COMP 9602: Convex Optimization

Convex Programs (II)

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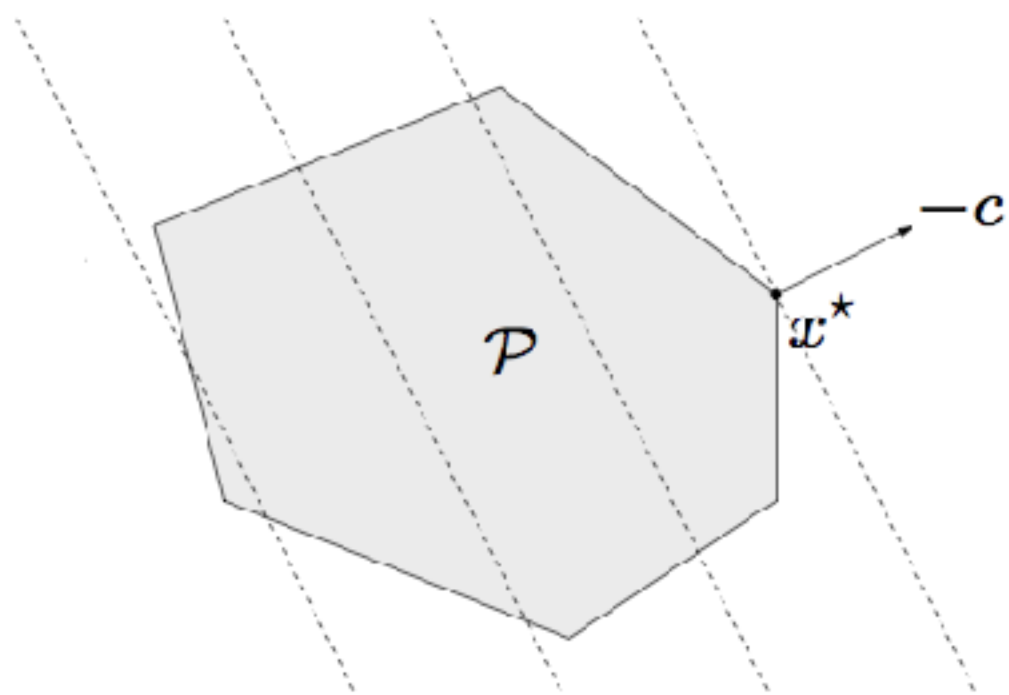
Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Standard form LP

- Standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$

- Canonical form

$$\text{minimize } c^T x$$

$$\text{subject to } Ax \succeq b, x \succeq 0$$

convertible to standard form

Examples

- The diet problem
- Network flow problems

discussed in the
first lecture

Examples

- Piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1,\ldots,m}(a_i^T x + b_i)$$

equivalent to an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

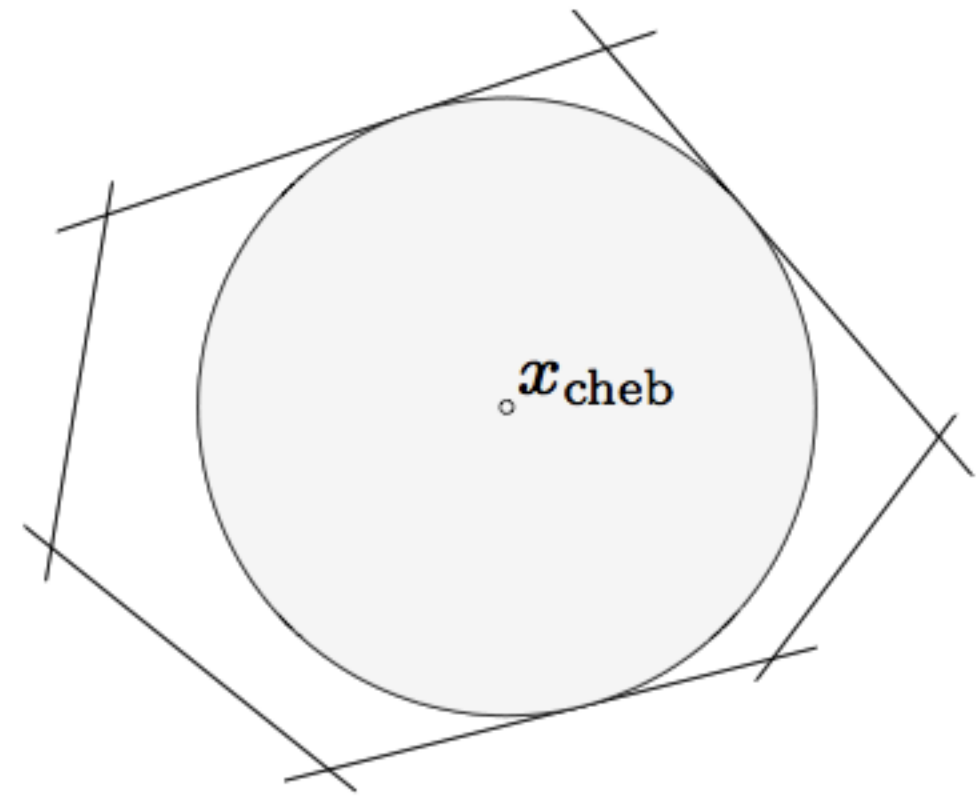
Examples

- Largest sphere inside a polyhedron

$$\mathcal{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, m\}$$

- Chebyshev center of a polyhedron: center of the largest sphere

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \le r\}$$



- hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \le b_i, \quad i = 1, \ldots, m \end{array}$$

Integer (linear) program (IP, ILP)

$$
\begin{array}{ll}
\text{minimize} & c^T x + d \\
\text{subject to} & Gx \preceq h \\
 & Ax = b \\
 & x \in \mathbf{Z}^n
\end{array}
$$

- Linear program with integer optimization variables
- Application of linear program to combinatorial optimization problems

Example

- Shortest path problem
  - find a path from s to t in directed graph G=(V,E) with the smallest total cost

$$\min \sum_{(i,j)\in E} c_{ij}x_{ij}$$

subject to:

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} 1, \text{for } i = s, \\ 0, \text{for all } i \in V - \{s,t\}, \\ -1, \text{for } i = t, \end{cases}$$

$$x_{ij} \in \{0,1\}, \forall (i,j) \in E$$

Example

- Minimum vertex cover
  - find the smallest subset of vertices in undirected graph G=(V,E) that contains at least one endpoint of every edge in the graph

$$\min \sum_{i \in V} x_i$$

subject to:

$$x_i + x_j \geq 1, \forall (i,j) \in E$$

$$x_i \in \{0,1\}, \forall i \in V$$

Example

- Maximum independent set
  - find the largest subset of vertices in undirected graph G=(V,E) such that no two vertices in the subset are connected by an edge in the graph

$$\max \sum_{i \in V} x_i$$

subject to:

$$x_i + x_j \leq 1, \forall (i,j) \in E$$

$$x_i \in \{0,1\}, \forall i \in V$$

Example

- Maximum weighted bipartite matching
  - find the subset of edges in a bipartite graph, that has the largest total weight, such that each vertex is incident to exactly 1 edge in the subset (perfect matching)

$$\max \sum_{(i,j)\in E} w_{ij} x_{ij}$$

subject to:

$$\sum_{j:(i,j)\in E} x_{ij} = 1, \forall i \in V$$

$$x_{ij} \in \{0,1\}, \forall (i,j) \in E$$

Example

- Maximum weight b-matching
  - find the subset of the edges in a graph that has the largest total weight, such that each vertex is incident to at most b edges in the subset

$$\max \sum_{e \in E} w_e x_e$$

subject to:

$$\sum_{\text{e incident on v}} x_e \leq b, \forall v \in V$$

$$x_e \in \{0, 1\}, \forall e \in E$$

NP hardness of integer programming

- Integer programming problems are generally NP-hard
  - ILPs that are NP hard

    minimum vertex cover, maximum independent set

    (they are complementary problems)
  - ILPs that are not NP hard

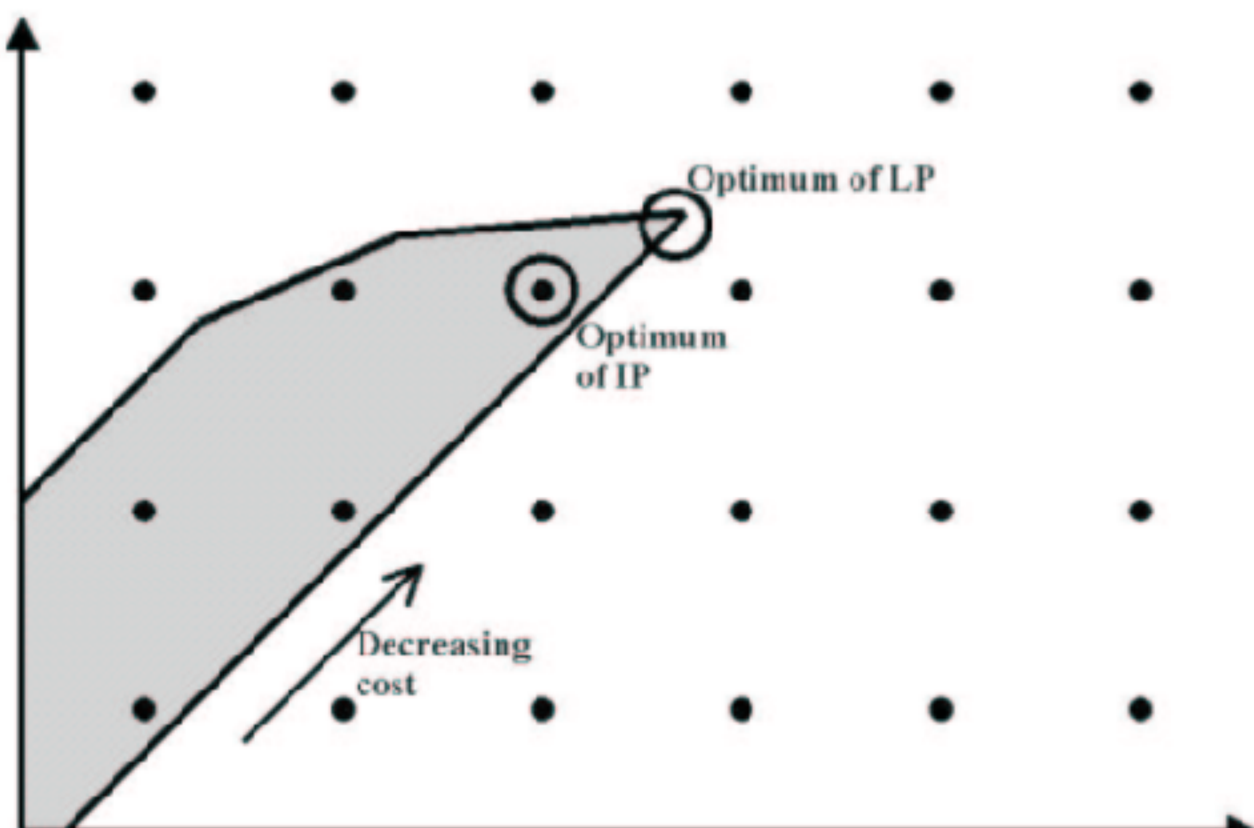
    shortest path problem, maximum weight bipartite matching, b matching

Solving integer programs

- One natural idea for solving ILPs: "relax" the integrality constraint, i.e., allow x to take on real values

  the resulting problem is an LP

  problem: the LP's optimum can be different from the optimum of the ILP



integrality gap: $\sup_I \frac{OPT(I)}{OPT_f(I)}$

OPT(I): exact optimal value of the ILP

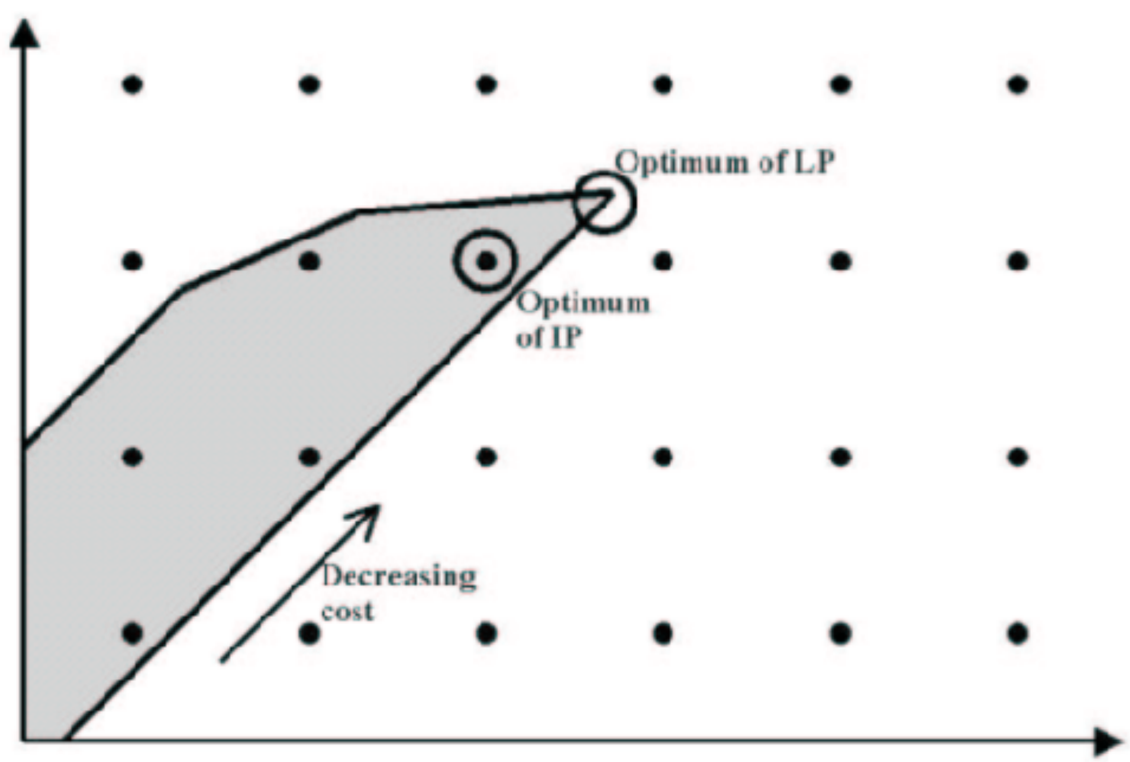
$OPT_f(I)$: optimal value of the relaxed LP

the max. ratio between OPT(I) and $OPT_f(I)$, over all instances (I) of the problem

Solving integer programs (cont'd)

solution: try to round the solution of the LP to a feasible integral point and take that as the solution to the ILP

- determining how to do rounding to achieve the optimum can be as hard as solving the original ILP itself
- develop approximation algorithms that go from the optimal point of the LP to a point that is nearby the optimal point of the ILP, and bound how far off it is from the true optimal point



approximation ratio:

$$\alpha = \sup_I \frac{A(I)}{OPT(I)}$$

A(I): approximate optimal value

OPT(I): exact optimal value of the integer program

Total unimodularity

There are special cases that the relaxation is exact,
i.e., the relaxed LP provides an integer optimal point

- Total unimodular matrix

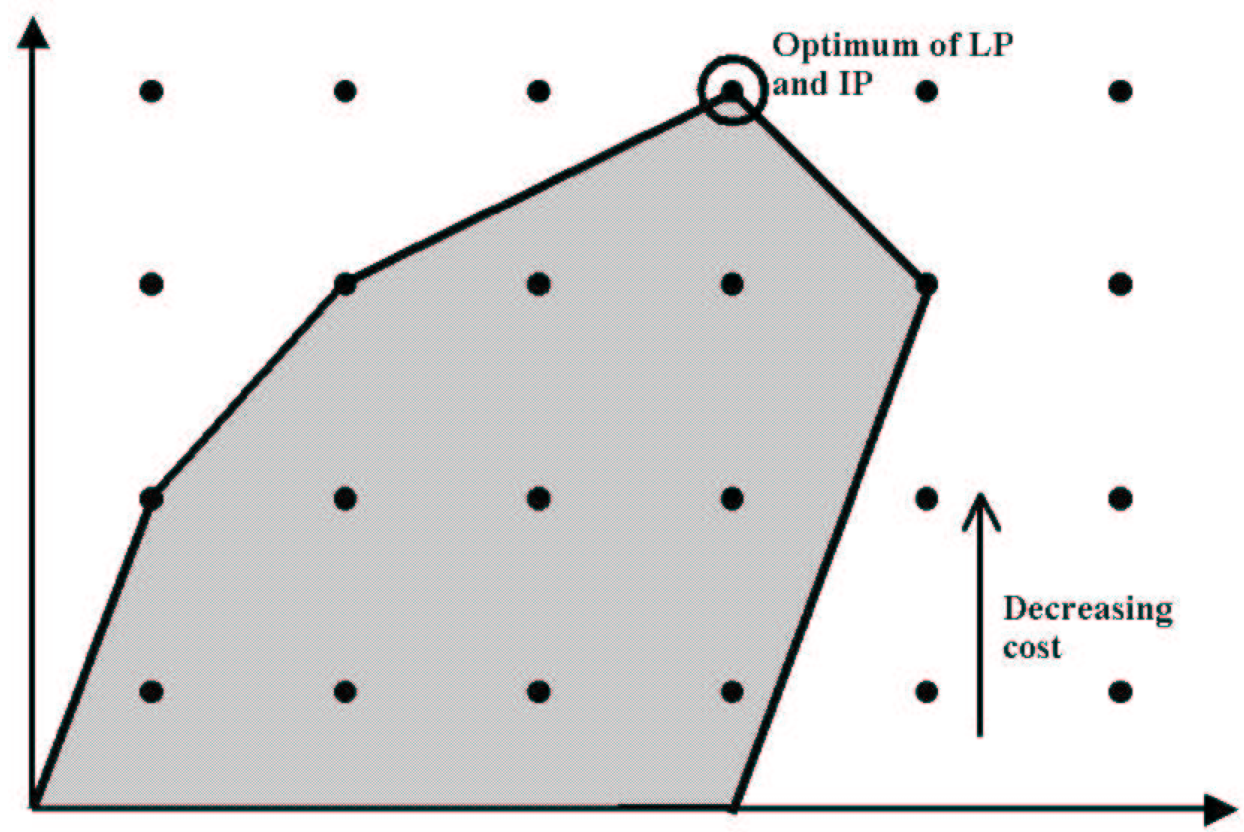
  An integer matrix is totally unimodular (TUM) if the determinant of every square submatrix of the matrix is either -1, 0, or 1.

  e.g., incidence matrix of a directed graph;

  incidence matrix of an undirected bipartite graph

Total unimodularity (cont'd)

- If A is TUM and b is an integer vector, all vertices of the polyhedron $P = \{x | Ax = b, x \succeq 0\}$ (or $P = \{x | Ax \succeq b, x \succeq 0\}$) are integer

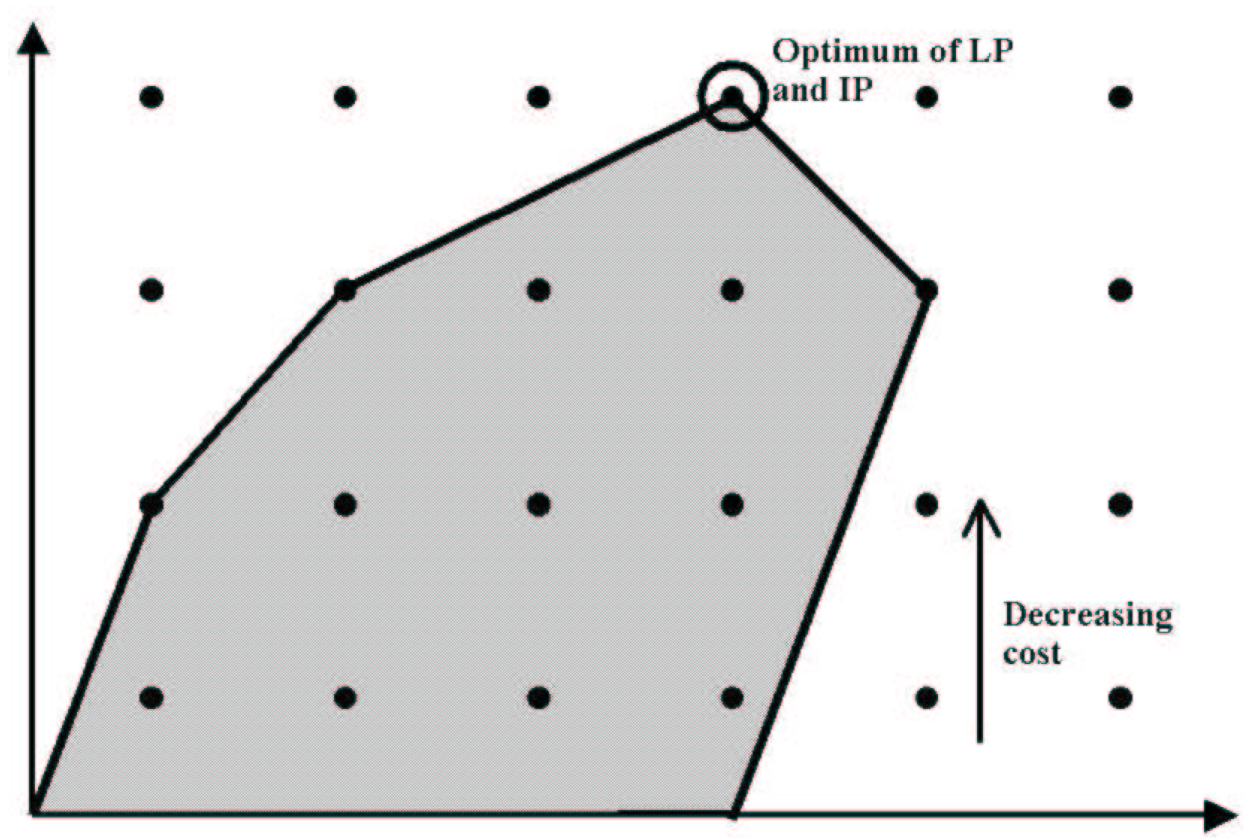


Total unimodularity (cont'd)

- An LP in the standard or canonical form where constraint matrix A is TUM and b is an integer vector, has an integer optimal point

So an ILP whose LP relaxation satisfies the above conditions can be solved exactly by solving its LP relaxation!

e.g., LP relaxation of shortest path problem, weighted bipartite matching



- Reference
  - Chapter 4.3, Convex Optimization.
  - For integer program: Chapter 13.1, Combinatorial Optimization: Algorithms and Complexity

- Acknowledgement
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