Dynamic Knapsack Problem

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Contents

1 Questions/Comments

1.1 Questions

- Priority need to be added

1.2 Comments

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2 Assumptions

- Assume all new patients are arriving at the beginning of the period
  - Assume arrivals follow poisson distribution
- It is assumed that there is no difference between how long a patient waits for their appointment within some limit (TL).
  - Let's assume our limit is 10 periods, there is no difference in terms of cost between patients who has been waiting for 1 period and patients waiting 10 period for their appointment (assuming they are the same category).
  - However, if a patient waits for 11 period for their appointment there should be some kind of penalty for the wait
  - The penalty comes from increasing patient complexity. Higher complexities require more resources and are thus more costly overall in the system
- We assume patient complexity transitions follow binary distribution - each period a patient can become more complex with a certain probability
  - While patient is waiting less that TL, the transition probability is 0.
  - After TL, transition probability is some arbitrary value
- It is assumed if a patient is scheduled in period one, it means they are served "immediately", regardless if in practice the appointment is at the beginning or end of the period.
- We distinguish between two types of reschedules: good and bad reschedules
  - Good reschedules are reschedules where a patient is rescheduled to an earlier period
  - Bad reschedules are reschedules where a patient is rescheduled to a later period
- We assume that only specific reschedules are allowed (to simplify the model, and remove redundancies)
  - Good reschedules are allowed from any period after 2 into period 1
  - Bad reschedules are only allowed from period 1 to period 2
- It is assumed that there is a certain default expected number of PPE units available for all periods
  - However, in the period 1, there is some random deviation from the expected number of units
  - This random deviation follows some uniform distribution
- We allow some violation of PPE units, in rare cases (with high cost), in order to accomodate changes due to variability.
  - If we have violation of PPE units - it reduces capacity in the future (Meaning violations are borrowed from future capacity)

3 MDP Model

3.1 Decision Epochs

Decisions are made at the beginning of each time period. There are 3 relevant time intervals to look at:

- Pre-decision state (S)
  - Pre-decision state defines the initial state on which a decision is required
  - This information drives decision making
- Post-decision state ($\hat{S}$)
  - Post-decision state is the time when the decision has been executed, but no new info came in
  - This state defines the immediate cost of an action
- Post-transition state (S')
  - This is the state when new information has come in (transition randomness). Defines next pre decision state
  - It is primarily used to generate expectation for the ADP

3.2 State Space

3.2.1 Description

State is defined by current available and used resources, patient waitlist/demand, and patients already scheduled

$$\vec{S} = (\vec{ue}, \vec{uu}, \vec{pw}, \vec{ps})$$

- $\vec{ue} = ue_{tp}$ - Expected unist for period t, and resource p
- $\vec{uu} = uu_{tp}$ - Used units for period t, and resource p
- $\vec{pw} = pw_{mdc}$ - Patients of complexity d, CPU c, on a wait list for m periods (m of 0 - just arrived)
- $\vec{ps} = ps_{tmdc}$ - Patients of complexity d, CPU c, scheduled to period t, waiting for m periods (m of 0 - just arrived)

3.3 Action Sets

3.3.1 Description

Decision consists of rescheduling currently scheduled patients, and scheduling patients on waitlist. There are also some goal and auxiliary variables

$$\vec{A} = (\vec{sc}, r\vec{sc}, \vec{uv}, \text{auxiliary variables})$$

- $\vec{sc} = sc_{tmdc}$ - Patients of complexity d, CPU c, waiting for m periods, to schedule in period t
- $r\vec{sc} = rsc_{tt'mdc}$ -Patients of complexity d, CPU c, waiting for m periods, to reschedule from period t to period t'
- $\vec{uv} = uv_{tp}$ - goal variable, violation on number of resources used for period t, of resource p

3.3.2 Auxiliary Variables

- $\hat{uu}_{tp}$ - post-decision units used
- $\hat{pw}_{mdc}$ - post-decision patients waiting
- $\hat{ps}_{tmdc}$ - post-decision patients scheduled

3.3.3 Auxiliary Variable Definition

$$\hat{uu}_{tp} = \sum_{mdc} U_{pdc} \hat{ps}_{tmdc} \qquad \forall tp \tag{1}$$

$$\hat{pw}_{mdc} = pw_{mdc} - \sum_{t} sc_{tmdc} \qquad \forall mdc \tag{2}$$

$$\hat{ps}_{tmdc} = ps_{tmdc} + sc_{tmdc} + \sum_{t} rsc_{tt'mdc} - \sum_{t'} rsc_{tt'mdc} \qquad \forall tmdc \tag{3}$$

3.3.4 State-Action Constraints

- Consistency Constraint

$$uu_{tp} = \sum_{mdc} U_{pdc} ps_{tmdc} \quad \forall tp \tag{4}$$

- Resource Usage Constraint

$$\hat{uu}_{tp} \leq ue_{tp} + uv_{tp} \quad \forall tp \tag{5}$$

- Custom bounds on when reschedules are allowed

$$rsc_{tt'mdc} = 0 \quad \forall t \in \{2...T\}, t' \in \{2...T\} mdc \tag{6}$$
$$rsc_{tt'mdc} = 0 \quad \forall t \in \{1\}, t' \in \{3...T\} mdc \tag{7}$$
$$rsc_{tt'mdc} = 0 \quad \forall tt'mdc, \text{where } t = t' \tag{8}$$

- Bounds on Schedules/Reschedules

$$\sum_{t'} rsc_{tt'mdc} \leq ps_{tmdc} \quad \forall tmdc \tag{9}$$
$$\sum_{t} sc_{tmdc} \leq pw_{mdc} \quad \forall mdc \tag{10}$$

- Bounds on states

$$ue_{tp} \leq uen_p + max(ued_p) \quad \forall tp \tag{11}$$
$$uu_{tp} \leq uen_p + max(ued_p) \quad \forall tp \tag{12}$$
$$pw_{mdc} \leq pea_{dc} * 4 \quad \forall mdc \tag{13}$$
$$ps_{tmdc} \leq pea_{dc} * 4 \quad \forall tmdc \tag{14}$$

3.4 Transition Probabilities

3.4.1 Uncertainty Sources

1. Number of patients arriving this period - pw_{0dc}
   - pea_{dc} - represents the random number of patients arriving. Follows a poisson distribution.
2. Transition between patient complexities within the wait list - $pw_{m>=1,dc}$
   - pwt_{mdc} represents the random number of patients of complexity d, CPU c, waiting for m period, who became more complex. Follows binomial distribution.
   - ptp_{mdc} represents transition probability to a higher complexity
3. Transition between patient complexities within the scheduled list - ps_{tmdc}
   - pst_{tmdc} represents the random number of patients of complexity d, CPU c, waiting for m period, scheduled into t period, who became more complex. Follows binomial distribution.
4. Amout of expected units of PPE resource for the next time period - ue_{1p}
   - ued_p represents the random deviation of resource p from the expectation. Follows uniform distribution.
   - uen_p expected units of resource p per period

3.4.2 Transition Constraints

1. Transition from $\vec{ue}$ to $\vec{ue'}$ - Expected Resources

$$ue'_{1p} = \overbrace{ue_{2p} + ued_p}^{\text{deviation}} + \overbrace{ue_{1p} - \hat{u}u_{1p}}^{\text{unused}} \quad \forall p \in \{\text{Carry Over Resources}\} \tag{15}$$
$$ue'_{1p} = ue_{2p} + ued_p \quad \forall p \in \{\text{Non Carry Over Resources}\} \tag{16}$$
$$ue'_{tp} = ue_{t+1,p} \quad \forall t \in \{2...T-1\} p \tag{17}$$
$$ue'_{Tp} = uen_p \quad \forall p \tag{18}$$

2. Transition from $\vec{uu}$ to $\vec{uu'}$ - Used Resources

$$
\begin{aligned}
uu'_{tp} &= \hat{uu}_{t+1,p} + \overbrace{\sum_{mdc} pst_{tmdc}(U_{pd+1c} - U_{pdc})}^{\text{change in complexities}} && \forall t \in \{1...T-1\} && (19)\\
uu'_{Tp} &= 0 && \forall p && (20)
\end{aligned}
$$

3. Transition from $\vec{pw}$ to $\vec{pw'}$ - Flow of patients on waitlist

$$
\begin{aligned}
pw'_{0dc} &= pea_{dc} && \forall dc && (21)\\
pw'_{mdc} &= \hat{pw}_{m-1,dc} + \overbrace{pwt_{m-1,d-1,c} - pwt_{m-1,dc}}^{\text{change in complexities}} && \forall m \in \{1...M-1\}dc && (22)\\
pw'_{Mdc} &= \sum_{M-1}^{M} \big(\hat{pw}_{mdc} + \overbrace{pwt_{md-1,c} - pwt_{mdc}}^{\text{change in complexities}}\big) && \forall dc && (23)
\end{aligned}
$$

4. Transition from $\vec{ps}$ to $\vec{ps'}$ - Flow of patiensts scheduled

$$
\begin{aligned}
ps'_{t0dc} &= 0 && \forall tdc && (24)\\
ps'_{Tmdc} &= 0 && \forall mdc && (25)\\
ps'_{tmdc} &= \hat{ps}_{t+1,m-1,dc} + \overbrace{pst_{t+1,m-1,d-1,c} - pst_{t+1,m-1,dc}}^{\text{change in complexities}} && \forall t \in \{1...T-1\} m \in \{1...M-1\}dc && (26)\\
ps'_{tMdc} &= \sum_{M-1}^{M} \big(\hat{ps}_{t+1mdc} + \overbrace{pst_{t+1,md-1,c} - pst_{t+1,mdc}}^{\text{change in complexities}}\big) && \forall t \in \{1...T-1\}dc && (27)
\end{aligned}
$$

3.5 Costs

Cost will come from 3 sources:

- waiting (cw)
- rescheduling ($cc = \lceil cw^M \rceil$)
- resource violation (M)

$$
\begin{aligned}
C = c(\vec{pw}, \vec{ps}, \vec{rsc}, \vec{uv}) = &\overbrace{\sum_{mdc} cw^m(\hat{pw}_{mdc})}^{\text{cost of waiting}} + \overbrace{\sum_{tdc} cw^M(\hat{ps}_{tMdc})}^{\text{prevent infinite reschedules}} + \\
&+ \overbrace{cc \sum_{\substack{tt'mdc \\ t'>t}} rsc_{tt'mdc}}^{\text{Bad Reschedule}} - \overbrace{cc \sum_{\substack{tt'mdc \\ t'<t}} rsc_{tt'mdc}}^{\text{Good Reschedule}} + M \sum_{tp} uv_{tp}
\end{aligned} \tag{28}
$$

4 LP Model

4.1 Full LP

Given a full MDP model, the equivalent LP would look as follows:

$$
\max_{\vec{v}} \sum \alpha(\vec{S}) v(\vec{S}) \tag{29}
$$

subject to

$$
c(\vec{pw}, \vec{ps}, \vec{rsc}, \vec{uv}) + \gamma \sum_{\vec{p}} p(\vec{pea}, \vec{pwt}, \vec{pst}, \vec{ued}) v(\vec{S'} | \vec{S}, \vec{A}, \vec{pea}, \vec{pwt}, \vec{pst}, \vec{ued}) \geq v(\vec{S}) \quad \forall \vec{S}\vec{A} \tag{30}
$$

5 Aproximate Dynamic Programming Model

5.1 Full ADP Model

Let's convert it into ADP. We do that by changing $v(\vec{S})$ to an approximation as follows:

$$\begin{aligned} v(\vec{ue}, \vec{uu}, \vec{pw}, \vec{ps}) = \beta^0 + \sum_{tp} \beta^{ue}_{tp} ue_{tp} + \sum_{tp} \beta^{uu}_{tp} uu_{tp} + \\ + \sum_{mdc} \beta^{pw}_{mdc} pw_{mdc} + \sum_{tmdc} \beta^{ps}_{tmdc} ps_{tmdc} \end{aligned} \tag{31}$$

This gives the following LP

$$\max_{\vec{\beta}} \sum \alpha(\vec{ue}, \vec{uu}, \vec{pw}, \vec{pe}, \vec{ps}) v(\vec{ue}, \vec{uu}, \vec{pw}, \vec{pe}, \vec{ps}) \tag{32}$$

Subject to:

$$\begin{aligned} & c(\vec{p\hat{w}}, \vec{p\hat{s}}, \vec{rsc}, \vec{uv}) + \gamma \sum_{\vec{p}} p \Bigg(\beta^0 + \\ & \left(\sum_p \beta^{ue}_{1p} (ue_{2p} + ued_p + ue_{1p} - \hat{u}u_{1p}) \right) + \left(\sum_{t=2,p}^{T-1} \beta^{ue}_{tp} ue_{t+1,p} \right) + \left(\sum_p \beta^{ue}_{Tp} uen_p \right) + \\ & \left(\sum_{tp}^{T-1} \beta^{uu}_{tp} (\hat{u}u_{t+1,p} + \sum_{mdc} pst_{tmdc} (U_{pd+1c} - U_{pdc})) \right) + \left(\sum_p \beta^{uu}_{Tp} * 0 \right) + \\ & \left(\sum_{dc} \beta^{pw}_{0dc} pea_{dc} \right) + \left(\sum_{m=1,dc}^{M-1} \beta^{pw}_{mdc} (\hat{pw}_{m-1dc} + pwt_{m-1d-1c} - pwt_{m-1dc}) \right) + \\ & \qquad \left(\sum_{dc} \beta^{pw}_{Mdc} (\sum_{M-1}^{M} \hat{pw}_{mdc} + pwt_{md-1c} - pwt_{mdc}) \right) + \\ & \left(\sum_{tdc} \beta^{ps}_{t0dc} 0 \right) + \left(\sum_{mdc} \beta^{ps}_{Tmdc} 0 \right) + \left(\sum_{t=1m=1dc}^{T-1M-1} \beta^{ps}_{tmdc} (\hat{ps}_{t+1m-1dc} + pst_{t+1m-1d-1c} - pst_{t+1m-1dc}) \right) + \\ & \left(\sum_{t=1Mdc}^{T-1} \beta^{ps}_{tmdc} (\sum_{M-1}^{M} (\hat{ps}_{t+1mdc} + pst_{t+1md-1c} - pst_{t+1mdc})) \right) \Bigg) \geq v(\vec{ue}, \vec{pw}, \vec{ps}) \quad \forall \vec{S}\vec{A} \end{aligned} \tag{33}$$

$$\vec{\beta} \geq 0 \quad \forall \beta \tag{34}$$

And all auxiliary constraints in "Auxiliary Variable Definition"

5.2 Expectation ADP Model

Steps for rearranging and converting the constraint (33) into expectation:

- $E[ued_p] = 0$
- $E[pwt_{mdc}] = ptp_{mdc} * \hat{pw}_{mdc}$
- $E[pst_{tmdc}] = ptp_{mdc} * \hat{ps}_{tmdc}$

$$\beta^0 : \qquad (1-\gamma)\beta^0 \tag{35}$$

$$\beta^{ue}_{tp} : \qquad \sum_p \beta^{ue}_{1p} \Big(ue_{1p} - \gamma(uen_p + ue_{1p} - \hat{u}u_{1p}) \Big) + \sum_{t=2p}^{T} \beta^{ue}_{tp} \Big(ue_{tp} - \gamma(uen_p) \Big) \tag{36}$$

$$\beta^{uu}_{tp} : \qquad \sum_{tp}^{T-1} \beta^{uu}_{tp} \Big(uu_{tp} - \gamma(\hat{u}u_{t+1p} + \sum_{mdc} E[pst_{t+1mdc}](U_{pd+1c} - U_{pdc})) \Big) + \sum_p \beta^{uu}_{Tp} \Big(uu_{Tp} \Big) \tag{37}$$

$$\beta^{pw}_{mdc}: \quad \sum_{dc}\beta^{pw}_{0dc}\Big(pw_{0dc}-\gamma(pea_{dc})\Big)+\sum_{m=1dc}^{M-1}\beta^{pw}_{mdc}\Big(pw_{mdc}-\gamma(\hat{pw}_{m-1dc}+E[pwt_{m-1d-1c}]-E[pwt_{m-1dc}])\Big)+$$
$$\sum_{dc}\beta^{pw}_{Mdc}\Big(pw_{Mdc}-\gamma\sum_{M-1}^{M}(\hat{pw}_{mdc}+E[pwt_{md-1c}]-E[pwt_{mdc}])\Big) \tag{38}$$

$$\beta^{ps}_{tmdc}: \quad \sum_{tdc}\beta^{ps}_{t0dc}\Big(ps_{t0dc}\Big)+\sum_{mdc}\beta^{ps}_{Tmdc}\Big(ps_{Tmdc}\Big)+$$
$$\sum_{t=1m=1dc}^{T-1M-1}\beta^{ps}_{tmdc}\Big(ps_{tmdc}-\gamma(\hat{ps}_{t+1m-1dc}+E[pst_{t+1m-1d-1c}]-E[pst_{t+1m-1dc}])\Big)+ \tag{39}$$
$$\sum_{t=1dc}^{T-1}\beta^{ps}_{tMdc}\Big(ps_{tMdc}-\gamma\sum_{M-1}^{M}(\hat{ps}_{t+1mdc}+E[pst_{t+1md-1c}]-E[pst_{t+1mdc}])\Big)$$

Let's say $E[V]$ is the addition of all parts above. Then the ADP model converted to expectation would look as follows:

$$\max_{\vec{\beta}}\Big(\sum\beta^0+\sum\beta^{ue}E[ue]+\sum\beta^{uu}E[uu]+\sum\beta^{uv}E[uv]+\sum\beta^{pw}E[pw]+\sum\beta^{ps}E[ps]\Big) \tag{40}$$

Subject to:

$$E[V]\le c(\vec{pw},\vec{sc},\vec{rsc},\vec{uv})\quad\forall\vec{S}\vec{A} \tag{41}$$

5.3 ADP Master Problem

Converting to Dual

$$\min_{w}\sum_{\vec{S}\vec{A}}w(\vec{S}\vec{A})c(\vec{S}\vec{A}) \tag{42}$$

Subject To:

$$\beta^0: \sum_{\vec{S}\vec{A}}\vec{w}(1-\gamma)=1 \tag{43}$$

$$\beta^{ue}: \sum_{\vec{S}\vec{A}}\vec{w}\Big(ue_{1p}-\gamma(uen_p+ue_{1p}-\hat{uu}_{1p})\Big)\ge E[ue_{tp}] \qquad \forall p \tag{44}$$

$$\sum_{\vec{S}\vec{A}}\vec{w}\Big(ue_{tp}-\gamma(uen_p)\Big)\ge E[ue_{tp}] \qquad \forall t\in(2...T)p \tag{45}$$

$$\beta^{uu}: \sum_{\vec{S}\vec{A}}\vec{w}\Big(uu_{tp}-\gamma(\hat{uu}_{t+1p}+\sum_{mdc}E[pst_{t+1mdc}](U_{pd+1c}-U_{pdc}))\Big)\ge E[uu_{tp}] \qquad \forall t\in\{1...T-1\}p \tag{46}$$

$$\sum_{\vec{S}\vec{A}}\vec{w}\Big(uu_{Tp}\Big)\ge E[uu_{Tp}] \qquad \forall p \tag{47}$$

$$\beta^{pw}: \sum_{\vec{S}\vec{A}}\vec{w}\Big(pw_{0dc}-\gamma(pea_{dc})\Big)\ge E[ps_{0dc}] \qquad \forall dc \tag{48}$$

$$\sum_{\vec{S}\vec{A}}\vec{w}\Big(pw_{mdc}-\gamma(\hat{pw}_{m-1dc}+E[pwt_{m-1d-1c}]-E[pwt_{m-1dc}])\Big)\ge E[pw_{mdc}] \quad \forall m\in\{1...M-1\}dc \tag{49}$$

$$\sum_{\vec{S}\vec{A}}\vec{w}\Big(pw_{Mdc}-\gamma\sum_{M-1}^{M}(\hat{pw}_{mdc}+E[pwt_{md-1c}]-E[pwt_{mdc}])\Big)\ge E[pw_{Mdc}] \quad \forall dc \tag{50}$$

$$\beta^{ps}: \sum_{\vec{S}\vec{A}}\vec{w}\Big(ps_{t0dc}\Big)\ge E[ps_{t0dc}] \qquad \forall tdc \tag{51}$$

$$\sum_{\vec{S}\vec{A}}\vec{w}\Big(ps_{Tmdc}\Big)\ge E[ps_{Tmdc}] \qquad \forall mdc \tag{52}$$

$$\sum_{\vec{S}\vec{A}}\vec{w}\Big(ps_{tmdc}-\gamma(\hat{ps}_{t+1m-1dc}+E[pst_{t+1m-1d-1c}]-E[pst_d])\Big)\ge E[ps_{tmdc}] \quad \forall\{1...T-1\}\{1...M-1\}dc \tag{53}$$

$$\sum_{\vec{S}\vec{A}} \vec{w}\Big(ps_{tMdc} - \gamma \sum_{M-1}^{M} (\hat{ps}_{t+1mdc} + E[pst_{t+1md-1c}] - E[pst_d])\Big) \geq E[ps_{tMdc}] \ \forall\{1...T-1\}dc \tag{54}$$

$$\text{State Action Bounds:} \qquad w \geq 0 \qquad \forall w \tag{55}$$

5.4 ADP Pricing Problem

$$\min_{(\vec{ue},\vec{uu},\vec{pw},\vec{pe},\vec{ps})\in S,(\vec{sc},r\vec{sc})\in A} c(\vec{pw},\vec{sc},r\vec{sc},\vec{uv}) - E[V] \tag{56}$$

Subject to: constraints in sections "State-Action Constraints" and "Auxiliary Variable Definition"

6 Solution Explanation

6.1 Algorithm for solving

To get $\vec{\beta}$ values, which will be used to generate an action follow steps below:

1. Perform a monte-carlo simulation (following some arbitrary policy) to get $E[ue], E[uu], E[pw]E[pw]$
2. Create an initial feasible set of state-action pairs - $\vec{w}$
3. Solve model in section "ADP Master Problem" where each state-action pairs in $\vec{w}$ corresponds to a variable and parameters for all the constraints for a specific action.
4. Solve model in section "ADP Pricing Problem", where duals from problem in step 3 correspond to $\vec{\beta}$ values.
    - If objective function is less than 0, add solution as a single state-action pair to $\vec{w}$ and go to step 3
    - If objective function is greater than 0, continue to next step
5. Duals from problem in step 3 correspond to final $\vec{\beta}$ values

6.2 Generating an Action

Once $\vec{\beta}$ values have been approximated, you may use the model below to generate a recommended action for a specific state.

$$\min_{\vec{A}} c(\vec{S},\vec{A}) - \gamma v(S')$$

$$\begin{aligned}
v(S') = \Bigg(\beta^0 &+ \Big(\sum_{p} \beta^{ue}_{1p}(uen_d + ue_{1p} - \hat{uu}_{1p})\Big) + \Big(\sum_{t=2,p}^{T} \beta^{ue}_{tp} uen_d\Big) + \\
&\Big(\sum_{tp}^{T-1} \beta^{uu}_{tp}(\hat{uu}_{t+1,p} + \sum_{mdc} E[pst_{tmdc}](U_{pd+1c} - U_{pdc}))\Big) + \\
&\Big(\sum_{dc} \beta^{pw}_{0dc} pea_{dc}\Big) + \Big(\sum_{m=1,dc}^{M-1} \beta^{pw}_{mdc}(\hat{pw}_{m-1dc} + E[pwt_{m-1d-1c}] - E[pwt_{m-1dc}])\Big) + \\
&\Big(\sum_{dc} \beta^{pw}_{Mdc}(\sum_{M-1}^{M} \hat{pw}_{mdc} + E[pwt_{md-1c}] - E[pwt_{mdc}])\Big) + \\
&\Big(\sum_{t=1m=1dc}^{T-1M-1} \beta^{ps}_{tmdc}(\hat{ps}_{t+1m-1dc} + E[pst_{t+1m-1d-1c}] - E[pst_{t+1m-1dc}])\Big) + \\
&\Big(\sum_{t=1Mdc}^{T-1} \beta^{ps}_{tmdc}(\sum_{M-1}^{M}(\hat{ps}_{t+1mdc} + E[pst_{t+1md-1c}] - E[pst_{t+1mdc}]))\Big)\Bigg)
\end{aligned} \tag{57}$$

Subject to: constraints in section "State-Action Constraints"