Mean Value Theorem Proof

The **mean value theorem** is one of the most useful tools in both differential and integral calculus. It has very important consequences in differential calculus and helps us to understand the identical behaviour of different functions.

The hypothesis and conclusion of the mean value theorem show some similarities to those of the intermediate value theorem. The mean value theorem is also known as Lagrange's mean value theorem. This theorem is abbreviated as MVT.

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Mean Value Theorem Statement

Suppose f(x) is a function that satisfies below conditions:

1. f(x) is continuous in [a,b]
2. f(x) is differentiable in (a,b)

Then, there exists a number c, such that a < c < b and

$f(b) - f(a) = f'(c)\,(b - a)$

Special Case:

When f(a) = f(b), then there exists at least one c with a < c < b such that f′(c) = 0. This case is known as ***Rolle's Theorem***.

Consider a line passing through the points (a, f(a)) and (b, f(b)). The equation of the line is

y – f(a) = {f(b) – f(a)}/(b-a) . (x – a)

or y = f(a)+ {f(b) – f(a)}/(b-a) . (x – a)

Let h be a function that defines the difference between any function f and the above line.

h(x) = f(x) – f(a) – {f(b)-f(a)}/(b-a) . (x – a)

Using "Rolle's theorem", we have

h'(x) = f'(x) – {f(b)-f(a)}/(b-a)

Or f(b) – f(a) = f'(x) (b – a). Hence, proved.

Physical Interpretation of Mean Value Theorem

Since (f(b)-f(c))/(b-a) is the average change in the function over [a, b], and f'(c) is the instantaneous change at 'c', the mean value theorem states that at some interior point, the instantaneous change is equal to the average change of the function over the interval.

Corollaries of Mean Value Theorem

Corollary 1: If f'(x) = 0 at each point of x of an open interval (a, b), then f(x) = C for all x in (a, b), where C is a constant.

Corollary 2: If f'(x) = g'(x) at each point x in an open interval (a, b), then there exists a constant C such that f(x) = g(x) + C.

The first corollary confirms that if the derivative of a function is zero, then the function is a constant function. The second corollary says that the graphs of functions with identical derivatives differ only by a vertical shift. This property is used to solve initial value problems in integral calculus.

Application of Mean Value Theorem

The mean value theorem is the relationship between the derivative of a function and the increasing or decreasing nature of the function. It basically defines the derivative of a differential and continuous function. Below are a few important results used in the mean value theorem.

1. Let the function be f such that it is continuous in interval [a,b] and differentiable on interval (a,b), then

f'(x) = 0, x ∈ (a,b), then f(x) is constant in [a,b].

2. Let f and g be functions such that f and g are continuous in interval [a,b] and differentiable on interval (a,b),

f'(x) = g'(x), x ∈ (a,b), then f(x) – g(x) is constant in [a,b].

3. Strictly increasing function

Let the function be f such that, continuous in interval [a, b] and differentiable in interval(a,b)

f'(x) > 0, x ∈ (a,b), then f(x) is a strictly increasing function in [a,b].

4. Strictly decreasing function

Let the function be f such that, continuous in interval [a,b] and differentiable in interval (a, b)

f'(x) < 0, x ∈ (a,b), then f(x) is a strictly decreasing function in [a,b].

and are differentiable in the open interval (a,b).

Then, there exists at least one point c ∈ (a,b) such that

$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

.

If we take g(x) = x for every x ∈ {a,b] in Cauchy's mean value theorem, we get

$\frac{f(b)-f(a)}{b-a} = f'(c)$

which is **Langrange's mean value theorem**. This is also called an extended mean value theorem.

Generalised Mean Value Theorem

If we have three functions f, g and h defined in such that,

f, g and h are continuous in [a,b],

f, g and h are derivable in (a,b).

$$D(x) = \begin{matrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{matrix}$$

Then, there exists a real number c ∈ (a,b) such that, D'(c) = 0

$$\begin{matrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{matrix} = 0$$

In the above, if we take g(x) = x and h(x) = 1, we obtain Langrange's mean value theorem, and if we take h(x) = 1, then we obtain Cauchy's mean value theorem.

Mean Value Theorem for Derivatives

With the help of the mean value theorem, we approximate the derivative of any function. Theorem can build a relationship between the slope of a tangent line and the secant line on a curve.

If f is differentiable over (a,b) and continuous over [a,b], then there exists a point

c in such a way that f'(c) = {f(b)-f(a)}/(b-a).

It shows that the actual slope is equal to the average slope at some point in the closed interval. Geometrically, we can say that between two endpoints of the curve, we have at least one point on the curve where the slope of the tangent line is equal to the slope of the secant line passing through A and B.

Also, read:

Theorems for Differentiation (https://byjus.com/jee/theorems-on-differentiation/)

Limits Continuity and Differentiability (https://byjus.com/jee/limits-continuity-and-differentiability/)

Mean Value Theorem Examples

Given below are examples of the mean value theorem for better understanding.

Question 1: Find the value or values of c, which satisfy the equation

$\frac{f(b)-f(a)}{b-c} = f'(c)$

, as stated in the mean value theorem for the function

$f(x) = \sqrt{(x-1)}$

in the interval [1, 3].

Solution:

$\frac{f(3)-f(1)}{3-1} = \frac{\sqrt{2}-0}{2} = \frac{\sqrt{2}}{2}$

.

Now, the derivative of the function can be found using the chain rule as

$f'(x) = \frac{1}{2\sqrt{x-1}}$

.

Hence, the equation can be formed as

$f'(c) = \frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}}{2}$

Cross multiplying and squaring the equation reduces to,

2(c − 1) = 1, which gives the solution as c = 3/2, which lies in the given interval [1, 3].

Question 2: Verify Rolle's theorem for the function f(x) = x^2 − 8x + 12 on (2, 6).

Solution:

Since a polynomial function is continuous and differentiable everywhere, f(x) is differentiable and continuous conditions of Rolle's theorem are satisfied.

f (2) = 2^2 − 8 (2) + 12 = 0

f (6) = 6^2 − 8(6) + 12 = 0

This implies, f(2) = f(3)

Therefore, Rolle's theorem is applicable for the given function f(x).

There must exist c ∈ (2, 6) such that f'(c) = 0

f'(x) = 2x − 8

f'(c) = 2c − 8

2c − 8 = 0

c = 4 ∈ (2,6)

Therefore, Rolle's theorem is verified.

Question 3: For the function f(x) = e^x, a = 0, b = 1, find the value of c in the mean value theorem.

Solution:

$\frac{f(b)-f(a)}{b-a} = f'(c)$

$\frac{e^b-e^a}{b-a} = f'(c)$

$\frac{e-1}{1-0} = e^c$

$\Rightarrow c = \log(e-1)$

Question 4: From mean value theorem,

$f(b) - f(a) = (b-a)f'(x_1); a < x_1 < b \text{ if } f(x) = \frac{1}{x}$

. Find the value of x_1.

Solution:

$f'(x_1) = \frac{-1}{x_1^2}$

$\therefore \frac{-1}{x_1^2} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$

$\Rightarrow x_1 = \sqrt{ab}$

Question 5: Let f(x) satisfy all the conditions of the mean value theorem in [0, 2]. If f (0) = 0 and |f'(x)| ≤ 1/2 for all x, in [0, 2], then

(A) f(x) ≤ 2

Solution:

$\frac{f(2)-f(0)}{2-0} = f'(x) \Rightarrow \frac{f(2)-0}{2} = f'(x)$

$\Rightarrow \frac{df(x)}{dx} = \frac{f(2)}{2} \Rightarrow f(x) = \frac{f(2)}{2}x + c$

So $f(0) = 0 \Rightarrow c = 0;$

$f(x) = \frac{f(2)}{2}x \ldots\ldots (i)$

Given

$|f'(x)| \leq \frac{1}{2} \Rightarrow \ \frac{f(2)}{2} \ \leq \frac{1}{2} \ldots (ii)$

$(i)|f(x)| = \ \frac{f(2)}{2}x \ = \ \frac{f(2)}{2} \ |x| \leq \frac{1}{2}|x|(from\ (ii))$

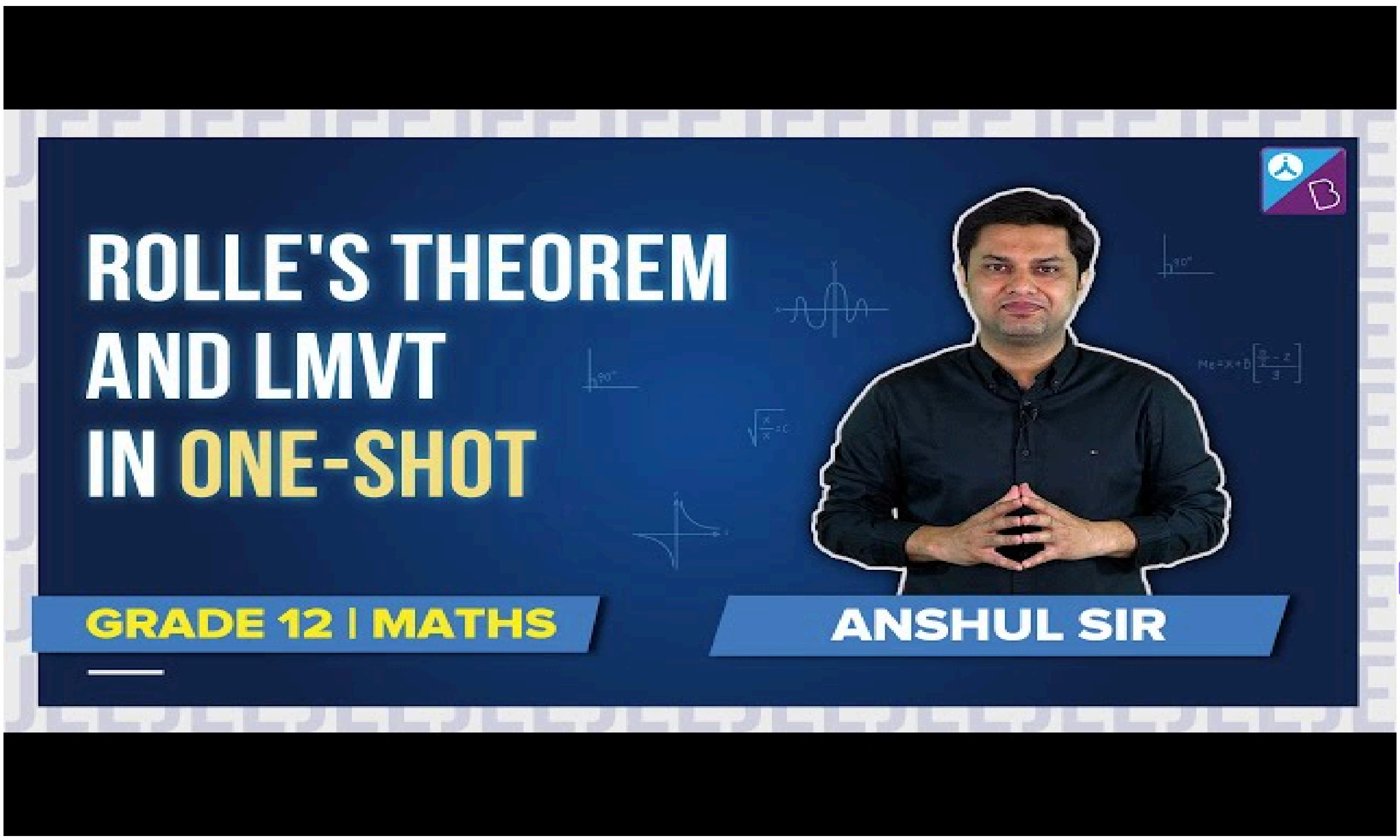
In [0, 2], for maximum

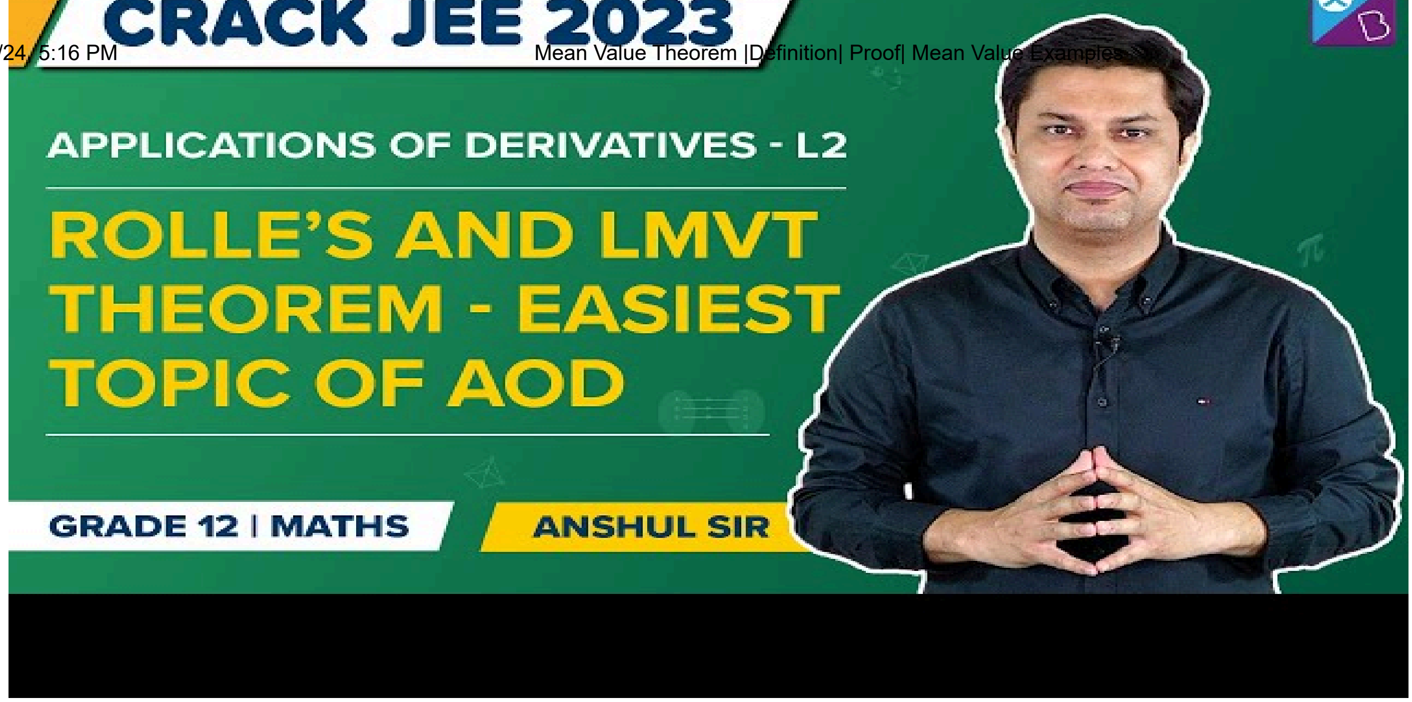
$x(x=2)|f(x)| \leq \frac{1}{2}. \ 2 \Rightarrow |f(x)| \leq 1$

.

Hence option B is the answer.

Rolle’s Theorem and Lagrange’s Mean Value Theorem – Important Topics





Frequently Asked Questions

Q1 **State the mean value theorem.**

The mean value theorem states that if f(x) is a function such that f(x) is continuous in [a,b] and f(x) is differentiable in (a,b), then there exists some c in (a, b), such that f'(c) = [f(b)–f(a)]/(b-a).

Q2 **State Rolle's theorem.**

Rolle's theorem states that let f(x) is a function continuous on [a, b] and differentiable on (a, b), such that f(a) = f(b), where a and b are some real numbers, then there exists some c in (a, b) such that f'(c) = 0.

Q3 **Give the mean value theorem formula.**

The mean value theorem formula is f'(c) = [f(b) – f(a)]/(b-a).

Test your Knowledge on Mean Value Theorem

Put your understanding of this concept to test by answering a few MCQs. Click 'Start Quiz' to begin!

Select the correct answer and click on the "Finish" button
Check your score and answers at the end of the quiz

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