

Introduction  
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Fisher & Bayesian games  
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Limiting cases  
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Statistical games  
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Philosophy & Future work  
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# Statistical Games

Playful approach to statistics

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# Why? Mathematician Answer:

- Why not?
- It might appeal to some.
- In the first part, established concepts:
  - Elementary calculations resulting sometimes surprising results.
- Can be continued to:
  - Optimization in Functional (Measure) spaces,
  - Delicate tools, expansions, inequalities for exploring limiting cases,
  - Quantum generalization (scenarios to quantum states).
- + Fun to design concepts and notation, grounded by their ability to simplify proofs and calculations.

### Why? Philosophical Answer:



- Pursuit for simplicity\* and coherence.
    - \* in thought, not necessarily in techniques.
  - Searching for an organizing principle in Statistics.
  - Formalization and (re-)introduction of Uncertainty.
  - Epistemology → Strategy.
  - Prior → Fictional Player.

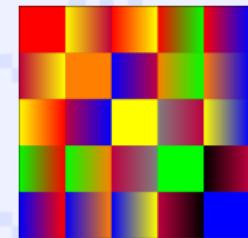
Figure: from @miniapeur on X

# Oversimplified introduction to Game Theory

Action sets for 2 Players:  $\mathcal{A}_1, \mathcal{A}_2$ .

Utility matrices:

$$\mathcal{A}_1 \times \mathcal{A}_2 \mapsto \mathcal{C}, \begin{cases} u_1 : \mathcal{C} \mapsto \mathbb{R}, \text{ or } (\mathcal{A}_1 \times \mathcal{A}_2 \mapsto \mathbb{R}) \\ u_2 : \mathcal{C} \mapsto \mathbb{R}, \text{ or } (\mathcal{A}_1 \times \mathcal{A}_2 \mapsto \mathbb{R}) \end{cases}$$



Strategies:

$$\sigma_1 \in \mathcal{P}(\mathcal{A}_1), \quad \sigma_2 \in \mathcal{P}(\mathcal{A}_2)$$

Expected utilities:  $EU_i = \mathbb{E}_{\alpha_1 \sim \sigma_1, \alpha_2 \sim \sigma_2} [u_i(\alpha_1, \alpha_2)]$

$$EU_1(a_1, \sigma_2) = \mathbb{E}_{\alpha_2 \sim \sigma_2} [u_1(a_1, \alpha_2)], \quad EU_2(\sigma_1, a_2) = \mathbb{E}_{\alpha_1 \sim \sigma_1} [u_2(\alpha_1, a_2)]$$

Figure:

Introduction to  
Game Theory by  
J. F. Nordstrom

# Oversimplified introduction to Game Theory

Nash equilibrium is a strategy profile:

$$\sigma^* = (\sigma_1^*, \sigma_2^*)$$

Where no player's expected utility can be improved by changing one's own strategy.

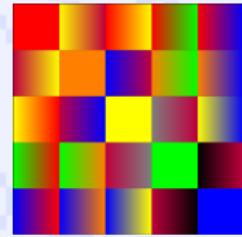


Figure:

Introduction to  
Game Theory by  
J. F. Nordstrom

Nash equilibrium always exist (for finite action sets), but it is not necessarily unique.

## Definition (Fisher game)

There are two players, Player 1 (Guesser) and Player 2 (Chooser). Player 2 needs to choose between scenario A or B first and then produce a binary sequence of length  $M$  containing precisely  $K_A$  or  $K_B$  number of 1-s. (Without losing generality, we will assume  $K_A \leq K_B$ .) Following this, Player 1 (not knowing the actions of Player 2) can sample  $N$  number of bits, and after observing their value, she guesses scenario A or B. If Player 1 guessed the scenario correctly, she wins the game (■) and loses otherwise (■). The above-defined Fisher game will be denoted as  $\text{Game}(N, K_A, K_B, M)$ .



## Fisher games

## Game Table: General description of Game( $N, K_A, K_B, M$ )

Player 1	Player 2
<ul style="list-style-type: none"><li>• Chooses <math>N</math> indices for sampling,<ul style="list-style-type: none"><li>• based on the bits in the chosen sample, guesses scenario A or B.</li></ul></li></ul>	<ul style="list-style-type: none"><li>• Chooses scenario A or B,<ul style="list-style-type: none"><li>• then chooses a binary sequence available for the chosen scenario.</li></ul></li></ul>

If Player 1 guessed the scenario correctly, she wins the game (■) and loses otherwise (□).

Statistically trivial example (Matching Pennies)

**Game**( $N = 0, K_A = 0, K_B = 0, M = 0$ ):

$$u_1 = \begin{bmatrix} \text{orange} \\ \text{blue} \\ \text{blue} \\ \text{orange} \end{bmatrix}, \quad u_2 = \begin{bmatrix} \text{blue} \\ \text{orange} \\ \text{orange} \\ \text{blue} \end{bmatrix}$$

Fisher games

## Statistically trivial example (Matching Pennies)

**Game**( $N = 0, K_A = 0, K_B = 0, M = 0$ ):

$$u_1 = \begin{bmatrix} \text{orange} \\ \text{blue} \\ \text{blue} \end{bmatrix}, \quad u_2 = \begin{bmatrix} \text{blue} \\ \text{orange} \\ \text{blue} \end{bmatrix}$$



## Figure: Hand Game

## Statistically trivial example (Matching Pennies)

The game has one unique Nash equilibrium: the players are guessing/choosing A or B with an equal 50% chance, and both players have a 50% chance to win or lose.

$$\sigma_1^* = (1/2, 1/2), \quad \sigma_2^* = (1/2, 1/2)$$

$$v^* = 1/2$$

## Fisher games

## Smallest nontrivial case

**Game**( $N = 1, K_A = 0, K_B = 1, M = 2$ ):

$$\mathcal{A}_2 = \{(A, (\square, \square)), (B, (\square, \blacksquare)), (B, (\blacksquare, \square))\}$$

$$\mathcal{A}'_1 = \{(1, \{(\square) \rightarrow A, (\blacksquare) \rightarrow B\}), (1, \{(\square) \rightarrow B, (\blacksquare) \rightarrow B\}), \\ (2, \{(\square) \rightarrow A, (\blacksquare) \rightarrow B\}), (2, \{(\square) \rightarrow B, (\blacksquare) \rightarrow B\})\}$$

## Fisher games

## Smallest nontrivial case

**Game**( $N = 1, K_A = 0, K_B = 1, M = 2$ ):

$$\mathcal{A}_2 = \{(A, (\square, \square)), (B, (\square, \blacksquare)), (B, (\blacksquare, \square))\}$$

$$\mathcal{A}'_1 = \{(1, \{(\Box) \rightarrow A, (\blacksquare) \rightarrow B\}), (1, \{(\Box) \rightarrow B, (\blacksquare) \rightarrow B\}), \\ (2, \{(\Box) \rightarrow A, (\blacksquare) \rightarrow B\}), (2, \{(\Box) \rightarrow B, (\blacksquare) \rightarrow B\})\}$$

$$\underline{\sigma}_1'^*(\lambda) = (1/3, \lambda/3, 1/3, (1-\lambda)/3), \quad \underline{\sigma}_2'^*(\lambda) = (1/3, 1/3, 1/3),$$

$$\underline{\sigma}_1'^* = \underline{\sigma}_1'^*S = (1/3, 1/6, 1/3, 1/6), \quad \underline{\sigma}_2'^* = \underline{\sigma}_2'^*S = (1/3, 1/3, 1/3)$$

## Fisher games

Game Table: Symmetric equilibrium strategy for  
 $\text{Game}(N = 1, K_A = 0, K_B = 1, M = 2)$

Player 1	Player 2
<ul style="list-style-type: none"><li>• Sample randomly from all possible indices uniformly<ul style="list-style-type: none"><li>• in case the sampled bit is □:<ul style="list-style-type: none"><li>• guess A with probability 2/3</li><li>• and B with probability 1/3</li></ul></li><li>• in case the sampled bit is ■:<ul style="list-style-type: none"><li>• guess B</li></ul></li></ul></li></ul>	<ul style="list-style-type: none"><li>• Choose scenario A with probability 1/3, and scenario B with probability 2/3<ul style="list-style-type: none"><li>• Choose uniformly from all different allowed sequences.</li></ul></li></ul>

## Fisher games

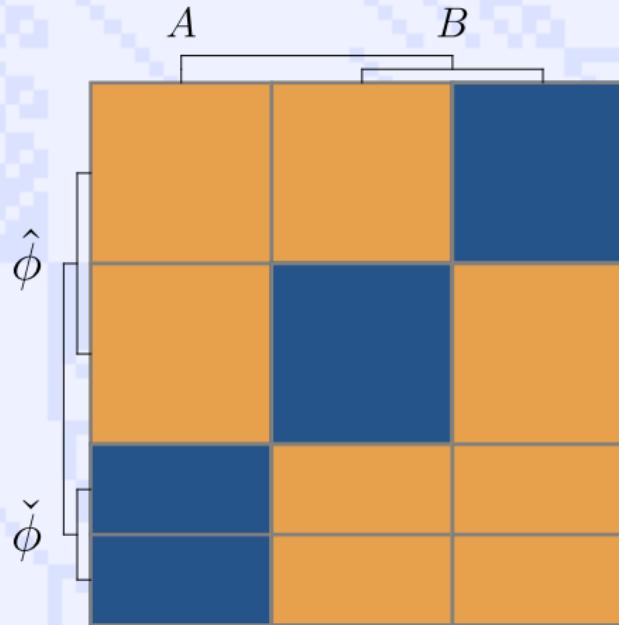


Figure: Strategy plot for the symmetric equilibrium of  
**Game**( $N = 1, K_A = 0, K_B = 1, M = 2$ ).

## Fisher games

## Ansatz

Game Table: **Ansatz** for the general  $\text{Game}(N, K_A, K_B, M)$  case,  
having 3 free variables:  $P$ ,  $k^*$  and  $\nu$ :

Player 1	Player 2
<ul style="list-style-type: none"><li>sample <math>N</math> bits randomly and uniformly from all available <math>M</math> bits,<ul style="list-style-type: none"><li>in case the number of ■-s <math>k &lt; k^*</math> guess A,</li><li>in case <math>k = k^*</math>:<ul style="list-style-type: none"><li>guess A with probability <math>\mu(\hat{\phi}) = \nu</math>,</li><li>guess B with probability <math>\mu(\check{\phi}) = 1 - \nu</math>,</li></ul></li><li>in case the number of ■-s <math>k &gt; k^*</math> guess B.</li></ul></li></ul>	<ul style="list-style-type: none"><li>choose scenario A with probability <math>\pi(A) = P</math>, or B with probability <math>\pi(B) = 1 - P</math>.<ul style="list-style-type: none"><li>Choose uniformly from all different allowed sequences.</li></ul></li></ul>

## Theorem (Symmetrical equilibrium)

*The parameters  $(k^*, \nu^*, P^*)$  can be determined from the parameters of the game  $(N, K_A, K_B, M)$ :*

$$p_k(A) = \frac{\binom{K_A}{k} \binom{M-K_A}{N-k}}{\binom{M}{N}}, \quad p_k(B) = \frac{\binom{K_B}{k} \binom{M-K_B}{N-k}}{\binom{M}{N}} \quad (1)$$

$$P^* = \frac{p_{k^*}(B)}{p_{k^*}(A) + p_{k^*}(B)} \quad (2)$$

$$\nu^* = \frac{\sum_{k \geq k^*} p_k(B) - \sum_{k < k^*} p_k(A)}{p_{k^*}(A) + p_{k^*}(B)} \quad (3)$$

*Finally,  $k^*$  is the smallest integer, for which the sum of probabilities becomes greater than 1:*

$$\sum_{k \leq k^*} p_k(A) + p_k(B) > 1, \quad \text{while} \quad \sum_{k < k^*} p_k(A) + p_k(B) \leq 1 \quad (4)$$



## Fisher games

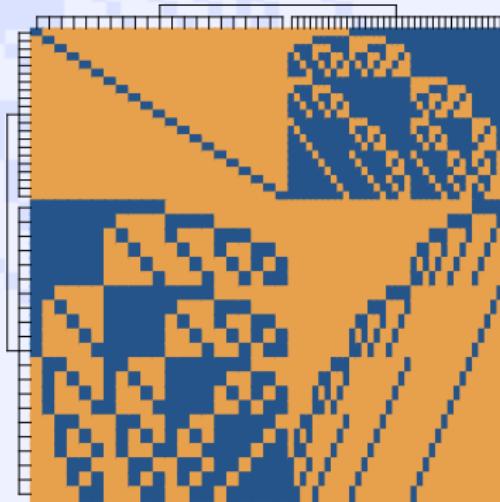
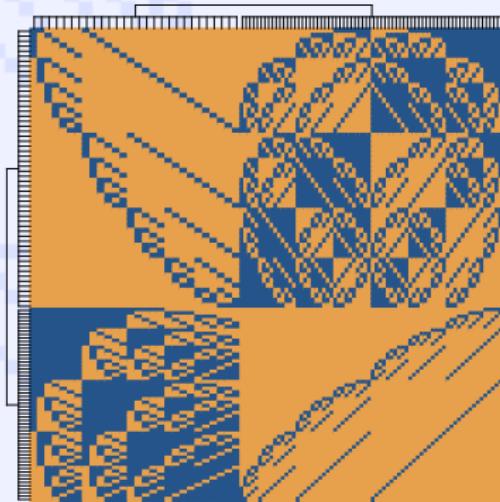
(a)  $\text{Game}(N = 2, K_A = 2, K_B = 4, M = 7)$ (b)  $\text{Game}(N = 3, K_A = 2, K_B = 4, M = 8)$ 

Figure: Strategy plots for symmetric equilibria

## Fisher games

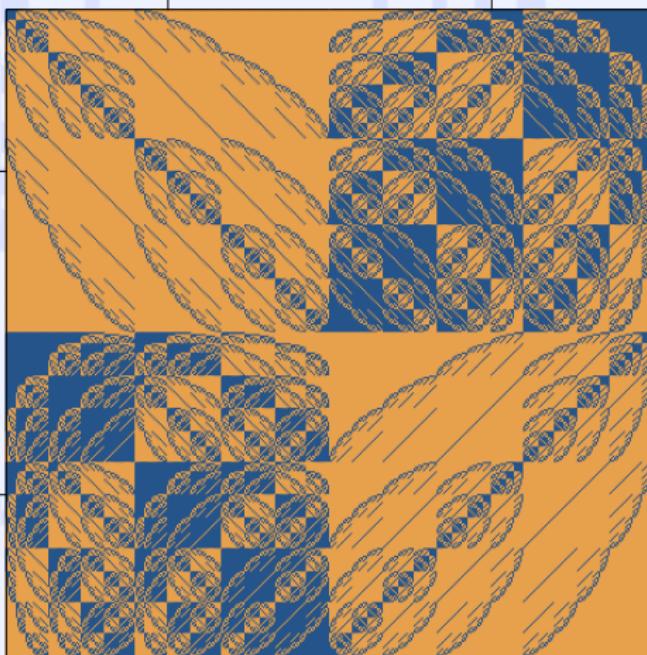
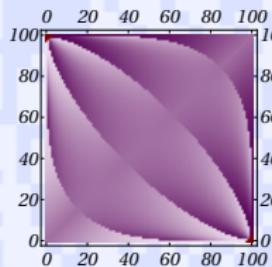
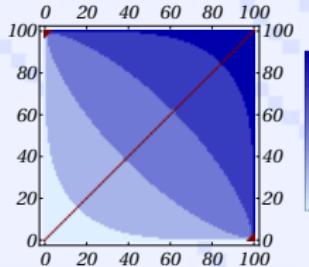
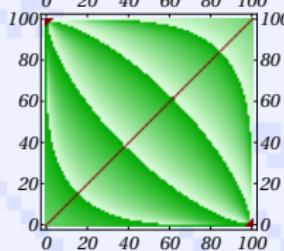
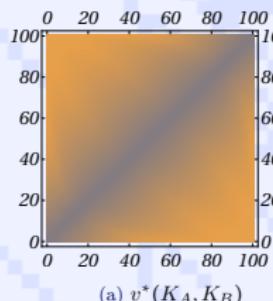
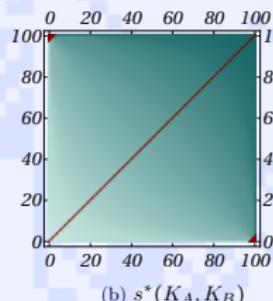


Figure: Strategy plot for the symmetric equilibrium of Game ( $N = 4, K_A = 4, K_B = 6, M = 10$ ).

## Fisher games

(a)  $P^*(K_A, K_B)$ (b)  $k^*(K_A, K_B)$ (c)  $v^*(K_A, K_B)$ Figure: Game( $N = 4, K_A, K_B, M = 100$ )(a)  $v^*(K_A, K_B)$ (b)  $s^*(K_A, K_B)$ Figure: Game( $N = 4, K_A, K_B, M = 100$ )

# Emerged Statistical concepts

- Sufficient statistics
- Type I and Type II errors
- Randomized sampling
- Emergence of probability distributions
- Randomized policies  
(appears here)
- Interpretation of mixed strategies (mostly appears here)



Figure: R. A. Fisher illustration by Rachelle Scarfó

# Type I and Type II errors

*We may now discuss what meaning could be given to the words: “a test independent of the probability law a priori.”*

*Definition A.* *The phrase might be defined as implying a choice of critical region  $w$  in such a way that the probability  $P_{\text{Error}}$  of making an error in testing  $H_0$  had a value independent of the probabilities  $\pi(A)$ ,  $\pi(B)$ .*

— *Definition A in Neyman, Pearson 1933b* <sup>1</sup>.

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<sup>1</sup>the notation has been slightly modified to harmonize with the notation used in this work

# Interpretation of mixed strategies

*Mixed strategy can alternatively be viewed as the belief held by all other players concerning a player's actions. A mixed strategy equilibrium is then an n-tuple of common knowledge expectations, which has the property that all the actions to which a strictly positive probability is assigned are optimal, given the beliefs. A player's behavior may be perceived by all the other players as the outcome of a random device even though this is not the case.*

— Ariel Rubinstein 1991.

# Bayesian betting

**Double or nothing:** Now, we enter a different casino, where instead of betting simply on scenario A or B, we can place some portion of our capital on each alternative. The portion we put on the winning scenario (chosen by Player 2) will be doubled, while the portion placed on the other scenario will be lost.



Figure: Image source



## Definition (Bayesian game)

There are two players, Player 1 and Player 2. Player 2 needs to choose between scenario A or B, then produce a binary sequence of length  $M$  containing precisely  $K_A$  or  $K_B$  number of 1-s.

(Without losing generality, we will assume  $K_A \leq K_B$ .) Following this, Player 1 (not knowing the actions of Player 2) can sample  $N$  number of bits. After observing their values, she determines what portion of her capital  $p'$  she places on scenario A (while the other  $1 - p'$  portion is placed on scenario B).

The portion Player 1 places on the scenario, chosen by Player 2, will be doubled, while the other part of her capital will be lost. For this specific game, we will assume that Player 1 has a logarithmic utility function and that Player 1 and Player 2 are playing a zero-sum game. The above-defined Bayesian game will be denoted as  $\mathbf{BGame}(N, K_A, K_B, M)$ .



## Bayesian games

Game Table: General description of  
 $\text{BGame}(N, K_A, K_B, M)$

Player 1

- Chooses  $N$  indices for sampling,
  - based on the bits in the chosen sample determines a continuous parameter  $p' \in [0, 1]$ ,
    - and places her capital's  $p'$  portion to scenario A and  $1 - p'$  portion to scenario B.

Player 2

- Chooses scenario A or B,
  - then chooses a binary sequence available for the chosen scenario.

The portion Player 1 places on the scenario chosen by Player 2 will be doubled,  
while the other part of her capital will be lost.

Player 1 has a logarithmic utility function, Player 1 and Player 2 are playing a  
zero-sum game.

## Bayesian games

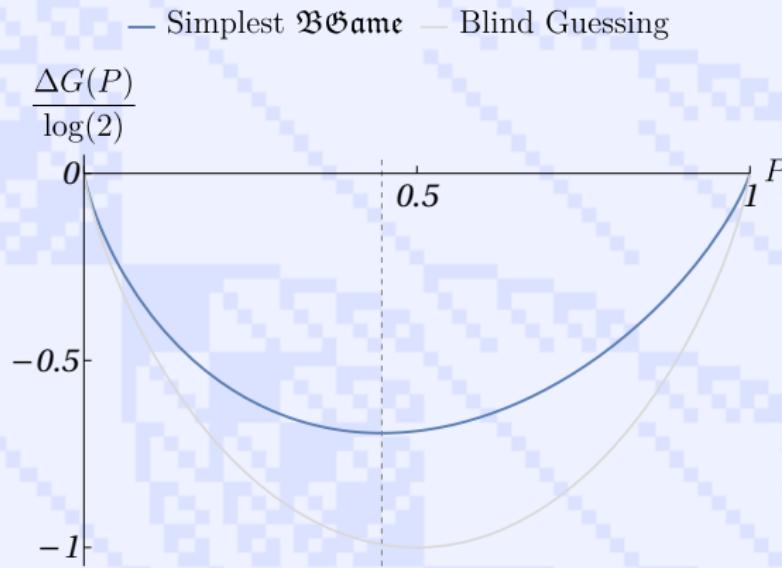


Figure: Visualization of the doubling factor difference (relative to a Sure winning game)  $\Delta G(P)/\log(2)$  for Player 1 in the game:  $\text{BGame}(N = 1, K_A = 0, K_B = 1, M = 2)$  and a Blind guessing game, as a function of Player 2's strategy of choosing A. The dashed grid line is placed to the equilibrium value of  $P^* = 1/\sqrt{5}$ .



## Theorem (Bayesian equilibrium)

**BGame**( $N, K_A, K_B, M$ ) has a unique Nash equilibrium, in which:

The parameters  $(P^*, \{p_k'\})$  can be determined from the parameters of the game  $(N, K_A, K_B, M)$ :

$$p_k(A) = \frac{\binom{K_A}{k} \binom{M-K_A}{N-k}}{\binom{M}{N}}, \quad p_k(B) = \frac{\binom{K_B}{k} \binom{M-K_B}{N-k}}{\binom{M}{N}} \quad (5)$$

$$p_k' = \frac{P^* p_k(A)}{P^* p_k(A) + (1 - P^*) p_k(B)} \quad (6)$$

while  $P^*$  is the unique minimum of the growth rate difference:

$$\begin{aligned} \Delta G(P) = & P \sum_{k \in \mathbb{K}_A} p_k(A) \log \left( \frac{P p_k(A)}{P p_k(A) + (1 - P) p_k(B)} \right) + \\ & (1 - P) \sum_{k \in \mathbb{K}_B} p_k(B) \log \left( \frac{(1 - P) p_k(B)}{P p_k(A) + (1 - P) p_k(B)} \right) \end{aligned} \quad (7)$$



## Examples

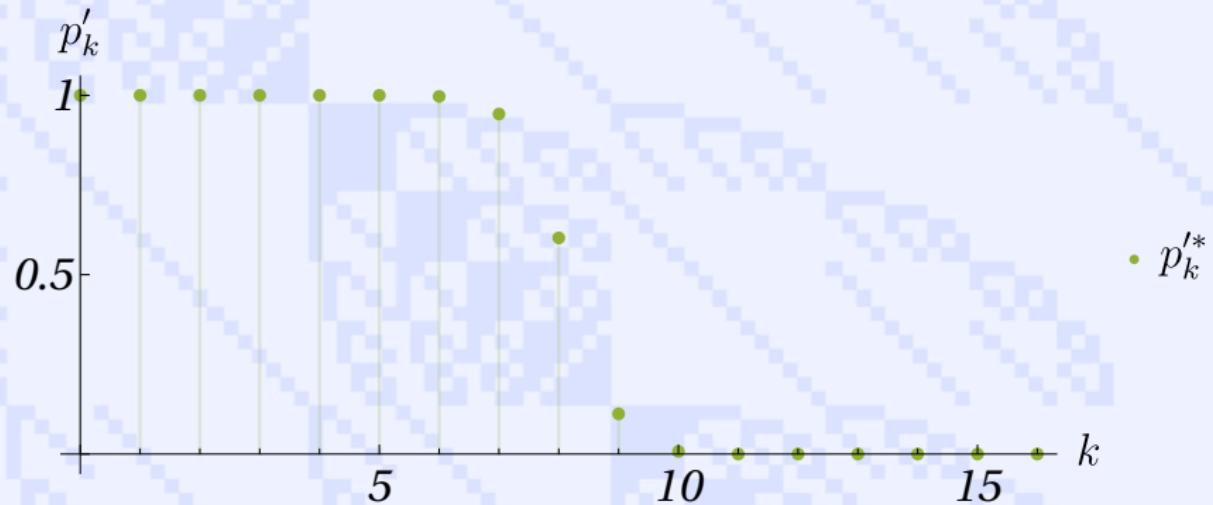
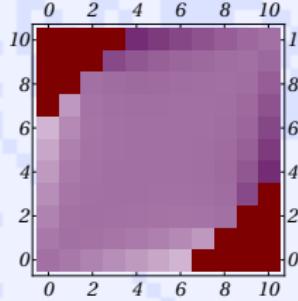
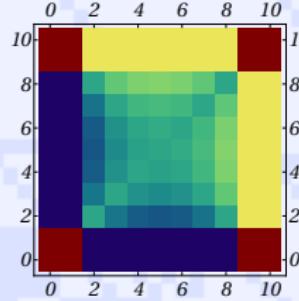
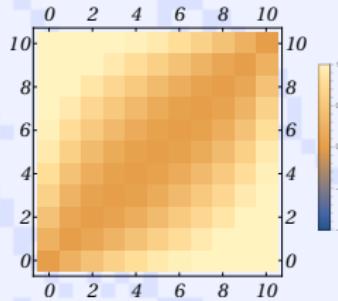


Figure: Illustration of  $p'^*_k$  for  $\text{BGame}(N = 17, K_A = 10, K_B = 16, M = 27)$ , in which case  $P^* \approx 0.4953$ .

## Bayesian games

## Examples

(a)  $P^*(K_A, K_B)$ (b)  $p'_{k=2}^*(K_A, K_B)$ (c)  $G^*(K_A, K_B)/\log(2)$ Figure:  $\mathfrak{BGame}(N = 4, K_A, K_B, M = 10)$

## Bayesian games

## Examples

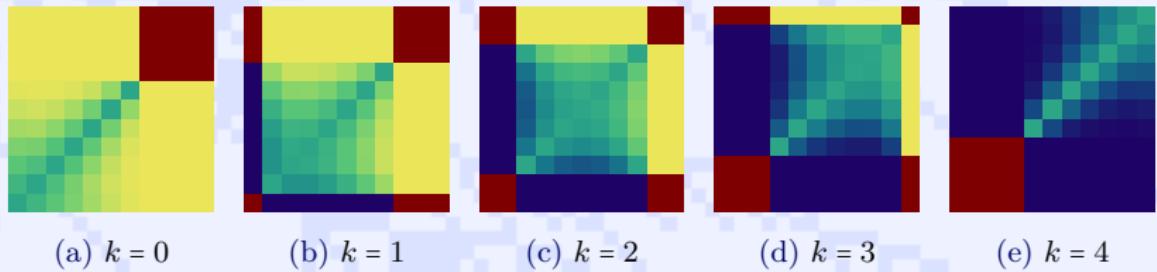
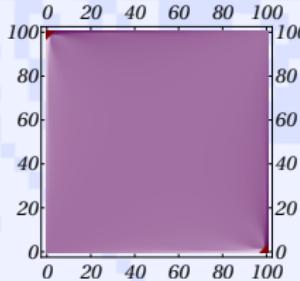
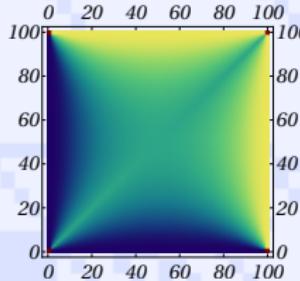
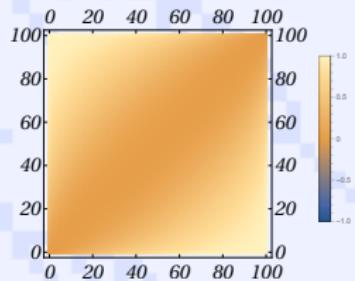


Figure:  $p_k''^*(K_A, K_B)$  for  $\texttt{BGame}(4, K_A, K_B)$ . (Axes and the colour scale have the same meaning as in figure 14b.)

## Bayesian games

## Examples

(a)  $P^*(K_A, K_B)$ (b)  $p'_{k=2}^*(K_A, K_B)$ (c)  $G^*(K_A, K_B)/\log(2)$ Figure:  $\mathfrak{BGame}(N = 4, K_A, K_B, M = 100)$

# Emerged Bayesian concepts

- Bayes' rule
- Exchangeability
- Shannon entropy
- Reference(-like) prior
- Emergence of probability distributions
- Interpretation of mixed strategies

$$P(H|E) = \frac{P(H) \cdot P(E|H)}{P(E)}$$



Figure: Bayes' theorem by  
3Blue1Brown

## Binomial games

## Binomial games

$M \rightarrow \infty$ , while  $K_A/M \rightarrow x_A \in (0, 1)$  and  $K_B/M \rightarrow x_B \in (0, 1)$ <sup>2</sup>.

This sequence of games will be denoted by:

$$\lim_{\substack{M \rightarrow \infty \\ K_A/M \rightarrow x_A \\ K_B/M \rightarrow x_B}} \mathbf{Game}(N, K_A, K_B, M) = \overline{\mathbf{Game}}(N, x_A, x_B) \quad (8)$$

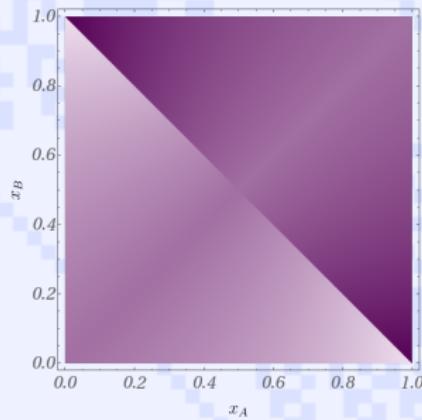
$$\lim_{\substack{M \rightarrow \infty \\ K_A/M \rightarrow x_A \\ K_B/M \rightarrow x_B}} \mathbf{BGame}(N, K_A, K_B, M) = \overline{\mathbf{BGame}}(N, x_A, x_B) \quad (9)$$

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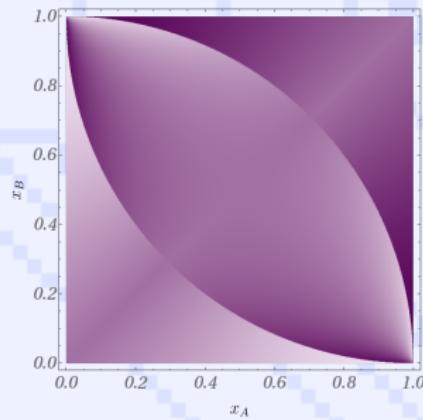
<sup>2</sup> $x_A, x_B$  are similar fractions as mole fractions in chemistry.

## Binomial games

## Examples



(a)  $\overline{\text{Game}}(1, x_A, x_B)$

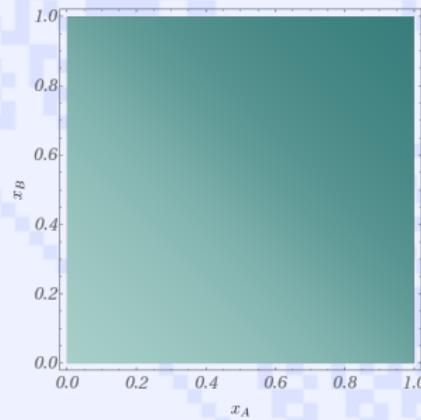
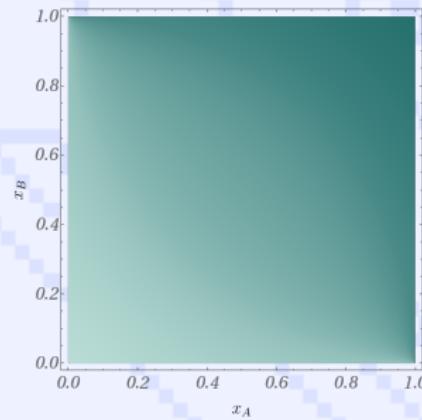


(b)  $\overline{\text{Game}}(2, x_A, x_B)$

Figure:  $P^*(x_A, x_B)$  for Binomial Fisher games.

## Binomial games

## Examples

(a)  $\overline{\text{Game}}(1, x_A, x_B)$ (b)  $\overline{\text{Game}}(2, x_A, x_B)$ Figure:  $s^*(x_A, x_B)$  for Binomial Fisher games.

## Binomial games

## Examples

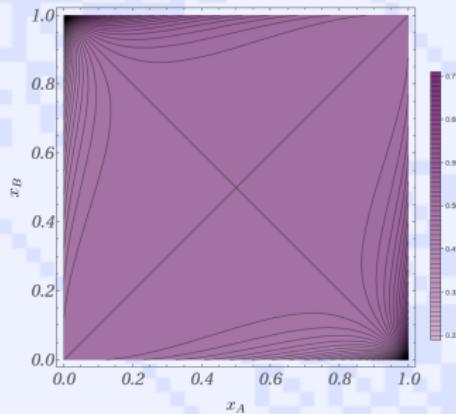
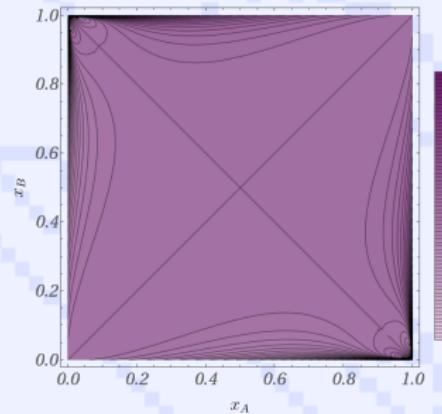
(a)  $\overline{\text{BGame}}(1, x_A, x_B)$ (b)  $\overline{\text{BGame}}(2, x_A, x_B)$ 

Figure:  $P^*(x_A, x_B)$  for Binomial Bayesian games. Contour lines show 1% difference.

## Binomial games

## Examples

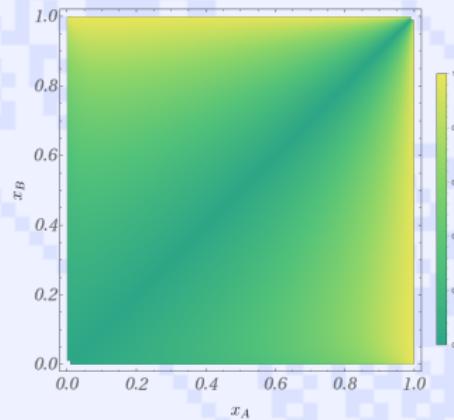
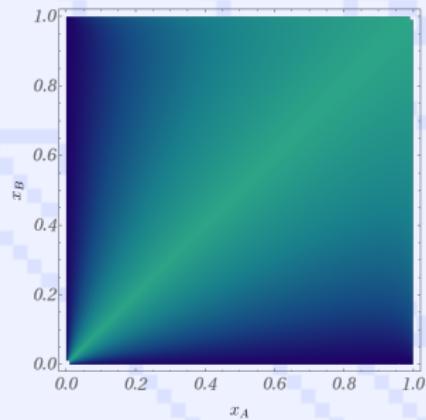
(a)  $k = 0$ (b)  $k = 1$ 

Figure:  $p_k^{**}(x_A, x_B)$  for  $\overline{\text{BGame}}(1, x_A, x_B)$ .

## Binomial games

## Examples

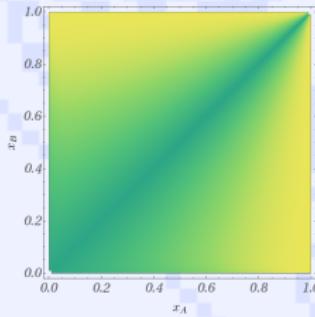
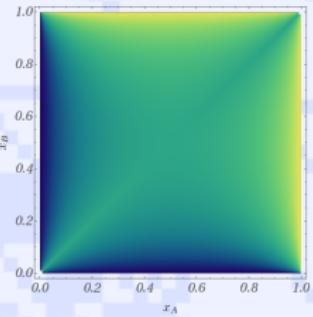
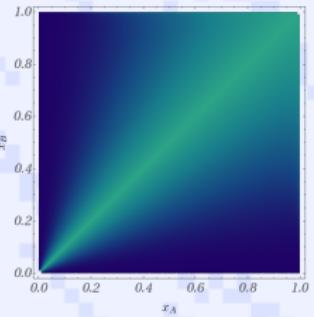
(a)  $k = 0$ (b)  $k = 1$ (c)  $k = 2$ 

Figure:  $p_k^*(x_A, x_B)$  for  $\overline{\text{BGame}}(2, x_A, x_B)$ .

Limiting policies for  $N \rightarrow \infty$ 

$$x_0^*(x_A, x_B) = \frac{\log\left(\frac{1-x_A}{1-x_B}\right)}{\log\left(\frac{(1-x_A)x_B}{(1-x_B)x_A}\right)} \quad (10)$$

$$\lim_{N \rightarrow \infty} s_N^*(x_A, x_B) = s_{\#}^*(x_A, x_B) = x_0^*(x_A, x_B) \quad (11)$$

meaning that for fixed  $0 < x_A < x_B < 1$ :

$$\lim_{N \rightarrow \infty} \frac{k_N^* + \nu_N^*}{N} = \lim_{N \rightarrow \infty} \frac{k_N^*}{N} = s_{\#}^* = x_0^* \quad (12)$$

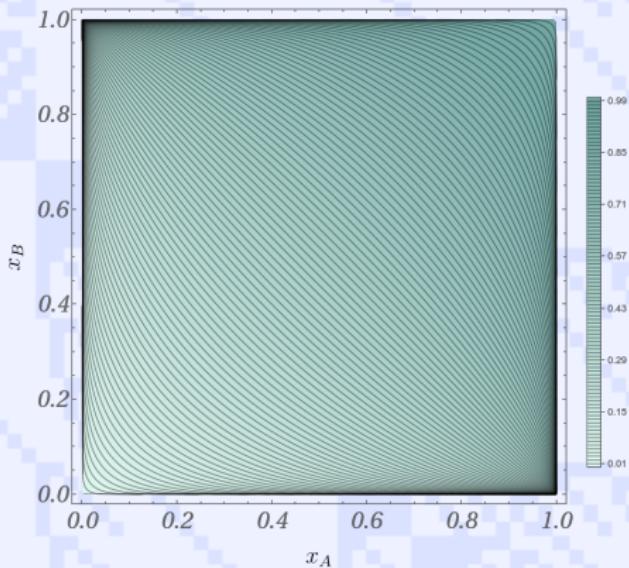
Limiting policies for  $N \rightarrow \infty$ 

Figure: Binomial Fisher policy limit  $s_{\text{FF}}^*(x_A, x_B)$ . Contour lines show 1% difference.

Limiting policies for  $N \rightarrow \infty$ 

$$P_{\text{approx}}^*(x_A, x_B) = \frac{\log\left(\frac{(1-x_0^*)x_B}{(1-x_B)x_0^*}\right)}{\log\left(\frac{(1-x_A)x_B}{(1-x_B)x_A}\right)}, \quad x_0^*(x_A, x_B) = \frac{\log\left(\frac{1-x_A}{1-x_B}\right)}{\log\left(\frac{(1-x_A)x_B}{(1-x_B)x_A}\right)}$$

$x_A \setminus x_B$	20%	30%	40%	50%	60%	70%	80%	90%
10%	0.4761	0.4651	0.4598	0.4584	0.4604	0.4661	0.4772	0.5000
20%		0.4887	0.4832	0.4816	0.4833	0.4889	0.5000	0.5228
30%			0.4944	0.4928	0.4945	0.5000	0.5111	0.5339
40%				0.4983	0.5000	0.5055	0.5167	0.5396
50%					0.5017	0.5072	0.5184	0.5416
60%						0.5056	0.5168	0.5402
70%							0.5113	0.5349
80%								0.5239

Table: Binomial Bayesian limit prior approximation  $P_{\text{approx}}^*(x_A, x_B)$  up to 4 digits.

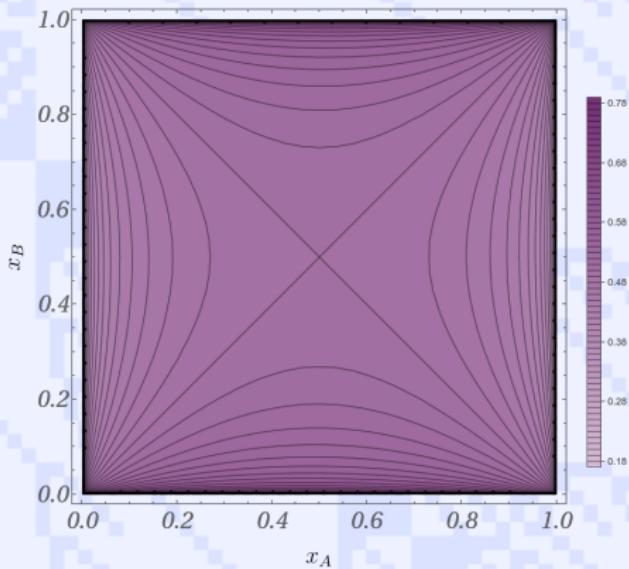
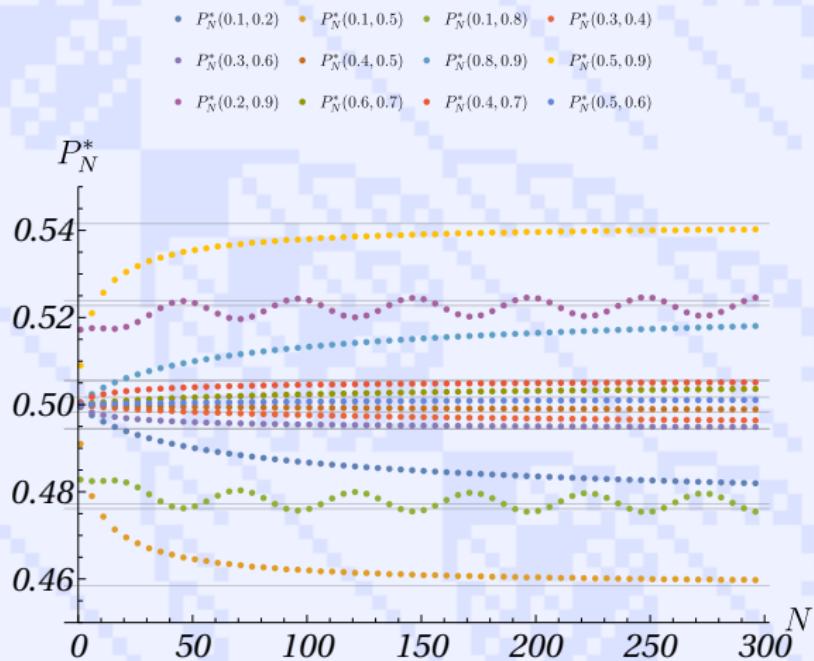
Bayesian game prior approximation for  $N \rightarrow \infty$ 

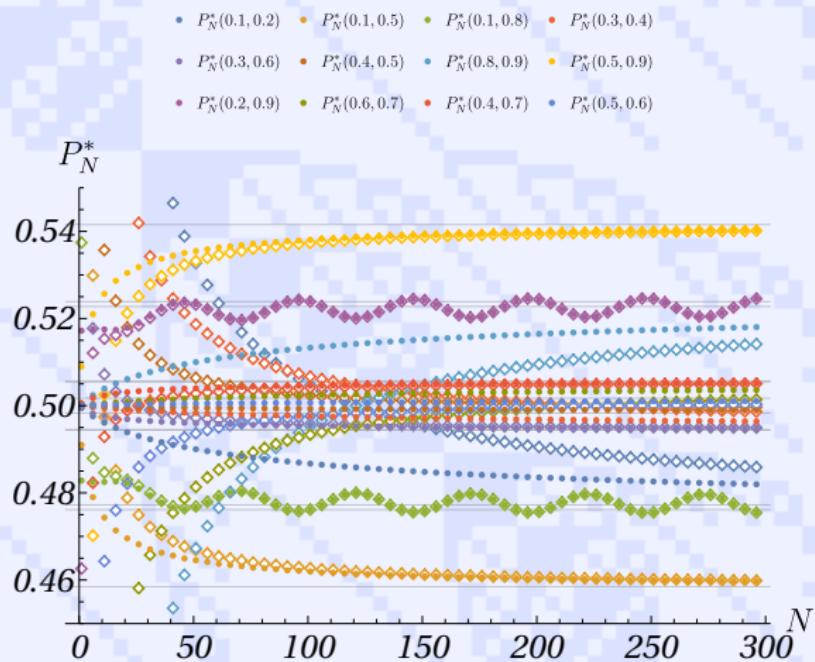
Figure: Binomial Bayesian limiting prior approximation  $P_{\approx}(x_A, x_B)$ .  
Contour lines show 1% difference.

$N \rightarrow \infty$  limit

## Numerical evidence



## Numerical evidence and Asymtotic expansion



First-order Asymptotic expansion of  $P_N(x_A, x_B)$ 

$$P_N^{*,(1)}(x_A, x_B) = \sigma \left( \vartheta_{\sharp}^{*,\varphi_N(x_A, x_B)}(x_A, x_B) + \frac{1}{N} \lambda^{\varphi_N(x_A, x_B)}(x_A, x_B) \right)$$

where  $\sigma(\cdot)$  stands for the sigmoid function  $\sigma(x) = 1/(1 + e^{-x})$ .

$$\varphi = 2\pi N x_0^* \mod 2\pi$$

For details see Appendix D in arXiv:2402.15892



## Definition (Statistical game)

Same as  $\mathfrak{BGame}(N, K_A, K_B, M)$ , but with isoelastic utility function:

$$u_\gamma(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma} \quad (13)$$

with relative risk aversion parameter  $\gamma > 0, \gamma \neq 1$ .

The above defined Statistical Game will be denoted as

$\mathfrak{SGame}(N, K_A, K_B, M, \gamma)$ .



For arguments why are isoelastic utilities special, see Appendix E in arXiv:2402.15892

## Statistical games

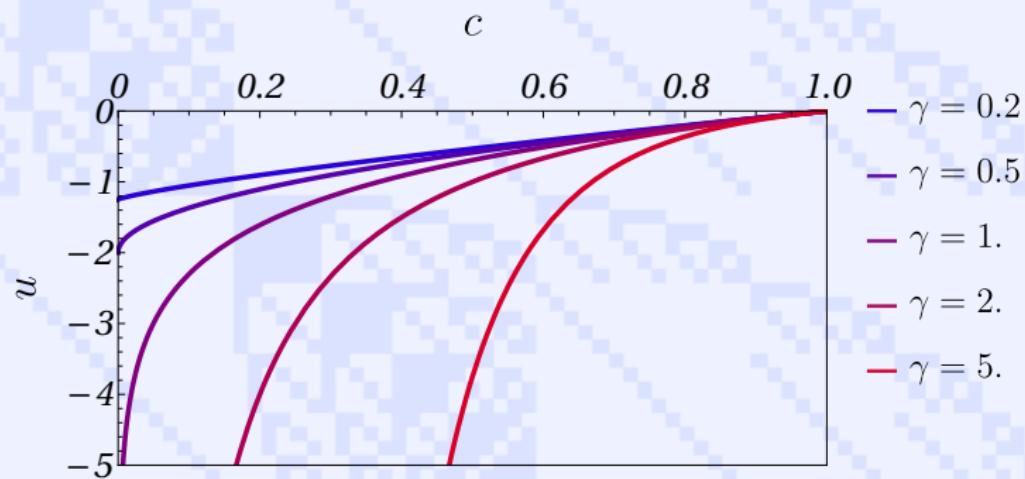


Figure: Isoelastic utility functions, for several relative risk aversion ( $\gamma$ ) parameters:  $u_\gamma(c)$ .

## Statistical games

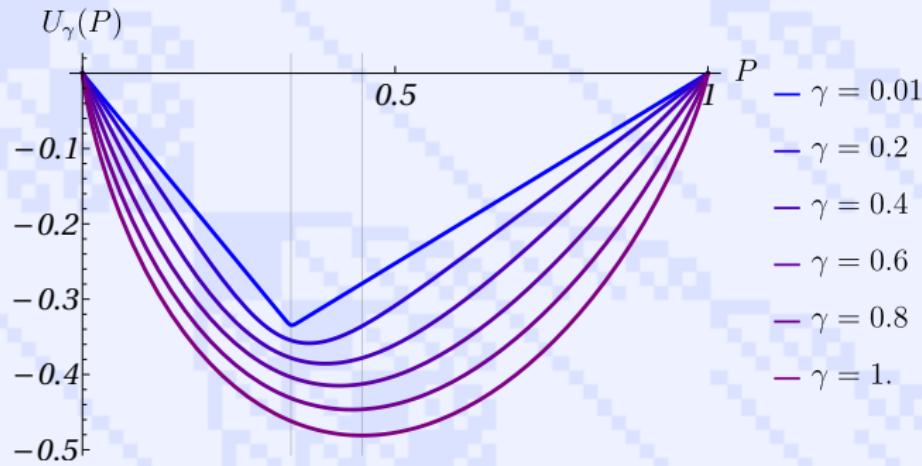


Figure: Expected utility for **SGame**( $N = 1, K_A = 0, K_B = 1, M = 2$ ) as a function of  $P$  for different  $\gamma$  (relative risk aversion) values. (Vertical gridlines are placed at  $1/3$  and  $1/\sqrt{5}$  values.)

## Statistical games

## Theorem (Isoelastic equilibrium)

**SGame**( $N, K_A, K_B, M, \gamma$ ) has a unique Nash equilibrium, in which:

$$p_k(A) = \frac{\binom{K_A}{k} \binom{M-K_A}{N-k}}{\binom{M}{N}}, \quad p_k(B) = \frac{\binom{K_B}{k} \binom{M-K_B}{N-k}}{\binom{M}{N}} \quad (14)$$

$$p'_k(P) = \frac{(P p_k(A))^{1/\gamma}}{(P p_k(A))^{1/\gamma} + ((1-P) p_k(B))^{1/\gamma}} \quad (15)$$

$$p'^*_k = p'_k(P^*_\gamma) \quad (16)$$

while  $P^*_\gamma$  is the unique minimum of the expected utility:

$$U_\gamma(P) = P \left( \sum_k p_k(A) u_\gamma(p'_k(P)) \right) + (1-P) \left( \sum_k p_k(B) u_\gamma(1-p'_k(P)) \right) \quad (17)$$

where the utility function for a given relative risk aversion  $\gamma > 0, \gamma \neq 1$  is:

$$u_\gamma(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma} \quad (18)$$



Statistical games

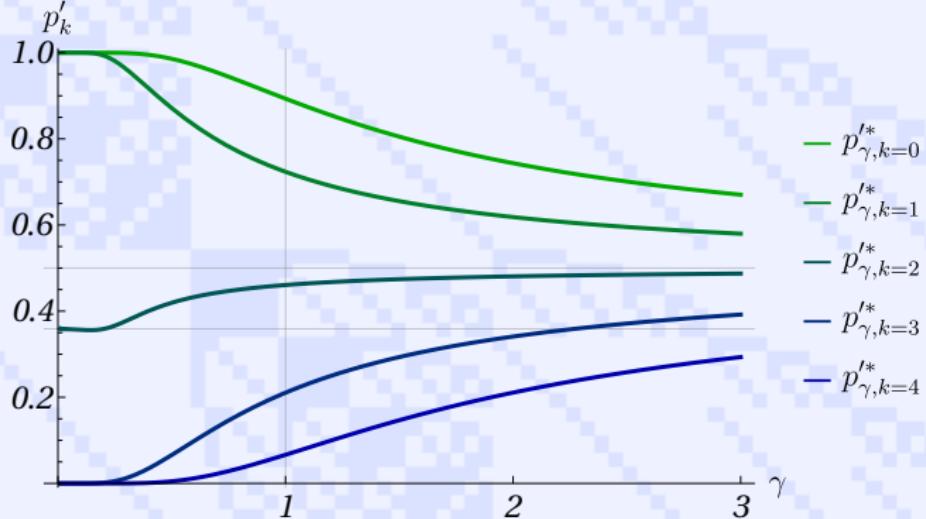


Figure: Splitting strategies,  $p_{\gamma,k}^{*}$  for **SGame**( $N = 4, K_A = 5, K_B = 8, M = 14$ ) as a function of  $\gamma$  (relative risk aversion parameter). Horizontal gridlines are placed at  $\{1/2, 14/39 \approx 0.359\}$ .

## Theorem

*The equilibrium strategies of a Statistical game*

**SGame**( $N, K_A, K_B, M, \gamma$ ) in the  $\gamma \rightarrow 0$  limit can be mapped to the symmetric equilibrium strategies of a Fisher game

**Game**( $N, K_A, K_B, M$ ) <sup>a</sup> with the following identification:

$$\lim_{\gamma \rightarrow 0} p'_{\gamma, k^*}^* = \nu^* \quad (19)$$

$$\lim_{\gamma \rightarrow 0} P_\gamma^* = P_0^* \quad (20)$$

Where  $(P_\gamma^*, \{p'_{\gamma, k}^*\})$  are the equilibrium parameters of **SGame**( $N, K_A, K_B, M, \gamma$ ), while  $(k^*, \nu^*, P_0^*)$  are the equilibrium parameters of **Game**( $N, K_A, K_B, M$ ).




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<sup>a</sup>if  $\nu^* \neq 0$

## Unification

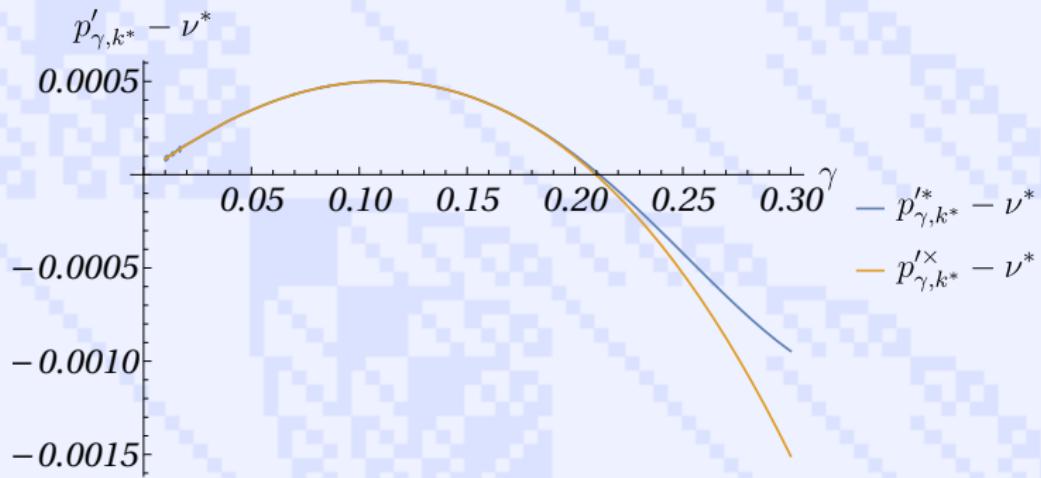


Figure: Illustration of  $p'^*_{\gamma,k^*}$  and  $p'^{\times}_{\gamma,k^*}$  for  $\text{SGame}(N = 2, K_A = 2, K_B = 7, M = 10)$ .

# Philosophical part

Game-theoretical framework for Decision making in the face of Uncertainty:

- Let us assume, the followings:
  - An Agent can restrict the possible states of the world to a finite set. We will denote this set by  $\Theta$  and call it the parameter set;
  - The Agent can consider only a finite set of possible actions. The set of actions will be denoted by  $\mathcal{A}$ ;
  - Lastly, the Agent can associate utilities (or rewards) to all potential consequences, which depends both on her action and the possible state of the world. This function (in the finite representable by a matrix) will be denoted by  $U : \mathcal{A} \times \Theta \mapsto \mathbb{R}$ .

## Philosophical part

Under these assumptions, the game-theoretic framework for statistics would suggest the following strategy for the Agent:

- Imagine that the unknown parameter  $\theta \in \Theta$  has been chosen by an opponent whose utility function is the regret of the Agent;
- Determine the Nash equilibrium for such a two-player non-cooperative game;
- Adopt the equilibrium strategy of this imagined game to choose an action from the action set  $\mathcal{A}$ .

# Future work and extension

- Assumptions about the unknown ( $\mathbb{M}$  vs.  $\mathbb{D}$ )
- Target of the Inference
  - “Platonian” Inference (inferential risk)
  - “Aristotelian” Inference (predictive risk)
  - Data compression
  - General
- Correspondence with Bayesians and Frequentists
- Effective approximative numerical methods (Blahut–Arimoto algorithm)
- Generalizing Game Theory with a player representing uncertainty
- Using the concept in Reinforcement Learning
- Foundations of Statistical Physics
- Quantum metrology
- Universal Inductive Inference
- ...



Figure: La Sagrada Família, Antoni Gaudí:  
“My client is in no hurry”.

# How is this related to AI?

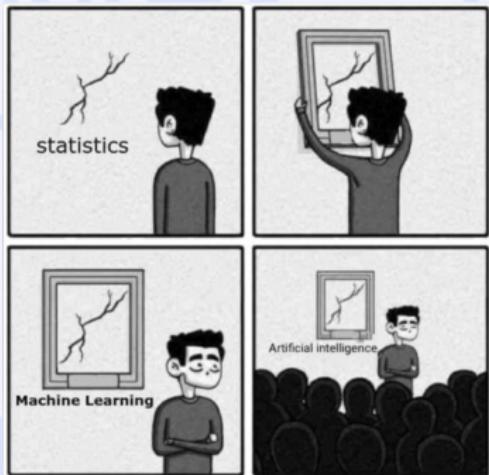


Figure: No, Machine Learning is not just glorified Statistics

- Statistics → Game theory
- Frame:
  - Effective approximations and Scaling → Goal engineering
- Machine Learning (ML) → Generative Adversarial Network (GAN)
- Artificial Intelligence (AI) → Better aligned AI, multi agent RL/AI

# Thank you for your attention!



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arXiv:2402.15892

Introduction  
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Fisher & Bayesian games  
oooooooooooo

Limiting cases  
ooooooo

Statistical games  
ooooo  
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Philosophy & Future work  
oooo●

