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NUMERICAL METHODS

Midterm

Instruction:

- **Read the questions!!!!**
- Midterm consists of two parts.
  - The theory part for 600 point
  - Practical part for 400+200 point bonus
- You only need to get 1000 points.
- You must finish the theory part(Problem 1-4) in class (4hr is given).
- You can finish the practical part as a take home due on the Sunday Midnight(Sunday evening) if you don't finish it in the given 4 hours.
- The exam is open books, open notes, open **your own** exercise/homework, open iPython, and open internet. Using other people's exercises/homework is not allowed. Other people's notes is ok though.
- Don't just look for the solution though.
- Anything sensible is allowed. We are grown ups here. Here is an example of the things that are **not** sensible.
  - Get help from others. Posting the question in a forum.
  - Look online specifically for the solution. Looking for the concept is OK.
  - If you feel that you are approaching the bound, please ask.
- For practical part, if you know what to do but you don't know how to put that into the program write it down what you plan to do. Partial credit will be given to the right idea.
- Read every single question before you go home. If you don't understand the question, ask me.
- You will find the exam very enjoyable.
- If you hand in iPython file put a comment clearly that this chunk of code belong to which question.

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1)(150 points) You should not lose any point on these

A))(50 points)Plot $y = x^2$ and $y = \exp x$ for x from -3 to 3 on the same figure using matplotlib make one line red and the other blue. Yeah free score.

B) (50 points) If we use Newton's method to solve this, what would be the next **four** guesses given that the initial guess is 1. Show your program and/or how you reach your answer!. You may use iPython as a calculator to calculate each step or as a programming platform and write a program to do this. (You may use your old code as starting point.)

$$e^{-x} = x - \sin(x)$$

C) (50 points) Suppose that we use bisection method to solve the following equation.

$$e^{-x} = x - \sin(x)$$

I) Is initial range of $[0, 1]$ a good initial range? Why? Find me a new one if it is not.

II) How many steps do we need to ensure that our guess is less than 10^{-8} away from the answer? Show your work.

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2) Newton's Method (150 points 50 each).

A) Pictorially explain Newton's method. Include How the next guess is calculated and show an example when Newton's Method does not converge. Just draw stuff and tell me what happen.

B) Give pros and cons of Newton's Method and Bisection Method. Which one converge faster(if it does) and which one is guaranteed to converge?

C) Plot the function. Then **numerically** calculate the first, and the second derivative and plot them for the following function:

$$f(x) = x\exp(x)$$

for x from 0 to 2 using matplotlib.

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3) Taylor's Theorem (150 points). (50 each)

A) Write down Taylor's theorem. (First three terms and the error term would suffice.) Yeah free points.

B) Computer the first 3 **non zero** terms of Taylor series for $f(x) = e^{-x^2}$ around $x = 0$. If we stop the taylor series at this first three term what would be the bound on the error term for estimating $e^{-0.1^2}$? Using Wolframalpha is OK.

C) Numerically compute the integral of this function using trapezoid rule with only 4 pieces. **Give me the area of each piece and also the sum**. You can do this by hand if needed.

$$\int_{x=1}^{x=3} \left[(x+1)^2 + 2\right] \ dx$$

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4) Gaussian Elimination(150 points) (75 each)

A) Perform **Gaussian Elimination** by hand to solve the following system of equations. Other methods will not be accepted. Use the back side of the paper if needed.

$$\begin{pmatrix} 1 & 0 & -1 \\ -3 & 1 & 1 \\ -2 & -2 & 3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

B) Do this either by hand or with a program. Suppose I went out and measure the weight and height of 10 students in my class and I found the following

Weight(kg)	45	47	50	80	70	55	60	62	62	75
Height(cm)	155.9	155.3	162.3	196.3	182.4	166.2	173.7	171.6	172.5	191

If I model the relation between the weight and the height with a straight line, what would be the prediction for the height for a student that weighs 65 kg. Show your work.

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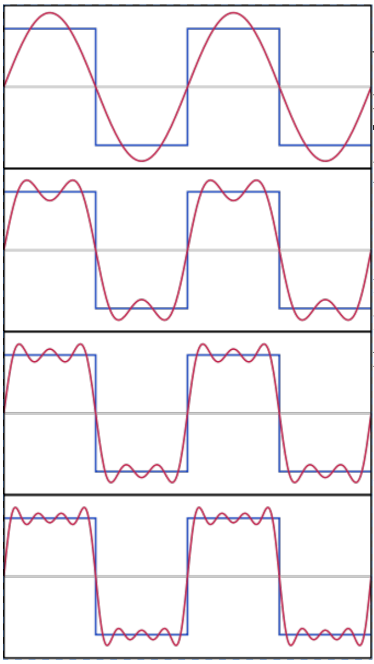
Application part:

5) Fourier Series(200 points) : We can approximate any well-behaved periodic function(function that repeats itself after certain period) by the sum of sine and cosine of varying period.

$$f(x) = \frac{a_0}{2} + a_1\cos(\pi x) + a_2\cos(2\pi x) + a_3\cos(3\pi x)\ldots + a_i\cos(i\pi x) + \ldots$$
$$+ b_1\sin(\pi x) + b_2\sin(2\pi x) + b_3\sin(3\pi x) + \ldots + b_i\sin(i\pi x) + \ldots$$

Like Taylor series, we may decide to terminate the series at any point. But the more terms we have the better approximation we have. For example, the figure below shows a Square wave function(blue) being approximated by sum of sine and cosine(red). As we add more terms the red line approximate the blue line better and better.



If you think about what this series does, it is quite amazing. We just take a periodic function and compress it to just a few numbers and the more numbers we have the more accurate we are. This is the basic idea of a lot of lossy compressions such as JPEG and MP3.

So, all we need to do to approximate a periodic function is to find all the a and b. They can be found by using the following formula. (The derivation is quite easy but don't worry about it for now). Here is how:

If $f(x)$ is a periodic function of period 2, then a_i and b_i is given by

$$a_i = \int_{-1}^{1} f(x)\cos(i\pi x)\,dx,$$

$$b_i = \int_{-1}^{1} f(x)\sin(i\pi x)\,dx$$

For example,

$a_0 = \int_{-1}^{1} f(x)\cos(0)\,dx$, $a_1 = \int_{-1}^{1} f(x)\cos(\pi x)\,dx$ and $a_2 = \int_{-1}^{1} f(x)\cos(2\pi x)\,dx$
$b_1 = \int_{-1}^{1} f(x)\sin(\pi x)\,dx$, $b_2 = \int_{-1}^{1} f(x)\sin(2\pi x)\,dx$, etc.

A) Consider a function given in the template file and find $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ (numerically) that approximate the function given.

B) Then make a plot of the function and the Fourier series approximation. (See the top equation). You should have something along the line of the plot above where you compare the fourier series approximation with the original function.

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6)(200 points) Recall the FWHM in the homework. In this problem we want to establish a famous relation that FWHM is in a linear relation with the width of gaussian distribution(σ).

We will do this by finding FWHM of gaussian distribution of varying width(σ). This will give us a bunch of data points for $(\sigma, FWHM)$. Then your job is to fit these data points and find the relation between FWHM and σ.

Specifically here is what I want you to do

A) Numerically find FWHM for these gaussian distribution.

$$f(x;\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

for $\sigma = 1, 2, 3, 4, 5, 6, 7, 8$

B) With data points of $(\sigma, FWHM)$ you got in part A. Find the linear relation between σ and FWHM.

$$\sigma = FWHM \times m + c$$

Find the value of m and c. (Be careful about how m and c is defined above)

You may use the code you use in the homework as starting point.

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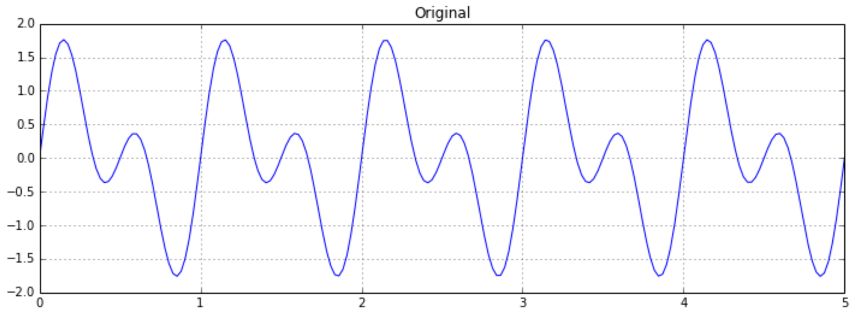
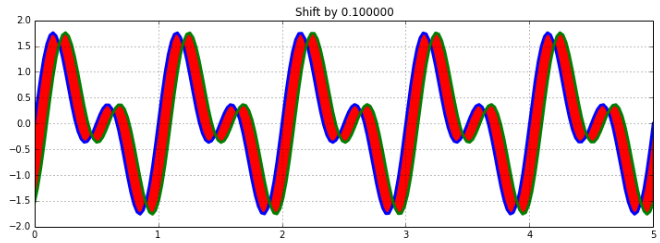
7) (200 points) **Bonus: Guitar Tuner.** Guitar tuner program is basically a program that takes a waveform and find out the main component of the frequency/period.

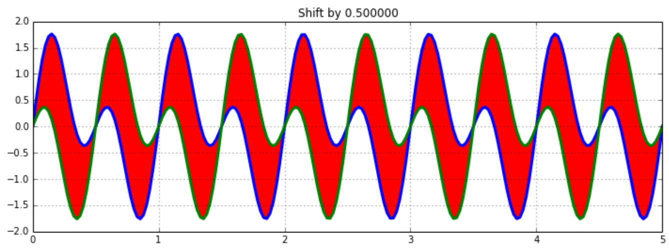
So here is how to do it.

Here is the waveform:



If we shift the graph by some random amount to the right and compare it with the original we will will have a bunch of area between the two graphs.





But, if we shift it by the “right” amount(aka the period), then the area is minimized.

So, to find the main period of the waveform, all you need to do is to find the amount of shift that minimize the area between the shifted curve and the original curve. This algorithm can be found on all guitar tuner program.

In the Midterm Template file, I give you a curve. Your job is to implement this algorithm and find the period of that curve.

You may find these useful
1) scipy.minimize may be useful (See example of how to use it in template file.)
2) If you don't know how to shift the graph. Try plot $g(x)$ and $h(x) = g(x-4)$ on the same figure.