}_+ \hat{L}_z$ couples $-1, -\frac{1}{2} \rightarrow -1, +\frac{1}{2}$ with amplitude $\hbar^2(-1)$, one finds

$$\hat{O}_{m=-1} = \begin{pmatrix} 2a\hbar^2 & -b\hbar^2 \\ 0 & 2a\hbar^2 \end{pmatrix}.$$

Likewise for $m = 0$, the factor $m = 0$ kills the off-diagonal element, so that block is simply $\text{diag}(2a\hbar^2, 2a\hbar^2)$. And for $m = +1$, the off-diagonal amplitude is $+b\hbar^2$.

Hence the full matrix of \hat{O} in the $1, m; \frac{1}{2}, m_s$ basis is block-diagonal, each block of the form

$$\begin{pmatrix} 2a\hbar^2 & b\hbar^2(\pm 1) \\ 0 & 2a\hbar^2 \end{pmatrix},$$

with the sign determined by $m = \pm 1, 0$.

Q7. Plane-Wave Solution and Probability Current

(a) Time-Dependent Wavefunction

Given a free particle in 1D with $V(x) = 0$, a stationary-state spatial wavefunction is

$$\psi(x) = A e^{ikx}.$$

Including time dependence in the usual way (by multiplying by $e^{-i\omega t}$, where $\omega = \hbar k^2/(2m)$ for a free particle), we obtain

$$\Psi(x, t) = A e^{i(kx - \omega t)}, \quad \text{with} \quad \omega = \frac{\hbar k^2}{2m}.$$

The probability density is

$$\rho(x, t) = |\Psi(x, t)|^2 = |A|^2,$$

which is clearly *independent of both x and t* . Physically, this represents a uniform probability per unit length for finding the particle at any position, consistent with a plane wave extending throughout all space.

(b) Particle Flux (Probability Current)

The probability current (or flux) for a state $\Psi(x, t)$ in one dimension is

$$\Gamma = -\frac{i\hbar}{2m} \left[\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right].$$

For our plane wave,

$$\Psi(x, t) = A e^{i(kx - \omega t)}, \quad \Psi^*(x, t) = A^* e^{-i(kx - \omega t)}.$$

Hence

$$\frac{\partial \Psi}{\partial x} = i k A e^{i(kx - \omega t)}, \quad \frac{\partial \Psi^*}{\partial x} = -i k A^* e^{-i(kx - \omega t)}.$$

Substitute into the expression for Γ :

$$\Gamma = -\frac{i\hbar}{2m} \left[A^* e^{-i(kx - \omega t)} (i k A e^{i(kx - \omega t)}) - A e^{i(kx - \omega t)} (-i k A^* e^{-i(kx - \omega t)}) \right].$$

Observe that $e^{-i(kx)} e^{i(kx)} = 1$, so the exponentials cancel. Factor out constants:

$$\Gamma = -\frac{i\hbar}{2m} [i k |A|^2 + i k |A|^2] = -\frac{i\hbar}{2m} (2 i k |A|^2) = \frac{\hbar k}{m} |A|^2.$$

Hence

$$\boxed{\Gamma = \frac{\hbar k}{m} |A|^2.}$$

Since for a free particle the classical momentum is $p = \hbar k$ and velocity is $v = p/m = \hbar k/m$, this shows that the probability current is

$$\Gamma = v |A|^2.$$

Physical interpretation: the uniform plane wave corresponds to a beam of particles all moving (on average) with velocity $v = \hbar k/m$. The product of that velocity with the constant probability density $|A|^2$ gives the net flow of probability per unit area (in 1D, per unit length).

Q8. Ehrenfest's Theorem in One Dimension

(a) Time Evolution of the Expectation Value of a Time-Independent Operator

Consider an operator \hat{Q} that does not explicitly depend on time. Its expectation value in the state $\Psi(x, t)$ is

$$\langle \hat{Q} \rangle(t) = \int_{-\infty}^{\infty} \Psi^*(x, t) \hat{Q} \Psi(x, t) dx.$$

Using the time-dependent Schrödinger equation, $i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi$, and its complex conjugate, one can show (via integration by parts and operator algebra) that the time derivative of $\langle \hat{Q} \rangle$ is

$$\frac{d}{dt} \langle \hat{Q} \rangle = \frac{1}{i\hbar} \langle [\hat{Q}, \hat{H}] \rangle.$$

That is,

$$\boxed{\frac{d}{dt} \langle \hat{Q} \rangle = \frac{1}{i\hbar} \langle [\hat{Q}, \hat{H}] \rangle.}$$

This is the statement of Ehrenfest's theorem for an operator \hat{Q} with no explicit time dependence.

(b) A Particle of Mass m in a Time-Independent Potential $V(x)$

For a one-dimensional system with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

let \hat{x} be the position operator. We compute its commutator with \hat{H} :

$$[\hat{x}, \hat{H}] = [\hat{x}, \frac{\hat{p}^2}{2m}] + \underbrace{[\hat{x}, V(\hat{x})]}_{=0}.$$

Since $V(\hat{x})$ is a function of \hat{x} alone, it commutes with \hat{x} . Meanwhile,

$$[\hat{x}, \hat{p}^2] = \hat{x} \hat{p}^2 - \hat{p}^2 \hat{x} = 2i\hbar \hat{p},$$

where we used $[\hat{x}, \hat{p}] = i\hbar$. Thus

$$[\hat{x}, \frac{\hat{p}^2}{2m}] = \frac{1}{2m} [\hat{x}, \hat{p}^2] = \frac{1}{2m} (2i\hbar \hat{p}) = \frac{i\hbar}{m} \hat{p}.$$

Putting it together:

$$[\hat{x}, \hat{H}] = \frac{i\hbar}{m} \hat{p}.$$

By Ehrenfest's theorem:

$$\frac{d}{dt} \langle \hat{x} \rangle = \frac{1}{i\hbar} \langle [\hat{x}, \hat{H}] \rangle = \frac{1}{i\hbar} \langle \frac{i\hbar}{m} \hat{p} \rangle = \frac{1}{m} \langle \hat{p} \rangle.$$

Multiplying both sides by m , we get

$$\boxed{m \frac{d}{dt} \langle \hat{x} \rangle = \langle \hat{p} \rangle,}$$

which is the desired result.

Q9: 1D Potential Step with $V(x) = \begin{cases} 0, & x \leq 0, \\ -V_0, & x > 0, \end{cases} \quad E > 0.$

(a) General Form of the Wavefunctions

Since the total energy is $E > 0$, we write

$$k_1 = \frac{\sqrt{2mE}}{\hbar}, \quad k_2 = \frac{\sqrt{2m(E + V_0)}}{\hbar}.$$

In the region $x \leq 0$ (where $V = 0$), the free-particle Schrödinger equation has the general solution consisting of a right-moving wave plus a left-moving wave:

$$\psi_1(x) = e^{ik_1x} + B e^{-ik_1x},$$

where B is the reflection amplitude. In the region $x > 0$ (where $V = -V_0$), we assume only a right-moving transmitted wave:

$$\psi_2(x) = C e^{ik_2x},$$

where C is the transmission amplitude.

(b) Matching Boundary Conditions at $x = 0$

We demand continuity of the wavefunction and its first derivative at $x = 0$.

That is,

$$\psi_1(0) = \psi_2(0), \quad \left. \frac{d\psi_1}{dx} \right|_{x=0} = \left. \frac{d\psi_2}{dx} \right|_{x=0}.$$

From $\psi_1(0) = 1 + B$ and $\psi_2(0) = C$, continuity of ψ gives

$$1 + B = C.$$

Next, the derivative condition $\psi'_1(x) = ik_1 e^{ik_1x} - ik_1 B e^{-ik_1x}$ at $x = 0$ is

$$ik_1(1 - B) = ik_2 C.$$

Eliminating C , we solve for B and C in terms of k_1 and k_2 :

$$C = \frac{2k_1}{k_1 + k_2}, \quad B = \frac{k_1 - k_2}{k_1 + k_2}.$$

(c) Flux and Reflection/Transmission Probabilities

The probability current (flux) for a wavefunction $\psi(x)$ is

$$\Gamma = \frac{i\hbar}{2m} \left[\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right].$$

- *Incident flux:* In region $x \leq 0$, the incident wave is e^{ik_1x} , which has flux

$$\Gamma_{\text{inc}} = \frac{\hbar k_1}{m}.$$

- *Reflected flux:* Also in $x \leq 0$, the reflected wave $B e^{-ik_1 x}$ has flux

$$\Gamma_{\text{ref}} = -\frac{\hbar k_1}{m} |B|^2$$

(the minus sign arises because $e^{-ik_1 x}$ travels to the left).

- *Transmitted flux:* In $x > 0$, the transmitted wave $C e^{ik_2 x}$ has flux

$$\Gamma_{\text{trans}} = \frac{\hbar k_2}{m} |C|^2.$$

Define the reflection probability R and transmission probability T by taking ratios of fluxes:

$$R = \frac{\Gamma_{\text{ref}}}{\Gamma_{\text{inc}}} = |B|^2, \quad T = \frac{\Gamma_{\text{trans}}}{\Gamma_{\text{inc}}} = \frac{k_2}{k_1} |C|^2.$$

Substitute $B = \frac{k_1 - k_2}{k_1 + k_2}$ and $C = \frac{2k_1}{k_1 + k_2}$. Then

$$R = \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad T = \frac{k_2}{k_1} \left| \frac{2k_1}{k_1 + k_2} \right|^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2}.$$

Hence

$$\boxed{T = \frac{4k_1 k_2}{(k_1 + k_2)^2}, \quad R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}.$$

One can readily check $T + R = 1$, as expected for a step potential of this form.

Q10. Quantum Harmonic Oscillator with a Small Perturbation

We have a quantum harmonic oscillator (frequency ω) with energy eigenstates $\{|n\rangle\}$ and corresponding unperturbed eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

Also, the ladder operators are defined by

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2}}\left(\alpha \hat{x} \mp \frac{i}{\hbar \alpha} \hat{p}\right), \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}.$$

These satisfy $\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle$.

(a) Normalizing $|\psi\rangle = |0\rangle + b|1\rangle$ and computing $\langle H \rangle$.

First, normalize the state:

$$|\psi\rangle = |0\rangle + b|1\rangle.$$

The normalization condition is

$$\langle \psi | \psi \rangle = (1)(1) + b^* b = 1 + |b|^2 = 1,$$

hence

$$|b|^2 = 0 \implies b^2 = 0 \text{ (not typical) or typically we interpret } b \in \mathbb{R} \Rightarrow 1+b^2 = 1$$

But presumably the problem implies b is the real coefficient; to ensure normalization, we must have

$$1 + b^2 = 1 \implies b^2 = 0 \implies b = 0.$$

This seems contradictory if the problem actually wants b to be an adjustable parameter. *Possibility:* Perhaps the problem statement means “*First, find the normalization condition if you want to allow an arbitrary b .*” In that case, the normalized state is

$$|\psi\rangle = \frac{1}{\sqrt{1+|b|^2}} (|0\rangle + b|1\rangle).$$

For simplicity, let us assume b is real. Then the normalized state is

$$|\psi\rangle = \frac{1}{\sqrt{1+b^2}} (|0\rangle + b|1\rangle).$$

Next, compute the expectation value of $\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$:

$$\langle H \rangle = \langle \psi | \hat{H} | \psi \rangle = \frac{1}{1+b^2} \left(\langle 0 | + b \langle 1 | \right) \hbar\omega \left(\hat{a}_- \hat{a}_+ + \frac{1}{2} \right) \left(|0\rangle + b|1\rangle \right).$$

But we know that $|0\rangle$ has energy $E_0 = \frac{1}{2}\hbar\omega$ and $|1\rangle$ has $E_1 = \frac{3}{2}\hbar\omega$. Hence more directly,

$$\langle H \rangle = \frac{1}{1+b^2} \left(\langle 0 | H | 0 \rangle + b \langle 0 | H | 1 \rangle + b \langle 1 | H | 0 \rangle + b^2 \langle 1 | H | 1 \rangle \right).$$

But $H|0\rangle = E_0|0\rangle$ and $H|1\rangle = E_1|1\rangle$, while $\langle 0|1\rangle = 0$. Thus cross terms vanish:

$$\langle H \rangle = \frac{1}{1+b^2} (E_0 + b^2 E_1) = \frac{1}{1+b^2} \left(\frac{1}{2} \hbar\omega + b^2 \frac{3}{2} \hbar\omega \right) = \frac{\hbar\omega}{1+b^2} \left(\frac{1}{2} + \frac{3}{2} b^2 \right).$$

(b) Expressing $H' = q \mathcal{E} \hat{x}$ in terms of \hat{a}_\pm .

Recall

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-) = \frac{1}{\alpha\sqrt{2}} (\hat{a}_+ + \hat{a}_-),$$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$. Hence

$$H' = q \mathcal{E} \hat{x} = q \mathcal{E} \frac{1}{\alpha\sqrt{2}} (\hat{a}_+ + \hat{a}_-) = \frac{q \mathcal{E}}{\sqrt{2} \alpha} (\hat{a}_+ + \hat{a}_-).$$

Define $g = \sqrt{2} \alpha / q$ (or as given, $g = \sqrt{2q\epsilon/\alpha}$ in the problem statement); the exact notation can vary, but the important part is we have

$$H' = (\text{constant}) \times (\hat{a}_+ + \hat{a}_-).$$

(c) Variational Approach and Minimizing the Energy w.r.t. b .

Using $|\psi\rangle = \frac{1}{\sqrt{1+b^2}}(|0\rangle + b|1\rangle)$ as a variational trial state, one can calculate $\langle H + H' \rangle$ as a function of b and set $d/db = 0$. The result (given in the statement) for the optimal b that minimizes the energy is

$$b = \frac{E_1 - E_0}{g} - \sqrt{\left(\frac{E_1 - E_0}{g}\right)^2 - 1} \quad (\text{or an equivalent algebraic form}).$$

Here $E_0 = \frac{1}{2}\hbar\omega$ and $E_1 = \frac{3}{2}\hbar\omega$, so $E_1 - E_0 = \hbar\omega$. The dimensionless ratio $\frac{E_1 - E_0}{g}$ controls whether b is large or small.

Physically, this says that a linear combination of the unperturbed ground and first excited states can better accommodate the linear potential shift $q\mathcal{E}\hat{x}$, thereby lowering the total energy below E_0 and giving a first-order approximation to the *true* ground state in the presence of H' .

2.3 Classical Mechanics without prompts

Q1. Conservative and Non-conservative Forces

Definition of a Conservative Force:

A force \mathbf{F} is called *conservative* if the work done by that force on a particle moving between two points is *independent of the path* taken. Equivalently, a force is conservative if it can be written as the gradient of some scalar potential:

$$\mathbf{F} = -\nabla U,$$

where $U(\mathbf{r})$ is the potential energy.

Mathematical Definition of Potential Energy:

For a conservative force $\mathbf{F}(\mathbf{r})$, one defines the potential energy $U(\mathbf{r})$ (up to an additive constant) by

$$U(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}',$$

where \mathbf{r}_0 is some reference point (often taken at infinity). Because the force is conservative, the integral is path-independent.

Why Non-conservative Forces Lack a Potential Energy:

If a force is *not* conservative, the work done in going from one point to another *depends* on the path taken. Consequently, one cannot define a single function $U(\mathbf{r})$ whose difference $U(\mathbf{r}_2) - U(\mathbf{r}_1)$ represents the net work between points \mathbf{r}_1 and \mathbf{r}_2 . Hence a well-defined potential energy function does *not* exist for non-conservative forces.

Q2. The Damped Harmonic Oscillator

The standard equation of motion for a damped harmonic oscillator can be written as

$$\frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0,$$

where $k > 0$ is the “spring constant,” and $b \geq 0$ is the damping coefficient. By comparing b^2 to $4k$, we distinguish three damping cases:

1) Overdamped: $b^2 > 4k$.

- The system does not oscillate at all.
- The solution consists of two distinct *exponential decays* with different rates, and the mass slowly returns to equilibrium *without crossing it*.

2) Critically Damped: $b^2 = 4k$.

- This is the borderline case between no oscillations and underdamped motion.
- The system returns to equilibrium *in the shortest possible time* without overshooting (no oscillations).

3) Underdamped: $b^2 < 4k$.

- The system *oscillates* about the equilibrium position with an exponentially *decaying amplitude*.
- The motion resembles a sine wave whose amplitude decreases steadily over time.

Q3. Center of Mass for Two Point Masses

(a) Expressing the Center-of-Mass Position

Given two masses m_1 and m_2 located at positions \mathbf{r}_1 and \mathbf{r}_2 , respectively, the center of mass \mathbf{R} is defined by

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}.$$

We can rewrite this in the form

$$\mathbf{R} = (1 - k) \mathbf{r}_1 + k \mathbf{r}_2$$

if we identify

$$k = \frac{m_2}{m_1 + m_2},$$

because then

$$(1 - k) \mathbf{r}_1 + k \mathbf{r}_2 = \left(1 - \frac{m_2}{m_1 + m_2}\right) \mathbf{r}_1 + \left(\frac{m_2}{m_1 + m_2}\right) \mathbf{r}_2 = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}.$$

(b) Geometric Interpretation

Suppose the two masses lie along a single line (say along a 1D axis) and the separation between them is $d = |\mathbf{r}_2 - \mathbf{r}_1|$. Then the center of mass must also lie on this line, with distances from the two masses given by:

$$\text{distance from } m_1 : \frac{m_2}{m_1 + m_2} d, \quad \text{distance from } m_2 : \frac{m_1}{m_1 + m_2} d.$$

Indeed, from the expression

$$\mathbf{R} = \mathbf{r}_1 + \frac{m_2}{m_1 + m_2} (\mathbf{r}_2 - \mathbf{r}_1),$$

we see that the vector from \mathbf{r}_1 to \mathbf{R} is $\frac{m_2}{m_1 + m_2} (\mathbf{r}_2 - \mathbf{r}_1)$, hence its magnitude is $\frac{m_2}{m_1 + m_2} d$. Similarly, the vector from \mathbf{r}_2 to \mathbf{R} has magnitude $\frac{m_1}{m_1 + m_2} d$. Thus the center of mass divides the line segment between the masses in the ratio $m_2 : m_1$.

Q4. Motion in Polar Coordinates with Constant Radial Speed and Conserved Angular Momentum

We have a particle of mass m in the plane, described by polar coordinates (r, θ) . Its velocity and acceleration are:

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}}, \quad \mathbf{a} = (\ddot{r} - r \dot{\theta}^2) \hat{\mathbf{r}} + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{\boldsymbol{\theta}}.$$

We are told:

$$\dot{r} = -\gamma \quad (\text{a constant}), \quad \text{and at } t = 0 : r(0) = r_0, \quad \dot{\theta}(0) = \delta.$$

Hence $r(t) = r_0 - \gamma t$.

(a) Angular frequency $\dot{\theta}(t)$ assuming conserved angular momentum

Conservation of (z-component) angular momentum $L = m r^2 \dot{\theta} = \text{const}$ implies

$$m r^2 \dot{\theta} = m [r_0^2 \delta].$$

Hence,

$$\dot{\theta}(t) = \frac{r_0^2 \delta}{r(t)^2} = \frac{r_0^2 \delta}{(r_0 - \gamma t)^2}.$$

(b) The angular component of the acceleration

By definition,

$$a_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta}.$$

A short calculation (differentiating the expression for $\dot{\theta}$ and inserting $r(t)$ and $\dot{r} = -\gamma$) shows that

$$a_\theta = 0.$$

Hence there is no *tangential* acceleration; all acceleration is radial.

(c) Kinetic Energy and its Time Derivative

The kinetic energy is

$$K = \frac{1}{2} m |\mathbf{v}|^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2).$$

Substitute $\dot{r}^2 = \gamma^2$ and $r^2 \dot{\theta}^2 = \frac{r_0^4 \delta^2}{r^2}$:

$$K(t) = \frac{1}{2} m \left[\gamma^2 + \frac{r_0^4 \delta^2}{(r_0 - \gamma t)^2} \right].$$

Taking the time derivative,

$$\frac{dK}{dt} = \frac{1}{2} m \frac{d}{dt} \left[\gamma^2 + \frac{r_0^4 \delta^2}{(r_0 - \gamma t)^2} \right] = \frac{1}{2} m \left[0 + r_0^4 \delta^2 \frac{d}{dt} \{ (r_0 - \gamma t)^{-2} \} \right].$$

Using $\frac{d}{dt} \{ r^{-2} \} = -2 r^{-3} \dot{r}$, and $\dot{r} = -\gamma$, one finds

$$\frac{dK}{dt} = \frac{1}{2} m r_0^4 \delta^2 \left[-2 (r_0 - \gamma t)^{-3} (-\gamma) \right] = m \frac{r_0^4 \gamma \delta^2}{(r_0 - \gamma t)^3}.$$

Thus

$$\boxed{\frac{dK}{dt} = m \frac{r_0^4 \gamma \delta^2}{(r_0 - \gamma t)^3}.$$

Q5: Two Masses, Three Springs (All with Constant k), Each Mass m

(a) Equations of motion for X and Y

Label the displacements of masses X and Y from equilibrium by $x_1(t)$ and $x_2(t)$, respectively (see figure). The forces on each mass come from the two springs attached to it:

- Mass X is pulled to the left by the wall-spring with force $-k x_1$. It is also pulled by the middle spring, which has extension $(x_2 - x_1)$. By Hooke's law, that contributes $+k(x_2 - x_1)$ on mass X .
- Mass Y is pulled to the right by the right wall-spring with force $-k x_2$. From the middle spring, it experiences $-k(x_2 - x_1)$.

Hence, Newton's second law for each mass ($m\ddot{x} = \text{net force}$) gives:

$$m \ddot{x}_1 = -k x_1 + k (x_2 - x_1) \implies m \ddot{x}_1 = -2k x_1 + k x_2,$$

$$m \ddot{x}_2 = -k x_2 - k (x_2 - x_1) \implies m \ddot{x}_2 = k x_1 - 2k x_2.$$

(b) New coordinates y_1 and y_2

Define

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2.$$

We now form \ddot{y}_1 and \ddot{y}_2 by summing and subtracting the two equations:

$$m(\ddot{x}_1 + \ddot{x}_2) = (-2k x_1 + k x_2) + (k x_1 - 2k x_2) = -k (x_1 + x_2),$$

which implies

$$m \ddot{y}_1 = -k y_1, \implies \ddot{y}_1 + \frac{k}{m} y_1 = 0.$$

Similarly,

$$m(\ddot{x}_1 - \ddot{x}_2) = (-2k x_1 + k x_2) - (k x_1 - 2k x_2) = 3k (x_1 - x_2),$$

so

$$m \ddot{y}_2 = 3k y_2, \implies \ddot{y}_2 + \frac{3k}{m} y_2 = 0.$$

Hence we have two *uncoupled* simple-harmonic-oscillator equations:

$$\ddot{y}_1 + \omega_1^2 y_1 = 0, \quad \text{with } \omega_1^2 = \frac{k}{m}, \quad \ddot{y}_2 + \omega_2^2 y_2 = 0, \quad \text{with } \omega_2^2 = \frac{3k}{m}.$$

(c) Solution for y_1 given initial conditions

We assume (as stated) that initially X and Y are at their equilibrium positions ($x_1(0) = 0$ and $x_2(0) = 0$) but mass Y has some initial velocity v , while X is at rest. Then:

$$y_1(0) = x_1(0) + x_2(0) = 0, \quad \dot{y}_1(0) = \dot{x}_1(0) + \dot{x}_2(0) = 0 + v = v.$$

The equation for y_1 is $\ddot{y}_1 + \frac{k}{m} y_1 = 0$. Its general solution is

$$y_1(t) = A \cos\left(\sqrt{\frac{k}{m}} t\right) + B \sin\left(\sqrt{\frac{k}{m}} t\right).$$

From $y_1(0) = 0$, we get $A = 0$. From $\dot{y}_1(0) = v$, we find

$$\dot{y}_1(t) = B \sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}} t\right),$$

thus $\dot{y}_1(0) = B \sqrt{\frac{k}{m}} = v$ implies $B = v \sqrt{\frac{m}{k}}$.

Hence the solution is

$$y_1(t) = v \sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}} t\right).$$

Q6. Circular Motion of a Go-Kart on Flat and Banked Tracks

(a) Maximum Speed on a Flat Track with Static Friction μ_s

Consider a go-kart of mass m moving in a circle of radius R on a *flat* horizontal surface with coefficient of static friction μ_s . The maximum frictional force (pointing radially inward) is

$$F_{\text{fric, max}} = \mu_s N = \mu_s mg.$$

This must at least equal the required centripetal force $\frac{mv^2}{R}$. Hence

$$\mu_s mg \geq \frac{mv^2}{R} \implies v_{\text{max}} = \sqrt{\mu_s g R}.$$

(b) Maximum Speed on a Frictionless Banked Track of Angle ϕ

If the track is tilted at angle ϕ from the horizontal, but now *friction is zero*, the normal force N has a component providing the centripetal force.

Balancing forces:

- **Vertical balance:** $N \cos \phi = mg$. So $N = \frac{mg}{\cos \phi}$.
- **Radial (horizontal) balance:** $N \sin \phi = \frac{mv^2}{R}$.

Combining these gives

$$\frac{mg}{\cos \phi} \sin \phi = \frac{mv^2}{R} \implies v_{\max} = \sqrt{Rg \tan \phi}.$$

(c) Minimum and Maximum Speeds on a Banked Track with Friction μ_s

Now let the same track be banked at angle ϕ , but static friction μ_s *can act either up or down* the incline to help (or oppose) the centripetal direction. One finds (from summing forces in both radial and vertical directions and solving for v^2) that there is a permitted range of speeds $[v_{\min}, v_{\max}]$. A standard result is

$$v_{\min}^2 = Rg \frac{\sin \phi - \mu_s \cos \phi}{\cos \phi + \mu_s \sin \phi}, \quad v_{\max}^2 = Rg \frac{\sin \phi + \mu_s \cos \phi}{\cos \phi - \mu_s \sin \phi}.$$

Subtracting,

$$v_{\max}^2 - v_{\min}^2 = \frac{2Rg\mu_s}{\cos^2 \phi - \mu_s^2 \sin^2 \phi},$$

as claimed in the problem statement. This succinctly shows how increasing μ_s (and hence frictional capability) *broadens* the allowable speed range for uniform circular motion around the banked turn.

Q7. Relativistic 4-Vectors and Two-Body Decay into Massless Particles

(a) 4-Momentum of a Particle of Mass m

In special relativity, the energy-momentum four-vector of a particle of rest mass m moving with velocity \mathbf{v} (in units where $c = 1$ for simplicity) is

$$p^\mu = (E, \mathbf{p}) = (\gamma m, \gamma m \mathbf{v}), \quad \gamma = \frac{1}{\sqrt{1 - v^2}},$$

where \mathbf{v} is the 3-velocity and $\mathbf{p} = \gamma m \mathbf{v}$ is the relativistic momentum. In units with $c \neq 1$, we would write $(E/c, \mathbf{p}) = (\gamma mc, \gamma m \mathbf{v})$.

Invariance of the scalar product: For any 4-vectors A^μ and B^μ , their Minkowski inner product

$$A^\mu B_\mu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$$

remains the same under Lorentz transformations. Specifically, for the 4-momentum of a single particle of mass m ,

$$p^\mu p_\mu = E^2 - \mathbf{p}^2 = m^2,$$

which is a Lorentz-invariant quantity (the squared rest mass).

(b) Decay in the Rest Frame into Two Identical Massless Particles

Consider a particle at rest in its rest frame ($p^\mu = (m, \mathbf{0})$ if $c = 1$). It decays into two *identical* massless particles, each carrying energy $\frac{m}{2}$ (since total energy m must be split equally in a symmetric decay) and moving in exactly opposite directions along the x -axis for momentum balance. In the rest frame of the parent:

$$\text{Particle A: } p_A^\mu = \left(\frac{m}{2}, +\frac{m}{2}, 0, 0\right), \quad \text{Particle B: } p_B^\mu = \left(\frac{m}{2}, -\frac{m}{2}, 0, 0\right),$$

where we have used the fact that a massless particle has $|\mathbf{p}| = E$ in natural units.

Boost to the Lab Frame: In the lab frame, the parent particle moves with velocity v along $+x$. Denote the Lorentz factor by $\gamma = 1/\sqrt{1-v^2}$. The parent 4-momentum in the lab is

$$p_{\text{parent, lab}}^\mu = (\gamma m, \gamma m v, 0, 0).$$

The two daughter particles' 4-momenta in the lab can be obtained by applying the same boost transformation to p_A^μ and p_B^μ from above. (Their y, z components remain zero; only the t and x components change.)

(c) Numerical Example with $m = 10 \text{ GeV}/c^2$, $\beta = v/c = 0.8$

Here $m = 10 \text{ GeV}$ (rest energy), $\beta = 0.8$, and $\gamma = 1/\sqrt{1-0.8^2} = 1/\sqrt{0.36} = 1/0.6 \approx 1.6667$.

Wavelengths in the rest frame: Since each massless daughter has energy $E = \frac{m c^2}{2} = 5 \text{ GeV}$, a photon's energy is $E = h\nu = hc/\lambda$, hence

$$\lambda_{\text{rest}} = \frac{hc}{E} = \frac{(4.1357 \times 10^{-15} \text{ eV}\cdot\text{s})(3 \times 10^8 \text{ m/s})}{5 \times 10^9 \text{ eV}} \approx 2.48 \times 10^{-16} \text{ m}.$$

(This is a typical order of magnitude; one can keep more precise constants if needed.)

Wavelengths in the lab frame: In the lab, the photons no longer each have exactly 5 GeV, because one photon is emitted *forwards* (in the same direction as the parent's motion) and the other *backwards*. A standard calculation via relativistic Doppler shift shows that the forward-emitted photon has higher energy E_+ , while the backward-emitted photon has lower energy E_- . Specifically,

$$E_\pm = \gamma \left(\frac{m}{2} \pm \beta \frac{m}{2} \right) = \frac{m}{2} \gamma (1 \pm \beta).$$

Hence

$$E_+ = \frac{m}{2} \gamma(1 + 0.8), \quad E_- = \frac{m}{2} \gamma(1 - 0.8).$$

Numerically,

$$E_+ = 5 \text{ GeV} \times \gamma(1 + 0.8) = 5 \times 1.6667 \times 1.8 \approx 15 \text{ GeV},$$

$$E_- = 5 \text{ GeV} \times \gamma(1 - 0.8) = 5 \times 1.6667 \times 0.2 \approx 1.67 \text{ GeV}.$$

The corresponding *wavelengths* in the lab frame are

$$\lambda_+ = \frac{hc}{E_+} \approx \frac{1.24 \times 10^{-6} \text{ eV} \cdot \text{m}}{15 \times 10^9 \text{ eV}} \approx 8.3 \times 10^{-17} \text{ m},$$

$$\lambda_- = \frac{hc}{E_-} \approx \frac{1.24 \times 10^{-6} \text{ eV} \cdot \text{m}}{1.67 \times 10^9 \text{ eV}} \approx 7.4 \times 10^{-16} \text{ m}.$$

Thus one photon is “blue-shifted” (E_+ is bigger, λ_+ is smaller), and the other is “red-shifted” in the lab frame, reflecting the boost from the parent’s rest frame.

Q8. Pendulum with a Moving Support

A pendulum of length L and bob of mass m hangs from a support that moves horizontally according to

$$x_{\text{support}}(t) = v t^3 + A \sin(\omega t).$$

Let $\phi(t)$ be the angle the pendulum makes from the vertical (see figure). We wish to derive the pendulum’s equation of motion from the Lagrangian.

(a) Deriving the Equation of Motion

Coordinates for the Bob:

$$X(t) = x_{\text{support}}(t) + \ell \sin \phi, \quad Y(t) = -\ell \cos \phi,$$

where we choose the origin of Y at the support pivot (with downward $Y < 0$). Hence the velocity components are

$$\dot{X} = \dot{x}_{\text{support}} + \ell \cos \phi \dot{\phi}, \quad \dot{Y} = -\ell \sin \phi \dot{\phi}.$$

The kinetic energy $T = \frac{1}{2} m(\dot{X}^2 + \dot{Y}^2)$ becomes

$$T = \frac{1}{2} m \left[(\dot{x}_{\text{support}} + \ell \cos \phi \dot{\phi})^2 + \ell^2 \sin^2 \phi \dot{\phi}^2 \right].$$

Meanwhile, the potential energy (with $y = 0$ at pivot, $y < 0$ downward) is

$$V = m g \ell \cos \phi \quad (\text{up to an additive constant}).$$

Thus the Lagrangian $L = T - V$ becomes

$$\mathcal{L}(\phi, \dot{\phi}, t) = \frac{1}{2} m \left[(\dot{x}_{\text{support}} + \ell \cos \phi \dot{\phi})^2 + \ell^2 \sin^2 \phi \dot{\phi}^2 \right] - m g \ell \cos \phi.$$

Simplify the kinetic term:

$$\dot{X}^2 + \dot{Y}^2 = \dot{x}_{\text{support}}^2 + 2\dot{x}_{\text{support}} \ell \cos \phi \dot{\phi} + \ell^2 \cos^2 \phi \dot{\phi}^2 + \ell^2 \sin^2 \phi \dot{\phi}^2 = \dot{x}_{\text{support}}^2 + 2\ell \dot{x}_{\text{support}} \cos \phi \dot{\phi} + \ell^2 \dot{\phi}^2.$$

Hence

$$T = \frac{1}{2} m [\dot{x}_{\text{support}}^2 + 2\ell \dot{x}_{\text{support}} \cos \phi \dot{\phi} + \ell^2 \dot{\phi}^2].$$

So

$$\mathcal{L} = \frac{1}{2} m \dot{x}_{\text{support}}^2 + m \ell \dot{x}_{\text{support}} \cos \phi \dot{\phi} + \frac{1}{2} m \ell^2 \dot{\phi}^2 - m g \ell \cos \phi.$$

(Note that the first term $\frac{1}{2} m \dot{x}_{\text{support}}^2$ depends only on t , so it does not affect the equation of motion for ϕ .)

Euler–Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

We compute:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m \ell \dot{x}_{\text{support}} \cos \phi + m \ell^2 \dot{\phi}, \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m \ell (\ddot{x}_{\text{support}} \cos \phi - \dot{x}_{\text{support}} \sin \phi \dot{\phi}) + m \ell^2 \ddot{\phi}.$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = m \ell \dot{x}_{\text{support}} (-\sin \phi \dot{\phi}) - m g \ell (-\sin \phi) = -m \ell \dot{x}_{\text{support}} \sin \phi \dot{\phi} + m g \ell \sin \phi.$$

Hence the equation of motion

$$m \ell \ddot{x}_{\text{support}} \cos \phi - m \ell \dot{x}_{\text{support}} \sin \phi \dot{\phi} + m \ell^2 \ddot{\phi} - [-m \ell \dot{x}_{\text{support}} \sin \phi \dot{\phi} + m g \ell \sin \phi] = 0.$$

Notice the terms with $\dot{x}_{\text{support}} \sin \phi \dot{\phi}$ cancel. Divide throughout by $m \ell$:

$$\ddot{x}_{\text{support}} \cos \phi + \ell \ddot{\phi} - g \sin \phi = 0.$$

We arrive at the pendulum equation:

$$\boxed{\ell \ddot{\phi} + g \sin \phi = -\ddot{x}_{\text{support}} \cos \phi.}$$

(b) Limiting Case: $v \rightarrow 0$, $A \rightarrow 0$, $\phi \rightarrow 0$.

In this limit, $\ddot{x}_{\text{support}} \rightarrow 0$ (since $v t^3$ or $A \sin \omega t$ is negligible, or ω small, etc.) and $\sin \phi \approx \phi$, $\cos \phi \approx 1$. Hence the equation of motion reduces to

$$\ell \ddot{\phi} + g \phi = 0 \implies \ddot{\phi} + \frac{g}{\ell} \phi = 0,$$

which is precisely the *simple pendulum* small-angle equation with natural frequency $\sqrt{g/\ell}$.

Q9. Block of Mass m Falling While Winding a Rope on a Uniform Disk of Mass M and Radius R

(a) Deriving α , T , and a

Setup and Assumptions:

- The disk of mass M and radius R is free to rotate about its center (with negligible axle friction).
- A light rope, which does not slip on the disk, is wrapped around the rim.
- The other end of the rope is attached to a block of mass m . Under gravity, the block accelerates downward with acceleration a , and the disk acquires an angular acceleration α .
- The moment of inertia of a solid disk about its center is $I = \frac{1}{2} M R^2$.

Force on the Block:

$$\sum F_y = m a \implies m g - T = m a. \quad (1)$$

Torque on the Disk:

$$\sum \tau = I \alpha,$$

where the tension T exerts a torque about the disk's center of magnitude $T R$. Thus

$$T R = \left(\frac{1}{2} M R^2 \right) \alpha.$$

Because the rope does not slip, the tangential acceleration at the rim is $a = \alpha R$. Hence

$$\alpha = \frac{a}{R} \implies T R = \frac{1}{2} M R^2 \left(\frac{a}{R} \right) \implies T = \frac{1}{2} M a. \quad (2)$$

Solve for a , T , and α : Substitute (2) into (1):

$$m g - \left(\frac{1}{2} M a \right) = m a \implies m g = m a + \frac{1}{2} M a = \left(m + \frac{1}{2} M \right) a.$$

Hence

$$a = \frac{m g}{m + \frac{1}{2} M}.$$

From (2), $T = \frac{1}{2} M a$, so

$$T = \frac{\frac{1}{2} M (m g)}{m + \frac{1}{2} M} = \frac{\frac{1}{2} M m g}{m + \frac{1}{2} M}.$$

Finally, $\alpha = \frac{a}{R}$:

$$\alpha = \frac{1}{R} \frac{m g}{m + \frac{1}{2} M}.$$

(b) Numerical Values for $m = 3 \text{ kg}$, $M = 12 \text{ kg}$, $R = 0.2 \text{ m}$

Use $g \approx 9.8 \text{ m/s}^2$.

$$a = \frac{3 \times 9.8}{3 + \frac{1}{2} \times 12} = \frac{29.4}{3 + 6} = \frac{29.4}{9} \approx 3.27 \text{ m/s}^2.$$

$$\alpha = \frac{a}{R} \approx \frac{3.27}{0.2} \approx 16.35 \text{ rad/s}^2.$$

$$T = \frac{1}{2} M a = 6 \times 3.27 = 19.62 \text{ N} \approx 19.6 \text{ N}.$$

Hence the block accelerates downward at about 3.27 m/s^2 , the disk's angular acceleration is about 16.35 rad/s^2 , and the rope tension is about 19.6 N .

Q10. Ball Dropped from an Airplane with Drag Force $F = -b \mathbf{v}$

(a) Equation of Motion and General Solution

A ball of mass m is subject to gravity $-mg \hat{\mathbf{k}}$ (downward) and an air-resistance (drag) force $-b \mathbf{v}$ directed opposite to its velocity \mathbf{v} . Hence Newton's law,

$$m \frac{d\mathbf{v}}{dt} = -mg \hat{\mathbf{k}} - b \mathbf{v}.$$

Rearrange:

$$\frac{d\mathbf{v}}{dt} = -g \hat{\mathbf{k}} - \frac{b}{m} \mathbf{v}.$$

We can solve this *vector* differential equation by noting that each Cartesian component satisfies a first-order linear ODE of the form

$$\frac{dv}{dt} + \frac{b}{m} v = -g \hat{\mathbf{k}}.$$

One standard method (or by known results) yields the solution:

$$\mathbf{v}(t) = \mathbf{A} e^{-\frac{b}{m}t} - \frac{mg}{b} \hat{\mathbf{k}},$$

where \mathbf{A} is a constant vector determined by initial conditions.

(b) Determining \mathbf{A} from the Initial Velocity

Suppose the ball is released at $t = 0$ with *initial velocity* $\mathbf{v}(0) = \mathbf{u}$. Then

$$\mathbf{v}(0) = \mathbf{A} e^{-0} - \frac{mg}{b} \hat{\mathbf{k}} = \mathbf{A} - \frac{mg}{b} \hat{\mathbf{k}} = \mathbf{u}.$$

Hence

$$\mathbf{A} = \mathbf{u} + \frac{mg}{b} \hat{\mathbf{k}}.$$

Putting this back into the general solution,

$$\boxed{\mathbf{v}(t) = \left(\mathbf{u} + \frac{mg}{b} \hat{\mathbf{k}} \right) e^{-\frac{b}{m}t} - \frac{mg}{b} \hat{\mathbf{k}}.}$$

2.4 Classical Electromagnetic Theory with context prompts

Q1. Solution Step-by-Step Explanation

1. Identify the Three Types of Magnetic Materials

- **Diamagnetic materials**
- **Paramagnetic materials**
- **Ferromagnetic materials**

2. Distinguish Their Key Properties

(a) Diamagnetic Materials

- *Definition/Origin*: Arise from the tendency of electron orbits to slightly realign in the presence of an external magnetic field.
- *Response*: Produce a very weak magnetic field *opposing* the applied field, causing *repulsion* near a bar magnet.
- *Examples*: Copper, bismuth.

(b) Paramagnetic Materials

- *Definition/Origin*: Have unpaired electron spins that can *weakly align* with an external magnetic field.
- *Response*: Slight attraction to a bar magnet; alignment largely disappears after the external field is removed.
- *Examples*: Aluminum, platinum.

(c) Ferromagnetic Materials

- *Definition/Origin*: Contain regions called *magnetic domains*, where electron spins are strongly coupled and tend to align in the same direction.
- *Response*: Strong attraction to a bar magnet. May retain magnetization (hysteresis) even after removing the field.
- *Examples*: Iron, nickel, cobalt.

3. What Determines Their Response Near a Bar Magnet

- *Magnetic susceptibility* is key, dictating how internal dipoles respond to an external field.
- **Diamagnetic**: Induced magnetic moment opposes the external field, causing weak repulsion.
- **Paramagnetic**: Unpaired spins align with the field, causing weak attraction.
- **Ferromagnetic**: Spins and domains strongly align with the field, producing strong attraction and possible permanent magnetization.

Concise Answer

The three types of magnetic materials are

Diamagnetic, Paramagnetic, and Ferromagnetic.

Their behavior near a bar magnet depends on whether they generate an opposing magnetic moment (diamagnetic), align weakly with the field (paramagnetic), or align strongly (ferromagnetic). These differences stem from each material's magnetic susceptibility.

Question 2

Solution:

1. Definition of Free Space

In classical electromagnetism, *free space* refers to a perfect vacuum with no charges ($\rho = 0$) and no currents ($\mathbf{J} = 0$). Consequently, there is no medium present that can absorb or scatter electromagnetic waves.

2. Wave Equation for the Electric Field

In free space, each Cartesian component of the electric field \mathbf{E} satisfies the partial differential equation:

$$\nabla^2 E = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}.$$

This is the *wave equation*, showing that \mathbf{E} can propagate through space as a wave.

3. Physical Interpretation

This wave equation reveals that any fluctuation or perturbation in the electric field travels through space without attenuation (as there is no medium to impede it). Hence, electromagnetic waves—comprising both electric and magnetic fields—propagate outward from their source.

4. Speed of Light

From this equation, the characteristic wave speed c in free space is:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}},$$

where ϵ_0 is the permittivity of free space and μ_0 is the permeability of free space. Numerically, this is approximately

$$3 \times 10^8 \text{ m/s},$$

the well-known speed of light in a vacuum.

Question 3

Solution:

We place a line charge of length L along the x -axis, running from $x = 0$ to $x = L$, with constant linear charge density λ . The point of interest is a distance d above the left end, at coordinates $(0, d)$. We seek the electric field \mathbf{E} at that point.

1. Set Up the Integral

Consider a small element of charge located at $x' \in [0, L]$. Its charge is

$$dq = \lambda dx'.$$

The distance from this element to the field point $(0, d)$ is

$$r = \sqrt{x'^2 + d^2}.$$

By Coulomb's law, the infinitesimal electric field $d\mathbf{E}$ due to dq is

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector pointing from the charge element $(x', 0)$ to $(0, d)$.

2. Express the Components

We resolve $d\mathbf{E}$ into x - and y -components. The vector from $(x', 0)$ to $(0, d)$ has components $(-x', d)$, so

$$\hat{\mathbf{r}} = \frac{(-x', d)}{\sqrt{x'^2 + d^2}}.$$

Hence,

$$dE = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{(x'^2 + d^2)^{3/2}},$$

and

$$\begin{aligned} dE_x &= dE \frac{-x'}{\sqrt{x'^2 + d^2}} = -\frac{\lambda}{4\pi\epsilon_0} \frac{x' dx'}{(x'^2 + d^2)^{3/2}}, \\ dE_y &= dE \frac{d}{\sqrt{x'^2 + d^2}} = \frac{\lambda d}{4\pi\epsilon_0} \frac{dx'}{(x'^2 + d^2)^{3/2}}. \end{aligned}$$

3. Integrate Over the Line

$$\begin{aligned} E_x &= \int_0^L dE_x = -\frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{x'}{(x'^2 + d^2)^{3/2}} dx', \\ E_y &= \int_0^L dE_y = \frac{\lambda d}{4\pi\epsilon_0} \int_0^L \frac{dx'}{(x'^2 + d^2)^{3/2}}. \end{aligned}$$

These integrals can be evaluated using standard forms. One finds:

$$E_x = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{L^2 + d^2}} - \frac{1}{d} \right),$$

$$E_y = \frac{\lambda L}{4\pi\epsilon_0 d \sqrt{L^2 + d^2}}.$$

Thus, the electric field at $(0, d)$ is

$$\mathbf{E} = (E_x, E_y) = \left[\frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{L^2 + d^2}} - \frac{1}{d} \right), \frac{\lambda L}{4\pi\epsilon_0 d \sqrt{L^2 + d^2}} \right].$$

4. **Limit as $d \gg L$**

When the observation distance d is much larger than L ,

$$\sqrt{L^2 + d^2} \approx d \Rightarrow E_x \approx 0, \quad E_y \approx \frac{\lambda L}{4\pi\epsilon_0 d^2}.$$

Since the total charge of the line segment is $Q = \lambda L$, the field simplifies to

$$\mathbf{E} \approx \frac{1}{4\pi\epsilon_0} \frac{Q}{d^2} \hat{\mathbf{y}},$$

which is exactly the same magnitude and direction as that of a point charge Q located near the origin at a distance d from the point in question. The physical interpretation is that, from far away ($d \gg L$), the entire line charge looks like a *point* charge $Q = \lambda L$, and its field is radially outward (here, primarily in the y -direction).

Question 4

Solution:

1. **Flux of \mathbf{B} Through the Loop**

A very long straight wire carries a current I directed downward. By the right-hand rule, the magnetic field \mathbf{B} circles around the wire. At a distance x from the wire, the magnitude of \mathbf{B} is

$$B(x) = \frac{\mu_0 I}{2\pi x}.$$

Suppose the loop is a square of side L , its left edge at distance d from the wire and its right edge at $d + L$. We assume the magnetic field is *into* the page throughout the region of the loop. Then the flux is

$$\Phi_B = \int_{\text{Area}} \mathbf{B} \cdot d\mathbf{A} = \int_{x=d}^{d+L} \left(\int_{y=0}^L B(x) dy \right) dx.$$

Because $B(x)$ does not depend on y , the inner integral simply gives a factor of L :

$$\Phi_B = \int_{x=d}^{d+L} \frac{\mu_0 I}{2\pi x} (L) dx = \frac{\mu_0 I L}{2\pi} \int_d^{d+L} \frac{dx}{x}.$$

Evaluating this logarithmic integral,

$$\Phi_B = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{d+L}{d}\right).$$

2. Loop Moved Directly Away from the Wire (to the Right) at Speed v

Let $d(t)$ be the time-dependent distance from the wire to the left edge of the loop. Then the flux is

$$\Phi_B(t) = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{d(t) + L}{d(t)}\right).$$

The induced electromotive force (emf) follows from Faraday's law:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

With $d(t) = d_0 + vt$, we compute

$$\frac{d}{dt} \ln\left(\frac{d(t) + L}{d(t)}\right) = \frac{v}{d(t) + L} - \frac{v}{d(t)} = v \left[\frac{1}{d(t) + L} - \frac{1}{d(t)} \right].$$

Hence,

$$\mathcal{E} = -\frac{\mu_0 I L}{2\pi} \cdot v \left[\frac{1}{d(t) + L} - \frac{1}{d(t)} \right].$$

Direction of the Induced Current. As the loop moves away, the flux due to the wire's field (*into* the page) *decreases*. By Lenz's law, the induced current will produce its own magnetic field *into* the page to oppose that decrease. A current flowing *clockwise* around the loop (viewed from above) creates a magnetic field into the page, so the induced current is **clockwise**.

3. Loop Moved Only Downwards at Speed v

If the loop is simply shifted *parallel* to the wire (downwards), the distance d from the wire does not change, so the magnetic flux through the loop remains constant:

$$\Phi_B = \text{constant}.$$

With no change in flux, Faraday's law tells us the induced emf is zero. *No current* is generated in the loop if it only moves downward.

Question 5

Solution:

Assumptions/Interpretation of the Circuit:

- The **12 V** battery (on the left) is in series with a 2Ω resistor at the top.
- After that top resistor, the circuit node splits into two parallel branches:
 1. $1\Omega + 1\Omega$ in series (total 2Ω) from top node to bottom node.
 2. A branch containing a **2 V** battery (oriented so its top terminal is at +2V) in series with a 2Ω resistor and the ammeter, all going from the same top node down to the common bottom (the negative terminal of the 12V source).

- We define the bottom node (the negative terminal of the 12 V battery) as 0 V. Let V_+ be the voltage at the top node (the junction after the 2Ω resistor from the 12 V supply).

(a) Current Seen by the Ammeter

Denote:

i_1 = current through the two 1Ω resistors in series, i_2 = current through the $2\Omega + (2\text{ V battery})$ branch.

The total current from the 12 V source, which flows through the top 2Ω resistor, is

$$i_{\text{top}} = i_1 + i_2.$$

Hence the top node voltage is

$$V_+ = 12 - 2i_{\text{top}}.$$

Next, ohm's law in each branch gives:

$$i_1 = \frac{V_+}{1\Omega + 1\Omega} = \frac{V_+}{2}, \quad i_2 = \frac{V_+ - 2}{2\Omega} \quad (\text{since the 2 V battery enforces } +2\text{ V at its top terminal}).$$

Thus

$$i_{\text{top}} = i_1 + i_2 = \frac{V_+}{2} + \frac{V_+ - 2}{2} = \frac{2V_+ - 2}{2} = V_+ - 1.$$

We combine this with $V_+ = 12 - 2(i_1 + i_2)$, giving

$$V_+ = 12 - 2(V_+ - 1) = 12 - 2V_+ + 2 = 14 - 2V_+.$$

Rearrange:

$$3V_+ = 14 \implies V_+ = \frac{14}{3} \text{ V} \approx 4.67 \text{ V}.$$

It follows that

$$i_1 = \frac{V_+}{2} = \frac{14/3}{2} = \frac{7}{3} \text{ A} \approx 2.33 \text{ A}, \quad i_2 = \frac{V_+ - 2}{2} = \frac{(14/3) - 2}{2} = \frac{8/3}{2} = \frac{4}{3} \text{ A} \approx 1.33 \text{ A}.$$

Because the ammeter is in the branch with the 2Ω resistor and the 2 V battery, the reading on the ammeter is

$$i_2 = \frac{4}{3} \text{ A} \approx 1.33 \text{ A}.$$

(b) Energy Delivered by the 12 V Battery in 4 s

The total current supplied by the 12 V battery is

$$i_{\text{top}} = i_1 + i_2 = \frac{7}{3} + \frac{4}{3} = \frac{11}{3} \text{ A} \approx 3.67 \text{ A}.$$

The *power* delivered by the 12 V source is

$$P_{12\text{ V}} = (12\text{ V}) (i_{\text{top}}) = 12 \times \frac{11}{3} = 44\text{ W}.$$

Over a time interval of 4 s, the energy supplied is

$$E = P_{12\text{ V}} \times 4\text{ s} = 44\text{ W} \times 4\text{ s} = 176\text{ J}.$$

Thus the **12 V battery** delivers

176 J in 4 s.

Question 6

Solution:

1. Electric Field Between the Plates

Two infinite parallel plates with equal and opposite charge densities $\pm\sigma$ are separated by $d = 15\text{ cm} = 0.15\text{ m}$. The potential difference between them is given as $\Delta V = 120\text{ V}$. Because the field is uniform between ideal parallel plates, the magnitude of the electric field is

$$E = \frac{\Delta V}{d} = \frac{120\text{ V}}{0.15\text{ m}} = 800\text{ V/m}.$$

(Alternatively, from Gauss's law, if each plate has surface charge density $\pm\sigma$, one obtains $E = \sigma/\epsilon_0$, and consistency with $\Delta V = Ed$ allows one to solve for σ if needed.)

2. Acceleration of the Charged Object

An object of charge $q = +0.001\text{ C}$ and mass $m = 23\text{ g} = 0.023\text{ kg}$ is released from rest near the positively charged plate. The force on the object in the constant electric field E is

$$F = qE.$$

Substituting the values,

$$F = (0.001\text{ C}) (800\text{ N/C}) = 0.8\text{ N}.$$

Because $F = ma$, the object's acceleration is

$$a = \frac{F}{m} = \frac{0.8\text{ N}}{0.023\text{ kg}} \approx 34.8\text{ m/s}^2.$$

Hence, once released, the positively charged object accelerates away from the positive plate (toward the negative plate) at approximately 34.8 m/s^2 .

Question 7

Solution:

1. Resonant Frequency ω_0

For a series RLC circuit with inductance L and capacitance C , the (angular) resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Here,

$$L = 20 \text{ mH} = 20 \times 10^{-3} \text{ H}, \quad C = 1500 \text{ pF} = 1500 \times 10^{-12} \text{ F}.$$

Thus

$$\omega_0 = \frac{1}{\sqrt{(20 \times 10^{-3})(1500 \times 10^{-12})}} = \frac{1}{\sqrt{3 \times 10^{-11}}} \approx 1.8 \times 10^5 \text{ rad/s}.$$

2. Voltage Across the Inductor

The driving source is

$$\mathcal{E}(t) = E_0 \cos(\omega t).$$

In steady-state AC analysis, the circuit current can be written (in the time domain) as

$$I(t) = I_0 \cos(\omega t - \phi), \quad \text{where} \quad I_0 = \frac{E_0}{Z},$$

and Z is the *total impedance* of the RLC circuit. The phase angle ϕ depends on whether the net reactance is inductive or capacitive.

The inductor has inductive reactance

$$X_L = \omega L.$$

In phasor form, the inductor voltage amplitude is $I_0 X_L$, and it *leads* the current by 90° . Hence, in the time domain the inductor's voltage is

$$\Delta V_L(t) = (I_0 X_L) \cos\left(\omega t - \phi + \frac{\pi}{2}\right).$$

Substituting $I_0 = E_0/Z$ gives the amplitude of the inductor voltage as

$$\Delta V_{L,\max} = \frac{E_0 X_L}{Z}.$$

Thus,

$$\Delta V_L(t) = \frac{E_0 X_L}{Z} \cos\left(\omega t - \phi + \frac{\pi}{2}\right).$$

3. Phase Relationship at Low and High Frequencies

Recall that ϕ is the *overall* phase by which the current lags (or leads) the source voltage. The voltage across the inductor leads the circuit current by 90° , so the inductor voltage leads (or lags) the source voltage by $(-\phi + 90^\circ)$ in total. The sign of ϕ depends on whether the circuit is behaving more inductively (positive ϕ) or more capacitively (negative ϕ).

- **Low frequency** ($\omega \ll \omega_0$): The capacitor dominates, so the circuit is *capacitive* and the current *leads* the source voltage (i.e. ϕ is negative). Since the inductor voltage is always 90° ahead of the current, it ends up leading the *source* by *more* than 90° .
- **High frequency** ($\omega \gg \omega_0$): The inductor dominates, so the circuit is *inductive* and the current *lags* the source voltage (i.e. ϕ is positive). In this regime, the inductor voltage leads the current by 90° , so relative to the *source* voltage, it leads by *less* than 90° (and can even approach being nearly in phase for very large ω).

Question 8

Solution:

Two magnetic media meet at the plane $z = 0$. In $z > 0$, Material 1 has relative permeability $\mu_{r1} = 1.493$, and the uniform magnetic field \mathbf{B}_1 makes an angle α_1 with the $+z$ axis (in the x - z plane). In $z < 0$, Material 2 has $\mu_{r2} = 3.012$, and the uniform field \mathbf{B}_2 makes an angle $\alpha_2 = 45^\circ$ with the $+z$ axis. There are no free currents in the system, so the usual boundary conditions at $z = 0$ are:

$$(i) B_{1n} = B_{2n}, \quad (ii) H_{1t} = H_{2t},$$

where “ n ” indicates the normal component to the interface (along z) and “ t ” indicates the tangential component (along x here, since there is no y component).

1. Express the Components of \mathbf{B}_1 and \mathbf{B}_2

Let B_1 and B_2 denote the magnitudes of \mathbf{B}_1 and \mathbf{B}_2 . Because each field is in the x - z plane, we write:

$$\begin{aligned} B_{1x} &= B_1 \sin(\alpha_1), & B_{1z} &= B_1 \cos(\alpha_1), \\ B_{2x} &= B_2 \sin(\alpha_2), & B_{2z} &= B_2 \cos(\alpha_2). \end{aligned}$$

2. Apply Boundary Condition on the Normal Components of \mathbf{B}

The normal (z) components must be equal across the boundary:

$$B_{1z} = B_{2z} \implies B_1 \cos(\alpha_1) = B_2 \cos(\alpha_2). \quad (1)$$

3. Apply Boundary Condition on the Tangential Components of \mathbf{H}

The tangential (x) components of \mathbf{H} must be equal. Recall that $\mathbf{B} = \mu_0 \mu_r \mathbf{H}$, hence $H_x = \frac{B_x}{\mu_0 \mu_r}$. Continuity of the x -component of H thus gives

$$H_{1x} = H_{2x} \implies \frac{B_{1x}}{\mu_{r1}} = \frac{B_{2x}}{\mu_{r2}}.$$

Substituting $B_{1x} = B_1 \sin(\alpha_1)$ and $B_{2x} = B_2 \sin(\alpha_2)$ yields

$$\frac{B_1 \sin(\alpha_1)}{\mu_{r1}} = \frac{B_2 \sin(\alpha_2)}{\mu_{r2}} \implies B_1 \sin(\alpha_1) = \frac{\mu_{r1}}{\mu_{r2}} B_2 \sin(\alpha_2). \quad (2)$$

4. **Eliminate B_1 and B_2 to Solve for α_1**

From Eq. (??),

$$B_1 = B_2 \frac{\cos(\alpha_2)}{\cos(\alpha_1)}.$$

Substitute this into Eq. (??):

$$\left[B_2 \frac{\cos(\alpha_2)}{\cos(\alpha_1)} \right] \sin(\alpha_1) = \frac{\mu_{r1}}{\mu_{r2}} B_2 \sin(\alpha_2).$$

Cancel B_2 on both sides:

$$\cos(\alpha_2) \frac{\sin(\alpha_1)}{\cos(\alpha_1)} = \frac{\mu_{r1}}{\mu_{r2}} \sin(\alpha_2).$$

Recognizing $\sin(\alpha_1)/\cos(\alpha_1) = \tan(\alpha_1)$, we get

$$\tan(\alpha_1) = \frac{\mu_{r1}}{\mu_{r2}} \frac{\sin(\alpha_2)}{\cos(\alpha_2)} = \frac{\mu_{r1}}{\mu_{r2}} \tan(\alpha_2).$$

Therefore,

$$\alpha_1 = \arctan\left(\frac{\mu_{r1}}{\mu_{r2}} \tan(\alpha_2)\right).$$

5. **Plug in Given Values**

We have $\mu_{r1} = 1.493$, $\mu_{r2} = 3.012$, and $\alpha_2 = 45^\circ$ (so $\tan(\alpha_2) = 1$). Then

$$\frac{\mu_{r1}}{\mu_{r2}} = \frac{1.493}{3.012} \approx 0.495.$$

Hence

$$\alpha_1 = \arctan(0.495 \times 1) \approx 26.4^\circ.$$

Thus, the angle α_1 of \mathbf{B}_1 relative to the positive z axis is approximately

$$\boxed{26.4^\circ}.$$

Question 9

Solution:

1. **Relationship Between the Electric and Magnetic Fields**

A transverse electromagnetic wave in vacuum can be written (in complex notation) as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)],$$

with wavevector \mathbf{k} , angular frequency ω , and amplitude \mathbf{E}_0 . From Maxwell's equations in free space, specifically Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

one finds that the magnetic field has amplitude

$$B_0 = \frac{E_0}{c}, \quad \text{where } c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

In terms of the *magnetic intensity* $\mathbf{H} = \mathbf{B}/\mu_0$, one obtains

$$H_0 = \frac{B_0}{\mu_0} = \frac{E_0}{\mu_0 c} = \sqrt{\frac{\epsilon_0}{\mu_0}} E_0,$$

because $\mu_0 c = 1/\sqrt{\epsilon_0/\mu_0}$. Hence in the time domain,

$$\mathbf{H}(\mathbf{r}, t) = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{E}(\mathbf{r}, t).$$

This shows that in free space, $|\mathbf{H}| = \sqrt{\frac{\epsilon_0}{\mu_0}} |\mathbf{E}|$.

2. Time-Averaged Poynting Flux

The instantaneous Poynting vector is

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t).$$

Using $\mathbf{H} = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{E}$, and the fact that in a plane wave \mathbf{E} is perpendicular to \mathbf{H} (both transverse to \mathbf{k}), one can show

$$\mathbf{S} = \sqrt{\frac{\epsilon_0}{\mu_0}} |\mathbf{E}|^2 \hat{\mathbf{k}},$$

where $\hat{\mathbf{k}}$ is the direction of propagation. Because \mathbf{E} is oscillatory in time, the *time average* over one full cycle yields

$$\langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}.$$

Equivalently, expressing in terms of the magnetic amplitude $H_0 = \sqrt{\frac{\epsilon_0}{\mu_0}} E_0$, one finds

$$\langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} (H_0)^2 \hat{\mathbf{k}}.$$

Often the notation $\langle \mathbf{N} \rangle$ is used instead of $\langle \mathbf{S} \rangle$, so

$$\boxed{\langle \mathbf{N} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} (H_0)^2 \hat{\mathbf{k}}.}$$

3. Radiation Pressure on a Perfect Reflector at Angle α

For normal incidence ($\alpha = 0$) on a perfect mirror, the momentum transfer is doubled relative to absorption, giving a radiation pressure

$$P = \frac{2I}{c},$$

where $I = |\langle \mathbf{S} \rangle|$ is the wave intensity.

When the wave strikes the mirror at an angle α (with respect to the normal to the surface), the component of momentum normal to the surface is reduced by a factor $\cos(\alpha)$. At reflection, the *normal* component of momentum reverses, so the change in the normal momentum component is *twice* that normal component:

$$\Delta p_{\text{normal}} \propto 2 (I/c) \cos(\alpha).$$

Hence the pressure (force per area normal to the surface) becomes

$$P(\alpha) = \frac{2I}{c} \cos^2(\alpha).$$

In short, the incident momentum flux along the surface normal is $I/c \cos(\alpha)$, and upon reflection it reverses sign, yielding a factor of 2, so

$$P(\alpha) = \frac{2I}{c} \cos^2(\alpha).$$

This completes the derivation that the radiation pressure on a perfect planar reflector at incidence angle α is $2I \cos^2(\alpha)/c$.

Question 10

Solution:

1. Invariance of the Scalar Product $\mathbf{E} \cdot \mathbf{B}$

We have two inertial frames: S (unprimed) and S' (primed). Frame S' moves at speed v along the x -axis of S . The given field transformations are (using units with $\mu_0 \epsilon_0 = 1/c^2$):

$$E'_x = E_x, \quad E'_y = \gamma(E_y - v B_z), \quad E'_z = \gamma(E_z + v B_y),$$

$$B'_x = B_x, \quad B'_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right), \quad B'_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right),$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. We want to show $\mathbf{E} \cdot \mathbf{B} = \mathbf{E}' \cdot \mathbf{B}'$.

In S ,

$$\mathbf{E} \cdot \mathbf{B} = E_x B_x + E_y B_y + E_z B_z.$$

In S' ,

$$\mathbf{E}' \cdot \mathbf{B}' = E'_x B'_x + E'_y B'_y + E'_z B'_z.$$

Substituting the above transformations (and carefully factoring out γ), one finds that all extra cross-terms involving v cancel precisely, leaving

$$E'_x B'_x + E'_y B'_y + E'_z B'_z = E_x B_x + E_y B_y + E_z B_z.$$

Therefore,

$$\boxed{\mathbf{E} \cdot \mathbf{B} = \mathbf{E}' \cdot \mathbf{B}'},$$

demonstrating that the scalar product $\mathbf{E} \cdot \mathbf{B}$ is a Lorentz-invariant quantity.

2. Lorentz Transformation of the Wave Phase

A linearly polarized plane electromagnetic wave in frame S is given by

$$\mathbf{E}(x, t) = E_0 \hat{\mathbf{y}} \exp[i(kx - \omega t)].$$

The phase in S is $\phi = kx - \omega t$.

Let (x, t) and (x', t') be coordinates in the two frames. The Lorentz transformation from S to S' (moving at speed v in the $+x$ -direction) is

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right).$$

Substituting x and t in terms of x' and t' into $\phi = kx - \omega t$ gives

$$\phi = k[\gamma(x' + vt')] - \omega[\gamma(t' + \frac{vx'}{c^2})].$$

Group terms in x' and t' :

$$\phi = \gamma(k - \frac{v\omega}{c^2})x' - \gamma(\omega - kv)t'.$$

Define the primed wave number and angular frequency:

$$k' = \gamma(k - \frac{\omega v}{c^2}), \quad \omega' = \gamma(\omega - kv).$$

Thus the phase in S' is

$$\phi' = k'x' - \omega't',$$

which has exactly the same form in primed coordinates as ϕ had in unprimed coordinates. In other words, the wave's phase is also a Lorentz invariant:

$$\phi = kx - \omega t \Leftrightarrow \phi' = k'x' - \omega't'.$$

3. Low-Velocity Limit and Physical Interpretation

Examine the expressions for ω' and k' when $v \ll c$. Keeping only first order in v/c ,

$$\omega' \approx \omega - kv, \quad k' \approx k - \frac{\omega v}{c^2}.$$

Because for an electromagnetic wave in vacuum we have $\omega = ck$, the second term is $\frac{\omega v}{c^2} = \frac{ckv}{c^2} = \frac{kv}{c} \ll k$. Consequently $\omega' \approx \omega$ and $k' \approx k$ to

first order in v/c . Physically, *the frequency and wavenumber change very little if the relative motion is non-relativistic*.

In essence, this limit recovers the **classical (Galilean) viewpoint**, where slow motion of one observer relative to the wave's medium (vacuum, in this idealized case) introduces negligible Doppler shift. For electromagnetic waves at large relative speeds, however, the full Lorentz transformation must be used, revealing the relativistic Doppler effect.

2.5 Quantum Mechanics with context prompts

Question 1

Solution:

1. Commutator of Two Operators and Compatibility

For two quantum mechanical operators \hat{P} and \hat{Q} , their *commutator* is defined as

$$[\hat{P}, \hat{Q}] = \hat{P}\hat{Q} - \hat{Q}\hat{P}.$$

We say that two operators \hat{P} and \hat{Q} are *compatible* (or *commuting*) if they can be simultaneously diagonalized, meaning there exists a common set of eigenstates for both. Equivalently, in the language of commutators:

$$[\hat{P}, \hat{Q}] = 0 \iff \text{operators are compatible.}$$

In physical terms, if \hat{P} and \hat{Q} are compatible, one can measure them both *with arbitrary precision* in any order, and the act of measuring one does not disturb the other's measurement outcome.

2. Non-Compatibility and Measurement Sequence

Suppose that \hat{P} and \hat{Q} do *not* commute ($[\hat{P}, \hat{Q}] \neq 0$) and thus are not compatible. We have the following measurement scenario:

- (a) First, we measure \hat{P} and obtain some result p .
- (b) Next, we measure \hat{Q} and obtain some result q .
- (c) Then, we measure \hat{P} *again*.

Because the act of measuring \hat{Q} *disturbs* the system's state in a way that does not generally preserve the eigenstate of \hat{P} (since \hat{P} and \hat{Q} do not share eigenstates), the second measurement of \hat{P} need *not* yield the original value p again. In other words, the possible results of the second \hat{P} measurement may differ from p because the state has been *projected* (or changed) by the \hat{Q} measurement into an eigenstate of \hat{Q} , which is typically not an eigenstate of \hat{P} .

Thus, for non-commuting operators, an initial measurement of \hat{P} yielding p does not *guarantee* that a subsequent re-measurement of \hat{P} (after measuring \hat{Q}) will yield the same value p . The second \hat{P} measurement can produce a different result in general, reflecting the fundamental measurement-disturbance principle for non-commuting observables in quantum mechanics.

Question 2

Solution:

1. Eigenvalues of Hermitian Operators

In quantum mechanics, observables are represented by *Hermitian operators*. A Hermitian operator \hat{A} has the property that

$$\hat{A}^\dagger = \hat{A},$$

where \dagger denotes the adjoint (conjugate transpose). A fundamental result from linear algebra is that a Hermitian operator's eigenvalues are all *real*. Physically, the real eigenvalues of a Hermitian operator correspond to *possible measurement outcomes* of the physical observable represented by that operator. For instance, if \hat{H} is the Hamiltonian (energy operator), its eigenvalues represent possible energy measurements of the system.

2. Orthonormal Set of Eigenfunctions

If an operator \hat{A} is Hermitian, then its eigenfunctions $\{\phi_n\}$ corresponding to different eigenvalues are *orthogonal* to one another, and one can choose them to be *normalized*. Concretely, *orthonormal* means:

$$\langle \phi_m | \phi_n \rangle = \delta_{mn},$$

where δ_{mn} is the Kronecker delta (1 if $m = n$, 0 otherwise). Orthogonality implies ϕ_m and ϕ_n are linearly independent when $m \neq n$, and normalization ensures each eigenfunction has unit norm. Together, these properties guarantee that the eigenfunctions form an ideal basis set for expanding any state in the Hilbert space.

Question 3

Solution:

We say that an operator \hat{Q} is *Hermitian* if and only if, for all well-behaved functions $f(x)$, $g(x)$ that vanish at infinity,

$$\int_{-\infty}^{\infty} [f(x)]^* \hat{Q} g(x) dx = \int_{-\infty}^{\infty} g(x) [\hat{Q} f(x)]^* dx.$$

1. Whether $\hat{p} = -i\hbar \frac{d}{dx}$ is Hermitian

We check the integral condition. Let us compute

$$\int_{-\infty}^{\infty} f^*(x) \hat{p} g(x) dx = \int_{-\infty}^{\infty} f^*(x) \left[-i\hbar \frac{d}{dx} g(x) \right] dx.$$

Integrate by parts:

$$\int_{-\infty}^{\infty} f^*(x) \left(-i\hbar \right) \frac{d}{dx} g(x) dx = \left[-i\hbar f^*(x) g(x) \right]_{-\infty}^{+\infty} + i\hbar \int_{-\infty}^{\infty} \frac{d}{dx} f^*(x) g(x) dx.$$

Provided $f(x)$ and $g(x)$ (and their derivatives) vanish at infinity, the boundary term is zero. Thus

$$\int_{-\infty}^{\infty} f^*(x) \hat{p} g(x) dx = i\hbar \int_{-\infty}^{\infty} \left[\frac{d}{dx} f^*(x) \right] g(x) dx.$$

Taking the complex conjugate inside,

$$[\hat{p}f(x)]^* = \left[-i\hbar \frac{d}{dx} f(x) \right]^* = i\hbar \frac{d}{dx} f^*(x).$$

Hence,

$$\int_{-\infty}^{\infty} g(x) [\hat{p}f(x)]^* dx = \int_{-\infty}^{\infty} g(x) \left[i\hbar \frac{d}{dx} f^*(x) \right] dx = i\hbar \int_{-\infty}^{\infty} \frac{d}{dx} f^*(x) g(x) dx,$$

which matches the expression we found for $\int f^*(x) \hat{p} g(x) dx$. Therefore,

$$\int_{-\infty}^{\infty} f^*(x) \hat{p} g(x) dx = \int_{-\infty}^{\infty} g(x) [\hat{p}f(x)]^* dx,$$

and $\hat{p} = -i\hbar \frac{d}{dx}$ is indeed a *Hermitian* operator (assuming sufficiently well-behaved wavefunctions vanishing at infinity).

2. Whether $\hat{Q} = \frac{d^2}{dx^2}$ is Hermitian

Consider

$$\int_{-\infty}^{\infty} f^*(x) \hat{Q} g(x) dx = \int_{-\infty}^{\infty} f^*(x) \frac{d^2}{dx^2} g(x) dx.$$

Perform integration by parts twice:

$$\int_{-\infty}^{\infty} f^*(x) \frac{d^2}{dx^2} g(x) dx = \left[f^*(x) \frac{d}{dx} g(x) \right]_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} f^*(x) \frac{d}{dx} g(x) dx.$$

Integrate by parts again on the remaining term:

$$- \int_{-\infty}^{\infty} \frac{d}{dx} f^*(x) \frac{d}{dx} g(x) dx = - \left[\frac{d}{dx} f^*(x) g(x) \right]_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} \frac{d^2}{dx^2} f^*(x) g(x) dx.$$

Hence,

$$\int_{-\infty}^{\infty} f^*(x) \frac{d^2}{dx^2} g(x) dx = \left[f^*(x) \frac{d}{dx} g(x) - \frac{d}{dx} f^*(x) g(x) \right]_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} g(x) \frac{d^2}{dx^2} f^*(x) dx.$$

So,

$$\int_{-\infty}^{\infty} f^*(x) \hat{Q} g(x) dx = \int_{-\infty}^{\infty} g(x) [\hat{Q} f(x)]^* dx + \underbrace{\left[f^*(x) \frac{d}{dx} g(x) - \frac{d}{dx} f^*(x) g(x) \right]_{-\infty}^{+\infty}}_{\text{boundary term}}$$

$$\text{where } [\hat{Q} f(x)]^* = \left(\frac{d^2}{dx^2} f(x) \right)^* = \frac{d^2}{dx^2} f^*(x).$$

For $\hat{Q} = d^2/dx^2$ to be Hermitian, we need the boundary term to be zero for *all* well-behaved $f(x), g(x)$ that vanish at infinity. We see that this requires not only f, g to vanish at $\pm\infty$ but also (in some treatments) that *their first derivatives* vanish as well, or that certain boundary conditions are imposed (e.g., periodic boundary conditions in a finite domain).

In standard quantum mechanics on $(-\infty, +\infty)$, d^2/dx^2 can be *essentially Hermitian* (or self-adjoint) for wavefunctions that vanish sufficiently fast at infinity so that all boundary terms vanish. More precisely, one needs to specify a suitable domain of functions (like the Sobolev space H^2 with appropriate boundary conditions) so that the boundary terms go to zero.

Hence, under appropriate boundary conditions (i.e. wavefunctions and derivatives going to zero at infinity, or periodic boundary conditions on a finite interval), the operator

$$\hat{Q} = \frac{d^2}{dx^2}$$

is *Hermitian*. But one must be careful about the domain of definition to ensure the boundary terms vanish.

Question 4

Solution:

We define the x -component of the angular momentum operator as

$$\hat{L}_x = \hat{Y} \hat{P}_z - \hat{Z} \hat{P}_y.$$

Recall the canonical commutation relations:

$$[\hat{X}_i, \hat{X}_j] = 0, \quad [\hat{P}_i, \hat{P}_j] = 0, \quad [\hat{X}_i, \hat{P}_j] = i \hbar \delta_{ij},$$

and note that δ_{ij} is the Kronecker delta (1 if $i = j$, 0 otherwise).

We derive each of the given commutators using $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$ repeatedly, along with the above canonical relations.

$$1. [\hat{L}_x, \hat{X}] = 0$$

Expand:

$$[\hat{L}_x, \hat{X}] = [\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y, \hat{X}].$$

Distribute:

$$[\hat{Y}\hat{P}_z, \hat{X}] - [\hat{Z}\hat{P}_y, \hat{X}].$$

Using the product rule for commutators:

$$[\hat{Y}\hat{P}_z, \hat{X}] = \hat{Y}[\hat{P}_z, \hat{X}] + [\hat{Y}, \hat{X}]\hat{P}_z.$$

But $[\hat{P}_z, \hat{X}] = 0$ (since X is not Z) and $[\hat{Y}, \hat{X}] = 0$, so $[\hat{Y}\hat{P}_z, \hat{X}] = 0$. A similar argument shows $[\hat{Z}\hat{P}_y, \hat{X}] = 0$. Hence,

$$[\hat{L}_x, \hat{X}] = 0.$$

$$2. [\hat{L}_x, \hat{P}_x] = 0$$

Again,

$$[\hat{L}_x, \hat{P}_x] = [\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y, \hat{P}_x].$$

Expand each term:

$$[\hat{Y}\hat{P}_z, \hat{P}_x] - [\hat{Z}\hat{P}_y, \hat{P}_x].$$

Using the product rule:

$$[\hat{Y}\hat{P}_z, \hat{P}_x] = \hat{Y}[\hat{P}_z, \hat{P}_x] + [\hat{Y}, \hat{P}_x]\hat{P}_z.$$

But $[\hat{P}_z, \hat{P}_x] = 0$ and $[\hat{Y}, \hat{P}_x] = 0$, so $[\hat{Y}\hat{P}_z, \hat{P}_x] = 0$. Likewise, $[\hat{Z}\hat{P}_y, \hat{P}_x] = 0$. Thus

$$[\hat{L}_x, \hat{P}_x] = 0.$$

$$3. [\hat{L}_x, \hat{Y}] = i\hbar\hat{Z}$$

Expand:

$$[\hat{L}_x, \hat{Y}] = [\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y, \hat{Y}].$$

Break it into two parts:

$$[\hat{Y}\hat{P}_z, \hat{Y}] - [\hat{Z}\hat{P}_y, \hat{Y}].$$

First term:

$$[\hat{Y}\hat{P}_z, \hat{Y}] = \hat{Y}[\hat{P}_z, \hat{Y}] + [\hat{Y}, \hat{Y}]\hat{P}_z.$$

But $[\hat{Y}, \hat{Y}] = 0$, so

$$[\hat{Y}\hat{P}_z, \hat{Y}] = \hat{Y}[\hat{P}_z, \hat{Y}].$$

From the canonical commutation relations, $[\hat{P}_z, \hat{Y}] = 0$ (different axes), so the first term is zero.

For the second term,

$$[\hat{Z}\hat{P}_y, \hat{Y}] = \hat{Z}[\hat{P}_y, \hat{Y}] + [\hat{Z}, \hat{Y}]\hat{P}_y.$$

Here $[\hat{Z}, \hat{Y}] = 0$, but $[\hat{P}_y, \hat{Y}] = -i\hbar$ (since $[\hat{Y}, \hat{P}_y] = i\hbar$, reversing the order gives a minus sign). Thus

$$[\hat{Z}\hat{P}_y, \hat{Y}] = \hat{Z}(-i\hbar) = -i\hbar\hat{Z}.$$

Putting it together,

$$[\hat{L}_x, \hat{Y}] = 0 - [-i\hbar\hat{Z}] = i\hbar\hat{Z}.$$

$$4. [\hat{L}_x, \hat{P}_y] = i\hbar\hat{P}_z$$

Write

$$\hat{L}_x = \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y, \quad [\hat{L}_x, \hat{P}_y] = [\hat{Y}\hat{P}_z, \hat{P}_y] - [\hat{Z}\hat{P}_y, \hat{P}_y].$$

The second commutator vanishes immediately because $[\hat{P}_y, \hat{P}_y] = 0$ (and also $[\hat{Z}, \hat{P}_y] = 0$ from different coordinates). So we focus on $[\hat{Y}\hat{P}_z, \hat{P}_y]$:

$$[\hat{Y}\hat{P}_z, \hat{P}_y] = \hat{Y}[\hat{P}_z, \hat{P}_y] + [\hat{Y}, \hat{P}_y]\hat{P}_z.$$

Because $[\hat{P}_z, \hat{P}_y] = 0$, only the second term remains:

$$[\hat{Y}, \hat{P}_y] = i\hbar, \quad \text{so} \quad [\hat{Y}\hat{P}_z, \hat{P}_y] = i\hbar\hat{P}_z.$$

Consequently,

$$[\hat{L}_x, \hat{P}_y] = i\hbar\hat{P}_z.$$

$$5. [\hat{L}_x, \hat{P}^2] = 0$$

Finally, define $\hat{P}^2 = \hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2$. Then

$$[\hat{L}_x, \hat{P}^2] = [\hat{L}_x, \hat{P}_x^2] + [\hat{L}_x, \hat{P}_y^2] + [\hat{L}_x, \hat{P}_z^2].$$

We already know $[\hat{L}_x, \hat{P}_x] = 0$ from (b), so $[\hat{L}_x, \hat{P}_x^2] = 0$. Also, from the standard angular momentum commutators, $[\hat{L}_x, \hat{P}_y] = i\hbar\hat{P}_z$ and $[\hat{L}_x, \hat{P}_z] = -i\hbar\hat{P}_y$. One can check that these lead to $[\hat{L}_x, \hat{P}_y^2] = \hat{P}_y[\hat{L}_x, \hat{P}_y] + [\hat{L}_x, \hat{P}_y]\hat{P}_y = i\hbar(\hat{P}_y\hat{P}_z + \hat{P}_z\hat{P}_y)$, and similarly $[\hat{L}_x, \hat{P}_z^2] = -i\hbar(\hat{P}_z\hat{P}_y + \hat{P}_y\hat{P}_z)$. Adding these two contributions cancels exactly. Hence the sum of all three is zero:

$$[\hat{L}_x, \hat{P}^2] = 0.$$

More simply, one recognizes that \hat{P}^2 is rotationally invariant (it's a scalar under rotations), so it must commute with any component of angular momentum.

Question 5

Solution:

In the unperturbed infinite square well of width L , the normalized eigenfunctions and energies are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2m L^2}, \quad n = 1, 2, 3, \dots$$

for $0 \leq x \leq L$, and $\phi_n(x) = 0$ otherwise.

1. First-Order Correction to the Ground-State Energy: $H' = V_0 \sin\left(\frac{2\pi x}{L}\right)$

The time-independent, first-order perturbation theory tells us that the shift in the n th level is

$$E_n^{(1)} = \langle \phi_n | H' | \phi_n \rangle = \int_0^L \phi_n(x) H' \phi_n(x) dx.$$

For the ground state ($n = 1$), we have

$$E_1^{(1)} = \int_0^L \left[\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \right] V_0 \sin\left(\frac{2\pi x}{L}\right) \left[\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \right] dx.$$

This becomes

$$E_1^{(1)} = \frac{2V_0}{L} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx.$$

One may show (by trigonometric identities or direct integration) that

$$\int_0^L \sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx = 0,$$

since the integrand is orthogonal over $[0, L]$. Therefore,

$$\boxed{E_1^{(1)} = 0 \quad \text{for the perturbation } H' = V_0 \sin\left(\frac{2\pi x}{L}\right).}$$

2. First-Order Correction for Any Level: $H' = L \alpha \delta\left(x - \frac{L}{2}\right)$

Now the perturbation is a Dirac delta function located at $x = L/2$. The first-order shift in the n th energy level is

$$E_n^{(1)} = \langle \phi_n | H' | \phi_n \rangle = L \alpha \int_0^L \phi_n^*(x) \delta\left(x - \frac{L}{2}\right) \phi_n(x) dx.$$

Exploiting the sifting property of the delta function, only the value of $\phi_n(x)$ at $x = L/2$ matters:

$$E_n^{(1)} = L \alpha |\phi_n(\frac{L}{2})|^2.$$

Since

$$\phi_n(\frac{L}{2}) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} \frac{L}{2}\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{2}\right),$$

we get

$$|\phi_n(\frac{L}{2})|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi}{2}\right).$$

Hence,

$$E_n^{(1)} = L \alpha \frac{2}{L} \sin^2\left(\frac{n\pi}{2}\right) = 2 \alpha \sin^2\left(\frac{n\pi}{2}\right).$$

Therefore,

$$E_n^{(1)} = 2 \alpha \sin^2\left(\frac{n\pi}{2}\right) \quad \text{for } H' = L \alpha \delta\left(x - \frac{L}{2}\right).$$

In particular, $\sin^2(\frac{n\pi}{2})$ vanishes for even n (giving zero shift) and is 1 for odd n , so odd levels get a shift of 2α while even levels remain unshifted.

Question 6

Solution:

$$1. \text{ Proving } \hat{\mathbf{J}}^2 = \hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2 \hat{L}_z \hat{S}_z$$

We define $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$, where $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ are the orbital and spin angular momentum operators, respectively. Then

$$\hat{\mathbf{J}}^2 = (\hat{\mathbf{L}} + \hat{\mathbf{S}}) \cdot (\hat{\mathbf{L}} + \hat{\mathbf{S}}) = \hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2 + 2 \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}.$$

Recall the identity for raising/lowering operators:

$$\hat{L}_{\pm} = \hat{L}_x \pm i \hat{L}_y, \quad \hat{S}_{\pm} = \hat{S}_x \pm i \hat{S}_y.$$

and

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \hat{L}_z \hat{S}_z + \frac{1}{2}(\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+).$$

Combining these yields

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2 \hat{L}_z \hat{S}_z,$$

which is the desired form.

$$2. \text{ Eigenstate of } \hat{\mathbf{J}}^2 \text{ with } |l, -l; s, -s\rangle$$

We have the state

$$|l, m; s, m_s\rangle \equiv |l, m\rangle \otimes |s, m_s\rangle,$$

where $\hat{L}^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle$ and $\hat{L}_z |l, m\rangle = \hbar m |l, m\rangle$, while $\hat{S}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$ and $\hat{S}_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$.

In particular, consider the combined ket $|l, -l; s, -s\rangle$. Applying $\hat{\mathbf{J}}^2 = (\hat{\mathbf{L}} + \hat{\mathbf{S}})^2$ to this state, we note

$$\hat{L}_z |l, -l\rangle = -\hbar l |l, -l\rangle, \quad \hat{S}_z |s, -s\rangle = -\hbar s |s, -s\rangle.$$

Hence,

$$\hat{J}_z |l, -l; s, -s\rangle = (\hat{L}_z + \hat{S}_z) |l, -l; s, -s\rangle = -\hbar(l+s) |l, -l; s, -s\rangle.$$

One can similarly check that the ladder operators $\hat{J}_\pm = \hat{L}_\pm + \hat{S}_\pm$ annihilate this state (since raising $-l$ or $-s$ by 1 would produce mismatch in the combined m_J). Thus $|l, -l; s, -s\rangle$ is indeed an eigenket of $\hat{\mathbf{J}}^2$ with total angular momentum quantum number $j = l + s$, so

$$\hat{\mathbf{J}}^2 |l, -l; s, -s\rangle = \hbar^2 [l + s(l + s + 1)] |l, -l; s, -s\rangle.$$

3. Matrix Representation of $\hat{O} = a\hat{L}^2 + b\hat{S}_+ \hat{L}_z$ in a Basis $|l = 1, m; s = \frac{1}{2}, m_s\rangle$

We choose the combined basis with $l = 1$ and $s = \frac{1}{2}$. Then m can be $-1, 0, +1$ and m_s can be $-\frac{1}{2}, +\frac{1}{2}$. Altogether, there are $3 \times 2 = 6$ basis states:

$$|1, m; \frac{1}{2}, m_s\rangle \text{ where } m \in \{-1, 0, +1\}, \quad m_s \in \{-\frac{1}{2}, +\frac{1}{2}\}.$$

Thus \hat{O} has a 6×6 matrix in this basis.

First, note that $\hat{L}^2 |1, m\rangle = \hbar^2 \cdot 1 \cdot (1 + 1) |1, m\rangle = 2\hbar^2 |1, m\rangle$. Hence on the combined state,

$$\hat{L}^2 |1, m; \frac{1}{2}, m_s\rangle = 2\hbar^2 |1, m; \frac{1}{2}, m_s\rangle.$$

Thus the matrix elements of $a\hat{L}^2$ are simply

$$\langle 1, m'; \frac{1}{2}, m'_s | a\hat{L}^2 | 1, m; \frac{1}{2}, m_s \rangle = a(2\hbar^2) \delta_{m', m} \delta_{m'_s, m_s}.$$

That is, $a\hat{L}^2$ contributes a diagonal block $2a\hbar^2$ (the same for all 6 basis states).

Next, consider $b\hat{S}_+ \hat{L}_z$. We know

$$\hat{S}_+ |s = \frac{1}{2}, m_s = -\frac{1}{2}\rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)} | \frac{1}{2}, +\frac{1}{2} \rangle = \hbar \sqrt{\frac{1}{2}(\frac{1}{2}+1) - (-\frac{1}{2})(-\frac{1}{2}+1)} | \frac{1}{2}, +\frac{1}{2} \rangle$$

Numerically, that factor is $\hbar \sqrt{\frac{1}{2} \cdot \frac{3}{2} - (-\frac{1}{2}) \cdot \frac{1}{2}} = \hbar \sqrt{\frac{3}{4} - (-\frac{1}{4})} = \hbar \sqrt{1} = \hbar$. And \hat{S}_+ acting on $| \frac{1}{2}, +\frac{1}{2} \rangle$ gives 0.

Meanwhile,

$$\hat{L}_z |1, m\rangle = \hbar m |1, m\rangle.$$

Therefore,

$$\hat{S}_+ \hat{L}_z |1, m; \frac{1}{2}, m_s\rangle = \hat{S}_+ (\hbar m |1, m; \frac{1}{2}, m_s\rangle) = \hbar m (\hat{S}_+ | \frac{1}{2}, m_s \rangle) \otimes |1, m\rangle.$$

From the explicit action of \hat{S}_+ , we see that m_s must be $-\frac{1}{2}$ for a nonzero result:

$$\hat{S}_+ | \frac{1}{2}, -\frac{1}{2} \rangle = \hbar | \frac{1}{2}, +\frac{1}{2} \rangle, \quad \hat{S}_+ | \frac{1}{2}, +\frac{1}{2} \rangle = 0.$$

Hence in the basis $|1, m; \frac{1}{2}, m_s\rangle$, the operator $\hat{S}_+\hat{L}_z$ has only off-diagonal elements that connect $|1, m; \frac{1}{2}, -\frac{1}{2}\rangle$ to $|1, m; \frac{1}{2}, +\frac{1}{2}\rangle$, with matrix element

$$\langle 1, m; \frac{1}{2}, +\frac{1}{2} | \hat{S}_+\hat{L}_z | 1, m; \frac{1}{2}, -\frac{1}{2} \rangle = \hbar m \hbar = \hbar^2 m.$$

All other matrix elements of $\hat{S}_+\hat{L}_z$ vanish. Multiplying by b just scales those off-diagonal entries by b .

Collecting these results, the 6×6 matrix for $\hat{O} = a\hat{L}^2 + b\hat{S}_+\hat{L}_z$ in the chosen basis has the block-diagonal form with:

- A diagonal part $2a\hbar^2$ for all basis states (due to $a\hat{L}^2$).
- Off-diagonal couplings *only* between $m_s = -\frac{1}{2}$ and $m_s = +\frac{1}{2}$ states with the same m , of magnitude $b\hbar^2 m$.

Hence one can write out the matrix (arranging basis in order of $m = \pm 1, 0$ and $m_s = -\frac{1}{2}, +\frac{1}{2}$) explicitly. Symbolically, one sees each m subspace yields a 2×2 block of the form

$$\begin{pmatrix} 2a\hbar^2 & b\hbar^2 m \\ 0 & 2a\hbar^2 \end{pmatrix},$$

except that we must be careful with the ordering of $m_s = -\frac{1}{2}, +\frac{1}{2}$ in the rows/columns. This fully specifies the matrix representation of \hat{O} .

Question 7

Solution:

1. Time-Dependent Solution and Probability Density

We have a plane-wave solution in one dimension:

$$\psi(x) = A e^{i k x},$$

where A is a normalization factor (often taken over some large normalization length), and k is the wave number. In free space with $V(x) = 0$, the corresponding *time-dependent* wavefunction solves the free Schrödinger equation

$$\Psi(x, t) = A e^{i[kx - \omega t]},$$

where $\omega = \frac{\hbar k^2}{2m}$ in nonrelativistic quantum mechanics.

The probability density is

$$\rho(x, t) = |\Psi(x, t)|^2 = |A|^2 |e^{i(kx - \omega t)}|^2 = |A|^2.$$

Because the exponential is of unit magnitude, $\rho(x, t)$ is constant in both space and time. Thus the probability per unit length of finding the particle is *uniform* (does not depend on x or t).

2. Particle Flux and Physical Interpretation

The quantum probability current (or flux) for a one-dimensional wave-function $\Psi(x, t)$ is given by

$$\Gamma = -\frac{i\hbar}{2m} \left[\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right].$$

Substituting $\Psi(x, t) = A e^{i(kx - \omega t)}$, we compute

$$\frac{\partial \Psi}{\partial x} = i k A e^{i(kx - \omega t)} = i k \Psi, \quad \frac{\partial \Psi^*}{\partial x} = -i k \Psi^*.$$

Hence,

$$\Psi^* \frac{\partial \Psi}{\partial x} = \Psi^* (i k \Psi) = i k |\Psi|^2, \quad \Psi \frac{\partial \Psi^*}{\partial x} = \Psi (-i k \Psi^*) = -i k |\Psi|^2.$$

Therefore,

$$\Gamma = -\frac{i\hbar}{2m} [i k |\Psi|^2 - (-i k |\Psi|^2)] = -\frac{i\hbar}{2m} [2 i k |\Psi|^2] = \frac{\hbar k}{m} |\Psi|^2.$$

Recall $|\Psi|^2 = |A|^2$ is the spatially uniform probability density. Identifying the classical momentum $p = \hbar k$ and velocity $v = p/m = \hbar k/m$, we can write

$$\Gamma = v |\Psi|^2 = \frac{\hbar k}{m} |A|^2.$$

Physical Interpretation: The flux Γ is a constant, reflecting a uniform flow of probability to the right (assuming $k > 0$). In other words, this is a free particle moving with velocity $v = \frac{\hbar k}{m}$, and the probability distribution for its position is uniform in space. Thus the result $\Gamma = v |\Psi|^2$ matches the intuitive picture of a beam of non-interacting particles, each traveling with the same velocity v .

Question 8

Solution:

1. Rate of Change of the Expectation Value of an Operator

We have an operator \hat{Q} that does not explicitly depend on time. Its expectation value in quantum mechanics is

$$\langle \hat{Q} \rangle = \int_{-\infty}^{+\infty} \Psi^*(x, t) \hat{Q} \Psi(x, t) dx.$$

Differentiating w.r.t. time,

$$\frac{d}{dt} \langle \hat{Q} \rangle = \int_{-\infty}^{+\infty} \frac{\partial \Psi^*(x, t)}{\partial t} \hat{Q} \Psi(x, t) dx + \int_{-\infty}^{+\infty} \Psi^*(x, t) \hat{Q} \frac{\partial \Psi(x, t)}{\partial t} dx.$$

Using the time-dependent Schrödinger equation in one dimension (and its complex conjugate),

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi, \quad -i\hbar \frac{\partial \Psi^*}{\partial t} = \Psi^* \hat{H},$$

we substitute

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar} \hat{H} \Psi, \quad \frac{\partial \Psi^*}{\partial t} = \frac{i}{\hbar} \Psi^* \hat{H}.$$

Plug these into the integrals:

$$\frac{d}{dt} \langle \hat{Q} \rangle = \int_{-\infty}^{+\infty} \left(\frac{i}{\hbar} \Psi^* \hat{H} \right) \hat{Q} \Psi dx + \int_{-\infty}^{+\infty} \Psi^* \hat{Q} \left(-\frac{i}{\hbar} \hat{H} \Psi \right) dx.$$

Factor out $\frac{i}{\hbar}$, combine, and rearrange:

$$\frac{d}{dt} \langle \hat{Q} \rangle = \frac{i}{\hbar} \int \Psi^* [\hat{H} \hat{Q} - \hat{Q} \hat{H}] \Psi dx = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle.$$

Introducing the usual factor of $1/i = -i$, we get the often-quoted form

$$\boxed{\frac{d}{dt} \langle \hat{Q} \rangle = \frac{1}{i\hbar} \langle [\hat{Q}, \hat{H}] \rangle.}$$

This is the Heisenberg equation of motion for the expectation value of \hat{Q} .

2. Ehrenfest's Theorem for the Position Operator

Consider a particle of mass m in a time-independent potential $V(x)$. The Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

We want to compute $m \frac{d}{dt} \langle \hat{x} \rangle$ and show that it equals $\langle \hat{p} \rangle$. From the result in part (a), or from the canonical commutator approach:

$$\hat{x} \hat{p} - \hat{p} \hat{x} = i\hbar,$$

we have

$$[\hat{x}, \hat{H}] = [\hat{x}, \frac{\hat{p}^2}{2m}] + [\hat{x}, V(\hat{x})].$$

But $[\hat{x}, V(\hat{x})] = 0$ since $V(\hat{x})$ is purely a function of \hat{x} . Meanwhile,

$$[\hat{x}, \frac{\hat{p}^2}{2m}] = \frac{1}{2m} [\hat{x}, \hat{p}^2] = \frac{1}{2m} (\hat{p} [\hat{x}, \hat{p}] + [\hat{x}, \hat{p}] \hat{p}).$$

Since $[\hat{x}, \hat{p}] = i\hbar$, this becomes

$$[\hat{x}, \frac{\hat{p}^2}{2m}] = \frac{1}{2m} (\hat{p} i\hbar + i\hbar \hat{p}) = \frac{i\hbar}{2m} (2\hat{p}) = \frac{i\hbar}{m} \hat{p}.$$

Hence,

$$[\hat{x}, \hat{H}] = \frac{i\hbar}{m} \hat{p}.$$

By the general formula,

$$\frac{d}{dt} \langle \hat{x} \rangle = \frac{1}{i\hbar} \langle [\hat{x}, \hat{H}] \rangle = \frac{1}{i\hbar} \langle \frac{i\hbar}{m} \hat{p} \rangle = \frac{1}{m} \langle \hat{p} \rangle.$$

Multiplying both sides by m gives

$$\boxed{m \frac{d}{dt} \langle \hat{x} \rangle = \langle \hat{p} \rangle},$$

which is precisely Ehrenfest's theorem for the average momentum. It is the quantum analogue of Newton's law $m d\langle x \rangle / dt = \langle p \rangle$.

Question 9

Solution:

Potential Step:

$$V(x) = \begin{cases} 0 & x \leq 0, \\ -V_0 & x > 0, \end{cases} \quad \text{with } E > 0 \text{ (particle energy).}$$

Define:

$$k_1 = \frac{\sqrt{2mE}}{\hbar}, \quad k_2 = \frac{\sqrt{2m(E + V_0)}}{\hbar}.$$

These follow from the time-independent Schrödinger equation $\frac{p^2}{2m} + V(x) = E$ in each region.

1. General Solutions in Each Region

Region I: $x \leq 0, V(x) = 0$. A general solution with incident and reflected waves can be written:

$$\psi_1(x) = e^{ik_1 x} + B e^{-ik_1 x}, \quad \text{where } B \text{ is the reflection amplitude.}$$

Often one sets the incident amplitude to 1, and calls that wave $e^{ik_1 x}$. (In your statement, $A = 1$ is taken implicitly, and the reflection amplitude is B .)

Region II: $x > 0, V(x) = -V_0$. Here the solution is a single transmitted wave traveling to the right (since $x > 0$ extends to $+\infty$ and we assume no incoming wave from $+\infty$). That wave is

$$\psi_2(x) = C e^{ik_2 x}, \quad \text{where } C \text{ is the transmission amplitude.}$$

Hence the full piecewise wavefunction is

$$\psi(x) = \begin{cases} e^{i k_1 x} + B e^{-i k_1 x}, & x \leq 0, \\ C e^{i k_2 x}, & x > 0. \end{cases}$$

This matches the statement (up to naming conventions and signs of exponents).

2. Boundary Conditions and Solving for B and C

The boundary conditions at $x = 0$ require:

$$\psi_1(0) = \psi_2(0), \quad \left. \frac{d\psi_1}{dx} \right|_{x=0} = \left. \frac{d\psi_2}{dx} \right|_{x=0}.$$

From continuity of the wavefunction at $x = 0$:

$$1 + B = C. \tag{1}$$

From continuity of the derivative at $x = 0$:

$$i k_1 (1 - B) = i k_2 C. \tag{2}$$

Solving (1) and (2) simultaneously for B and C gives

$$C = \frac{2 k_1}{k_1 + k_2}, \quad B = \frac{k_1 - k_2}{k_1 + k_2}.$$

(Details: from (2) one has $k_1(1 - B) = k_2 C$, and from (1) one has $C = 1 + B$. Solve algebraically.)

3. Incident, Reflected, and Transmitted Flux

The flux (probability current) for a wavefunction $\psi(x)$ is

$$\Gamma = -\frac{i \hbar}{2 m} \left[\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} \right].$$

For a plane wave of form $A e^{i k x}$, the flux is $\Gamma = \frac{\hbar k}{m} |A|^2$.

Region I (Incident + Reflected):

$$\psi_1(x) = e^{i k_1 x} + B e^{-i k_1 x}.$$

The incident wave is $e^{i k_1 x}$, flux

$$\Gamma_{\text{inc}} = \frac{\hbar k_1}{m}.$$

The reflected wave is $B e^{-i k_1 x}$, flux

$$\Gamma_{\text{ref}} = -\frac{\hbar k_1}{m} |B|^2$$

(the minus sign arises because $k_1 < 0$ for the wave traveling in the $-x$ direction, or more precisely from substituting $e^{-ik_1 x}$ in the flux formula).

Region II (Transmitted):

$$\psi_2(x) = C e^{ik_2 x},$$

flux

$$\Gamma_{\text{trans}} = \frac{\hbar k_2}{m} |C|^2.$$

Transmission and Reflection Probabilities:

By definition,

$$T = \frac{\Gamma_{\text{trans}}}{\Gamma_{\text{inc}}}, \quad R = \frac{|\Gamma_{\text{ref}}|}{\Gamma_{\text{inc}}}.$$

Here $\Gamma_{\text{inc}} = \hbar k_1/m$, so

$$T = \frac{\hbar k_2/m |C|^2}{\hbar k_1/m} = \frac{k_2}{k_1} |C|^2, \quad R = \frac{\hbar k_1/m |B|^2}{\hbar k_1/m} = |B|^2.$$

Substitute the expressions for B and C :

$$C = \frac{2k_1}{k_1 + k_2}, \quad B = \frac{k_1 - k_2}{k_1 + k_2}.$$

Hence

$$|C|^2 = \left(\frac{2k_1}{k_1 + k_2} \right)^2 = \frac{4k_1^2}{(k_1 + k_2)^2}, \quad |B|^2 = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2.$$

Thus

$$T = \frac{k_2}{k_1} \frac{4k_1^2}{(k_1 + k_2)^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2}, \quad R = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2.$$

These are the final forms:

$$\boxed{T = \frac{4k_1 k_2}{(k_1 + k_2)^2}, \quad R = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2.}$$

One can also check that $T+R=1$, consistent with probability conservation in this simple step potential problem.

Question 10

Solution:

Quantum Harmonic Oscillator: The unperturbed energy eigenstates are $\{|n\rangle\}_{n=0,1,2,\dots}$ with eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

We use the *annihilation* and *creation* operators \hat{a}_- , \hat{a}_+ (often denoted \hat{a} , \hat{a}^\dagger), defined by

$$\hat{a}_\pm = \frac{1}{\sqrt{2}} \left(\alpha \hat{x} \pm \frac{i}{\alpha \hbar} \hat{p} \right), \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}.$$

These satisfy

$$\hat{a}_- |n\rangle = \sqrt{n} |n-1\rangle, \quad \hat{a}_+ |n\rangle = \sqrt{n+1} |n+1\rangle.$$

Also, we recall that

$$\hat{H}_0 = \hbar \omega \left(\hat{a}_+ \hat{a}_- + \frac{1}{2} \right), \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_- + \hat{a}_+).$$

1. Part (a): Normalizing $|\psi\rangle = |0\rangle + b|1\rangle$

We have a trial state

$$|\psi\rangle = |0\rangle + b|1\rangle,$$

where b is a constant (possibly real for simplicity). To normalize $|\psi\rangle$, we require

$$\langle\psi|\psi\rangle = (\langle 0| + b^*\langle 1|) (|0\rangle + b|1\rangle) = \langle 0|0\rangle + b\langle 0|1\rangle + b^*\langle 1|0\rangle + |b|^2 \langle 1|1\rangle.$$

Because $\{|n\rangle\}$ are orthonormal, $\langle 0|1\rangle = 0$ and $\langle 0|0\rangle = \langle 1|1\rangle = 1$. Thus

$$1 = \langle\psi|\psi\rangle = 1 + |b|^2.$$

Hence

$$|b|^2 = 0 \implies b = 0 \quad (\text{trivial case, or}) \quad \text{or if the problem suggests } b \text{ is a free parameter,}$$

Actually, for the general normalization condition we get

$$1 + |b|^2 = 1 \implies |b|^2 = 0 \implies b = 0,$$

which suggests the state is simply $|0\rangle$ if we strictly impose $\langle\psi|\psi\rangle = 1$ with the given form.

However, it is more common that the state was intended to be written

$$|\psi\rangle = \frac{1}{\sqrt{1 + |b|^2}} (|0\rangle + b|1\rangle),$$

so that the normalization condition is automatically satisfied. We often rename b to be the amplitude ratio, in which case the normalized state becomes

$$|\psi\rangle_{\text{norm}} = \frac{1}{\sqrt{1 + |b|^2}} (|0\rangle + b|1\rangle).$$

Expectation Value of the Hamiltonian:

$$\langle H_0 \rangle = \langle\psi|\hat{H}_0|\psi\rangle.$$

If we assume the normalized version, then

$$\langle \psi | \hat{H}_0 | \psi \rangle = \frac{1}{1 + |b|^2} \left(\langle 0 | + b^* \langle 1 | \right) \hat{H}_0 \left(|0\rangle + b |1\rangle \right).$$

Because $|n\rangle$ is an eigenstate with eigenvalue $E_n = \hbar\omega(n + 1/2)$,

$$\hat{H}_0 |0\rangle = \hbar\omega\left(\frac{1}{2}\right) |0\rangle, \quad \hat{H}_0 |1\rangle = \hbar\omega\left(\frac{3}{2}\right) |1\rangle.$$

Thus

$$\hat{H}_0 \left(|0\rangle + b |1\rangle \right) = \frac{1}{2} \hbar\omega |0\rangle + b \frac{3}{2} \hbar\omega |1\rangle.$$

Substituting,

$$\langle H_0 \rangle = \frac{1}{1 + |b|^2} \left[\langle 0 | \left(\frac{1}{2} \hbar\omega |0\rangle \right) + \langle 0 | \left(b \frac{3}{2} \hbar\omega |1\rangle \right) + b^* \langle 1 | \left(\frac{1}{2} \hbar\omega |0\rangle \right) + b^* \langle 1 | \left(b \frac{3}{2} \hbar\omega |1\rangle \right) \right].$$

The cross terms $\langle 0 | 1 \rangle$ and $\langle 1 | 0 \rangle$ vanish, so only the diagonal pieces remain:

$$\langle H_0 \rangle = \frac{1}{1 + |b|^2} \left[\frac{1}{2} \hbar\omega + |b|^2 \frac{3}{2} \hbar\omega \right] = \frac{\hbar\omega}{1 + |b|^2} \left(\frac{1}{2} + \frac{3}{2} |b|^2 \right) = \hbar\omega \frac{\frac{1}{2} + \frac{3}{2} |b|^2}{1 + |b|^2}.$$

Simplify:

$$\boxed{\langle H_0 \rangle = \hbar\omega \frac{1/2 + (3/2) |b|^2}{1 + |b|^2}.$$

2. Part (b): Expressing $H' = q \mathcal{E} \hat{x}$ in Terms of \hat{a}_\pm

The perturbation is

$$H' = q \mathcal{E} \hat{x}.$$

Recall

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_- + \hat{a}_+) = \frac{1}{\alpha\sqrt{2}} (\hat{a}_- + \hat{a}_+) \quad \text{since } \alpha = \sqrt{\frac{m\omega}{\hbar}}.$$

Thus

$$H' = q \mathcal{E} \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_- + \hat{a}_+) = q \mathcal{E} \frac{1}{\alpha\sqrt{2}} (\hat{a}_- + \hat{a}_+).$$

That is,

$$\boxed{H' = \frac{q \mathcal{E}}{\sqrt{2} \alpha} (\hat{a}_- + \hat{a}_+).$$

3. Part (c): Variational Principle and the Optimal b

We are told that by applying the variational principle (with $|\psi\rangle = |0\rangle + b |1\rangle$) to minimize the total energy $E_0 + E_1$ + the perturbation, one obtains

$$b = \frac{E_1 - E_0}{g} - \sqrt{\left(\frac{E_1 - E_0}{g} \right)^2 - 1}, \quad \text{where } g = \sqrt{2} q \mathcal{E} / \alpha.$$

The steps to derive this usually involve forming the *expectation value* $\langle H_0 + H' \rangle$ in the trial state, differentiating w.r.t. b^* (and b), and setting that derivative to zero. Solving the resulting nonlinear equation gives the expression for b .

Without going through the full derivation, the final result is

$$b = \frac{E_1 - E_0}{g} - \sqrt{\left(\frac{E_1 - E_0}{g}\right)^2 - 1}, \quad \text{where } g = \sqrt{\frac{2q\mathcal{E}}{\alpha}} \quad \text{and} \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

This b (which can be real) makes the energy functional $\langle \psi | (H_0 + H') | \psi \rangle$ a minimum under the constraint of normalization.

2.6 Classical Mechanics with context prompts

Question 1

Solution:

Definition of a Conservative Force

A force \mathbf{F} is called *conservative* if it can be written as the gradient (negative gradient in physics convention) of a scalar function $U(\mathbf{r})$, commonly called the *potential energy*:

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}).$$

Equivalently, a force is conservative if (1) its work done around any *closed* path is zero, or (2) the work done by the force depends only on the end points (initial and final positions), and not on the path taken:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad (\text{for any closed loop } C).$$

Potential Energy of a Conservative Force

If \mathbf{F} is conservative, the potential energy function $U(\mathbf{r})$ is defined (up to an additive constant) by choosing a reference point \mathbf{r}_0 and writing

$$U(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}'.$$

By construction, the force then follows as

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}).$$

Since the work done is path-independent, the line integral from \mathbf{r}_0 to \mathbf{r} is well-defined and single-valued, making $U(\mathbf{r})$ a valid function.

Why Non-Conservative Forces Cannot Have a Potential Energy

For a *non-conservative* force, the work done around a closed path is non-zero, or equivalently, the net work depends on the path itself rather than just the end points. This makes it impossible to define a single-valued potential energy function $U(\mathbf{r})$ such that $\mathbf{F} = -\nabla U$, since taking different paths between the same two points would yield different values of $\int \mathbf{F} \cdot d\mathbf{r}$. Hence no consistent scalar function $U(\mathbf{r})$ can be assigned to reproduce a non-conservative force via $-\nabla U$.

Question 2

Solution:

Consider the damped harmonic oscillator described by

$$\frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0.$$

Let the undamped natural frequency be $\omega_0 = \sqrt{\frac{k}{m}}$, and let the damping coefficient be $b = \frac{2m\gamma}{2m}$ so that $\gamma = \frac{b}{2m}$. The three cases of damping depend on the discriminant $\Delta = \gamma^2 - \omega_0^2$:

1. **Overdamped** ($\gamma^2 > \omega_0^2$)

In this case, the damping is strong enough that the system does *not* oscillate. Instead, the solution $x(t)$ exponentially decays to equilibrium without crossing the equilibrium position. The motion is a sum of two exponential decays with different decay rates.

2. **Critically Damped** ($\gamma^2 = \omega_0^2$)

The system is *just* at the threshold between overdamped and underdamped. The motion returns to equilibrium as quickly as possible *without* oscillation. The solution is a product of an exponential and a polynomial in t , leading to the fastest non-oscillatory decay back to equilibrium.

3. **Underdamped** ($\gamma^2 < \omega_0^2$)

Here, the damping is *weak* enough that the system still oscillates, but with a frequency $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$ lower than the natural frequency ω_0 . The amplitude of the oscillations decays exponentially over time due to the damping term.

Question 3

Solution:

1. **Center of Mass Position Vector**

Two objects of masses m_1 and m_2 have position vectors \mathbf{r}_1 and \mathbf{r}_2 , respectively. The center of mass (COM) \mathbf{R} is defined by:

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}.$$

We can rearrange this into the form

$$\mathbf{R} = (1 - k) \mathbf{r}_1 + k \mathbf{r}_2$$

if we solve for k . Comparing,

$$\frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = (1 - k) \mathbf{r}_1 + k \mathbf{r}_2,$$

this implies

$$(1 - k) = \frac{m_1}{m_1 + m_2}, \quad k = \frac{m_2}{m_1 + m_2}.$$

Thus the parameter k is $m_2/(m_1 + m_2)$.

2. Distances from Each Mass

Suppose \mathbf{r}_1 and \mathbf{r}_2 lie along some line (we can treat this as a one-dimensional vector scenario or a straight-line geometry in higher dimensions). If their separation is $d = \|\mathbf{r}_2 - \mathbf{r}_1\|$, then the center of mass \mathbf{R} also lies on this line.

Indeed, because $\mathbf{R} = (1 - k) \mathbf{r}_1 + k \mathbf{r}_2$, it must be somewhere between \mathbf{r}_1 and \mathbf{r}_2 . Specifically:

$$\|\mathbf{R} - \mathbf{r}_1\| = \|k(\mathbf{r}_2 - \mathbf{r}_1)\| = k d = \frac{m_2}{m_1 + m_2} d.$$

Thus the center of mass is $\frac{m_2}{m_1 + m_2}$ of the way from \mathbf{r}_1 to \mathbf{r}_2 . Equivalently, measuring from the other mass,

$$\|\mathbf{r}_2 - \mathbf{R}\| = \|(1 - k)(\mathbf{r}_2 - \mathbf{r}_1)\| = (1 - k) d = \frac{m_1}{m_1 + m_2} d.$$

Hence we see that the distance from mass m_1 to the COM is $\frac{m_2}{m_1 + m_2} d$, and the distance from mass m_2 to the COM is $\frac{m_1}{m_1 + m_2} d$.

Question 4

Solution:

Given Setup in 2D Polar Coordinates

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}}, \quad \mathbf{a} = (\ddot{r} - r \dot{\theta}^2) \hat{\mathbf{r}} + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{\boldsymbol{\theta}}.$$

A particle of mass m has the radial equation $\dot{r} = -\gamma$ (constant $\gamma > 0$ implies r is shrinking linearly in time). At $t = 0$, the particle has $r(0) = r_0$ and angular velocity $\dot{\theta}(0) = \delta$. We assume *angular momentum* is conserved, so $L = m r^2 \dot{\theta} = \text{const.}$

1. **Expression for Angular Frequency $\dot{\theta}(t)$ under Angular Momentum Conservation**

Because $L = m r^2 \dot{\theta}$ is constant in time, and at $t = 0$ it equals $m r_0^2 \delta$, we have

$$L = m r^2(t) \dot{\theta}(t) = m r_0^2 \delta.$$

But $r(t) = r_0 - \gamma t$. Hence

$$r^2(t) \dot{\theta}(t) = r_0^2 \delta, \implies \dot{\theta}(t) = \frac{r_0^2 \delta}{(r_0 - \gamma t)^2}.$$

This is the time-dependent angular frequency. Summarizing,

$$\boxed{\dot{\theta}(t) = \frac{r_0^2 \delta}{(r_0 - \gamma t)^2}.$$

2. **Angular Component of the Acceleration**

From the general polar-coordinate formula,

$$a_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta}.$$

Since $r(t) = r_0 - \gamma t$, we compute $\dot{r} = -\gamma$ and

$$\dot{\theta}(t) = \frac{r_0^2 \delta}{(r_0 - \gamma t)^2}.$$

We also need $\ddot{\theta}(t)$:

$$\ddot{\theta}(t) = \frac{d}{dt} \dot{\theta}(t) = \frac{d}{dt} \left[\frac{r_0^2 \delta}{(r_0 - \gamma t)^2} \right] = r_0^2 \delta \frac{d}{dt} (r_0 - \gamma t)^{-2}.$$

Using the chain rule,

$$\ddot{\theta}(t) = r_0^2 \delta (-2) (r_0 - \gamma t)^{-3} (-\gamma) = \frac{2 \gamma r_0^2 \delta}{(r_0 - \gamma t)^3}.$$

Therefore,

$$a_\theta(t) = (r_0 - \gamma t) \frac{2 \gamma r_0^2 \delta}{(r_0 - \gamma t)^3} + 2 (-\gamma) \frac{r_0^2 \delta}{(r_0 - \gamma t)^2}.$$

Factor out common powers of $(r_0 - \gamma t)$:

$$a_\theta(t) = \frac{2 \gamma r_0^2 \delta}{(r_0 - \gamma t)^2} - \frac{2 \gamma r_0^2 \delta}{(r_0 - \gamma t)^2} = 0.$$

Interestingly, these two terms cancel exactly, implying

$$\boxed{a_\theta(t) = 0.}$$

The angular acceleration (the $\hat{\theta}$ component of \mathbf{a}) vanishes, which is consistent with the direct statement that angular momentum is conserved *and* $r(t)$ is changing linearly.

3. Kinetic Energy and Its Time Derivative

The particle's kinetic energy:

$$K = \frac{1}{2} m |\mathbf{v}|^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2).$$

Substitute $\dot{r} = -\gamma$ and $\dot{\theta}(t) = \frac{r_0^2 \delta}{(r_0 - \gamma t)^2}$. Then

$$K(t) = \frac{1}{2} m \left[(-\gamma)^2 + (r_0 - \gamma t)^2 \left(\frac{r_0^2 \delta}{(r_0 - \gamma t)^2} \right)^2 \right] = \frac{1}{2} m \left[\gamma^2 + r_0^4 \delta^2 \frac{1}{(r_0 - \gamma t)^2} \right].$$

Simplify:

$$K(t) = \frac{1}{2} m \gamma^2 + \frac{1}{2} m \frac{r_0^4 \delta^2}{(r_0 - \gamma t)^2}.$$

The statement to be shown is

$$\frac{dK}{dt} = \frac{m r_0^4 \gamma \delta^2}{(r_0 - \gamma t)^3} \quad (\text{or an equivalent form}).$$

Let us differentiate explicitly. Note that the first term $\frac{1}{2} m \gamma^2$ is constant in time, so its derivative is 0. The second term:

$$\frac{d}{dt} \left[\frac{1}{2} m \frac{r_0^4 \delta^2}{(r_0 - \gamma t)^2} \right] = \frac{1}{2} m r_0^4 \delta^2 \frac{d}{dt} \left[(r_0 - \gamma t)^{-2} \right].$$

Using the chain rule,

$$\frac{d}{dt} (r_0 - \gamma t)^{-2} = -2 (r_0 - \gamma t)^{-3} (-\gamma) = \frac{2\gamma}{(r_0 - \gamma t)^3}.$$

Hence

$$\frac{dK}{dt} = \frac{1}{2} m r_0^4 \delta^2 \frac{2\gamma}{(r_0 - \gamma t)^3} = \frac{m r_0^4 \gamma \delta^2}{(r_0 - \gamma t)^3}.$$

Which matches the desired expression. In a more compact form:

$$\boxed{\frac{dK}{dt} = \frac{m r_0^4 \gamma \delta^2}{(r_0 - \gamma t)^3}}.$$

This shows how the kinetic energy grows in time as r shrinks (due to conservation of angular momentum).

Question 5

Solution:

System Setup: Two identical masses m (objects X and Y) are connected by a spring of constant k and each mass is also connected to a fixed wall by a spring of the same constant k , as shown:

$$(\text{Wall})\text{---}k\text{---}[X]\text{---}k\text{---}[Y]\text{---}k\text{---}(\text{Wall}).$$

Denote the small displacements of X and Y from their respective equilibrium positions by $x_1(t)$ and $x_2(t)$.

1. Equations of Motion for X and Y

For object X:

- The left spring (wall-X) exerts a restoring force $-k x_1$ on X.
- The middle spring (X-Y) is extended by $(x_2 - x_1)$; hence the force on X due to that spring is

$$-k(x_1 - x_2) = k(x_2 - x_1).$$

Therefore, the total force on X is

$$F_X = -k x_1 + k(x_2 - x_1) = -2k x_1 + k x_2.$$

Newton's second law, $m \ddot{x}_1 = F_X$, gives

$$m \ddot{x}_1 = -2k x_1 + k x_2. \implies \ddot{x}_1 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2.$$

For object Y:

- The middle spring (X-Y) exerts a force $-k(x_2 - x_1)$ on Y (pulling it toward X).
- The right spring (Y-wall) exerts $-k x_2$ on Y.

Hence,

$$F_Y = -k(x_2 - x_1) - k x_2 = k x_1 - 2k x_2.$$

Then $m \ddot{x}_2 = F_Y$ yields

$$\ddot{x}_2 = \frac{k}{m} x_1 - \frac{2k}{m} x_2.$$

2. Normal-Mode Coordinates and Decoupling

Define the new coordinates:

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2.$$

We want to express \ddot{y}_1 and \ddot{y}_2 in terms of $\{x_1, x_2\}$.

Compute:

$$\ddot{y}_1 = \ddot{x}_1 + \ddot{x}_2, \quad \ddot{y}_2 = \ddot{x}_1 - \ddot{x}_2.$$

From the original equations:

$$\ddot{x}_1 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2, \quad \ddot{x}_2 = \frac{k}{m} x_1 - \frac{2k}{m} x_2.$$

Add them to get \ddot{y}_1 :

$$\ddot{y}_1 = (\ddot{x}_1 + \ddot{x}_2) = \left[-\frac{2k}{m} x_1 + \frac{k}{m} x_2 \right] + \left[\frac{k}{m} x_1 - \frac{2k}{m} x_2 \right] = -\frac{2k}{m} (x_1 + x_2) + \frac{k}{m} (x_2 + x_1) = -\frac{k}{m} (x_1 + x_2).$$

Notice $x_1 + x_2 = y_1$, so

$$\ddot{y}_1 = -\frac{k}{m} y_1.$$

Similarly, subtracting the second from the first yields \ddot{y}_2 :

$$\ddot{y}_2 = \ddot{x}_1 - \ddot{x}_2 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2 - \left[\frac{k}{m} x_1 - \frac{2k}{m} x_2 \right] = -\frac{3k}{m} x_1 + \frac{3k}{m} x_2 = -\frac{3k}{m} (x_1 - x_2).$$

But $x_1 - x_2 = y_2$, so

$$\ddot{y}_2 = -\frac{3k}{m} y_2.$$

Hence the motion in y_1 and y_2 is uncoupled:

$$\boxed{\ddot{y}_1 + \frac{k}{m} y_1 = 0, \quad \ddot{y}_2 + \frac{3k}{m} y_2 = 0.}$$

These describe two independent SHM modes with angular frequencies $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{3k}{m}}$.

3. Solution for $y_1(t)$ Under the Given Initial Conditions

Initially, $x_1(0) = 0$ (mass X at rest at equilibrium), $x_2(0) = 0$ (assume Y is also initially at equilibrium position but has velocity), and $\dot{x}_2(0) = v$ (the given initial velocity). Then

$$y_1(0) = x_1(0) + x_2(0) = 0 + 0 = 0, \quad \dot{y}_1(0) = \dot{x}_1(0) + \dot{x}_2(0) = 0 + v = v.$$

From the y_1 -equation,

$$\ddot{y}_1 + \left(\frac{k}{m}\right) y_1 = 0,$$

the general solution is $y_1(t) = A \sin(\omega_1 t) + B \cos(\omega_1 t)$ with $\omega_1 = \sqrt{k/m}$. The initial conditions imply

$$y_1(0) = B = 0, \quad \dot{y}_1(0) = A \omega_1 = v \implies A = \frac{v}{\omega_1} = v \sqrt{\frac{m}{k}}.$$

Hence

$$\boxed{y_1(t) = \left(v \sqrt{\frac{m}{k}}\right) \sin\left(\sqrt{\frac{k}{m}} t\right) = v \sqrt{\frac{m}{k}} \sin\left(t \sqrt{\frac{k}{m}}\right).}$$

This is exactly the stated result for $y_1(t)$.

Question 6

Solution:

(a) Maximum Speed on a Flat Track (Friction Present)

A go-kart travels around a flat circular track of radius R . The only force providing the centripetal acceleration is friction, whose maximum (static) value is

$$f_{\max} = \mu_s N,$$

where N is the normal force. On flat ground with no vertical acceleration,

$$N = mg.$$

Hence

$$f_{\max} = \mu_s mg.$$

This frictional force must equal the required centripetal force $m \frac{v^2}{R}$ (if the go-kart is just on the verge of slipping). Thus

$$m \frac{v^2}{R} \leq \mu_s mg \implies v_{\max} = \sqrt{\mu_s g R}.$$

Hence

$$\boxed{v_{\max} = \sqrt{\mu_s g R} \quad (\text{flat track}).}$$

(b) Maximum Speed on a Banked (Frictionless) Curve

Now the track is tilted at an angle ϕ w.r.t. the horizontal, and the track is frictionless. In this scenario, the centripetal force arises from components of the normal force N and gravity mg .

- *Radial (horizontal, toward center):* The component $N \sin(\phi)$ provides centripetal acceleration $m \frac{v^2}{R}$.
- *Vertical:* $N \cos(\phi) = mg$ (no vertical acceleration).

From $N \cos(\phi) = mg$, we get $N = \frac{mg}{\cos(\phi)}$. Hence in the radial direction,

$$N \sin(\phi) = \frac{mg}{\cos(\phi)} \sin(\phi) = mg \tan(\phi) = m \frac{v^2}{R}.$$

Solve for v :

$$v = \sqrt{g R \tan(\phi)}.$$

Thus the largest speed (which simply means the speed needed to remain in circular motion without sliding up or down, given no friction) is

$$\boxed{v_{\max} = \sqrt{g R \tan(\phi)} \quad (\text{banked, frictionless}).}$$

(c) Banked Curve with Friction (μ_s) and Speeds v_{\min} , v_{\max}

We combine both friction and banking. The normal force N again balances a portion of mg but now friction f can act *up* or *down* the plane, depending on whether the go-kart's speed is too large or too small. Let us choose a coordinate system with axes:

$$\text{Radial (horizontal, center-seeking)} = +\hat{r}, \quad \text{Vertical} = +\hat{z}.$$

Force balance in vertical direction:

$$N \cos(\phi) - f \sin(\phi) = mg \quad (1)$$

Force balance in radial (horizontal) direction:

$$N \sin(\phi) + f \cos(\phi) = m \frac{v^2}{R} \quad (2)$$

Here we have $|f| \leq \mu_s N$. The sign of f (i.e. up or down the incline) depends on whether the cart's speed tries to push it up or down the bank.

Maximum Speed, v_{\max} : If the speed is very large, friction must act *down* the incline to help provide additional center-seeking force. Thus f points down-slope, meaning in eqn. (1) we have $-f \sin(\phi)$, and in eqn. (2) $+f \cos(\phi)$. Also, $f = \mu_s N$ in magnitude at the threshold.

So set $f = \mu_s N$ and rewrite (1) and (2):

$$\begin{cases} N \cos(\phi) - \mu_s N \sin(\phi) = mg, \\ N \sin(\phi) + \mu_s N \cos(\phi) = m \frac{v_{\max}^2}{R}. \end{cases}$$

Factor out N :

$$\begin{cases} N [\cos(\phi) - \mu_s \sin(\phi)] = mg, \\ N [\sin(\phi) + \mu_s \cos(\phi)] = m \frac{v_{\max}^2}{R}. \end{cases}$$

Divide the second by the first to eliminate N :

$$\frac{\sin(\phi) + \mu_s \cos(\phi)}{\cos(\phi) - \mu_s \sin(\phi)} = \frac{\frac{v_{\max}^2}{R}}{g}.$$

Hence

$$v_{\max}^2 = g R \frac{\sin(\phi) + \mu_s \cos(\phi)}{\cos(\phi) - \mu_s \sin(\phi)}.$$

Minimum Speed, v_{\min} : If the speed is too small, friction must act *up* the incline to keep the go-kart from sliding down. In eqn. (1), that means the friction term has $+f \sin(\phi)$, and in eqn. (2) $-f \cos(\phi)$. At the threshold, $f = \mu_s N$.

So the system becomes

$$\begin{cases} N \cos(\phi) + \mu_s N \sin(\phi) = mg, \\ N \sin(\phi) - \mu_s N \cos(\phi) = m \frac{v_{\min}^2}{R}. \end{cases}$$

Factor out N :

$$\begin{cases} N(\cos \phi + \mu_s \sin \phi) = mg, \\ N(\sin \phi - \mu_s \cos \phi) = m \frac{v_{\min}^2}{R}. \end{cases}$$

Divide the second by the first to eliminate N :

$$\frac{\sin \phi - \mu_s \cos \phi}{\cos \phi + \mu_s \sin \phi} = \frac{\frac{v_{\min}^2}{R}}{g}.$$

Hence

$$v_{\min}^2 = g R \frac{\sin(\phi) - \mu_s \cos(\phi)}{\cos(\phi) + \mu_s \sin(\phi)}.$$

Showing the Identity

$$v_{\max}^2 - v_{\min}^2 = \frac{2 R g \mu_s}{\cos^2(\phi) - \mu_s^2 \sin^2(\phi)}.$$

We observe:

$$v_{\max}^2 = g R \frac{\sin(\phi) + \mu_s \cos(\phi)}{\cos(\phi) - \mu_s \sin(\phi)}, \quad v_{\min}^2 = g R \frac{\sin(\phi) - \mu_s \cos(\phi)}{\cos(\phi) + \mu_s \sin(\phi)}.$$

Compute the difference:

$$v_{\max}^2 - v_{\min}^2 = g R \left[\frac{\sin \phi + \mu_s \cos \phi}{\cos \phi - \mu_s \sin \phi} - \frac{\sin \phi - \mu_s \cos \phi}{\cos \phi + \mu_s \sin \phi} \right].$$

Put each fraction over a common denominator (the product $(\cos \phi - \mu_s \sin \phi)(\cos \phi + \mu_s \sin \phi)$):

$$\frac{(\sin \phi + \mu_s \cos \phi)(\cos \phi + \mu_s \sin \phi) - (\sin \phi - \mu_s \cos \phi)(\cos \phi - \mu_s \sin \phi)}{(\cos \phi - \mu_s \sin \phi)(\cos \phi + \mu_s \sin \phi)}.$$

Expand numerator:

$$\text{Num} = (\sin \phi \cos \phi + \mu_s \sin^2 \phi + \mu_s \cos^2 \phi + \mu_s^2 \sin \phi \cos \phi) - (\sin \phi \cos \phi - \mu_s \sin^2 \phi - \mu_s \cos^2 \phi + \mu_s^2 \sin \phi \cos \phi).$$

Group like terms carefully:

$$\text{Num} = \sin \phi \cos \phi + \mu_s \sin^2 \phi + \mu_s \cos^2 \phi + \mu_s^2 \sin \phi \cos \phi - \sin \phi \cos \phi + \mu_s \sin^2 \phi + \mu_s \cos^2 \phi - \mu_s^2 \sin \phi \cos \phi.$$

We see some cancellations: $\sin \phi \cos \phi$ terms cancel, as do $\mu_s^2 \sin \phi \cos \phi$ terms (with a sign difference). So

$$\text{Num} = \mu_s \sin^2 \phi + \mu_s \cos^2 \phi + \mu_s \sin^2 \phi + \mu_s \cos^2 \phi = 2 \mu_s (\sin^2 \phi + \cos^2 \phi) = 2 \mu_s.$$

Hence the entire difference becomes

$$v_{\max}^2 - v_{\min}^2 = g R \frac{2 \mu_s}{(\cos \phi - \mu_s \sin \phi)(\cos \phi + \mu_s \sin \phi)}.$$

Meanwhile, the denominator factor is

$$(\cos \phi - \mu_s \sin \phi)(\cos \phi + \mu_s \sin \phi) = \cos^2 \phi - \mu_s^2 \sin^2 \phi.$$

So,

$$v_{\max}^2 - v_{\min}^2 = g R \frac{2 \mu_s}{\cos^2 \phi - \mu_s^2 \sin^2 \phi}.$$

Or equivalently,

$$v_{\max}^2 - v_{\min}^2 = \frac{2 g R \mu_s}{\cos^2(\phi) - \mu_s^2 \sin^2(\phi)}.$$

which matches the stated form.

Question 7

Solution:

(a) Energy-Momentum 4-Vector and Its Invariance

A particle of mass m and velocity \mathbf{v} (with $|\mathbf{v}| = v$) in the lab frame has energy

$$E = \gamma m c^2, \quad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Its momentum is

$$\mathbf{p} = \gamma m \mathbf{v}.$$

We collect these into a 4-vector,

$$P^\mu = \left(\frac{E}{c}, \mathbf{p} \right) = (\gamma m c, \gamma m \mathbf{v}).$$

The *Lorentz-invariant* scalar (Minkowski) product of P^μ with itself is

$$P^\mu P_\mu = \left(\frac{E}{c} \right)^2 - |\mathbf{p}|^2 = \left(\frac{\gamma m c}{1} \right)^2 - (\gamma m v)^2 = \gamma^2 m^2 (c^2 - v^2) = m^2 c^2,$$

independent of \mathbf{v} . Hence $P^\mu P_\mu = m^2 c^2$ is the same in all inertial frames.

(b) Two-Body Decay in the Particle's Rest Frame and the Lab Frame

Suppose the original particle of mass m is at rest in its own rest frame. It decays into two identical *massless* particles emerging along $\pm x$ directions.

- *In the rest frame of the decaying particle:*

$$P_{(\text{initial})}^\mu = (m c, \mathbf{0}).$$

After decay, each daughter is massless with energy E_γ (we often call them photons if they are truly massless). Conservation of 4-momentum implies

$$P_{(\text{initial})}^\mu = P_{(1)}^\mu + P_{(2)}^\mu.$$

Because they are identical and back-to-back in $\pm x$, each has momentum magnitude $p_\gamma = E_\gamma/c$ (for massless). The total momentum must be zero, so one photon's momentum is $+p_\gamma \hat{x}$ and the other's is $-p_\gamma \hat{x}$. Thus

$$P_{(1)}^\mu = \left(\frac{E_\gamma}{c}, +p_\gamma, 0, 0\right), \quad P_{(2)}^\mu = \left(\frac{E_\gamma}{c}, -p_\gamma, 0, 0\right).$$

Adding these gives $(\frac{2E_\gamma}{c}, 0, 0, 0)$, which must match $(mc, 0, 0, 0)$. Hence $2E_\gamma = mc^2$, so each photon has $E_\gamma = \frac{1}{2}mc^2$.

- *In the lab frame:* the decaying particle moves with velocity v along, say, the $+x$ axis. Its 4-momentum in the lab is

$$P_{(\text{initial})}^\mu = (\gamma mc, \gamma mv, 0, 0).$$

Because each daughter is massless, each has $p^0 = E/c = |\mathbf{p}|$. One photon (call it #1) is emitted “forward” (in the same sense as the parent’s motion), and the other (#2) is “backward.” Their 4-momenta $P_{(1)}^\mu, P_{(2)}^\mu$ must sum to $P_{(\text{initial})}^\mu$. Solving explicitly for $P_{(1)}^\mu$ and $P_{(2)}^\mu$ in closed form is standard but somewhat messy:

$$P_{(1)}^\mu = \left(\frac{E_1}{c}, +E_1/c, 0, 0\right), \quad P_{(2)}^\mu = \left(\frac{E_2}{c}, -E_2/c, 0, 0\right),$$

with $E_1 + E_2 = \gamma mc^2$, and $E_1 - E_2 = \gamma mv c$. Solving these simultaneously gives

$$E_1 = \frac{\gamma mc^2}{2} (1 + \beta), \quad E_2 = \frac{\gamma mc^2}{2} (1 - \beta), \quad \text{where } \beta = \frac{v}{c}.$$

Hence the 4-momenta of the two massless particles in the lab are

$$P_{(1)}^\mu = \left(\frac{E_1}{c}, +\frac{E_1}{c}, 0, 0\right), \quad P_{(2)}^\mu = \left(\frac{E_2}{c}, -\frac{E_2}{c}, 0, 0\right).$$

with E_1, E_2 as above.

(c) Numerical Example: $m = 10 \text{ GeV}/c^2$, $\beta = 0.8$, and the decay products are photons

- *Wavelengths in the rest frame of the decaying particle:*

Each photon in the rest frame has energy $E_\gamma = \frac{1}{2}mc^2 = 5 \text{ GeV}$. The corresponding momentum is $p_\gamma = E_\gamma/c = 5 \text{ GeV}/c$. Hence

$$\lambda_{\text{rest}} = \frac{hc}{E_\gamma} = \frac{(4.1357 \times 10^{-15} \text{ eV} \cdot \text{s}) (3 \times 10^8 \text{ m/s})}{5 \times 10^9 \text{ eV}} \approx 2.48 \times 10^{-16} \text{ m}.$$

(If one prefers $\hbar c = 0.197 \text{ GeV fm}$ or similar units, one can do a direct conversion.)

- *Wavelengths in the lab frame:*

In the lab, one photon has energy $E_1 = \frac{1}{2} \gamma m c^2 (1+\beta)$, and the other $E_2 = \frac{1}{2} \gamma m c^2 (1-\beta)$. With $m = 10 \text{ GeV}/c^2$, $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-0.64} = 1/\sqrt{0.36} \approx 1.6667$, and $\beta = 0.8$. Then

$$E_1 = \frac{1}{2} (1.6667) (10 \text{ GeV}) (1+0.8) = 0.5 \times 1.6667 \times 10 \times 1.8 \text{ GeV} \approx 15 \text{ GeV}.$$

$$E_2 = \frac{1}{2} (1.6667) (10 \text{ GeV}) (1-0.8) = 0.5 \times 1.6667 \times 10 \times 0.2 \text{ GeV} \approx 1.6667 \text{ GeV}.$$

So one photon is “boosted” to 15 GeV in the lab, while the other is only 1.67 GeV. Their wavelengths are

$$\lambda_1 = \frac{hc}{E_1} \approx \frac{4.1357 \times 10^{-15} \text{ eV} \cdot \text{s} \cdot 3 \times 10^8 \text{ m/s}}{15 \times 10^9 \text{ eV}} \approx 8.3 \times 10^{-17} \text{ m},$$

$$\lambda_2 = \frac{hc}{E_2} \approx \frac{4.1357 \times 10^{-15} \text{ eV} \cdot \text{s} \cdot 3 \times 10^8 \text{ m/s}}{1.6667 \times 10^9 \text{ eV}} \approx 7.5 \times 10^{-16} \text{ m}.$$

Question 8

Solution:

System Description:

A pendulum of mass m and fixed length L is attached to a support that moves horizontally according to

$$x_{\text{support}}(t) = v t^3 + A \sin(\omega t),$$

where v , A , and ω are constants. Let $\phi(t)$ be the instantaneous angle of the pendulum from the vertical.

1. (a) Deriving the Equation of Motion from the Lagrangian

Coordinates: Choose the origin at the pivot, but note the pivot itself moves in the x -direction. At any time t , the bob’s coordinates become

$$X(t) = x_{\text{support}}(t) + L \sin(\phi), \quad Y(t) = -L \cos(\phi),$$

assuming the downward vertical is $Y < 0$ and $\phi = 0$ means straight down.

Kinetic Energy:

$$T = \frac{1}{2} m (\dot{X}^2 + \dot{Y}^2).$$

Compute \dot{X} , \dot{Y} :

$$\dot{X} = \dot{x}_{\text{support}}(t) + L \cos(\phi) \dot{\phi}, \quad \dot{Y} = -[-L \sin(\phi) \dot{\phi}] = L \sin(\phi) \dot{\phi}.$$

Hence

$$\dot{X}^2 + \dot{Y}^2 = \left(\dot{x}_{\text{support}} + L \cos \phi \dot{\phi} \right)^2 + \left(L \sin \phi \dot{\phi} \right)^2.$$

Therefore

$$T = \frac{1}{2} m \left[\dot{x}_{\text{support}}^2 + 2 \dot{x}_{\text{support}} L \cos \phi \dot{\phi} + L^2 \cos^2 \phi \dot{\phi}^2 + L^2 \sin^2 \phi \dot{\phi}^2 \right].$$

Combine the $\cos^2 \phi + \sin^2 \phi = 1$ terms:

$$T = \frac{1}{2} m \left[\dot{x}_{\text{support}}^2 + 2 \dot{x}_{\text{support}} L \cos \phi \dot{\phi} + L^2 \dot{\phi}^2 \right].$$

Potential Energy: We take $U = 0$ at $Y = 0$ for convenience, or more standardly we set $U = mg \times (\text{vertical displacement})$. Since $Y = -L \cos(\phi)$, the bob's height relative to $Y = 0$ is $-L \cos(\phi)$, so

$$U = m g (\text{height}) = m g (L \cos \phi) \quad (\text{up to an additive constant}).$$

Thus

$$U(\phi) = m g L \cos(\phi).$$

Lagrangian: $\mathcal{L} = T - U$.

$$\mathcal{L}(\phi, \dot{\phi}, t) = \frac{1}{2} m \left[\dot{x}_{\text{support}}^2 + 2 \dot{x}_{\text{support}} L \cos \phi \dot{\phi} + L^2 \dot{\phi}^2 \right] - m g L \cos(\phi).$$

Note that $\dot{x}_{\text{support}}^2$ does not depend on ϕ or $\dot{\phi}$, so it contributes no force or torque on the pendulum (though it does matter if we wanted the total energy).

Euler-Lagrange Equation:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Compute each piece carefully:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2} m \left[2 \dot{x}_{\text{support}} L \cos \phi + 2 L^2 \dot{\phi} \right] = m \left[\dot{x}_{\text{support}} L \cos \phi + L^2 \dot{\phi} \right].$$
 Taking the time derivative,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m \left[\frac{d}{dt} (\dot{x}_{\text{support}}) L \cos \phi - \dot{x}_{\text{support}} L \sin \phi \dot{\phi} + 2 L^2 \dot{\phi} \ddot{\phi} \right].$$

But $\frac{d}{dt} (\dot{x}_{\text{support}}) = \ddot{x}_{\text{support}}(t)$. So

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m \left[L \cos \phi \ddot{x}_{\text{support}} - L \dot{x}_{\text{support}} \sin \phi \dot{\phi} + 2 L^2 \dot{\phi} \ddot{\phi} \right].$$

Meanwhile,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{1}{2} m \left[2 \dot{x}_{\text{support}} L (-\sin \phi) \dot{\phi} \right] - (m g L (-\sin \phi)),$$

since $\frac{\partial}{\partial \phi}(L^2 \dot{\phi}^2) = 0$ and $\frac{\partial}{\partial \phi}(\dot{x}_{\text{support}}^2) = 0$. Hence

$$\frac{\partial \mathcal{L}}{\partial \phi} = m \dot{x}_{\text{support}} L (-\sin \phi) \dot{\phi} + m g L \sin \phi.$$

Thus the Euler-Lagrange equation $\frac{d}{dt}(\partial \mathcal{L} / \partial \dot{\phi}) - (\partial \mathcal{L} / \partial \phi) = 0$ becomes

$$m \left[L \cos \phi \ddot{x}_{\text{support}} - L \dot{x}_{\text{support}} \sin \phi \dot{\phi} + 2 L^2 \dot{\phi} \ddot{\phi} \right] - \left[m \dot{x}_{\text{support}} L (-\sin \phi) \dot{\phi} + m g L \sin \phi \right] = 0.$$

Factor out $m L$:

$$m L \left[\cos \phi \ddot{x}_{\text{support}} - \dot{x}_{\text{support}} \sin \phi \dot{\phi} + 2 L \dot{\phi} \ddot{\phi} \right] - m L \left[\dot{x}_{\text{support}} (-\sin \phi) \dot{\phi} + g \sin \phi \right] = 0.$$

Simplify terms involving $\dot{x}_{\text{support}} \sin \phi \dot{\phi}$:

$$-\dot{x}_{\text{support}} \sin \phi \dot{\phi} - \dot{x}_{\text{support}} (-\sin \phi) \dot{\phi} = 0.$$

They cancel. So we get

$$m L \left[\cos \phi \ddot{x}_{\text{support}} + 2 L \dot{\phi} \ddot{\phi} - g \sin \phi \right] = 0.$$

Divide out $m L$:

$$\cos \phi \ddot{x}_{\text{support}} + 2 L \dot{\phi} \ddot{\phi} - g \sin \phi = 0.$$

Rearrange to an ordinary differential equation in $\phi(t)$:

$$2 L \dot{\phi} \ddot{\phi} = g \sin \phi - \cos \phi \ddot{x}_{\text{support}}(t).$$

Or more standardly we might write:

$$2 \dot{\phi} \ddot{\phi} = \frac{g}{L} \sin \phi - \frac{\cos \phi}{L} \ddot{x}_{\text{support}}(t).$$

One can also do a bit more algebra to isolate $\ddot{\phi}$. Either form is the EoM for $\phi(t)$ with a time-dependent forcing from $\ddot{x}_{\text{support}}(t)$.

2. (b) **Small-Angle Limit with $v \rightarrow 0$ and $A \rightarrow 0$**

In the limit that the pivot is *nearly stationary* ($\dot{x}_{\text{support}} \approx 0$, $\ddot{x}_{\text{support}} \approx 0$), and also ϕ is small so $\sin \phi \approx \phi$, $\cos \phi \approx 1$, the equation of motion reduces to

$$2 L \dot{\phi} \ddot{\phi} \approx g \phi.$$

In typical small-oscillation expansions, if the amplitude is also small, one might approximate $2 \dot{\phi} \ddot{\phi} \approx \frac{d}{dt}(\dot{\phi}^2)$ or simply proceed with the usual linearized pendulum form $\ddot{\phi} + \frac{g}{L} \phi = 0$.

Indeed, the *classic* pendulum equation for small angles is

$$\ddot{\phi} + \frac{g}{L} \phi = 0,$$

which yields simple harmonic motion of frequency $\sqrt{\frac{g}{L}}$. All extra terms vanish because the support's motion is negligible ($\dot{x}_{\text{support}} \rightarrow 0$) and $\sin \phi \approx \phi$.

Question 9

Solution:

A rope is wrapped around a uniform disk of mass M and radius R , and a block of mass m is attached to the free end of the rope. As the block falls, the rope unwinds without slipping, causing the disk to rotate. The disk has moment of inertia

$$I = \frac{1}{2} M R^2.$$

We want to find:

1. The angular acceleration α of the disk.
2. The tension T in the rope.
3. The linear acceleration a of the block.

(a) Deriving the Formulas

We assume the rope does not slip on the disk, so there is a one-to-one relation between the disk's angular acceleration α and the block's linear acceleration a :

$$a = \alpha R.$$

Forces and Torques:

- On the block: The forces are tension T (upward) and gravity mg (downward). Hence

$$m a = m g - T. \quad (1)$$

(Taking downward as positive, so a is the downward acceleration.)

- On the disk: The rope exerts tension T tangentially at radius R . This tension provides a torque $\tau = T R$ (taking the sign so that a positive T yields a clockwise angular acceleration). Because $I = \frac{1}{2} M R^2$, the rotational equation is

$$I \alpha = \tau = T R. \quad (2)$$

So

$$\frac{1}{2} M R^2 \alpha = T R \implies T = \frac{1}{2} M R \alpha.$$

But we also know $a = \alpha R$, or $\alpha = \frac{a}{R}$. Substitute $\alpha = \frac{a}{R}$ into $T = \frac{1}{2} M R \alpha$:

$$T = \frac{1}{2} M R \left(\frac{a}{R} \right) = \frac{1}{2} M a.$$

Now use this in equation (1). That is

$$m a = m g - \left(\frac{1}{2} M a \right).$$

Rearrange to solve for a :

$$m a + \frac{1}{2} M a = m g \implies \left(m + \frac{1}{2} M \right) a = m g.$$

Hence

$$a = \frac{m}{m + \frac{1}{2}M} g = \frac{2m}{2m + M} g.$$

Then the tension $T = \frac{1}{2} M a$ becomes

$$T = \frac{1}{2} M \frac{2m}{2m + M} g = \frac{Mm}{2m + M} g.$$

And finally, the disk's angular acceleration is

$$\alpha = \frac{a}{R} = \frac{1}{R} \frac{2m}{2m + M} g.$$

Summarizing succinctly:

$$a = \frac{2m}{2m + M} g, \quad T = \frac{mM}{2m + M} g, \quad \alpha = \frac{1}{R} \frac{2m}{2m + M} g.$$

(b) Numerical Values for $m = 3 \text{ kg}$, $M = 12 \text{ kg}$, $R = 0.20 \text{ m}$

$$a = \frac{2m}{2m + M} g = \frac{2 \times 3 \text{ kg}}{2 \times 3 + 12} 9.8 \text{ m/s}^2 = \frac{6}{6 + 12} 9.8 = \frac{6}{18} 9.8 = 0.3333 \times 9.8 = 3.27 \text{ m/s}^2 \approx 3.27.$$

The tension

$$T = \frac{mM}{2m + M} g = \frac{3 \times 12}{6 + 12} 9.8 = \frac{36}{18} 9.8 = 2 \times 9.8 = 19.6 \text{ N}.$$

And the angular acceleration

$$\alpha = \frac{a}{R} = \frac{3.27}{0.20} = 16.35 \text{ rad/s}^2 \approx 16.4.$$

$$a \approx 3.27 \text{ m/s}^2, \quad T \approx 19.6 \text{ N}, \quad \alpha \approx 16.4 \text{ rad/s}^2.$$

Question 10

Solution:

We have a ball of mass m dropped from an airplane with an initial horizontal velocity \mathbf{u} . The drag force on the ball is

$$\mathbf{F} = -b\mathbf{v},$$

where $b > 0$ is a constant and \mathbf{v} is the velocity of the ball.

1. (a) Equation of Motion and Its General Solution

Since gravity acts downward (say along $-\hat{\mathbf{k}}$) with magnitude mg , the net force on the ball is

$$m \frac{d\mathbf{v}}{dt} = -mg\hat{\mathbf{k}} - b\mathbf{v}.$$

Rearrange:

$$m \frac{d\mathbf{v}}{dt} + b \mathbf{v} = -m g \hat{\mathbf{k}}. \quad (1)$$

This is a first-order linear ODE in $\mathbf{v}(t)$. We can solve by standard methods (e.g. integrating factor). The vector form indicates that each Cartesian component satisfies a similar scalar differential equation. The known result (or by directly solving) is

$$\mathbf{v}(t) = \mathbf{A} e^{-\frac{b}{m}t} - \frac{m g}{b} \hat{\mathbf{k}},$$

where \mathbf{A} is a constant vector determined by initial conditions.

2. (b) Determining \mathbf{A} From the Initial Condition

Let $t = 0$ be the time the ball is released. At that instant, $\mathbf{v}(0) = \mathbf{u}$ (the airplane's horizontal velocity). So

$$\mathbf{v}(0) = \mathbf{A} e^0 - \frac{m g}{b} \hat{\mathbf{k}} = \mathbf{A} - \frac{m g}{b} \hat{\mathbf{k}} = \mathbf{u}.$$

Hence

$$\mathbf{A} = \mathbf{u} + \frac{m g}{b} \hat{\mathbf{k}}.$$

Thus the complete velocity solution is

$$\mathbf{v}(t) = \left(\mathbf{u} + \frac{m g}{b} \hat{\mathbf{k}} \right) e^{-\frac{b}{m}t} - \frac{m g}{b} \hat{\mathbf{k}}.$$

The first term decays exponentially, so eventually the velocity approaches the terminal velocity $-\frac{m g}{b} \hat{\mathbf{k}}$ (straight downward with constant speed).