COMP9602: Convex Optimization

Introduction

Dr. C Wu

Department of Computer Science
The University of Hong Kong

Topics

- Theory of convex optimization
  - convex sets and functions
  - linear programming
  - quadratic programming
  - semidefinite programming
  - integer programming
  - vector optimization
  - ...
  - duality theory (primal-dual, Lagrange multiplier, KKT conditions)
- Algorithms to solve convex optimization problems
  - gradient descent algorithm
  - Newton's method
  - interior point method
  - Lagrangian relaxation and subgradient method
  - localization method
  - decomposition methods
  - ...
- Application of theory & algorithms
  - interpolation and regression
  - portfolio optimization
  - network flow problems (max flow, shortest path)
  - zero-sum game
  - ...

A taste of mathematical modeling using optimization techniques

Course information

- Lecturer:
  - Dr. Chuan Wu (Room 427, Chow Yei Ching Building)
  - Consultation hours:

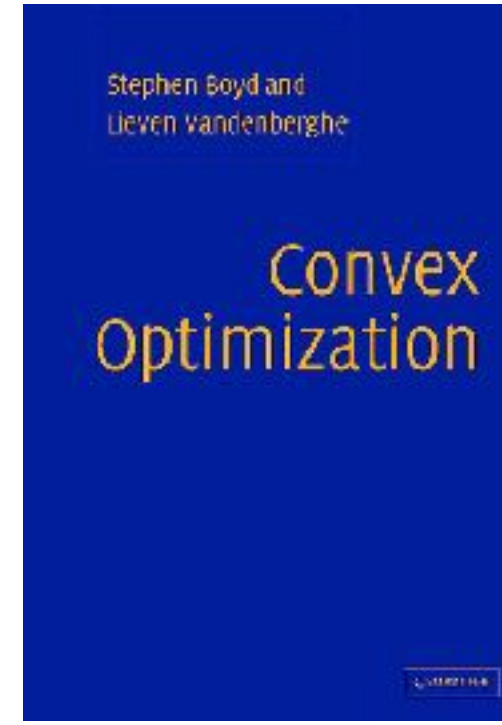
    Monday 3–4pm, Wednesday 3-4pm
- Teaching assistant:
  - Miss Yixin Bao (Room 412, Chow Yei Ching Building)
  - Consultation hours: Friday 10am-12noon
- Lecture hours
  - 1:30-2:50pm, Mondays & Wednesdays

Course available on Moodle

Course information

- **Textbook**

  Convex Optimization
  Stephen Boyd & Lieven Vandenberghe,
  Cambridge University Press, 2004
  pdf version available http://www.stanford.edu/~boyd/cvxbook/



- **Reference**

  Combinatorial Optimization: Algorithms and Complexity
  Christos H. Papadimitriou & Kenneth Steiglitz, Dover Publications, 1998.

  Convex Optimization Theory
  Dimitri P. Bertsekas, Athena Scientific, 2009

  Convex Optimization Algorithms
  Dimitri P. Bertsekas, Athena Scientific, 2015

  Nonlinear Programming (3rd edition)
  Dimitri P. Bertsekas, Athena Scientific, 2016

Learning Outcomes

Outcome 1 [Basic Concepts]: be able to master the key concepts on convex sets, convex functions and convex programs.

Outcome 2 [Key Theory]: be able to master the key theory on duality.

Outcome 3 [Representative Algorithms]: be able to master the representative algorithms to solve convex programs.

Outcome 4 [Problem Solving]: be able to model practical problems which you identify into convex programs, design/apply algorithms to solve them and present them in the format of academic papers.

Assessment and grading

- 3 assignments
  - 12% each
  - problem set, MATLAB code
- 1 project
  - 24%
  - modeling of an optimization problem in your research area

    literature study of optimization problems in your area

    find an optimization problem you would like to solve

    model the (convex) optimization problem

    design the solution algorithm

    test the solution algorithm using Matlab

    write a report (paper)
  - 1-2 student(s) per group
- 1 final examination
  - 40%
  - based on lectures in the entire semester
  - allowed to bring all lecture slides and handwritten notes

Academic policies

Write your own assignments

- You must reach your own understanding of the problem, discover a path to its solution
- Copying any portion of an assignment is a plagiarism

  Definition of plagiarism at http://www.hku.hk/plagiarism

  Both the copier and the being copied will get "0"

- **Late policy**

  15% deduction per day, maximum 2 days;

  no submission accepted after 2 days

Tentative schedule

Week	Date	Lecture Topic
1	Sep. 3	Introduction to convex optimization
	Sep. 5	Convex set
2	Sep. 10, 12	Convex function
3	Sep. 17	Convex function; standard forms of optimization problems, convex program
	Sep. 19	Equivalent programs
4	Sep. 24	Quasi-convex program, linear program integer linear program
	Sep. 26	Quadratic program, etc.
5	Oct. 1	National Day
	Oct. 3	Geometric program, semidefinite program
6	Oct. 8	Vector optimization, duality
	Oct. 10	Duality
7	Oct. 15	Duality [the lecture will be in theatre C, Chow Yei Ching Building]
	Oct. 17	Reading Week
8	Oct. 22, 24	Duality
9	Oct. 29	Duality
	Oct. 31	Algorithms for unconstrained optimization
10	Nov. 5	Algorithms for equality constrained optimization
	Nov. 7	No lecture
11	Nov. 12, 14	Interior-point methods
12	Nov. 19	Subgradient methods
	Nov. 21	Localization methods
13	Nov. 26	Decomposition methods
	Nov. 28	Decomposition methods; branch and bound methods; other methods for nonconvex optimization

Assignment 1

Introduction to mathematical optimization (program)

The diet problem

- A young lady wishes to be as skinny as possible while still keeping her healthy. There are n different food types available, the j th food containing $c_j \in R$ calories per milligram, $1 \le j \le n$, and $a_{ij} \in R$ milligrams of vitamin i per milligram, $1 \le i \le m$. The lady requires at least $b_i \in R$ milligrams of vitamin i to stay healthy. Given that the goal is to minimize caloric intake while having enough of each vitamin, how should she diet?

Let x_j be the number of milligrams of food j the lady eats.

The total amount of calories $\sum_{j=1}^{n} c_j x_j$

Intake of each vitamin i $\sum_{j=1}^{n} a_{ij} x_j$

The optimization problem

$$\min \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i, \forall i = 1, \ldots, m$$

$$x_j \ge 0, \forall j = 1, \ldots, n$$

Linear Program

The diet problem (cont'd)

- If the young lady wishes to maximize the total amount of vitamin intake (the goal), while keeping the overall caloric intake under a value (the constraint), how should she diet?

Let x_j be the number of milligrams of food j the lady eats.

The total amount of calories $\sum_{j=1}^{n} c_j x_j$

Intake of each vitamin i $\sum_{j=1}^{n} a_{ij} x_j$

The optimization problem

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j$$

subject to:

$$\sum_{j=1}^{n} c_j x_j \le d$$

$$x_j \geq 0, \forall j = 1, \ldots, n$$

Linear Program

Optimization problem

- An optimization problem consists of a set D, called the domain, and a real-valued function $f : D \to R$, called the objective function.
  - minimization problem: find an $x \in D$ such that $f(x) \leq f(y)$ for all $y \in D$
  - maximization problem: find an $x \in D$ such that $f(y) \leq f(x)$ for all $y \in D$

Mathematical optimization

- Mathematical programming (optimization)
  - a *mathematical program* consists of

    an objective function $f_0 : \mathbb{R}^n \to \mathbb{R}$

    a set of m constraint functions $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$

    and a constant vector $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}^m$

  - optimization variables $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\min f_0(\mathbf{x})$$

subject to:

$$f_i(\mathbf{x}) \leq b_i, i = 1, \ldots, m,$$

$$\mathbf{x} \in \mathbb{R}^n$$

  - optimal solution: $\mathbf{x}^*$ with the smallest value of f_0 among all vectors $\mathbf{x}$ satisfying the constraints
  - the course deals with sub-class of the problems with f_0 and f_i are convex

Solving optimization problems

- General optimization problem
  - very difficult to solve

    very long computation time

    or cannot always find the exact solution
- Exceptions
  - certain problem classes can be solved efficiently and reliably

    convex optimization problems

The great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity - R. Tyrrell Rockafellar (SIAM Review '93)

Convex optimization

$$\min f_0(\mathbf{x})$$
subject to:
$$f_i(\mathbf{x}) \leq b_i, i = 1, \ldots, m,$$
$$\mathbf{x} \in \mathbb{R}^n$$

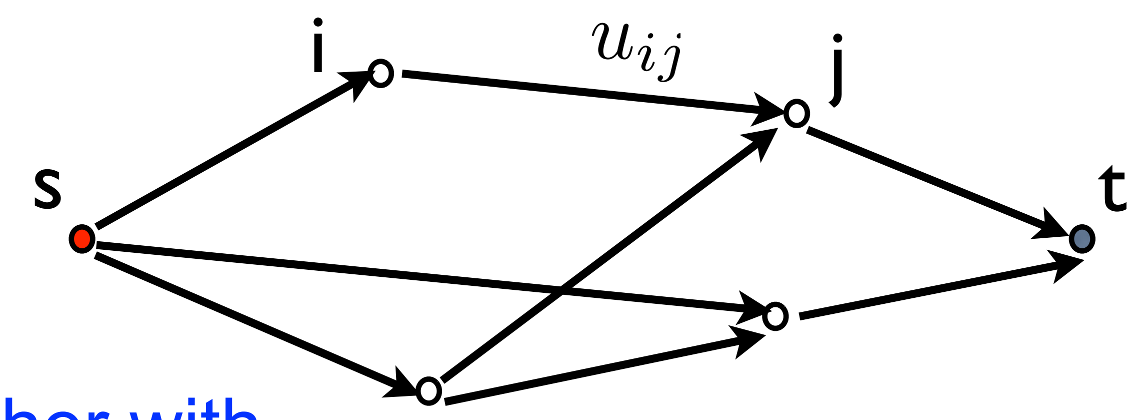
- Convex program
  - objective and constraint functions are convex
- $f_i(\mathbf{x})$ is convex if its domain is convex set and

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y) \quad \text{if } \alpha + \beta = 1,\ \alpha \geq 0,\ \beta \geq 0$$

- Example convex functions
  - linear functions are convex $\quad f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$
  - some quadratic functions $\quad f(x) = x^2$
- Special case: linear programming problems, least-square problems

Examples of convex program

- Network flow problem



- Given a directed graph G=(V, E) together with a capacity u_{ij} on each edge $(i, j) \in E$, and two distinguished vertices $s, t \in V$ (s is the source and t is the sink)

- A flow x_{ij}, for all $(i, j) \in E$, satisfies

non-negativity: for all $(i, j) \in E$, $x_{ij} \geq 0$

capacity constraints: for all $(i, j) \in E$, $x_{ij} \leq u_{ij}$

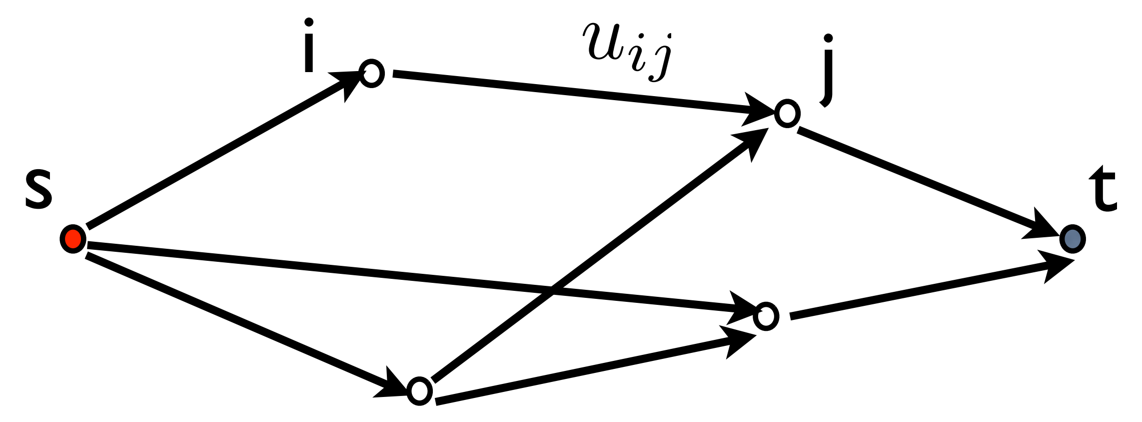
flow conservation: for all $i \in V\text{-}\{s,t\}$, $\sum_{j:(i,j)\in E} x_{ij} = \sum_{j:(j,i)\in E} x_{ji}$

- Maximum flow problem
- Min-cost flow problem

Examples of convex program

- Maximum flow problem

  v: the amount of end-to-end flow from s to t; an optimization variable



$$\max v$$

subject to

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} v, \text{for } i = s, \\ 0, \text{for all } i \in V - \{s,t\}, \\ -v, \text{for } i = t, \end{cases}$$

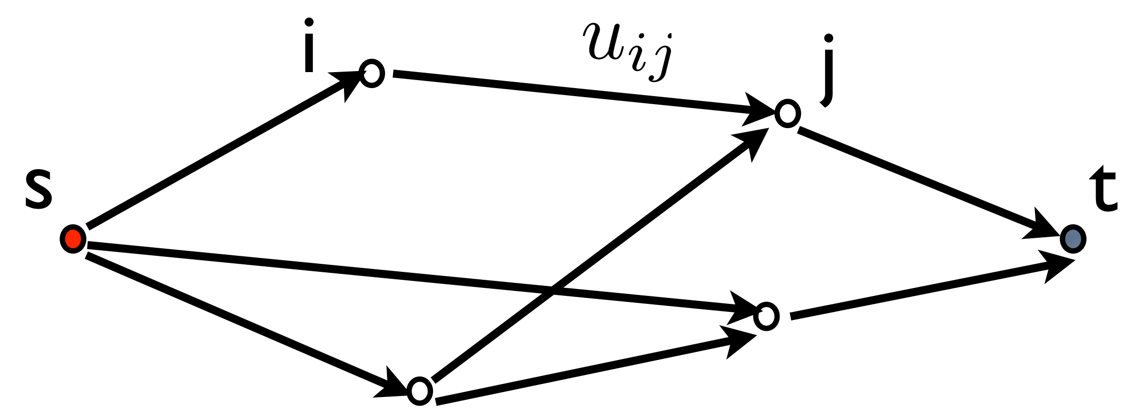
$$0 \le x_{ij} \le u_{ij}, \forall (i,j) \in E$$

Examples of convex program

- Min-cost flow problem

  given c_{ij} as cost of flow on $(i, j) \in E$,

  given v, solve for x_{ij}, for all $(i, j) \in E$



$$\min \sum_{(i,j)\in E} c_{ij} x_{ij}$$

subject to:

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} v, \text{for } i = s, \\ 0, \text{for all } i \in V - \{s, t\}, \\ -v, \text{for } i = t, \end{cases}$$

$$0 \le x_{ij} \le u_{ij}, \forall (i, j) \in E$$

v=1, xij∈ {0,1} => shortest path problem!

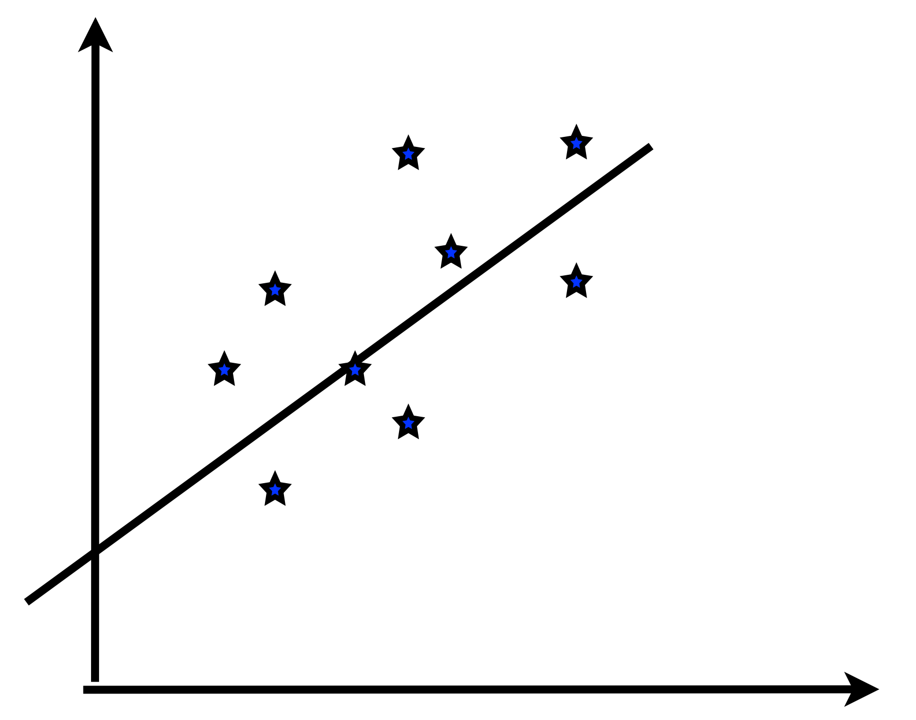
Examples of convex program

- Least square problems
  - line fitting

    given measured points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$, find out the line $\hat{y} = ax + b$, that fits the points best

$$\min \sum_{i=1}^{N} (y_i - \hat{y}_i)^2$$

$$where\ \hat{y}_i = ax_i + b$$



Examples of convex program

- Maximum likelihood estimation
  - given a family of probability distributions, with densities $p_x(\cdot)$, indexed by $x \in R^n$
  - a random variable Y
  - estimate $p_x(Y)$ from n observed values: y1, y2, ...yn
  - log-likelihood function $$l(x) = \sum_{i=1}^{n} \log p_x(y_i)$$
  - maximum likelihood estimation

$$\max \sum_{i=1}^{n} \log p_x(y_i)$$

subject to: $x \in C$

  - a convex optimization problem if $\log p_x(y_i)$ is concave on x

Solving convex optimization problems

$$\min f_0(\mathbf{x})$$

subject to:

$$f_i(\mathbf{x}) \le b_i, i = 1, \dots, m,$$

$$\mathbf{x} \in \mathbb{R}^n$$

- In general, no analytical solution
- Reliable and efficient algorithms to derive numerical solutions
- Computation time (roughly) proportional to $\max\{n^3, n^2 m, F\}$
  - where F is cost of evaluating fi's and their first and second derivatives

- Reference
  - Chapter 1, Convex Optimization.
- Acknowledgement
  - Some materials are extracted from the slides created by Prof. Stephen Boyd for the textbook
  - Some materials are extracted from the lecture notes of Convex Optimization by Prof. Wei Yu at the University of Toronto