

Annette J. Dobson

An Introduction to Generalized Linear Models  
Solutions

by Konstantinos Kirillov

# CHAPTER 1

## Exercise 1.1:

Let  $Y_1$  and  $Y_2$  be independent random variables with  $Y_1 \sim N(1, 3)$  and  $Y_2 \sim N(2, 5)$ . If  $W_1 = Y_1 + 2Y_2$  and  $W_2 = 4Y_1 - Y_2$  what is the joint distribution of  $W_1$  and  $W_2$ ?

## SOLUTION:

A reminder from the book:

### 1.4.1 Normal distributions:

1. If the random variable  $Y$  has the Normal distribution with mean  $\mu$  and variance  $\sigma^2$ , its probability density function is:

$$f(y; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{1}{2} \left( \frac{y - \mu}{\sigma} \right)^2 \right]$$

We denote this by  $Y \sim N(\mu, \sigma^2)$ .

2. The Normal distribution with  $\mu = 0$  and  $\sigma^2 = 1$ ,  $Y \sim N(0, 1)$ , is called the **standard Normal distribution**.
3. Let  $Y_1, \dots, Y_n$  denote Normally distributed random variables with  $Y_i \sim N(\mu_i, \sigma_i^2)$  for  $i = 1, \dots, n$  and let the covariance of  $Y_i$  and  $Y_j$  be denoted by:

$$\text{cov}(Y_i, Y_j) = \rho_{ij} \sigma_i \sigma_j,$$

where  $\rho_{ij}$  is the correlation coefficient for  $Y_i$  and  $Y_j$ . Then the joint distribution of the  $Y_i$ 's is the **multivariate Normal distribution** with mean vector  $\mu = [\mu_1, \dots, \mu_n]^T$  and variance-covariance matrix  $V$  with diagonal elements  $\sigma_i^2$  and non-diagonal elements  $\rho_{ij} \sigma_i \sigma_j$  for  $i \neq j$ . We write this as:

$$\mathbf{y} \sim N(\mu, V), \text{ where } \mathbf{y} = [Y_1, \dots, Y_n]^T$$

4. Suppose the random variables  $Y_1, \dots, Y_n$  are independent and normally distributed with the distributions  $Y_i \sim N(\mu_i, \sigma_i^2)$  for  $i = 1, \dots, n$ . If

$$W = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n,$$

where the  $a_i$ 's are constants. Then  $W$  is also Normally distributed, so that:

$$W = \sum_{i=1}^n a_i Y_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$$

It seems that the joint distribution of two normally distributed variables is yet another normal distribution. In this exercise, in order to find the joint distribution of  $W_1$  and  $W_2$ , we first need to determine the mean, the variance and the covariance of  $W_1$  and  $W_2$  and then use those to derive the joint distribution.

Given that:

$$Y_1 \sim N(1, 3)$$

$$Y_2 \sim N(2, 5)$$

First, let us find the means of  $W_1$  and  $W_2$ :

$$E(W_1) = E(Y_1 + 2Y_2) = E(Y_1) + 2E(Y_2) = 1 + 2 \cdot 2 = 5 \Rightarrow \mathbf{E(W_1) = 5}$$

$$E(W_2) = E(4Y_1 - Y_2) = 4E(Y_1) - E(Y_2) = 4 \cdot 1 - 2 = 2 \Rightarrow \mathbf{E(W_2) = 2}$$

Next, let us calculate the variances of  $W_1$  and  $W_2$ :

$$Var(W_1) = Var(Y_1 + 2Y_2) = Var(Y_1) + 2^2 \cdot Var(Y_2) = 3 + 4 \cdot 5 = 23 \Rightarrow \mathbf{Var(W_1) = 23}$$

$$Var(W_2) = Var(4Y_1 - Y_2) = 4^2 \cdot Var(Y_1) + Var(Y_2) = 16 \cdot 3 + 5 = 53 \Rightarrow \mathbf{Var(W_2) = 53}$$

And finally, let us also compute the covariance between  $W_1$  and  $W_2$ :

$$\begin{aligned} Cov(W_1, W_2) &= Cov(Y_1 + 2Y_2, 4Y_1 - Y_2) = \\ &= Cov(Y_1, 4Y_1) + Cov(Y_1, -Y_2) + Cov(2Y_2, 4Y_1) + Cov(2Y_2, -Y_2) = \\ &= 4Var(Y_1) - Cov(Y_1, Y_2) + 8Cov(Y_2, Y_1) - 2Var(Y_2) = \\ &= 4 \cdot 3 - 0 + 8 \cdot 0 - 2 \cdot 5 = 2 \Rightarrow \\ &\Rightarrow \mathbf{Cov(W_1, W_2) = 2} \end{aligned}$$

Therefore, the joint distribution will be:

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \sim N \left[ \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 23 & 2 \\ 2 & 53 \end{pmatrix} \right]$$

The correlation coefficient between  $W_1$  and  $W_2$  in this case shall be:

$$\rho = \frac{Cov(W_1, W_2)}{\sigma_{W_1} \cdot \sigma_{W_2}} = \frac{Cov(W_1, W_2)}{\sqrt{Var(W_1)} \cdot \sqrt{Var(W_2)}} = \frac{2}{\sqrt{23} \cdot \sqrt{53}} \approx \frac{2}{4.8 \cdot 7.3} \approx 0.057 \Rightarrow \rho = \mathbf{0.057}$$

Therefore, another way to express the joint distribution, would be:

$$\begin{aligned} f(W_1, W_2) &= \frac{1}{2 \cdot \pi \cdot \sigma_{W_1} \cdot \sigma_{W_2} \cdot \sqrt{1 - \rho^2}} \exp \left[ -\frac{Z_{W_1}^2 - 2 \cdot Z_{W_1} \cdot Z_{W_2} + Z_{W_2}^2}{2 \cdot \sqrt{1 - \rho^2}} \right] \Rightarrow \\ \Rightarrow f(W_1, W_2) &= \frac{1}{2 \cdot \pi \cdot 4.8 \cdot 7.3 \cdot \sqrt{1 - 0.057^2}} \exp \left[ -\frac{Z_{W_1}^2 - 2 \cdot Z_{W_1} \cdot Z_{W_2} + Z_{W_2}^2}{2 \cdot \sqrt{1 - 0.057^2}} \right] \end{aligned}$$

Where:

$$Z_{W_1} = \frac{W_1 - \mu_{W_1}}{\sigma_{W_1}}$$

$$Z_{W_2} = \frac{W_2 - \mu_{W_2}}{\sigma_{W_2}}$$

**Exercise 1.2:**

Let  $Y_1$  and  $Y_2$  be independent random variables with  $Y_1 \sim N(0, 1)$  and  $Y_2 \sim N(3, 4)$ .

a. What is the distribution of  $Y_1^2$ ?

b. If  $y = \begin{bmatrix} Y_1 \\ (Y_2 - 3)/2 \end{bmatrix}$ , obtain an expression for  $y^T y$ . What is its distribution?

c. If  $y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$  and its distribution is  $y \sim N(\mu, V)$ , obtain an expression for  $y^T V^{-1} y$ . What is its distribution?

**SOLUTION:**

A reminder from the book:

1.4.2 Chi-squared distribution:

1. The **central chi-squared distribution** with  $n$  degrees of freedom is defined as the sum of squares of  $n$  independent random variables  $Z_1, \dots, Z_n$  each with the standard Normal distribution. It is denoted by:

$$X^2 = \sum_{i=1}^n Z_i^2 \sim \chi_n^2$$

In matrix notation, if  $\mathbf{z} = [Z_1, \dots, Z_n]^T$  then  $\mathbf{z}^T \mathbf{z} = \sum_{i=1}^n Z_i^2$  so that  $X^2 = \mathbf{z}^T \mathbf{z} \sim \chi_n^2$ .

2. If  $X^2$  has the distribution  $\chi_n^2$ , then its expected value is  $E(X^2) = n$  and its variance is  $\text{Var}(X^2) = 2n$ .
3. If  $Y_1, \dots, Y_n$  are independent Normally distributed random variables each with the distribution  $Y_i \sim N(\mu_i, \sigma_i^2)$  then:

$$X^2 = \sum_{i=1}^n \left( \frac{Y_i - \mu_i}{\sigma_i} \right)^2 \sim \chi_n^2$$

because each of the variables  $Z_i = (Y_i - \mu_i)/\sigma_i$  has the standard Normal distribution  $N(0, 1)$ .

4. Let  $Z_1, \dots, Z_n$  be independent random variables each with the distribution  $N(0, 1)$  and let  $Y_i = Z_i + \mu_i$ , where at least one of the  $\mu_i$ 's is non-zero. Then the distribution of:

$$\sum Y_i^2 = \sum (Z_i + \mu_i)^2 = \sum Z_i^2 + 2 \sum Z_i \mu_i + \sum \mu_i^2$$

has larger mean  $n + \lambda$  and larger variance  $2n + 4\lambda$  than  $\chi_n^2$  where  $\lambda = \sum \mu_i^2$ . This is called the **non-central chi-squared distribution** with  $n$  degrees of freedom and **non-centrality parameter**  $\lambda$ . It is denoted by  $\chi_n^2(\lambda)$ .

5. Suppose that the  $Y_i$ 's are not necessarily independent and the vector  $\mathbf{y} = [Y_1, \dots, Y_n]^T$  has the multivariate normal distribution  $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$  where the variance-covariance matrix  $\mathbf{V}$  is non-singular and its inverse is  $\mathbf{V}^{-1}$ . Then:

$$X^2 = (\mathbf{y} - \boldsymbol{\mu})^T \mathbf{V}^{-1} (\mathbf{y} - \boldsymbol{\mu}) \sim \chi_n^2$$

6. More generally if  $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$  then the random variable  $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$  has the non-central chi-squared distribution  $\chi_n^2(\lambda)$  where  $\lambda = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu}$ .
7. If  $X_1^2, \dots, X_m^2$  are  $m$  independent random variables with the chi-squared distributions  $X_i^2 \sim \chi_{n_i}^2(\lambda_i)$ , which may or may not be central, then their sum also has a chi-squared distribution with  $\sum n_i$  degrees of freedom and non-centrality parameter  $\sum \lambda_i$ , i.e.,

$$\sum_{i=1}^m X_i^2 \sim \chi_{\sum_{i=1}^m n_i}^2 \left( \sum_{i=1}^m \lambda_i \right)$$

This is called the **reproductive property** of the chi-squared distribution.

8. Let  $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$ , where  $\mathbf{y}$  has  $n$  elements but the  $Y_i$ 's are not independent so that  $\mathbf{V}$  is singular with rank  $k < n$  and the inverse of  $\mathbf{V}$  is not uniquely defined. Let  $\mathbf{V}^-$  denote a generalized inverse of  $\mathbf{V}$ . Then the random variable  $\mathbf{y}^T \mathbf{V}^- \mathbf{y}$  has the non-central chi-squared distribution with  $k$  degrees of freedom and non-centrality parameter  $\lambda = \boldsymbol{\mu}^T \mathbf{V}^- \boldsymbol{\mu}$ .

**a.** As property 1 from above would suggest, the chi-squared distribution with  $n$  degrees of freedom,  $\chi_n^2$ , is the distribution of the sum of the squares of  $n$  independent standard normal random variables. If  $Y_1$  is a random variable following a normal distribution with mean  $\mu = 0$  and variance  $\sigma^2 = 1$  ( $Y_1 \sim N(0, 1)$ ), then the distribution of  $Y_1^2$  is a special case of the **chi-squared distribution with one degree of freedom,  $\chi_1^2$** . Meaning that:

$$Y_1^2 \sim \chi_1^2$$

The chi-squared distribution with 1 degree of freedom is sometimes referred to as the exponential distribution with rate parameter  $\lambda = 2$  ( $mean = 1/\lambda = 1/2$ ,  $variance = 1/\lambda^2 = 1/4$ ).

So, the distribution of  $Y_1^2$  is  $\chi_1^2$  or equivalently, an exponential distribution with rate parameter  $\lambda = 2$ .

b. The expression  $y^T y$  is the dot product of the vector  $y$  with itself. So:

$$y^T y = \begin{bmatrix} Y_1 & \frac{Y_2 - 3}{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ \frac{Y_2 - 3}{2} \end{bmatrix} = Y_1^2 + \left( \frac{Y_2 - 3}{2} \right)^2 \Rightarrow y^T y = Y_1^2 + \frac{Y_2^2 - 6 \cdot Y_2 + 9}{4}$$

We know that  $Y_1 \sim N(0,1)$  and  $Y_2 \sim N(3,4)$ , and that they are independent. We also know (from a) that  $Y_1^2$  is a special case of the chi-squared distribution with one degree of freedom,  $\chi_1^2$ , or in other words:  $Y_1^2 \sim \chi_1^2$ .

Furthermore, we are given that:  $Y_2 \sim N(3,4)$ , thus:

$$Y_2 \sim N(3,4) \Rightarrow Y_2 - 3 \sim N(0,4) \Rightarrow \frac{Y_2 - 3}{2} \sim N(0,1) \Rightarrow \left( \frac{Y_2 - 3}{2} \right)^2 \sim \chi_1^2$$

Since both  $Y_1^2$  and  $\left( \frac{Y_2 - 3}{2} \right)^2$  are independent and follow a chi-squared distribution with 1 degree of freedom, then it follows that their sum will also follow the chi-squared distribution, but with two degrees of freedom, that are coming from the two terms combined. Therefore (and also according to property 3):

$$y^T y = Y_1^2 + \frac{Y_2^2 - 6 \cdot Y_2 + 9}{4} \sim \chi_2^2$$

c. We know that  $Y_1 \sim N(0,1)$  and  $Y_2 \sim N(3,4)$ . Given that:  $y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$  and its distribution is  $y \sim N(\mu, V)$ , we have that:

The mean vector  $\mu$  of  $y$ , is:

$$\mu = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

While the Variance-Covariance matrix  $V$ , is a diagonal matrix, because  $Y_1$  and  $Y_2$  are independent and it is:

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Let us also compute the inverse of Variance-Covariance matrix  $\mathbf{V}$ ,  $\mathbf{V}^{-1}$  as it will be used:

$$\mathbf{V}^{-1} = \frac{1}{1 \cdot 4 - 0 \cdot 0} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathbf{V}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$$

Now, an expression for  $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$ , will be:

$$\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = [Y_1 \quad Y_2] \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = [Y_1 \quad Y_2/4] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \Rightarrow \mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = Y_1^2 + \frac{Y_2^2}{4}$$

As it was already shown above (in a),  $Y_1^2 \sim \chi_1^2$ . Now, it was also shown (in b) that  $\left(\frac{Y_2-3}{2}\right)^2 \sim \chi_1^2$ , and thus  $\frac{Y_2^2}{4} \sim \chi_1^2$ , plus a non-centrality parameter  $\lambda$ , which from property 6, is the following:

$$\lambda = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu} = [0 \quad 3] \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = [0 \quad 3/4] \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow \lambda = \frac{9}{4}$$

And therefore, since we are adding two chi-squared distributed variables, with one degree of freedom each, it follows that (again from property 6):

$$\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = Y_1^2 + \frac{Y_2^2}{4} \sim \chi_2^2 \left( \frac{9}{4} \right)$$



**Exercise 1.3:**

Let the joint distribution of  $Y_1$  and  $Y_2$  be  $N(\mu, V)$  with:

$$\mu = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } V = \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}$$

- a. Obtain an expression for  $(y - \mu)^T V^{-1} (y - \mu)$ . What is its distribution?
- b. Obtain an expression for  $y^T V^{-1} y$ . What is its distribution?

**SOLUTION:**

