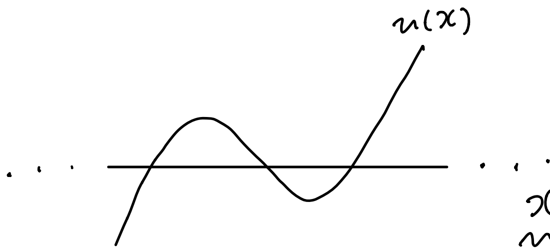
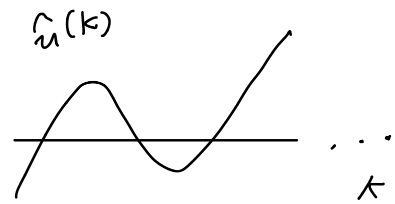
# Ch.2 {Unbounded / Descritized} Grids




$\longleftrightarrow$




$u(x)$ . continuous
unbounded
연속 , 무한한

$\underset{\sim}{x} \in \mathbb{R}$
실수
physical space

$\hat{u}(k)$ . continuous
unbounded
연속 , 무한한

Fourier space

$k \in \mathbb{R}$
파수 실수

「Forward」 → 푸리에

$$\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x)dx$$

「Inverse」 → 역 푸리에

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} \hat{u}(k)\, dk$$

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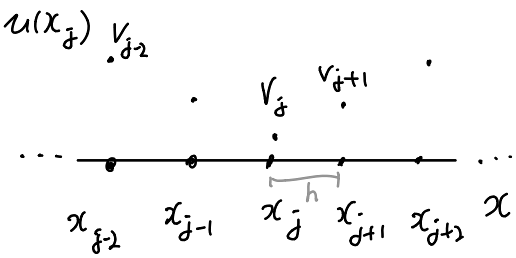
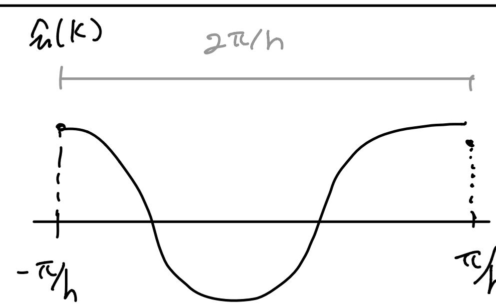



$\longleftrightarrow$




$V_j = u(x_j)$

↳ $u(x)$ . Discrete
. Unbonded
이산 , 무한한

$\begin{cases} j \in \mathbb{Z} \\ x_j \in h\mathbb{Z} \end{cases}$

정수(-1,0,1,2…)

$x_j, x_{j+1}$ : 정수형자료
h : 정수

$k_{max} = \frac{2\pi}{h}$ , Nyquist . Continuous

$k \in [-\pi/h, \pi/h] \rightarrow$ 실수 . bounded

* Nyquist 이론 : 신호의 가장 높은 주파수의 2배의 빈도로 샘플링하면 원래 신호 복원가능.

「Forward」

$$\hat{u}(k) = h\sum_{j=-\infty}^{\infty} e^{-ikx_j} \cdot v_j$$

「Inverse」

$$v_j = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j} \hat{u}(k)\, dk$$

Physical space(x) : discrete, ~~un~~bounded : $x \in h\mathbb{Z}$

Fourier space(k) : bounded, Continuous : $k \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

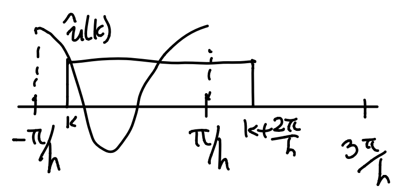
discrete

「Aliasing」 앨리어싱. → 잘못된 주파수 성분이 나온다.

$$\hat{u}(k) = h\sum_{-\infty}^{\infty} e^{-ikh\cdot j}\, v_j, \quad x_j = hj$$

$$\to \hat{u}(k+2\pi/h) = h\sum_{-\infty}^{\infty} e^{-i(k+2\pi/h)h\cdot j} \cdot v_j$$

$$= h\sum_{-\infty}^{\infty} e^{-ikhj}\, v_j = \hat{u}(k)$$

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「Spectral Differentiation」

Recall : $u(x) \simeq p(x)$ → Polynomial / cos, sin의 함수

$v_j = p(x_j)$

$u'(x) \simeq p'(x)$

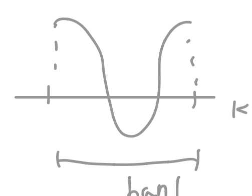
$$v_j = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j}\,\hat{u}(k)\,dk$$

↳ each value

$$p(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\,\hat{u}(k)\,dK$$

($x$: any x value.)

↓ function




$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\hat{p}(k)\,dk$$

$$\left(\text{where } \hat{p}(k) = \begin{cases} \hat{u}(k), & (-\pi/h, \pi/h) \\ 0, & \text{otherwise.} \end{cases}\right.$$

band-limited interpolant.

→ 포함하지 않음.

$$\boxed{\hat{u}(k) = \hat{v}(k)}$$

↳ 혼용 중

$u'(x) \simeq p'(x)$

$$p'(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx} \underbrace{(ik\cdot\hat{u}(k))}_{\hat{w}(k)}\, dk \qquad \Rightarrow w_j = p'(x_j)$$

$$w_j = p'(x_j) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j}\, \hat{w}(k)\, dk$$

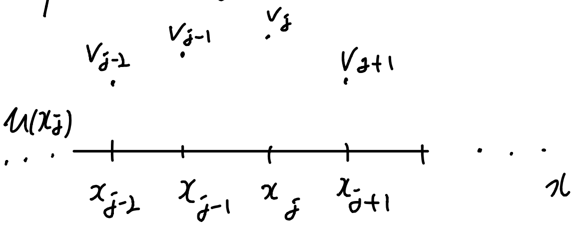
「Method. 1.」

- Given $V_j$, determine $P(x)$ : band-limited interpolunt.
- set $w_j = p'(x_j)$

「Method. 2」

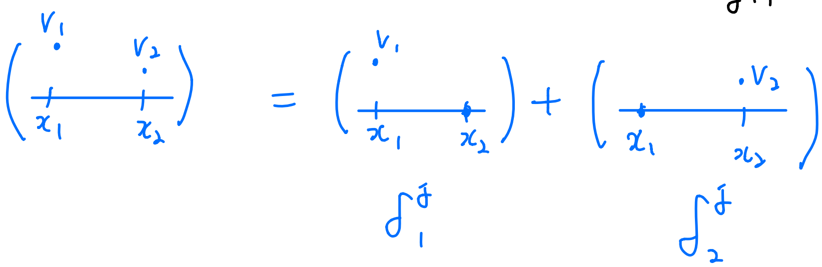
- Given $V_j \rightarrow \hat{V}(k)$
- $\hat{w}(k) = ik\,\hat{v}(k)$
- Inverse of $\hat{w}(k) \rightarrow w_j$

「Spectral Differentiation Matrix」




, Define $\delta_j^m = \begin{cases} 1, & m=j \\ 0, & \text{otherwise} \end{cases}$

$$u(x_j) = \cdots + V_{j-1}\delta_{j-1}^{j} + V_j \delta_j^j + V_{j+1}\delta_{j+1}^{j} + \cdots = \sum_{m=-\infty}^{\infty} V_m \delta_m^j$$




$$\hat{u}(k) = \cdots + V_j \hat{\delta}_j(k) + \cdots = \sum_{m=-\infty}^{\infty} V_m \hat{\delta}_m(k)$$

$$\hat{\delta}_m(k) \to h\sum_{j=-\infty}^{\infty} e^{-ikx_j}\delta_m^j$$

Inverse

$$\delta_m^j = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j}\hat{\delta}_m(k)\,dk$$

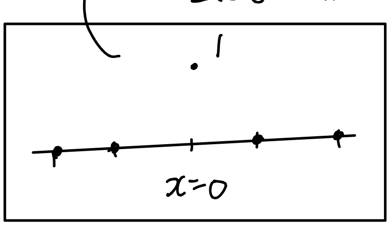
let $x_j = j\cdot h \to x_0 = 0$

$$\hat{\delta}_0(k) = h\sum_{j=-\infty}^{\infty} e^{-ikx_j}\cdot\delta_0^j = h$$

$$P_{\delta_0}(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\hat{\delta}_0(k)\,dk$$

$$= \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\cancel{\hat{\delta}_0}(k)\,dk = \frac{\sin(\pi x/h)}{\pi x/h} = S_h(x)$$

Sinc function




로피탈정리

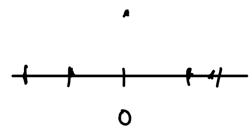
$$V_j = p(x_j) \to \delta_0^j = P_\delta(x_j) = \begin{cases} 1, & j=0 \\ 0, & \text{otherwise} \end{cases}$$

$\hookrightarrow \; S_h(x_j) = \begin{cases} 1, & j=0 \\ 0, & \text{otherwise} \end{cases}$

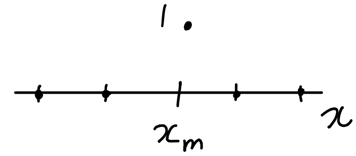
「For $j \neq 0$」

$\delta_m^j = S_h(x_j - x_m)$

↳ shift라고 생각



shift $\Rightarrow$



$$V_j = \sum_{m=-\infty}^{\infty} V_m \delta_m^j \longleftrightarrow p(x) = \sum_{m=-\infty}^{\infty} V_m S_h(x - x_m)$$

$$\hookrightarrow = \sum_{m=-\infty}^{\infty} V_m S_h(x_j - x_m)$$

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$V_j = p(x_j) \rightarrow w_j = p'(x_j)$ matrix

$$= \sum_{m=-\infty}^{\infty} V_m S_h'(x - x_m)\Big|_{x=x_j}$$

Given

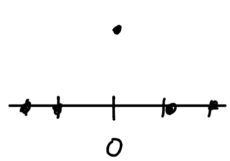
$$S_h'(x_j) = \begin{cases} 0, & j=0 \\ \dfrac{(-1)^j}{jh}, & j \neq 0 \end{cases} \quad (2.12)$$

(1.4)

$u''(x_j) \simeq p''(x_j) = \sum V_m S_h''(x_j - x_m)$ $\quad (2.14) \Rightarrow S_h''(x_j)$

$\boxed{\text{Exercise } 2.1, 2.2}$

「Example」

output 3. ①  ②  ③