COMP9602: Convex Optimization

Convex Set

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Convex set

line segment between x_1 and x_2: all points

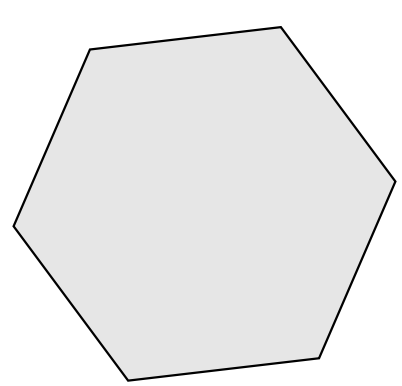
$$x = \theta x_1 + (1 - \theta) x_2$$

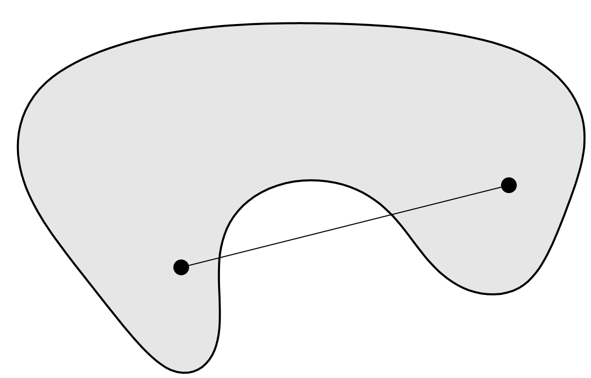
with $0 \leq \theta \leq 1$

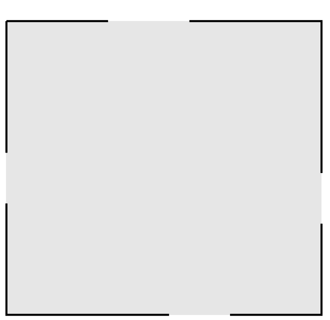
convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1 - \theta) x_2 \in C$$

examples (convex and nonconvex sets):



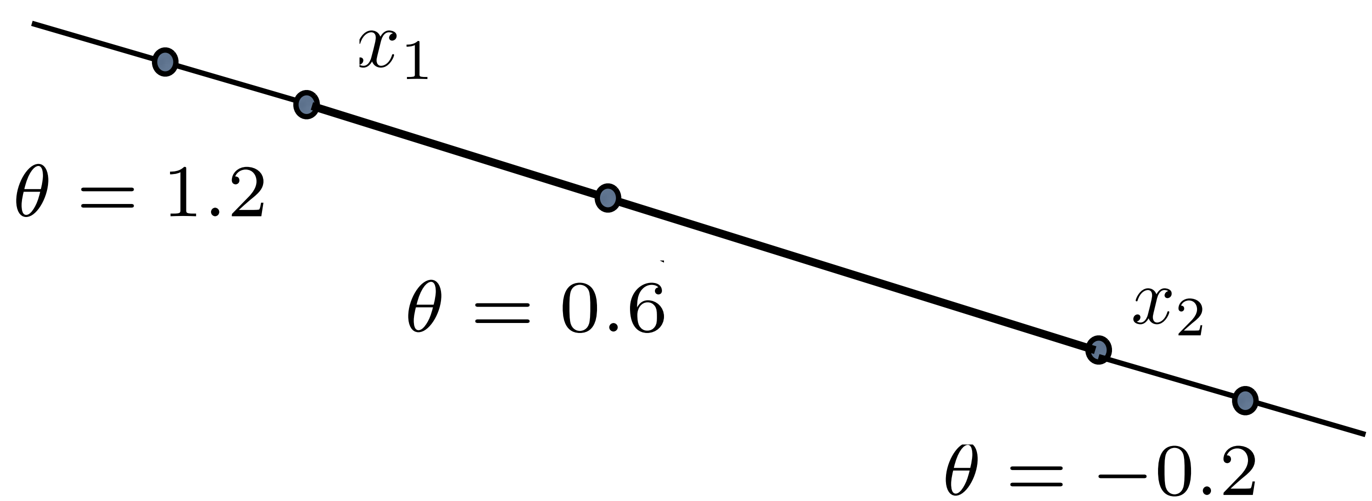




Affine set

line through x_1, x_2: all points

$$x = \theta x_1 + (1-\theta)x_2 \qquad (\theta \in \mathbf{R})$$



affine set: contains the line through any two distinct points in the set

examples: solution set of linear equations $\{x \mid Ax = b\}$

Convex hull

convex combination of $x_1, \ldots, x_k$: any point x of the form

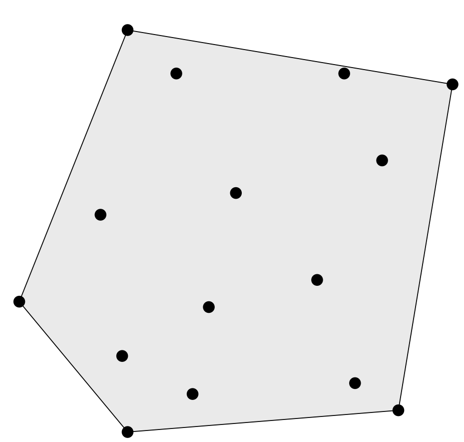
$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

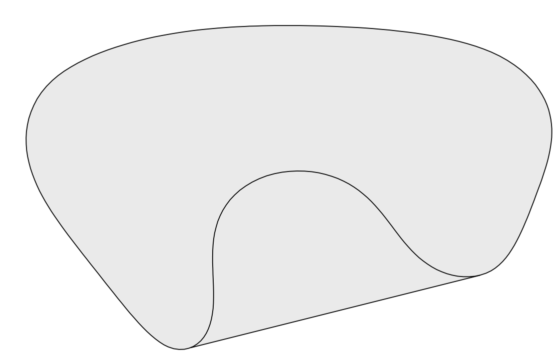
with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S

$$\mathbf{conv}\, S = \{\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n | x_i \in S, \theta_i \geq 0, \sum_{i=1}^{n} \theta_i = 1\}$$

examples:

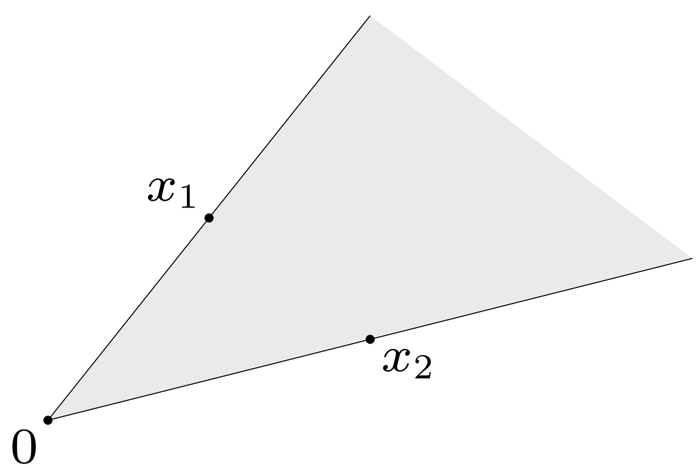


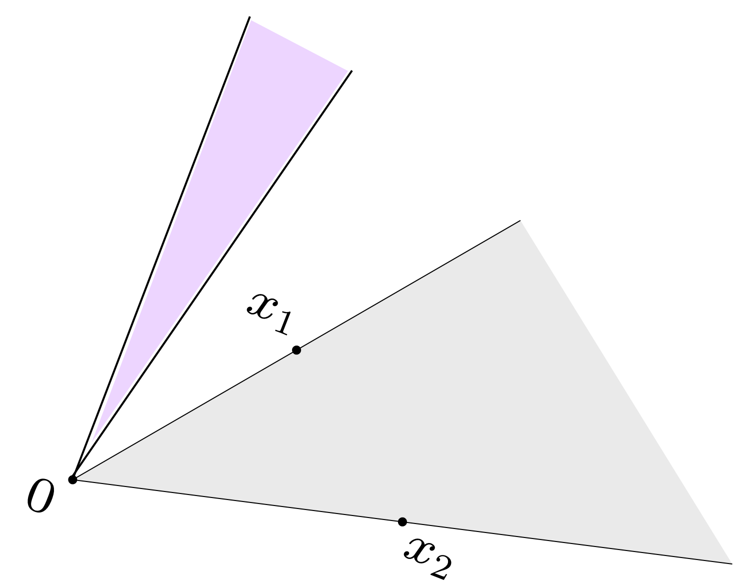


Cones

cones: a set C is a cone, if for every $x \in C$ and $\theta \geq 0$, we have $\theta x \in C$.

convex cones: a set C is a convex cone, if it is a cone and it is convex, i.e., for any $x_1, x_2 \in C$ and $\theta_1 \geq 0, \theta_2 \geq 0$, we have $\theta_1 x_1 + \theta_2 x_2 \in C$





conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

Proper cone

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line) i.e., x ∈ K, -x∈ K => x=0

examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$

Positive semidefinite cone

$\mathbf{S}^n$ is set of symmetric $n \times n$ matrices

$\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

$\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

Property: $\mathbf{S}^n_+$ is a proper cone

Generalized inequalities

inequality in $\mathbf{R}_+ = \{x | x \geq 0\}$: $x \leq y \iff y - x \in \mathbf{R}_+$

generalized inequality defined by a proper cone K:

$$x \preceq_K y \quad \iff \quad y - x \in K \qquad\qquad x \prec_K y \quad \iff \quad y - x \in \textbf{int}\, K$$

Examples:

componentwise inequality $(K = \mathbf{R}^n_+)$

$$x \preceq_{\mathbf{R}^n_+} y \quad \iff \quad x_i \leq y_i, \quad i = 1, \ldots, n$$

matrix inequality $(K = \mathbf{S}^n_+)$

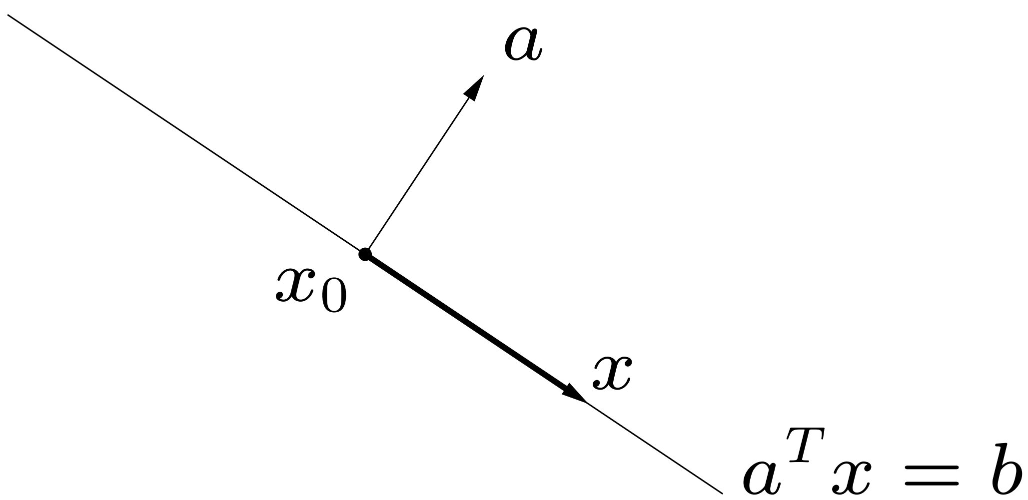
$$X \preceq_{\mathbf{S}^n_+} Y \quad \iff \quad Y - X \text{ positive semidefinite}$$

Note: these two types are so common that we usually drop the subscript in $\preceq_K$

Generalized inequality defined by a proper cone K is in general not a total ordering (i.e., it is not true either $x \preceq_K y$ or $y \preceq_K x$)

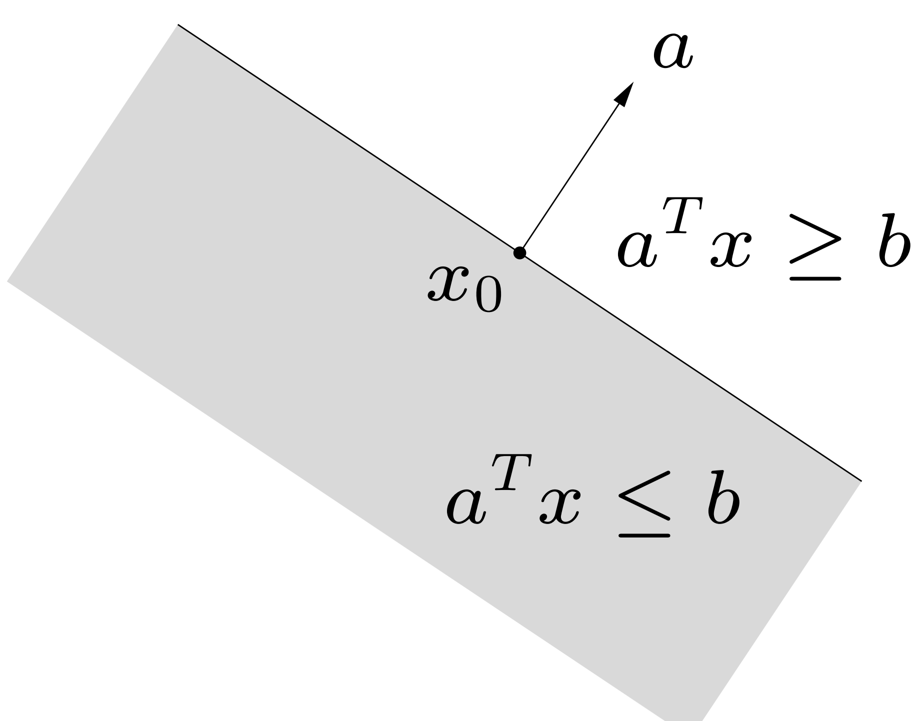
E.g., positive semidefinte cone example: it is not always either "$X \preceq Y$" or "$Y \preceq X$"

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex

halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- halfspaces are convex

Euclidean balls and ellipsoids

(Euclidean) ball with center x_c and radius r:

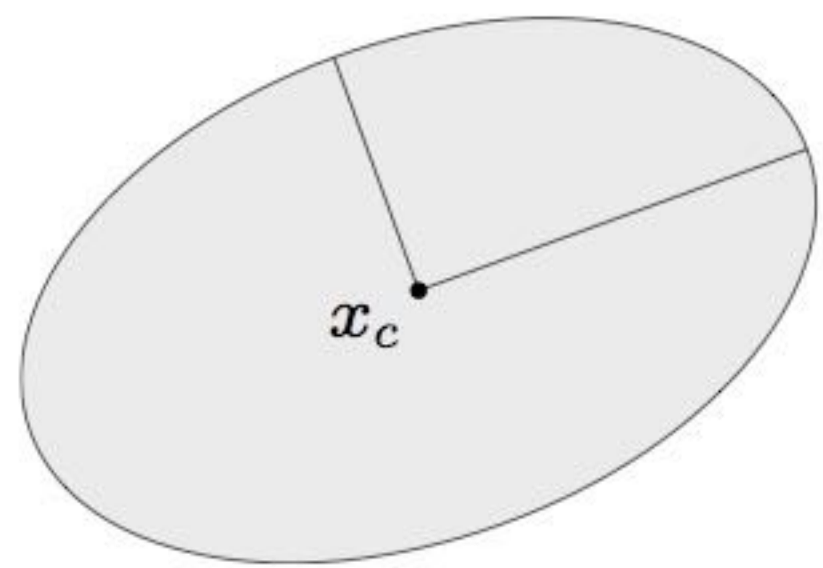
$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)

lengths of semi-axes: $\sqrt{\lambda_i}$,
(λ_i eigenvalues of P)



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

$(A = P^{1/2})$

Norm

norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\,\|x\|$ for $t \in \mathbf{R}$ (homogeneous)
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

l_p-norm: $\|\mathbf{x}\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}, p \geq 1$

l_1-norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$

$$l_\infty - norm : \|\mathbf{x}\|_\infty = \lim_{p->\infty} (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} = \lim_{p->\infty} (\max_i |x_i|^p)^{\frac{1}{p}}$$
$$= \max_i |x_i|$$

Property: $B(x_c, r)$ is convex for all norms.

Proof as exercise

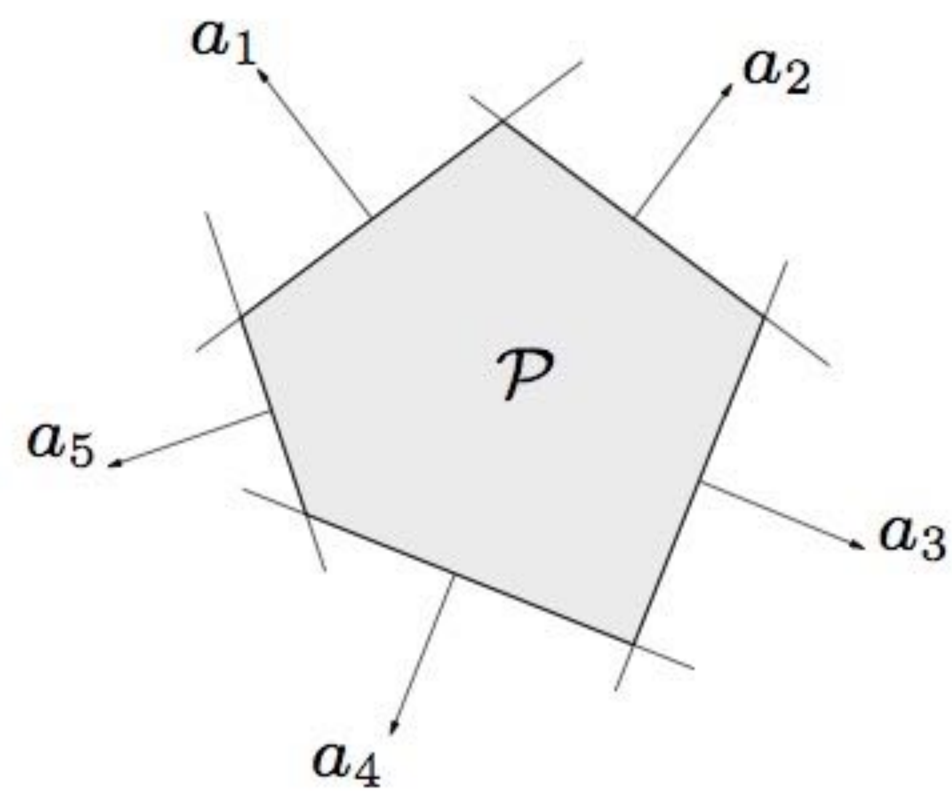
Polyhedron and polytope

- **Polyhedron** is solution set of finitely many linear inequalities and equalities

$$P = \{x | \mathbf{a}_j^T \mathbf{x} \le \mathbf{b}_j, j = 1, \ldots, m, \\ \mathbf{c}_j^T \mathbf{x} = \mathbf{d}_j, j = 1, \ldots, k\}$$

  - polyhedron is intersection of finite number of halfspaces and hyperplanes
  - polyhedron is convex

- **Polytope** is a bounded polyhedron



Methods for establishing convexity of a set

1. apply definition:

$$x_1, x_2 \in C,\ 0 \le \theta \le 1 \Rightarrow \theta x_1 + (1 - \theta) x_2 \in C$$

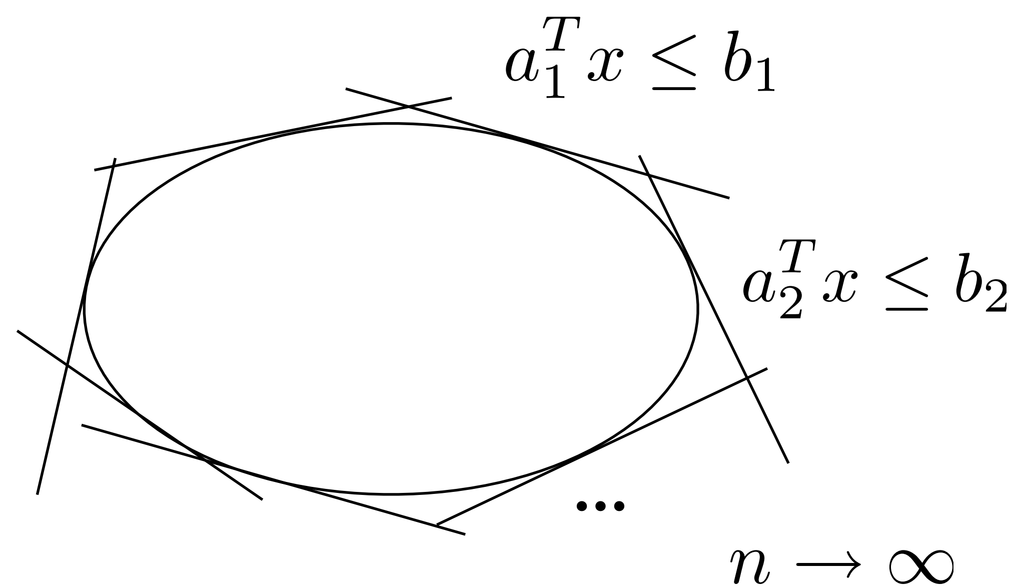
2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity

   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

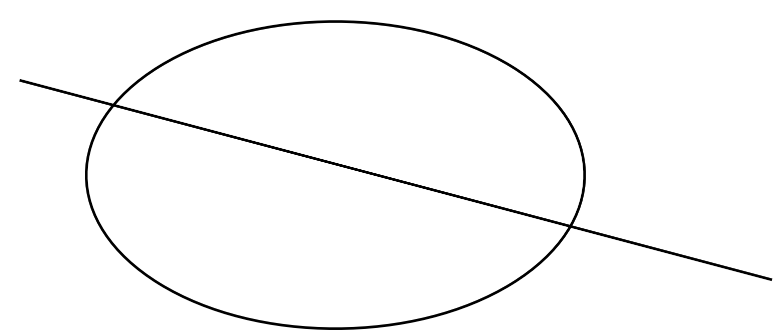
Intersection

If S_i is (affine, convex, convex cone), for $i \in A$, then $\cap_{i\in A} S_i$ is (affine, convex, convex cone)

★ intersection need not be finite: e.g., a convex set is intersection of infinite halfspaces



Union?



Affine function

Affine function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$

$$f(x) = Ax + b \text{ with } A \in \mathbf{R}^{m \times n},\ b \in \mathbf{R}^m$$

★If S is convex, $f(S)$ is also convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

★If C is convex, $f^{-1}(C)$ is also convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

example:

- The ellipsoid $\mathcal{E} = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$ is the image of the unit ball $\{u \mid \|u\|_2 \leq 1\}$ under the affine mapping $f(u) = P^{1/2} u + x_c$. It is also the inverse image of the unit ball under the affine mapping $g(x) = P^{-1/2}(x - x_c)$.

Perspective function & linear-fractional function

perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

★ images and inverse images of convex set under these functions are convex

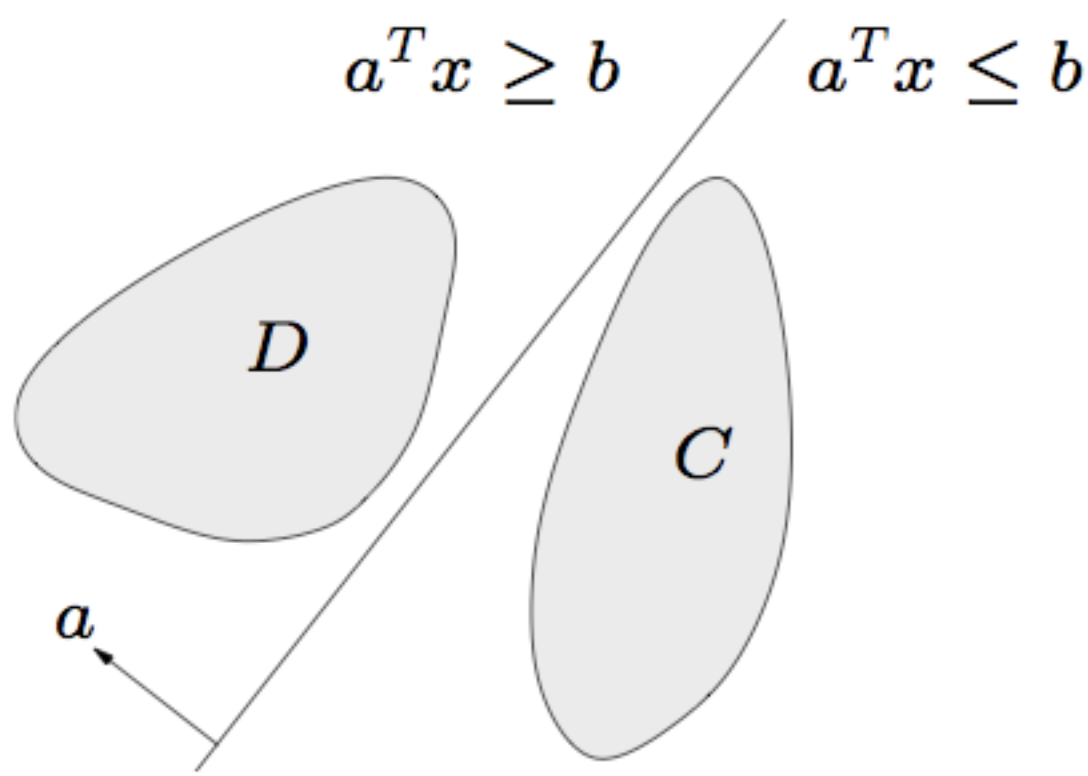
★ perspective functions preserve lines => preserve convexity

Separating hyperplane theorem

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

the hyperplane $\{x \mid a^T x = b\}$ separates C and D

i.e., $a^T x \leq b$ for $x \in C$, $a^T x \geq b$ for $x \in D$



Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

- Reference
  - Chapter 2, Appendix A, Convex Optimization.
- Acknowledgement
  - Some materials are extracted from the slides created by Prof. Stephen Boyd for the textbook
  - Some materials are extracted from the lecture notes of Convex Optimization by Prof. Wei Yu at the University of Toronto