Dynamic Knapsack Problem

AS, WM, KV

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Contents

1 Questions/Comments

1.1 Questions

- Should ul definition include units violated?? it seems a constraint might be violated otherwise

1.2 Comments

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2 Assumptions

- Assume all new patients are arriving at the beginning of the period
  - Assume arrivals follow poisson distribution
- It is assumed that patients do not change complexities within some limit (TL).
  - Let's assume our limit is 10 periods, patients will never change complexities within the 10 periods.
  - However, on their 10th period, they can increase complexity
- We assume patient complexity transitions follow binary distribution - each period a patient can become more complex with a static probability
  - While patient is waiting less that TL, the transition probability is 0.
  - After TL, a patient has some probability they move to a higher complexity in that period
- It is assumed if a patient is scheduled in period one, it means they are served "immediately", regardless if in practice the appointment is at the beginning or end of the period.
- We distinguish between two types of reschedules: good and bad reschedules
  - Good reschedules are reschedules where a patient is rescheduled to an earlier period
  - Bad reschedules are reschedules where a patient is rescheduled to a later period
- We assume that only specific reschedules are allowed (to simplify the model, and remove redundancies)
  - Good reschedules are allowed from any period after 2 into period 1
  - Bad reschedules are only allowed from period 1 to any period after 2
- It is assumed that there is a certain default expected number of PPE units available for all periods
  - However, in the period 1, there is some random deviation from the expected number of units
  - This random deviation follows some uniform distribution
- We allow some violation of PPE units, in rare cases (with high cost), in order to accomodate changes due to variability.
  - If we have violation of PPE units - the extra capacity comes from external source

3 MDP Model

3.1 Decision Epochs

Decisions are made at the beginning of each time period. There are 3 relevant time intervals to look at:

- Pre-decision state (S)
  - Pre-decision state defines the initial state on which a decision is required
  - This information drives decision making
- Post-decision state ($\hat{S}$)

- Post-decision state is the time when the decision has been executed, but no new info came in
- This state defines the immediate cost of an action

- Post-transition state (S')
  - This is the state when new information has come in (transition randomness). Defines next pre decision state
  - It is primarily used to generate expectation for the ADP

3.2 State Space

3.2.1 Description

State is defined by current available and used resources, patient waitlist/demand, and patients already scheduled

$$\vec{S} = (\vec{ul}, \vec{pw}, \vec{ps})$$

- $\vec{ul} = ul_p$ - Units left over from previous period for resource p plus deviation for that unit for period 1. This is only for resources that can be carried over
- $\vec{pw} = pw_{mdc}$ - Patients of complexity d, CPU c, on a wait list for m periods (m of 0 - just arrived)
- $\vec{ps} = ps_{tmdc}$ - Patients of complexity d, CPU c, scheduled to period t, waiting for m periods (m of 0 - just arrived)

3.3 Action Sets

3.3.1 Description

Decision consists of rescheduling currently scheduled patients, and scheduling patients on waitlist. There are also some goal and auxiliary variables

$$\vec{A} = (\vec{sc}, r\vec{sc}, \vec{uv}, \text{auxiliary variables})$$

- $\vec{sc} = sc_{tmdc}$ - Patients of complexity d, CPU c, waiting for m periods, to schedule in period t
- $r\vec{sc} = rsc_{tt'mdc}$ -Patients of complexity d, CPU c, waiting for m periods, to reschedule from period t to period t'
- $\vec{uv} = uv_{tp}$ - goal variable, violation on number of resources used for period t, of resource p

3.3.2 Auxiliary Variables

- uvb_{tp} - binary variable to enforce uv variable without objective function
- $\hat{ul}_p$ - post-decision unit leftover at period 1
- ulb_p - binary variable to enforce ul variable without objective function
- $\hat{uu}_{tp}$ - post-decision units used
- $\hat{pw}_{mdc}$ - post-decision patients waiting
- $\hat{ps}_{tmdc}$ - post-decision patients scheduled

3.3.3 Auxiliary Variable Definition

$$\hat{uu}_{tp} = \sum_{mdc} U_{pdc} \hat{ps}_{tmdc} \qquad \forall tp \tag{1}$$

$$\hat{pw}_{mdc} = pw_{mdc} - \sum_{t} sc_{tmdc} \qquad \forall mdc \tag{2}$$

$$\hat{ps}_{tmdc} = ps_{tmdc} + sc_{tmdc} + \sum_{t} rsc_{tt'mdc} - \sum_{t'} rsc_{tt'mdc} \qquad \forall tmdc \tag{3}$$

- Define Resource Violation Variable

$$\hat{uv}_{tp} \leq M(uvb_{tp}) \quad \forall tp \tag{4}$$
$$\hat{uv}_{1p} \leq (\hat{uu}_{1p} - uen_{1p} - ul_p) + M(1 - uvb_{tp}) \quad \forall p \tag{5}$$
$$\hat{uv}_{tp} \leq (\hat{uu}_{tp} - uen_{tp}) + M(1 - uvb_{tp}) \quad \forall t \in \{2..T\}p \tag{6}$$
$$\tag{7}$$

- Define Units Left Over Variable

$$\hat{ul}_p \geq 0 \quad \forall p \tag{8}$$
$$\hat{ul}_p \geq uen_{1p} + ul_p - \hat{uu}_{1p} \quad \forall p \tag{9}$$
$$\hat{ul}_p \leq M(ulb_p) \quad \forall p \tag{10}$$
$$\hat{ul}_p \leq (uen_{1p} + ul_p - \hat{uu}_{1p}) + M(1 - ulb_p) \quad \forall p \tag{11}$$

3.3.4 State-Action Constraints

- Resource Usage Constraint

$$\hat{uu}_{1p} \leq uen_{1p} + ul_p + uv_{1p} \quad \forall p \tag{12}$$
$$\hat{uu}_{tp} \leq uen_{tp} + uv_{tp} \quad \forall t \in \{2..T\}p \tag{13}$$

- Custom bounds on when reschedules are allowed

$$rsc_{tt'mdc} = 0 \quad \forall t \in \{2...T\}, t' \in \{2...T\}mdc \tag{14}$$
$$rsc_{tt'mdc} = 0 \quad \forall tt'mdc, \text{where } t = t' = 1 \tag{15}$$

- Bounds on Schedules/Reschedules

$$\sum_{t'} rsc_{tt'mdc} \leq ps_{tmdc} \quad \forall tmdc \tag{16}$$
$$\sum_{t} sc_{tmdc} \leq pw_{mdc} \quad \forall mdc \tag{17}$$

- Bounds on states

$$ul_p \leq uen_p * 3 \quad \forall p \tag{18}$$
$$pw_{mdc} \leq pea_{dc} * 20 \quad \forall mdc \tag{19}$$
$$ps_{tmdc} \leq pea_{dc} * 4 \quad \forall tmdc \tag{20}$$

3.4 Transition Probabilities

3.4.1 Uncertainty Sources

1. Number of patients arriving this period - pw_{0dc}
   - pea_{dc} - represents the random number of patients arriving. Follows a poisson distribution.
2. Transition between patient complexities within the wait list - pw_{mdc}
   - pwt_{mdc} represents the random number of patients of complexity d, CPU c, waiting for m period, who became more complex. Follows binomial distribution.
   - ptp_{dc} represents transition probability to a higher complexity
   - TL_{dc} represents the time at which this transition probability starts to have an effect (must be less than M)
3. Transition between patient complexities within the scheduled list - ps_{tmdc}
   - pst_{tmdc} represents the random number of patients of complexity d, CPU c, waiting for m period, scheduled into t period, who became more complex. Follows binomial distribution.
4. Amout of expected units of PPE resource for the next time period - ue_{1p}
   - ued_p represents the random deviation of resource p from the expectation. Follows uniform distribution.
   - uen_p expected units of resource p per period

3.4.2 Transition Constraints

1. Transition from $\vec{ue}$ to $\vec{ue'}$ - Resource Carry Over

$$ul'_p = \hat{ul}_p + ued_p \quad \forall p \in \{\text{Carry Over}\} \tag{21}$$

$$ul'_p = ued_p \quad \forall p \in \{\text{Non Carry Over}\} \tag{22}$$

2. Transition from $\vec{pw}$ to $\vec{pw'}$ - Flow of patients on waitlist

$$pw'_{0dc} = pea_{dc} \quad \forall dc \tag{23}$$

$$pw'_{mdc} = \hat{pw}_{m-1,dc} \quad \forall m \in \{1...(TL_{dc}-1)\}dc \tag{24}$$

$$pw'_{mdc} = \hat{pw}_{m-1,dc} + \overbrace{pwt_{m-1,d-1,c} - pwt_{m-1,dc}}^{\text{change in complexities}} \quad \forall m \in \{TL_{dc}...M-1\}dc \tag{25}$$

$$pw'_{Mdc} = \sum_{M-1}^{M} \left(\hat{pw}_{mdc} + \overbrace{pwt_{md-1,c} - pwt_{mdc}}^{\text{change in complexities}}\right) \quad \forall dc \tag{26}$$

3. Transition from $\vec{ps}$ to $\vec{ps'}$ - Flow of patiensts scheduled

$$ps'_{t0dc} = 0 \quad \forall tdc \tag{27}$$

$$ps'_{Tmdc} = 0 \quad \forall mdc \tag{28}$$

$$ps'_{tmdc} = \hat{ps}_{t+1,m-1,dc} \quad \forall t \in \{1...T-1\} m \in \{1...(TL_{dc}-1)\}dc \tag{29}$$

$$ps'_{tmdc} = \hat{ps}_{t+1,m-1,dc} + \overbrace{pst_{t+1,m-1,d-1,c} - pst_{t+1,m-1,dc}}^{\text{change in complexities}} \quad \forall t \in \{1...T-1\} m \in \{TL_{dc}...M-1\}dc \tag{30}$$

$$ps'_{tMdc} = \sum_{M-1}^{M} \left(\hat{ps}_{t+1mdc} + \overbrace{pst_{t+1,md-1,c} - pst_{t+1,mdc}}^{\text{change in complexities}}\right) \quad \forall t \in \{1...T-1\}dc \tag{31}$$

3.5 Costs

Cost will come from 4 sources:

- waiting (cw)
- rescheduling (cc)
- scheduling ($cs_t = \sum_t cw^t$)
- resource violation (M)

$$\begin{aligned} C = c(\hat{\vec{pw}}, \hat{\vec{ps}}, \vec{rsc}, \vec{uv}) = & \overbrace{\sum_{mdc} cw(\hat{pw}_{mdc})}^{\text{cost of waiting}} + \overbrace{\sum_{tmdc} cs_t(sc_{tmdc})}^{\text{Prefer earlier appointments}} \\ & + \overbrace{\sum_{\substack{tt`mdc \\ t'>t}} (cs_{t'-t} + cc) * (rsc_{tt'mdc})}^{\text{Bad Reschedule}} - \overbrace{\sum_{\substack{tt`mdc \\ t'<t}} (cs_{t-t'} - cc) * (rsc_{tt'mdc})}^{\text{Good Reschedule}} + M \sum_{tp} uv_{tp} \end{aligned} \tag{32}$$

4 LP Model

4.1 Full LP

Given a full MDP model, the equivalent LP would look as follows:

$$\max_{\vec{v}} \sum \alpha(\vec{S}) v(\vec{S}) \tag{33}$$

subject to

$$c(\hat{\vec{pw}}, \hat{\vec{ps}}, \vec{rsc}, \vec{uv}) + \gamma \sum_{\vec{p}} p(\vec{pea}, \vec{pwt}, \vec{pst}, \vec{ued}) v(\vec{S'} | \vec{S}, \vec{A}, \vec{pea}, \vec{pwt}, \vec{pst}, \vec{ued}) \geq v(\vec{S}) \quad \forall \vec{S}\vec{A} \tag{34}$$

5 Aproximate Dynamic Programming Model

5.1 Full ADP Model

Let's convert it into ADP. We do that by changing $v(\vec{S})$ to an approximation as follows:

$$v(\vec{ul}, \vec{pw}, \vec{ps}) = \beta^0 + \sum_p \beta^{ul}_p ul_p + \sum_{mdc} \beta^{pw}_{mdc} pw_{mdc} + \sum_{tmdc} \beta^{ps}_{tmdc} ps_{tmdc} \tag{35}$$

This gives the following LP

$$\max_{\vec{\beta}} \sum \alpha(\vec{ul}, \vec{pw}, \vec{ps}) v(\vec{ul}, \vec{pw}, \vec{ps}) \tag{36}$$

Subject to:

$$\begin{aligned}
&c(\vec{pw}, \vec{ps}, \vec{rsc}, \vec{uv}) + \gamma \sum_{\vec{p}} p \Bigg(\beta^0 + \Big(\beta^{ul}_{p\in \text{co}} (\hat{ul}_p + uen_d) \Big) + \Big(\beta^{ul}_{p\in \text{nco}} (uen_d) \Big) + \\
&\Big(\sum_{dc} \beta^{pw}_{0dc} pea_{dc} \Big) + \Big(\sum_{m=1,dc}^{M-1} \beta^{pw}_{mdc} (\hat{pw}_{m-1dc} + pwt_{m-1d-1c} - pwt_{m-1dc}) \Big) + \\
&\qquad \Big(\sum_{dc} \beta^{pw}_{Mdc} (\sum_{M-1}^{M} \hat{pw}_{mdc} + pwt_{md-1c} - pwt_{mdc}) \Big) + \\
&\Big(\sum_{tdc} \beta^{ps}_{t0dc} 0 \Big) + \Big(\sum_{mdc} \beta^{ps}_{Tmdc} 0 \Big) + \Big(\sum_{t=1 m=1 dc}^{T-1 M-1} \beta^{ps}_{tmdc} (\hat{ps}_{t+1m-1dc} + pst_{t+1m-1d-1c} - pst_{t+1m-1dc}) \Big) + \\
&\Big(\sum_{t=1Mdc}^{T-1} \beta^{ps}_{tmdc} (\sum_{M-1}^{M} (\hat{ps}_{t+1mdc} + pst_{t+1md-1c} - pst_{t+1mdc})) \Big) \Bigg) \geq v(\vec{ul}, \vec{pw}, \vec{ps}) \quad \forall \vec{S} \vec{A}
\end{aligned} \tag{37}$$

$$\vec{\beta} \geq 0 \quad \forall \beta \tag{38}$$

And all auxiliary constraints in "Auxiliary Variable Definition"

5.2 Expectation ADP Model

Steps for rearranging and converting the constraint (37) into expectation:

- $E[ued_p] = 0$
- $E[pwt_{mdc}] = ptp_{mdc} * \hat{pw}_{mdc}$
- $E[pst_{tmdc}] = ptp_{mdc} * \hat{ps}_{tmdc}$

$$\beta^0 : \qquad (1-\gamma)\beta^0 \tag{39}$$

$$\beta^{ul}_p : \qquad \sum_{p\in \text{co}} \beta^{ul}_p \Big(ul_p - \gamma(\hat{ul}_p) \Big) + \sum_{p\in \text{nco}} \beta^{ul}_p (ul_p) \tag{40}$$

$$\begin{aligned}
\beta^{pw}_{mdc} : \qquad &\sum_{dc} \beta^{pw}_{0dc} \Big(pw_{0dc} - \gamma(pea_{dc}) \Big) + \sum_{m=1dc}^{M-1} \beta^{pw}_{mdc} \Big(pw_{mdc} - \gamma(\hat{pw}_{m-1dc} + E[pwt_{m-1d-1c}] - E[pwt_{m-1dc}]) \Big) + \\
&\sum_{dc} \beta^{pw}_{Mdc} \Big(pw_{Mdc} - \gamma \sum_{M-1}^{M} (\hat{pw}_{mdc} + E[pwt_{md-1c}] - E[pwt_{mdc}]) \Big)
\end{aligned} \tag{41}$$

$$\begin{aligned}
\beta^{ps}_{tmdc} : \qquad &\sum_{tdc} \beta^{ps}_{t0dc} \Big(ps_{t0dc} \Big) + \sum_{mdc} \beta^{ps}_{Tmdc} \Big(ps_{Tmdc} \Big) + \\
&\sum_{t=1m=1dc}^{T-1M-1} \beta^{ps}_{tmdc} \Big(ps_{tmdc} - \gamma(\hat{ps}_{t+1m-1dc} + E[pst_{t+1m-1d-1c}] - E[pst_{t+1m-1dc}]) \Big) + \\
&\sum_{t=1dc}^{T-1} \beta^{ps}_{tMdc} \Big(ps_{tMdc} - \gamma \sum_{M-1}^{M} (\hat{ps}_{t+1mdc} + E[pst_{t+1md-1c}] - E[pst_{t+1mdc}]) \Big)
\end{aligned} \tag{42}$$

Let's say $E[V]$ is the addition of all parts above. Then the ADP model converted to expectation would look as follows:

$$\max_{\vec{\beta}} \Big(\sum \beta^0 + \sum \beta^{ut} E[ut] + \sum \beta^{pw} E[pw] + \sum \beta^{ps} E[ps] \Big) \tag{43}$$

Subject to:

$$E[V] \leq c(\vec{pw}, \vec{sc}, r\vec{sc}, \vec{uv}) \quad \forall \vec{S}\vec{A} \tag{44}$$

5.3 ADP Master Problem

Converting to Dual

$$\min_{w} \sum_{\vec{S}\vec{A}} w(\vec{S}\vec{A}) c(\vec{S}\vec{A}) \tag{45}$$

Subject To:

$$\beta^0 : \sum_{\vec{S}\vec{A}} \vec{w}(1-\gamma) = 1 \tag{46}$$

$$\beta^{ul} : \sum_{\vec{S}\vec{A}} \vec{w}\Big(ul_p - \gamma(\hat{ul}_p)\Big) \geq E[ul_p] \qquad \forall p \in \{\text{carry over}\} \tag{47}$$

$$\sum_{\vec{S}\vec{A}} \vec{w}(ul_p) \geq E[ul_p] \qquad \forall p \in \{\text{non carry over}\} \tag{48}$$

$$\beta^{pw} : \sum_{\vec{S}\vec{A}} \vec{w}\Big(pw_{0dc} - \gamma(pea_{dc})\Big) \geq E[ps_{0dc}] \qquad \forall dc \tag{49}$$

$$\sum_{\vec{S}\vec{A}} \vec{w}\Big(pw_{mdc} - \gamma(\hat{pw}_{m-1dc} + E[pwt_{m-1d-1c}] - E[pwt_{m-1dc}])\Big) \geq E[pw_{mdc}] \;\forall m \in \{1...M-1\}dc \tag{50}$$

$$\sum_{\vec{S}\vec{A}} \vec{w}\Big(pw_{Mdc} - \gamma \sum_{M-1}^{M} (\hat{pw}_{mdc} + E[pwt_{md-1c}] - E[pwt_{mdc}])\Big) \geq E[pw_{Mdc}] \quad \forall dc \tag{51}$$

$$\beta^{ps} : \sum_{\vec{S}\vec{A}} \vec{w}\Big(ps_{t0dc}\Big) \geq E[ps_{t0dc}] \qquad \forall tdc \tag{52}$$

$$\sum_{\vec{S}\vec{A}} \vec{w}\Big(ps_{Tmdc}\Big) \geq E[ps_{Tmdc}] \qquad \forall mdc \tag{53}$$

$$\sum_{\vec{S}\vec{A}} \vec{w}\Big(ps_{tmdc} - \gamma(\hat{ps}_{t+1m-1dc} + E[pst_{t+1m-1d-1c}] - E[pst_d])\Big) \geq E[ps_{tmdc}] \;\forall \{1...T-1\}\{1...M-1\}dc \tag{54}$$

$$\sum_{\vec{S}\vec{A}} \vec{w}\Big(ps_{tMdc} - \gamma \sum_{M-1}^{M} (\hat{ps}_{t+1mdc} + E[pst_{t+1md-1c}] - E[pst_d])\Big) \geq E[ps_{tMdc}] \;\forall \{1...T-1\}dc \tag{55}$$

$$\text{State Action Bounds:} \qquad w \geq 0 \qquad \forall w \tag{56}$$

5.4 ADP Pricing Problem

$$\min_{(\vec{ue}, \vec{uu}, \vec{pw}, \vec{pe}, \vec{ps}) \in S, (\vec{sc}, r\vec{sc}) \in A} c(\vec{pw}, \vec{sc}, r\vec{sc}, \vec{uv}) - E[V] \tag{57}$$

Subject to: constraints in sections "State-Action Constraints" and "Auxiliary Variable Definition"

6 Solution Explanation

6.1 Algorithm for solving

To get $\vec{\beta}$ values, which will be used to generate an action follow steps below:

1. Perform a monte-carlo simulation (following some arbitrary policy) to get $E[ue], E[uu], E[pw]E[pw]$

2. Create an initial feasible set of state-action pairs - $\vec{w}$
3. Solve model in section "ADP Master Problem" where each state-action pairs in $\vec{w}$ corresponds to a variable and parameters for all the constraints for a specific action.
4. Solve model in section "ADP Pricing Problem", where duals from problem in step 3 correspond to $\vec{\beta}$ values.
   - If objective function is less than 0, add solution as a single state-action pair to $\vec{w}$ and go to step 3
   - If objective function is greater than 0, continue to next step
5. Duals from problem in step 3 correspond to final $\vec{\beta}$ values

6.2 Generating an Action

Once $\vec{\beta}$ values have been approximated, you may use the model below to generate a recommended action for a specific state.

$$\min_{\vec{A}} c(\vec{S}, \vec{A}) + \gamma v(S')$$

$$\begin{aligned} v(S') = \Bigg(& \beta^0 + \Big(\sum_p \beta^{ue}_{1p}(uen_d + ue_{1p} - \hat{uu}_{1p} + uv_{1p}) \Big) + \Big(\sum_{t=2,p}^{T} \beta^{ue}_{tp} uen_d \Big) + \\ & \Big(\sum_{tp}^{T-1} \beta^{uu}_{tp}(\hat{uu}_{t+1,p} + \sum_{mdc} E[pst_{tmdc}](U_{pd+1c} - U_{pdc})) \Big) + \\ & \Big(\sum_{dc} \beta^{pw}_{0dc} pea_{dc} \Big) + \Big(\sum_{m=1,dc}^{M-1} \beta^{pw}_{mdc}(\hat{pw}_{m-1dc} + E[pwt_{m-1d-1c}] - E[pwt_{m-1dc}]) \Big) + \\ & \Big(\sum_{dc} \beta^{pw}_{Mdc}(\sum_{M-1}^{M} \hat{pw}_{mdc} + E[pwt_{md-1c}] - E[pwt_{mdc}]) \Big) + \\ & \Big(\sum_{t=1m=1dc}^{T-1M-1} \beta^{ps}_{tmdc}(\hat{ps}_{t+1m-1dc} + E[pst_{t+1m-1d-1c}] - E[pst_{t+1m-1dc}]) \Big) + \\ & \Big(\sum_{t=1Mdc}^{T-1} \beta^{ps}_{tmdc}(\sum_{M-1}^{M} (\hat{ps}_{t+1mdc} + E[pst_{t+1md-1c}] - E[pst_{t+1mdc}])) \Big) \Bigg) \end{aligned} \tag{58}$$

Subject to: constraints in section "State-Action Constraints"