COMP 9602: Convex Optimization

Convex Programs (I)

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Where we are

Theory	convex set convex function standard forms of optimization problems, quasi-convex optimization linear program quadratic program geometric program vector optimization integer program Duality
Algorithm	unconstrained optimization equality constrained optimization interior-point method localization methods subgradient method decomposition methods etc.

Optimization problem in standard form

Standard form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

optimal value:

$$p^\star = \inf\{f_0(x) \mid f_i(x) \le 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Feasible and optimal points

x is **feasible** if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints

optimal point: a feasible x is **optimal** if $f_0(x) = p^\star$

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize (over } z\text{)} & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \leq R \end{array}$$

Examples:

- $f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$
- $f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$
- $f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$

Feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

Convex optimization problem

Standard form convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array} \quad \text{<=>} \quad \begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

$f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine

important property: feasible set of a convex optimization problem is convex

Generalized inequalities are also ok

- e.g.,

$$\min \mathbf{c}^T \mathbf{x}$$

subject to:

$$A_0 + A_1 x_1 + A_2 x_2 + \ldots + A_n x_n \preceq 0$$

Local and global optima

- Any locally optimal point of a convex program is (globally) optimal
  - proof:

Optimality criteria for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$

- **unconstrained problem**: x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad \nabla f_0(x) = 0$$

- **equality constrained problem**

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad Ax = b$$

x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom}\, f_0, \qquad Ax = b, \qquad \nabla f_0(x) + A^T\nu = 0$$

Optimality criteria for differentiable f_0 (cont'd)

- **minimization over nonnegative orthant**

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad x \succeq 0$$

x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad x \succeq 0, \qquad \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

Equivalent convex programs

two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

- **eliminating equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

find

- x_0: a particular solution of Ax=b
- matrix F: whose range is the nullspace of A, i.e. R(F)=N(A)

is equivalent to

$$\begin{array}{ll} \text{minimize (over } z\text{)} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \le 0, \quad i = 1, \dots, m \end{array}$$

Equivalent convex programs

- **introducing equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(A_0x + b_0) \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x\text{, } y_i) & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\ & y_i = A_ix + b_i, \quad i = 0, 1, \ldots, m \end{array}$$

Equivalent convex programs

- **introducing slack variables for linear inequalities**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, s) & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\ & s_i \ge 0, \quad i = 1, \ldots m \end{array}$$

Equivalent convex programs

- **epigraph form**: standard form convex problem is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x, t) & t \\
\text{subject to} & f_0(x) \leq t \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- **minimizing over some variables**

$$
\begin{array}{ll}
\text{minimize} & f_0(x_1, x_2) \\
\text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize} & \tilde{f}_0(x_1) \\
\text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{array}
$$

where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

- Small perturbation of the problem makes it very hard (potentially)
  - max convex or minimize concave
  - non-convex constraints
  - convex equality constraints

Quasiconvex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

with $f_0 : \mathbf{R}^n \to \mathbf{R}$ quasiconvex, $f_1, \ldots, f_m$ convex

can have locally optimal points that are not (globally) optimal

- example

Quasiconvex optimization

- Quasiconvex function $f_0(x)$ can be represented by a family of convex functions $\phi_t(x)$, indexed by $t \in R$

if f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

- $\phi_t(x)$ is convex in x for fixed t
- t-sublevel set of f_0 is 0-sublevel set of ϕ_t, *i.e.*, $f_0(x) \leq t \quad \Longleftrightarrow \quad \phi_t(x) \leq 0$

Quasiconvex optimization

- Such a representation always exists and not unique

e.g.:

(1) $$\phi_t(x) = \begin{cases} 0, f_0(x) \le t, \\ \infty, otherwise, \end{cases}$$

(2) $$\phi_t(x) = dist(x, \{z | f_0(z) \le t\})$$

(3) Convex over concave functions:

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $q(x) > 0$ on $\mathbf{dom}\, f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \ge 0$, ϕ_t convex in x
- $p(x)/q(x) \le t$ if and only if $\phi_t(x) \le 0$

Quasiconvex optimization via convex feasibility problems

Let p* denote the optimal value of the quasiconvex program

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

Solve the feasibility problem:

$$
\begin{array}{ll}
\text{find} & x \\
\text{subject to:} & \phi_t(x) \leq 0 \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m, \\
& Ax = b
\end{array}
$$

if feasible, $p^* \leq t$; otherwise, $p^* > t$

=> solve the quasiconvex problem using bisection, solving a convex feasibility problem at each step

Quasiconvex optimization via convex feasibility problems

Start with an interval [l,u] known to contain p*

Bisection method for quasiconvex optimization

given $l \leq p^\star$, $u \geq p^\star$, tolerance $\epsilon > 0$.

repeat

1. $t := (l + u)/2$.
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, $u := t$; **else** $l := t$.

until $u - l \leq \epsilon$.

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requires exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations

- Reference
  - Chapter 4.1— 4.2, Convex Optimization.
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