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An Introduction to Generalized Linear Models
Solutions

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CHAPTER 1

Exercise 1.1:

Let Y_1 and Y_2 be independent random variables with $Y_1 \sim N(1, 3)$ and $Y_2 \sim N(2, 5)$. If $W_1 = Y_1 + 2Y_2$ and $W_2 = 4Y_1 - Y_2$ what is the joint distribution of W_1 and W_2 ?

SOLUTION:

A reminder from the book:

1.4.1 Normal distributions:

1. If the random variable Y has the Normal distribution with mean μ and variance σ^2 , its probability density function is:

$$f(y; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2} \left(\frac{y - \mu}{\sigma} \right)^2 \right]$$

We denote this by $Y \sim N(\mu, \sigma^2)$.

2. The Normal distribution with $\mu = 0$ and $\sigma^2 = 1$, $Y \sim N(0, 1)$, is called the **standard Normal distribution**.
3. Let Y_1, \dots, Y_n denote Normally distributed random variables with $Y_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \dots, n$ and let the covariance of Y_i and Y_j be denoted by:

$$\text{cov}(Y_i, Y_j) = \rho_{ij} \sigma_i \sigma_j,$$

where ρ_{ij} is the correlation coefficient for Y_i and Y_j . Then the joint distribution of the Y_i 's is the **multivariate Normal distribution** with mean vector $\mu = [\mu_1, \dots, \mu_n]^T$ and variance-covariance matrix V with diagonal elements σ_i^2 and non-diagonal elements $\rho_{ij} \sigma_i \sigma_j$ for $i \neq j$. We write this as:

$$\mathbf{y} \sim N(\mu, V), \text{ where } \mathbf{y} = [Y_1, \dots, Y_n]^T$$

4. Suppose the random variables Y_1, \dots, Y_n are independent and normally distributed with the distributions $Y_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \dots, n$. If

$$W = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n,$$

where the a_i 's are constants. Then W is also Normally distributed, so that:

$$W = \sum_{i=1}^n a_i Y_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$$

It seems that the joint distribution of two normally distributed variables is yet another normal distribution. In this exercise, in order to find the joint distribution of W_1 and W_2 , we first need to determine the mean, the variance and the covariance of W_1 and W_2 and then use those to derive the joint distribution.

Given that:

$$Y_1 \sim N(1, 3)$$

$$Y_2 \sim N(2, 5)$$

First, let us find the means of W_1 and W_2 :

$$E(W_1) = E(Y_1 + 2Y_2) = E(Y_1) + 2E(Y_2) = 1 + 2 \cdot 2 = 5 \Rightarrow \mathbf{E(W_1) = 5}$$

$$E(W_2) = E(4Y_1 - Y_2) = 4E(Y_1) - E(Y_2) = 4 \cdot 1 - 2 = 2 \Rightarrow \mathbf{E(W_2) = 2}$$

Next, let us calculate the variances of W_1 and W_2 :

$$Var(W_1) = Var(Y_1 + 2Y_2) = Var(Y_1) + 2^2 \cdot Var(Y_2) = 3 + 4 \cdot 5 = 23 \Rightarrow \mathbf{Var(W_1) = 23}$$

$$Var(W_2) = Var(4Y_1 - Y_2) = 4^2 \cdot Var(Y_1) + Var(Y_2) = 16 \cdot 3 + 5 = 53 \Rightarrow \mathbf{Var(W_2) = 53}$$

And finally, let us also compute the covariance between W_1 and W_2 :

$$\begin{aligned} Cov(W_1, W_2) &= Cov(Y_1 + 2Y_2, 4Y_1 - Y_2) = \\ &= Cov(Y_1, 4Y_1) + Cov(Y_1, -Y_2) + Cov(2Y_2, 4Y_1) + Cov(2Y_2, -Y_2) = \\ &= 4Var(Y_1) - Cov(Y_1, Y_2) + 8Cov(Y_2, Y_1) - 2Var(Y_2) = \\ &= 4 \cdot 3 - 0 + 8 \cdot 0 - 2 \cdot 5 = 2 \Rightarrow \\ &\Rightarrow \mathbf{Cov(W_1, W_2) = 2} \end{aligned}$$

Therefore, the joint distribution will be:

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \sim N \left[\begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 23 & 2 \\ 2 & 53 \end{pmatrix} \right]$$

The correlation coefficient between W_1 and W_2 in this case shall be:

$$\rho = \frac{Cov(W_1, W_2)}{\sigma_{W_1} \cdot \sigma_{W_2}} = \frac{Cov(W_1, W_2)}{\sqrt{Var(W_1)} \cdot \sqrt{Var(W_2)}} = \frac{2}{\sqrt{23} \cdot \sqrt{53}} \approx \frac{2}{4.8 \cdot 7.3} \approx 0.057 \Rightarrow \rho = \mathbf{0.057}$$

Therefore, another way to express the joint distribution, would be:

$$\begin{aligned} f(W_1, W_2) &= \frac{1}{2 \cdot \pi \cdot \sigma_{W_1} \cdot \sigma_{W_2} \cdot \sqrt{1 - \rho^2}} \exp \left[-\frac{Z_{W_1}^2 - 2 \cdot Z_{W_1} \cdot Z_{W_2} + Z_{W_2}^2}{2 \cdot \sqrt{1 - \rho^2}} \right] \Rightarrow \\ \Rightarrow f(W_1, W_2) &= \frac{1}{2 \cdot \pi \cdot 4.8 \cdot 7.3 \cdot \sqrt{1 - 0.057^2}} \exp \left[-\frac{Z_{W_1}^2 - 2 \cdot Z_{W_1} \cdot Z_{W_2} + Z_{W_2}^2}{2 \cdot \sqrt{1 - 0.057^2}} \right] \end{aligned}$$

Where:

$$Z_{W_1} = \frac{W_1 - \mu_{W_1}}{\sigma_{W_1}}$$

$$Z_{W_2} = \frac{W_2 - \mu_{W_2}}{\sigma_{W_2}}$$

Exercise 1.2:

Let Y_1 and Y_2 be independent random variables with $Y_1 \sim N(0, 1)$ and $Y_2 \sim N(3, 4)$.

a. What is the distribution of Y_1^2 ?

b. If $y = \begin{bmatrix} Y_1 \\ (Y_2 - 3)/2 \end{bmatrix}$, obtain an expression for $y^T y$. What is its distribution?

c. If $y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ and its distribution is $y \sim N(\mu, V)$, obtain an expression for $y^T V^{-1} y$. What is its distribution?

SOLUTION:

A reminder from the book:

1.4.2 Chi-squared distribution:

1. The **central chi-squared distribution** with n degrees of freedom is defined as the sum of squares of n independent random variables Z_1, \dots, Z_n each with the standard Normal distribution. It is denoted by:

$$X^2 = \sum_{i=1}^n Z_i^2 \sim \chi_n^2$$

In matrix notation, if $\mathbf{z} = [Z_1, \dots, Z_n]^T$ then $\mathbf{z}^T \mathbf{z} = \sum_{i=1}^n Z_i^2$ so that $X^2 = \mathbf{z}^T \mathbf{z} \sim \chi_n^2$.

2. If X^2 has the distribution χ_n^2 , then its expected value is $E(X^2) = n$ and its variance is $\text{Var}(X^2) = 2n$.
3. If Y_1, \dots, Y_n are independent Normally distributed random variables each with the distribution $Y_i \sim N(\mu_i, \sigma_i^2)$ then:

$$X^2 = \sum_{i=1}^n \left(\frac{Y_i - \mu_i}{\sigma_i} \right)^2 \sim \chi_n^2$$

because each of the variables $Z_i = (Y_i - \mu_i)/\sigma_i$ has the standard Normal distribution $N(0, 1)$.

4. Let Z_1, \dots, Z_n be independent random variables each with the distribution $N(0, 1)$ and let $Y_i = Z_i + \mu_i$, where at least one of the μ_i 's is non-zero. Then the distribution of:

$$\sum Y_i^2 = \sum (Z_i + \mu_i)^2 = \sum Z_i^2 + 2 \sum Z_i \mu_i + \sum \mu_i^2$$

has larger mean $n + \lambda$ and larger variance $2n + 4\lambda$ than χ_n^2 where $\lambda = \sum \mu_i^2$. This is called the **non-central chi-squared distribution** with n degrees of freedom and **non-centrality parameter** λ . It is denoted by $\chi_n^2(\lambda)$.

5. Suppose that the Y_i 's are not necessarily independent and the vector $\mathbf{y} = [Y_1, \dots, Y_n]^T$ has the multivariate normal distribution $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$ where the variance-covariance matrix \mathbf{V} is non-singular and its inverse is \mathbf{V}^{-1} . Then:

$$X^2 = (\mathbf{y} - \boldsymbol{\mu})^T \mathbf{V}^{-1} (\mathbf{y} - \boldsymbol{\mu}) \sim \chi_n^2$$

6. More generally if $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$ then the random variable $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$ has the non-central chi-squared distribution $\chi_n^2(\lambda)$ where $\lambda = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu}$.
7. If X_1^2, \dots, X_m^2 are m independent random variables with the chi-squared distributions $X_i^2 \sim \chi_{n_i}^2(\lambda_i)$, which may or may not be central, then their sum also has a chi-squared distribution with $\sum n_i$ degrees of freedom and non-centrality parameter $\sum \lambda_i$, i.e.,

$$\sum_{i=1}^m X_i^2 \sim \chi_{\sum_{i=1}^m n_i}^2 \left(\sum_{i=1}^m \lambda_i \right)$$

This is called the **reproductive property** of the chi-squared distribution.

8. Let $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$, where \mathbf{y} has n elements but the Y_i 's are not independent so that \mathbf{V} is singular with rank $k < n$ and the inverse of \mathbf{V} is not uniquely defined. Let \mathbf{V}^- denote a generalized inverse of \mathbf{V} . Then the random variable $\mathbf{y}^T \mathbf{V}^- \mathbf{y}$ has the non-central chi-squared distribution with k degrees of freedom and non-centrality parameter $\lambda = \boldsymbol{\mu}^T \mathbf{V}^- \boldsymbol{\mu}$.

a. As property 1 from above would suggest, the chi-squared distribution with n degrees of freedom, χ_n^2 , is the distribution of the sum of the squares of n independent standard normal random variables. If Y_1 is a random variable following a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$ ($Y_1 \sim N(0, 1)$), then the distribution of Y_1^2 is a special case of the **chi-squared distribution with one degree of freedom, χ_1^2** . Meaning that:

$$Y_1^2 \sim \chi_1^2$$

The chi-squared distribution with 1 degree of freedom is sometimes referred to as the exponential distribution with rate parameter $\lambda = 2$ ($mean = 1/\lambda = 1/2$, $variance = 1/\lambda^2 = 1/4$).

So, the distribution of Y_1^2 is χ_1^2 or equivalently, an exponential distribution with rate parameter $\lambda = 2$.

b. The expression $y^T y$ is the dot product of the vector y with itself. So:

$$y^T y = \begin{bmatrix} Y_1 & \frac{Y_2 - 3}{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ \frac{Y_2 - 3}{2} \end{bmatrix} = Y_1^2 + \left(\frac{Y_2 - 3}{2} \right)^2 \Rightarrow y^T y = Y_1^2 + \frac{Y_2^2 - 6 \cdot Y_2 + 9}{4}$$

We know that $Y_1 \sim N(0,1)$ and $Y_2 \sim N(3,4)$, and that they are independent. We also know (from a) that Y_1^2 is a special case of the chi-squared distribution with one degree of freedom, χ_1^2 , or in other words: $Y_1^2 \sim \chi_1^2$.

Furthermore, we are given that: $Y_2 \sim N(3,4)$, thus:

$$Y_2 \sim N(3,4) \Rightarrow Y_2 - 3 \sim N(0,4) \Rightarrow \frac{Y_2 - 3}{2} \sim N(0,1) \Rightarrow \left(\frac{Y_2 - 3}{2} \right)^2 \sim \chi_1^2$$

Since both Y_1^2 and $\left(\frac{Y_2 - 3}{2} \right)^2$ are independent and follow a chi-squared distribution with 1 degree of freedom, then it follows that their sum will also follow the chi-squared distribution, but with two degrees of freedom, that are coming from the two terms combined. Therefore (and also according to property 3):

$$y^T y = Y_1^2 + \frac{Y_2^2 - 6 \cdot Y_2 + 9}{4} \sim \chi_2^2$$

c. We know that $Y_1 \sim N(0,1)$ and $Y_2 \sim N(3,4)$. Given that: $y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ and its distribution is $y \sim N(\mu, V)$, we have that:

The mean vector μ of y , is:

$$\mu = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

While the Variance-Covariance matrix V , is a diagonal matrix, because Y_1 and Y_2 are independent and it is:

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Let us also compute the inverse of Variance-Covariance matrix \mathbf{V} , \mathbf{V}^{-1} as it will be used:

$$\mathbf{V}^{-1} = \frac{1}{1 \cdot 4 - 0 \cdot 0} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathbf{V}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$$

Now, an expression for $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$, will be:

$$\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = [Y_1 \quad Y_2] \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = [Y_1 \quad Y_2/4] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \Rightarrow \mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = Y_1^2 + \frac{Y_2^2}{4}$$

As it was already shown above (in a), $Y_1^2 \sim \chi_1^2$. Now, it was also shown (in b) that $\left(\frac{Y_2-3}{2}\right)^2 \sim \chi_1^2$, and thus $\frac{Y_2^2}{4} \sim \chi_1^2$, plus a non-centrality parameter λ , which from property 6, is the following:

$$\lambda = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu} = [0 \quad 3] \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = [0 \quad 3/4] \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow \lambda = \frac{9}{4}$$

And therefore, since we are adding two chi-squared distributed variables, with one degree of freedom each, it follows that (again from property 6):

$$\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = Y_1^2 + \frac{Y_2^2}{4} \sim \chi_2^2 \left(\frac{9}{4} \right)$$

Exercise 1.3:

Let the joint distribution of Y_1 and Y_2 be $N(\mu, V)$ with:

$$\mu = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } V = \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}$$

a. Obtain an expression for $(y - \mu)^T V^{-1} (y - \mu)$. What is its distribution?

b. Obtain an expression for $y^T V^{-1} y$. What is its distribution?

SOLUTION:

A reminder from the book:

1.4.2 Chi-squared distribution:

1. The **central chi-squared distribution** with n degrees of freedom is defined as the sum of squares of n independent random variables Z_1, \dots, Z_n each with the standard Normal distribution. It is denoted by:

$$X^2 = \sum_{i=1}^n Z_i^2 \sim \chi_n^2$$

In matrix notation, if $\mathbf{z} = [Z_1, \dots, Z_n]^T$ then $\mathbf{z}^T \mathbf{z} = \sum_{i=1}^n Z_i^2$ so that $X^2 = \mathbf{z}^T \mathbf{z} \sim \chi_n^2$.

2. If X^2 has the distribution χ_n^2 , then its expected value is $E(X^2) = n$ and its variance is $Var(X^2) = 2n$.
3. If Y_1, \dots, Y_n are independent Normally distributed random variables each with the distribution $Y_i \sim N(\mu_i, \sigma_i^2)$ then:

$$X^2 = \sum_{i=1}^n \left(\frac{Y_i - \mu_i}{\sigma_i} \right)^2 \sim \chi_n^2$$

because each of the variables $Z_i = (Y_i - \mu_i)/\sigma_i$ has the standard Normal distribution $N(0, 1)$.

4. Let Z_1, \dots, Z_n be independent random variables each with the distribution $N(0, 1)$ and let $Y_i = Z_i + \mu_i$, where at least one of the μ_i 's is non-zero. Then the distribution of:

$$\sum Y_i^2 = \sum (Z_i + \mu_i)^2 = \sum Z_i^2 + 2 \sum Z_i \mu_i + \sum \mu_i^2$$

has larger mean $n + \lambda$ and larger variance $2n + 4\lambda$ than χ_n^2 where $\lambda = \sum \mu_i^2$. This is called the **non-central chi-squared distribution** with n degrees of freedom and **non-centrality parameter** λ . It is denoted by $\chi_n^2(\lambda)$.

5. Suppose that the Y_i 's are not necessarily independent and the vector $\mathbf{y} = [Y_1, \dots, Y_n]^T$ has the multivariate normal distribution $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$ where the variance-covariance matrix \mathbf{V} is non-singular and its inverse is \mathbf{V}^{-1} . Then:

$$X^2 = (\mathbf{y} - \boldsymbol{\mu})^T \mathbf{V}^{-1} (\mathbf{y} - \boldsymbol{\mu}) \sim \chi_n^2$$

6. More generally if $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$ then the random variable $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$ has the non-central chi-squared distribution $\chi_n^2(\lambda)$ where $\lambda = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu}$.
7. If X_1^2, \dots, X_m^2 are m independent random variables with the chi-squared distributions $X_i^2 \sim \chi_{n_i}^2(\lambda_i)$, which may or may not be central, then their sum also has a chi-squared distribution with $\sum n_i$ degrees of freedom and non-centrality parameter $\sum \lambda_i$, i.e.,

$$\sum_{i=1}^m X_i^2 \sim \chi_{\sum_{i=1}^m n_i}^2 \left(\sum_{i=1}^m \lambda_i \right)$$

This is called the **reproductive property** of the chi-squared distribution.

8. Let $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$, where \mathbf{y} has n elements but the Y_i 's are not independent so that \mathbf{V} is singular with rank $k < n$ and the inverse of \mathbf{V} is not uniquely defined. Let \mathbf{V}^- denote a generalized inverse of \mathbf{V} . Then the random variable $\mathbf{y}^T \mathbf{V}^- \mathbf{y}$ has the non-central chi-squared distribution with k degrees of freedom and non-centrality parameter $\lambda = \boldsymbol{\mu}^T \mathbf{V}^- \boldsymbol{\mu}$.

a. First, let us compute the inverse of Variance-Covariance matrix \mathbf{V} , \mathbf{V}^{-1} as it will be needed.
So:

$$\mathbf{V}^{-1} = \frac{1}{4 \cdot 9 - 1 \cdot 1} \begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix} \Rightarrow \mathbf{V}^{-1} = \begin{bmatrix} 9/35 & -1/35 \\ -1/35 & 4/35 \end{bmatrix}$$

Since $y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ and $\mu = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then their difference shall be:

$$y - \mu = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} Y_1 - 2 \\ Y_2 - 3 \end{bmatrix}$$

And therefore, the joint distribution, will have the following form:

$$\begin{aligned} (y - \mu)^T V^{-1} (y - \mu) &= [Y_1 - 2 \quad Y_2 - 3] \begin{bmatrix} 9/35 & -1/35 \\ -1/35 & 4/35 \end{bmatrix} \begin{bmatrix} Y_1 - 2 \\ Y_2 - 3 \end{bmatrix} = \\ &= \left[\frac{9}{35} (Y_1 - 2) - \frac{1}{35} (Y_2 - 3) \quad -\frac{1}{35} (Y_1 - 2) + \frac{4}{35} (Y_2 - 3) \right] \begin{bmatrix} Y_1 - 2 \\ Y_2 - 3 \end{bmatrix} = \\ &= \frac{9}{35} (Y_1 - 2)^2 - \frac{1}{35} (Y_2 - 3)(Y_1 - 2) - \frac{1}{35} (Y_1 - 2)(Y_2 - 3) + \frac{4}{35} (Y_2 - 3)^2 \Rightarrow \\ &\Rightarrow (y - \mu)^T V^{-1} (y - \mu) = \frac{9}{35} (Y_1 - 2)^2 - \frac{2}{35} (Y_1 - 2)(Y_2 - 3) + \frac{4}{35} (Y_2 - 3)^2 \end{aligned}$$

From property 5, we know that for a multivariate normal distribution $y \sim N(\mu, V)$, the quadratic form $(y - \mu)^T V^{-1} (y - \mu)$ follows a chi-squared distribution with degrees of freedom equal to the dimension of y (and in this case, we have only two dimensions), and therefore:

$$(y - \mu)^T V^{-1} (y - \mu) = \frac{9}{35} (Y_1 - 2)^2 - \frac{2}{35} (Y_1 - 2)(Y_2 - 3) + \frac{4}{35} (Y_2 - 3)^2 \sim \chi_2^2$$

b. From the previous question (a), we already know that:

$$V^{-1} = \begin{bmatrix} 9/35 & -1/35 \\ -1/35 & 4/35 \end{bmatrix}$$

And therefore, the expression for $y^T V^{-1} y$ shall be:

$$\begin{aligned} y^T V^{-1} y &= [Y_1 \quad Y_2] \begin{bmatrix} 9/35 & -1/35 \\ -1/35 & 4/35 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \left[\frac{9}{35} \cdot Y_1 - \frac{1}{35} \cdot Y_2 \quad -\frac{1}{35} \cdot Y_1 + \frac{4}{35} \cdot Y_2 \right] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \\ &= \frac{9}{35} \cdot Y_1^2 - \frac{1}{35} \cdot Y_1 \cdot Y_2 - \frac{1}{35} \cdot Y_2 \cdot Y_1 + \frac{4}{35} \cdot Y_2^2 \Rightarrow \end{aligned}$$

$$\Rightarrow \mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = \frac{9}{35} \cdot Y_1^2 - \frac{2}{35} \cdot Y_1 \cdot Y_2 + \frac{4}{35} \cdot Y_2^2$$

Now, the distribution of $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$ is a more general case of the one described in the previous question (a) and thus follows property 6, meaning that: “if $\mathbf{y} \sim \mathbf{N}(\boldsymbol{\mu}, \mathbf{V})$ then the random variable $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$ has the non-central chi-squared distribution $\chi_n^2(\lambda)$ where $\lambda = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu}$.”

In our case, \mathbf{y} can be written as $\mathbf{y} = \boldsymbol{\mu} + \mathbf{Z}$, where: $\mathbf{Z} \sim \mathbf{N}(0, \mathbf{V})$. So if we expanded on this, we would have:

$$\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = (\boldsymbol{\mu} + \mathbf{Z})^T \mathbf{V}^{-1} (\boldsymbol{\mu} + \mathbf{Z}) = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu} + 2\boldsymbol{\mu}^T \mathbf{V}^{-1} \mathbf{Z} + \mathbf{Z}^T \mathbf{V}^{-1} \mathbf{Z}$$

with:

- $\mathbf{Z}^T \mathbf{V}^{-1} \mathbf{Z} \sim \chi_2^2$, because $\mathbf{Z} \sim \mathbf{N}(0, \mathbf{V})$, and the quadratic form of a multivariate normal distribution follows a chi-squared distribution with degrees of freedom equal to the dimension of \mathbf{Z} (which is 2).
- $2\boldsymbol{\mu}^T \mathbf{V}^{-1} \mathbf{Z}$ is normally distributed with a mean of 0.

Thus $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$ is a sum of a chi-squared distribution and a normal distribution. This means that $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$ follows a non-central chi-squared distribution with 2 degrees of freedom and a non-centrality parameter $\lambda = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu}$, which is:

$$\begin{aligned} \lambda = \boldsymbol{\mu}^T \mathbf{V}^{-1} \boldsymbol{\mu} &= \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 9/35 & -1/35 \\ -1/35 & 4/35 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 18/35 - 3/35 & -2/35 + 12/35 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \\ &= \begin{bmatrix} 15/35 & 10/35 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{30}{35} + \frac{30}{35} = \frac{60}{35} \Rightarrow \lambda = \frac{12}{7} \end{aligned}$$

Therefore, in conclusion:

$$\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y} = \frac{9}{35} \cdot Y_1^2 - \frac{2}{35} \cdot Y_1 \cdot Y_2 + \frac{4}{35} \cdot Y_2^2 \sim \chi_2^2\left(\frac{12}{7}\right)$$

Exercise 1.4:

Let Y_1, \dots, Y_n be independent random variables each with the distribution $N(\mu, \sigma^2)$. Let:

$$\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i \quad \text{and} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2.$$

a. What is the distribution of \bar{Y} ?

b. Show that $S^2 = \frac{1}{n-1} [\sum_{i=1}^n (Y_i - \mu)^2 - n(\bar{Y} - \mu)^2]$.

c. From (b) it follows that $\sum (Y_i - \mu)^2 / \sigma^2 = (n-1) S^2 / \sigma^2 + [(\bar{Y} - \mu)^2 n / \sigma^2]$. How does this allow you to deduce that \bar{Y} and S^2 are independent?

d. What is the distribution of $\frac{(n-1)S^2}{\sigma^2}$?

e. What is the distribution of $\frac{\bar{Y} - \mu}{S/\sqrt{n}}$?

SOLUTION:

a. Since the Y_i are independent and each has the distribution $N(\mu, \sigma^2)$, the expectation of \bar{Y} is:

$$E(\bar{Y}) = E\left[\frac{1}{n} \sum_{i=1}^n Y_i\right] = \frac{1}{n} \sum_{i=1}^n E[Y_i] = \frac{1}{n} \cdot n \cdot \mu = \mu$$

while its variance is:

$$Var(\bar{Y}) = Var\left(\frac{1}{n} \sum_{i=1}^n Y_i\right) = \frac{1}{n^2} \sum_{i=1}^n Var(Y_i) = \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

We know that \bar{Y} consists of a linear combination of independent, normally distributed variables and therefore it is itself normally distributed. Thus, the distribution of \bar{Y} shall be:

$$\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

b. Let us start from the definition of the sample variance:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 = \frac{1}{n-1} \sum_{i=1}^n [(Y_i - \mu) - (\bar{Y} - \mu)]^2 =$$

$$\begin{aligned}
&= \frac{1}{n-1} \sum_{i=1}^n [(Y_i - \mu)^2 - 2 \cdot (Y_i - \mu)(\bar{Y} - \mu) + (\bar{Y} - \mu)^2] = \\
&= \frac{1}{n-1} \left[\sum_{i=1}^n (Y_i - \mu)^2 - \sum_{i=1}^n 2 \cdot (Y_i - \mu)(\bar{Y} - \mu) + \sum_{i=1}^n (\bar{Y} - \mu)^2 \right] = \\
&= \frac{1}{n-1} \left[\sum_{i=1}^n (Y_i - \mu)^2 - 2 \cdot (\bar{Y} - \mu) \sum_{i=1}^n (Y_i - \mu) + \sum_{i=1}^n (\bar{Y} - \mu)^2 \right] = \\
&= \frac{1}{n-1} \left[\sum_{i=1}^n (Y_i - \mu)^2 - 2 \cdot (\bar{Y} - \mu) \cdot n \cdot (\bar{Y} - \mu) + n \cdot (\bar{Y} - \mu)^2 \right] = \\
&= \frac{1}{n-1} \left[\sum_{i=1}^n (Y_i - \mu)^2 - 2 \cdot n \cdot (\bar{Y} - \mu)^2 + n \cdot (\bar{Y} - \mu)^2 \right] \Rightarrow \\
&\Rightarrow S^2 = \frac{1}{n-1} \left[\sum_{i=1}^n (Y_i - \mu)^2 - n \cdot (\bar{Y} - \mu)^2 \right]
\end{aligned}$$

c. We are given the following expression:

$$\frac{\sum_{i=1}^n (Y_i - \mu)^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} + \frac{n(\bar{Y} - \mu)^2}{\sigma^2}$$

So, why are \bar{Y} and S^2 independent? Let us look at the two right hand terms one by one.

Firstly, let us discuss the term:

$$\frac{(n-1)S^2}{\sigma^2}$$

Here S^2 is the sample variance, which is defined as:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2$$

Therefore:

$$\frac{(n-1)S^2}{\sigma^2} = \frac{n-1}{\sigma^2} \cdot \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 = \frac{1}{\sigma^2} \sum_{i=1}^n (Y_i - \bar{Y})^2$$

The sample variance measures the spread of the individual Y_i 's around the sample mean \bar{Y} . This involves $n - 1$ degrees of freedom because the calculation of S^2 depends on n data points, but the sample mean \bar{Y} is used to estimate the center of the data, reducing the degrees of freedom by 1.

Thus, under the assumption that the Y_i 's are normally distributed, the sum of squares, which was defined above, follows a chi-squared distribution with $n - 1$ degrees of freedom:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

Secondly, let us discuss the term:

$$\frac{n(\bar{Y} - \mu)^2}{\sigma^2}$$

And from question **a**, we already know that:

$$\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Therefore:

$$\frac{n(\bar{Y} - \mu)^2}{\sigma^2} = \left(\frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim Z^2$$

where Z is a standard normal random variable, $Z \sim N(0,1)$. And hence:

$$\frac{n(\bar{Y} - \mu)^2}{\sigma^2} \sim \chi_1^2$$

Since the total sum of squares can be split into two independent components, one involving \bar{Y} and the other involving S^2 , then by Cochran's Theorem, the chi-squared terms must be independent. More formally, the independence of χ_1^2 and χ_{n-1}^2 implies that **\bar{Y} and S^2 are independent.**

Cochran's Theorem:

Cochran's Theorem provides a way to decompose sums of squared normal random variables into independent chi-squared distributions. Specifically, if you have a set of independent normal random variables Y_1, Y_2, \dots, Y_n drawn from $N(\mu, \sigma^2)$. In our example, Cochran's Theorem states that the total sum of squares:

$$\sum_{i=1}^n (Y_i - \mu)^2$$

can be decomposed into two independent components:

$$\frac{n(\bar{Y} - \mu)^2}{\sigma^2} \sim \chi_1^2$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

This result is a key property of normal distributions and is a consequence of the fact that the sample mean \bar{Y} and sample variance S^2 capture independent aspects of the data. \bar{Y} captures location (center), while S^2 captures spread (variability) around the center.

