1 Normal Random Variables

1.1 Problem 1

Let X and Y be two normal random variables, with means 0 and 3 , respectively, and variances 1 and 16 , respectively.
We have $X \sim \mathcal{N}(0,\, 1^2)$ and $Y \sim \mathcal{N}(3,\, 4^2)$

- $P(X > -1) = P(X < 1) = 0.8413$
- $P(X \leq -2) = 1 - P(X < 2) = 0.023$
- Let $V = \frac{4-Y}{3}$. Find the mean and the variance of V
  $E[V] = E\left[\frac{4-Y}{3}\right] = \frac{4}{3} - E\left[\frac{Y}{3}\right] = \frac{1}{3}$
  $Var(V) = Var(\frac{4-Y}{3}) = Var\left(\frac{Y}{3}\right) = \frac{Var(Y)}{9} = \frac{16}{9}$
- $P(-2 < Y \leq 2) = P\left(\frac{-2-3}{\sqrt{16}} < \frac{Y-3}{\sqrt{16}}\right) \leq \frac{2-3}{\sqrt{16}} = P(-5/4 < Z \leq -1/4) = 0.2957$

2 A Joint PDF Given By a Simple Formula

2.1 Problem 2

The random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,\, y) \begin{cases} ax^2, & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le x, \\ 0, & \text{otherwise} \end{cases}$$

1. Find the constant a.

$$\int_1^2 \int_0^x ax^2 \, dydx = \frac{15}{4}$$

   We have $a = 4/15$

2. Determine the marginal PDF $f_Y(y)$

   (a) if $0 \le y \le 1$

$$\begin{aligned} f_Y(y) &= \int_0^1 ax^2 \, dx \\ &= a\frac{x^3}{3}\bigg|_0^1 \\ &= \frac{28}{45} \end{aligned}$$

   (b) if $1 < y \le 2$

$$\begin{aligned} f_Y(y) &= \int_y^2 ax^3 \, dx \\ &= \frac{32 - 4y^3}{45} \end{aligned}$$

3. Determine the conditional expectation of $1/(X^2Y)$, given that $Y = 5/4$

$$E\left[\frac{1}{X^2Y} \,\Big|\, Y = \frac{5}{4}\right] = \int_{-\infty}^{\infty} \frac{1}{x^2 y} f_{X\,|\,Y}\left(x \,\Big|\, \frac{5}{4}\right) \, dx$$

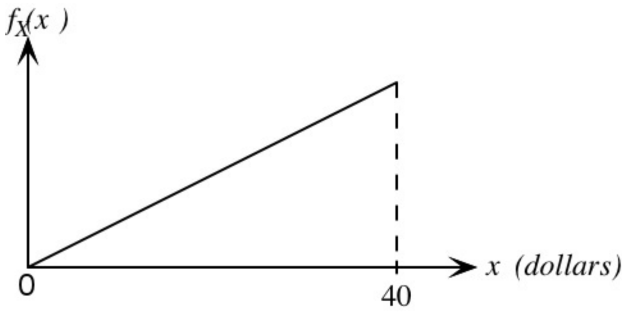
$$\begin{aligned} f_{X\,|\,Y}\left(x \,\Big|\, \frac{5}{4}\right) &= \frac{f_{X,Y}(x,\, 5/4)}{f_Y\left(\frac{5}{4}\right)} \\ &= \frac{64x^2}{129}, \quad \text{for } 5/4 \le x \le 2 \end{aligned}$$

$$\begin{aligned} E\left[\frac{1}{X^2Y} \,\Big|\, Y = \frac{5}{4}\right] &= \int_{-\infty}^{\infty} \frac{4}{5x^2} \cdot \frac{f_{X,Y}(x,\, 5/4)}{f_Y\left(\frac{5}{4}\right)} \, dx \\ &= \int_{5/4}^{2} \frac{256}{645} \, dx \\ &= \frac{64}{215} \end{aligned}$$

3 Sophia's Vocation

3.1 Problem 3

Sophia is vacationing in Monte Carlo. On any given night, she takes X dollars to the casino and returns with Y dollars. The random variable X has the PDF shown in the figure. Conditional on $X = x$, the continuous random variable Y is uniformly distributed between zero and $3x$.



1. Determine the joint PDF $f_{X,Y}(x,\, y)$

   if $0 < x < 40$ and $0 < y < 3x$ Based on the above image, $f_X(x) = ax$ where a is the weight constant.

$$1 = \int_0^{40} ax\, dx = 800a \longrightarrow a = \frac{1}{800}$$

$$\begin{aligned} f_{X,Y}(x,\, y) &= f_X(x) f_{Y\,|\,X}(y\,|\,x) \\ &= \frac{x}{800} \cdot \frac{1}{3x} \\ &= \frac{1}{2400} \end{aligned}$$

   if $y < 0$ or $y > 3x$

$$f_{X,Y}(x, y) = 0 \text{ as } f_{Y\,|\,X}(y\,|\,x) = 0$$

2. On any particular night, Sophia makes a profit $Z = YX$ dollars. Find the probability that Sophia makes a positive profit, that is, find $P(Z > 0)$.

   In order to have positive profit, we need $Y > X$. For any $x \in (0, 40)$, we need $y \in (x, 3x)$. Thus,

$$P(Z > 0) = P(Y > X) = \int_0^{40} \int_x^{3x} f_{X,Y}(x,\, y)\, dydx = \frac{2}{3}$$

3. Find the PDF of Z As $Z = Y - X$ where Y is uniformly distributed on $[0, 3x]$ given $X = x$, so $Z = Y - x$ is uniformly distributed on $[-x, 2x]$.

$$f_{X,Z}(x,\, z) = f_X(x) f_{Z\,|\,X}(z\,|\,x) = \frac{x}{800} \cdot \frac{1}{3x} = \frac{1}{2400}, \text{ for } 0 < x < 40 \text{ and } -x \leq z \leq 2x$$

if $-40 < z < 0$

$$f_Z(x) = \int_{-z}^{40} f_{X,\,Z}(x,\,z)\,dx = \frac{40+z}{2400}$$

if $0 < z < 80$

$$f_Z(x) = \int_{z/2}^{40} f_{X,\,Z}(x,\,z)\,dx = \frac{80-z}{4800}$$

if $z < -40$ or $z > 80$

$$f_Z(z) = 0 \text{ as } f_{Z\,|\,X}(z\,|\,x) = 0$$

4. What is $E[Z]$ Since $Z = Y - X$, by linearity of expectation, $E[Z] = E[Y] - E[X]$.

   Note that $E[Y\,|\,X = x] = 3x/2$ for any $x \in (0, 40)$. Using the total expectation theorem,

$$\begin{aligned} E[Y] &= \int_0^{40} E[Y\,|\,X = x] f_X(x)\,dx \\ &= \frac{3}{2}\int_0^{40} x f_X(x)\,dx \\ &= \frac{3}{2}E[X] \end{aligned}$$

   So $E[Z] = 1/2E[X]$.

$$\begin{aligned} E[Z] = \frac{1}{2}\cdot E[X] &= \frac{1}{2}\int_0^{40} x f_X(x)\,dx \\ &= \frac{1}{2}\int_0^{40} \frac{x^2}{800}\,dx \\ &= \frac{80}{3} \end{aligned}$$

4 True or False

4.1 Problem 4

Determine whether each of the following statement is true (i.e., always true) or false (i.e., not always true).

1. Let X be a random variable that takes values between 0 and c only, for some $c \geq 0$, so that $P(0 \leq X \leq c) = 1$. Then $Var(X) \leq c^2/4$.

$$\begin{aligned}
Var(X) &= E(X^2) - (E(X))^2 \\
&= E(X \cdot X) - (E(X))^2 \\
&\leq E(c \cdot X) - (E(X))^2 \\
&= c \cdot E(X) - (E(X))^2 \\
&= c^2 \cdot \left(\frac{E(X)}{c}\right) - c^2\left(\frac{E(X)}{c}\right)^2 \\
&= c^2\alpha(1-\alpha), \text{ where } \alpha = E(X)/c \\
&\leq \frac{c^2}{4}
\end{aligned}$$

2. Let X and Y be continuous random variables. If $X \sim \mathcal{N}(\mu, \sigma^2)$, $Y = aX + b$, and $a > 0$, then $Y \sim \mathcal{N}(a\mu + b, a\sigma^2)$.

   This is a **False** statement.

$$Var(Y) = Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

3. The expected value of a non-negative continuous random variable X, which is defined by

$$E[X] = \int_0^\infty x f_X(x)\, dx, \text{ also satisfies } E[X] = \int_0^\infty P(X > t)\, dt$$

$$\begin{aligned}
E[X] &= \int_0^\infty P(X > t)\, dt \\
&= \int_0^\infty \int_t^\infty f_X(x)\, dx dt \\
&= \int_0^\infty \int_0^x f_X(x)\, dt dx \\
&= \int_0^\infty x f_X(x)\, dx
\end{aligned}$$

5 Bayes' Rule

5.1 Problem 5

Let K be a discrete random variable with PMF

$$p_K(k) = \begin{cases} 1/4, & \text{if } k = 1, \\ 1/2, & \text{if } k = 2, \\ 1/4, & \text{if } k = 3, \\ 0, & \text{otherwise} \end{cases}$$

Conditional on $K = 1, 2$, or 3, random variable Y is exponentially distributed with parameter $1, 1/2$, or $1/3$, respectively.
Using Bayes' rule, find the conditional PMF $p_{K\,|\,Y}(2\,|\,y)$ when $y \geq 0$.
Applying Bayes's rule, we have

$$p_{K\,|\,Y}(2\,|\,y) = \frac{p_K(2) f_{Y\,|\,K}(y\,|\,2)}{f_Y(y)}$$

By the total probability theorem,

$$f_Y(y) = \sum_{k=1}^{3} p_K(k) f_{Y\,|\,K}(y\,|\,k) = \frac{1}{4}e^{-y} + \frac{1}{2} \cdot \frac{1}{2}e^{-y/2} + \frac{1}{4} \cdot \frac{1}{3}e^{-y/3}$$

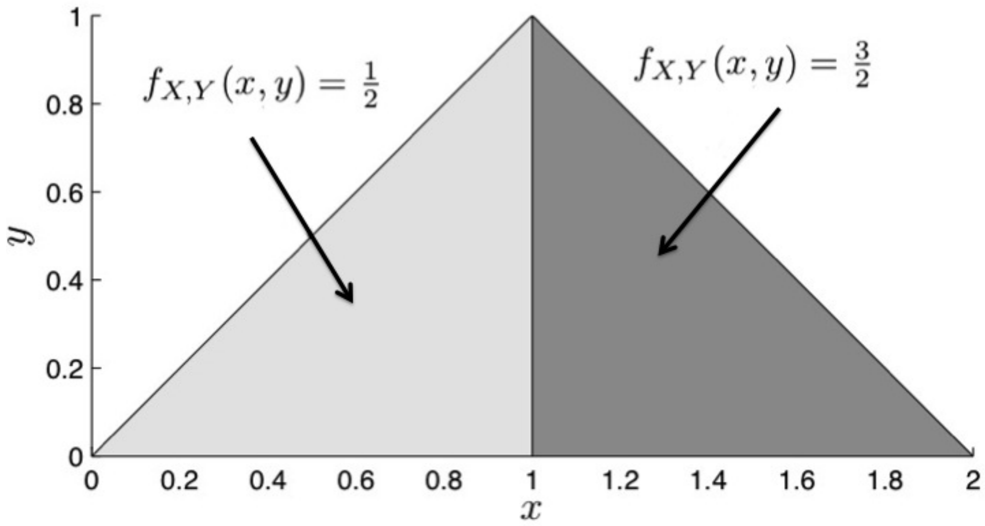
Thus,

$$p_{K\,|\,Y}(2\,|\,y) = \frac{e^{-y/2}}{e^{-y} + e^{-y/2} + 1/3e^{-y/3}}$$

6 A Joint PDF on a Triangular Region

6.1 Problem 6

This figure below describes the joint PDF of the random variables X and Y. These random variables take values in $[0, 2$ and $[0, 1]$, respectively. At $x = 1$, the value of the joint PDF is 1/2.



1. Are X and Y independent?

   X and Y are not independent. For example, if $X < 0.5$, we can infer that $Y < 0.5$ based on the above figure.

2. Find $f_X(x)$.

$$f_X(x) = \begin{cases} \int_0^x \frac{1}{2}\, dy & \text{if } 0 \leq x \leq 1 \\ \int_0^{2-x} \frac{3}{2}\, dy & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq 1 \\ \frac{6-3x}{2} & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

3. Find $f_{Y\,|\,X}(y\,|\,0.5)$.

   Given that $X = 0.5$, Y is uniformly distributed between 0 and 1/2. Thus,

$$f_{Y\,|\,X}(y\,|\,0.5) = \begin{cases} 2, & \text{if } 0 \leq y \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

4. Find $f_{X \mid Y}(x \mid 0.5)$.

   Given that $Y = 0.5$, the conditional distribution of X is a piece-wise constant,

$$f_{X \mid Y}(x \mid 0.5) = \begin{cases} 1/2, & \text{if } 1/2 \le x \le 1 \\ 3/2, & \text{if } 1 < x \le 1.5 \\ 0, & \text{otherwise} \end{cases}$$

5. Let $R = XY$ and A be the event that $\{X < 0.5\}$. Find $E[R \mid A]$.

   Under event A, the pair (X, Y) takes values in a triangle region with sides of length $1/2$ and area $1/8$. The conditional PDF is uniform and this tells us $f_{X,Y \mid A}(x, y) = 8$. Thus,

$$\begin{aligned} E[R \mid A] &= E[XY \mid A] \\ &= \int_0^{1/2} \int_y^{0.5} xy f_{X,Y \mid A}(x, y)\, dxdy \\ &= \int_0^{1/2} \int_y^{0.5} 8xy\, dxdy \\ &= \frac{1}{16} \end{aligned}$$