Essay on uncertainty

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Opening

THE subject of this essay might seem hopelessly vast and ungraspable, almost by definition. However, I do not attempt to discuss uncertainty itself, instead I will examine mainly the possible strategies by which one can cope with uncertain factors.

In general, it is very hard – perhaps impossible – to say or suggest anything meaningful or well-founded about the "unknown". If we take an honest look at the possibilities in a real-world scenario, then we have to conclude that we have no real good arguments by which we could rule out any shape into which reality might fold out[1]. We might have good heuristics – such as our own experience, scientific findings, and possibly other guiding principles – which would be faulty to disregard, but in my view, none of these has the power to bring certainty. (This does not mean, however, that operating in an uncertain world would be impossible. Organisms make decisions and act continually. Realizing that thinking can not bring certainty only emphasizes that acting in the real world is more of an art than a science.)

To overcome this both disturbing and liberating relationship with reality, it is safer to start discussing simpler, idealized decision-making problems for the sake of constructing a theoretical framework. Although these examples suffer from idealization, they hopefully can function as prototypes, leading to reasonable and useful heuristics. Adopting a self restrictive definition[2] of "probability" has been used several times to grapple with the concept. Games of chance, repeatable mass phenomena, situations where our experience allows us to formulate a degree of belief and the case of specific physical systems (such as chaotic/ergodic systems and quantum states) have all been used to restrict the domain of uncertainty for the sake of model making[3].

In this essay, I attempt to impose a little less restriction on the concept than in previous works. Because of the broader scope, the introduced framework can be viewed as the generalization of previous frameworks, even if it is still very far from a general theory of the uncertain.

Proposed framework

Probably the best way to introduce the proposed framework, is to give a simple and general example, together with the suggested strategy. Let us assume, the followings:

- $\star$ There is an "Agent" who is part of the "World". The Agent can only partially model the whole World, she can develop heuristics, and, most importantly, perform actions;
- The Agent considers only a finite number of possible actions. The set of actions will be denoted by $\mathcal{A}$;

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[1]Some might feel that I declared the solution to the "problem of induction" [69] too boldly. Let me try to clarify my statement by taking an imagined ideal casino as an example. I would make the same statement for gamblers playing Hazard or Craps in this casino. I cannot see any good argument that could rule out any result from a throw of a pair of dice. However, not being able to rule something out does not mean that there are no better and worse heuristics for gambling. Furthermore, it does not mean we can not have some advantage at the Blackjack table by, for example, counting cards.

In fact, the primary purpose of this essay is to introduce and discuss such heuristics, hopefully, applicable not only in ideal casinos but also in real-world scenarios.

[2]See Von Mises [110] for elaborating on "analytic" and "synthetic" definitions of "probability", referring back to Kant.

[3]I suggest the following references for the listed approaches (in the same order): *Philosophical Essay on Probabilities* by Laplace [8]; *Probability, statistics, and truth* by Von Mises [110]; *Probability theory: the logic of science* by Jaynes [43], *Truth and Probability* by Ramsey [131]; *Ergodic Theory* by Sinai, Cornfeld and Fomin [79]; *Mathematical Foundations of Quantum Mechanics* by Von Neumann [113].

- The Agent can (or is willing to) restrict the possible states of the world to a finite set. We will denote this set by Θ and call it the parameter set;
- Lastly, the Agent can associate utilities (or rewards) to all potential consequences, which depends both on her action and the state of the world. This function (in the finite case representable by table or a matrix) will be denoted by $U : \mathcal{A} \times \Theta \mapsto \mathbb{R}$.

Under these assumptions, the game-theoretic framework for decision-making and statistics suggests the following strategy for the Agent:

- Imagine that the unknown parameter $\theta \in \Theta$ has been chosen by an opponent whose utility function is the regret of the Agent;
- Determine the Nash equilibrium for such a two-player non-cooperative game: $\langle \sigma^*; \pi^* \rangle$;
- Adopt the equilibrium strategy σ^* of this imagined game to choose an action from the action set $\mathcal{A}$.

The primary goal of the following treatise is to elaborate on this procedure and show the viability of the above-suggested decision-making strategy.

Objections, defences and remarks

Discussion of the assumptions

First, I should discuss the assumptions of the proposed framework.

Separating the Agent from the World: How is it possible for the Agent to maintain a separate identity from the World? How do the Agent's preferences come to be? What does it mean to "choose" for the Agent?

In my view, these are all valid questions[4] , but in this essay, I will not attempt to clarify or investigate the emergence or the workings of the willing, thinking and acting faculties of the Agent from the "Naturalistic perspective". I will assume that such an Agent or Organism is given – or at least that this description can characterise some existing structures well. If I am honest, I need to admit that this is an idealisation, hopefully a useful one.

Where are the preferences coming from? In this framework, it is assumed that the Agent *has* preferences. Moreover, we assume that these preferences are characterizable by a utility function. To address the question, preferences may stem from or be shaped by various sources. For humans, they might arise from evolutionary pressures, cultural influences, psychological tendencies or conditioning, or any mixture of mentioned or further factors.

This framework can be viewed as a goal-agnostic way of addressing uncertainty in decision-making situations. This might signal that it aims to be "only" an effective theory. Still, I think this is a safer, pragmatic way to construct a framework that does not take firm ideological positions prematurely and can be refined by acceptable preferences separately. Finding and formalizing what "acceptable preferences" might mean seems to be a delicate matter, which is extremely important. However, because it can not get the deserved care and depth here, it has to be outside the scope of this essay. I can only hope that those who apply this theory, will adopt – by choosing or inheriting – their values and preferences wisely.

[4] Most philosophical and religious frameworks assume or recognise a concept of "self" [121, 152, 53] as a separate – or at least not identical – entity with the "World". (Several mystical branches arrive at a view of Unity (or non-duality). However, these frameworks are often considered as advanced – and often controversial – teachings that are not comprehensible or suitable for everybody or in every stage of life or role in society. A few related concepts might include: Nature as the one substance by Spinoza [155, 101], Godhead by Meister Eckhart [139, 58, 104], Wahdat al-Wujud often attributed to Ibn 'Arabi [25, 44, 26], Advaita Vedanta by Shankara [37, 72], Sunyavada by Nagarjuna [168, 167], etc.)

Even without believing in the independent, True existence of an individual "self" or agency, one can still find such a framework useful merely as an effective theory. I see the game-theoretic framework as a favourable plateau from which we can have a better view and which offers a more robust and more consistent framework before continuing our journey of understanding. For a possible path where individuals emerge – instead of being postulated –, see, for example, the paper of Krakauer et al. *The information theory of individuality* [90].

This framework aims to help navigate the uncertain world for those who are – or at least play the role of being – in the state of separation. However, I hope that suggesting strategies in imagined games will contribute more to enhance the overall conditions for enjoying the play, then solidifying the illusion of the game itself.

I think we can stay just as perplexed about the intricacies of the "self" (or Agents in our context) as Isaac Newton was on "causes hitherto unknown" [118, 117, 151] for action at a distance in his law of universal gravitation [1, 42]; unless a framework assuming individuals can go near the fruitfulness of the success of the inverse square law in celestial mechanics. By not excluding the possibility of further generalisation and reinterpretation of the theory as emergent, this hopefully does not prevent deeper understanding in the future.

Possibilities without probabilities: Some might question the third assumption, or more precisely, the lack of – subjective – prior probabilities associated with the possible states of the world: Θ.

This objection[5] is not easy to answer completely, but it seems to me that there are good arguments for why an Agent should have decision-making processes in her repertoire for cases when no prior is justified. I agree that if the Agent has plentiful credible experience or information about the possible states of the world, this should be incorporated and then continue with standard (for instance, Bayesian) decision-making process. However, I disagree with the suggestion that model-making always requires (or implies) a conditional prior, given all knowledge the Agent has. In the light of a more general framework, requiring a prior might seem like forcing the Agent to pick unfounded beliefs prematurely.

Insights from black-and-white thinking: Giving a definitive counter-example that shows the absurdity of requiring a prior from the Agent seems impossible to me. I could list more or less convincing examples, such as decision problems on other planets, in constructed virtual worlds, predicting the results of future physical experiments, which are outside of the domain of validity of successful present theories; but ultimately, none of these examples seems unobjectionable for me. Instead of searching and constructing more elaborate examples, I aim to show that even a Bayesian can not convince only by a simple counter-example, an Agent who is committed to black-and-white thinking.

Perhaps this more extreme requirement can illustrate how premature judgements can cause "hard to accept" decisions: Imagine that a "subjective logicist" believes that all propositions must ultimately be either true or false[6]. Therefore, model-making requires picking a definite non-contradicting set of truth values for propositions, i.e. believing and choosing one definite state from Θ. This might look like an extreme requirement and can lead to an extreme decision-making process, but formally, it seems just as hard to rule out as demanding a subjective prior [7]. Agents who follow a "subjective logicist" doctrine can sometimes win, and if they guess the state of the world correctly, they may be more successful than more cautious players. This example shows that even in extreme cases, there are no obvious arguments by which one could definitely rule out an Agent's decision-making doctrine.

However, there are weaker arguments:

- The "subjective logicist" framework can be reinterpreted as a particular case of a subjective Bayesian framework because if the Agent has very strong background knowledge, then her prior can be concentrated on one single state from Θ, and her choice of action become indistinguishable from a "subjective logicist's" behaviour (if both happened to believe in the same state).
- There might be situations where the doctrine of a "subjective logicist" "feels" extreme, and the definite choice of a state forced and premature.

I do not see much better arguments for considering a framework which does not demand the Agent to develop a prior for the possible states. One can show that in a multi-level game setting, as the amount of collected data from other rounds goes to infinity, the collected data can be incorporated into further decision-making as a prior, i.e. the Bayesian framework can be interpreted as a particular limiting case of the game-theoretic framework. Further, one can "feel" that there are some decision-making problems in which no specific prior is well founded, and a prior choice would seem forced and premature.

Finiteness: An objection against both the second and third assumptions about the finiteness of the sets of states and actions is that, sometimes, they are better modelled by continuous quantities. As a result, a finite framework might seem inadequate in such cases.

The main defense to this objection is that continuous models can be defined as appropriate limits of finite models. Being able to address discrete models can be viewed as an advantage of the framework compared to some "objective Bayesian" frameworks, such as parameterization invariant priors[8] which can be defined naturally on continuous parameter sets.

Of course, reaching a continuum limit from finite quantities can open new questions of convergence. There are general theorems guaranteeing the existence of a Nash equilibrium for continuous games[9], but there are examples of discontinuous continuum games with no equilibrium solution[10]. However, naturally

[5] Maybe closest to the subjective Bayesian position [97], most famously advocated by Bruno de Finetti [47].

[6] Also known as "Dichotomous Thinking" in psychology [3].

[7] One might think that whenever a "subjective logicist's" firm belief turns out to be false, she might need to give up her decision-making doctrine. However, she can simply admit that she has chosen the wrong axioms, which turned out to be inconsistent with the facts, but then she picks another non-excluded set of axioms, i.e. she can continue following her black-and-white decision-making process. This shows that the "subjective logicist" is not incapable of learning from experience, and this partial flexibility makes it harder to challenge her doctrine.

[8] Also known as Jeffreys prior [43, 82].

[9] See for example [56, 156] for reference.

[10] See "the game without value" as an example [39].

occurring examples seem to have well-behaving continuum limit[11] (even if the requirements of the general theorems on continuous games are not satisfied, usually one can find unique equilibrium points by direct calculation). Perhaps it is possible to construct pathological examples, but this can also be done in standard probability theory[12]. Exploring the sufficient requirements for a well-defined continuum model remains a task for future mathematical investigations, but it seems like a reachable goal which can be achieved gradually, just as probability theory as a mathematical discipline based on measure theory has been established and then further generalized to stochastic processes.

Assigning utilities: Introducing the concept of utility, and requiring the Agent to assign subjective utility to all possible outcomes, might seem like an added complication to statistics and a significant burden for the Agent.

However, the concept of utility is essential in most other decision-making processes[13], aided by statistics and/or probability theory. The standard assumption is that a "rational" Agent maximizes her Expected Utility[14], where the expectation is taken with respect to probabilities determined by statistics and/or a probabilistic/stochastic model. The decision-making process is typically modular. First, stochastic models, often supported by statistics, generate probabilities for possible states independent of the outcomes' utilities. Next, expected utilities are calculated as the weighted sum of the outcome utilities for all actions. Finally, the action with the highest expected utility is chosen.

To use probability and statistics for decision-making, one must determine the utility of possible outcomes. The reason why these concepts can decouple is the form of Expected Utility, which is defined as the simple bilinear expression of the probabilities and utilities ($\mathrm{EU}(a) = \sum_\theta U(a;\theta) \cdot \pi(\theta)$).

The main difference compared to traditional frameworks is that in the game-theoretic framework, the main focus is on the actions and strategies, not beliefs or asymptotic frequencies. In traditional frameworks, the estimation or calculation of beliefs and utilities can be decoupled, while in the game-theoretic framework, the whole decision-making process is interpreted as one unified strategy.

Discussion of the suggested strategy

After discussing the assumptions, we can take a closer look at the suggested decision-making process.

Imaginary opponent: Imagining that the unknown states are chosen by a strategic player is perhaps the most controversial part of the suggested strategy. It might seem like an unfounded metaphysical claim or a form of animism[15].

However, this imaginary opponent is not something the Agent has to wholeheartedly believe or accept. It is only a metaphor, an analogy, which can be adopted—not because of any claim about the unknown, but because such construction can provide "reasonable" and generalizable strategies.

To put this aspect into perspective, it is good to notice that no other frameworks are free of metaphysical claims and constructions either. The classical theory of probability postulates that all possibilities have to be assigned with the same degree of belief – known as the principle of insufficient reason – and often introduces idealized devices of chance (such as coins and dice), which are embodiments of stochastic behaviour. (While the associated probabilities are often determined by symmetry arguments.) In the frequentist framework, it is often postulated that some experiments have inherently stochastic behaviour and that no pattern can be found in their outcomes. The possibility of infinite repetition, the assumption of independence, and the convergence of frequency ratios are all further very strong claims and assumptions appearing in this framework. In the Bayesian framework, the concept of prior has an unspecified interpretation and might be viewed as a non-strictly empirical component of the theory.

Potential values of science: Why can't we rely solely on objective scientific methods to determine truth and then use that knowledge to guide us in the best possible direction?

The scientific worldview favours objective formulations of "natural laws" and "natural phenomena" that are not biased by the observer's beliefs or values. Deducing patterns and principles that are not subservient to popular ideologies has been a very successful project in various fields of science, but science alone can not advise Agents how to act. In my view, the practical application of scientific knowledge

[11] To see how game-theoretic equilibrium calculations can be performed on continuous spaces, I suggest reading the following papers [84, 85, 103, 4].

[12] To see a collection of counterexamples see [157, 50, 148].

[13] Modern decision theory is concerned with "goal-directed behaviour in the presence of options" [63].

[14] More on Expected Utility theory see [20].

[15] Animism is not meant to be a derogatory term here. Animistic belief systems assumes independent personality and motives to non-human beings [67], while in the proposed framework the "motives" of the unknown depends only on the Agent's preferences.

necessitates the incorporation of personal values; therefore, a purely scientific, value-free decision-making process is hardly imaginable.

Moreover, to a large extent, science is an empirical discipline that relies on real-world data to construct models or match parameters. Statistics is the bridge, which is used to connect the finite amount of observation with abstract values, by separating the "noise" from the "phenomena". Because of this unique position, connecting the complex and mysterious real world with the constructed abstractions of the scientist, a framework of statistics and probability might be judged differently than abstract scientific theories. (Simply put, if we need statistics to test scientific theories, then how could we test statistics itself?) Therefore, it could be acceptable – or even be necessary – that such frameworks contain metaphysical elements, including values or utilities.

However, it might be possible to keep the scientific project separate from subjective values and ideologies if we spell out scientific values. A natural scientific value might be a form of simplicity (quantitatively the length in bits of the model and the data[16]), predictive power (the success rate of guessing the outcome of not yet performed experiments[17]), and possibly other information-related measures. In this way, science does not need to maintain that it is value-free, but it has to spell out its ideology-free "neutral" or "scientific" values.

A family of "natural" utility functions has been adopted and analysed in a separate, more technical, and more mathematically oriented manuscript[18]. Because of the obtained results – which seem robust and reasonable – these could be suggested as default utilities for a standard statistical procedure and might be good candidates for "neutral" or "scientific" utility functions.

Why scientific values matter? To illustrate why choosing values wisely is important, and how they can change behaviour lets consider an example:

Imagine a mathematician whose aim is to list as many true theorems with exact proofs as possible. Her most effective strategy is to take a non-contradicting set of axioms Γ, and then apply the *rules of inference* (or *logical axioms*) in a *deductive system*[19] to generate true theorems together with their proof in a mechanistic way.

Surely this can be turned to a theorem and proof producing factory, but this activity would hardly resemble what real mathematicians do. The actual behaviour of mathematicians can be understood only if we assume that they see utility not only in the sheer number of true theorems and valid proofs, but that they have a taste i.e. theorems and proofs can have nontrivial values beside correctness[20].

Regret over negative utility: A further important detail in this metaphysical construction is that we assume that the imagined player controlling the unknown parameters "feeds on" the Agent's regret. The Agent's regret[21] associated with a specific outcome $(a;\theta)$ is the difference between the utility of that outcome and the highest possible utility for the same parameter θ. Formally: $R(a;\theta) = \max_b U(b;\theta) - U(a;\theta)$. Historically, Abraham Wald pioneered a construction where "Nature" – the player controlling the unknown parameters – is in a zero-sum game with the "Experimenter"[22] – the Agent in our framework. One might ask: Why is it better to imagine that an opponent is preferring to maximize our regret instead of our negative utility?

The difference between regret and negative utility might seem small, but in some cases, this difference can cause a dramatic change in behaviour. The problems with a minimax decision theory were pointed out by Leonard Savage, who half-heartedly suggested the use of regret instead of utilities[23].

To illustrate the difference between zero-sum and regret-based attitudes, let us consider a fictional toy example: Imagine a prehistoric band searching for a new home in Europe or Asia. They have wandered for a long time, reaching a different habitat from which they came. Once they discover a cave opening that might be suitable for the group. However, they know that cave bears exist, which prefer the same kind of caves suitable for humans, and that they vigorously defend their territory against intruders[24]. Fighting a cave bear is not hopeless, but it takes lengthy and costly preparation, and the possible encounter is inherently dangerous and risky. The decision-making problem of the band standing at the cave opening can be simplified in the following way: The band has to decide whether to start a costly preparation for fighting a cave bear or just quickly send in a small group without wasting any precious time. The

[16]This is the core idea of the Minimum Description Length (MDL) principle [59, 124].
[17]This aim coupled with algorithmic probability [163] can lead to the Solomonoff induction formula [132, 77].
[18]See the paper on Statistical Games [89] and for further resources the related GitHub repository.
[19]For reference see for example [147, 111, 52].
[20]For more details on what these values might be, see the paper of Terence Tao on *Good mathematics* [159].
[21]Regret is well known concept in decision theory [108], economics [99] and game theory [87].
[22]The concept was most explicitly described in Wald's 1950 book, *Statistical decision functions* [165] which is a highly technical and abstract introduction to the topic.
[23]See Savage's book *The Foundations of Statistics* [142] for authentic details.
[24]The story is fictional and allegoric, but cave bear encounters might contain a grain of truth [14, 93, 94].

outcome of their choice naturally depends very much on the presence or absence of a cave bear. There are four possible outcomes for which they can associate subjective utilities[25]:

Band \ Bear	Present	Absent
Prepared but slow	Bad (-10 ℔)	Inconvenient (-1 ℔)
Swift but unprepared	Very bad (-50 ℔)	Convenient ($+10$ ℔)

Table 1: Utility matrix of the band. Utilities are estimated in utils (denoted as ℔).

Now, let us see what difference in behaviour it would make if a band imagines a zero-sum game or assumes that the unknown factors are chosen by a force that feeds on their regret.

Zero-sum band and Regret band: The Zero-sum band's conclusion and behaviour are straightforward to determine with standard game theoretical argumentation. From the cave bear's point of view – as the Zero-sum band imagines it – being Present is always better than being Absent because it is worse for the band regardless of the band's action. In technical terms, being Absent is strictly dominated by being Present for the cave bear. If we know – or believe – that the bear is surely present, then we conclude that we always have to Prepare. So, for the Zero-sum band, their metaphysical construction leads to a strategy where they will always prepare to fight the bear.

Now, let us see what a band focusing on Regret would do.

Band \ Bear	Present	Absent
Prepared but slow	Best action if the bear is Present (0 ℔)	Inconvenient instead of Convenient (11 ℔)
Swift but unprepared	Very bad instead of Bad (40 ℔)	Best action if the bear is Absent (0 ℔)

Table 2: Regret matrix of the band. Regrets are determined as $R(a, \theta) = \max_b U(b, \theta) – U(a, \theta)$, $\theta \in \{\text{Present}, \text{Absent}\}$, $a, b \in \{\text{Prepared}, \text{Unprepared}\}$; and are measured in the same utils as utility (denoted as ℔).

In an imagined game, where the band is trying to maximize its expected utility $U(a, \theta)$, while the bear is trying to maximize the regret of the band $R(a, \theta)$, has only a mixed strategy equilibrium, where both the band and the bear are choosing their action randomly. A straightforward calculation[26] shows that in equilibrium, the band will more probably Prepare with a 78.4% chance and will rush into the cave only with a 21.6% chance.

The equilibrium calculation also determines the bear's strategy; however, the band's own strategy is of primary importance, and the behaviour of the imagined bear is secondary (if not completely irrelevant). In equilibrium, the imaginary bear is present with probability 21.6% and absent with 78.4%.

Exploring some variations: To see more clearly how the strategies of the bands match our intuition, let us consider a varied example in which the unprepared bear encounter has much worse consequences:

Band \ Bear	Present	Absent
Prepared but slow	Bad (-10 ℔)	Inconvenient (-1 ℔)
Swift but unprepared	Catastrophic (-500 ℔)	Convenient ($+10$ ℔)

Table 3: Utility matrix of the band in a more dangerous situation. Utilities are estimated in utils (denoted as ℔).

We could repeat the same steps which we made previously and arrive at the following results: The behaviour of the Zero-sum band does not change; they will always Prepare and enter slowly. The behaviour of the Regret band changes: now they will prepare with a much higher chance, with probability 97.8% (and rush into only with 2.2% chance).

Comparing the bands' behaviour: The difference between these two approaches might not be striking at first, and some might even prefer to belong to the more cautious band because it is "better to be safe than sorry"[27].

However, the overly cautious nature of the Zero-sum band becomes apparent when we consider that before entering the cave, band members can search for clues and evidence about the bear's presence or

[25] Utility has no rigorously defined and widely accepted unit, however it is common to measure it in "utils". There is no need to introduce an absolute scale of utilities in the game theoretic framework, because any positive affine transformation of utilities leaves the equilibrium strategies invariant. However, for internal purposes we introduce a util unit, which we will denote by ℔ symbol. The symbol stylized version of the abbreviation for "libra pondo" [74] for which the original notation morphed gradually to Octothorpe.

[26] Which can be calculated on paper, or alternatively computed by standard computational software, such as Online Bimatrix Game Solver, Nashpy, Sage or Gambit etc.

[27] As a well known proverb suggests [102].

absence. If there are no tracks, fur, or other typical clue, then the chances of a bear encounter decrease. However, there is no consistent mechanism or argument on how this could alter the Zero-sum band's belief that there must be a bear in the cave. No matter the weak probabilistic evidence, until there is no proof that finding a bear in the cave is impossible, they will remain in the same mindset and prepare for the encounter. Technically, this means that a band with such metaphysical construction can not incorporate new evidence and change its strategy accordingly, i.e., it can not learn.

On the contrary, the band, assuming that a trickster is trying to cause them regret by the unknown parameters, will be able to incorporate such side information (given that they can connect weakly or probabilistically the clues with the presence or absence of bears) and change their strategy accordingly. Even if they can not prove that there is no bear in the cave, enough amount of evidence can convince them, that they can choose preparation only very rarely.

To illustrate the difference, let us take a fictional example: The band finds a cave in a snow-covered territory. They know the last snowfall was one week ago and that cave bears are not hibernated yet. They carefully approach the cave opening and find no tracks or evidence of a large animal. Even if the elders know that for a cave bear, not leaving the cave for a week before hibernation is very improbable, the two different bands would have very different strategies. The Zero-sum band is convinced that there is a cave bear because this would harm the band most, so they prepare before entering. While the other band, assuming a trickster-like opponent interested in their regret, will incorporate this new information, enter more swiftly, and make the costly preparation only when they "feel exceptionally unlucky".

Conclusion: What can this fictional story teach us? I think the strongest message is that subtle differences in metaphysical constructions can dramatically change agents' behaviour. Some constructions can be safer, but result rigid beliefs derived from pessimism, which prevent learning and adopting to a less dangerious situation. Other constructions can promote more risk-taking; they generally require randomized or mixed strategies and can change their behaviour if new information is available.

Being adaptive or able to learn can be a meta-requirement for policies, but this does not mean that such policies will perform better in all environments. Learning requires making mistakes, and it can happen that an adaptive Agent needs to pay more during her adaptation than a rigid player, who happened to choose a beneficial strategy from the beginning. (In biology, for example, both approaches can make a species successful: they can be adaptive on the individual level by learning and accommodating, or individuals can follow rigid action patterns, which can be changed only by mutation and recombination of genes in the next generation. Naturally, in real organisms, both can be present, but the proportions and importance can vary.)

Compatibility with Bayesian decision making: An important aspect of a formal system is the symmetries it respects. It is natural to ask: What symmetries does the suggested framework have?

Game theoretic decision theory and Bayesian decision theory do not suggest the same strategies in general, but curiously, the two frameworks respect the same symmetries.

Assume we have a Bayesian collective, or jury, where each member has a different prior π_α, with α possibly being a continuous index for the decision-makers. The prior is a probability distribution on the world's possible states, i.e., the parameter space Θ. Bayesian decision-making suggests that an Agent should determine the expected utilities by weighting the utilities for any action with the prior probabilities of the possible states: $EU_\alpha(a) = \sum_{\theta\in\Theta} U(a;\theta)\cdot\pi_\alpha(\theta)$. Then, simply choose the action that yields the highest expected utility.

It is easy to see that if we add a constant to every utility in the $U(a;\theta)$ matrix or multiply the entries by a positive number, the chosen action remains unchanged for every Bayesian decision-maker α. We can phrase this observation in a way that Bayesian decision-making is invariant under positive affine transformations of the utility matrix, i.e. this is a symmetry of the decision-making process.

A less obvious symmetry transformation is the addition of action-independent constants to the utility matrix: $U'(a;\theta) = U(a;\theta) + c(\theta)$. This changes the expected utilities only by an action-independent constant: $EU'_\alpha(a) = EU_\alpha(a) + \sum_{\theta\in\Theta} c(\theta)\pi_\alpha(\theta)$. This constant shift of the expected utilities does not change, which action maximizes the expression for any Bayesian agent. This means that if two utility matrices can be made equal by adding action-independent constants and multiplying by an appropriate positive number, then for Bayesian decision-makers, these two decision problems are equivalent. If a Bayesian decision maker chooses action a for one utility matrix, then she will choose the same action a for an equivalent decision problem.

Remarkably, the game theoretic decision-making doctrine respects the same symmetry group, i.e. game theoretic decision makers, imagining regret maximizing fictional opponents will use the same strategies for decision-making problems, which are equivalent for Bayesian decision makers. (The strategy of an Agent following the game theoretic doctrine and Bayesian decision-making is not going to be the same always, but for two problems U and U' if all Bayesian decision-makers act in the same way, then game-theoretic agents would also follow the same strategy for U and U'.)

It is important to point out that this property is not automatically satisfied for fictional two-person games. An Agent imagining a zero-sum game would not realize the same symmetry (only a restricted symmetry, the positive affine transformation group).

Why Nash equilibrium? Some may object to using Nash equilibrium[28] as the solution concept for the imagined game.

To take the objection against Nash equilibrium seriously, it might be useful to list a few alternatives:

- An interesting alternative might be the so-called Berge equilibrium[29], which can be viewed as a formalization of the "golden rule", assuming that the agents are more concerned about the other's expected utility than their own.
- Another alternative might be the "logit equilibrium" (which could be alternatively called a Boltzmann-Nash equilibrium)[30], in which expected utilities play the role of (negative-) energy, while a temperature-like factor can be used to tune the randomization of the actions. (In general, the "temperatures" could be different for different players.)
- Correlated equilibrium[31] could be another candidate to replace Nash equilibrium solutions.

There might be myriad ways to define and formulate an equilibrium concept, however this does not alter the general arguments for choosing Nash equilibrium.

Simplicity: The first argument is that Nash equilibrium is a simple concept. It is a pair of (possibly randomized) strategies in which no player can win more by unilaterally changing her strategy.

Being descriptive in biology: It seems to describe sufficiently well a wide variety of biological and animal behaviour[32], signalling that biological organisms might have heuristics and neural – or other kind of – faculties which can understand and react to "games". Thus, it can be expected that an organism with a long evolutionary history reached an evolutionary stable Nash equilibrium point. (Technically, evolutionary adaptation and adaptation using cognitive skills differ, but Nash equilibrium can similarly describe the observed behaviour in a given situation.) This argument strengthens the case to adopt Nash equilibrium as a descriptive theory, but if such decision theory can be formalized and has "desirable properties", then it might serve as a natural starting point for decision-making doctrines.

Sound consequences: However, in my view, the most important feature of an equilibrium concept is the decision-making behaviour it induces. This will be analyzed in more detail in the followings.

Defense of choosing Nash equilibrium: None of these properties excludes other equilibrium alternatives; however, it is reasonable to start constructing a theory from the simplest building blocks and complicate the framework only if we find its consequences unacceptable[33] (or maybe if alternative frameworks might have much greater descriptive power or desirability). In the current stage, no serious internal inconsistency forces us to abandon the concept of Nash equilibrium, and no other alternative seems to be much more general or compelling.

Randomization in mixed strategies: What does randomization in mixed strategies really mean, and how can the Agent take a randomized action?

I think this question can lead to very complex concepts, but it could also be answered very simply: random acts are what we expect and do when we play rock-paper-scissors (or alternatively Matching Pennies[34]). When the rules are set for such simple games, then even before the first round, we have a belief about what we expect from our opponent. I suggest promoting this expectation for the yet unknown choice to the definition of a random choice. On the other hand, for our own strategy, randomization means that we might follow a very complex algorithm, incorporate data from external sources, or take

[28] Nash equilibrium is a central concept of Game Theory [87, 107, 31, 115], corresponding to a set of strategies in which no player's expected outcome can be improved by changing one's own strategy.

[29] For reference see [109]. A weakness of this solution concept is that it does not always exist [129].

[30] To give references from the scientific literature I might point to [7, 173] to provide starting points for further exploration of the concept.

[31] See [10] for foundational reference and [87] for a concise introduction.

[32] The discipline applying game theory to biological (or other replicating) systems is called Evolutionary Game Theory [153, 160] (for publicly available resources see [5, 36]).

[33] Ironically the strongest criticism of "Rational Choice Theory" comes from sociology [18], despite that historically game theory and utility theory was constructed to model human behaviour which believed to be "rational". There is criticism coming from economics as well; an alternative approach being Behavioral economics [161, 38] and Bounded rationality [169]. In this essay, I do not attempt to reconcile concepts from game theory with real human behaviour, but maybe some of the difficulties could be explained by assuming that humans are not "elementary Agents" characterizable with a single utility function but "composite Agents" hosting a multitude of action influencing elements or aspects (which all can have different, often conflicting preferences).

[34] Also known as "Guess The Hand" or "Hand Game" [35, 164] which can be traced back to prehistoric times.

any necessary measures to make our choice of action as unpredictable as possible. Strictly speaking, we have two different concepts: randomization and expecting a random choice.

I think the concrete realization of randomization is the one which leads to deeper questions. In practice, agents or players will use various methods to choose from their actions in games that have mixed equilibrium. This might include pseudorandom numbers[35] or the result of any complex computation or algorithm[36]. However, what might look complex for one player might look regular for another, and if the regularity can be spotted, then it can be exploited. Are there sequences that can not be exploited? Or, more precisely, are there sequences of actions that would discourage any opponent from searching for patterns and expecting to take advantage?

Complex versus random sequences[37]*:* Personally, I think that no realized finite sequence has this discouraging effect because for any finite sequence, we can find a finite algorithm (or computable program) which matches the known sequence perfectly and can continue indefinitely. Naturally (and provably), most sequences need a long algorithm to match exactly[38]. However, I do not see any objective criteria which might separate "regular sequences" from "random sequences", at least in the finite case.

If there would be a criterion, let's say in the form of a test T, applicable to n-long binary sequences, then this would lead to a little paradox: Let's say we can bet on the outcome of an n-long random sequence. If randomness means that the sequence needs to belong to the subset of all possibilities where T is true, that would exclude some "regular sequences" and increase the bettor's winning prospects. I think this clearly shows that if you want to play Matching Pennies, you must not exclude "too regular sequences" because this can make you more predictable.

The situation might be different for infinite sequences. In the infinite case, non-computable sequences such as Chaitin omega number(s)[39] can be defined, which might discourage any opponent from taking advantage. Personally, I think this is a fascinating branch of mathematics, but I do not think it is necessary to think and reason about constructions and strategies involving randomness.

Beside performing complex computations, the Agent might also be able to collect "high entropy" data from its environment[40], and use it directly, or as a seed (or salt[41]) to randomize. However, there is no unquestionable external source of randomness either[42].

Therefore, I suggest grounding the concept of randomness in our experience with simple games. Allowing the agents to perform the randomization as they wish.

Correspondence principle and limiting cases: Does the suggested framework align with standard decision-making processes in various limiting cases?

There are several viewpoints from which the behaviour emerging from a metaphysical construction can be analyzed and evaluated. An important test for any theory with a broader domain of validity is to explore its limiting cases, where we expect to get back the results of previously successful, narrower frameworks. The suggested decision-making framework has several limiting cases:

1. A case when the utilities of some consequences go to extreme values;
2. The limit where the number of possible states of the world goes to infinity;
3. The case where the number of possible actions for the Agent goes to infinity;
4. The limit where the amount of available "relevant data" goes to infinity.

A few general meta requirements can be collected, such as:

[35] See a standard reference by Knuth [88].

[36] The concepts of "computational irreducibility" [171, 172, 170, 80, 144, 81] and the "no-shortcut principle" [91, 98] can give further insight to what can be produced even by classical computation.

[37] "Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin. For, as has been pointed out several times, there is no such thing as a random number – there are only methods to produce random numbers, and a strict arithmetic procedure of course is not such a method." as Neumann stated in [114].

[38] Technically this means that most strings have high Kolmogorov complexity i.e. they are not well-compressible [163].

[39] For definition and reference see [22, 21, 163, 135]. For a relatively concise survey see [12].

[40] There were several methods, sometimes in the form of divination [112] which were used for – random – decision-making in prehistoric and antique times. Books have been published containing high-quality random numbers [32, 34], websites are dedicated to generate random numbers [62, 95, 166], – even lava lamps are standing guard to enhance entropy if needed [27] – addressing the need for randomization.

[41] In the context of cryptography [2, 60].

[42] Maybe quantum measurements can be mentioned as the strongest candidates for the source of "true randomness". It is random in the context of the Copenhagen interpretation of Quantum mechanics, but there are several other interpretations and for instance non-locality could make the requirement of randomness unnecessary (conspiracy can never be totally ruled out [73]). To date Quantum Random Generators [70, 125] are one of the most reliable entropy sources, but their status is theoretically not unquestionable.

Evade catastrophes: A natural requirement for the 1. case might be that an Agent should avoid actions that can lead to catastrophic outcomes (if there are other actions that do not bring such calamities).

The game-theoretic framework seems to fulfil this requirement. Firstly, by excluding "dominated strategies" (actions for which there is another dominating action or strategy which performs better in any possible state of the world). Secondly, it can be proven that even if an action is not dominated, but there is a state of the world in which case the utility of the outcome converges to negative infinity, then in equilibrium, the probability of choosing this action converges to 0. (This phenomenon can be spotted in the previous bear story when the consequence of an unprepared bear encounter changed from "Very bad" to "Catastrophic".)

Continuouty and Smoothness: Much weaker requirements can be listed for the 2. and 3. cases: Reaching a well-defined continuum limit might be a desirable property of a metaphysical construction (if the Agent acquires well-behaving utility functions in the limit). However, continuity or smoothness does not seem to be a property, which, in its absence, could exclude metaphysical constructions. Analytic properties might be reassuring but not essential.

It has been shown in a previous more technical paper that the calculation of an equilibrium with continuous action space is feasible[43]. There are other works demonstrating that handling continuous state spaces is also possible[44].

Correspondence with Bayesian and "Frequentist" decision making: The 4. limiting case can give much stricter requirements for acceptable behaviours in the face of uncertainty: This is where the general correspondence principle[45] can be effectively utilized to connect the proposed framework with decision-making under risk (where the Agent can associate probabilities with the possible states of the world). In particular, it seems a reasonable requirement that if the amount of "relevant data" goes to infinity, the behaviour should converge to both "Frequentist" and Bayesian decision-making strategies.

It has been shown that – at least in a simplest toy model – this decision-making framework can mimic both "Frequentist" and Bayesian methods by tuning a parameter of the utility function which can be associated with relative risk aversion[46]. In the limit where the amount of data goes to infinity, the Bayesian version of the betting game can be characterized by a subset of priors in the context of Bayesian decision-making, while the risk-neutral betting (or alternatively the scenario guessing "Fisher game") approximates a decision-making process based on maximum likelihood.

A nuanced question in the case of convergence is the metric – or topology – in which we investigate convergence. There might be metrics in which two heuristics are convergent, while in other metrics they might be divergent. For example, the equilibrium priors for Bayesian betting games do not converge to any specific prior for a general Statistical Game as the amount of observed data goes to infinity. However, if we compare heuristics by measuring their success rate, i.e. their expected utility for some distribution of scenarios, then in the limit, all Bayesian decision makers (with non-extreme priors) can expect the same 100% success rate. Comparing a Bayesian bettor using the game-theoretic framework to any Agent using orthodox Bayesian decision-making (with a non-extreme prior) shows convergence in this metric. A more suitable and sophisticated technique to measure the "distance" (or "divergence"[47]) of two decision-making heuristics might be the "Delegation premium" or alternatively the "Independence tax". This could measure how much tax a decision maker with given preferences would be willing to pay to follow her own optimal heuristics instead of an alternative decision-making process. Using this distance, it seems that the game theoretic decision-making – with logarithmic utility function – converges to any Bayesian decision-making (with non-extreme prior) as the amount of observed relevant data goes to infinity, i.e. game-theoretic Agents would pay less and less tax to use their own strategies instead of Bayesian methods (with some fixed non-extreme prior). The asymptotic comparison of decision-making strategies is an important and interesting future direction, but the first results seem promising.

One dimensional, real-valued utilities: Is it enough to associate one real number to consequences measuring their "utility"?

I think the claim that a one-dimensional scale to compare consequences might not be sufficient could be taken seriously. Salty food might make us less hungry but more thirsty, or another outcome might

[43] See the technical paper on Statistical Games [89] and for further resources the related GitHub repository.

[44] For examples see [84, 85] and an example in a slightly modified setting can be found in [103, 4].

[45] In theoretical physics the correspondence principle was most famously utilized by Bohr in the context of quantum theory [17, 138], however as a general principle it can be applied to any new theories: "The most important heuristic restriction is the General Correspondence Principle. Roughly speaking, this is the requirement that any acceptable new theory L should account for the success of its predecessor S by 'degenerating' into that theory under those conditions under which S has been well confirmed by tests." [61].

[46] For details see [89] and for references for power utilities, also known as isoelastic utilities, and relative risk aversion parameter see [99, 9, 128].

[47] As in statistics or in information geometry [6]. A well-known example is Kullback–Leibler (KL) divergence, also known as relative entropy [119, 6, 92].

increase our reproductive chances but decrease our safety. It could be conceivable that we associate separate "flavoured utilities" with several "aspects" that might be important for an Agent. This might reflect an Agent's internal/psychological states more faithfully, but it seems hard to derive strategies based on such multidimensional utilities.

This conflict between actionability and personal experiences might be bridged by assuming that decision makers (humans, for instance) are not "elementary Agents" (which can be characterized with a one-dimensional utility function) but "composite Agent", which are composed of multiple elementary Agents having different utilities choosing their own internal actions[48]. In this framework a composite Agent with external action set $\mathcal{A}$ and consequence set $\mathcal{C}$ (and consequence matrix $C : \mathcal{A} \times \Theta \mapsto \mathcal{C}$) could be described by $\aleph_1, \ldots, \aleph_n$ internal agents, a decision rule (a mapping from the internal action profile to the external actions, $D : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \mapsto \mathcal{A}$) and separate elementary utility functions for each internal agents $u_i : \mathcal{A}_1 \times \ldots \mathcal{A}_n \times \mathcal{C} \mapsto \mathbb{R}$. This construction might capture a "composite Agent's" internal life and can yield a framework in which different aspects can potentially play a role in determining the final strategy of a non-elementary or composite Agent.

Capricious behaviour: In this framework, the seemingly capricious behaviour of some Agents could be understood as a composite Agent composed of highly non-aligned elementary agents and non-authoritarian decision rule.

Dual descriptions: This might open an interesting direction for understanding real-world Agents. One could start to search for a finer and finer "dissection" of composite Agents to elementary Agents. Although this is not unheard of in Philosophy, Religious metaphysics, and Psychology, associating agency to smaller and smaller constituents of organisms is not the "Naturalistic" or "Materialistic" viewpoint. (There are, however, framings which have a similar flavour, such as the Free will theorem by Conway and Kochen[49] (although not associating preferences to elementary particles).) I am not convinced that such an animated description of the world and decision-makers is the "right" approach toward a unified theory. However, it is conceivable for me that it might yield a dual description besides the Naturalistic framework. (By duality, I mean that a dictionary between two theories or frameworks can be found that can describe the "same thing" in a "different language"[50]. However, some things might be more suitable for one language, and other things might be more suitable for another.)

In "continental philosophy" (called this way by "analytical philosophers"), phenomenology advocated by Husserl and Merleau-Ponty[51] might be considered to be such a dual description. (In natural sciences, Ernst Mach's *Analysis of Sensations*[52] might also belong here.) Starting with the Agent's Experiences, referring to "Natural laws" only as organizing principles. I think this can be a perfectly valid dual understanding of the world, where reality is defined as an equivalence class of subjective experiences. I do not see a huge difference between an Agent adopting a Naturalistic or Phenomenological viewpoint because I do not think this would necessarily cause dramatic differences in the Agent's behaviour. (I think actions are what really separate metaphysical constructions, not the actual beliefs (or the chosen language to articulate these beliefs).)

Randomized decisions: Some may feel uneasy about a theory that sometimes suggests randomized decision-making[53].

I think that possible aversion against randomization can be addressed differently if it is present naturally in the reader or if the criticism comes from academic circles.

Arguments for randomization: In games, randomization is essential. Not just because it makes games more fun, but because adopting mixed strategies offers real strategic advantages. Our choice of actions in games can be understood as decision-making. Therefore, we might find a special case – when we know that we face an adaptive opponent – in which randomization is justified and familiar.

Another argument in favour of randomization comes from statistics. Fisher and many others advo-

[48] Similar concepts are not new in psychology, usually referred to as "psychoanalytical" [33, 28] or "psychodynamical" models. However, these models (versions proposed by Freud, Jung, Adler, Lacan, Erikson, etc.) introduce claims and structures that are often more pervasive than factual. (In non-western traditions, the concept of "Tantra" [45] can be viewed as a somewhat related understanding of body and mind.) With the proposed theoretical building blocks, I try not to commit myself to a fixed inner structure of "drives", "desires" or mechanisms governing how these form, change or extinguish, but give a general template which can be populated and fine-tuned based on empirical evidence. This is, of course, not an easily falsifiable theory but a general framework. (In itself, it is relatively empty and general, such as a statement: "human behaviour can be better understood by casting individuals to categories" would be.) It could become a scientific theory if a concrete structure would be picked.

[49] See further details in the following papers: [29, 30].

[50] For more context and examples for dualities in Physics and Mathematics see [120]. (A prominent example in theoretical physics is anti-de Sitter/conformal field theory (AdS/CFT) correspondence [127, 13].)

[51] For reference see [150], and the *Phenomenology of Perception* by Merleau-Ponty [105].

[52] See [126, 100] as reference.

[53] See an early criticism from Chernoff [24].

cated strongly[54] for actively selecting random samples from the population in question, therefore going beyond the role of a passive observer. (There are, of course, counterarguments[55], but still, randomization is an essential part of mainstream statistics.) Statistical decision-making can be embedded into general decision-making simply by including the sample selection in the possible actions and identifying all possible configurations of the population with the set of states of the world. (The action set and the set of states can become enormously big – due to combinatorial factors – but still finite.) Therefore, if we accept randomization in statistics, then we accept randomization in – at least some – decision-making processes.

An attitude against choosing actions sometimes randomly might come from the assumption that "there always has to be one best option". Maybe there is, but we might have to be omniscient to know which of our actions will be the best one. However, decision-making in the face of uncertainty is the opposite of being omniscient, therefore we have to approach the problem differently as an all-knowing being would.

Academic criticism: It seems to me that the aversion to randomized acts in academia is mainly due to historical reasons[56], going back to the works of Chernoff and Savage in 1954. The aversion seems to be related to the axiomatization project of preferences under uncertainty because a certain set of axioms (or postulates) imply that: "We shall now consider the implications of the theorems of the preceding section. Theorem 1 states that if a rational solution exists for our simplified formulation, randomized strategies are unnecessary." as Chernoff writes in his 1954 paper[57]. I think the assumption – elevated to the level of axiom – leading to this view is the exclusion of so-called *Menu dependence*, which will be discussed in detail in the next paragraph.

However, in my view, framing decision theory as a study of preferences – i.e. assuming a pairwise preference relation from the Agent on the set of actions (or possibly mixed strategies) – is not the most suitable conceptual frame and is prone to demand or reject properties which might be natural for preferences, but overly restrictive for realistic decision-making. Untangling the web and history of various popular "axioms" for decision-making appearing in scientific literature and reevaluating the feasibility of these axioms and their consequences is a laborious project, which is out of the scope of this essay.

Menu dependent choices: One might fear that the suggested framework might not satisfy the requirement of the so-called "independence of irrelevant alternatives"[58].

Prototypical games: If decision-making is an art, then why would anyone want to construct a theory out of it?

Even if every real-life situation is different, there might be some typical decision-making problems that occur frequently and can give good starting points for developing our own heuristics that suit our own situation. I will mention three main branches of prototypical decision-making problems and one simple modification generalizing the Target (which might be in a very different domain from which we gathered the Data or where the parameters of the model are understood).

While I vigorously defend the Agent's right to define her own utility function in a way she finds most appropriate, I think it is illuminating to list a few default utility functions and make suggestions about "natural", "neutral" or "scientific" utility functions which might be suitable for "purely scientific" inquiry.

Contest view: ...

Reinforcement learning as a testing ground: ...

[54]For an original reference see [48] and for a retrospective paper see [146].

[55]See for example Jaynes [43] arguing against randomization.

[56]The academic environment is good in preserving ideas, so much so that sometimes erroneous knowledge or outdated theories keep reappearing in the literature. For a famous example see [71, 46].

[57]For reference see [24] and see Uzawa's refinement from 1957 [162].

[58]Savage wrote in 1954 [142]: "Suppose, to take a striking case, that one f_r, say $f_{r'}$ is the unique minimax for the narrow problem and a different one, $f_{r''}$ is the unique minimax for the wide problem. It is absurd, as Chernoff says in effect, to recommend $f_{r'}$, as the best act among the f_r's when only the f_r's are available and then to recommend $f_{r''}$ as the best for an even wider class of possibilities. Fancy saying to the butcher, 'Seeing that you have geese, I'll take a duck instead of a chicken or a ham.' ".

Further, Chernoff in 1954 introduced an axiom (or postulate) for decision making excluding such choice functions [24].

Concepts and their Interpretation

Probability

Game theoretic interpretation: The interpretation of probability in the suggested game theoretic framework is intimately intertwined with the concept of randomization and mixed strategies.

> *Mixed strategy can alternatively be viewed as the belief held by all other players concerning a player's actions. A mixed strategy equilibrium is then an n-tuple of common knowledge expectations, which has the property that all the actions to which a strictly positive probability is assigned are optimal, given the beliefs. A player's behavior may be perceived by all the other players as the outcome of a random device even though this is not the case.*
>
> — Ariel Rubinstein[59] 1991.

Although in standard introductions to game theory, the concept of mixed strategies is built on the historically earlier formalisation of probability theory, and standard devices of chance such as dice and coins are used to ground the concept, in my view, the direction of grounding could be reversed. I suggest using simple games as the starting point where probability and randomization emerge, and viewing other concepts in probability theory as abstractions of these game-theoretic situations. For example, the concept: "a random choice where the probability of choosing Heads or Tails is $50\% - 50\%$" could mean a process that an agent would consider when playing Matching Pennies against an unknown opponent. "A belief or expectation that Heads or Tails will happen with $50\% - 50\%$ chance" means a similar expectation an Agent has when thinking about the potential moves of an unknown opponent in a Matching Pennies game.

One can construct prototypical games for finite probability distributions that can serve as a grounding for these distributions. The reader might be concerned that this grounding is not deep enough or that it is not grounded in some "natural law". To this concern, I might answer that surely this interpretation of probability is not trying to portray chance as an objectively occurring phenomenon in the world but as an emergent concept, which is not out there in the world but a concept invented and used by the Agent. Naturally, it might be an interesting question that how complex an organism should be to see patterns in its behaviour, which can be described as decision heuristics aided by probabilistic models. However, I think that grounding probability on this level can be at least as fruitful as the grounding of "frequentist" probability theory with idealised devices of chance. The traditional grounding stops at the level of standard gambling devices and does not need to dig deeper into the physics and dynamics of dice throwing and coin flipping. (It is possible to investigate such processes further[60], and explore chaotic and/or ergodic dynamical systems, but it is not desperately needed to harvest the fruits of probability theory built on this shallower grounding.)

Exploring the dynamical systems traditionally used as devices of chance is, in spirit, similar to exploring simple organisms, which have started to adopt simple heuristics to function in complex environments. It is intellectually exciting and can reveal interesting details, but it does not seem to be needed to construct an idealised theoretical framework for the concept of probability.

"Almost surely" grounding: Some might argue that the concept of probability can have rigorous meaning only if it is close to 0 or 1. "If something has a very low probability, then we can practically neglect the chance of its occurrence"[61].

In my view, this grounding of probability does not completely lack interpretive power, but we can do much better to grasp the concept. To me, this looks a little like saying that "velocity is such a property of an object that if the velocity is very small, then we can consider the object practically static". I am not trying to ridicule this grounding by the mechanical example; I think that many nontrivial statements could be derived and interpreted using only the previous understanding of "velocity", however, I suspect that we might miss or at least arrive at an overcomplicated version of mechanical concepts and calculations if we would trace back every statement to the appearance of a static body.

In the case of probability, an Agent can associate non-extreme values (not close to 0 or 1) to events, and this belief or this mental construction can have observable consequences in her actions. (Splitting ratios in gambles, choices of alternatives, etc.) To me, this indicates that not only extreme values of probability can be discussed and inferred.

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[59] See the paper at [140].

[60] See for instance a paper by Diaconis at al. from 2007 [41].

[61] "[...] Events with zero or low probability are unlikely to occur. [...] If an event with zero or low probability does occur, then we can reject the game as a model of the world." stated as part of the Fundamental Interpretative Hypothesis in [55] (for further exposition of the concepts in the book see [54]).

Another problem with 0-1 grounding is the threshold of smallness in the interpretive definition. How small should a probability be to ignore it safely? 1%? 0.1%? One in a million, trillion? One atom in a mol? In my view, this depends very much on the potential consequences of our judgment. I think eliminating possibilities based on some threshold on estimated/calculated/assumed probabilities can not be done without considering the damage that excluding a given possibility might cause.

I agree that if probabilities are close to 0 or 1 for an Agent, then "subjective factors" such as utilities start to influence the Agent's behaviour less and less. However, taking only extreme probabilities seriously to ground the concept seems overly restrictive to me.

Utility

In my view, the concept of utility is most useful if we keep it inherently subjective and do not try to ground it in any objectively measurable quantity of the consequences or the Agent[62]. In this way, "utility" is more of an organizing principle, which can be inferred from the Agent's strategies under different circumstances.

(An analogy can be drawn with hidden Markov models[63], where the states are not necessarily "physical" or correspond to anything in reality but can be very useful models to predict or reason about real-world phenomena.)

Generalization to multiple Agents

A single Agent's decision theory in an environment containing unknown parameters might seem sufficient to construct a broader framework for probability theory and statistics. However, even to demonstrate a natural symmetry of the framework, we are forced to generalize it to a decision theory for multiple Agents. Formally, this will lead us to a generalization of Game Theory, extended by uncertain parameters.

Formal definition of the framework for multiple Agents

Structure of the Game: We assume that we have N, finite number of Agents, labelled with an index $i \in \{1, \ldots, N\}$, having $\mathcal{A}_1, \ldots, \mathcal{A}_N$ – possibly different – set of actions. All Agents agree that the possible set of parameters (i.e. possible states of the world) are in Θ. (At this point, we assume that all action and parameter sets are finite.) All players performed a complete scenario analysis, i.e. they constructed their utility arrays: $U_i : \mathcal{A}_1 \times \cdots \times \mathcal{A}_N \times \Theta \mapsto \mathbb{R}$

Solution concept: Given this structure, we can define an extended strategy profile, which contains "real" and "imaginary" strategies: $\langle \sigma_1, \ldots, \sigma_N; \pi_1, \ldots, \pi_N \rangle$ where these strategies are probability distributions on appropriate sets: $\sigma_i \in \Delta(\mathcal{A}_i)$, $\pi_i \in \Delta(\Theta)$.

Given a (real-)strategy profile, we can introduce the following "average regret" based utility – or rather regret – arrays:

$$R_i(a_1, \ldots, a_N; \theta|\underline{\sigma}) = \max_{b_i} \left(\sum_{a_1, \ldots, a_N} U_i(a_1, \ldots, a_i \leftarrow b_i, \ldots, a_N; \theta)\ \sigma_1(a_1) \ldots \sigma_N(a_N) \right) - U_i(a_1, \ldots, a_N; \theta)$$

(Where $a \leftarrow b$ is a notation for value setting, i.e. $f(a \leftarrow b) = f(b)$.)

Introducing the "expected" (relative to both "real" strategies and "imaginary" strategies i.e. subjective priors) utilities:

$$EU_i = \sum_{a_1, \ldots, a_N, \theta} U_i(a_1, \ldots, a_N; \theta)\ \sigma_1(a_1) \ldots \sigma_N(a_N) \pi_i(\theta)$$

and "expected" average regrets:

$$ER_i = \sum_{a_1, \ldots, a_N, \theta} R_i(a_1, \ldots, a_N; \theta|\underline{\sigma})\ \sigma_1(a_1) \ldots \sigma_N(a_N) \pi_i(\theta)$$

An *extended equilibrium* strategy profile $\langle \underline{\sigma}^*; \underline{\pi}^* \rangle$ is defined by the following properties:

[62] Possible objective measure candidates might include: calories [134, 83] for living beings with metabolism; fitness, i.e. expected future reproductive success in the context of evolutionary game theory and Darwinian dynamics; wasted time as a universal cost (related to discount factors); chance of dying, i.e. associating micromorts [49, 106] to consequences; amount of chemicals part of the internal reward system such as Dopamine for biological organisms with brain.

[63] See the Machine Learning textbook by Bishop for reference [16].

$$\forall i,\ \forall b_i \in \mathcal{A}_i,\ EU_i \geq \sum_{a_1,\ldots,a_N;\theta} (U_i(a_1,\ldots,a_i \leftarrow b_i,\ldots,a_N;\theta))\ \sigma_1^*(a_1)\ldots\sigma_N^*(a_N)\pi_i^*(\theta)$$

$$\forall i,\ \forall \theta' \in \Theta,\ ER_i \geq \sum_{a_1,\ldots,a_N;\theta} (R_i(a_1,\ldots,a_N;\theta \leftarrow \theta'|\underline{\sigma}^*))\ \sigma_1^*(a_1)\ldots\sigma_N^*(a_N)\pi_i^*(\theta)$$

Such an extended equilibrium profile always exists (at least for finite action sets). Naturally, this statement has to be proven rigorously – I plan to make the proof accessible in a separate paper –, but intuition and claims from other papers back up the statement[64].

In less formal language, the following collective thought process can illuminate the structure of the equilibrium strategy profile: Each player assumes that the unknown player is governed by an imaginary player who wants to maximize their regret in some way. However, the Agents do not assume that the uncertain is conspiring with other "real" Agents. This is the reasoning behind calculating the regret of an expected utility with respect to other Agents' strategy[65].

The regret of all Agents can vary as a function of their actions and the parameter. Therefore, they will all potentially project or imagine different fictional players controlling the uncertain parameter. Therefore, all Agents can – and generally will – imagine different side games with the parameters, resulting in different π_i imaginary (or ghost) priors for the parameter.

We call a set of strategies and beliefs an extended equilibrium strategy profile, if no player and no fictional player can increase their subjective (partially imagined) expected utility by unilaterally changing their strategy.

The construction can be viewed as a generalization of Nash equilibrium to games, where one player is not "rational" but represents an uncertain parameter. (The special "uncertainty player" influences other players' outcomes but has no well-defined utility function.)

Subgame symmetry of decision-making problems

Constructing a paradox: Let's take the simplest example where a paradox can be constructed. Let us have a two by two dilemma $\mathfrak{G}$, with action set $\mathcal{A}_1 = \{\uparrow,\downarrow\}$ and parameter set $\Theta = \{A, B\}$, where can be three different consequences: $\mathcal{C} = \{\mathfrak{A}, \mathfrak{B}, \mathfrak{O}\}$, appearing in a consequence matrix in a following way:

$$C = \begin{bmatrix} \mathfrak{A} & \mathfrak{O} \\ \mathfrak{O} & \mathfrak{B} \end{bmatrix}$$

Now let assume, that the consequence $\mathfrak{B}$ means entering to a subgame $\mathfrak{G}'$ with only $\mathcal{C}' = \{\mathfrak{A}, \mathfrak{O}\}$ possible outcomes and action sets $\mathcal{A}_1' = \{\downarrow +, \downarrow -\}$, $\Theta' = \{B+, B-\}$ which can be arranged to the following matrix:

$$C' = \begin{bmatrix} \mathfrak{A} & \mathfrak{O} \\ \mathfrak{O} & \mathfrak{A} \end{bmatrix}$$

If the Agent prefers the outcome $\mathfrak{A}$ more then $\mathfrak{O}$, then without loss of generality we can assume that $u_1(\mathfrak{A}) = 2$, $u_1(\mathfrak{O}) = 0$.

Now we can conclude by following the original decision-making process for the simpler subgame that can be mapped to Matching Pennies, and it has $1/2\ u_1(\mathfrak{A}) + 1/2\ u_1(\mathfrak{O}) = 1$ expected utility in equilibrium. Therefore we derived that $u_1(\mathfrak{B}) = 1$, which yields the following two by two utility matrix:

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

However, instead of solving the subgame, we can simply extend the original dilemma to a three-by-three game containing the subgame ($\mathcal{A}_1^E = \{\uparrow, \downarrow +, \downarrow -\}$, $\Theta^E = \{A, B+, B-\}$):

$$C^E = \begin{bmatrix} \mathfrak{A} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{O} & \mathfrak{A} & \mathfrak{O} \\ \mathfrak{O} & \mathfrak{O} & \mathfrak{A} \end{bmatrix}$$

Having the extended utility matrix:

[64] See the following paper for a similar construction, in which the authors are claiming the existence of such equilibrium [78].

[65] Not taking the full regret and then taking the expectation of it. We define the best-case scenario as $\max_{b_i}\left(\sum_{a_1,\ldots,a_N} U_i(a_1,\ldots,a_i \leftarrow b_i,\ldots,a_N,\theta)\ \sigma_1(a_1)\ldots\sigma_N(a_N)\right)$ and not as $\sum_{a_1,\ldots,a_N}\left(\max_{b_i} U_i(a_1,\ldots,a_i \leftarrow b_i,\ldots,a_N,\theta)\right)\ \sigma_1(a_1)\ldots\sigma_N(a_N)$.

$$U^E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

However, suppose we faithfully follow the steps of our decision-making doctrine. This leads to a paradox: the equilibrium strategies of the expanded and contracted forms of the decision-making problem differ.

$$\sigma^* = (2/3, 1/3), \quad \pi^* = (1/3, 2/3)$$

However, for the extended problem, the equilibrium is:

$$\sigma^{E*} = (1/3, 1/3, 1/3), \quad \pi^{E*} = (1/3, 1/3, 1/3)$$

The contradiction immediately manifests if we read the probability of choosing $\uparrow$ in the equilibrium strategies:

$$\sigma^*(\uparrow) = 2/3 \neq \sigma^{E*}(\uparrow) = 1/3$$

Resolving the paradox: This paradox could be almost detrimental to the framework because if the suggested strategy depends on a superficial representation of our decision-making dilemma, then virtually any result can be achieved by modifying the representation.

Fortunately, the paradox disappears if we specify a subgame more carefully and apply the extended framework.

If we imagine the subgame $\mathfrak{G}'$ is not as a sub-decision-making dilemma with an uncertain two-state parameter, but is an actual game with an adversarial player, then the extended game becomes practically three-player game[66].

$$C^{E'}(.,+,.) = \begin{bmatrix} \mathfrak{A} & \mathfrak{O} \\ \mathfrak{O} & \mathfrak{A}' \\ \mathfrak{O} & \mathfrak{O}' \end{bmatrix}, \quad C^{E'}(.,-,.) = \begin{bmatrix} \mathfrak{A} & \mathfrak{O} \\ \mathfrak{O} & \mathfrak{O}' \\ \mathfrak{O} & \mathfrak{A}' \end{bmatrix}$$

$$U_1^{E'}(.,+,.) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad U_1^{E'}(.,-,.) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$U_2^{E'}(.,+,.) = \begin{bmatrix} y & x \\ x & 0 \\ x & 2 \end{bmatrix}, \quad U_2^{E'}(.,-,.) = \begin{bmatrix} y & x \\ x & 2 \\ x & 0 \end{bmatrix}$$

Direct calculation shows that

$$\sigma_1^* = (2/3, 1/6, 1/6), \quad \sigma_2^* = (1/2, 1/2), \quad \pi_1^* = (1/3, 2/3), \quad \pi_2^* = (p, 1-p)$$

is an extended equilibrium of this multi-player game, restoring the consistency of the framework.

Conclusions: If some consequence is to enter into a real subgame, then the utility of that consequence can be substituted by the expected utility of the subgame and the suggested strategy is representation invariant in the extended framework.

However, if some consequence is a subdilemma, containing uncertain parameters, then it can not be substituted by their expected utility at the equilibrium of the subdilemma. This can be viewed as a general property of expected utilities of decision-making problems in the face of uncertainty. (It seems intuitive that the consistency can be restored if we substitute some appropriate value between the minimum and maximum of the possible utility of an outcomes of a subdilemma. However, this statement has to be properly proven.)

[66] By faithfully following the construction this would involve four players (two real and two imaginary), but the prior for the uncertain parameter viewed by the second player does not influence any strategy at all.

Optimism

The proposed framework – essentially a minimax regret decision rule – has been called "pragmatic"[67] and "less pessimistic"[68]. One might wonder what Optimistic people (or Agents) ought to do.

What is Optimism? To answer this concern, first, we need to define what "Optimism"[69] might mean in formal terms: In psychology and the colloquial use of the phrase, optimism seems to be an umbrella term depicting multiple similar but distinct concepts. One meaning appearing in popular writings (such as "technological optimism")[70] might be simply put: A belief that conditions can be improved, and the determination to make progress happen. In this sense, our framework is optimistic if the Agent is able to imagine scenarios where "conditions can improve". (This is not trivially true for any decision-making framework in which the Agent imagines a fictional player controlling the unknown parameters. An imagined zero-sum game could rule out possibilities which hold greater opportunities.)

Another meaning of optimism seems to be confused with higher risk-taking. A person willing to risk a bigger portion of her capital besides known chances might seem more optimistic. If we focus on this meaning of optimism, then the framework is suitable for both cautious and brave gamblers because this feature of the Agent can be tuned by the utility function (more precisely by the relative risk aversion of the associated utility function).

However, in my view, optimism can have a more abstract and deeper meaning than an ability to see opportunities, attribute failures to a lack of knowledge and be less risk-averse. Mainly in social settings, optimism is – I believe – more about imagining allies behind the unknown factors. Thinking, or believing that there are others – real or fictional – who are aligned with our cause and will help us to achieve our goals.[71]

Formalization of Optimism: A naive or extreme optimist might imagine a fictional player controlling the parameters who has exactly the same preferences and utilities as her own. In the case of a nontrivial utility function, this is an extremely friendly and reassuring outlook to the unknown factors, but (unless the utility functions are not chosen or adopted very-very wisely), this leads to an extreme behaviour: The Agent will always choose the action, which yields the highest reward, while she assumes that the unknown is always cooperates fully and tunes the uncertain parameter to make her possible to win the greatest reward[72].

Because of the behaviour it induces, I think extreme optimism is not a viable alternative to the proposed regret-based framework. However, it does not mean that there could not be a more tempered way to increase the agent's "optimism"[73].

Two-staged fictional game: To formalise a tempered optimistic attitude, we can imagine a game where the unknown player has a "type". The first stage can be viewed as "Bayesian game" à la Harsányi[74]. We can imagine two types of fictional players: an Ally, who is totally aligned with us, and a Trickster, who plays a regret-maximising strategy (just as how I suggested to model uncertainty in general). The Agent does not know in advance if she will face an Ally or a Trickster. The type of her opponent will be chosen by a "higher level" trickster, who tries to cause regret relative to an insider Agent (who knows the type of her opponent in advance) consisting of the second stage of the game[75]. If we assume that the chances of seeing an Ally is α' and seeing a Trickster is $1 - \alpha'$, then we might have a somewhat similar construction to Hurwitz criteria[76], which can interpolate between extremely optimistic and regret-based approaches.

A nontrivial statement is that this two-staged game generally has a non-extreme solution ($\alpha' \in (0, 1)$) and seems to induce a behaviour corresponding to moderate optimism.

Summary: I am positive that the framework can be modified to show increased levels of (dispositional-) optimism. However, these enhancements do not seem essential, and the effect on behaviour can also be achieved by simply modifying the available possibilities (e.g. in social settings, explicitly including the possibility that we might have allies whose actions influence unknown factors).

However, introducing fictional Ally players has consequences: it breaks, for instance, the equivalence

[67] See for example [158, 130].

[68] Abraham Wald's original minimax formulation [165] has been called "prudential" by R. B. Braithwaite [57, 19] and has been called "pessimistic" by many (see for example [133]).

[69] For the psychologist perspective see some collected works here [23].

[70] For reference see [40, 116], where David Deutsch coins another definition: "All evils are caused by insufficient knowledge." as "The Principle of Optimism".

[71] Dispositional optimism is the psychological term describing related attitude [143].

[72] Also known as "maximising the maximums" strategy of an "unbridled optimist" in decision theory [133]

[73] An example, known in decision theory might by the Hurwitz criteria [108, 76]. "Optimism" is tunable by an $\alpha \in [0, 1]$ parameter in that framework.

[74] For reference see [65, 64, 66, 174].

[75] Resembling the construction of hierarchical Bayesian models [51].

[76] Following the conventions in [108]. However, the Hurwitz criteria interpolates between "Optimism" and "Pessimism", while in our two-step approach, we rather interpolate between "Optimism" and "Pragmatism".

classes – i.e. the symmetry – of the proposed regret-based framework shared with Bayesian decision-making. Nevertheless, this is a nuanced property, which many decision-makers may easily sacrifice if it yields more successful heuristics.

As in the case of the "black or white thinking", I see no unquestionable arguments to rule out decision-making heuristics, which are constructed to be more optimistic[77]. Still, because of its simplicity and sound induced behaviour, I think the original regret-based approach does not need a severe optimism enhancement to remain suggestible when we face uncertain factors.

Virtues of a metaphysical framework

A compelling aspect of science is that its claims are testable by experiments. Of course, there are further "theoretical virtues" of scientific theories[78] some of which I will list and discuss.

The biggest obstacle to applying scientific techniques to evaluate the suggested framework is that its elements and aim are inherently not scientific. We can not talk scientifically about the "intentions behind unknown factors" or "strategies of Nature" because these can not be contrasted with experiments. "Properties of the unknown", by definition, disappear – as a mirage – immediately when we gain knowledge by performing an experiment. Therefore, what we are contemplating here is not a scientific or a physical theory, but a metaphysical construction[79]. However, even if we might lose a very important tool in science, experimental testability or the possibility of falsification, we might keep or transfer the remaining theoretical virtues from science and mathematics[80] to judge and compare metaphysical constructions.

- Experimental validity (not applicable)
- Empirical accuracy (not applicable)
- Internal consistency
- Scope: unifying power
- Simplicity
- Fruitfulness
- Beauty
- Applicability
- + Independent discovery or application
- + Sustainability
- + Performance on historical data (compared to alternatives)
- + Performance in synthetic worlds (compared to alternatives)

It is not easy to define what the virtues of a metaphysical theory might indicate. In the case of science, the standard interpretation — or belief — is that theories with more virtues might be closer to the Truth. Even if we leave criticism aside of this interpretation, we can not say the same for virtues metaphysical constructions. They can not be True in the sense that scientific models or mathematical theorems can. However, instead, – I think – they can be Fixed points that stay in the Agent's worldview indefinitely. A big difference between seeking the truth and looking for fixed points is that fixed points do not need to be unique. Finding one fixed point does not exclude the existence of others. (However, this does not mean that there has to be multiple fixed points; there may exist only one, or maybe none.)[81]

[77] Primarily in some social situations, enhanced optimism might yield a more adaptive behaviour. I feel a close analogy with successful behavioural strategies in iterated evolutionary games, which can get more and more forgiving as malicious agents vanish and cooperative agents start to dominate the population. For reference, see the book of Robert Axelrod [11].

[78] See references in the context of the philosophy of science [86, 145].

[79] Maybe I need to clarify what I mean by "metaphysical construction" here. To function and operate in the world, agents require not only data about their environment but also some further structure, which they use to organise and translate their empirical knowledge into actions. I call these extra structures in Agents' model-making factions metaphysical constructions because they are not derived directly from empirical data. Therefore, metaphysical constructions are created and used by the Agents to navigate in the world and fill the gaps in their knowledge, to acquire meaning or goals, but most crucially, to translate experience to action.

[80] For an inexhaustive list of aspects of "good mathematics" see the following paper [159].

[81] In fact, it is conceivable that there is a whole ecology of subjective metaphysical beliefs, just as strategies – which can be viewed as very simple subjective metaphysical constructions – in evolutionary game theory can coexist or have complicated dynamics [141].

Another severe complication of judging metaphysical constructions appears to be the circumstance that we always need a metaphysical construction to make judgements. Therefore, there seems to be no outer, objective view of metaphysical systems. What we can do is only analyse a system $\mathscr{M}_X$ from another system's point of view $\mathscr{M}_Y$ and make relative assessments $\mathbb{A}(\mathscr{M}_Y||\mathscr{M}_X)$. (In a previous section, the "Independence tax" was an example of such a relative assessment.)

Independent appearance and empirical evidence

The suggested framework is not primarily a descriptive theory of decision-makers but rather a metaphysical construction leading to action – or strategies – in uncertain situations. Therefore, empirical evidence does not validate or invalidate the framework's durability (fixed pointness). Still, finding independently appearing examples for its use can be viewed as a theoretical virtue.

Human behaviour

Randomization: It is known that humans generally are not good at generating random numbers or choices, i.e., they are not maximally unpredictable. However, "experts", for example, professional players in sports, can effectively randomize on the field[82]. There are other studies investigating the neural processes in humans while performing mixed strategies in games[83]. This shows that humans can acquire this essential ingredient of mixed strategies, which are often the equilibrium solutions of games and decision-making problems in the suggested framework.

However, humans often rely on external tools or practices to generate random choices. There is a huge variety of practices from ancient times to contemporary cryptography applications that people and communities use to take unpredictable actions.

Besides strategic decision-making, people used randomization devices for fun since prehistoric times in the form of children's play and gambling. Even if gambling has been often discouraged, the interaction and fascination with unpredictable outcomes seems to be deeply embedded in many cultures all over time and space[84].

Ellsberg paradox: *Cognitive science of decision making* Brain studies show that humans might have dedicated brain areas involved in decision-making and accessing uncertain factors[85].

Allais paradox: ...

Animal behaviour

I have to confess that I was reluctant to mention and cite the results of biological experiments because most of the time, I read the papers with feeling sorrow for the animals. I convinced myself to write about the results found in these papers only because I might have a chance to promote the expansion of the circle of compassion[86]. (My concern is not primarily focused on suffering – although this makes the matter more painful to comprehend – but the alienation of living beings, treating them as subjects or sometimes as objects, and therefore preventing them from realizing their potential and engaging fully in the dance of life and being. I don't think that it is only harmful to "them"; I believe it is harmful to us if we distance ourselves from the majority of living beings because the price for alienation – and the capacity of being cruel – is the feeling of isolation and boundedness. I am not advocating for treating animals and other living beings in the same way we treat humans. I promote practices which don't cause the evaporation of compassion and where animals are not viewed as subjects but as partners by whom we can engage and hopefully foster a mutually beneficial relationship. On the other hand, if we expand our circle of compassion to a broader range of living beings, then we enrich ourselves and can feel more connected and more at home in the world. Because compassion and love are not getting less if given; it is just getting more.)

[82] See experimental results for the randomization of soccer (or football) players both on the field and in a laboratory setting [122, 123, 96].
[83] See for example [154].
[84] See an ethnographic and historical investigation of gambling in [15].
[85] For reference see [75].
[86] Circle of altruism is a similar concept in Peter Singer's book [149].

Playing a Nash equilibrium: Primates (rhesus monkeys) randomize their choices according to Nash equilibrium[87].

Ambiguity aversion: Ambiguity aversion has been observed in primates[88]

Epilogue

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To be continued.

[87]See [154].

[88]See for example [137, 136, 68]

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