Gamma Distribution

The probability density function of Gamma distribution is:

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\beta \cdot x}$$

Where, $a > 0$ is the shape parameter and $\beta > 0$ is the rate parameter. The likelihood function will be:

$$L(a, \beta | x) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot {x_i}^{\alpha-1} \cdot e^{-\beta \cdot x_i}$$

And therefore, the log-likelihood function, is:

$$l(a, b | x) = ln[L(a, \beta | x)] = \sum_{i=1}^{n} ln\left(\frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot {x_i}^{\alpha-1} \cdot e^{-\beta \cdot x_i}\right) =$$

$$= \sum_{i=1}^{n} ln\left(\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right) + \sum_{i=1}^{n} ln({x_i}^{\alpha-1}) + \sum_{i=1}^{n} ln\left(e^{-\beta \cdot x_i}\right) =$$

$$= \sum_{i=1}^{n} a \cdot l\,n(\beta) - \sum_{i=1}^{n} ln\big(\Gamma(\alpha)\big) + \sum_{i=1}^{n} (\alpha - 1) \cdot \ln\,(x_i) + \sum_{i=1}^{n} (-\beta \cdot x_i) \cdot l\,n(e) =$$

$$= n \cdot a \cdot l\,n(\beta) - n \cdot ln\big(\Gamma(\alpha)\big) + (\alpha - 1) \cdot \sum_{i=1}^{n} \ln(x_i) - \beta \cdot \sum_{i=1}^{n} x_i$$

Now this is if we want to maximize our function, if instead we want to minimize it (like for example, if we use the $optim()$ function in R), then we will use the following log-likelihood function:

$$-l(a, b | x) = -ln[L(a, \beta | x)] =$$

$$= -\left[n \cdot a \cdot l\,n(\beta) - n \cdot ln\big(\Gamma(\alpha)\big) + (\alpha - 1) \cdot \sum_{i=1}^{n} \ln(x_i) - \beta \cdot \sum_{i=1}^{n} x_i\right] =$$

$$= -n \cdot a \cdot l\,n(\beta) + n \cdot ln\big(\Gamma(\alpha)\big) - (\alpha - 1) \cdot \sum_{i=1}^{n} \ln(x_i) + \beta \cdot \sum_{i=1}^{n} x_i$$