

# ChatGPT o1 Marked Responses

Konstantinos Doran

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## 1 Introduction

This document catalogues some of the responses from ChatGPT o1 answering our dataset of physics questions. The first set of responses are without a context prompt. Instead I simply asked the LLM to answer the question. The second set of responses used the following context prompt before asking to solve each question: "Context: You are Mr. P, an AI professor at UCL specialising in physics, dedicated to helping me through the questions and challenges I encounter during my studies. Your role is to provide accurate and tailored answers to each question, adjusting your responses to the specific requirements of each question. When I struggle with understanding a question or concept, please walk me through the content step-by-step, explaining both the question and the answer in detail.". All have been marked given at the beginning of each question.

## 2 Main Contents

### 2.1 Classical Electromagnetic Theory without prompts

#### 2.1.1 Question 2:

**5/5 Marks Definition of Free Space:**In classical electromagnetism, "free space" refers to a region devoid of any charges and currents, and is assumed to be a perfect vacuum with the permittivity  $\varepsilon_0$  and the permeability  $\mu_0$ . In this idealized medium, the electric and magnetic fields are free to propagate without attenuation due to matter. **The Wave Equation:**In free space, each

Cartesian component of the electric field  $\mathbf{E}$  satisfies the wave equation

$$\nabla^2 E = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}.$$

This equation asserts that the spatial variation (via  $\nabla^2$ ) of the field component is proportional to its second derivative in time. It is precisely the form of a wave equation, indicating that disturbances in the field propagate

through space. **Physical Interpretation:** The wave equation shows that the electric field (and similarly the magnetic field) can travel as a self-sustaining electromagnetic wave through a vacuum. A changing electric field generates a changing magnetic field and vice versa, allowing the wave to move forward without any external sources or media. **Speed of Light:** From Maxwell's equations, the wave speed in free space emerges as

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

This constant  $c$  is the *speed of light in a vacuum* (approximately  $3.0 \times 10^8 \text{ m s}^{-1}$ ). Thus, the wave equation not only describes the propagation of electromagnetic waves but also reveals the fundamental speed limit for all massless energy transfer in the universe.

#### 2.1.2 Question 4:

##### 8/11 Marks

A very long, straight wire carries a steady current  $I$  downward. A square loop of side  $L$  is placed a distance  $d$  to the right of the wire (see figure). We take “into the page” as the negative flux direction.

##### (a) Flux of $B$ through the loop:

**5/5 Marks** From the Biot–Savart law (or Ampère's law), the magnetic field a distance  $r$  from a long straight wire (carrying current  $I$ ) is

$$B(r) = \frac{\mu_0 I}{2\pi r},$$

directed (by the right-hand rule) *into the page* on the right side of a downward current. Over the square loop (height  $L$ ),  $r$  varies from  $x = d$  to  $x = d + L$ . Because  $B$  is (approximately) uniform vertically, we can slice the loop into vertical strips of width  $dx$  at horizontal coordinate  $x$ . Each strip has area  $L dx$ , and the flux element is  $d\Phi = B(x) L dx$  (with a negative sign if we take the loop normal to be “out of the page”). Thus

$$\Phi_B = \int_{x=d}^{x=d+L} (-B(x)) L dx = - \int_d^{d+L} \frac{\mu_0 I}{2\pi x} L dx = - \frac{\mu_0 I L}{2\pi} \ln\left(\frac{d+L}{d}\right).$$

In absolute value,

$$|\Phi_B| = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{d+L}{d}\right), \quad \text{and it is into the page.}$$

##### (b) Loop moved outward (to the right) at speed $v$ :

**2/5 Marks** As the loop moves away,  $d$  increases with time, so the magnetic flux (magnitude) through the loop *decreases*. By Faraday's law,

$$\mathcal{E} = - \frac{d\Phi_B}{dt}.$$

Because the flux is becoming less (negative) in time,  $-d\Phi_B/dt$  is positive, meaning the induced current attempts to maintain the “into the page” flux. Hence by Lenz’s law, the loop’s induced field must also be *into* the page, which requires a *clockwise* current in the loop. A more explicit calculation (treating  $d(t) = d_0 + v t$ ) shows

$$\Phi_B(t) = -\frac{\mu_0 I L}{2\pi} \ln\left[\frac{d(t)+L}{d(t)}\right], \quad \text{thus} \quad \mathcal{E}(t) = \frac{\mu_0 I L v}{2\pi} \left[ \frac{1}{d(t)+L} - \frac{1}{d(t)} \right].$$

The sign (and hence direction) of the induced current is clockwise.

**(c) Loop moved *downward* at speed  $v$ :**

**1/1 Marks** If the loop simply moves vertically downward (parallel to the wire) without changing its horizontal distance  $d$ , the field at each part of the loop remains the same, and the total flux through the loop *does not change*. Therefore, no emf is induced and no current flows.

**2.1.3 Question 5:**

**1/11 Marks**

**(a) Current through the ammeter**

**0/8 Marks** From inspection of the diagram, the two sources (12 V and 2 V) appear in series with the five resistors whose values sum to  $6\,\Omega$  (three resistors of  $2\,\Omega$ ,  $2\,\Omega$ , and  $2\,\Omega$  when the two  $1\,\Omega$  resistors are in series, plus the two remaining  $2\,\Omega$  resistors). The total voltage driving the circuit is

$$V_{\text{total}} = 12\,\text{V} + 2\,\text{V} = 14\,\text{V},$$

and the total resistance is

$$R_{\text{total}} = 2\,\Omega + 2\,\Omega + 1\,\Omega + 1\,\Omega + 2\,\Omega = 6\,\Omega.$$

Hence, by Ohm’s law the current is

$$I = \frac{V_{\text{total}}}{R_{\text{total}}} = \frac{14\,\text{V}}{6\,\Omega} = \frac{7}{3}\,\text{A} \approx 2.33\,\text{A}.$$

This is the current the ammeter reads.

**(b) Energy delivered by the 12 V battery in 4 s**

**1/3 Marks** The power supplied by the 12 V source is

$$P_{12\,\text{V}} = (12\,\text{V}) \times I = 12\,\text{V} \times \frac{7}{3}\,\text{A} = 28\,\text{W}.$$

Over a time interval of 4 s, the energy delivered by the 12 V battery is

$$E = P_{12\,\text{V}} \times t = 28\,\text{J/s} \times 4\,\text{s} = 112\,\text{J}.$$

### 2.1.4 Question 7:

#### 6/6 Marks

- $R = 600\ \Omega$ ,  $C = 1500\ \text{pF} = 1500 \times 10^{-12}\ \text{F}$ ,  
 $L = 20\ \text{mH} = 20 \times 10^{-3}\ \text{H}$ .
- Driving source:  $E(t) = E_0 \cos(\omega t)$  (series RLC).

#### (a) Resonant angular frequency $\omega_0$

At resonance, the inductive reactance  $X_L = \omega L$  cancels the capacitive reactance  $X_C = 1/(\omega C)$ . The resonant angular frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(20 \times 10^{-3})(1500 \times 10^{-12})}} \approx 1.8 \times 10^5\ \text{rad s}^{-1}.$$

(Equivalently, the resonant frequency in hertz is  $f_0 = \omega_0/(2\pi) \approx 2.9 \times 10^4\ \text{Hz}$ .)

#### (b) Voltage drop across the inductor, $\Delta V_L$

In a series RLC circuit, the same current  $i(t)$  flows through  $R$ ,  $L$ , and  $C$ . The total impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{where} \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}.$$

The current amplitude is

$$I_0 = \frac{E_0}{Z}.$$

Phasor analysis shows that the inductor voltage leads the current by  $90^\circ$ , so if we write

$$i(t) = I_0 \cos(\omega t - \phi),$$

then

$$\Delta V_L(t) = I_0 X_L \cos(\omega t - \phi + 90^\circ).$$

Substitute  $I_0 = E_0/Z$ , to get

$$\Delta V_L(t) = \frac{E_0 X_L}{Z} \cos(\omega t - \phi + 90^\circ).$$

Here,  $\phi$  is the overall phase by which the current lags (or leads) the source voltage.

#### (c) Phase relationship at low and high frequency

- *Low frequency* ( $\omega \ll \omega_0$ ): The circuit behaves predominantly like a capacitor (because  $X_C = 1/(\omega C)$  is large, so the current leads the source). Consequently, the inductor's voltage, which itself leads the current by  $90^\circ$ , ends up nearly  $180^\circ$  out of phase with the source voltage.
- *High frequency* ( $\omega \gg \omega_0$ ): The circuit is predominantly inductive ( $X_L = \omega L$  is large, so the current lags the source). Here the inductor's voltage is only about  $0^\circ$  to  $10^\circ$  out of phase with the source (more precisely, it becomes almost *in phase* with the driving voltage as  $\omega \rightarrow \infty$ ).

### 2.1.5 Question 9:

13/15 Marks

**Given:** A transverse electromagnetic wave traveling in free space, described (in complex notation) by

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)],$$

where  $\mathbf{E}_0$  and  $\mathbf{k}$  are real vectors. We wish to show:

- (a) The (real) magnetic field amplitude satisfies  $H(\mathbf{r}, t) = \sqrt{\frac{\varepsilon_0}{\mu_0}} E(\mathbf{r}, t)$ .
- (b) The cycle-averaged Poynting flux is  $\langle \mathbf{N} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} H_0^2 \hat{\mathbf{k}}$ .
- (c) If this wave (with intensity  $I$ ) strikes a *perfectly reflecting* mirror at an incidence angle  $\alpha$ , the resulting radiation pressure on the mirror is  $P = \frac{2I}{c} \cos^2 \alpha$ .

#### (a) Relation between $\mathbf{E}$ and $\mathbf{H}$ via Faraday's Law 6/7 Marks

In free space (with  $\rho_{\text{free}} = 0$ ,  $\mathbf{J}_{\text{free}} = 0$ ), Maxwell's curl equations are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t},$$

where  $\mathbf{D} = \varepsilon_0 \mathbf{E}$  and  $\mathbf{B} = \mu_0 \mathbf{H}$  in vacuum.

For a plane wave traveling along  $\hat{\mathbf{k}}$ ,  $\mathbf{E}$  is perpendicular to  $\mathbf{k}$ , and  $\mathbf{B}$  (or  $\mathbf{H}$ ) is perpendicular to both  $\mathbf{k}$  and  $\mathbf{E}$ . Using  $\mathbf{B} = \mu_0 \mathbf{H}$  and  $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ , one finds that in vacuum the magnitudes of  $\mathbf{E}$  and  $\mathbf{H}$  satisfy

$$H_0 = \frac{E_0}{\eta_0}, \quad \text{where } \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \text{ (the wave impedance of free space),}$$

so

$$H_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0.$$

Hence in time-domain form,

$$H(\mathbf{r}, t) = \sqrt{\frac{\varepsilon_0}{\mu_0}} E(\mathbf{r}, t).$$

#### (b) Time-Averaged Poynting Flux 6/6 Marks

The instantaneous Poynting vector is

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t).$$

For a linearly polarized plane wave with  $E_0 \perp H_0 \perp \hat{\mathbf{k}}$ , its *time-averaged* magnitude over one full cycle is

$$\langle S \rangle = \frac{1}{2} E_0 H_0,$$

because  $\langle \cos^2(\omega t) \rangle = \frac{1}{2}$ . Substituting  $H_0 = \sqrt{\varepsilon_0/\mu_0} E_0$ ,

$$\langle S \rangle = \frac{1}{2} E_0 \left( \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \right) = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2.$$

Sometimes this is written in terms of  $H_0$ :

$$\langle S \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} H_0^2.$$

In vector form (directed along  $\hat{\mathbf{k}}$ ):

$$\langle \mathbf{N} \rangle = \langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} H_0^2 \hat{\mathbf{k}}.$$

**(c) Radiation Pressure for a Perfect Reflector at Angle  $\alpha$  1/2 Marks**

If a wave of *intensity*  $I = \langle S \rangle$  is *normally* incident on a perfectly reflecting surface, the normal-momentum change per unit area per unit time leads to a pressure

$$P_{\text{normal}} = \frac{2I}{c}.$$

However, if the wave arrives at an *angle*  $\alpha$  (measured from the surface normal), two geometric effects reduce the normal momentum flux:

- The component of the wave's momentum *normal* to the surface is  $\cos \alpha$  times the total.
- On reflection, the momentum normal to the surface reverses direction, giving a factor of 2, so net normal impulse is  $(2 \cos \alpha) \times (\text{incident momentum})$ .
- But the intensity that actually acts on that area is still  $I$  (we assume the cross-section is oriented for reflection). The net effect is an extra factor of  $\cos \alpha$  from the momentum component normal to the surface.

Thus, the resulting pressure is

$$P = \frac{2I}{c} \cos^2 \alpha.$$

For  $\alpha = 0$  (normal incidence), this reduces to  $2I/c$ , as expected.

**Answer Summary:**

(a)  $H(\mathbf{r}, t) = \sqrt{\frac{\varepsilon_0}{\mu_0}} E(\mathbf{r}, t).$

(b)  $\langle \mathbf{N} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} (H_0)^2 \hat{\mathbf{k}}.$

(c)  $P = \frac{2I}{c} \cos^2 \alpha$  for a perfectly reflecting surface at incidence angle  $\alpha$ .

### 2.1.6 Question 10:

6/7 Marks

(a) Invariance of the scalar product  $(\mathbf{E} \cdot \mathbf{B})$

**3/4 Marks** Two inertial frames  $S$  and  $S'$  are arranged so that  $S'$  moves with speed  $v$  along the  $+x$ -direction relative to  $S$ . The Lorentz transformation for the fields (in SI units,  $c = 1/\sqrt{\varepsilon_0 \mu_0}$ ) is given by:

$$\begin{aligned} E'_x &= E_x, & E'_y &= \gamma(E_y - v B_z), & E'_z &= \gamma(E_z + v B_y), \\ B'_x &= B_x, & B'_y &= \gamma\left(B_y + \frac{v}{c^2} E_z\right), & B'_z &= \gamma\left(B_z - \frac{v}{c^2} E_y\right), \end{aligned}$$

where  $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ .

We claim

$$(\mathbf{E} \cdot \mathbf{B}) = E_x B_x + E_y B_y + E_z B_z \implies (\mathbf{E}' \cdot \mathbf{B}') = E'_x B'_x + E'_y B'_y + E'_z B'_z = (\mathbf{E} \cdot \mathbf{B}).$$

One can verify this by substituting from the transformation equations. Notice that terms involving  $v B_z$  mix with  $v E_y/c^2$  (and similarly for the  $z$ -component), and after collecting like terms with the appropriate factors of  $\gamma$ , the two dot products are equal. Therefore, the quantity  $\mathbf{E} \cdot \mathbf{B}$  is Lorentz-invariant.

(b) Plane Electromagnetic Wave in Frames  $S$  and  $S'$

**2/2 Marks** Consider in frame  $S$  a plane EM wave propagating along  $+x$  in vacuum:

$$\mathbf{E}(x, t) = E_0 \hat{\mathbf{y}} \exp[i(kx - \omega t)].$$

Its phase is  $\phi = kx - \omega t$ . To find the corresponding phase in  $S'$ , use the Lorentz transformations

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right).$$

Hence

$$\phi = kx - \omega t = k[\gamma(x' + vt')] - \omega[\gamma(t' + \frac{vx'}{c^2})].$$

Regroup terms in  $x'$  and  $t'$ :

$$\phi = \gamma\left(k - \frac{\omega v}{c^2}\right)x' - \gamma(\omega - kv)t'.$$

We define

$$k' = \gamma \left( k - \frac{\omega v}{c^2} \right), \quad \omega' = \gamma (\omega - k v).$$

Thus in  $S'$ , the same wave is expressed as

$$\mathbf{E}'(x', t') = E'_0 \hat{\mathbf{y}}' \exp[i(k' x' - \omega' t')],$$

where  $k' x' - \omega' t'$  is exactly the Lorentz-transformed phase.

### (c) Relationship of Frequencies in the Limit $v/c \rightarrow 0$

**1/1 Marks** From

$$\omega' = \gamma (\omega - k v),$$

observe that as  $v \rightarrow 0$ ,  $\gamma \rightarrow 1$  and  $k v \rightarrow 0$ . Thus

$$\omega' \longrightarrow \omega.$$

Similarly,  $k' \rightarrow k$ . Physically, this means that in the limit of negligible relative motion, the two observers measure the *same* frequency (and wavenumber).

More generally, for finite  $v$ , these relations represent the *relativistic Doppler shift* of the wave frequency and wave vector between moving inertial frames.

In the low-velocity (nonrelativistic) limit, it reproduces the classical Doppler effect; as  $v/c \rightarrow 0$ , the shift vanishes, so  $\omega' \approx \omega$ .

## 2.2 Quantum Mechanics without prompts

### 2.2.1 Q1. Commutators and Compatibility in Quantum Mechanics

**5/5 Marks**

#### (a) Commutator and Compatible Operators

##### **3/3 Marks Definition of the Commutator:**

For two operators  $\hat{P}$  and  $\hat{Q}$ , the *commutator* is defined as

$$[\hat{P}, \hat{Q}] = \hat{P} \hat{Q} - \hat{Q} \hat{P}.$$

##### **Compatible Operators:**

Two observables (represented by Hermitian operators) are said to be *compatible* if they can be simultaneously measured with definite values.

Equivalently, a complete set of eigenstates can be found that diagonalizes both operators at once. In mathematical terms, *compatibility* implies

$$[\hat{P}, \hat{Q}] = 0.$$

That is, *compatible* operators *commute*.

#### (b) Effect of Non-Compatibility on Measurements

**2/2 Marks**



If  $\hat{P}$  and  $\hat{Q}$  *do not commute* (i.e.  $[\hat{P}, \hat{Q}] \neq 0$ ), then they are *not* compatible observables. Suppose that the system is measured first for  $\hat{P}$  and found to have some definite value  $p$ . Immediately afterward,  $\hat{Q}$  is measured, giving a definite value  $q$ . Because  $\hat{Q}$  does not commute with  $\hat{P}$ , this second measurement in general *disturbs* the state in a way that it will *not* remain an eigenstate of  $\hat{P}$ . As a result, if one measures  $\hat{P}$  *again* immediately, the outcome need *not* be  $p$ . The act of measuring  $\hat{Q}$  has changed the quantum state so that it is no longer an eigenstate of  $\hat{P}$ . Hence the final measurement of  $\hat{P}$  may give a range of possible results, subject to the (new) post-measurement state determined by the earlier  $\hat{Q}$  measurement.

### 2.2.2 Q3. Checking Hermiticity by Integration by Parts

8/9 Marks

**Definition (1D):** An operator  $\hat{Q}$  is Hermitian if, for all sufficiently well-behaved functions  $f(x)$  and  $g(x)$  that vanish at  $\pm\infty$ ,

$$\int_{-\infty}^{\infty} f^*(x) [\hat{Q} g(x)] dx = \int_{-\infty}^{\infty} [\hat{Q} f(x)]^*(x) g(x) dx.$$

(a) **Momentum Operator**  $\hat{p} = -i\hbar \frac{d}{dx}$

4/4 Marks

We test:

$$\int f^*(x) \left( -i\hbar \frac{d}{dx} g(x) \right) dx \stackrel{?}{=} \int \left[ -i\hbar \frac{d}{dx} f(x) \right]^*(x) g(x) dx.$$

Focus on the left integral. By integration by parts,

$$\int_{-\infty}^{\infty} f^*(x) \left( -i\hbar g'(x) \right) dx = \left[ -i\hbar f^*(x) g(x) \right]_{-\infty}^{\infty} + i\hbar \int_{-\infty}^{\infty} f'^*(x) g(x) dx.$$

Since  $f(x)$  and  $g(x)$  both vanish at  $\pm\infty$ , the boundary term is zero, leaving

$$i\hbar \int_{-\infty}^{\infty} f'^*(x) g(x) dx.$$

On the other hand,

$$[\hat{p} f(x)]^*(x) = \left[ -i\hbar \frac{d}{dx} f(x) \right]^* = +i\hbar \frac{d}{dx} f^*(x).$$

Hence

$$\int [\hat{p} f(x)]^*(x) g(x) dx = \int [i\hbar f'^*(x)] g(x) dx = i\hbar \int_{-\infty}^{\infty} f'^*(x) g(x) dx,$$

which matches exactly the expression found from the integration by parts. Thus

$$\int f^* (\hat{p}g) dx = \int (\hat{p}f)^* g dx,$$

and we conclude  $\hat{p} = -i\hbar \frac{d}{dx}$  is indeed a Hermitian operator (provided the wavefunctions vanish sufficiently fast at  $\pm\infty$ ).

**(b) Operator  $\hat{Q} = \frac{d^2}{dx^2}$**

**4/5 Marks**

Similarly, consider

$$\int_{-\infty}^{\infty} f^*(x) \left( \frac{d^2}{dx^2} g(x) \right) dx \stackrel{?}{=} \int_{-\infty}^{\infty} \left( \frac{d^2}{dx^2} f(x) \right)^* (x) g(x) dx.$$

Apply integration by parts *twice*. The boundary terms at  $\pm\infty$  will vanish (again assuming  $f, g$  and their derivatives vanish at  $\pm\infty$ ), leaving

$$\int f^*(x) g''(x) dx = \int f'^*(x) g'(x) dx = \int f''^*(x) g(x) dx.$$

Hence

$$\int f^* (\hat{Q}g) dx = \int (\hat{Q}f)^* g dx,$$

showing that  $\hat{Q} = \frac{d^2}{dx^2}$  is also Hermitian (with suitable boundary conditions).

### 2.2.3 Q5. First-Order Energy Corrections in an Infinite Square Well

**7/7 Marks**

We have a particle of mass  $m$  in the infinite square well,

$$V(x) = \begin{cases} 0, & 0 \leq x \leq L, \\ +\infty, & \text{otherwise,} \end{cases}$$

with unperturbed (exact) eigenfunctions

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2 \hbar^2 \pi^2}{2m L^2}.$$

**(a) Correction for  $\hat{H}' = V_0 \sin(\frac{2\pi x}{L})$  (Ground State)**

**4/4 Marks**

In time-independent perturbation theory, the first-order energy correction to the  $n$ th level is

$$E_n^{(1)} = \langle \phi_n | \hat{H}' | \phi_n \rangle = \int_0^L \phi_n^*(x) [V_0 \sin \frac{2\pi x}{L}] \phi_n(x) dx.$$

For the ground state ( $n = 1$ ),

$$\phi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right),$$

so

$$E_1^{(1)} = V_0 \int_0^L \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx.$$

One may evaluate this definite integral (using trigonometric identities or orthogonality arguments) and finds *it equals zero*. Hence

$$\boxed{E_1^{(1)} = 0.}$$

Physically,  $\sin(\frac{2\pi x}{L})$  is orthogonal in just the right way to  $\sin^2(\frac{\pi x}{L})$  over  $[0, L]$ .

**(b) General First-Order Correction for  $\hat{H}' = L\alpha\delta(x - \frac{L}{2})$**   
**3/3 Marks**

Here, the perturbation is a Dirac delta function at the midpoint  $x = L/2$ . For the  $n$ th level,

$$E_n^{(1)} = \int_0^L \phi_n^*(x) \left[ L\alpha\delta\left(x - \frac{L}{2}\right) \right] \phi_n(x) dx = L\alpha \left| \phi_n\left(\frac{L}{2}\right) \right|^2.$$

Since  $\phi_n\left(\frac{L}{2}\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{2}\right)$ , we get

$$\left| \phi_n\left(\frac{L}{2}\right) \right|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi}{2}\right).$$

Hence

$$E_n^{(1)} = L\alpha \frac{2}{L} \sin^2\left(\frac{n\pi}{2}\right) = 2\alpha \sin^2\left(\frac{n\pi}{2}\right).$$

In concise form,

$$\boxed{E_n^{(1)} = 2\alpha \sin^2\left(\frac{n\pi}{2}\right).}$$

Notably,  $\sin^2(n\pi/2)$  alternates between 0 and 1 depending on  $n$ . For even  $n$ ,  $E_n^{(1)} = 0$ , and for odd  $n$ ,  $E_n^{(1)} = 2\alpha$ .

**2.2.4 Q6: Angular Momentum and Spin Operators**

**14/17 Marks**

**(a) Showing  $\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2\hat{L}_z \hat{S}_z$ .**

**4/4 Marks**

Define  $\hat{J} = \hat{L} + \hat{S}$ . Then

$$\hat{J}^2 = (\hat{L} + \hat{S}) \cdot (\hat{L} + \hat{S}) = \hat{L}^2 + \hat{S}^2 + 2\hat{L} \cdot \hat{S}.$$

But  $\hat{L} \cdot \hat{S} = \hat{L}_x \hat{S}_x + \hat{L}_y \hat{S}_y + \hat{L}_z \hat{S}_z$ . Using ladder operators  $\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y$  and  $\hat{S}_\pm = \hat{S}_x \pm i \hat{S}_y$ , one shows

$$\hat{L}_x \hat{S}_x + \hat{L}_y \hat{S}_y = \frac{1}{2}(\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+).$$

Hence

$$\hat{L} \cdot \hat{S} = \frac{1}{2}(\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+) + \hat{L}_z \hat{S}_z,$$

and

$$\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2 \hat{L}_z \hat{S}_z.$$

**(b) The state  $|l, -l; s, -s\rangle$  is an eigenvector of  $\hat{J}^2$ .**

**4/4 Marks**

When combining orbital ( $\hat{L}^2, \hat{L}_z$ ) and spin ( $\hat{S}^2, \hat{S}_z$ ), the total angular momentum is labeled by quantum numbers  $\{j, m_j\}$ , where  $\hat{J}^2 |j, m_j\rangle = \hbar^2 j(j+1) |j, m_j\rangle$ . For given  $l$  and  $s$ , the maximum possible  $j$  is  $l+s$ . The state with  $m = -l$  and  $m_s = -s$  has total  $m_j = -(l+s)$ , which is the *lowest* rung in the  $j = l+s$  multiplet. One can show by applying  $\hat{J}_-$  that no lower  $m_j$  exists in that subspace, hence it must be the  $|j = l+s, m_j = -(l+s)\rangle$  eigenstate. Thus

$$\hat{J}^2 |l, -l; s, -s\rangle = \hbar^2 (l+s)(l+s+1) |l, -l; s, -s\rangle.$$

**(c) Matrix of  $\hat{O} = a \hat{L}^2 + b \hat{S}_+ \hat{L}_z$  in the basis  $l = 1, m; s = \frac{1}{2}, m_s$ .**

**6/9 Marks**

We take the six-dimensional subspace with  $l = 1$  ( $m = -1, 0, 1$ ) and  $s = \frac{1}{2}$  ( $m_s = \pm \frac{1}{2}$ ).

- $\hat{L}^2$  acts as  $\hbar^2 l(l+1) = 2\hbar^2$  times the identity on all these states (since  $l = 1$ ).
- $\hat{S}_+ \hat{L}_z$  first multiplies  $1, m$  by  $\hbar m$  (from  $\hat{L}_z$ ) and then raises the spin from  $m_s$  to  $m_s + 1$  (provided  $m_s + 1$  is in range). In particular for  $s = \frac{1}{2}$ ,  $\hat{S}_+ \frac{1}{2}, m_s$  is nonzero *only* if  $m_s = -\frac{1}{2}$ , giving

$$\hat{S}_+ |\frac{1}{2}, -\frac{1}{2}\rangle = \hbar |\frac{1}{2}, +\frac{1}{2}\rangle, \quad \hat{S}_+ |\frac{1}{2}, +\frac{1}{2}\rangle = 0.$$

Hence  $\hat{S}_+ \hat{L}_z$  is *off-diagonal* in the spin- $\frac{1}{2}$  subspace, connecting  $m_s = -\frac{1}{2}$  to  $m_s = +\frac{1}{2}$  with amplitude  $\hbar^2 m$ .

Label the basis as

$$|m, m_s\rangle = |l = 1, m; s = \frac{1}{2}, m_s\rangle, \quad m = -1, 0, 1; m_s = \pm \frac{1}{2}.$$

Then  $\hat{O}$  has a  $6 \times 6$  block-diagonal form with three  $2 \times 2$  blocks (one block for each  $m$ ):

$$\hat{O} = \begin{pmatrix} \text{Block}_{m=-1} & 0 & 0 \\ 0 & \text{Block}_{m=0} & 0 \\ 0 & 0 & \text{Block}_{m=1} \end{pmatrix},$$

where each  $2 \times 2$  block corresponds to  $m_s = \{-\frac{1}{2}, +\frac{1}{2}\}$ .

Example:  $m = -1$  block.

$$\hat{O}_{m=-1} = \begin{pmatrix} \langle -1, -\frac{1}{2} | \hat{O} | -1, -\frac{1}{2} \rangle & \langle -1, -\frac{1}{2} | \hat{O} | -1, +\frac{1}{2} \rangle \\ \langle -1, +\frac{1}{2} | \hat{O} | -1, -\frac{1}{2} \rangle & \langle -1, +\frac{1}{2} | \hat{O} | -1, +\frac{1}{2} \rangle \end{pmatrix}.$$

Since  $\hat{L}^2 = 2\hbar^2$  on all states and  $\hat{S}_+ \hat{L}_z$  couples  $-1, -\frac{1}{2} \rightarrow -1, +\frac{1}{2}$  with amplitude  $\hbar^2(-1)$ , one finds

$$\hat{O}_{m=-1} = \begin{pmatrix} 2a\hbar^2 & -b\hbar^2 \\ 0 & 2a\hbar^2 \end{pmatrix}.$$

Likewise for  $m = 0$ , the factor  $m = 0$  kills the off-diagonal element, so that block is simply  $\text{diag}(2a\hbar^2, 2a\hbar^2)$ . And for  $m = +1$ , the off-diagonal amplitude is  $+b\hbar^2$ .

**Hence the full matrix of  $\hat{O}$  in the  $1, m; \frac{1}{2}, m_s$  basis is block-diagonal, each block of the form**

$$\begin{pmatrix} 2a\hbar^2 & b\hbar^2(\pm 1) \\ 0 & 2a\hbar^2 \end{pmatrix},$$

with the sign determined by  $m = \pm 1, 0$ .

## 2.2.5 Q9: 1D Potential Step with

17/17 Marks

$$V(x) = \begin{cases} 0, & x \leq 0, \\ -V_0, & x > 0, \end{cases} \quad E < 0.$$

**(a) General Form of the Wavefunctions**

5/5 Marks

Since the total energy is  $E > 0$ , we write

$$k_1 = \frac{\sqrt{2mE}}{\hbar}, \quad k_2 = \frac{\sqrt{2m(E + V_0)}}{\hbar}.$$

In the region  $x \leq 0$  (where  $V = 0$ ), the free-particle Schrödinger equation has the general solution consisting of a right-moving wave plus a left-moving wave:

$$\psi_1(x) = e^{ik_1x} + B e^{-ik_1x},$$

where  $B$  is the reflection amplitude. In the region  $x > 0$  (where  $V = -V_0$ ), we assume only a right-moving transmitted wave:

$$\psi_2(x) = C e^{ik_2 x},$$

where  $C$  is the transmission amplitude.

**(b) Matching Boundary Conditions at  $x = 0$**

**5/5 Marks**

We demand continuity of the wavefunction and its first derivative at  $x = 0$ .

That is,

$$\psi_1(0) = \psi_2(0), \quad \left. \frac{d\psi_1}{dx} \right|_{x=0} = \left. \frac{d\psi_2}{dx} \right|_{x=0}.$$

From  $\psi_1(0) = 1 + B$  and  $\psi_2(0) = C$ , continuity of  $\psi$  gives

$$1 + B = C.$$

Next, the derivative condition  $\psi'_1(x) = ik_1 e^{ik_1 x} - ik_1 B e^{-ik_1 x}$  at  $x = 0$  is

$$ik_1(1 - B) = ik_2 C.$$

Eliminating  $C$ , we solve for  $B$  and  $C$  in terms of  $k_1$  and  $k_2$ :

$$C = \frac{2k_1}{k_1 + k_2}, \quad B = \frac{k_1 - k_2}{k_1 + k_2}.$$

**(c) Flux and Reflection/Transmission Probabilities**

**7/7 Marks**

The probability current (flux) for a wavefunction  $\psi(x)$  is

$$\Gamma = \frac{i\hbar}{2m} \left[ \psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right].$$

- *Incident flux:* In region  $x \leq 0$ , the incident wave is  $e^{ik_1 x}$ , which has flux

$$\Gamma_{\text{inc}} = \frac{\hbar k_1}{m}.$$

- *Reflected flux:* Also in  $x \leq 0$ , the reflected wave  $B e^{-ik_1 x}$  has flux

$$\Gamma_{\text{ref}} = -\frac{\hbar k_1}{m} |B|^2$$

(the minus sign arises because  $e^{-ik_1 x}$  travels to the left).

- *Transmitted flux:* In  $x > 0$ , the transmitted wave  $C e^{ik_2 x}$  has flux

$$\Gamma_{\text{trans}} = \frac{\hbar k_2}{m} |C|^2.$$

Define the reflection probability  $R$  and transmission probability  $T$  by taking ratios of fluxes:

$$R = \frac{\Gamma_{\text{ref}}}{\Gamma_{\text{inc}}} = |B|^2, \quad T = \frac{\Gamma_{\text{trans}}}{\Gamma_{\text{inc}}} = \frac{k_2}{k_1} |C|^2.$$

Substitute  $B = \frac{k_1 - k_2}{k_1 + k_2}$  and  $C = \frac{2k_1}{k_1 + k_2}$ . Then

$$R = \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad T = \frac{k_2}{k_1} \left| \frac{2k_1}{k_1 + k_2} \right|^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2}.$$

Hence

$$T = \frac{4k_1 k_2}{(k_1 + k_2)^2}, \quad R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}.$$

One can readily check  $T + R = 1$ , as expected for a step potential of this form.

### 2.2.6 Q10. Quantum Harmonic Oscillator with a Small Perturbation

11/16 Marks

We have a quantum harmonic oscillator (frequency  $\omega$ ) with energy eigenstates  $\{|n\rangle\}$  and corresponding unperturbed eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

Also, the ladder operators are defined by

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2}}\left(\alpha \hat{x} \mp \frac{i}{\hbar \alpha} \hat{p}\right), \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}.$$

These satisfy  $\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$  and  $\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle$ .

**(a) Normalizing  $|\psi\rangle = |0\rangle + b|1\rangle$  and computing  $\langle H \rangle$ .**

5/5 Marks

First, normalize the state:

$$|\psi\rangle = |0\rangle + b|1\rangle.$$

The normalization condition is

$$\langle \psi | \psi \rangle = (1)(1) + b^*b = 1 + |b|^2 = 1,$$

hence

$$|b|^2 = 0 \implies b^2 = 0 \text{ (not typical) or typically we interpret } b \in \mathbb{R} \implies 1 + b^2 = 1$$

But presumably the problem implies  $b$  is the real coefficient; to ensure normalization, we must have

$$1 + b^2 = 1 \implies b^2 = 0 \implies b = 0.$$

This seems contradictory if the problem actually wants  $b$  to be an adjustable parameter. *Possibility:* Perhaps the problem statement means “*First, find the normalization condition if you want to allow an arbitrary  $b$ .*” In that case, the normalized state is

$$|\psi\rangle = \frac{1}{\sqrt{1+|b|^2}} (|0\rangle + b|1\rangle).$$

For simplicity, let us assume  $b$  is real. Then the normalized state is

$$|\psi\rangle = \frac{1}{\sqrt{1+b^2}} (|0\rangle + b|1\rangle).$$

Next, compute the expectation value of  $\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$ :

$$\langle H \rangle = \langle \psi | \hat{H} | \psi \rangle = \frac{1}{1+b^2} \left( \langle 0 | + b \langle 1 | \right) \hbar\omega \left( \hat{a}_- \hat{a}_+ + \frac{1}{2} \right) \left( |0\rangle + b|1\rangle \right).$$

But we know that  $|0\rangle$  has energy  $E_0 = \frac{1}{2}\hbar\omega$  and  $|1\rangle$  has  $E_1 = \frac{3}{2}\hbar\omega$ . Hence more directly,

$$\langle H \rangle = \frac{1}{1+b^2} \left( \langle 0 | H | 0 \rangle + b \langle 0 | H | 1 \rangle + b \langle 1 | H | 0 \rangle + b^2 \langle 1 | H | 1 \rangle \right).$$

But  $H|0\rangle = E_0|0\rangle$  and  $H|1\rangle = E_1|1\rangle$ , while  $\langle 0|1\rangle = 0$ . Thus cross terms vanish:

$$\langle H \rangle = \frac{1}{1+b^2} (E_0 + b^2 E_1) = \frac{1}{1+b^2} \left( \frac{1}{2} \hbar\omega + b^2 \frac{3}{2} \hbar\omega \right) = \frac{\hbar\omega}{1+b^2} \left( \frac{1}{2} + \frac{3}{2} b^2 \right).$$

**(b) Expressing  $H' = q \mathcal{E} \hat{x}$  in terms of  $\hat{a}_\pm$ .**

**2/2 Marks**

Recall

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-) = \frac{1}{\alpha\sqrt{2}} (\hat{a}_+ + \hat{a}_-),$$

where  $\alpha = \sqrt{\frac{m\omega}{\hbar}}$ . Hence

$$H' = q \mathcal{E} \hat{x} = q \mathcal{E} \frac{1}{\alpha\sqrt{2}} (\hat{a}_+ + \hat{a}_-) = \frac{q \mathcal{E}}{\sqrt{2} \alpha} (\hat{a}_+ + \hat{a}_-).$$

Define  $g = \sqrt{2} \alpha / q$  (or as given,  $g = \sqrt{2q\epsilon/\alpha}$  in the problem statement); the exact notation can vary, but the important part is we have

$$H' = (\text{constant}) \times (\hat{a}_+ + \hat{a}_-).$$

**(c) Variational Approach and Minimizing the Energy w.r.t.  $b$ .**

**4/9 Marks**



Using  $|\psi\rangle = \frac{1}{\sqrt{1+b^2}}(|0\rangle + b|1\rangle)$  as a variational trial state, one can calculate  $\langle H + H' \rangle$  as a function of  $b$  and set  $d/db = 0$ . The result (given in the statement) for the optimal  $b$  that minimizes the energy is

$$b = \frac{E_1 - E_0}{g} - \sqrt{\left(\frac{E_1 - E_0}{g}\right)^2 - 1} \quad (\text{or an equivalent algebraic form}).$$

Here  $E_0 = \frac{1}{2}\hbar\omega$  and  $E_1 = \frac{3}{2}\hbar\omega$ , so  $E_1 - E_0 = \hbar\omega$ . The dimensionless ratio  $\frac{E_1 - E_0}{g}$  controls whether  $b$  is large or small.

Physically, this says that a linear combination of the unperturbed ground and first excited states can better accommodate the linear potential shift  $q\mathcal{E}\hat{x}$ , thereby lowering the total energy below  $E_0$  and giving a first-order approximation to the *true* ground state in the presence of  $H'$ .

## 2.3 Classical Mechanics without prompts

### 2.3.1 Q2. The Damped Harmonic Oscillator

#### 6/6 Marks

The standard equation of motion for a damped harmonic oscillator can be written as

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

where  $k > 0$  is the “spring constant,” and  $b \geq 0$  is the damping coefficient. By comparing  $b^2$  to  $4k$ , we distinguish three damping cases:

#### 1) Overdamped: $b^2 > 4k$ .

- The system does not oscillate at all.
- The solution consists of two distinct *exponential decays* with different rates, and the mass slowly returns to equilibrium *without crossing it*.

#### 2) Critically Damped: $b^2 = 4k$ .

- This is the borderline case between no oscillations and underdamped motion.
- The system returns to equilibrium *in the shortest possible time* without overshooting (no oscillations).

#### 3) Underdamped: $b^2 < 4k$ .

- The system *oscillates* about the equilibrium position with an exponentially *decaying amplitude*.
- The motion resembles a sine wave whose amplitude decreases steadily over time.

**2.3.2 Q5: Two Masses, Three Springs (All with Constant  $k$ ), Each Mass  $m$**

**9/9 Marks**

**(a) Equations of motion for X and Y**

**2/2 Marks**

Label the displacements of masses  $X$  and  $Y$  from equilibrium by  $x_1(t)$  and  $x_2(t)$ , respectively (see figure). The forces on each mass come from the two springs attached to it:

- Mass  $X$  is pulled to the left by the wall-spring with force  $-k x_1$ . It is also pulled by the middle spring, which has extension  $(x_2 - x_1)$ . By Hooke's law, that contributes  $+k(x_2 - x_1)$  on mass  $X$ .
- Mass  $Y$  is pulled to the right by the right wall-spring with force  $-k x_2$ . From the middle spring, it experiences  $-k(x_2 - x_1)$ .

Hence, Newton's second law for each mass ( $m\ddot{x} = \text{net force}$ ) gives:

$$m \ddot{x}_1 = -k x_1 + k (x_2 - x_1) \implies m \ddot{x}_1 = -2k x_1 + k x_2,$$

$$m \ddot{x}_2 = -k x_2 - k (x_2 - x_1) \implies m \ddot{x}_2 = k x_1 - 2k x_2.$$

**(b) New coordinates  $y_1$  and  $y_2$**

**4/4 Marks**

Define

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2.$$

We now form  $\ddot{y}_1$  and  $\ddot{y}_2$  by summing and subtracting the two equations:

$$m(\ddot{x}_1 + \ddot{x}_2) = (-2k x_1 + k x_2) + (k x_1 - 2k x_2) = -k (x_1 + x_2),$$

which implies

$$m \ddot{y}_1 = -k y_1, \implies \ddot{y}_1 + \frac{k}{m} y_1 = 0.$$

Similarly,

$$m(\ddot{x}_1 - \ddot{x}_2) = (-2k x_1 + k x_2) - (k x_1 - 2k x_2) = 3k (x_1 - x_2),$$

so

$$m \ddot{y}_2 = 3k y_2, \implies \ddot{y}_2 + \frac{3k}{m} y_2 = 0.$$

Hence we have two *uncoupled* simple-harmonic-oscillator equations:

$$\ddot{y}_1 + \omega_1^2 y_1 = 0, \quad \text{with } \omega_1^2 = \frac{k}{m}, \quad \ddot{y}_2 + \omega_2^2 y_2 = 0, \quad \text{with } \omega_2^2 = \frac{3k}{m}.$$

**(c) Solution for  $y_1$  given initial conditions**

### 3/3 Marks

We assume (as stated) that initially  $X$  and  $Y$  are at their equilibrium positions ( $x_1(0) = 0$  and  $x_2(0) = 0$ ) but mass  $Y$  has some initial velocity  $v$ , while  $X$  is at rest. Then:

$$y_1(0) = x_1(0) + x_2(0) = 0, \quad \dot{y}_1(0) = \dot{x}_1(0) + \dot{x}_2(0) = 0 + v = v.$$

The equation for  $y_1$  is  $\ddot{y}_1 + \frac{k}{m} y_1 = 0$ . Its general solution is

$$y_1(t) = A \cos\left(\sqrt{\frac{k}{m}} t\right) + B \sin\left(\sqrt{\frac{k}{m}} t\right).$$

From  $y_1(0) = 0$ , we get  $A = 0$ . From  $\dot{y}_1(0) = v$ , we find

$$\dot{y}_1(t) = B \sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}} t\right),$$

thus  $\dot{y}_1(0) = B \sqrt{\frac{k}{m}} = v$  implies  $B = v \sqrt{\frac{m}{k}}$ .

Hence the solution is

$$y_1(t) = v \sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}} t\right).$$

### 2.3.3 Q7. Relativistic 4-Vectors and Two-Body Decay into Massless Particles

#### 10/13 Marks

#### (a) 4-Momentum of a Particle of Mass $m$

##### 3/4 Marks

In special relativity, the energy-momentum four-vector of a particle of rest mass  $m$  moving with velocity  $\mathbf{v}$  (in units where  $c = 1$  for simplicity) is

$$p^\mu = (E, \mathbf{p}) = (\gamma m, \gamma m \mathbf{v}), \quad \gamma = \frac{1}{\sqrt{1 - v^2}},$$

where  $\mathbf{v}$  is the 3-velocity and  $\mathbf{p} = \gamma m \mathbf{v}$  is the relativistic momentum. In units with  $c \neq 1$ , we would write  $(E/c, \mathbf{p}) = (\gamma mc, \gamma m \mathbf{v})$ .

*Invariance of the scalar product:* For any 4-vectors  $A^\mu$  and  $B^\mu$ , their Minkowski inner product

$$A^\mu B_\mu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$$

remains the same under Lorentz transformations. Specifically, for the 4-momentum of a single particle of mass  $m$ ,

$$p^\mu p_\mu = E^2 - \mathbf{p}^2 = m^2,$$

which is a Lorentz-invariant quantity (the squared rest mass).

**(b) Decay in the Rest Frame into Two Identical Massless Particles**  
**3/5 Marks**

Consider a particle at rest in its rest frame ( $p^\mu = (m, \mathbf{0})$  if  $c = 1$ ). It decays into two *identical* massless particles, each carrying energy  $\frac{m}{2}$  (since total energy  $m$  must be split equally in a symmetric decay) and moving in exactly opposite directions along the  $x$ -axis for momentum balance. In the rest frame of the parent:

$$\text{Particle A: } p_A^\mu = \left(\frac{m}{2}, +\frac{m}{2}, 0, 0\right), \quad \text{Particle B: } p_B^\mu = \left(\frac{m}{2}, -\frac{m}{2}, 0, 0\right),$$

where we have used the fact that a massless particle has  $|\mathbf{p}| = E$  in natural units.

*Boost to the Lab Frame:* In the lab frame, the parent particle moves with velocity  $v$  along  $+x$ . Denote the Lorentz factor by  $\gamma = 1/\sqrt{1-v^2}$ . The parent 4-momentum in the lab is

$$p_{\text{parent, lab}}^\mu = (\gamma m, \gamma m v, 0, 0).$$

The two daughter particles' 4-momenta in the lab can be obtained by applying the same boost transformation to  $p_A^\mu$  and  $p_B^\mu$  from above. (Their  $y, z$  components remain zero; only the  $t$  and  $x$  components change.)

**(c) Numerical Example with  $m = 10 \text{ GeV}/c^2$ ,  $\beta = v/c = 0.8$**   
**4/4 Marks**

Here  $m = 10 \text{ GeV}$  (rest energy),  $\beta = 0.8$ , and  $\gamma = 1/\sqrt{1-0.8^2} = 1/\sqrt{0.36} = 1/0.6 \approx 1.6667$ .

*Wavelengths in the rest frame:* Since each massless daughter has energy  $E = \frac{m c^2}{2} = 5 \text{ GeV}$ , a photon's energy is  $E = h\nu = hc/\lambda$ , hence

$$\lambda_{\text{rest}} = \frac{hc}{E} = \frac{(4.1357 \times 10^{-15} \text{ eV}\cdot\text{s})(3 \times 10^8 \text{ m/s})}{5 \times 10^9 \text{ eV}} \approx 2.48 \times 10^{-16} \text{ m}.$$

(This is a typical order of magnitude; one can keep more precise constants if needed.)

*Wavelengths in the lab frame:* In the lab, the photons no longer each have exactly 5 GeV, because one photon is emitted *forwards* (in the same direction as the parent's motion) and the other *backwards*. A standard calculation via relativistic Doppler shift shows that the forward-emitted photon has higher energy  $E_+$ , while the backward-emitted photon has lower energy  $E_-$ . Specifically,

$$E_\pm = \gamma \left( \frac{m}{2} \pm \beta \frac{m}{2} \right) = \frac{m}{2} \gamma (1 \pm \beta).$$

Hence

$$E_+ = \frac{m}{2} \gamma (1 + 0.8), \quad E_- = \frac{m}{2} \gamma (1 - 0.8).$$

Numerically,

$$E_+ = 5 \text{ GeV} \times \gamma(1 + 0.8) = 5 \times 1.6667 \times 1.8 \approx 15 \text{ GeV},$$

$$E_- = 5 \text{ GeV} \times \gamma(1 - 0.8) = 5 \times 1.6667 \times 0.2 \approx 1.67 \text{ GeV}.$$

The corresponding *wavelengths* in the lab frame are

$$\lambda_+ = \frac{hc}{E_+} \approx \frac{1.24 \times 10^{-6} \text{ eV}\cdot\text{m}}{15 \times 10^9 \text{ eV}} \approx 8.3 \times 10^{-17} \text{ m},$$

$$\lambda_- = \frac{hc}{E_-} \approx \frac{1.24 \times 10^{-6} \text{ eV}\cdot\text{m}}{1.67 \times 10^9 \text{ eV}} \approx 7.4 \times 10^{-16} \text{ m}.$$

Thus one photon is “blue-shifted” ( $E_+$  is bigger,  $\lambda_+$  is smaller), and the other is “red-shifted” in the lab frame, reflecting the boost from the parent’s rest frame.

### 2.3.4 Q8. Pendulum with a Moving Support

#### 7/10 Marks

A pendulum of length  $L$  and bob of mass  $m$  hangs from a support that moves horizontally according to

$$x_{\text{support}}(t) = vt^3 + A \sin(\omega t).$$

Let  $\phi(t)$  be the angle the pendulum makes from the vertical (see figure). We wish to derive the pendulum’s equation of motion from the Lagrangian.

#### (a) Deriving the Equation of Motion

#### 6/9 Marks

Coordinates for the Bob:

$$X(t) = x_{\text{support}}(t) + \ell \sin \phi, \quad Y(t) = -\ell \cos \phi,$$

where we choose the origin of  $Y$  at the support pivot (with downward  $Y < 0$ ). Hence the velocity components are

$$\dot{X} = \dot{x}_{\text{support}} + \ell \cos \phi \dot{\phi}, \quad \dot{Y} = -\ell \sin \phi \dot{\phi}.$$

The kinetic energy  $T = \frac{1}{2} m(\dot{X}^2 + \dot{Y}^2)$  becomes

$$T = \frac{1}{2} m \left[ (\dot{x}_{\text{support}} + \ell \cos \phi \dot{\phi})^2 + \ell^2 \sin^2 \phi \dot{\phi}^2 \right].$$

Meanwhile, the potential energy (with  $y = 0$  at pivot,  $y < 0$  downward) is

$$V = m g \ell \cos \phi \quad (\text{up to an additive constant}).$$

Thus the Lagrangian  $L = T - V$  becomes

$$\mathcal{L}(\phi, \dot{\phi}, t) = \frac{1}{2} m \left[ (\dot{x}_{\text{support}} + \ell \cos \phi \dot{\phi})^2 + \ell^2 \sin^2 \phi \dot{\phi}^2 \right] - m g \ell \cos \phi.$$

Simplify the kinetic term:

$$\dot{X}^2 + \dot{Y}^2 = \dot{x}_{\text{support}}^2 + 2 \dot{x}_{\text{support}} \ell \cos \phi \dot{\phi} + \ell^2 \cos^2 \phi \dot{\phi}^2 + \ell^2 \sin^2 \phi \dot{\phi}^2 = \dot{x}_{\text{support}}^2 + 2 \ell \dot{x}_{\text{support}} \cos \phi \dot{\phi} + \ell^2 \dot{\phi}^2.$$

Hence

$$T = \frac{1}{2} m [\dot{x}_{\text{support}}^2 + 2 \ell \dot{x}_{\text{support}} \cos \phi \dot{\phi} + \ell^2 \dot{\phi}^2].$$

So

$$\mathcal{L} = \frac{1}{2} m \dot{x}_{\text{support}}^2 + m \ell \dot{x}_{\text{support}} \cos \phi \dot{\phi} + \frac{1}{2} m \ell^2 \dot{\phi}^2 - m g \ell \cos \phi.$$

(Note that the first term  $\frac{1}{2} m \dot{x}_{\text{support}}^2$  depends only on  $t$ , so it does not affect the equation of motion for  $\phi$ .)

Euler–Lagrange equation:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

We compute:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m \ell \dot{x}_{\text{support}} \cos \phi + m \ell^2 \dot{\phi}, \quad \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m \ell (\ddot{x}_{\text{support}} \cos \phi - \dot{x}_{\text{support}} \sin \phi \dot{\phi}) + m \ell^2 \ddot{\phi}.$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = m \ell \dot{x}_{\text{support}} (-\sin \phi \dot{\phi}) - m g \ell (-\sin \phi) = -m \ell \dot{x}_{\text{support}} \sin \phi \dot{\phi} + m g \ell \sin \phi.$$

Hence the equation of motion

$$m \ell \ddot{x}_{\text{support}} \cos \phi - m \ell \dot{x}_{\text{support}} \sin \phi \dot{\phi} + m \ell^2 \ddot{\phi} - [-m \ell \dot{x}_{\text{support}} \sin \phi \dot{\phi} + m g \ell \sin \phi] = 0.$$

Notice the terms with  $\dot{x}_{\text{support}} \sin \phi \dot{\phi}$  cancel. Divide throughout by  $m \ell$ :

$$\ddot{x}_{\text{support}} \cos \phi + \ell \ddot{\phi} - g \sin \phi = 0.$$

We arrive at the pendulum equation:

$$\boxed{\ell \ddot{\phi} + g \sin \phi = -\ddot{x}_{\text{support}} \cos \phi.}$$

**(b) Limiting Case:**  $v \rightarrow 0$ ,  $A \rightarrow 0$ ,  $\phi \rightarrow 0$ . **1/1 Marks**

In this limit,  $\ddot{x}_{\text{support}} \rightarrow 0$  (since  $v t^3$  or  $A \sin \omega t$  is negligible, or  $\omega$  small, etc.) and  $\sin \phi \approx \phi$ ,  $\cos \phi \approx 1$ . Hence the equation of motion reduces to

$$\ell \ddot{\phi} + g \phi = 0 \implies \ddot{\phi} + \frac{g}{\ell} \phi = 0,$$

which is precisely the *simple pendulum* small-angle equation with natural frequency  $\sqrt{g/\ell}$ .

**2.3.5 Q9. Block of Mass  $m$  Falling While Winding a Rope on a Uniform Disk of Mass  $M$  and Radius  $R$**

**7/7 Marks**

**(a) Deriving  $\alpha$ ,  $T$ , and  $a$  6/6 Marks**

*Setup and Assumptions:*

- The disk of mass  $M$  and radius  $R$  is free to rotate about its center (with negligible axle friction).
- A light rope, which does not slip on the disk, is wrapped around the rim.
- The other end of the rope is attached to a block of mass  $m$ . Under gravity, the block accelerates downward with acceleration  $a$ , and the disk acquires an angular acceleration  $\alpha$ .
- The moment of inertia of a solid disk about its center is  $I = \frac{1}{2} M R^2$ .

**Force on the Block:**

$$\sum F_y = m a \implies m g - T = m a. \quad (1)$$

**Torque on the Disk:**

$$\sum \tau = I \alpha,$$

where the tension  $T$  exerts a torque about the disk's center of magnitude  $T R$ . Thus

$$T R = \left( \frac{1}{2} M R^2 \right) \alpha.$$

Because the rope does not slip, the tangential acceleration at the rim is  $a = \alpha R$ . Hence

$$\alpha = \frac{a}{R} \implies T R = \frac{1}{2} M R^2 \left( \frac{a}{R} \right) \implies T = \frac{1}{2} M a. \quad (2)$$

**Solve for  $a$ ,  $T$ , and  $\alpha$ :** Substitute (??) into (??):

$$m g - \left( \frac{1}{2} M a \right) = m a \implies m g = m a + \frac{1}{2} M a = \left( m + \frac{1}{2} M \right) a.$$

Hence

$$a = \frac{m g}{m + \frac{1}{2} M}.$$

From (2),  $T = \frac{1}{2} M a$ , so

$$T = \frac{\frac{1}{2} M (m g)}{m + \frac{1}{2} M} = \frac{\frac{1}{2} M m g}{m + \frac{1}{2} M}.$$

Finally,  $\alpha = \frac{a}{R}$ :

$$\alpha = \frac{1}{R} \frac{m g}{m + \frac{1}{2} M}.$$

**(b) Numerical Values for  $m = 3$  kg,  $M = 12$  kg,  $R = 0.2$  m 1/1 Marks**

Use  $g \approx 9.8$  m/s<sup>2</sup>.

$$a = \frac{3 \times 9.8}{3 + \frac{1}{2} \times 12} = \frac{29.4}{3 + 6} = \frac{29.4}{9} \approx 3.27 \text{ m/s}^2.$$

$$\alpha = \frac{a}{R} \approx \frac{3.27}{0.2} \approx 16.35 \text{ rad/s}^2.$$

$$T = \frac{1}{2} M a = 6 \times 3.27 = 19.62 \text{ N} \approx 19.6 \text{ N}.$$

Hence the block accelerates downward at about 3.27 m/s<sup>2</sup>, the disk's angular acceleration is about 16.35 rad/s<sup>2</sup>, and the rope tension is about 19.6 N.

### 2.3.6 Q10. Ball Dropped from an Airplane with Drag Force

$$F = -b \mathbf{v}$$

**6/8 Marks**

**(a) Equation of Motion and General Solution**

**4/6 Marks**

A ball of mass  $m$  is subject to gravity  $-mg \hat{\mathbf{k}}$  (downward) and an air-resistance (drag) force  $-b \mathbf{v}$  directed opposite to its velocity  $\mathbf{v}$ . Hence Newton's law,

$$m \frac{d\mathbf{v}}{dt} = -mg \hat{\mathbf{k}} - b \mathbf{v}.$$

Rearrange:

$$\frac{d\mathbf{v}}{dt} = -g \hat{\mathbf{k}} - \frac{b}{m} \mathbf{v}.$$

We can solve this *vector* differential equation by noting that each Cartesian component satisfies a first-order linear ODE of the form

$$\frac{d\mathbf{v}}{dt} + \frac{b}{m} \mathbf{v} = -g \hat{\mathbf{k}}.$$

One standard method (or by known results) yields the solution:

$$\mathbf{v}(t) = \mathbf{A} e^{-\frac{b}{m}t} - \frac{mg}{b} \hat{\mathbf{k}},$$

where  $\mathbf{A}$  is a constant vector determined by initial conditions.

**(b) Determining  $\mathbf{A}$  from the Initial Velocity**

**2/2 Marks**

Suppose the ball is released at  $t = 0$  with *initial velocity*  $\mathbf{v}(0) = \mathbf{u}$ . Then

$$\mathbf{v}(0) = \mathbf{A} e^{-0} - \frac{mg}{b} \hat{\mathbf{k}} = \mathbf{A} - \frac{mg}{b} \hat{\mathbf{k}} = \mathbf{u}.$$

Hence

$$\mathbf{A} = \mathbf{u} + \frac{mg}{b} \hat{\mathbf{k}}.$$



Putting this back into the general solution,

$$\mathbf{v}(t) = \left( \mathbf{u} + \frac{m g}{b} \hat{\mathbf{k}} \right) e^{-\frac{b}{m} t} - \frac{m g}{b} \hat{\mathbf{k}}.$$

## 2.4 Classical Electromagnetic Theory with context prompts

### 2.4.1 Question 2

4/5 Marks

**Solution:**

#### 1. Definition of Free Space

In classical electromagnetism, *free space* refers to a perfect vacuum with no charges ( $\rho = 0$ ) and no currents ( $\mathbf{J} = 0$ ). Consequently, there is no medium present that can absorb or scatter electromagnetic waves.

#### 2. Wave Equation for the Electric Field

In free space, each Cartesian component of the electric field  $\mathbf{E}$  satisfies the partial differential equation:

$$\nabla^2 E = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}.$$

This is the *wave equation*, showing that  $\mathbf{E}$  can propagate through space as a wave.

#### 3. Physical Interpretation

This wave equation reveals that any fluctuation or perturbation in the electric field travels through space without attenuation (as there is no medium to impede it). Hence, electromagnetic waves—comprising both electric and magnetic fields—propagate outward from their source.

#### 4. Speed of Light

From this equation, the characteristic wave speed  $c$  in free space is:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}},$$

where  $\epsilon_0$  is the permittivity of free space and  $\mu_0$  is the permeability of free space. Numerically, this is approximately

$$3 \times 10^8 \text{ m/s},$$

the well-known speed of light in a vacuum.

### 2.4.2 Question 4

10/11 Marks

**Solution:**

#### 1. Flux of $\mathbf{B}$ Through the Loop 5/5 Marks

A very long straight wire carries a current  $I$  directed downward. By the right-hand rule, the magnetic field  $\mathbf{B}$  circles around the wire. At a distance  $x$  from the wire, the magnitude of  $\mathbf{B}$  is

$$B(x) = \frac{\mu_0 I}{2\pi x}.$$

Suppose the loop is a square of side  $L$ , its left edge at distance  $d$  from the wire and its right edge at  $d + L$ . We assume the magnetic field is *into* the page throughout the region of the loop. Then the flux is

$$\Phi_B = \int_{\text{Area}} \mathbf{B} \cdot d\mathbf{A} = \int_{x=d}^{d+L} \left( \int_{y=0}^L B(x) dy \right) dx.$$

Because  $B(x)$  does not depend on  $y$ , the inner integral simply gives a factor of  $L$ :

$$\Phi_B = \int_{x=d}^{d+L} \frac{\mu_0 I}{2\pi x} (L) dx = \frac{\mu_0 I L}{2\pi} \int_d^{d+L} \frac{dx}{x}.$$

Evaluating this logarithmic integral,

$$\Phi_B = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{d+L}{d}\right).$$

#### 2. Loop Moved Directly Away from the Wire (to the Right) at Speed $v$ 4/5 Marks

Let  $d(t)$  be the time-dependent distance from the wire to the left edge of the loop. Then the flux is

$$\Phi_B(t) = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{d(t)+L}{d(t)}\right).$$

The induced electromotive force (emf) follows from Faraday's law:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

With  $d(t) = d_0 + vt$ , we compute

$$\frac{d}{dt} \ln\left(\frac{d(t)+L}{d(t)}\right) = \frac{v}{d(t)+L} - \frac{v}{d(t)} = v \left[ \frac{1}{d(t)+L} - \frac{1}{d(t)} \right].$$

Hence,

$$\mathcal{E} = -\frac{\mu_0 I L}{2\pi} \cdot v \left[ \frac{1}{d(t) + L} - \frac{1}{d(t)} \right].$$

*Direction of the Induced Current.* As the loop moves away, the flux due to the wire's field (*into* the page) *decreases*. By Lenz's law, the induced current will produce its own magnetic field *into* the page to oppose that decrease. A current flowing *clockwise* around the loop (viewed from above) creates a magnetic field into the page, so the induced current is **clockwise**.

### 3. Loop Moved Only Downwards at Speed $v$ 1/1 Marks

If the loop is simply shifted *parallel* to the wire (downwards), the distance  $d$  from the wire does not change, so the magnetic flux through the loop remains constant:

$$\Phi_B = \text{constant}.$$

With no change in flux, Faraday's law tells us the induced emf is zero. *No current* is generated in the loop if it only moves downward.

### 2.4.3 Question 5

9/11 Marks

#### Solution:

*Assumptions/Interpretation of the Circuit:*

- The **12 V** battery (on the left) is in series with a  $2\Omega$  resistor at the top.
- After that top resistor, the circuit node splits into two parallel branches:
  1.  $1\Omega + 1\Omega$  in series (total  $2\Omega$ ) from top node to bottom node.
  2. A branch containing a **2 V** battery (oriented so its top terminal is at  $+2\text{V}$ ) in series with a  $2\Omega$  resistor and the ammeter, all going from the same top node down to the common bottom (the negative terminal of the 12 V source).
- We define the bottom node (the negative terminal of the 12 V battery) as 0 V. Let  $V_+$  be the voltage at the top node (the junction after the  $2\Omega$  resistor from the 12 V supply).

#### (a) Current Seen by the Ammeter 6/8 Marks

Denote:

$i_1$  = current through the two  $1\Omega$  resistors in series,  $i_2$  = current through the  $2\Omega + (2\text{V battery})$  branch.

The total current from the 12 V source, which flows through the top  $2\Omega$  resistor, is

$$i_{\text{top}} = i_1 + i_2.$$

Hence the top node voltage is

$$V_+ = 12 - 2i_{\text{top}}.$$

Next, ohm's law in each branch gives:

$$i_1 = \frac{V_+}{1\Omega + 1\Omega} = \frac{V_+}{2}, \quad i_2 = \frac{V_+ - 2}{2\Omega} \quad (\text{since the } 2\text{ V battery enforces } +2\text{ V at its top terminal}).$$

Thus

$$i_{\text{top}} = i_1 + i_2 = \frac{V_+}{2} + \frac{V_+ - 2}{2} = \frac{2V_+ - 2}{2} = V_+ - 1.$$

We combine this with  $V_+ = 12 - 2(i_1 + i_2)$ , giving

$$V_+ = 12 - 2(V_+ - 1) = 12 - 2V_+ + 2 = 14 - 2V_+.$$

Rearrange:

$$3V_+ = 14 \implies V_+ = \frac{14}{3}\text{ V} \approx 4.67\text{ V}.$$

It follows that

$$i_1 = \frac{V_+}{2} = \frac{14/3}{2} = \frac{7}{3}\text{ A} \approx 2.33\text{ A}, \quad i_2 = \frac{V_+ - 2}{2} = \frac{(14/3) - 2}{2} = \frac{8/3}{2} = \frac{4}{3}\text{ A} \approx 1.33\text{ A}.$$

Because the ammeter is in the branch with the  $2\Omega$  resistor and the  $2\text{ V}$  battery, the reading on the ammeter is

$$i_2 = \frac{4}{3}\text{ A} \approx 1.33\text{ A}.$$

**(b) Energy Delivered by the 12 V Battery in 4 s 3/3 Marks**

The total current supplied by the  $12\text{ V}$  battery is

$$i_{\text{top}} = i_1 + i_2 = \frac{7}{3} + \frac{4}{3} = \frac{11}{3}\text{ A} \approx 3.67\text{ A}.$$

The *power* delivered by the  $12\text{ V}$  source is

$$P_{12\text{ V}} = (12\text{ V}) (i_{\text{top}}) = 12 \times \frac{11}{3} = 44\text{ W}.$$

Over a time interval of  $4\text{ s}$ , the energy supplied is

$$E = P_{12\text{ V}} \times 4\text{ s} = 44\text{ W} \times 4\text{ s} = 176\text{ J}.$$

Thus the **12 V battery** delivers

$$176\text{ J in } 4\text{ s}.$$

### 2.4.4 Question 7

5/6 Marks

**Solution:**

#### 1. Resonant Frequency $\omega_0$

For a series RLC circuit with inductance  $L$  and capacitance  $C$ , the (angular) resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Here,

$$L = 20 \text{ mH} = 20 \times 10^{-3} \text{ H}, \quad C = 1500 \text{ pF} = 1500 \times 10^{-12} \text{ F}.$$

Thus

$$\omega_0 = \frac{1}{\sqrt{(20 \times 10^{-3})(1500 \times 10^{-12})}} = \frac{1}{\sqrt{3 \times 10^{-11}}} \approx 1.8 \times 10^5 \text{ rad/s}.$$

#### 2. Voltage Across the Inductor

The driving source is

$$\mathcal{E}(t) = E_0 \cos(\omega t).$$

In steady-state AC analysis, the circuit current can be written (in the time domain) as

$$I(t) = I_0 \cos(\omega t - \phi), \quad \text{where} \quad I_0 = \frac{E_0}{Z},$$

and  $Z$  is the *total impedance* of the RLC circuit. The phase angle  $\phi$  depends on whether the net reactance is inductive or capacitive.

The inductor has inductive reactance

$$X_L = \omega L.$$

In phasor form, the inductor voltage amplitude is  $I_0 X_L$ , and it *leads* the current by  $90^\circ$ . Hence, in the time domain the inductor's voltage is

$$\Delta V_L(t) = (I_0 X_L) \cos\left(\omega t - \phi + \frac{\pi}{2}\right).$$

Substituting  $I_0 = E_0/Z$  gives the amplitude of the inductor voltage as

$$\Delta V_{L,\max} = \frac{E_0 X_L}{Z}.$$

Thus,

$$\Delta V_L(t) = \frac{E_0 X_L}{Z} \cos\left(\omega t - \phi + \frac{\pi}{2}\right).$$

### 3. Phase Relationship at Low and High Frequencies

Recall that  $\phi$  is the *overall* phase by which the current lags (or leads) the source voltage. The voltage across the inductor leads the circuit current by  $90^\circ$ , so the inductor voltage leads (or lags) the source voltage by  $(-\phi + 90^\circ)$  in total. The sign of  $\phi$  depends on whether the circuit is behaving more inductively (positive  $\phi$ ) or more capacitively (negative  $\phi$ ).

- **Low frequency** ( $\omega \ll \omega_0$ ): The capacitor dominates, so the circuit is *capacitive* and the current *leads* the source voltage (i.e.  $\phi$  is negative). Since the inductor voltage is always  $90^\circ$  ahead of the current, it ends up leading the *source* by *more* than  $90^\circ$ .
- **High frequency** ( $\omega \gg \omega_0$ ): The inductor dominates, so the circuit is *inductive* and the current *lags* the source voltage (i.e.  $\phi$  is positive). In this regime, the inductor voltage leads the current by  $90^\circ$ , so relative to the *source* voltage, it leads by *less* than  $90^\circ$  (and can even approach being nearly in phase for very large  $\omega$ ).

#### 2.4.5 Question 9

12/15 Marks

**Solution:**

##### 1. Relationship Between the Electric and Magnetic Fields 7/7 Marks

A transverse electromagnetic wave in vacuum can be written (in complex notation) as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)],$$

with wavevector  $\mathbf{k}$ , angular frequency  $\omega$ , and amplitude  $\mathbf{E}_0$ . From Maxwell's equations in free space, specifically Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

one finds that the magnetic field has amplitude

$$B_0 = \frac{E_0}{c}, \quad \text{where } c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

In terms of the *magnetic intensity*  $\mathbf{H} = \mathbf{B}/\mu_0$ , one obtains

$$H_0 = \frac{B_0}{\mu_0} = \frac{E_0}{\mu_0 c} = \sqrt{\frac{\epsilon_0}{\mu_0}} E_0,$$

because  $\mu_0 c = 1/\sqrt{\epsilon_0/\mu_0}$ . Hence in the time domain,

$$\mathbf{H}(\mathbf{r}, t) = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{E}(\mathbf{r}, t).$$

This shows that in free space,  $|\mathbf{H}| = \sqrt{\frac{\epsilon_0}{\mu_0}} |\mathbf{E}|$ .

## 2. Time-Averaged Poynting Flux 4/6 Marks

The instantaneous Poynting vector is

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t).$$

Using  $\mathbf{H} = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{E}$ , and the fact that in a plane wave  $\mathbf{E}$  is perpendicular to  $\mathbf{H}$  (both transverse to  $\mathbf{k}$ ), one can show

$$\mathbf{S} = \sqrt{\frac{\epsilon_0}{\mu_0}} |\mathbf{E}|^2 \hat{\mathbf{k}},$$

where  $\hat{\mathbf{k}}$  is the direction of propagation. Because  $\mathbf{E}$  is oscillatory in time, the *time average* over one full cycle yields

$$\langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}.$$

Equivalently, expressing in terms of the magnetic amplitude  $H_0 = \sqrt{\frac{\epsilon_0}{\mu_0}} E_0$ , one finds

$$\langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} (H_0)^2 \hat{\mathbf{k}}.$$

Often the notation  $\langle \mathbf{N} \rangle$  is used instead of  $\langle \mathbf{S} \rangle$ , so

$$\langle \mathbf{N} \rangle = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} (H_0)^2 \hat{\mathbf{k}}.$$

## 3. Radiation Pressure on a Perfect Reflector at Angle $\alpha$ 1/2 Marks

For normal incidence ( $\alpha = 0$ ) on a perfect mirror, the momentum transfer is doubled relative to absorption, giving a radiation pressure

$$P = \frac{2I}{c},$$

where  $I = |\langle \mathbf{S} \rangle|$  is the wave intensity.

When the wave strikes the mirror at an angle  $\alpha$  (with respect to the normal to the surface), the component of momentum normal to the surface is reduced by a factor  $\cos(\alpha)$ . At reflection, the *normal* component of momentum reverses, so the change in the normal momentum component is *twice* that normal component:

$$\Delta p_{\text{normal}} \propto 2 \left( \frac{I}{c} \right) \cos(\alpha).$$

Hence the pressure (force per area normal to the surface) becomes

$$P(\alpha) = \frac{2I}{c} \cos^2(\alpha).$$

In short, the incident momentum flux along the surface normal is  $I/c \cos(\alpha)$ , and upon reflection it reverses sign, yielding a factor of 2, so

$$P(\alpha) = \frac{2I}{c} \cos^2(\alpha).$$

This completes the derivation that the radiation pressure on a perfect planar reflector at incidence angle  $\alpha$  is  $2I \cos^2(\alpha)/c$ .

#### 2.4.6 Question 10

7/7 Marks

**Solution:**

##### 1. Invariance of the Scalar Product $\mathbf{E} \cdot \mathbf{B}$

4/4 Marks

We have two inertial frames:  $S$  (unprimed) and  $S'$  (primed). Frame  $S'$  moves at speed  $v$  along the  $x$ -axis of  $S$ . The given field transformations are (using units with  $\mu_0 \epsilon_0 = 1/c^2$ ):

$$\begin{aligned} E'_x &= E_x, & E'_y &= \gamma(E_y - v B_z), & E'_z &= \gamma(E_z + v B_y), \\ B'_x &= B_x, & B'_y &= \gamma\left(B_y + \frac{v}{c^2} E_z\right), & B'_z &= \gamma\left(B_z - \frac{v}{c^2} E_y\right), \end{aligned}$$

where  $\gamma = 1/\sqrt{1 - v^2/c^2}$ . We want to show  $\mathbf{E} \cdot \mathbf{B} = \mathbf{E}' \cdot \mathbf{B}'$ .

In  $S$ ,

$$\mathbf{E} \cdot \mathbf{B} = E_x B_x + E_y B_y + E_z B_z.$$

In  $S'$ ,

$$\mathbf{E}' \cdot \mathbf{B}' = E'_x B'_x + E'_y B'_y + E'_z B'_z.$$

Substituting the above transformations (and carefully factoring out  $\gamma$ ), one finds that all extra cross-terms involving  $v$  cancel precisely, leaving

$$E'_x B'_x + E'_y B'_y + E'_z B'_z = E_x B_x + E_y B_y + E_z B_z.$$

Therefore,

$$\boxed{\mathbf{E} \cdot \mathbf{B} = \mathbf{E}' \cdot \mathbf{B}'},$$

demonstrating that the scalar product  $\mathbf{E} \cdot \mathbf{B}$  is a Lorentz-invariant quantity.

##### 2. Lorentz Transformation of the Wave Phase

2/2 Marks

A linearly polarized plane electromagnetic wave in frame  $S$  is given by

$$\mathbf{E}(x, t) = E_0 \hat{\mathbf{y}} \exp[i(kx - \omega t)].$$



The phase in  $S$  is  $\phi = kx - \omega t$ .

Let  $(x, t)$  and  $(x', t')$  be coordinates in the two frames. The Lorentz transformation from  $S$  to  $S'$  (moving at speed  $v$  in the  $+x$ -direction) is

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right).$$

Substituting  $x$  and  $t$  in terms of  $x'$  and  $t'$  into  $\phi = kx - \omega t$  gives

$$\phi = k[\gamma(x' + vt')] - \omega[\gamma(t' + \frac{vx'}{c^2})].$$

Group terms in  $x'$  and  $t'$ :

$$\phi = \gamma(k - \frac{v\omega}{c^2})x' - \gamma(\omega - kv)t'.$$

Define the primed wave number and angular frequency:

$$k' = \gamma(k - \frac{\omega v}{c^2}), \quad \omega' = \gamma(\omega - kv).$$

Thus the phase in  $S'$  is

$$\phi' = k'x' - \omega't',$$

which has exactly the same form in primed coordinates as  $\phi$  had in unprimed coordinates. In other words, the wave's phase is also a Lorentz invariant:

$$\phi = kx - \omega t \Leftrightarrow \phi' = k'x' - \omega't'.$$

### 3. Low-Velocity Limit and Physical Interpretation 1/1 Marks

Examine the expressions for  $\omega'$  and  $k'$  when  $v \ll c$ . Keeping only first order in  $v/c$ ,

$$\omega' \approx \omega - kv, \quad k' \approx k - \frac{\omega v}{c^2}.$$

Because for an electromagnetic wave in vacuum we have  $\omega = ck$ , the second term is  $\frac{\omega v}{c^2} = \frac{ckv}{c^2} = \frac{kv}{c} \ll k$ . Consequently  $\omega' \approx \omega$  and  $k' \approx k$  to first order in  $v/c$ . Physically, *the frequency and wavenumber change very little if the relative motion is non-relativistic.*

In essence, this limit recovers the **classical (Galilean) viewpoint**, where slow motion of one observer relative to the wave's medium (vacuum, in this idealized case) introduces negligible Doppler shift. For electromagnetic waves at large relative speeds, however, the full Lorentz transformation must be used, revealing the relativistic Doppler effect.

## 2.5 Quantum Mechanics with context prompts

### 2.5.1 Question 1

5/5 Marks

**Solution:**

#### 1. Commutator of Two Operators and Compatibility

For two quantum mechanical operators  $\hat{P}$  and  $\hat{Q}$ , their *commutator* is defined as

$$[\hat{P}, \hat{Q}] = \hat{P}\hat{Q} - \hat{Q}\hat{P}.$$

We say that two operators  $\hat{P}$  and  $\hat{Q}$  are *compatible* (or *commuting*) if they can be simultaneously diagonalized, meaning there exists a common set of eigenstates for both. Equivalently, in the language of commutators:

$$[\hat{P}, \hat{Q}] = 0 \iff \text{operators are compatible.}$$

In physical terms, if  $\hat{P}$  and  $\hat{Q}$  are compatible, one can measure them both *with arbitrary precision* in any order, and the act of measuring one does not disturb the other's measurement outcome.

#### 2. Non-Compatibility and Measurement Sequence

Suppose that  $\hat{P}$  and  $\hat{Q}$  do *not* commute ( $[\hat{P}, \hat{Q}] \neq 0$ ) and thus are not compatible. We have the following measurement scenario:

- (a) First, we measure  $\hat{P}$  and obtain some result  $p$ .
- (b) Next, we measure  $\hat{Q}$  and obtain some result  $q$ .
- (c) Then, we measure  $\hat{P}$  *again*.

Because the act of measuring  $\hat{Q}$  *disturbs* the system's state in a way that does not generally preserve the eigenstate of  $\hat{P}$  (since  $\hat{P}$  and  $\hat{Q}$  do not share eigenstates), the second measurement of  $\hat{P}$  need *not* yield the original value  $p$  again. In other words, the possible results of the second  $\hat{P}$  measurement may differ from  $p$  because the state has been *projected* (or changed) by the  $\hat{Q}$  measurement into an eigenstate of  $\hat{Q}$ , which is typically not an eigenstate of  $\hat{P}$ .

Thus, for non-commuting operators, an initial measurement of  $\hat{P}$  yielding  $p$  does not *guarantee* that a subsequent re-measurement of  $\hat{P}$  (after measuring  $\hat{Q}$ ) will yield the same value  $p$ . The second  $\hat{P}$  measurement can produce a different result in general, reflecting the fundamental measurement-disturbance principle for non-commuting observables in quantum mechanics.

### 2.5.2 Question 3

9/9 Marks

#### Solution:

We say that an operator  $\hat{Q}$  is *Hermitian* if and only if, for all well-behaved functions  $f(x)$ ,  $g(x)$  that vanish at infinity,

$$\int_{-\infty}^{\infty} [f(x)]^* \hat{Q} g(x) dx = \int_{-\infty}^{\infty} g(x) [\hat{Q} f(x)]^* dx.$$

#### 1. Whether $\hat{p} = -i\hbar \frac{d}{dx}$ is Hermitian 4/4 Marks

We check the integral condition. Let us compute

$$\int_{-\infty}^{\infty} f^*(x) \hat{p} g(x) dx = \int_{-\infty}^{\infty} f^*(x) \left[ -i\hbar \frac{d}{dx} g(x) \right] dx.$$

Integrate by parts:

$$\int_{-\infty}^{\infty} f^*(x) \left( -i\hbar \right) \frac{d}{dx} g(x) dx = \left[ -i\hbar f^*(x) g(x) \right]_{-\infty}^{+\infty} + i\hbar \int_{-\infty}^{\infty} \frac{d}{dx} f^*(x) g(x) dx.$$

Provided  $f(x)$  and  $g(x)$  (and their derivatives) vanish at infinity, the boundary term is zero. Thus

$$\int_{-\infty}^{\infty} f^*(x) \hat{p} g(x) dx = i\hbar \int_{-\infty}^{\infty} \left[ \frac{d}{dx} f^*(x) \right] g(x) dx.$$

Taking the complex conjugate inside,

$$[\hat{p} f(x)]^* = \left[ -i\hbar \frac{d}{dx} f(x) \right]^* = i\hbar \frac{d}{dx} f^*(x).$$

Hence,

$$\int_{-\infty}^{\infty} g(x) [\hat{p} f(x)]^* dx = \int_{-\infty}^{\infty} g(x) \left[ i\hbar \frac{d}{dx} f^*(x) \right] dx = i\hbar \int_{-\infty}^{\infty} \frac{d}{dx} f^*(x) g(x) dx,$$

which matches the expression we found for  $\int f^*(x) \hat{p} g(x) dx$ . Therefore,

$$\int_{-\infty}^{\infty} f^*(x) \hat{p} g(x) dx = \int_{-\infty}^{\infty} g(x) [\hat{p} f(x)]^* dx,$$

and  $\hat{p} = -i\hbar \frac{d}{dx}$  is indeed a *Hermitian* operator (assuming sufficiently well-behaved wavefunctions vanishing at infinity).

#### 2. Whether $\hat{Q} = \frac{d^2}{dx^2}$ is Hermitian 5/5 Marks

Consider

$$\int_{-\infty}^{\infty} f^*(x) \hat{Q} g(x) dx = \int_{-\infty}^{\infty} f^*(x) \frac{d^2}{dx^2} g(x) dx.$$

Perform integration by parts twice:

$$\int_{-\infty}^{\infty} f^*(x) \frac{d^2}{dx^2} g(x) dx = \left[ f^*(x) \frac{d}{dx} g(x) \right]_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} f^*(x) \frac{d}{dx} g(x) dx.$$

Integrate by parts again on the remaining term:

$$- \int_{-\infty}^{\infty} \frac{d}{dx} f^*(x) \frac{d}{dx} g(x) dx = - \left[ \frac{d}{dx} f^*(x) g(x) \right]_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} \frac{d^2}{dx^2} f^*(x) g(x) dx.$$

Hence,

$$\int_{-\infty}^{\infty} f^*(x) \frac{d^2}{dx^2} g(x) dx = \left[ f^*(x) \frac{d}{dx} g(x) - \frac{d}{dx} f^*(x) g(x) \right]_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} g(x) \frac{d^2}{dx^2} f^*(x) dx.$$

So,

$$\int_{-\infty}^{\infty} f^*(x) \hat{Q} g(x) dx = \int_{-\infty}^{\infty} g(x) [\hat{Q} f(x)]^* dx + \underbrace{\left[ f^*(x) \frac{d}{dx} g(x) - \frac{d}{dx} f^*(x) g(x) \right]_{-\infty}^{+\infty}}_{\text{boundary term}}$$

where  $[\hat{Q} f(x)]^* = \left( \frac{d^2}{dx^2} f(x) \right)^* = \frac{d^2}{dx^2} f^*(x)$ .

For  $\hat{Q} = d^2/dx^2$  to be Hermitian, we need the boundary term to be zero *for all* well-behaved  $f(x), g(x)$  that vanish at infinity. We see that this requires not only  $f, g$  to vanish at  $\pm\infty$  but also (in some treatments) that *their first derivatives* vanish as well, or that certain boundary conditions are imposed (e.g., periodic boundary conditions in a finite domain).

In standard quantum mechanics on  $(-\infty, +\infty)$ ,  $d^2/dx^2$  can be *essentially Hermitian* (or self-adjoint) for wavefunctions that vanish sufficiently fast at infinity so that all boundary terms vanish. More precisely, one needs to specify a suitable domain of functions (like the Sobolev space  $H^2$  with appropriate boundary conditions) so that the boundary terms go to zero.

Hence, under appropriate boundary conditions (i.e. wavefunctions and derivatives going to zero at infinity, or periodic boundary conditions on a finite interval), the operator

$$\hat{Q} = \frac{d^2}{dx^2}$$

is *Hermitian*. But one must be careful about the domain of definition to ensure the boundary terms vanish.

### 2.5.3 Question 5

7/7 Marks

#### Solution:

In the unperturbed infinite square well of width  $L$ , the normalized eigenfunctions and energies are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2m L^2}, \quad n = 1, 2, 3, \dots$$

for  $0 \leq x \leq L$ , and  $\phi_n(x) = 0$  otherwise.

#### 1. First-Order Correction to the Ground-State Energy: $H' = V_0 \sin\left(\frac{2\pi x}{L}\right)$ 4/4 Marks

The time-independent, first-order perturbation theory tells us that the shift in the  $n$ th level is

$$E_n^{(1)} = \langle \phi_n | H' | \phi_n \rangle = \int_0^L \phi_n(x) H' \phi_n(x) dx.$$

For the ground state ( $n = 1$ ), we have

$$E_1^{(1)} = \int_0^L \left[ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \right] V_0 \sin\left(\frac{2\pi x}{L}\right) \left[ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \right] dx.$$

This becomes

$$E_1^{(1)} = \frac{2V_0}{L} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx.$$

One may show (by trigonometric identities or direct integration) that

$$\int_0^L \sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx = 0,$$

since the integrand is orthogonal over  $[0, L]$ . Therefore,

$$\boxed{E_1^{(1)} = 0 \quad \text{for the perturbation } H' = V_0 \sin\left(\frac{2\pi x}{L}\right).}$$

#### 2. First-Order Correction for Any Level: $H' = L \alpha \delta\left(x - \frac{L}{2}\right)$ 3/3 Marks

Now the perturbation is a Dirac delta function located at  $x = L/2$ . The first-order shift in the  $n$ th energy level is

$$E_n^{(1)} = \langle \phi_n | H' | \phi_n \rangle = L \alpha \int_0^L \phi_n^*(x) \delta\left(x - \frac{L}{2}\right) \phi_n(x) dx.$$

Exploiting the sifting property of the delta function, only the value of  $\phi_n(x)$  at  $x = L/2$  matters:

$$E_n^{(1)} = L \alpha \left| \phi_n\left(\frac{L}{2}\right) \right|^2.$$

Since

$$\phi_n\left(\frac{L}{2}\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} \frac{L}{2}\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{2}\right),$$

we get

$$\left| \phi_n\left(\frac{L}{2}\right) \right|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi}{2}\right).$$

Hence,

$$E_n^{(1)} = L \alpha \frac{2}{L} \sin^2\left(\frac{n\pi}{2}\right) = 2 \alpha \sin^2\left(\frac{n\pi}{2}\right).$$

Therefore,

$$E_n^{(1)} = 2 \alpha \sin^2\left(\frac{n\pi}{2}\right) \quad \text{for } H' = L \alpha \delta\left(x - \frac{L}{2}\right).$$

In particular,  $\sin^2\left(\frac{n\pi}{2}\right)$  vanishes for even  $n$  (giving zero shift) and is 1 for odd  $n$ , so odd levels get a shift of  $2\alpha$  while even levels remain unshifted.

#### 2.5.4 Question 6

14/17 Marks

**Solution:**

1. **Proving  $\hat{\mathbf{J}}^2 = \hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2 \hat{L}_z \hat{S}_z$  4/4 Marks**

We define  $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ , where  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  are the orbital and spin angular momentum operators, respectively. Then

$$\hat{\mathbf{J}}^2 = (\hat{\mathbf{L}} + \hat{\mathbf{S}}) \cdot (\hat{\mathbf{L}} + \hat{\mathbf{S}}) = \hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2 + 2 \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}.$$

Recall the identity for raising/lowering operators:

$$\hat{L}_{\pm} = \hat{L}_x \pm i \hat{L}_y, \quad \hat{S}_{\pm} = \hat{S}_x \pm i \hat{S}_y.$$

and

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \hat{L}_z \hat{S}_z + \frac{1}{2} (\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+).$$

Combining these yields

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2 \hat{L}_z \hat{S}_z,$$

which is the desired form.

## 2. Eigenstate of $\hat{\mathbf{J}}^2$ with $|l, -l; s, -s\rangle$ 4/4 Marks

We have the state

$$|l, m; s, m_s\rangle \equiv |l, m\rangle \otimes |s, m_s\rangle,$$

where  $\hat{L}^2|l, m\rangle = \hbar^2 l(l+1)|l, m\rangle$  and  $\hat{L}_z|l, m\rangle = \hbar m|l, m\rangle$ , while  $\hat{S}^2|s, m_s\rangle = \hbar^2 s(s+1)|s, m_s\rangle$  and  $\hat{S}_z|s, m_s\rangle = \hbar m_s|s, m_s\rangle$ .

In particular, consider the combined ket  $|l, -l; s, -s\rangle$ . Applying  $\hat{\mathbf{J}}^2 = (\hat{\mathbf{L}} + \hat{\mathbf{S}})^2$  to this state, we note

$$\hat{L}_z|l, -l\rangle = -\hbar l|l, -l\rangle, \quad \hat{S}_z|s, -s\rangle = -\hbar s|s, -s\rangle.$$

Hence,

$$\hat{J}_z|l, -l; s, -s\rangle = (\hat{L}_z + \hat{S}_z)|l, -l; s, -s\rangle = -\hbar(l+s)|l, -l; s, -s\rangle.$$

One can similarly check that the ladder operators  $\hat{J}_{\pm} = \hat{L}_{\pm} + \hat{S}_{\pm}$  annihilate this state (since raising  $-l$  or  $-s$  by 1 would produce mismatch in the combined  $m_J$ ). Thus  $|l, -l; s, -s\rangle$  is indeed an eigenket of  $\hat{\mathbf{J}}^2$  with total angular momentum quantum number  $j = l + s$ , so

$$\hat{\mathbf{J}}^2|l, -l; s, -s\rangle = \hbar^2[l + s(l + s + 1)]|l, -l; s, -s\rangle.$$

## 3. Matrix Representation of $\hat{O} = a\hat{L}^2 + b\hat{S}_+\hat{L}_z$ in a Basis $|l = 1, m; s = \frac{1}{2}, m_s\rangle$ 6/9 Marks

We choose the combined basis with  $l = 1$  and  $s = \frac{1}{2}$ . Then  $m$  can be  $-1, 0, +1$  and  $m_s$  can be  $-\frac{1}{2}, +\frac{1}{2}$ . Altogether, there are  $3 \times 2 = 6$  basis states:

$$|1, m; \frac{1}{2}, m_s\rangle \text{ where } m \in \{-1, 0, +1\}, \quad m_s \in \{-\frac{1}{2}, +\frac{1}{2}\}.$$

Thus  $\hat{O}$  has a  $6 \times 6$  matrix in this basis.

First, note that  $\hat{L}^2|1, m\rangle = \hbar^2 \cdot 1 \cdot (1+1)|1, m\rangle = 2\hbar^2|1, m\rangle$ . Hence on the combined state,

$$\hat{L}^2|1, m; \frac{1}{2}, m_s\rangle = 2\hbar^2|1, m; \frac{1}{2}, m_s\rangle.$$

Thus the matrix elements of  $a\hat{L}^2$  are simply

$$\langle 1, m'; \frac{1}{2}, m'_s | a\hat{L}^2 | 1, m; \frac{1}{2}, m_s \rangle = a(2\hbar^2) \delta_{m', m} \delta_{m'_s, m_s}.$$

That is,  $a\hat{L}^2$  contributes a diagonal block  $2a\hbar^2$  (the same for all 6 basis states).

Next, consider  $b\hat{S}_+\hat{L}_z$ . We know

$$\hat{S}_+|s = \frac{1}{2}, m_s = -\frac{1}{2}\rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)}|\frac{1}{2}, +\frac{1}{2}\rangle = \hbar \sqrt{\frac{1}{2}(\frac{1}{2}+1) - (-\frac{1}{2})(-\frac{1}{2}+1)}|\frac{1}{2}, +\frac{1}{2}\rangle$$

Numerically, that factor is  $\hbar \sqrt{\frac{1}{2} \cdot \frac{3}{2} - (-\frac{1}{2}) \cdot \frac{1}{2}} = \hbar \sqrt{\frac{3}{4} - (-\frac{1}{4})} = \hbar \sqrt{1} = \hbar$ . And  $\hat{S}_+$  acting on  $|\frac{1}{2}, +\frac{1}{2}\rangle$  gives 0.

Meanwhile,

$$\hat{L}_z |1, m\rangle = \hbar m |1, m\rangle.$$

Therefore,

$$\hat{S}_+ \hat{L}_z |1, m; \frac{1}{2}, m_s\rangle = \hat{S}_+ (\hbar m |1, m; \frac{1}{2}, m_s\rangle) = \hbar m (\hat{S}_+ |\frac{1}{2}, m_s\rangle) \otimes |1, m\rangle.$$

From the explicit action of  $\hat{S}_+$ , we see that  $m_s$  must be  $-\frac{1}{2}$  for a nonzero result:

$$\hat{S}_+ |\frac{1}{2}, -\frac{1}{2}\rangle = \hbar |\frac{1}{2}, +\frac{1}{2}\rangle, \quad \hat{S}_+ |\frac{1}{2}, +\frac{1}{2}\rangle = 0.$$

Hence in the basis  $|1, m; \frac{1}{2}, m_s\rangle$ , the operator  $\hat{S}_+ \hat{L}_z$  has only off-diagonal elements that connect  $|1, m; \frac{1}{2}, -\frac{1}{2}\rangle$  to  $|1, m; \frac{1}{2}, +\frac{1}{2}\rangle$ , with matrix element

$$\langle 1, m; \frac{1}{2}, +\frac{1}{2} | \hat{S}_+ \hat{L}_z | 1, m; \frac{1}{2}, -\frac{1}{2} \rangle = \hbar m \hbar = \hbar^2 m.$$

All other matrix elements of  $\hat{S}_+ \hat{L}_z$  vanish. Multiplying by  $b$  just scales those off-diagonal entries by  $b$ .

Collecting these results, the  $6 \times 6$  matrix for  $\hat{O} = a \hat{L}^2 + b \hat{S}_+ \hat{L}_z$  in the chosen basis has the block-diagonal form with:

- A diagonal part  $2a\hbar^2$  for all basis states (due to  $a \hat{L}^2$ ).
- Off-diagonal couplings *only* between  $m_s = -\frac{1}{2}$  and  $m_s = +\frac{1}{2}$  states with the same  $m$ , of magnitude  $b\hbar^2 m$ .

Hence one can write out the matrix (arranging basis in order of  $m = \pm 1, 0$  and  $m_s = -\frac{1}{2}, +\frac{1}{2}$ ) explicitly. Symbolically, one sees each  $m$  subspace yields a  $2 \times 2$  block of the form

$$\begin{pmatrix} 2a\hbar^2 & b\hbar^2 m \\ 0 & 2a\hbar^2 \end{pmatrix},$$

except that we must be careful with the ordering of  $m_s = -\frac{1}{2}, +\frac{1}{2}$  in the rows/columns. This fully specifies the matrix representation of  $\hat{O}$ .

### 2.5.5 Question 9

17/17 Marks

**Solution:**



**Potential Step:**

$$V(x) = \begin{cases} 0 & x \leq 0, \\ -V_0 & x > 0, \end{cases} \quad \text{with } E > 0 \text{ (particle energy).}$$

Define:

$$k_1 = \frac{\sqrt{2mE}}{\hbar}, \quad k_2 = \frac{\sqrt{2m(E + V_0)}}{\hbar}.$$

These follow from the time-independent Schrödinger equation  $\frac{p^2}{2m} + V(x) = E$  in each region.

**1. General Solutions in Each Region 5/5 Marks**

*Region I:*  $x \leq 0, V(x) = 0$ . A general solution with incident and reflected waves can be written:

$$\psi_1(x) = e^{i k_1 x} + B e^{-i k_1 x}, \quad \text{where } B \text{ is the reflection amplitude.}$$

Often one sets the incident amplitude to 1, and calls that wave  $e^{i k_1 x}$ . (In your statement,  $A = 1$  is taken implicitly, and the reflection amplitude is  $B$ .)

*Region II:*  $x > 0, V(x) = -V_0$ . Here the solution is a single transmitted wave traveling to the right (since  $x > 0$  extends to  $+\infty$  and we assume no incoming wave from  $+\infty$ ). That wave is

$$\psi_2(x) = C e^{i k_2 x}, \quad \text{where } C \text{ is the transmission amplitude.}$$

Hence the full piecewise wavefunction is

$$\psi(x) = \begin{cases} e^{i k_1 x} + B e^{-i k_1 x}, & x \leq 0, \\ C e^{i k_2 x}, & x > 0. \end{cases}$$

This matches the statement (up to naming conventions and signs of exponents).

**2. Boundary Conditions and Solving for  $B$  and  $C$  5/5 Marks**

The boundary conditions at  $x = 0$  require:

$$\psi_1(0) = \psi_2(0), \quad \left. \frac{d\psi_1}{dx} \right|_{x=0} = \left. \frac{d\psi_2}{dx} \right|_{x=0}.$$

From continuity of the wavefunction at  $x = 0$ :

$$1 + B = C. \tag{1}$$

From continuity of the derivative at  $x = 0$ :

$$i k_1 (1 - B) = i k_2 C. \tag{2}$$

Solving (1) and (2) simultaneously for  $B$  and  $C$  gives

$$C = \frac{2k_1}{k_1 + k_2}, \quad B = \frac{k_1 - k_2}{k_1 + k_2}.$$

(Details: from (2) one has  $k_1(1 - B) = k_2C$ , and from (1) one has  $C = 1 + B$ . Solve algebraically.)

### 3. Incident, Reflected, and Transmitted Flux 7/7 Marks

The flux (probability current) for a wavefunction  $\psi(x)$  is

$$\Gamma = -\frac{i\hbar}{2m} \left[ \psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} \right].$$

For a plane wave of form  $Ae^{ikx}$ , the flux is  $\Gamma = \frac{\hbar k}{m}|A|^2$ .

*Region I (Incident + Reflected):*

$$\psi_1(x) = e^{ik_1x} + Be^{-ik_1x}.$$

The incident wave is  $e^{ik_1x}$ , flux

$$\Gamma_{\text{inc}} = \frac{\hbar k_1}{m}.$$

The reflected wave is  $Be^{-ik_1x}$ , flux

$$\Gamma_{\text{ref}} = -\frac{\hbar k_1}{m}|B|^2$$

(the minus sign arises because  $k_1 < 0$  for the wave traveling in the  $-x$  direction, or more precisely from substituting  $e^{-ik_1x}$  in the flux formula).

*Region II (Transmitted):*

$$\psi_2(x) = Ce^{ik_2x},$$

flux

$$\Gamma_{\text{trans}} = \frac{\hbar k_2}{m}|C|^2.$$

#### Transmission and Reflection Probabilities:

By definition,

$$T = \frac{\Gamma_{\text{trans}}}{\Gamma_{\text{inc}}}, \quad R = \frac{|\Gamma_{\text{ref}}|}{\Gamma_{\text{inc}}}.$$

Here  $\Gamma_{\text{inc}} = \hbar k_1/m$ , so

$$T = \frac{\hbar k_2/m|C|^2}{\hbar k_1/m} = \frac{k_2}{k_1}|C|^2, \quad R = \frac{\hbar k_1/m|B|^2}{\hbar k_1/m} = |B|^2.$$

Substitute the expressions for  $B$  and  $C$ :

$$C = \frac{2k_1}{k_1 + k_2}, \quad B = \frac{k_1 - k_2}{k_1 + k_2}.$$

Hence

$$|C|^2 = \left( \frac{2k_1}{k_1 + k_2} \right)^2 = \frac{4k_1^2}{(k_1 + k_2)^2}, \quad |B|^2 = \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2.$$

Thus

$$T = \frac{k_2}{k_1} \frac{4k_1^2}{(k_1 + k_2)^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2}, \quad R = \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2.$$

These are the final forms:

$$\boxed{T = \frac{4k_1 k_2}{(k_1 + k_2)^2}, \quad R = \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2.}$$

One can also check that  $T+R=1$ , consistent with probability conservation in this simple step potential problem.

### 2.5.6 Question 10

11/16 Marks

**Solution:**

**Quantum Harmonic Oscillator:** The unperturbed energy eigenstates are  $\{|n\rangle\}_{n=0,1,2,\dots}$  with eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

We use the *annihilation* and *creation* operators  $\hat{a}_-$ ,  $\hat{a}_+$  (often denoted  $\hat{a}$ ,  $\hat{a}^\dagger$ ), defined by

$$\hat{a}_\pm = \frac{1}{\sqrt{2}}\left(\alpha \hat{x} \pm \frac{i}{\alpha \hbar} \hat{p}\right), \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}.$$

These satisfy

$$\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle.$$

Also, we recall that

$$\hat{H}_0 = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right), \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_- + \hat{a}_+).$$

#### 1. Part (a): Normalizing $|\psi\rangle = |0\rangle + b|1\rangle$ 5/5 Marks

We have a trial state

$$|\psi\rangle = |0\rangle + b|1\rangle,$$

where  $b$  is a constant (possibly real for simplicity). To normalize  $|\psi\rangle$ , we require

$$\langle\psi|\psi\rangle = (\langle 0| + b^*\langle 1|)(|0\rangle + b|1\rangle) = \langle 0|0\rangle + b\langle 0|1\rangle + b^*\langle 1|0\rangle + |b|^2\langle 1|1\rangle.$$

Because  $\{|n\rangle\}$  are orthonormal,  $\langle 0|1\rangle = 0$  and  $\langle 0|0\rangle = \langle 1|1\rangle = 1$ . Thus

$$1 = \langle\psi|\psi\rangle = 1 + |b|^2.$$

Hence

$$|b|^2 = 0 \implies b = 0 \quad (\text{trivial case, or}) \quad \text{or if the problem suggests } b \text{ is a free parameter,}$$

Actually, for the general normalization condition we get

$$1 + |b|^2 = 1 \implies |b|^2 = 0 \Rightarrow b = 0,$$

which suggests the state is simply  $|0\rangle$  if we strictly impose  $\langle\psi|\psi\rangle = 1$  with the given form.

However, it is more common that the state was intended to be written

$$|\psi\rangle = \frac{1}{\sqrt{1 + |b|^2}}(|0\rangle + b|1\rangle),$$

so that the normalization condition is automatically satisfied. We often rename  $b$  to be the amplitude ratio, in which case the normalized state becomes

$$|\psi\rangle_{\text{norm}} = \frac{1}{\sqrt{1 + |b|^2}}(|0\rangle + b|1\rangle).$$

*Expectation Value of the Hamiltonian:*

$$\langle H_0 \rangle = \langle\psi|\hat{H}_0|\psi\rangle.$$

If we assume the normalized version, then

$$\langle\psi|\hat{H}_0|\psi\rangle = \frac{1}{1 + |b|^2}(\langle 0| + b^*\langle 1|)\hat{H}_0(|0\rangle + b|1\rangle).$$

Because  $|n\rangle$  is an eigenstate with eigenvalue  $E_n = \hbar\omega(n + 1/2)$ ,

$$\hat{H}_0|0\rangle = \hbar\omega\left(\frac{1}{2}\right)|0\rangle, \quad \hat{H}_0|1\rangle = \hbar\omega\left(\frac{3}{2}\right)|1\rangle.$$

Thus

$$\hat{H}_0(|0\rangle + b|1\rangle) = \frac{1}{2}\hbar\omega|0\rangle + b\frac{3}{2}\hbar\omega|1\rangle.$$

Substituting,

$$\langle H_0 \rangle = \frac{1}{1 + |b|^2} \left[ \langle 0|(\frac{1}{2}\hbar\omega|0\rangle) + \langle 0|(b\frac{3}{2}\hbar\omega|1\rangle) + b^*\langle 1|(\frac{1}{2}\hbar\omega|0\rangle) + b^*\langle 1|(b\frac{3}{2}\hbar\omega|1\rangle) \right].$$

The cross terms  $\langle 0|1\rangle$  and  $\langle 1|0\rangle$  vanish, so only the diagonal pieces remain:

$$\langle H_0 \rangle = \frac{1}{1+|b|^2} \left[ \frac{1}{2} \hbar \omega + |b|^2 \frac{3}{2} \hbar \omega \right] = \frac{\hbar \omega}{1+|b|^2} \left( \frac{1}{2} + \frac{3}{2} |b|^2 \right) = \hbar \omega \frac{\frac{1}{2} + \frac{3}{2} |b|^2}{1+|b|^2}.$$

Simplify:

$$\boxed{\langle H_0 \rangle = \hbar \omega \frac{1/2 + (3/2) |b|^2}{1+|b|^2}.$$

## 2. Part (b): Expressing $H' = q \mathcal{E} \hat{x}$ in Terms of $\hat{a}_{\pm}$ 2/2 Marks

The perturbation is

$$H' = q \mathcal{E} \hat{x}.$$

Recall

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_- + \hat{a}_+) = \frac{1}{\alpha \sqrt{2}} (\hat{a}_- + \hat{a}_+) \quad \text{since } \alpha = \sqrt{\frac{m\omega}{\hbar}}.$$

Thus

$$H' = q \mathcal{E} \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_- + \hat{a}_+) = q \mathcal{E} \frac{1}{\alpha \sqrt{2}} (\hat{a}_- + \hat{a}_+).$$

That is,

$$\boxed{H' = \frac{q \mathcal{E}}{\sqrt{2} \alpha} (\hat{a}_- + \hat{a}_+).$$

## 3. Part (c): Variational Principle and the Optimal $b$ 4/9 Marks

We are told that by applying the variational principle (with  $|\psi\rangle = |0\rangle + b|1\rangle$ ) to minimize the total energy  $E_0 + E_1$  + the perturbation, one obtains

$$b = \frac{E_1 - E_0}{g} - \sqrt{\left(\frac{E_1 - E_0}{g}\right)^2 - 1}, \quad \text{where } g = \sqrt{2q\mathcal{E}/\alpha}.$$

The steps to derive this usually involve forming the *expectation value*  $\langle H_0 + H' \rangle$  in the trial state, differentiating w.r.t.  $b^*$  (and  $b$ ), and setting that derivative to zero. Solving the resulting nonlinear equation gives the expression for  $b$ .

Without going through the full derivation, the final result is

$$\boxed{b = \frac{E_1 - E_0}{g} - \sqrt{\left(\frac{E_1 - E_0}{g}\right)^2 - 1}, \quad \text{where } g = \sqrt{\frac{2q\mathcal{E}}{\alpha}} \quad \text{and} \quad E_n = \hbar \omega \left(n + \frac{1}{2}\right).$$

This  $b$  (which can be real) makes the energy functional  $\langle \psi | (H_0 + H') | \psi \rangle$  a minimum under the constraint of normalization.

## 2.6 Classical Mechanics with context prompts

### 2.6.1 Question 2

6/6 Marks

**Solution:**

Consider the damped harmonic oscillator described by

$$\frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0.$$

Let the undamped natural frequency be  $\omega_0 = \sqrt{\frac{k}{m}}$ , and let the damping coefficient be  $b = \frac{2m\gamma}{2m}$  so that  $\gamma = \frac{b}{2m}$ . The three cases of damping depend on the discriminant  $\Delta = \gamma^2 - \omega_0^2$ :

1. **Overdamped** ( $\gamma^2 > \omega_0^2$ )

In this case, the damping is strong enough that the system does *not* oscillate. Instead, the solution  $x(t)$  exponentially decays to equilibrium without crossing the equilibrium position. The motion is a sum of two exponential decays with different decay rates.

2. **Critically Damped** ( $\gamma^2 = \omega_0^2$ )

The system is *just* at the threshold between overdamped and underdamped. The motion returns to equilibrium as quickly as possible *without* oscillation. The solution is a product of an exponential and a polynomial in  $t$ , leading to the fastest non-oscillatory decay back to equilibrium.

3. **Underdamped** ( $\gamma^2 < \omega_0^2$ )

Here, the damping is *weak* enough that the system still oscillates, but with a frequency  $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$  lower than the natural frequency  $\omega_0$ . The amplitude of the oscillations decays exponentially over time due to the damping term.

### 2.6.2 Question 5

9/9 Marks

**Solution:**

**System Setup:** Two identical masses  $m$  (objects X and Y) are connected by a spring of constant  $k$  and each mass is also connected to a fixed wall by a spring of the same constant  $k$ , as shown:

$$(\text{Wall}) - k - [\text{X}] - k - [\text{Y}] - k - (\text{Wall}).$$

Denote the small displacements of X and Y from their respective equilibrium positions by  $x_1(t)$  and  $x_2(t)$ .

## 1. Equations of Motion for X and Y

For object X:

- The left spring (wall-X) exerts a restoring force  $-k x_1$  on X.
- The middle spring (X-Y) is extended by  $(x_2 - x_1)$ ; hence the force on X due to that spring is

$$-k(x_1 - x_2) = k(x_2 - x_1).$$

Therefore, the total force on X is

$$F_X = -k x_1 + k(x_2 - x_1) = -2k x_1 + k x_2.$$

Newton's second law,  $m \ddot{x}_1 = F_X$ , gives

$$m \ddot{x}_1 = -2k x_1 + k x_2. \implies \ddot{x}_1 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2.$$

For object Y:

- The middle spring (X-Y) exerts a force  $-k(x_2 - x_1)$  on Y (pulling it toward X).
- The right spring (Y-wall) exerts  $-k x_2$  on Y.

Hence,

$$F_Y = -k(x_2 - x_1) - k x_2 = k x_1 - 2k x_2.$$

Then  $m \ddot{x}_2 = F_Y$  yields

$$\ddot{x}_2 = \frac{k}{m} x_1 - \frac{2k}{m} x_2.$$

## 2. Normal-Mode Coordinates and Decoupling

Define the new coordinates:

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2.$$

We want to express  $\ddot{y}_1$  and  $\ddot{y}_2$  in terms of  $\{x_1, x_2\}$ .

Compute:

$$\ddot{y}_1 = \ddot{x}_1 + \ddot{x}_2, \quad \ddot{y}_2 = \ddot{x}_1 - \ddot{x}_2.$$

From the original equations:

$$\ddot{x}_1 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2, \quad \ddot{x}_2 = \frac{k}{m} x_1 - \frac{2k}{m} x_2.$$

Add them to get  $\ddot{y}_1$ :

$$\ddot{y}_1 = (\ddot{x}_1 + \ddot{x}_2) = \left[ -\frac{2k}{m} x_1 + \frac{k}{m} x_2 \right] + \left[ \frac{k}{m} x_1 - \frac{2k}{m} x_2 \right] = -\frac{2k}{m} (x_1 + x_2) + \frac{k}{m} (x_2 + x_1) = -\frac{k}{m} (x_1 + x_2).$$

Notice  $x_1 + x_2 = y_1$ , so

$$\ddot{y}_1 = -\frac{k}{m} y_1.$$

Similarly, subtracting the second from the first yields  $\ddot{y}_2$ :

$$\ddot{y}_2 = \ddot{x}_1 - \ddot{x}_2 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2 - \left[ \frac{k}{m} x_1 - \frac{2k}{m} x_2 \right] = -\frac{3k}{m} x_1 + \frac{3k}{m} x_2 = -\frac{3k}{m} (x_1 - x_2).$$

But  $x_1 - x_2 = y_2$ , so

$$\ddot{y}_2 = -\frac{3k}{m} y_2.$$

Hence the motion in  $y_1$  and  $y_2$  is uncoupled:

$$\boxed{\ddot{y}_1 + \frac{k}{m} y_1 = 0, \quad \ddot{y}_2 + \frac{3k}{m} y_2 = 0.}$$

These describe two independent SHM modes with angular frequencies  $\omega_1 = \sqrt{\frac{k}{m}}$  and  $\omega_2 = \sqrt{\frac{3k}{m}}$ .

### 3. Solution for $y_1(t)$ Under the Given Initial Conditions

Initially,  $x_1(0) = 0$  (mass X at rest at equilibrium),  $x_2(0) = 0$  (assume Y is also initially at equilibrium position but has velocity), and  $\dot{x}_2(0) = v$  (the given initial velocity). Then

$$y_1(0) = x_1(0) + x_2(0) = 0 + 0 = 0, \quad \dot{y}_1(0) = \dot{x}_1(0) + \dot{x}_2(0) = 0 + v = v.$$

From the  $y_1$ -equation,

$$\ddot{y}_1 + \left(\frac{k}{m}\right) y_1 = 0,$$

the general solution is  $y_1(t) = A \sin(\omega_1 t) + B \cos(\omega_1 t)$  with  $\omega_1 = \sqrt{k/m}$ . The initial conditions imply

$$y_1(0) = B = 0, \quad \dot{y}_1(0) = A \omega_1 = v \implies A = \frac{v}{\omega_1} = v \sqrt{\frac{m}{k}}.$$

Hence

$$\boxed{y_1(t) = \left(v \sqrt{\frac{m}{k}}\right) \sin\left(\sqrt{\frac{k}{m}} t\right) = v \sqrt{\frac{m}{k}} \sin\left(t \sqrt{\frac{k}{m}}\right).}$$

This is exactly the stated result for  $y_1(t)$ .



### 2.6.3 Question 7

13/13 Marks

**Solution:**

#### (a) Energy-Momentum 4-Vector and Its Invariance 4/4 Marks

A particle of mass  $m$  and velocity  $\mathbf{v}$  (with  $|\mathbf{v}| = v$ ) in the lab frame has energy

$$E = \gamma m c^2, \quad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Its momentum is

$$\mathbf{p} = \gamma m \mathbf{v}.$$

We collect these into a 4-vector,

$$P^\mu = \left( \frac{E}{c}, \mathbf{p} \right) = (\gamma m c, \gamma m \mathbf{v}).$$

The *Lorentz-invariant* scalar (Minkowski) product of  $P^\mu$  with itself is

$$P^\mu P_\mu = \left( \frac{E}{c} \right)^2 - |\mathbf{p}|^2 = \left( \frac{\gamma m c}{1} \right)^2 - (\gamma m v)^2 = \gamma^2 m^2 (c^2 - v^2) = m^2 c^2,$$

independent of  $\mathbf{v}$ . Hence  $P^\mu P_\mu = m^2 c^2$  is the same in all inertial frames.

#### (b) Two-Body Decay in the Particle's Rest Frame and the Lab Frame 5/5 Marks

Suppose the original particle of mass  $m$  is at rest in its own rest frame. It decays into two identical *massless* particles emerging along  $\pm x$  directions.

- *In the rest frame of the decaying particle:*

$$P_{(\text{initial})}^\mu = (m c, \mathbf{0}).$$

After decay, each daughter is massless with energy  $E_\gamma$  (we often call them photons if they are truly massless). Conservation of 4-momentum implies

$$P_{(\text{initial})}^\mu = P_{(1)}^\mu + P_{(2)}^\mu.$$

Because they are identical and back-to-back in  $\pm x$ , each has momentum magnitude  $p_\gamma = E_\gamma/c$  (for massless). The total momentum must be zero, so one photon's momentum is  $+p_\gamma \hat{x}$  and the other's is  $-p_\gamma \hat{x}$ . Thus

$$P_{(1)}^\mu = \left( \frac{E_\gamma}{c}, +p_\gamma, 0, 0 \right), \quad P_{(2)}^\mu = \left( \frac{E_\gamma}{c}, -p_\gamma, 0, 0 \right).$$

Adding these gives  $\left( \frac{2E_\gamma}{c}, 0, 0, 0 \right)$ , which must match  $(m c, 0, 0, 0)$ . Hence  $2E_\gamma = m c^2$ , so each photon has  $E_\gamma = \frac{1}{2} m c^2$ .

- *In the lab frame:* the decaying particle moves with velocity  $v$  along, say, the  $+x$  axis. Its 4-momentum in the lab is

$$P_{(\text{initial})}^{\mu} = (\gamma m c, \gamma m v, 0, 0).$$

Because each daughter is massless, each has  $p^0 = E/c = |\mathbf{p}|$ . One photon (call it #1) is emitted “forward” (in the same sense as the parent’s motion), and the other (#2) is “backward.” Their 4-momenta  $P_{(1)}^{\mu}$ ,  $P_{(2)}^{\mu}$  must sum to  $P_{(\text{initial})}^{\mu}$ . Solving explicitly for  $P_{(1)}^{\mu}$  and  $P_{(2)}^{\mu}$  in closed form is standard but somewhat messy:

$$P_{(1)}^{\mu} = \left( \frac{E_1}{c}, +E_1/c, 0, 0 \right), \quad P_{(2)}^{\mu} = \left( \frac{E_2}{c}, -E_2/c, 0, 0 \right),$$

with  $E_1 + E_2 = \gamma m c^2$ , and  $E_1 - E_2 = \gamma m v c$ . Solving these simultaneously gives

$$E_1 = \frac{\gamma m c^2}{2} (1 + \beta), \quad E_2 = \frac{\gamma m c^2}{2} (1 - \beta), \quad \text{where } \beta = \frac{v}{c}.$$

Hence the 4-momenta of the two massless particles in the lab are

$$P_{(1)}^{\mu} = \left( \frac{E_1}{c}, +\frac{E_1}{c}, 0, 0 \right), \quad P_{(2)}^{\mu} = \left( \frac{E_2}{c}, -\frac{E_2}{c}, 0, 0 \right).$$

with  $E_1, E_2$  as above.

**(c) Numerical Example:**  $m = 10 \text{ GeV}/c^2$ ,  $\beta = 0.8$ , and the decay products are photons **4/4 Marks**

- *Wavelengths in the rest frame of the decaying particle:*

Each photon in the rest frame has energy  $E_{\gamma} = \frac{1}{2} m c^2 = 5 \text{ GeV}$ . The corresponding momentum is  $p_{\gamma} = E_{\gamma}/c = 5 \text{ GeV}/c$ . Hence

$$\lambda_{\text{rest}} = \frac{h c}{E_{\gamma}} = \frac{(4.1357 \times 10^{-15} \text{ eV} \cdot \text{s}) (3 \times 10^8 \text{ m/s})}{5 \times 10^9 \text{ eV}} \approx 2.48 \times 10^{-16} \text{ m}.$$

(If one prefers  $\hbar c = 0.197 \text{ GeV fm}$  or similar units, one can do a direct conversion.)

- *Wavelengths in the lab frame:*

In the lab, one photon has energy  $E_1 = \frac{1}{2} \gamma m c^2 (1 + \beta)$ , and the other  $E_2 = \frac{1}{2} \gamma m c^2 (1 - \beta)$ . With  $m = 10 \text{ GeV}/c^2$ ,  $\gamma = 1/\sqrt{1 - \beta^2} = 1/\sqrt{1 - 0.64} = 1/\sqrt{0.36} \approx 1.6667$ , and  $\beta = 0.8$ . Then

$$E_1 = \frac{1}{2} (1.6667) (10 \text{ GeV}) (1 + 0.8) = 0.5 \times 1.6667 \times 10 \times 1.8 \text{ GeV} \approx 15 \text{ GeV}.$$

$$E_2 = \frac{1}{2} (1.6667) (10 \text{ GeV}) (1-0.8) = 0.5 \times 1.6667 \times 10 \times 0.2 \text{ GeV} \approx 1.6667 \text{ GeV}.$$

So one photon is “boosted” to 15 GeV in the lab, while the other is only 1.67 GeV. Their wavelengths are

$$\lambda_1 = \frac{hc}{E_1} \approx \frac{4.1357 \times 10^{-15} \text{ eV} \cdot \text{s} \cdot 3 \times 10^8 \text{ m/s}}{15 \times 10^9 \text{ eV}} \approx 8.3 \times 10^{-17} \text{ m},$$

$$\lambda_2 = \frac{hc}{E_2} \approx \frac{4.1357 \times 10^{-15} \text{ eV} \cdot \text{s} \cdot 3 \times 10^8 \text{ m/s}}{1.6667 \times 10^9 \text{ eV}} \approx 7.5 \times 10^{-16} \text{ m}.$$

### 2.6.4 Question 8

6/10 Marks

**Solution:**

**System Description:**

A pendulum of mass  $m$  and fixed length  $L$  is attached to a support that moves horizontally according to

$$x_{\text{support}}(t) = v t^3 + A \sin(\omega t),$$

where  $v$ ,  $A$ , and  $\omega$  are constants. Let  $\phi(t)$  be the instantaneous angle of the pendulum from the vertical.

#### 1. (a) Deriving the Equation of Motion from the Lagrangian

*Coordinates:* Choose the origin at the pivot, but note the pivot itself moves in the  $x$ -direction. At any time  $t$ , the bob's coordinates become

$$X(t) = x_{\text{support}}(t) + L \sin(\phi), \quad Y(t) = -L \cos(\phi),$$

assuming the downward vertical is  $Y < 0$  and  $\phi = 0$  means straight down.

*Kinetic Energy:*

$$T = \frac{1}{2} m (\dot{X}^2 + \dot{Y}^2).$$

Compute  $\dot{X}$ ,  $\dot{Y}$ :

$$\dot{X} = \dot{x}_{\text{support}}(t) + L \cos(\phi) \dot{\phi}, \quad \dot{Y} = -[-L \sin(\phi) \dot{\phi}] = L \sin(\phi) \dot{\phi}.$$

Hence

$$\dot{X}^2 + \dot{Y}^2 = \left( \dot{x}_{\text{support}} + L \cos \phi \dot{\phi} \right)^2 + (L \sin \phi \dot{\phi})^2.$$

Therefore

$$T = \frac{1}{2} m \left[ \dot{x}_{\text{support}}^2 + 2 \dot{x}_{\text{support}} L \cos \phi \dot{\phi} + L^2 \cos^2 \phi \dot{\phi}^2 + L^2 \sin^2 \phi \dot{\phi}^2 \right].$$

Combine the  $\cos^2 \phi + \sin^2 \phi = 1$  terms:

$$T = \frac{1}{2} m \left[ \dot{x}_{\text{support}}^2 + 2 \dot{x}_{\text{support}} L \cos \phi \dot{\phi} + L^2 \dot{\phi}^2 \right].$$

*Potential Energy:* We take  $U = 0$  at  $Y = 0$  for convenience, or more standardly we set  $U = mg \times (\text{vertical displacement})$ . Since  $Y = -L \cos(\phi)$ , the bob's height relative to  $Y = 0$  is  $-L \cos(\phi)$ , so

$$U = m g (\text{height}) = m g (L \cos \phi) \quad (\text{up to an additive constant}).$$

Thus

$$U(\phi) = m g L \cos(\phi).$$

*Lagrangian:*  $\mathcal{L} = T - U$ .

$$\mathcal{L}(\phi, \dot{\phi}, t) = \frac{1}{2} m \left[ \dot{x}_{\text{support}}^2 + 2 \dot{x}_{\text{support}} L \cos \phi \dot{\phi} + L^2 \dot{\phi}^2 \right] - m g L \cos(\phi).$$

Note that  $\dot{x}_{\text{support}}^2$  does not depend on  $\phi$  or  $\dot{\phi}$ , so it contributes no force or torque on the pendulum (though it does matter if we wanted the total energy).

*Euler-Lagrange Equation:*

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Compute each piece carefully:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2} m \left[ 2 \dot{x}_{\text{support}} L \cos \phi + 2 L^2 \dot{\phi} \right] = m \left[ \dot{x}_{\text{support}} L \cos \phi + L^2 \dot{\phi} \right].$$
 Taking the time derivative,

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m \left[ \frac{d}{dt} (\dot{x}_{\text{support}}) L \cos \phi - \dot{x}_{\text{support}} L \sin \phi \dot{\phi} + 2 L^2 \dot{\phi} \ddot{\phi} \right].$$

But  $\frac{d}{dt} (\dot{x}_{\text{support}}) = \ddot{x}_{\text{support}}(t)$ . So

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m \left[ L \cos \phi \ddot{x}_{\text{support}} - L \dot{x}_{\text{support}} \sin \phi \dot{\phi} + 2 L^2 \dot{\phi} \ddot{\phi} \right].$$

Meanwhile,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{1}{2} m \left[ 2 \dot{x}_{\text{support}} L (-\sin \phi) \dot{\phi} \right] - (m g L (-\sin \phi)),$$

since  $\frac{\partial}{\partial \phi} (L^2 \dot{\phi}^2) = 0$  and  $\frac{\partial}{\partial \phi} (\dot{x}_{\text{support}}^2) = 0$ . Hence

$$\frac{\partial \mathcal{L}}{\partial \phi} = m \dot{x}_{\text{support}} L (-\sin \phi) \dot{\phi} + m g L \sin \phi.$$

Thus the Euler-Lagrange equation  $\frac{d}{dt}(\partial\mathcal{L}/\partial\dot{\phi}) - (\partial\mathcal{L}/\partial\phi) = 0$  becomes

$$m \left[ L \cos \phi \ddot{x}_{\text{support}} - L \dot{x}_{\text{support}} \sin \phi \dot{\phi} + 2 L^2 \dot{\phi} \ddot{\phi} \right] - \left[ m \dot{x}_{\text{support}} L (-\sin \phi) \dot{\phi} + m g L \sin \phi \right] = 0.$$

Factor out  $m L$ :

$$m L \left[ \cos \phi \ddot{x}_{\text{support}} - \dot{x}_{\text{support}} \sin \phi \dot{\phi} + 2 L \dot{\phi} \ddot{\phi} \right] - m L \left[ \dot{x}_{\text{support}} (-\sin \phi) \dot{\phi} + g \sin \phi \right] = 0.$$

Simplify terms involving  $\dot{x}_{\text{support}} \sin \phi \dot{\phi}$ :

$$- \dot{x}_{\text{support}} \sin \phi \dot{\phi} - \dot{x}_{\text{support}} (-\sin \phi) \dot{\phi} = 0.$$

They cancel. So we get

$$m L \left[ \cos \phi \ddot{x}_{\text{support}} + 2 L \dot{\phi} \ddot{\phi} - g \sin \phi \right] = 0.$$

Divide out  $m L$ :

$$\cos \phi \ddot{x}_{\text{support}} + 2 L \dot{\phi} \ddot{\phi} - g \sin \phi = 0.$$

Rearrange to an ordinary differential equation in  $\phi(t)$ :

$$2 L \dot{\phi} \ddot{\phi} = g \sin \phi - \cos \phi \ddot{x}_{\text{support}}(t).$$

Or more standardly we might write:

$$2 \dot{\phi} \ddot{\phi} = \frac{g}{L} \sin \phi - \frac{\cos \phi}{L} \ddot{x}_{\text{support}}(t).$$

One can also do a bit more algebra to isolate  $\ddot{\phi}$ . Either form is the EoM for  $\phi(t)$  with a time-dependent forcing from  $\ddot{x}_{\text{support}}(t)$ .

## 2. (b) Small-Angle Limit with $v \rightarrow 0$ and $A \rightarrow 0$

In the limit that the pivot is *nearly stationary* ( $\dot{x}_{\text{support}} \approx 0$ ,  $\ddot{x}_{\text{support}} \approx 0$ ), and also  $\phi$  is small so  $\sin \phi \approx \phi$ ,  $\cos \phi \approx 1$ , the equation of motion reduces to

$$2 L \dot{\phi} \ddot{\phi} \approx g \phi.$$

In typical small-oscillation expansions, if the amplitude is also small, one might approximate  $2 \dot{\phi} \ddot{\phi} \approx \frac{d}{dt}(\dot{\phi}^2)$  or simply proceed with the usual linearized pendulum form  $\ddot{\phi} + \frac{g}{L} \phi = 0$ .

Indeed, the *classic* pendulum equation for small angles is

$$\ddot{\phi} + \frac{g}{L} \phi = 0,$$

which yields simple harmonic motion of frequency  $\sqrt{\frac{g}{L}}$ . All extra terms vanish because the support's motion is negligible ( $\ddot{x}_{\text{support}} \rightarrow 0$ ) and  $\sin \phi \approx \phi$ .

### 2.6.5 Question 9

7/7 Marks

**Solution:**

A rope is wrapped around a uniform disk of mass  $M$  and radius  $R$ , and a block of mass  $m$  is attached to the free end of the rope. As the block falls, the rope unwinds without slipping, causing the disk to rotate. The disk has moment of inertia

$$I = \frac{1}{2} M R^2.$$

We want to find:

1. The angular acceleration  $\alpha$  of the disk.
2. The tension  $T$  in the rope.
3. The linear acceleration  $a$  of the block.

**(a) Deriving the Formulas**

We assume the rope does not slip on the disk, so there is a one-to-one relation between the disk's angular acceleration  $\alpha$  and the block's linear acceleration  $a$ :

$$a = \alpha R.$$

**Forces and Torques:**

- On the block: The forces are tension  $T$  (upward) and gravity  $mg$  (downward). Hence

$$m a = m g - T. \quad (1)$$

(Taking downward as positive, so  $a$  is the downward acceleration.)

- On the disk: The rope exerts tension  $T$  tangentially at radius  $R$ . This tension provides a torque  $\tau = T R$  (taking the sign so that a positive  $T$  yields a clockwise angular acceleration). Because  $I = \frac{1}{2} M R^2$ , the rotational equation is

$$I \alpha = \tau = T R. \quad (2)$$

So

$$\frac{1}{2} M R^2 \alpha = T R \implies T = \frac{1}{2} M R \alpha.$$

But we also know  $a = \alpha R$ , or  $\alpha = \frac{a}{R}$ . Substitute  $\alpha = \frac{a}{R}$  into  $T = \frac{1}{2} M R \alpha$ :

$$T = \frac{1}{2} M R \left( \frac{a}{R} \right) = \frac{1}{2} M a.$$

Now use this in equation (1). That is

$$m a = m g - \left( \frac{1}{2} M a \right).$$

Rearrange to solve for  $a$ :

$$m a + \frac{1}{2} M a = m g \implies \left(m + \frac{1}{2} M\right) a = m g.$$

Hence

$$a = \frac{m}{m + \frac{1}{2} M} g = \frac{2m}{2m + M} g.$$

Then the tension  $T = \frac{1}{2} M a$  becomes

$$T = \frac{1}{2} M \frac{2m}{2m + M} g = \frac{M m}{2m + M} g.$$

And finally, the disk's angular acceleration is

$$\alpha = \frac{a}{R} = \frac{1}{R} \frac{2m}{2m + M} g.$$

Summarizing succinctly:

$$a = \frac{2m}{2m + M} g, \quad T = \frac{m M}{2m + M} g, \quad \alpha = \frac{1}{R} \frac{2m}{2m + M} g.$$

**(b) Numerical Values for  $m = 3 \text{ kg}$ ,  $M = 12 \text{ kg}$ ,  $R = 0.20 \text{ m}$**

$$a = \frac{2m}{2m + M} g = \frac{2 \times 3 \text{ kg}}{2 \times 3 + 12} 9.8 \text{ m/s}^2 = \frac{6}{6 + 12} 9.8 = \frac{6}{18} 9.8 = 0.3333 \times 9.8 = 3.27 \text{ m/s}^2 \approx 3.27.$$

The tension

$$T = \frac{m M}{2m + M} g = \frac{3 \times 12}{6 + 12} 9.8 = \frac{36}{18} 9.8 = 2 \times 9.8 = 19.6 \text{ N}.$$

And the angular acceleration

$$\alpha = \frac{a}{R} = \frac{3.27}{0.20} = 16.35 \text{ rad/s}^2 \approx 16.4.$$

$$a \approx 3.27 \text{ m/s}^2, \quad T \approx 19.6 \text{ N}, \quad \alpha \approx 16.4 \text{ rad/s}^2.$$

### 2.6.6 Question 10

**7/8 Marks**

**Solution:**

We have a ball of mass  $m$  dropped from an airplane with an initial horizontal velocity  $\mathbf{u}$ . The drag force on the ball is

$$\mathbf{F} = -b \mathbf{v},$$

where  $b > 0$  is a constant and  $\mathbf{v}$  is the velocity of the ball.

1. **(a) Equation of Motion and Its General Solution**

Since gravity acts downward (say along  $-\hat{\mathbf{k}}$ ) with magnitude  $mg$ , the net force on the ball is

$$m \frac{d\mathbf{v}}{dt} = -mg\hat{\mathbf{k}} - b\mathbf{v}.$$

Rearrange:

$$m \frac{d\mathbf{v}}{dt} + b\mathbf{v} = -mg\hat{\mathbf{k}}. \quad (1)$$

This is a first-order linear ODE in  $\mathbf{v}(t)$ . We can solve by standard methods (e.g. integrating factor). The vector form indicates that each Cartesian component satisfies a similar scalar differential equation. The known result (or by directly solving) is

$$\mathbf{v}(t) = \mathbf{A} e^{-\frac{b}{m}t} - \frac{mg}{b} \hat{\mathbf{k}},$$

where  $\mathbf{A}$  is a constant vector determined by initial conditions.

2. **(b) Determining  $\mathbf{A}$  From the Initial Condition**

Let  $t = 0$  be the time the ball is released. At that instant,  $\mathbf{v}(0) = \mathbf{u}$  (the airplane's horizontal velocity). So

$$\mathbf{v}(0) = \mathbf{A} e^0 - \frac{mg}{b} \hat{\mathbf{k}} = \mathbf{A} - \frac{mg}{b} \hat{\mathbf{k}} = \mathbf{u}.$$

Hence

$$\mathbf{A} = \mathbf{u} + \frac{mg}{b} \hat{\mathbf{k}}.$$

Thus the complete velocity solution is

$$\boxed{\mathbf{v}(t) = \left( \mathbf{u} + \frac{mg}{b} \hat{\mathbf{k}} \right) e^{-\frac{b}{m}t} - \frac{mg}{b} \hat{\mathbf{k}}.}$$

The first term decays exponentially, so eventually the velocity approaches the terminal velocity  $-\frac{mg}{b} \hat{\mathbf{k}}$  (straight downward with constant speed).