

Stanford CS224W: How Expressive are Graph Neural Networks?

CS224W: Machine Learning with Graphs
Jure Leskovec, Stanford University
<http://cs224w.stanford.edu>



ANNOUNCEMENTS

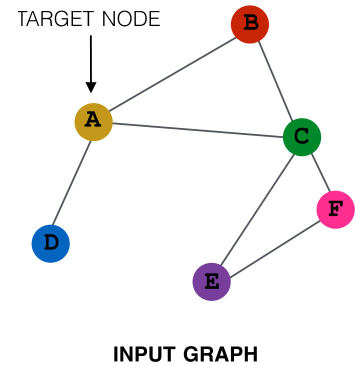
- Homework 1 due on **Thursday** (2/2)
- Based on course feedback, we will hold in-person OHs every week on Wednesday 9-11 AM PT. Location will be updated on the OH calendar.

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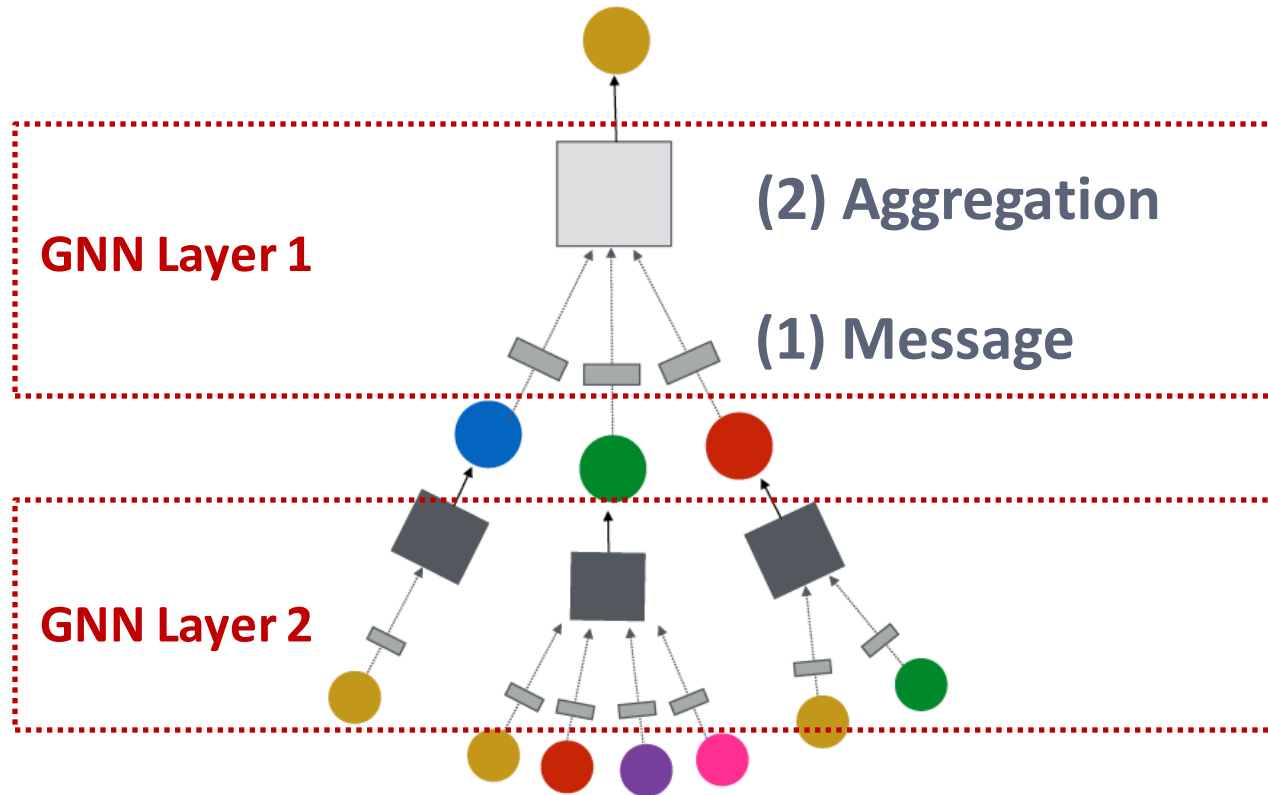


Recap: A General GNN Framework

(5) Learning objective

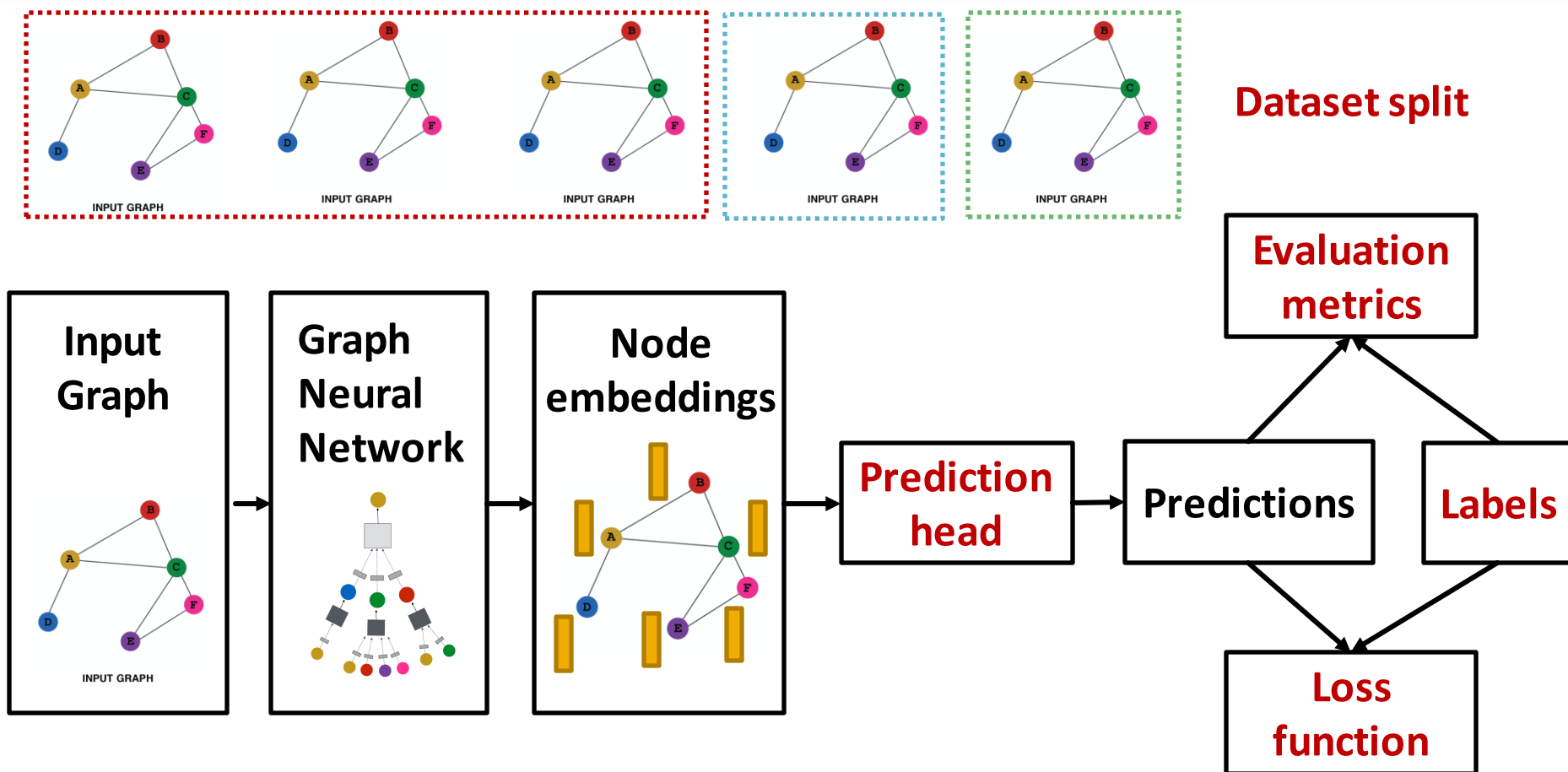


(3) Layer connectivity



(4) Graph augmentation

Recap: GNN Training Pipeline



Implementation resources:

[PyG](#) provides core modules for this pipeline

[GraphGym](#) further implements the full pipeline to facilitate GNN design

Theory of GNNs

How powerful are GNNs?

- Many GNN models have been proposed (e.g., GCN, GAT, GraphSAGE, design space).
- What is the expressive power (ability to distinguish different graph structures) of these GNN models?
- How to design a maximally expressive GNN model?

Background: A Single GNN Layer

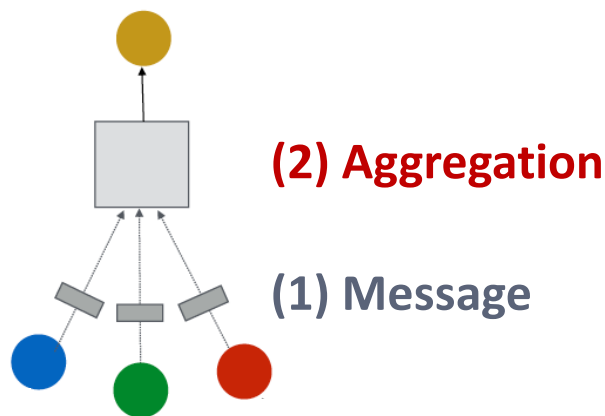
- We focus on message passing GNNs:

- **(1) Message**: each node computes a message

$$\mathbf{m}_u^{(l)} = \text{MSG}^{(l)} \left(\mathbf{h}_u^{(l-1)} \right), u \in \{N(v) \cup v\}$$

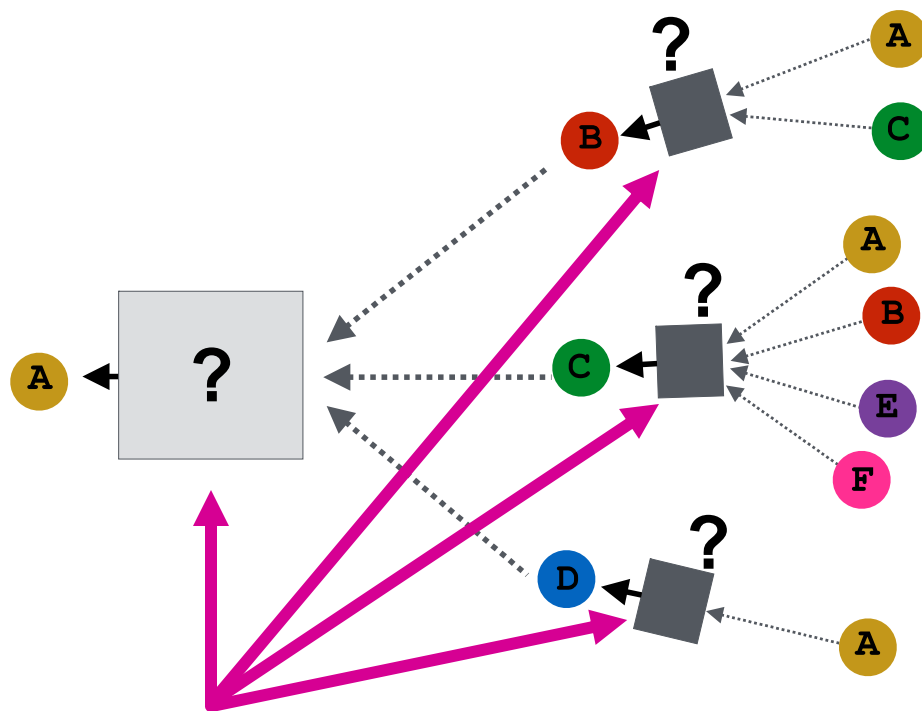
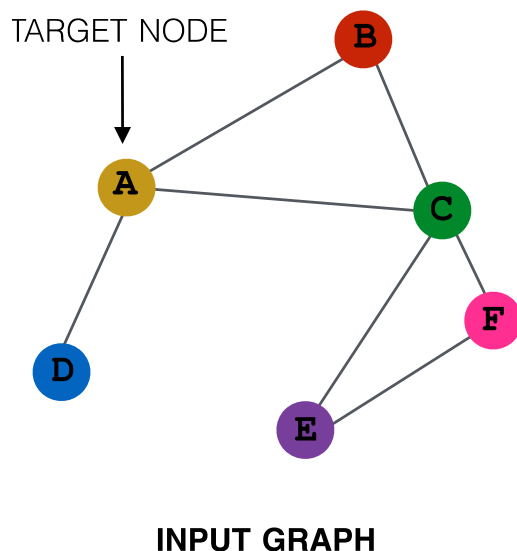
- **(2) Aggregation**: aggregate messages from neighbors

$$\mathbf{h}_v^{(l)} = \text{AGG}^{(l)} \left(\left\{ \mathbf{m}_u^{(l)}, u \in N(v) \right\}, \mathbf{m}_v^{(l)} \right)$$



Background: Many GNN Models

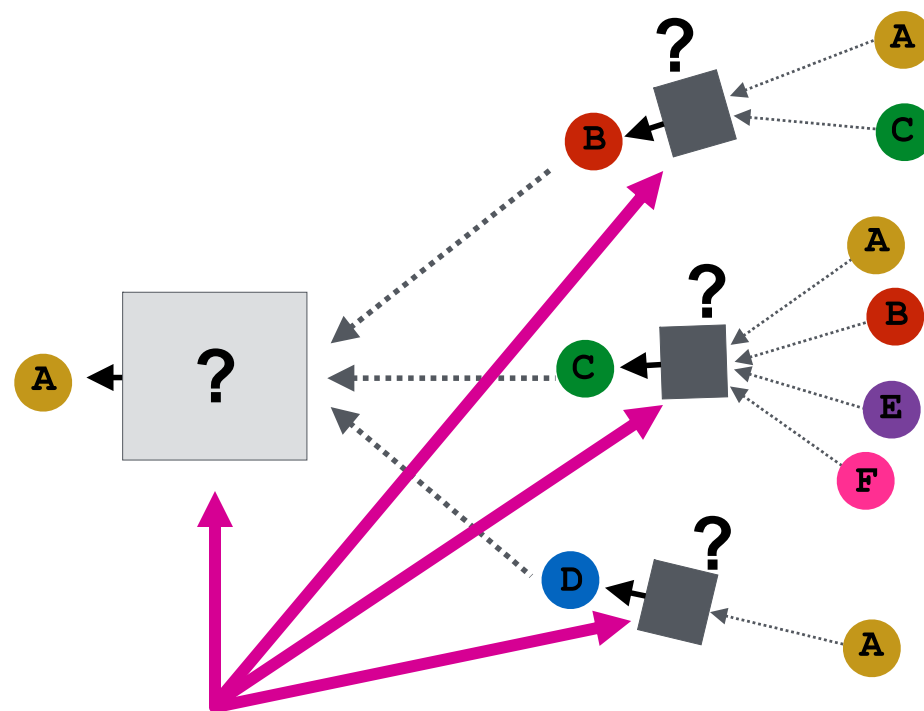
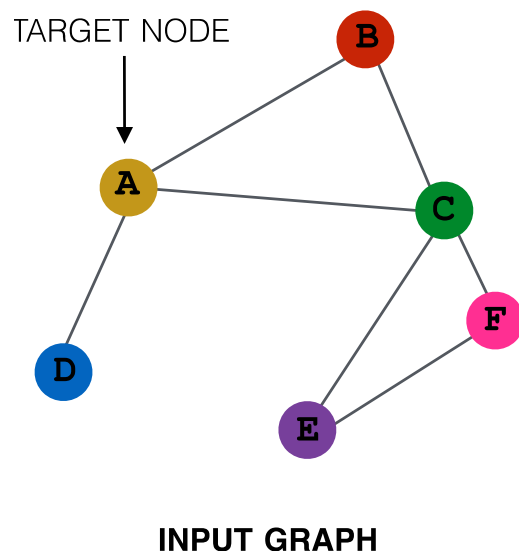
- Many GNN models have been proposed:
 - GCN, GraphSAGE, GAT, Design Space etc.



Different GNN models use different neural networks in the box

GNN Model Example (1)

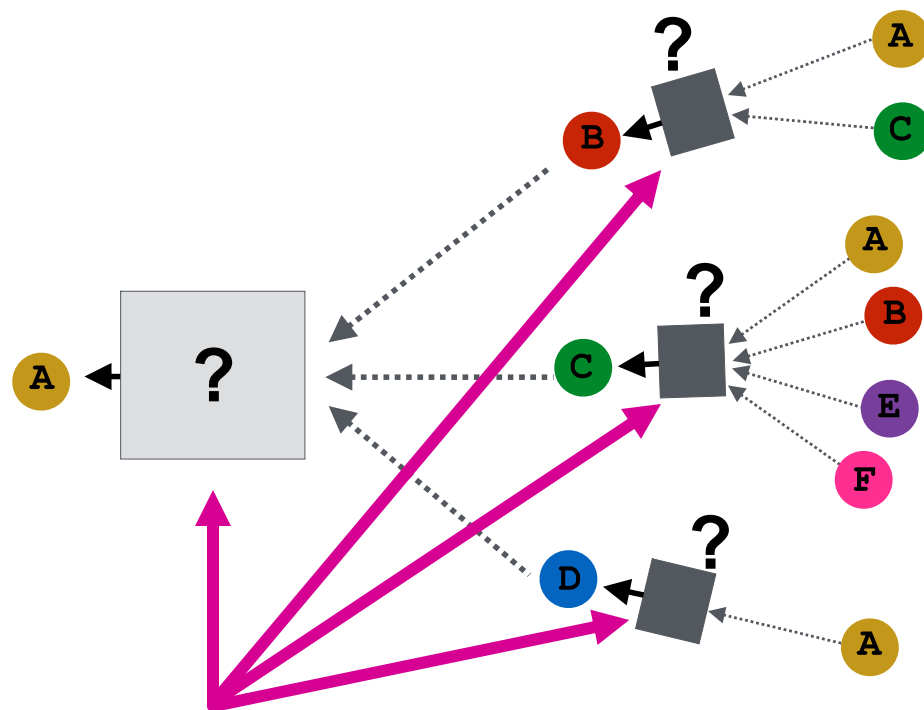
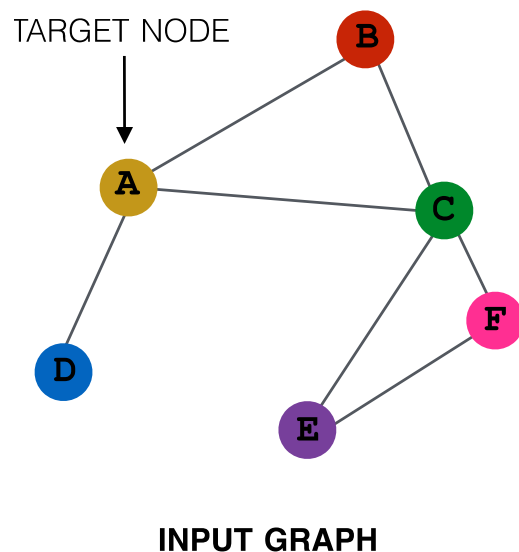
■ GCN (mean-pool) [Kipf and Welling ICLR 2017]



Element-wise mean pooling +
Linear + ReLU non-linearity

GNN Model Example (2)

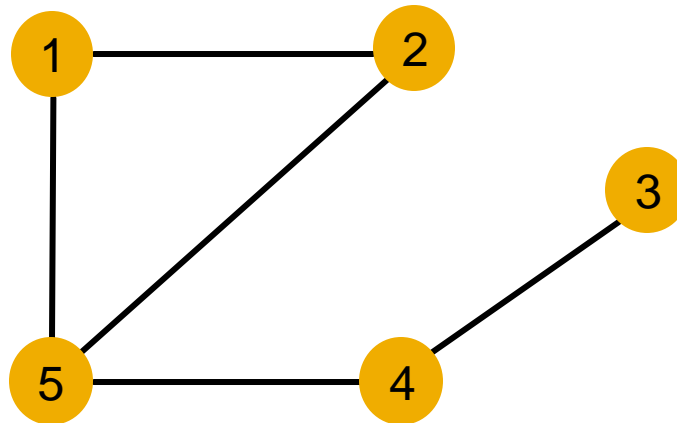
■ GraphSAGE (max-pool) [Hamilton et al. NeurIPS 2017]



MLP + element-wise max-pooling

Note: Node Colors

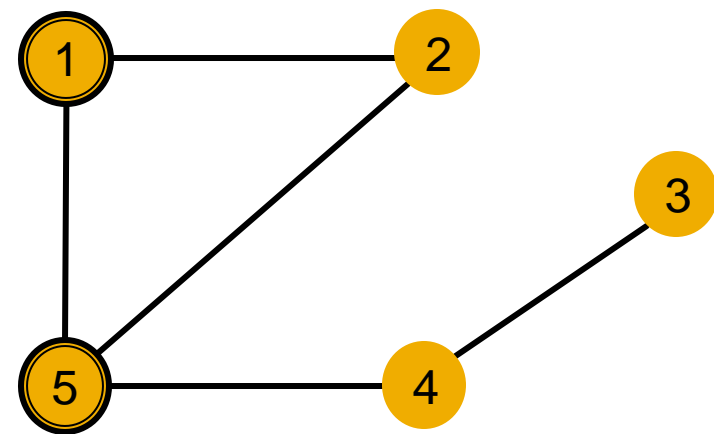
- We use node same/different **colors** to represent nodes with same/different features.
 - For example, the graph below assumes all the nodes share the same feature.



- **Key question:** How well can a GNN distinguish different graph structures?

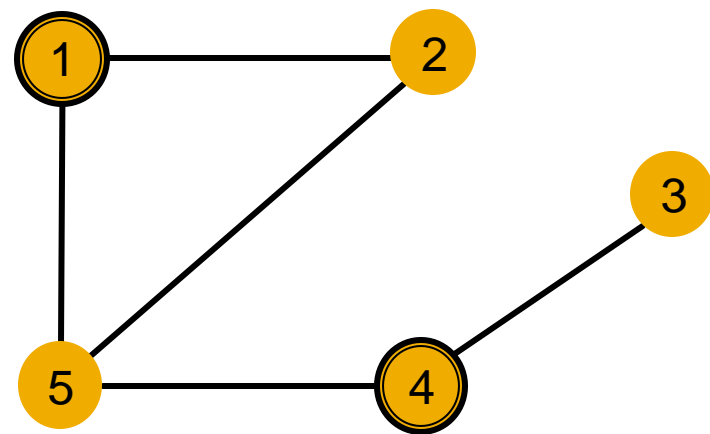
Local Neighborhood Structures

- We specifically consider **local neighborhood structures** around each node in a graph.
- **Example:** Nodes 1 and 5 have different neighborhood structures because they have different node degrees.



Local Neighborhood Structures

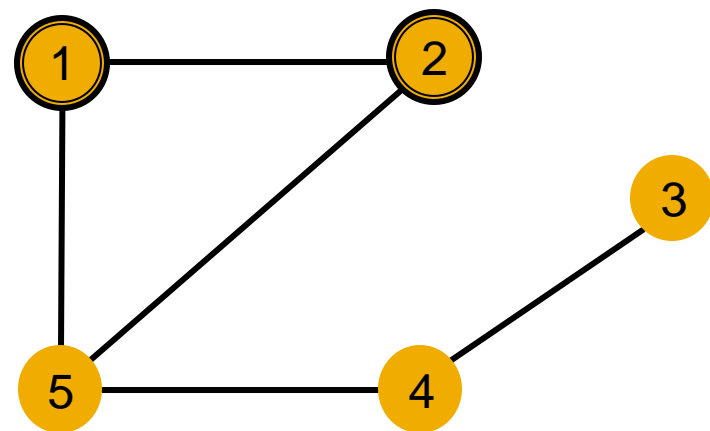
- We specifically consider **local neighborhood structures** around each node in a graph.
- **Example:** Nodes 1 and 4 both have the same node degree of 2. However, they still have **different** neighborhood structures because **their neighbors have different node degrees**.



Node 1 has neighbors of degrees 2 and 3.
Node 4 has neighbors of degrees 1 and 3.

Local Neighborhood Structures

- We specifically consider **local neighborhood structures** around each node in a graph.
- **Example:** Nodes 1 and 2 have the **same** neighborhood structure because **they are symmetric within the graph.**



Node 1 has neighbors of degrees 2 and 3.

Node 2 has neighbors of degrees 2 and 3.

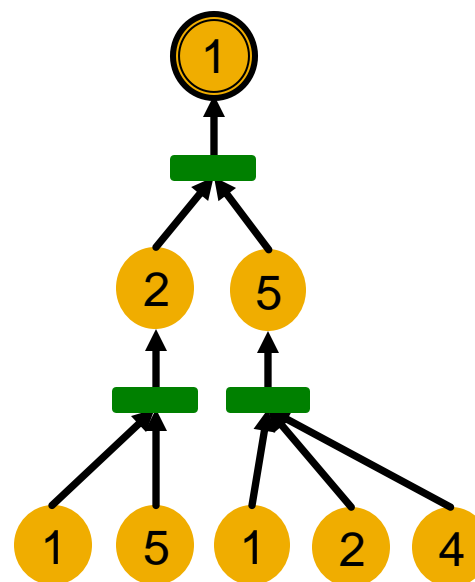
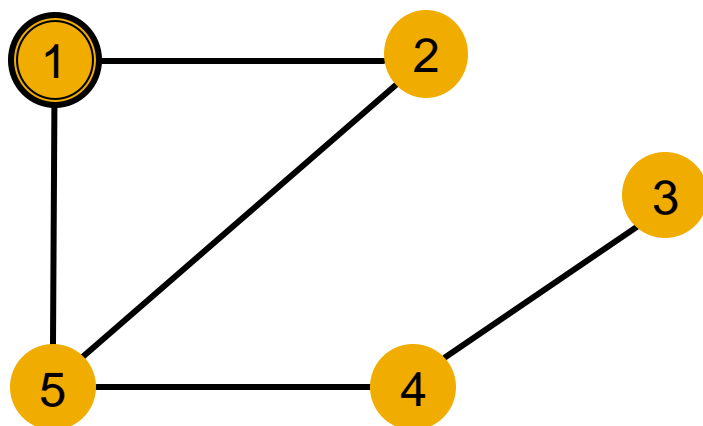
And even if we go a step deeper to 2nd hop neighbors, both nodes have the same degrees (Node 4 of degree 2)

Local Neighborhood Structures

- **Key question:** Can GNN node embeddings distinguish different node's local neighborhood structures?
 - If so, when? If not, when will a GNN fail?
- **Next:** We need to understand how a GNN captures local neighborhood structures.
 - Key concept: **Computational graph**

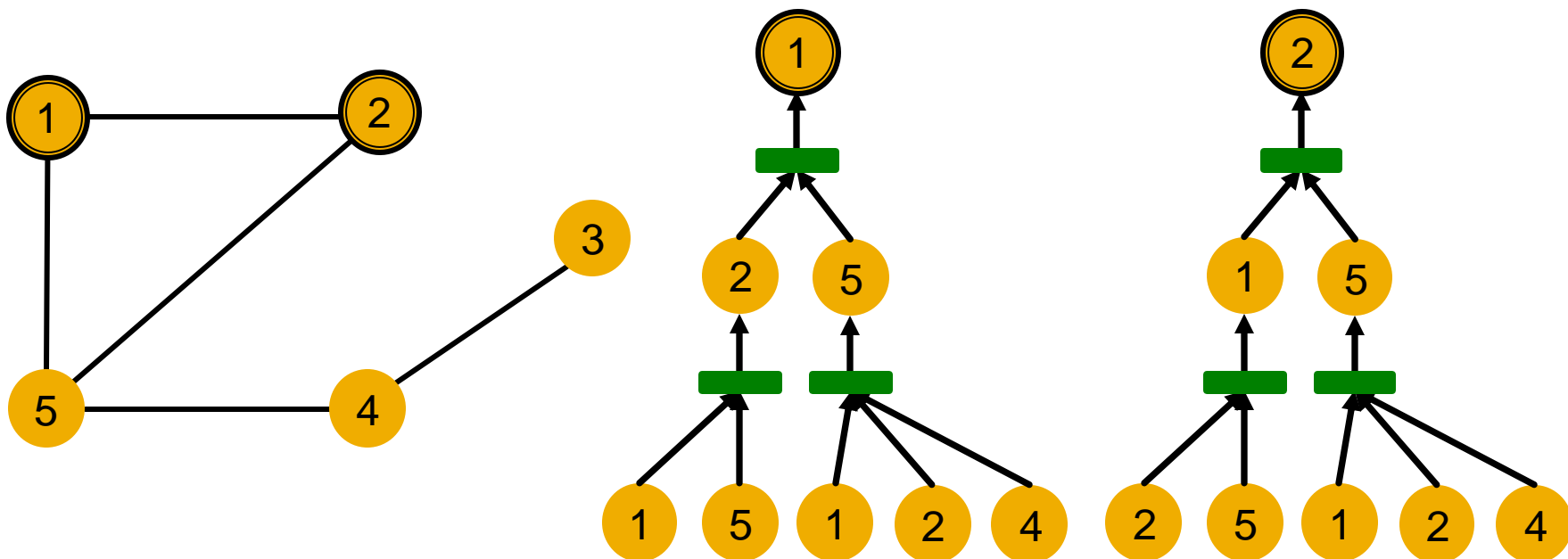
Computational Graph (1)

- In each layer, a GNN aggregates neighboring node embeddings.
- A GNN generates node embeddings through a **computational graph defined by the neighborhood**.
 - **Ex:** Node 1's computational graph (2-layer GNN)



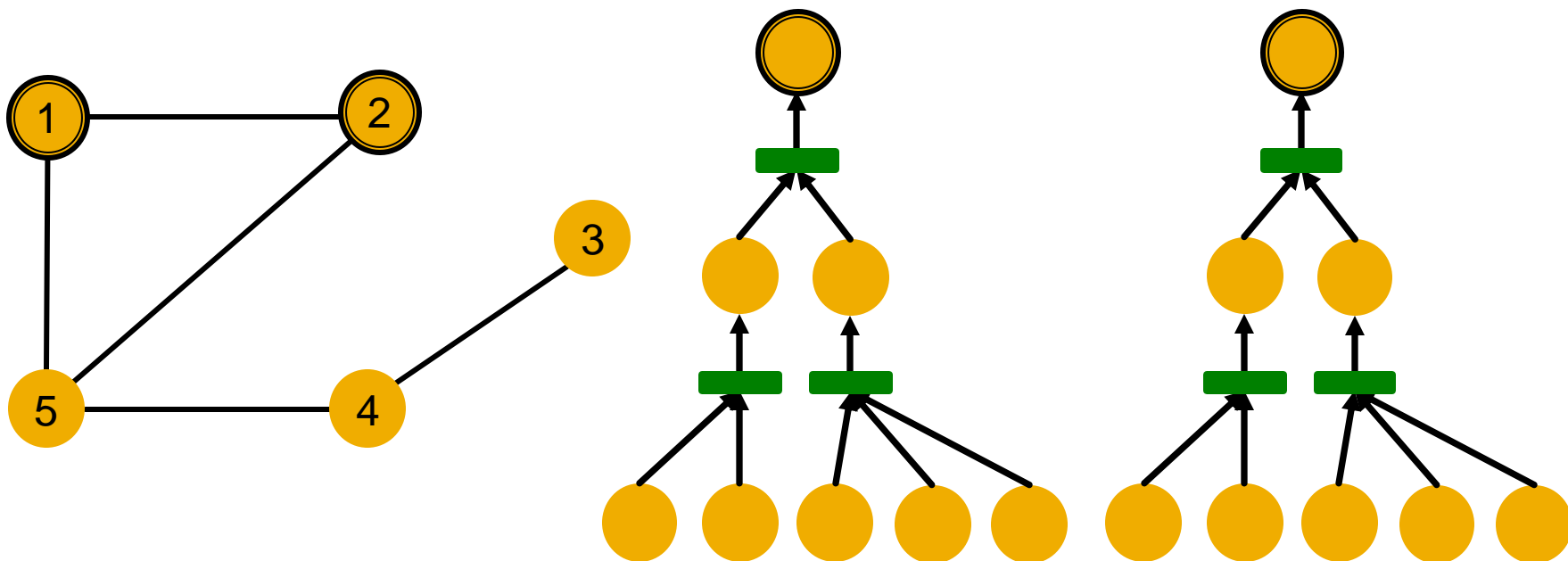
Computational Graph (2)

- **Ex:** Nodes 1 and 2's computational graphs.



Computational Graph (3)

- **Ex:** Nodes 1 and 2's computational graphs.
- **But GNN only sees node features (not IDs):**

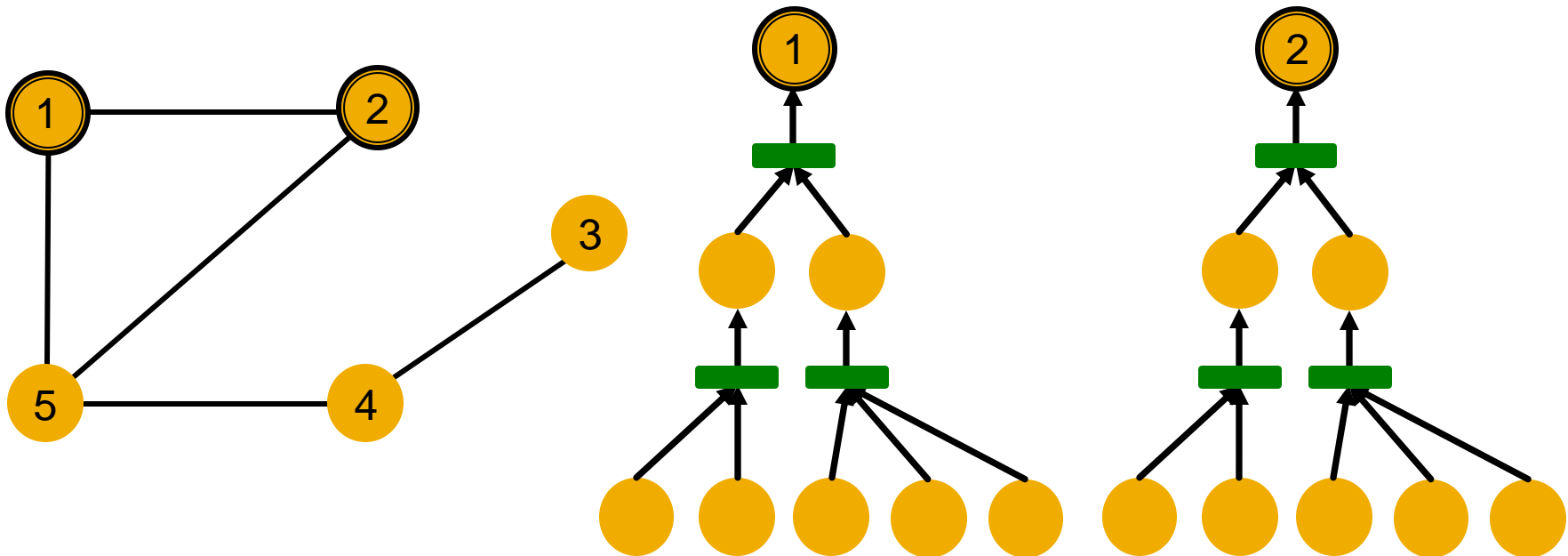


Computational Graph (4)

- A GNN will generate the same embedding for nodes 1 and 2 because:

- Computational graphs are the same.
- Node features (colors) are identical.

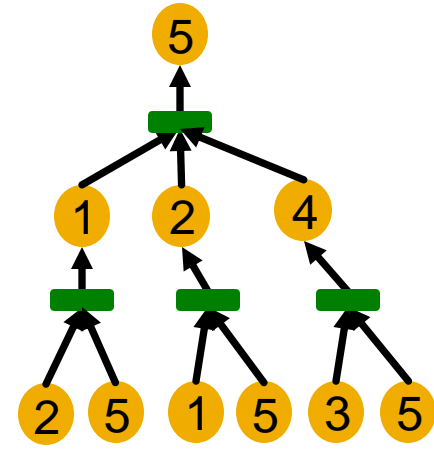
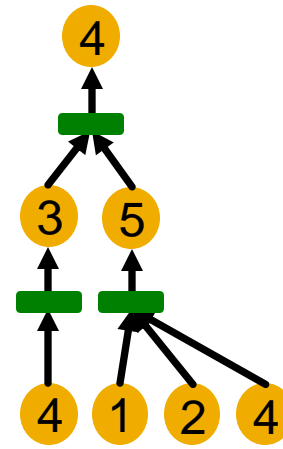
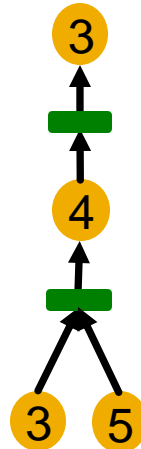
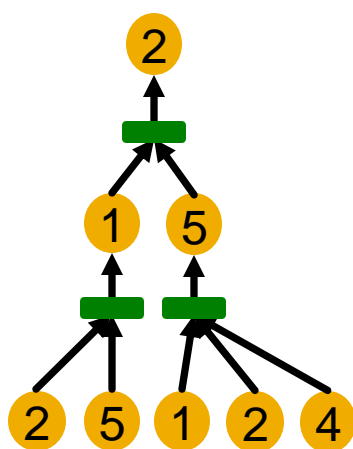
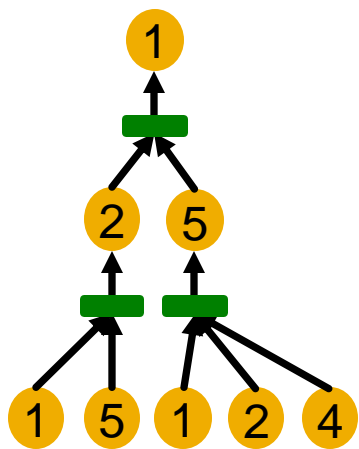
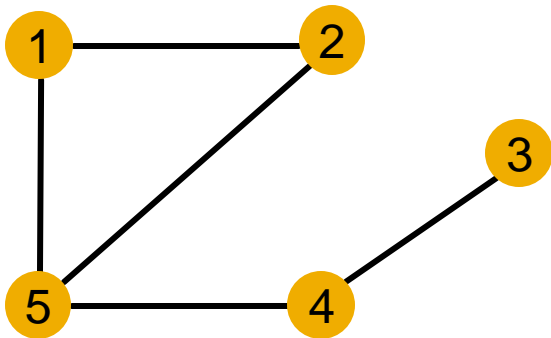
Note: GNN does not care about node ids, it just aggregates features vectors of different nodes.



GNN won't be able to distinguish nodes 1 and 2

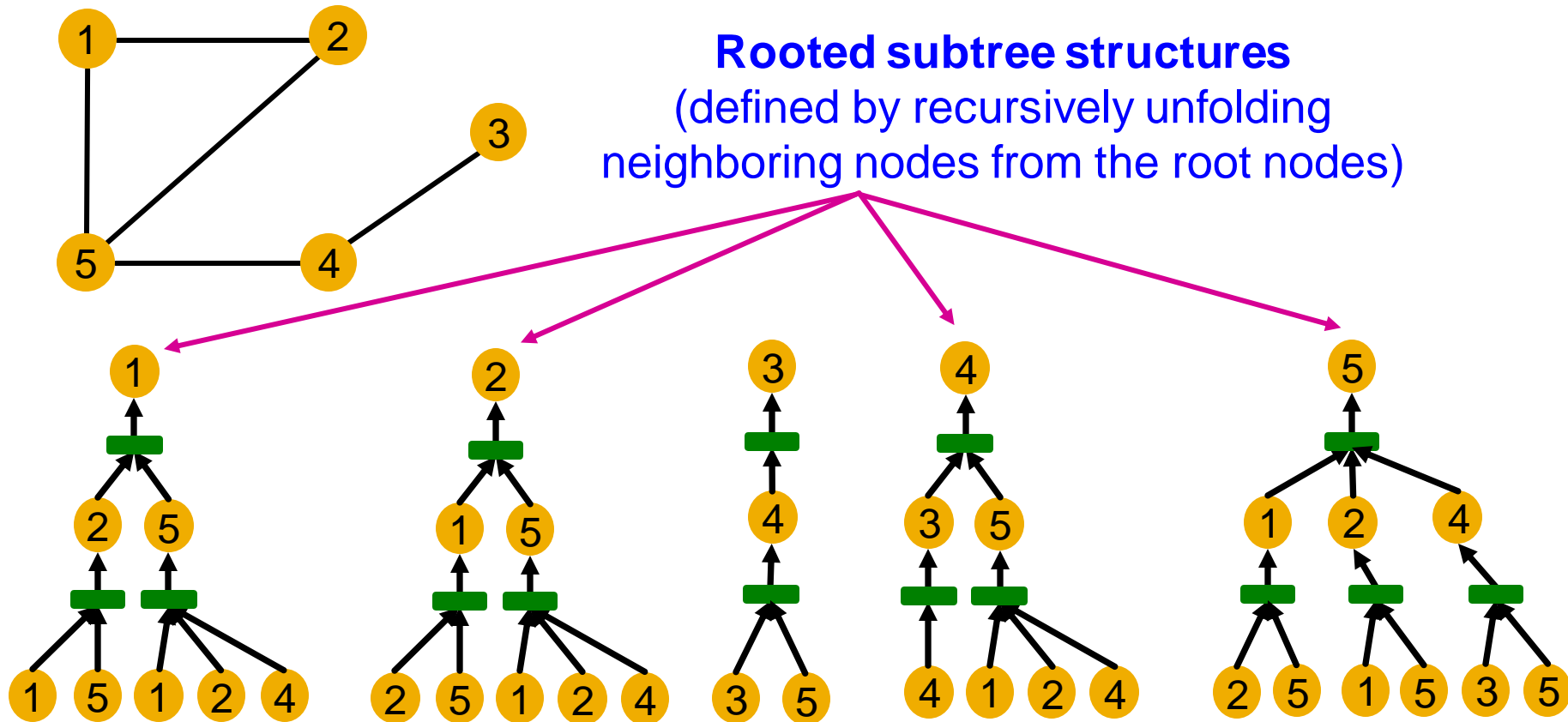
Computational Graph

- In general, different local neighborhoods define different computational graphs



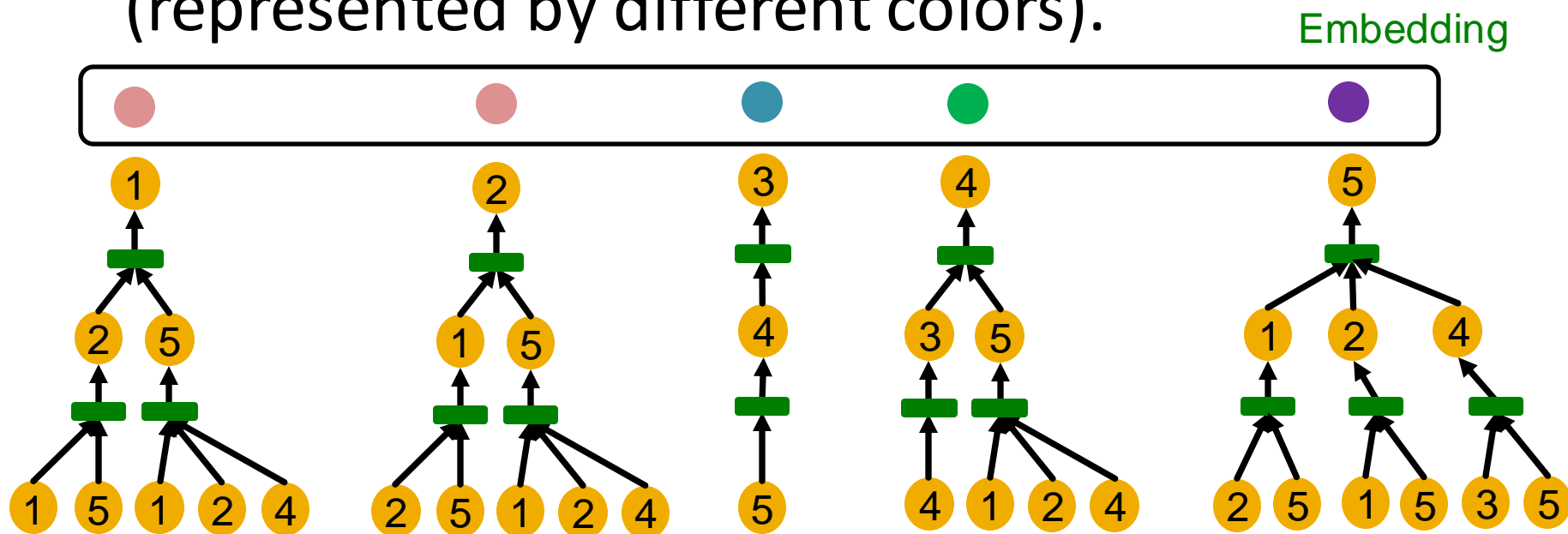
Computational Graph

- Computational graphs are identical to **rooted subtree structures** around each node.



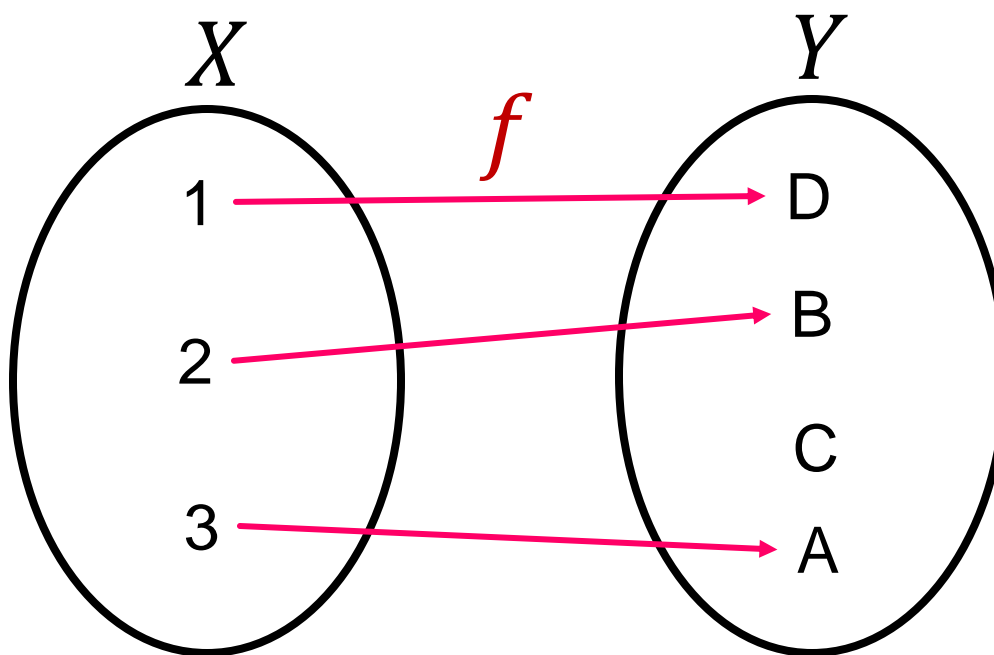
Computational Graph

- GNN's node embeddings capture **rooted subtree structures**.
- Most expressive GNN maps different **rooted subtrees** into different node embeddings (represented by different colors).



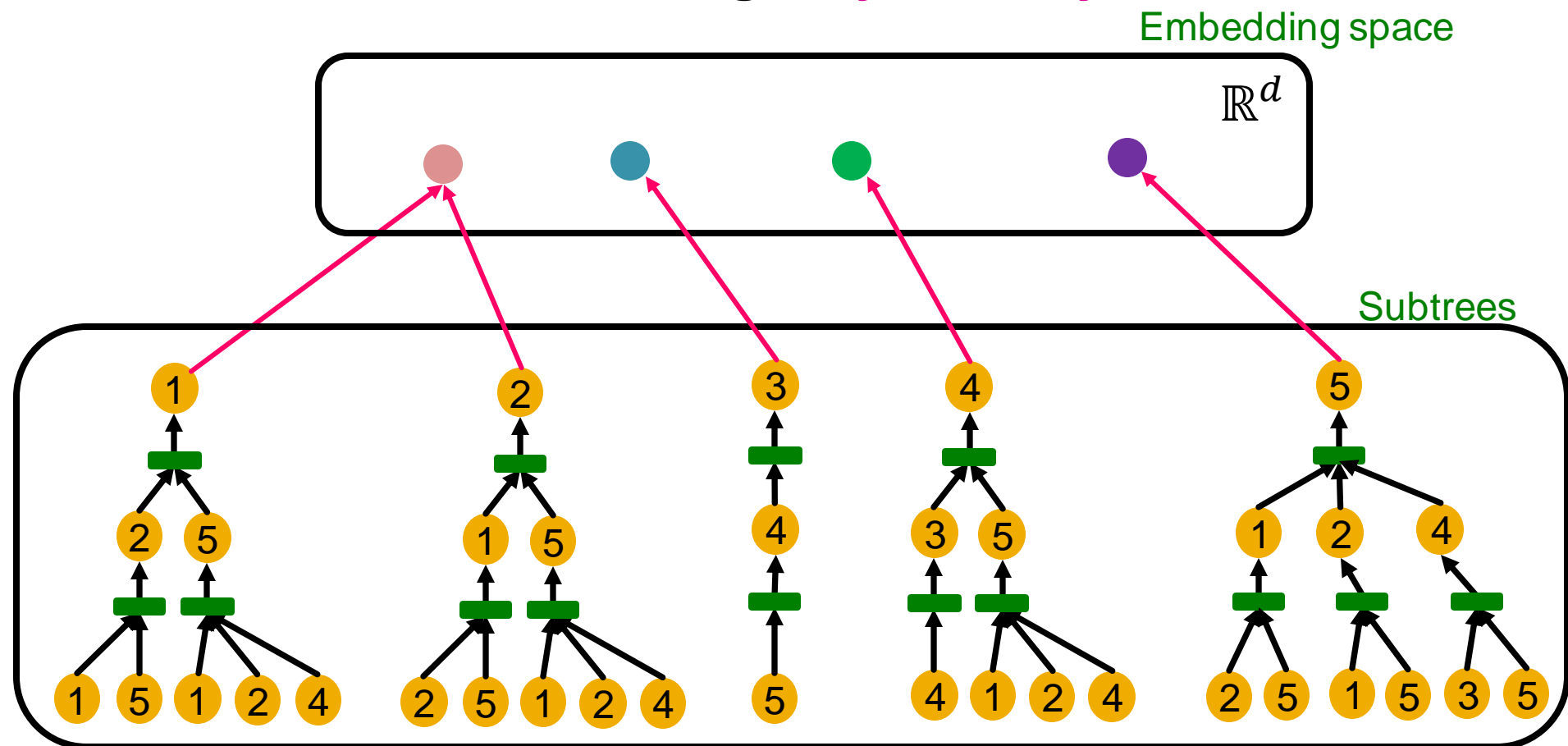
Recall: Injective Function

- **Function** $f: X \rightarrow Y$ is **injective** if it maps different elements into different outputs.
- **Intuition:** f retains all the information about input.



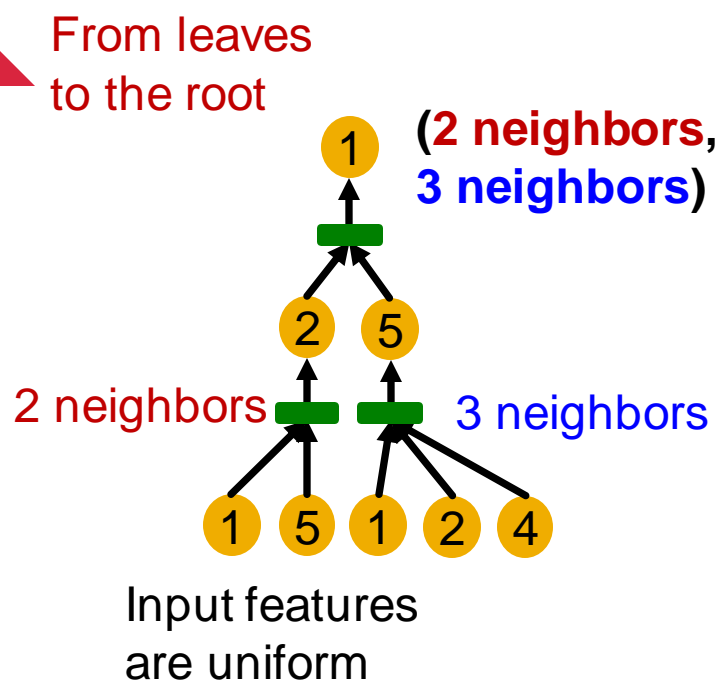
How Expressive is a GNN?

- Most expressive GNN should map subtrees to the node embeddings **injectively**.

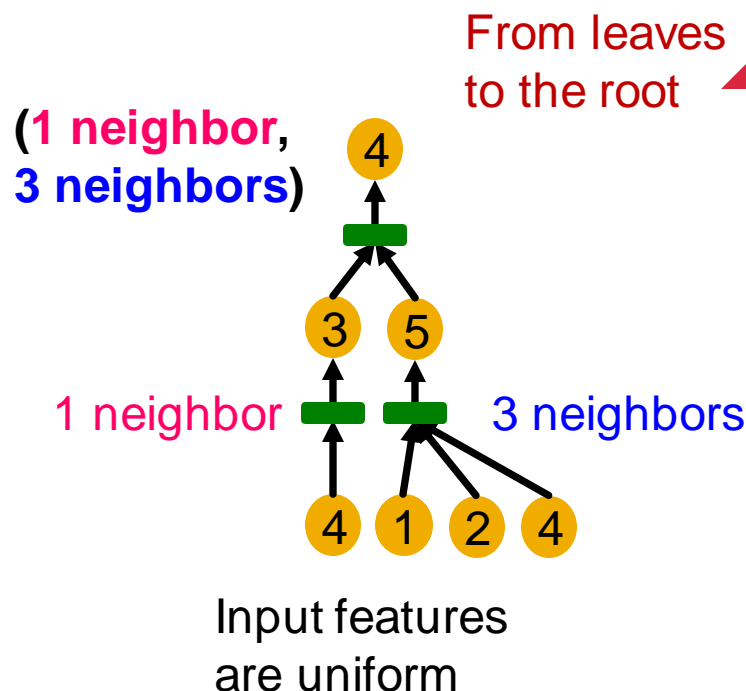


How Expressive is a GNN?

- **Key observation:** Subtrees of the same depth can be recursively characterized from the leaf nodes to the root nodes.

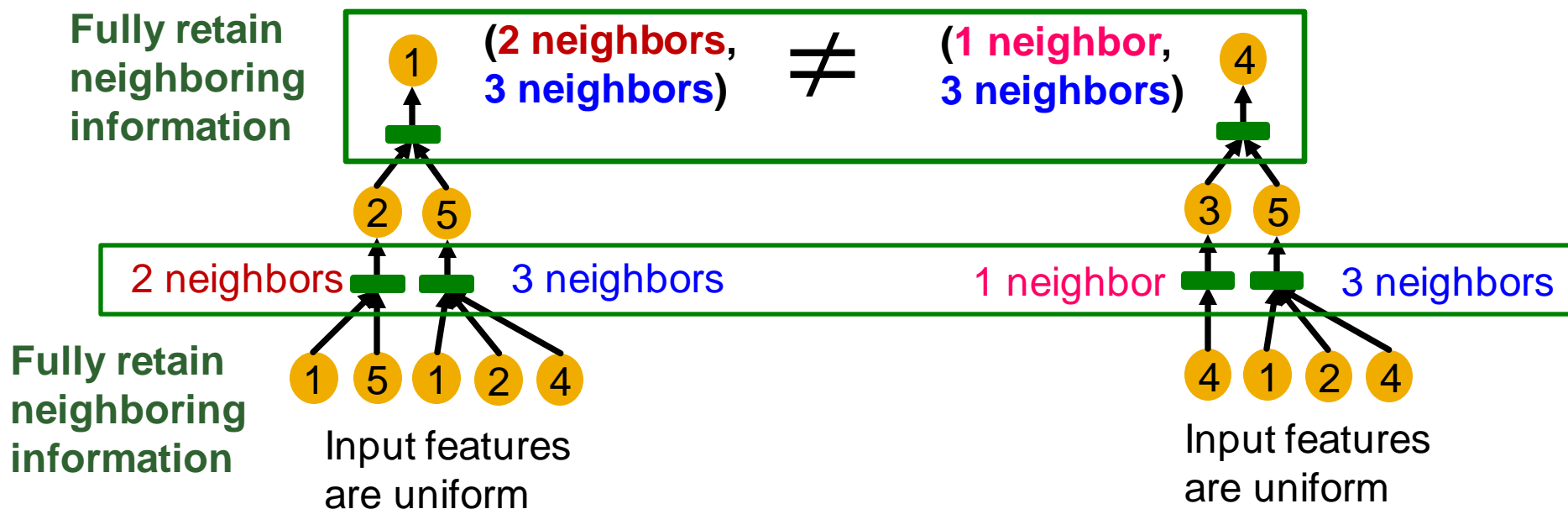


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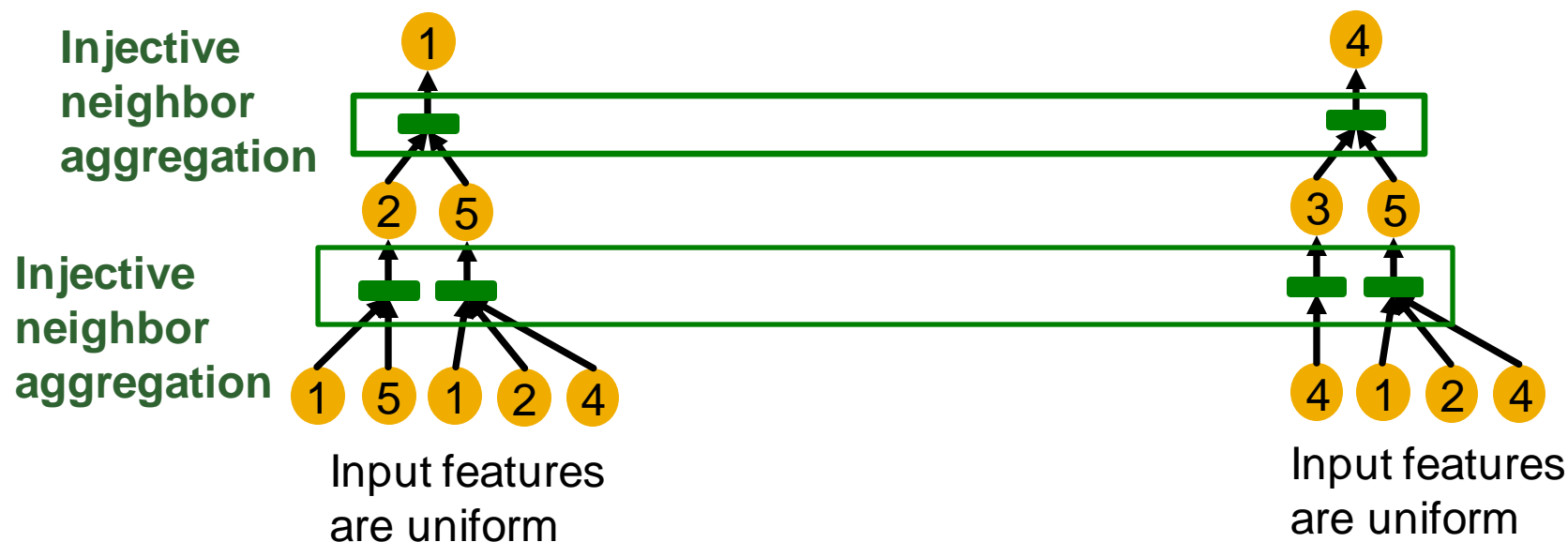
How Expressive is a GNN?

- If each step of GNN's aggregation **can fully retain the neighboring information**, the generated node embeddings can distinguish different rooted subtrees.



How Expressive is a GNN?

- In other words, most expressive GNN would use an **injective neighbor aggregation** function at each step.
 - Maps different neighbors to different embeddings.

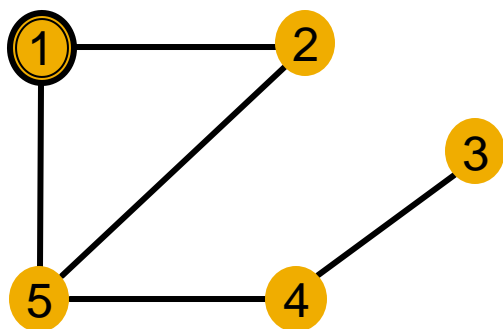


How Expressive is a GNN?

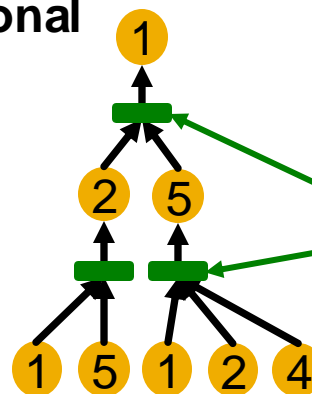
■ Summary so far

- To generate a node embedding, GNNs use a computational graph corresponding to a **subtree rooted around each node**.

Input graph



Computational graph
= **Rooted subtree**



Using injective neighbor aggregation
→ distinguish different subtrees

- GNN can fully distinguish different subtree structures if **every step of its neighbor aggregation is injective**.

Stanford CS224W: Designing the Most Powerful Graph Neural Network

CS224W: Machine Learning with Graphs
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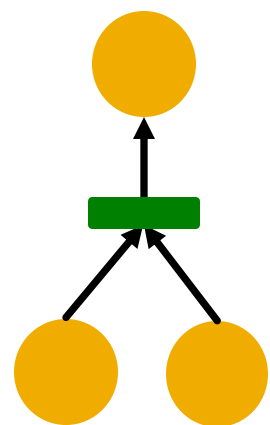


Expressive Power of GNNs

- **Key observation:** Expressive power of GNNs can be characterized by that of neighbor aggregation functions they use.
 - A more expressive aggregation function leads to a more expressive a GNN.
 - **Injective aggregation function** leads to the most expressive GNN.
- **Next:**
 - Theoretically analyze expressive power of aggregation functions.

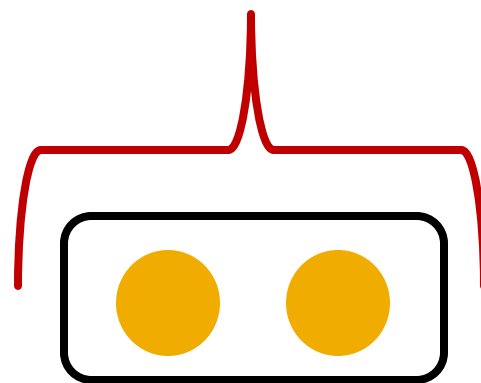
Neighbor Aggregation

- **Observation:** Neighbor aggregation can be abstracted as **a function over a multi-set** (a set with repeating elements).



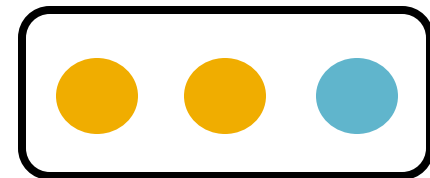
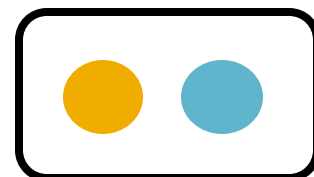
Neighbor aggregation

Equivalent



Multi-set function

Examples of multi-set



Same color indicates the same features.

Neighbor Aggregation

- **Next:** We analyze aggregation functions of two popular GNN models
 - **GCN** (mean-pool) [Kipf & Welling, ICLR 2017]
 - Uses **element-wise** mean pooling over neighboring node features

$$\text{Mean}(\{x_u\}_{u \in N(v)})$$

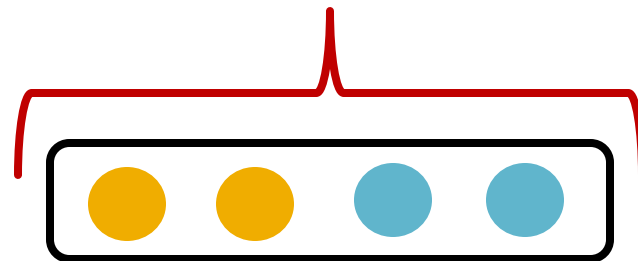
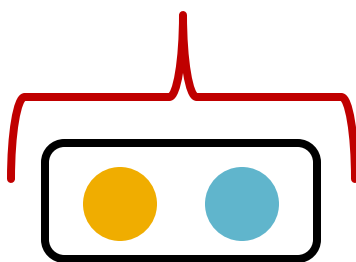
- **GraphSAGE** (max-pool) [Hamilton et al. NeurIPS 2017]
 - Uses **element-wise** max pooling over neighboring node features

$$\text{Max}(\{x_u\}_{u \in N(v)})$$

Neighbor Aggregation: Case Study

- **GCN (mean-pool)** [Kipf & Welling ICLR 2017]
 - Take **element-wise mean**, followed by linear function and ReLU activation, i.e., $\max(0, x)$.
 - **Theorem** [Xu et al. ICLR 2019]
 - GCN's aggregation **function cannot distinguish different multi-sets with the same color proportion.**

Failure case



■ **Why?**

Neighbor Aggregation

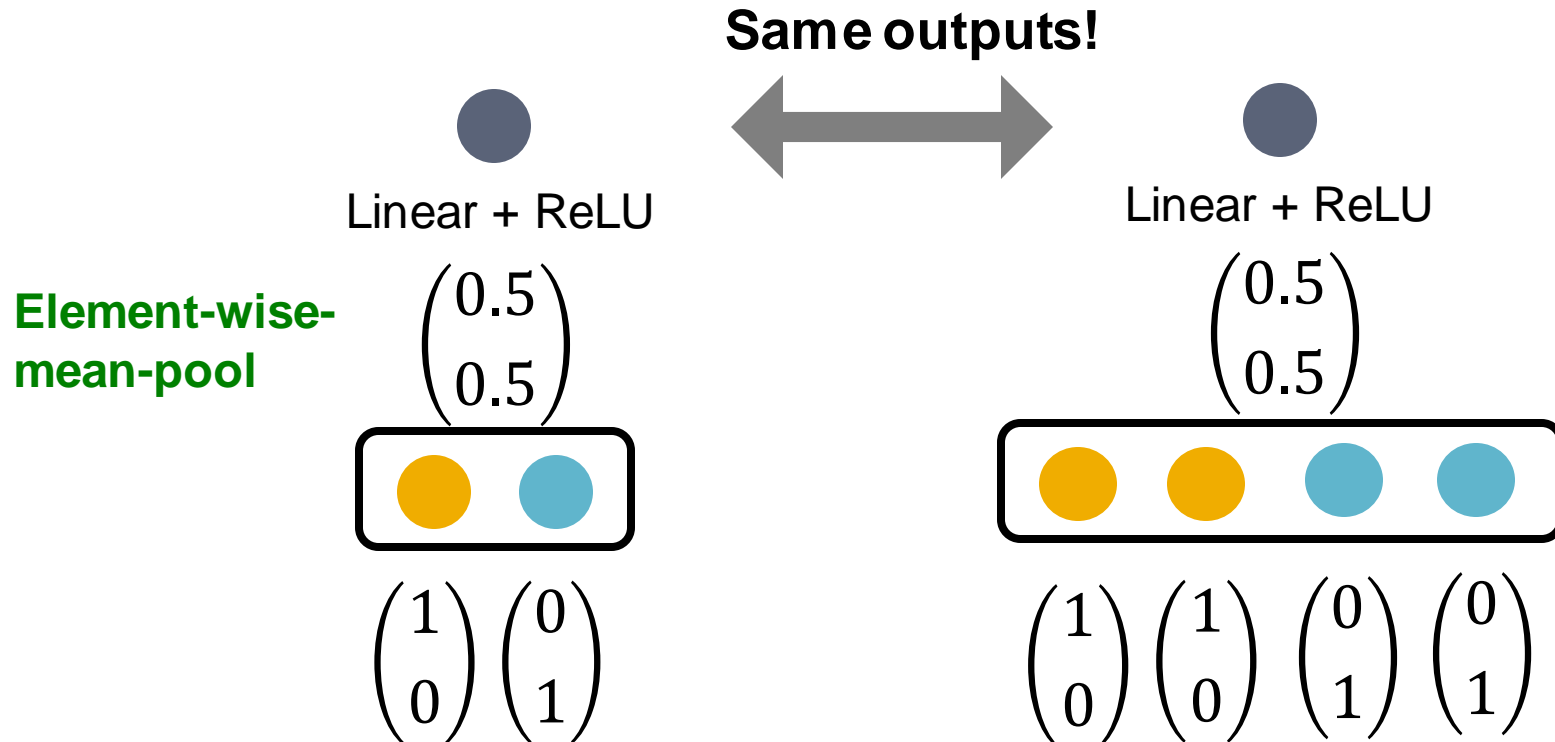
- For simplicity, we assume node features (colors) are represented by **one-hot encoding**.
 - **Example:** If there are two distinct colors:

$$\text{Yellow Circle} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Blue Circle} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- This assumption is sufficient to illustrate how GCN fails.

Neighbor Aggregation: Case Study

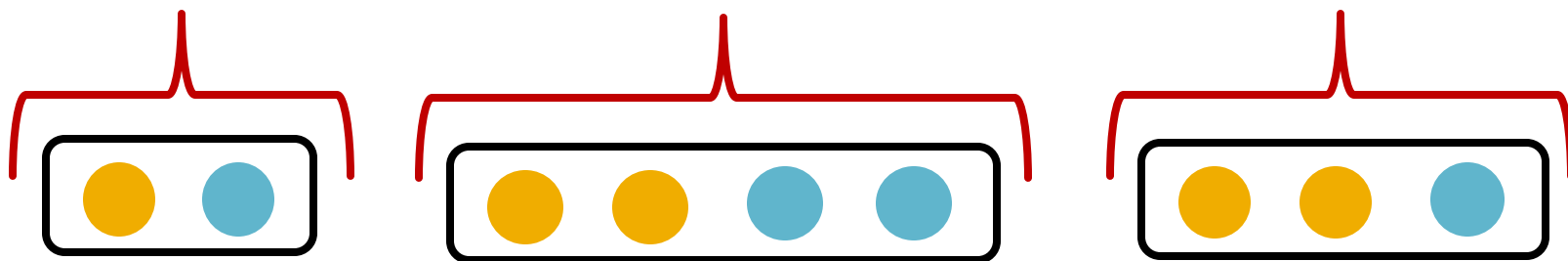
- **GCN (mean-pool)** [Kipf & Welling ICLR 2017]
 - **Failure case illustration**



Neighbor Aggregation: Case Study

- **GraphSAGE (max-pool)** [Hamilton et al. NeurIPS 2017]
 - Apply an MLP, then take **element-wise max**.
 - **Theorem** [Xu et al. ICLR 2019]
 - GraphSAGE's aggregation **function cannot distinguish different multi-sets with the same set of distinct colors.**

Failure case



- **Why?**

Neighbor Aggregation: Case Study

- **GraphSAGE (max-pool)** [Hamilton et al. NeurIPS 2017]

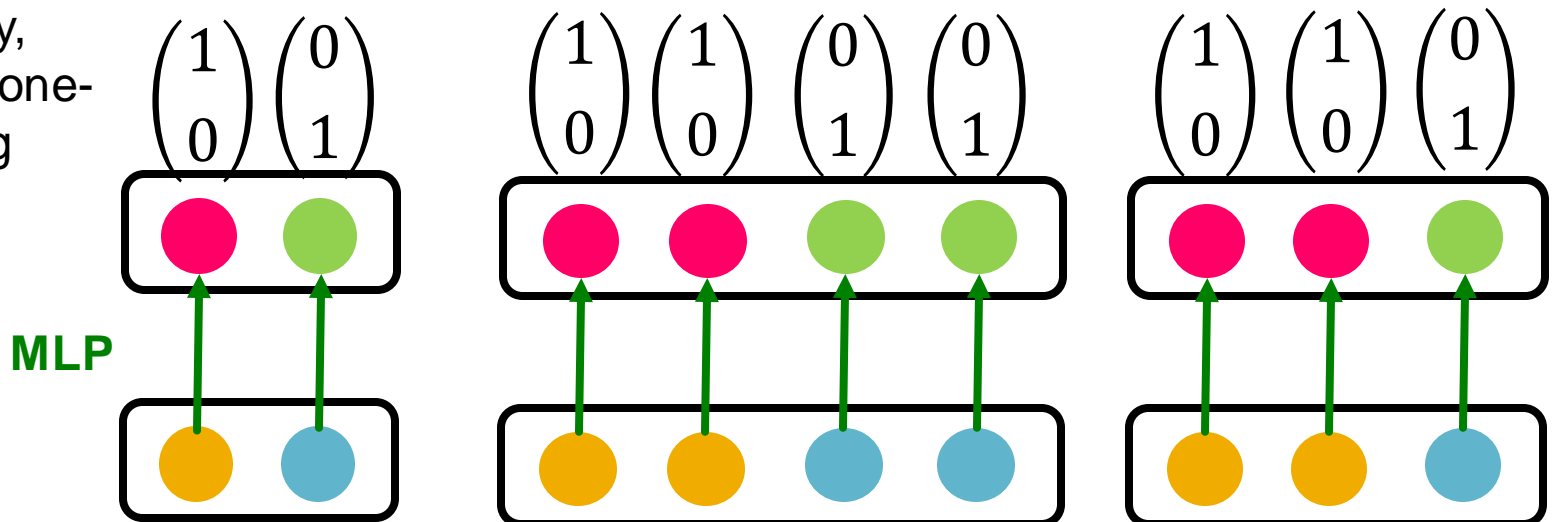
- **Failure case illustration**

The same outputs!

Element-wise-
max-pool

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For simplicity,
assume the one-
hot encoding
after **MLP**.



Summary So Far

- We analyzed the **expressive power of GNNs.**
- **Main takeaways:**
 - Expressive power of GNNs can be characterized by that of the neighbor aggregation function.
 - Neighbor aggregation is a function over multi-sets (sets with repeating elements)
 - GCN and GraphSAGE's aggregation functions fail to distinguish some basic multi-sets; hence **not injective**.
 - Therefore, GCN and GraphSAGE are **not** maximally powerful GNNs.

Designing Most Expressive GNNs

- **Our goal:** Design maximally powerful GNNs in the class of message-passing GNNs.
- This can be achieved by designing **injective** neighbor aggregation function over multisets.
- Here, we design a **neural network** that can model **injective** multiset function.

Injective Multi-Set Function

Theorem [Xu et al. ICLR 2019]

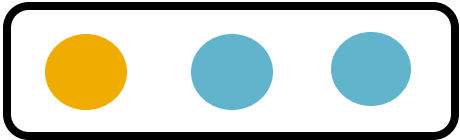
Any injective multi-set function can be expressed as:

Some non-linear function $\rightarrow \Phi \left(\sum_{x \in S} f(x) \right)$

Some non-linear function $\rightarrow f(x)$

Sum over multi-set $\rightarrow \sum_{x \in S}$

S : multi-set

 $\rightarrow \Phi \left(f(\text{yellow circle}) + f(\text{blue circle}) + f(\text{blue circle}) \right)$

Injective Multi-Set Function

✓ **Proof Intuition:** [Xu et al. ICLR 2019]

f produces one-hot encodings of colors. Summation of the one-hot encodings retains all the information about the input multi-set.

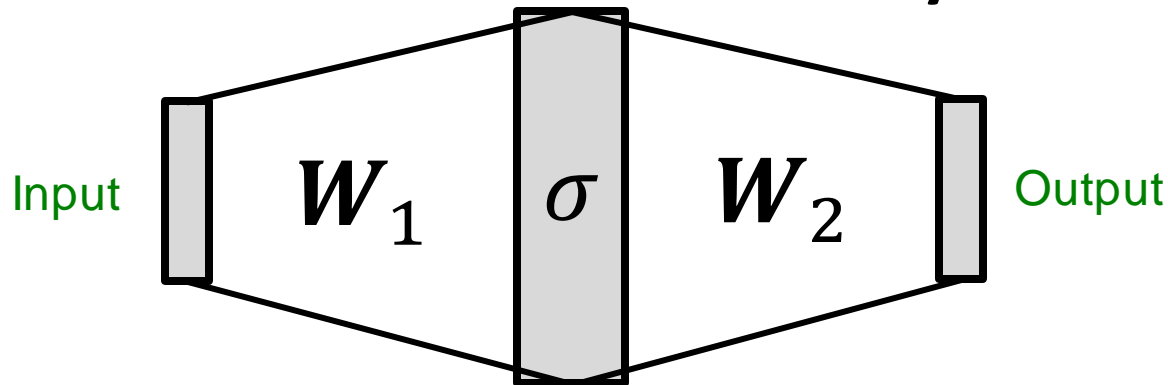
$$\Phi \left(\sum_{x \in S} f(x) \right)$$

Example: $\Phi \left[\underbrace{f(\text{yellow})}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \underbrace{f(\text{blue})}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} + \underbrace{f(\text{blue})}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right]$

One-hot $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Universal Approximation Theorem

- How to model Φ and f in $\Phi(\sum_{x \in \mathcal{S}} f(x))$?
- We use a Multi-Layer Perceptron (MLP).
- **Theorem: Universal Approximation Theorem**
[Hornik et al., 1989]
 - 1-hidden-layer MLP with sufficiently-large hidden dimensionality and appropriate non-linearity $\sigma(\cdot)$ (including ReLU and sigmoid) can **approximate any continuous function to an arbitrary accuracy**.



Injective Multi-Set Function

- ✓ ■ We have arrived at a **neural network** that can model any injective multiset function.

$$\text{MLP}_{\Phi} \left(\sum_{x \in S} \text{MLP}_f(x) \right)$$

- In practice, MLP hidden dimensionality of 100 to 500 is sufficient.

Most Expressive GNN

■ Graph Isomorphism Network (GIN) [Xu et al. ICLR 2019]


- Apply an MLP, element-wise **sum**, followed by another MLP.

$$\text{MLP}_{\Phi} \left(\sum_{x \in S} \text{MLP}_f(x) \right)$$

■ Theorem [Xu et al. ICLR 2019]

- GIN's neighbor aggregation function is injective.
- **No failure cases!**
- **GIN is THE most expressive GNN** in the class of message-passing GNNs we have introduced!

Full Model of GIN

- 
- **So far:** We have described the neighbor aggregation part of GIN.
 - We now describe the full model of GIN by relating it to **WL graph kernel** (traditional way of obtaining graph-level features).
 - We will see how GIN is a “neural network” version of the WL graph kernel.

Relation to WL Graph Kernel

Recall: Color refinement algorithm in WL kernel.

- **Given:** A graph G with a set of nodes V .
 - Assign an initial color $c^{(0)}(v)$ to each node v .
 - Iteratively refine node colors by

$$c^{(k+1)}(v) = \text{HASH} \left(c^{(k)}(v), \{c^{(k)}(u)\}_{u \in N(v)} \right),$$

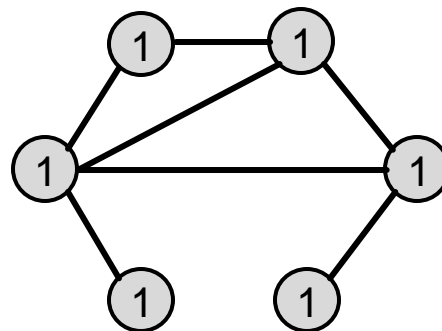
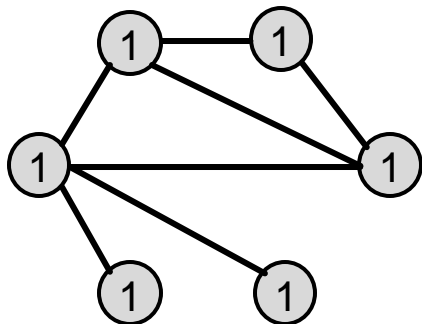
where HASH maps different inputs to different colors.

- After K steps of color refinement, $c^{(K)}(v)$ summarizes the structure of K -hop neighborhood

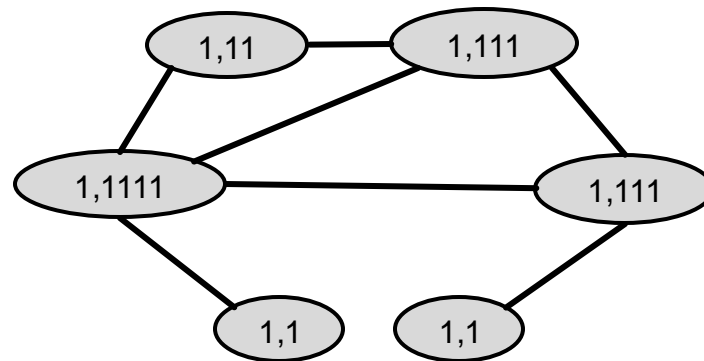
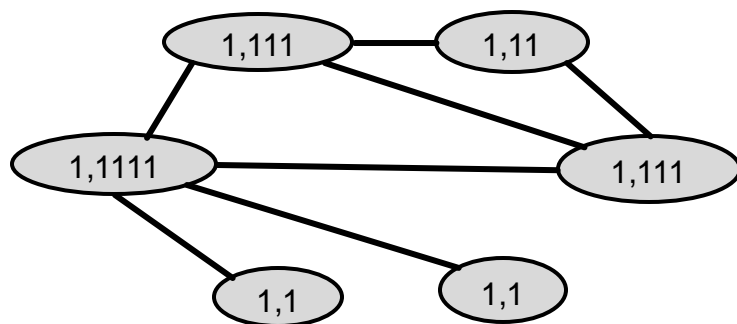
Color Refinement (1)

Example of color refinement given two graphs

- Assign initial colors



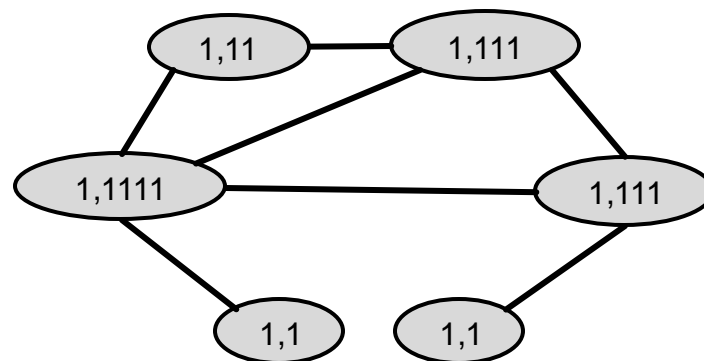
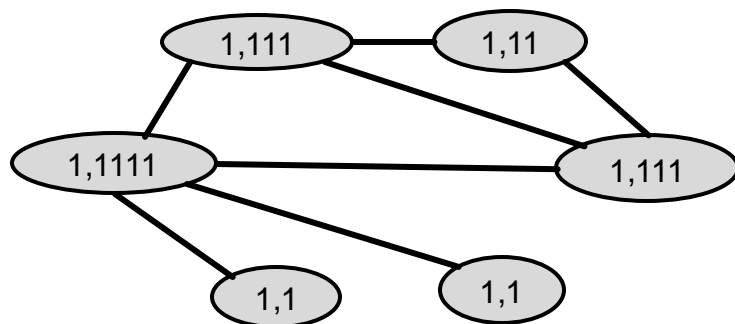
- Aggregate neighboring colors



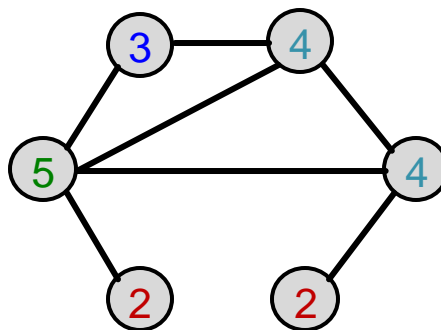
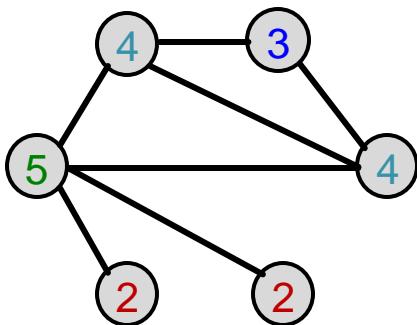
Color Refinement (2)

Example of color refinement given two graphs

- Aggregated colors:



- Injectively** HASH the aggregated colors



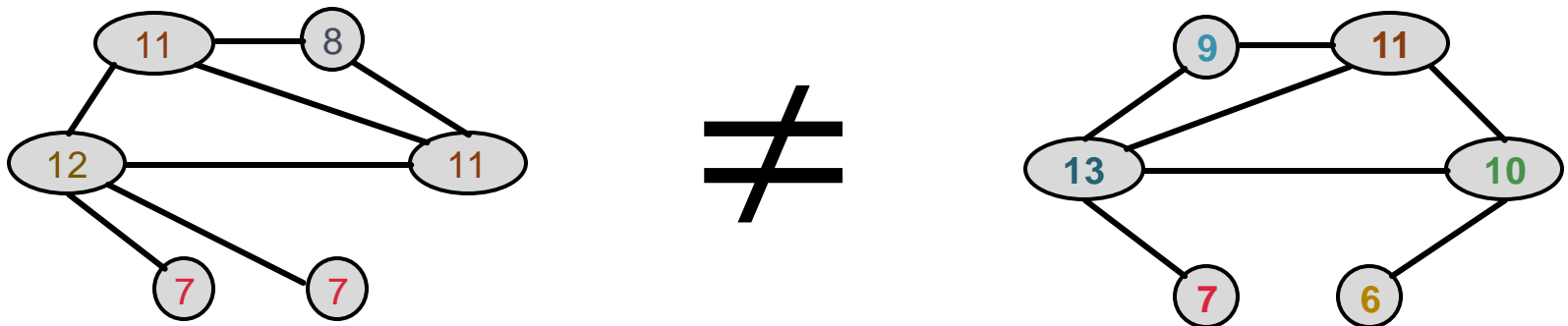
HASH table: **Injective!**

1,1	-->	2
1,11	-->	3
1,111	-->	4
1,1111	-->	5

Color Refinement (3)

Example of color refinement given two graphs

- Process continues until a stable coloring is reached
- Two graphs are considered **isomorphic** if they have the same set of colors.



The Complete GIN Model

- GIN uses a **neural network** to model the injective HASH function.

$$c^{(k+1)}(v) = \text{HASH} \left(c^{(k)}(v), \{c^{(k)}(u)\}_{u \in N(v)} \right)$$

- Specifically, we will model the injective function over the tuple:

$$(c^{(k)}(v), \{c^{(k)}(u)\}_{u \in N(v)})$$

Root node
features

Neighboring
node colors

The Complete GIN Model

✓ **Theorem** (Xu et al. ICLR 2019)

Any injective function over the tuple

Root node
feature

$(c^{(k)}(v), \{c^{(k)}(u)\}_{u \in N(v)})$

Neighboring
node features

can be modeled as

$$\text{MLP}_{\Phi} \left((1 + \epsilon) \cdot \text{MLP}_f(c^{(k)}(v)) + \sum_{u \in N(v)} \text{MLP}_f(c^{(k)}(u)) \right)$$

where ϵ is a learnable scalar.

The Complete GIN Model

- If input feature $c^{(0)}(v)$ is represented as one-hot, **direct summation is injective.**

Example: $\Phi \left[\underbrace{\text{yellow circle}}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \underbrace{\text{blue circle}}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} + \underbrace{\text{blue circle}}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right]$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- We only need Φ to ensure the injectivity.

$$\text{GINConv} \left(\underbrace{c^{(k)}(v)}_{\text{Root node features}}, \underbrace{\{c^{(k)}(u)\}_{u \in N(v)}}_{\text{Neighboring node features}} \right) = \text{MLP}_{\Phi} \left((1 + \epsilon) \cdot c^{(k)}(v) + \sum_{u \in N(v)} c^{(k)}(u) \right)$$

This MLP can provide “one-hot” input feature for the next layer.

The Complete GIN Model

- GIN's node embedding updates
- **Given:** A graph G with a set of nodes V .
 - Assign an **initial vector** $c^{(0)}(v)$ to each node v .
 - Iteratively update node vectors by

$$c^{(k+1)}(v) = \text{GINConv} \left(\left\{ c^{(k)}(v), \{c^{(k)}(u)\}_{u \in N(v)} \right\} \right),$$

Differentiable color HASH function

where **GINConv** maps different inputs to different embeddings.

- After K steps of GIN iterations, $c^{(K)}(v)$ summarizes the structure of K -hop neighborhood.

GIN and WL Graph Kernel

- GIN can be understood as differentiable neural version of the WL graph Kernel:

	Update target	Update function
WL Graph Kernel	Node colors (one-hot)	HASH
GIN	Node embeddings (low-dim vectors)	GINConv

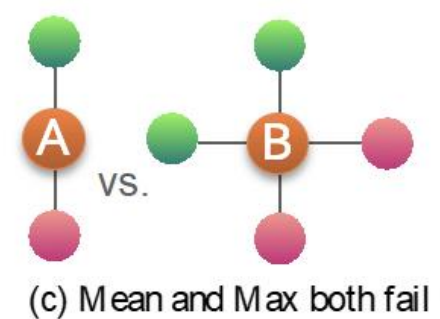
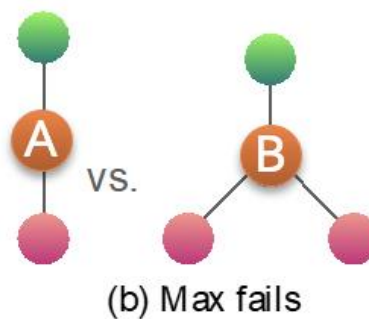
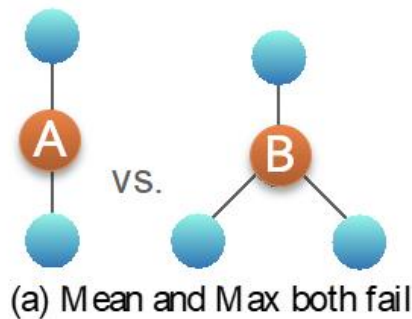
- **Advantages of GIN over the WL graph kernel are:**
 - Node embeddings are **low-dimensional**; hence, they can capture the fine-grained similarity of different nodes.
 - Parameters of the update function can be **learned for the downstream tasks**.

Expressive Power of GIN

- Because of the relation between GIN and the WL graph kernel, their expressive is exactly the same.
 - If two graphs can be distinguished by GIN, they can be also distinguished by the WL kernel, and vice versa.
- **How powerful is this?**
 - WL kernel has been both theoretically and empirically shown to distinguish most of the real-world graphs [Cai et al. 1992].
 - Hence, GIN is also powerful enough to distinguish most of the real graphs!

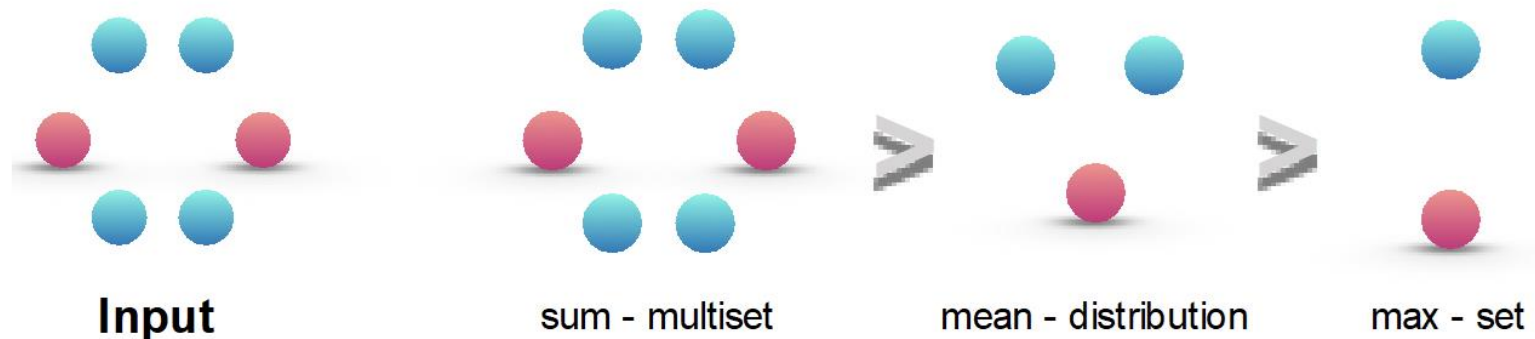
Discussion: The Power of Pooling

Failure cases for mean and max pooling:



Colors represent feature values

Ranking by discriminative power:

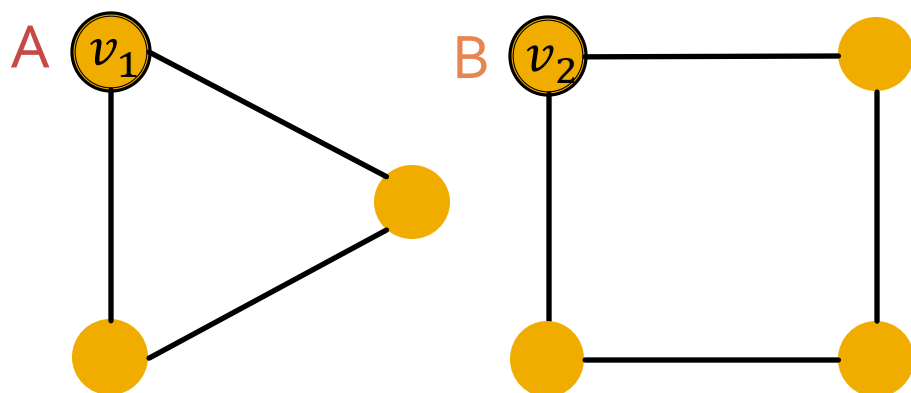


Improving GNNs' Power

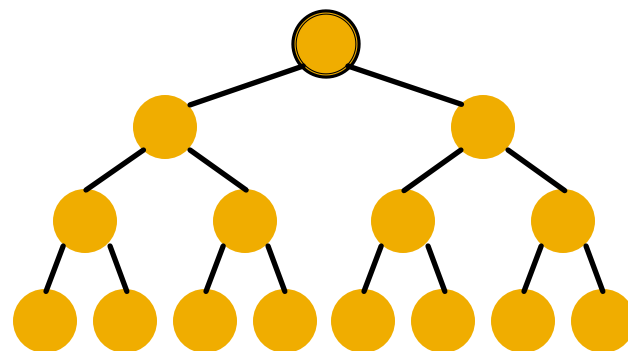
■ Can the expressive power of GNNs be improved?

- There are basic graph structures that existing GNN framework cannot distinguish, such as difference in cycles.

Graphs



Computational graphs for nodes v_1 and v_2 :



- GNNs' expressive power **can be improved** to resolve the above problem. [You et al. AAAI 2021, Li et al. NeurIPS 2020]
- Stay tuned for **Lecture 15: Advanced Topics in GNNs**

Summary of the Lecture

- We design a neural network that can model **injective multi-set function**.
- We use the neural network for neighbor aggregation function and arrive at **GIN---the most expressive GNN model**.
- The key is to use **element-wise sum pooling**, instead of mean-/max-pooling.
- GIN is closely related to the WL graph kernel.
- Both GIN and WL graph kernel can distinguish most of the real graphs!

Stanford CS224W: When Things Don't Go As Planned

CS224W: Machine Learning with Graphs
Jure Leskovec, Stanford University
<http://cs224w.stanford.edu>



General Tips

- **Data preprocessing is important:**
 - Node attributes can vary a lot! Use **normalization**
 - E.g. probability ranges $(0,1)$, but some inputs could have much larger range, say $(-1000, 1000)$
- **Optimizer:** ADAM is relatively robust to learning rate
- **Activation function**
 - ReLU activation function often works well
 - Other good alternatives: [LeakyReLU](#), [PReLU](#)
 - No activation function at your output layer
 - Include bias term in every layer
- **Embedding dimensions:**
 - 32, 64 and 128 are often good starting points

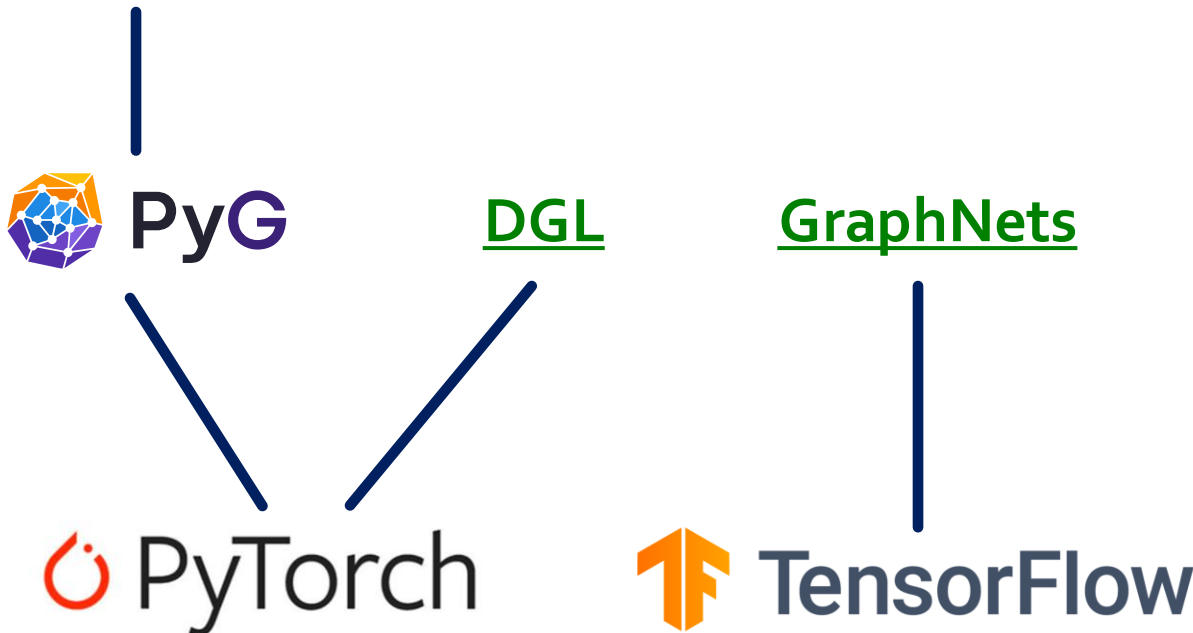
Debugging Deep Networks

- **Debug issues:** Loss/accuracy not converging during training
 - Check pipeline (e.g. in PyTorch we need [zero_grad](#))
 - Adjust hyperparameters such as learning rate
 - Pay attention to weight parameter [initialization](#)
 - **Scrutinize loss function!**
- **Important for model development:**
 - **Overfit on (part of) training data:**
 - With a small training dataset, loss should be essentially close to 0, with an expressive neural network
 - **Monitor the training & validation loss curve**

Resources on Graph Neural Networks

GraphGym:

Easy and flexible end-to-end GNN pipeline
based on PyTorch Geometric (PyG)



GNN frameworks:
Implements a variety
of GNN architectures

Auto-differentiation frameworks

Resources on Graph Neural Networks

Tutorials and overviews:

- Relational inductive biases and graph networks (Battaglia et al., 2018)
- Representation learning on graphs: Methods and applications (Hamilton et al., 2017)

Attention-based neighborhood aggregation:

- Graph attention networks (Hoshen, 2017; Velickovic et al., 2018; Liu et al., 2018)

Embedding entire graphs:

- Graph neural nets with edge embeddings (Battaglia et al., 2016; Gilmer et al., 2017)
- Embedding entire graphs (Duvenaud et al., 2015; Dai et al., 2016; Li et al., 2018) and graph pooling (Ying et al., 2018, Zhang et al., 2018)
- Graph generation and relational inference (You et al., 2018; Kipf et al., 2018)
- How powerful are graph neural networks (Xu et al., 2017)

Embedding nodes:

- Varying neighborhood: Jumping knowledge networks (Xu et al., 2018), GeniePath (Liu et al., 2018)
- Position-aware GNN (You et al. 2019)

Spectral approaches to graph neural networks:

- Spectral graph CNN & ChebNet (Bruna et al., 2015; Defferrard et al., 2016)
- Geometric deep learning (Bronstein et al., 2017; Monti et al., 2017)

Other GNN techniques:

- Pre-training Graph Neural Networks (Hu et al., 2019)
- GNNExplainer: Generating Explanations for Graph Neural Networks (Ying et al., 2019)