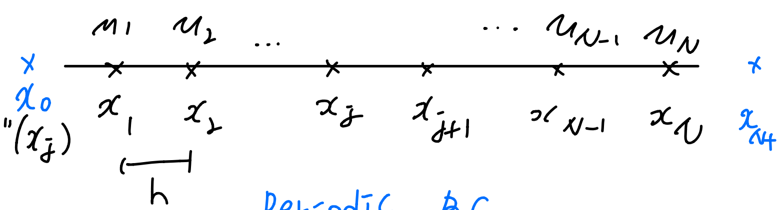
Ch.1. Grid point, approximate $u'(x_j)$?

1) Finite Differences.

테일러 전개

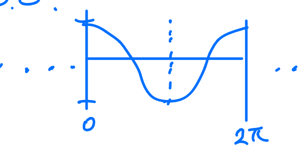


✓ $u(x_{j+1}) = u(x_j + h)$

$$= u(x_j) + h u'(x_j) + \frac{h^2}{2} u''(x_j)$$

Periodic B.C.

$u(x) = \cos x$ …



…

✓ $u(x_{j-1}) = u(x_j - h)$

$$- \quad = u(x_j) - h u'(x_j) + \frac{h^2}{2} u''(x_j)$$

$\underline{\frac{h^3}{3!} u'''(x_j)} \times 2$

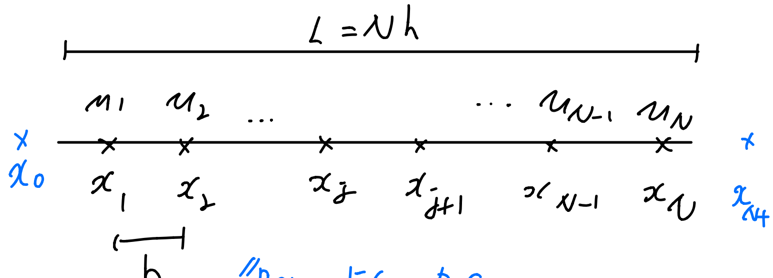
$$u(x_{j+1}) - u(x_{j-1}) = 0 + 2h u'(x_j) + 0 + \underline{\frac{h^3}{3} u'''(x_j) + O(h^4)}$$

↳ $u'(x_j) \simeq \frac{1}{2h}\left(u(x_{j+1}) - u(x_{j-1})\right)$ + (high term ignore)

$= w_j$ ✓

↳ centered Difference Formula (2nd-order accuracy)

$$\left.\begin{aligned} w_1 &= \frac{1}{2h}(u_2 - \cancel{u_0}^{\,u_N}) \\ w_2 &= \frac{1}{2h}(u_3 - u_1) \\ &\;\vdots \\ w_{N-1} &= \frac{1}{2h}(u_N - u_{N-2}) \\ w_N &= \frac{1}{2h}(\cancel{u_{N+1}}_{\,u_1} - u_{N-1}) \end{aligned}\right]$$



"Periodic B.C.

$$\begin{cases} u(x+L) = u(x) \\ u(x_0) = u(x_0 + Nh) = u(x_N) \\ u(x_{N+1}) = u(x_{N+1} - Nh) = u(x_1) \end{cases}$$

↳ Matrix

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{pmatrix} = \frac{1}{2h} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1\ 0 & 1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix}$$

$\hookrightarrow D_N = (N \times N)$: Toeplitz

a_{ij} depend only on $i-j$ · circulant

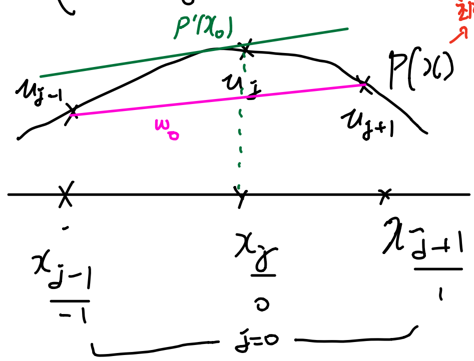
ex) $(i-j+1)$

$a_{i+1,j}$ $(i-j+1)-N$ $\Longleftarrow$ $\ulcorner a_{ij} = a_{i+1,j+1}$

$= a_{i,j-1+N}$ $\quad$ $N-j+1 = -(j-1-N)$ $\quad (i-j) \quad (i+1)-(j+1)$

?? $(= a_{i+N,j-1}$ $\quad = (i-j) \lrcorner$

최대오차 가?.



$w_j = p'(x_j)$,

where $p(x)$ is polynomical ≤ 2

$p(x) = a + bx + cx^2$

$= u_{-1}\alpha_{-1}(x-x_0)(x-x_1)$

$+ u_0\alpha_0(x-x_{-1})(x-x_1)$??

$+ u_1\alpha_1(x-x_{-1})(x-x_0)$

$x_{j+1} - x_j = h$

$u_{-1} = p(x_{-1}) = u_{-1}\alpha_{-1}(-h)(-2h)$

$\Rightarrow \alpha_{-1} = \frac{1}{2h^2}$

$u_0 = p(x_0) = u_0\alpha_0(h)(-h)$ $\quad \left\{ \alpha_{-1} = \frac{1}{2h^2},\ \alpha_0 = -\frac{1}{h^2},\ \alpha_1 = \frac{1}{2h^2} \right.$

$\Rightarrow \alpha_0 = -\frac{1}{h^2}$, $\quad \alpha_1 = \frac{1}{2h^2}$

$P(x) = a + bx + cx^2$

$x^2 - (x_0 + x_1)x + x_0x_1$

$= \frac{u_{-1}}{2h^2}(x-x_0)(x-x_1)$

$- \frac{u_0}{h^2}(x-x_{-1})(x-x_1)$

$+ \frac{u_1}{2h^2}(x-x_{-1})(x-x_0)$

$$p'(x) = \frac{u_{-1}}{2h^2}\Big(\underbrace{(x-x_1)}_{-h} + \cancel{(x-x_0)}\Big)$$

$$-\frac{u_0}{h^2}\Big(\underbrace{(x-x_1)}_{-h} + (x-x_{-1})\Big) \quad -h \to 0 \leftarrow h$$

If $x=x_0$

$$+\frac{u_1}{2h^2}\Big(\cancel{(x-x_0)} + \underbrace{(x-x_{-1})}_{h}\Big)$$

$$\Rightarrow \quad w_0 = p'(x_0)$$

$$= \frac{u_1}{2h^2}h + \frac{u_{-1}}{2h^2}(-h)$$

$$= \frac{u_1}{2h} - \frac{u_{-1}}{2h}$$

↳ Mean Value

① Finite differences

② Finite element

③ Spectral Method

$p(x)$ w' degree $\le N$

2nd-order : F.D. $w_j = \frac{1}{2h}(u_{j+1} - u_{j-1})$, acc, $O(h^2)$

4th- " : " . $w_j = \frac{u_{j-2} - u_{j+2}}{12h} + \frac{u_{j+1} - u_{j-1}}{3h/2}$, $O(h^4)$

$$p(x) = u_{-2}\alpha_{-2}(x-x_{-1})(x-x_0)(x-x_1)(x-x_2)$$

$$+ u_{-1}\alpha_{-1}(x-x_{-2}) \quad " \quad " \quad "$$

$$+ u_1\alpha_1 \quad " \quad (x-x_{-1})(x-x_0) \quad "$$

$$+ u_2\alpha_2 \quad " \quad " \quad (x-x_0)(x-x_1)$$

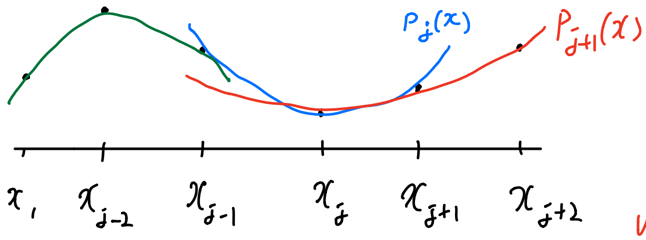
$p(x_2) = u_2 \to \alpha_2$

$p(x_1) = u_1 \to \alpha_1$

$p(x_{-1}) = u_{-1} \to \alpha_{-1}$

$p(x_{-2}) = u_{-2} \to \alpha_{-2}$

「$w_j = p'(x_j)$」



$w_j = P_j'(x_j)$, $O(h^2)$

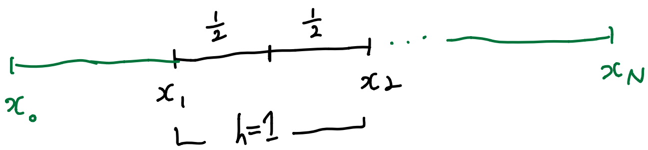
$w_{j+1} = P_{j+1}'(x_{j+1})$, $O(h^2)$

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{pmatrix} \quad \begin{pmatrix} & N \times N & \\ \frac{1}{12} & & \\ \frac{2}{3} & -\frac{1}{12} & \end{pmatrix}$$

, Eq.(1.3) $\Rightarrow$ 4th-order F.D.

「확장」

N th-order F.D, $O(h^N)$

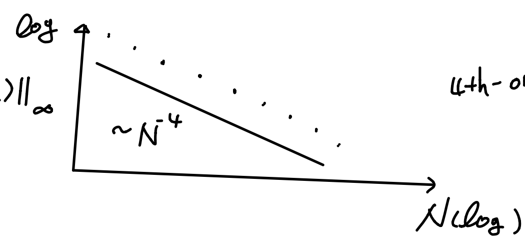


$\rightarrow \begin{cases} O(h^2) : \left(\frac{1}{2}\right)^2 \\ O(h^4) : \left(\frac{1}{2}\right)^4 \\ O(h^N) : \left(\frac{1}{2}\right)^N \end{cases}$

⌊ ... Total Length $= L$ ... ⌋

$= Nh$

$\hookrightarrow h = \frac{L}{N} \rightarrow \begin{cases} O(h^2) = O(N^{-2}) \\ O(h^4) = O(N^{-4}) \\ O(h^N) = O(N^{-N}) \end{cases}$



$P(x)$ Can be any single-valued function

-For periodic domain, $P(x)$ = Fourier Series.

$P(x) = \sum C_n e^{inx}$

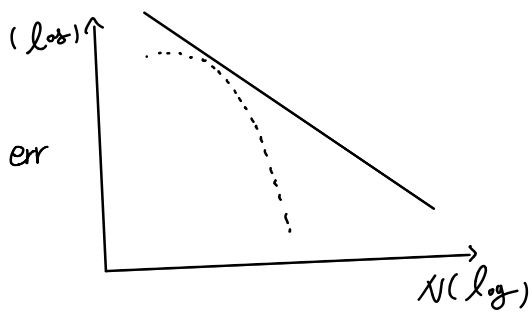
$P'(x) = \sum C_n \cdot in \cdot e^{inx}$

$= \sum (in \cdot C_n) e^{inx}$

- periodic Domain $D_N = \begin{pmatrix} N \times N \end{pmatrix}$, Eq. (1.5)

↳ Toeplitz & circulant

Prog. 2 & Ouput 2.



* Reproduce

MATLAB to PYTHON.