COMP 9602: Convex Optimization

Duality (II)

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Next

- Interpretations of duality
  - Geometric interpretation
  - Sensitivity interpretation
  - Saddle-point interpretation
  - Game interpretation

- Examples
  - usage of duality
  - non-convex problem with strong duality

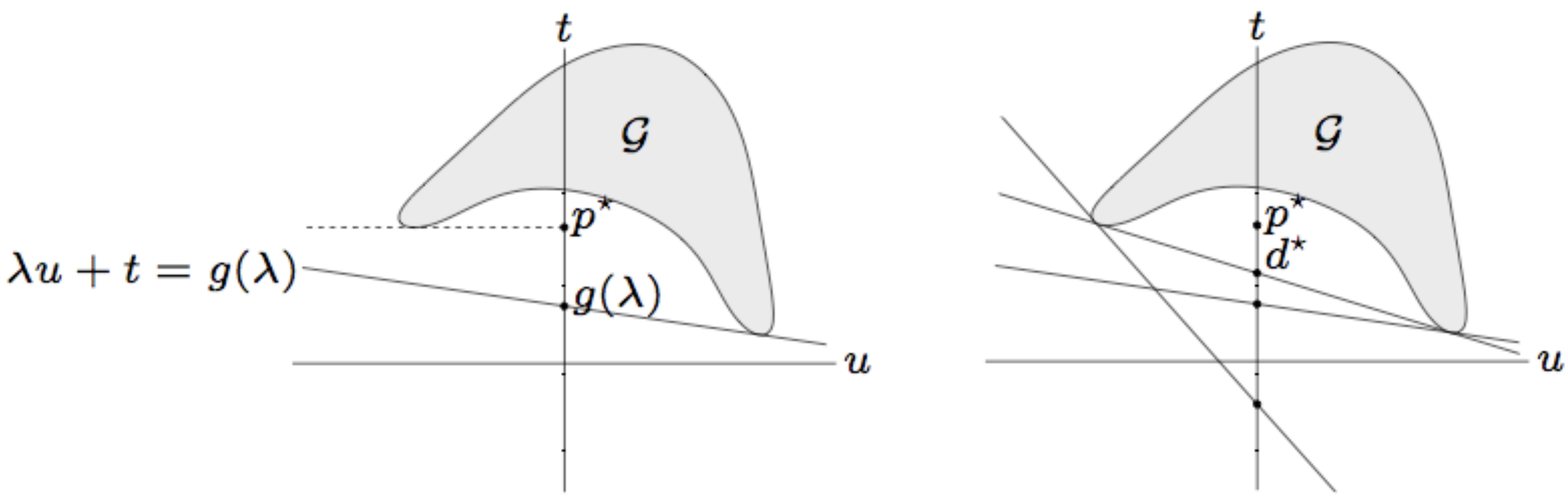
Geometric interpretation

for simplicity, consider problem with one constraint $f_1(x) \leq 0$

interpretation of dual function:

$$g(\lambda) = \inf_{(u,t) \in \mathcal{G}} (t + \lambda u), \qquad \text{where} \quad \mathcal{G} = \{(f_1(x), f_0(x)) \mid x \in \mathcal{D}\}$$

$$u = f_1(x), t = f_0(x)$$



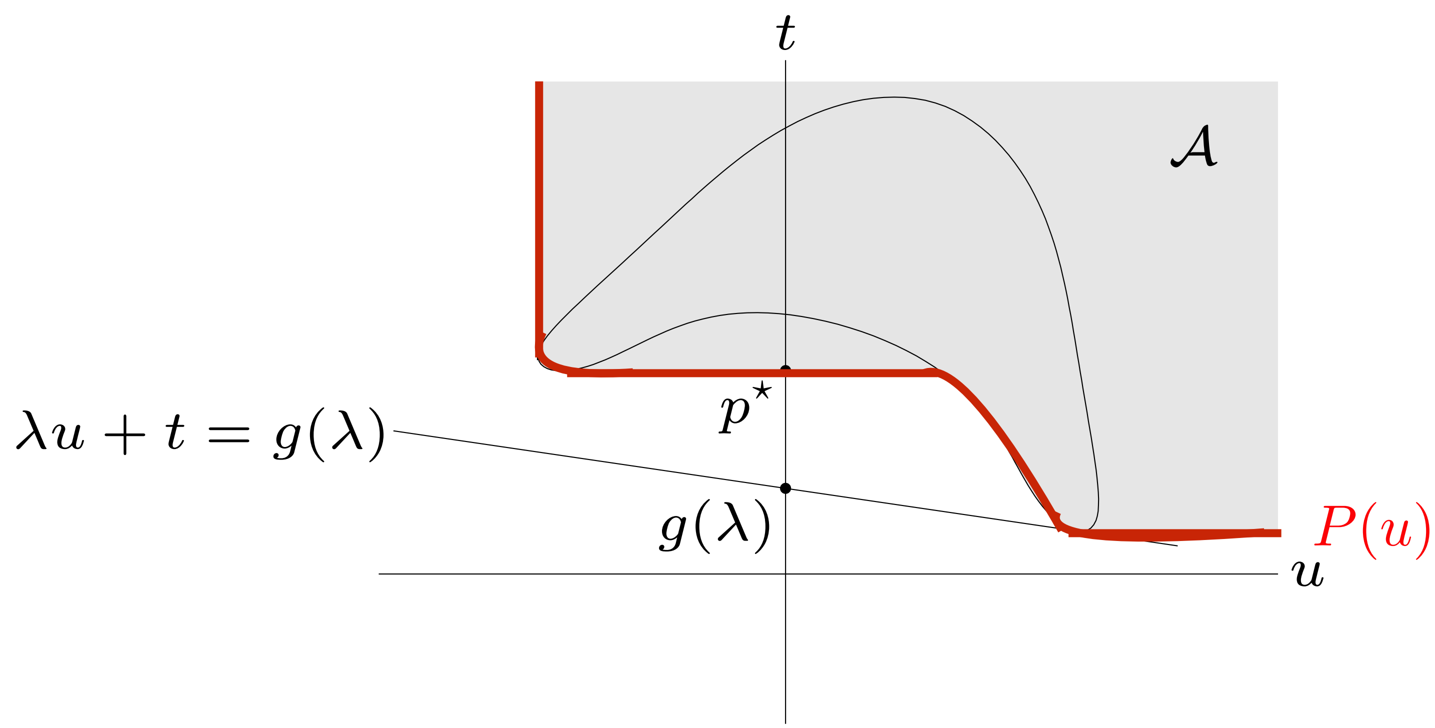
- $\lambda u + t = g(\lambda)$ is (non-vertical) supporting hyperplane to $\mathcal{G}$

  which intersects t-axis at $t = g(\lambda)$

Geometric interpretation (cont'd)

epigraph variation: same interpretation if $\mathcal{G}$ is replaced with

$$\mathcal{A} = \{(u, t) \mid f_1(x) \le u,\, f_0(x) \le t \text{ for some } x \in \mathcal{D}\}$$



$$P(u) = \min f_0(x) \quad \text{s.t. } f_1(x) \le u$$

strong duality

- holds if there is a non-vertical supporting hyperplane to $\mathcal{A}$ at $(0, p^\star)$
- for convex problem, $\mathcal{A}$ is convex, hence has supp. hyperplane at $(0, p^\star)$ P(u) is convex
- Slater's condition: if there exist $(\tilde{u}, \tilde{t}) \in \mathcal{A}$ with $\tilde{u} < 0$, then supporting hyperplanes at $(0, p^\star)$ must be non-vertical

Sensitivity interpretation

(unperturbed) optimization problem and its dual

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array} \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

perturbed problem and its dual

$$\begin{array}{lll} \text{min.} & f_0(x) & \\ \text{s.t.} & f_i(x) \le u_i, & i = 1, \dots, m \\ & h_i(x) = v_i, & i = 1, \dots, p \end{array} \qquad \begin{array}{ll} \text{max.} & g(\lambda, \nu) - u^T\lambda - v^T\nu \\ \text{s.t.} & \lambda \succeq 0 \end{array}$$

- $p^\star(u, v)$ is optimal value as a function of u, v

- **Local sensitivity**: if strong duality holds and p*(u,v) is differentiable at (0,0)

$$\lambda_i^\star = -\frac{\partial p^\star(0,0)}{\partial u_i}, \qquad \nu_i^\star = -\frac{\partial p^\star(0,0)}{\partial v_i}$$

Saddle-point interpretation

- Assume no equality constraints (results can be easily extended)

$$p^* = \inf_x \sup_{\lambda \succeq 0} L(x, \lambda) \qquad d^* = \sup_{\lambda \succeq 0} \inf_x L(x, \lambda)$$

- Weak duality: $\sup_{\lambda \succeq 0} \inf_x L(x, \lambda) \leq \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$

- Strong duality: $\sup_{\lambda \succeq 0} \inf_x L(x, \lambda) = \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$

- Max-min inequality generally holds:

$$\sup_{z \in Z} \inf_{w \in W} f(w, z) \leq \inf_{w \in W} \sup_{z \in Z} f(w, z) \qquad \text{for any } f, W, Z$$

- Strong max-min property (or saddle-point property) holds if

$$\sup_{z \in Z} \inf_{w \in W} f(w, z) = \inf_{w \in W} \sup_{z \in Z} f(w, z)$$

Saddle-point interpretation (cont'd)

- Saddle-point for f: a pair $\tilde{w} \in W, \tilde{z} \in Z$ that satisfy

$$f(\tilde{w}, z) \leq f(\tilde{w}, \tilde{z}) \leq f(w, \tilde{z}), \forall w \in W, z \in Z$$

<=> $f(\tilde{w}, \tilde{z}) = \inf_{w \in W} f(w, \tilde{z})$ $\qquad$ $f(\tilde{w}, \tilde{z}) = \sup_{z \in Z} f(\tilde{w}, z)$

<=> strong max-min property holds

- x^*, λ^* are primal and dual optimal points and strong duality holds
  <=> (x^*, λ^*) form a saddle-point for the Lagrangian

Game interpretation

$x \backslash y$	1	2	....	m
1	P_{11}	P_{12}	....	P_{1m}
2				
.	.	.		.
.				
n	P_{n1}	P_{n2}	....	P_{nm}

two-player zero-sum game

- Player x chooses strategy from 1, 2, ..., n
- Player y chooses strategy from 1, 2, ..., m
- P_{ij} is the amount x pays to y (payoff) when x plays strategy i, y plays strategy j

- mixed strategy:

  u_i: prob(player x chooses strategy i)

  v_j: prob(player y chooses strategy j)
- expected payoff: $u^T P v$

Game interpretation (cont'd)

- Suppose x fixes strategy u, then y plays (decides v) to maximize expected payoff:

$$\max u^T P v$$

s.t. $$\sum_{i=1}^{m} v_i = 1$$

$$v \succeq 0$$

=> optimal value: $\max_{i=1,\ldots,m} (P^T u)_i$ ith row

=> optimal point: $v_j = 1, j = \text{argmax}_i (P^T u)_i$

$v_i = 0, \forall i \neq j$

- So x must choose u to minimize $\max_{i=1,\ldots,m} (P^T u)_i$:

$$\min_u \max_{i=1,\ldots,m} (P^T u)_i$$

s.t. $$\sum_{i=1}^{n} u_i = 1$$

$$u \succeq 0$$

=>

$$\min t$$

s.t. $$P^T u \preceq t\mathbf{1}$$

$$u^T \mathbf{1} = 1$$

$$u \succeq 0$$

LP (1) with Optimal value: p_1^*

Game interpretation (cont'd)

- Suppose y plays first (v given), then x chooses u to minimize expected payoff:

$$\min u^T P v$$

s.t. $$\sum_{i=1}^{n} u_i = 1$$

$$u \succeq 0$$

=> optimal value: $\min_{i=1,\ldots,n} (Pv)_i$ ith row

=> optimal point: $u_j = 1, j = \mathrm{argmin}_i (Pv)_i$

$u_i = 0, \forall i \neq j$

- So y chooses v to maximize $\min_{i=1,\ldots,n} (Pv)_i$:

$$\max_v \min_{i=1,\ldots,n} (Pv)_i$$

s.t. $$\sum_{i=1}^{m} v_i = 1$$

$$v \succeq 0$$

=>

$$\max t$$

s.t. $$Pv \succeq t\mathbf{1}$$

$$v^T \mathbf{1} = 1$$

$$v \succeq 0$$

LP (2) with Optimal value: p_2^*

Game interpretation (cont'd)

LP (1) and LP (2) are duals of each other and thus have the same optimal values: $p_1^* = p_2^*$

Therefore, there is no advantage to play second, i.e., $p_1^* \cancel{\geq} p_2^*$

Game interpretation (cont'd)

- Consider payoff function $f(u,v) = u^T P v$, the optimum u* for LP (1) and the optimum v* for LP (2) form a saddle-point for f(u,v)

  - $f(u^*,v) \le f(u^*,v^*) \le f(u,v^*)$

  <=> $f(u^*,v^*) = \inf_u f(u,v^*)$ $f(u^*,v^*) = \sup_v f(u^*,v)$

- Nash equilibrium of the game: (u*,v*) such that

  u* is the best response of player x with respect to v*
  v* is the best response of player y with respect to u*

Usage example of duality

- Duality gives a way to analytically solve an optimization problem
  - example:

$$\min \| x \|_2^2$$

s.t.

$$Ax = b$$

Non-convex problem with strong duality

- Strong duality $\not\Rightarrow$ convex problem
  - example non-convex problem with strong duality:

$$\min x^T A x$$

s.t. $$x^T x = 1$$

$$A \in S^n$$

- Reference
  - Chapter 5.3 - 5.4, 5.6, Convex Optimization.
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