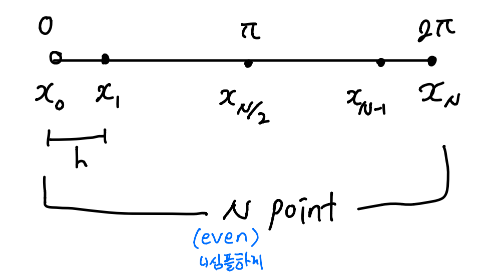
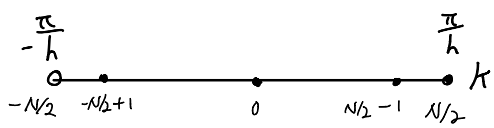
Ch.3 Bounded Periodic Grids.



$u(x) = u(x+2\pi)$
$u(x_0) = u(x_N)$, $h = 2\pi / N \rightarrow \frac{\pi}{h} = \frac{N}{2}$

⇒ physical space	Discrete	Bounded	$x \in \{h, 2h, 3h, \cdots Nh\}$ ($Nh = 2\pi$)
⇒ Fourier space	Bounded	Discrete	$k \in \{-\frac{N}{2}+1, -\frac{N}{2}+2 \cdots \frac{N}{2}\}$



$$\Delta k = \frac{2\pi}{h} \cdot \frac{1}{N} = 1$$

Given V_j, FT, $x_j = jh$

$$\hat{V}_k = h \sum_{j=1}^{N} e^{-ikx_j} V_j$$

$$k \in \{-\tfrac{N}{2}+1 \cdots \tfrac{N}{2}\}$$

Inverse FT

$$V_j = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}+1}^{k=\frac{N}{2}} \cdot e^{ikx_j} \hat{V}_k$$

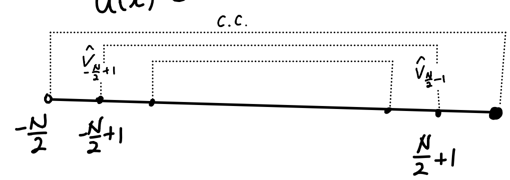
$$x \in \{h, 2h \cdots Nh\}$$

i) $P(x) = \frac{1}{2\pi} \sum_k e^{ikx} \hat{V}_k \rightarrow W_j = P'(x_j)$

ii) $\hat{w}_k = i k \hat{v}_k$ ✱ $\rightarrow W_j = IFT(\hat{w}_k)$

Exercise 2.2

$u(x) \in \mathbb{R} \rightarrow \hat{u}(k) = \hat{u}^*(-k)$



Let $\hat{V}_k = \begin{cases} 1, & k = \frac{N}{2} \\ 0, & |k| < \frac{N}{2} \end{cases}$

$$V_j = \frac{1}{2\pi} e^{i\frac{N}{2}x_j} = \frac{1}{2\pi} e^{i\pi j} \in \text{Real},$$

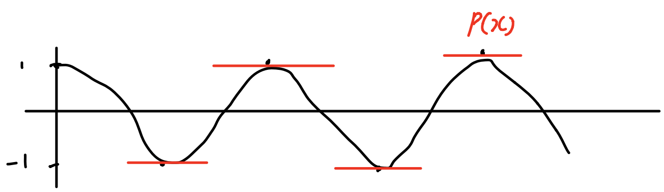
$$P(x) = \frac{1}{2\pi} e^{i\frac{N}{2}x} \in \text{Complex}, \quad W_j = P'(x_j) \in \text{imag}$$

Remedy. $\hat{V}_k' = \begin{cases} \frac{1}{2}, & k = \pm N/2 \\ 0, & |k| < \frac{N}{2} \end{cases}$

$$V_j = \frac{1}{2\pi}\cdot\frac{1}{2}\left(e^{i\frac{N}{2}x_j} + e^{-i\frac{N}{2}x_j}\right) = \frac{1}{2\pi}\cos\frac{N}{2}x_j$$

$$p(x) = \frac{1}{2\pi}\cos\frac{N}{2}x = \frac{1}{2\pi}\cos\pi j, \quad w_j = p'(x_j) = 0$$

「Fig 3.1」



$N=4$, $K=\frac{N}{2}$

Before $V_j = \frac{1}{2\pi}\sum_{k=-\frac{N}{2}+1}^{N/2} e^{ikx_j}\hat{V}_k$

Modified $V_j = \frac{1}{2\pi}\sum_{k=-\frac{N}{2}}^{N/2} e^{ikx_j}\hat{V}_k$

↓

$$\frac{1}{2\pi}\left(e^{i(-\frac{N}{2})x_j}\frac{\hat{V}_{N/2}}{2} + e^{i(-\frac{N}{2}+1)x_j}\hat{V}_{-\frac{N}{2}+1} + \cdots + e^{i\frac{N}{2}x_j}\frac{\hat{V}_{N/2}}{2}\right.$$

$$= \frac{1}{2\pi}\sum_{k=-\frac{N}{2}}^{N/2} e^{ikx_j}\hat{V}_k$$

$$p(x) = \frac{1}{2\pi}\sum_{k=-\frac{N}{2}}^{N/2} e^{ikx}\hat{V}_k$$

$V_j = \sum_{-\infty}^{\infty} V_m \delta_j^m$, $FT(\delta_j^0) \to \hat{\delta}_j(k) = h \to P_\delta(x) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx} \to$ sinc func

$$S_h(x) = \frac{\sin(\pi x/h)}{\pi x/h}$$

$$p(x) = \sum_{-\infty}^{\infty} V_m S_h(x - x_m)$$

$V_j = \sum_{j=1}^{N} V_m \delta_j^m$, $\hat{V}_k = h\sum_{j=1}^{N} e^{-ikx_j} V_j$, $\frac{h}{2\pi}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}} e^{ikx}$ (periodic)

$$\sum_{m=1}^{N} V_m S_N(x - x_m)$$

$$\boxed{S_N(x) = \frac{\sin(\pi x/h)}{(2\pi/h)\tan(x/2)} \quad (3.7)}$$

$$w_j = p'(x_j) = \sum_{m=1}^{N} V_m S_N'(x_j - x_m) \longrightarrow p''(x_j) = \sum_{m=1}^{N} V_m S_N''(x_j - x_m)$$

S_N' , S_N'' $\to$ (3.9) , (3.11)

$N \times N$

$$\begin{pmatrix} w_j \end{pmatrix} = \underbrace{\begin{pmatrix} S_N'(x_j - x_m) \end{pmatrix}}_{D_N} \begin{pmatrix} V_m \end{pmatrix}$$

$D_N \to$ (3.10)

Toeplitz & circulant.

$D_N^{(2)} \to$ (3.12)

Toeplitz & circulant

// Output 6 //

$$u_t = -\left(\frac{1}{5} + \sin^2(x-1)\right)$$

x : $u_x(x_j) \simeq (D_N V)_j$

$$t : u_t(t^n) \simeq \frac{u(t^{n+1}) - u(t^{n-1})}{2\Delta t}$$

Skip 6.

$$\frac{V_j^{(n+1)} - V_j^{(n-1)}}{2\Delta t} = -C(x)\left(D_N V^n\right)_j$$

Coe