

# Dynamic Knapsack Problem

AS, WM, KV

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# 1 Questions/Comments

## 1.1 Questions

- Should *ul* definition include units violated?? it seems a constraint might be violated otherwise

## 1.2 Comments

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# 2 Assumptions

- Assume all new patients are arriving at the beginning of the period
  - Assume arrivals follow poisson distribution
- It is assumed that patients do not change complexities within some limit ( $TL$ ).
  - Let's assume our limit is 10 periods, patients will never change complexities within the 10 periods.
  - However, if a patient waits for 11 periods there is a possibility they increase complexity
- We assume patient complexity transitions follow binary distribution - each period a patient can become more complex with a certain probability
  - While patient is waiting less than  $TL$ , the transition probability is 0.
  - After  $TL$ , transition probability is some arbitrary value
- It is assumed if a patient is scheduled in period one, it means they are served "immediately", regardless if in practice the appointment is at the beginning or end of the period.
- We distinguish between two types of reschedules: good and bad reschedules
  - Good reschedules are reschedules where a patient is rescheduled to an earlier period
  - Bad reschedules are reschedules where a patient is rescheduled to a later period
- We assume that only specific reschedules are allowed (to simplify the model, and remove redundancies)
  - Good reschedules are allowed from any period after 2 into period 1
  - Bad reschedules are only allowed from period 1 to any period after 2
- It is assumed that there is a certain default expected number of PPE units available for all periods
  - However, in the period 1, there is some random deviation from the expected number of units
  - This random deviation follows some uniform distribution
- We allow some violation of PPE units, in rare cases (with high cost), in order to accommodate changes due to variability.
  - If we have violation of PPE units - the extra capacity comes from external source

# 3 MDP Model

## 3.1 Decision Epochs

Decisions are made at the beginning of each time period. There are 3 relevant time intervals to look at:

- Pre-decision state ( $S$ )
  - Pre-decision state defines the initial state on which a decision is required
  - This information drives decision making
- Post-decision state ( $\hat{S}$ )

- Post-decision state is the time when the decision has been executed, but no new info came in
- This state defines the immediate cost of an action
- Post-transition state ( $S'$ )
  - This is the state when new information has come in (transition randomness). Defines next pre decision state
  - It is primarily used to generate expectation for the ADP

## 3.2 State Space

### 3.2.1 Description

State is defined by current available and used resources, patient waitlist/demand, and patients already scheduled

$$\vec{S} = (\vec{ul}, p\vec{w}, p\vec{s})$$

- $\vec{ul} = ul_p$  - Units left over from previous period for resource  $p$  plus deviation for that unit for period 1. This is only for resources that can be carried over
- $p\vec{w} = pw_{mdc}$  - Patients of complexity  $d$ , CPU  $c$ , on a wait list for  $m$  periods ( $m$  of 0 - just arrived)
- $p\vec{s} = ps_{tmdc}$  - Patients of complexity  $d$ , CPU  $c$ , scheduled to period  $t$ , waiting for  $m$  periods ( $m$  of 0 - just arrived)

## 3.3 Action Sets

### 3.3.1 Description

Decision consists of rescheduling currently scheduled patients, and scheduling patients on waitlist. There are also some goal and auxiliary variables

$$\vec{A} = (\vec{sc}, r\vec{sc}, \vec{uv}, \text{auxiliary variables})$$

- $\vec{sc} = sc_{tmdc}$  - Patients of complexity  $d$ , CPU  $c$ , waiting for  $m$  periods, to schedule in period  $t$
- $r\vec{sc} = rsc_{tt'mdc}$  - Patients of complexity  $d$ , CPU  $c$ , waiting for  $m$  periods, to reschedule from period  $t$  to period  $t'$
- $\vec{uv} = uv_{tp}$  - goal variable, violation on number of resources used for period  $t$ , of resource  $p$

### 3.3.2 Auxiliary Variables

- $uvb_{tp}$  - binary variable to enforce  $uv$  variable without objective function
- $\hat{ul}_p$  - post-decision unit leftover at period 1
- $ulb_p$  - binary variable to enforce  $ul$  variable without objective function
- $\hat{u}u_{tp}$  - post-decision units used
- $\hat{p}w_{mdc}$  - post-decision patients waiting
- $\hat{p}s_{tmdc}$  - post-decision patients scheduled

### 3.3.3 Auxiliary Variable Definition

$$\hat{u}u_{tp} = \sum_{mdc} U_{pdc} \hat{p}s_{tmdc} \quad \forall tp \quad (1)$$

$$\hat{p}w_{mdc} = pw_{mdc} - \sum_t sc_{tmdc} \quad \forall mdc \quad (2)$$

$$\hat{p}s_{tmdc} = ps_{tmdc} + sc_{tmdc} + \sum_t rsc_{tt'mdc} - \sum_{t'} rsc_{tt'mdc} \quad \forall tmdc \quad (3)$$

- Define Resource Violation Variable

$$\hat{u}v_{tp} \leq M(uvb_{tp}) \quad \forall tp \quad (4)$$

$$\hat{u}v_{1p} \leq (\hat{u}u_{1p} - uen_{1p} - ul_p) + M(1 - uvb_{tp}) \quad \forall p \quad (5)$$

$$\hat{u}v_{tp} \leq (\hat{u}u_{tp} - uen_{tp}) + M(1 - uvb_{tp}) \quad \forall t \in \{2..T\}p \quad (6)$$

$$(7)$$

- Define Units Left Over Variable

$$\hat{u}l_p \geq 0 \quad \forall p \quad (8)$$

$$\hat{u}l_p \geq uen_{1p} + ul_p - \hat{u}u_{1p} \quad \forall p \quad (9)$$

$$\hat{u}l_p \leq M(ulb_p) \quad \forall p \quad (10)$$

$$\hat{u}l_p \leq (uen_{1p} + ul_p - \hat{u}u_{1p}) + M(1 - ulb_p) \quad \forall p \quad (11)$$

### 3.3.4 State-Action Constraints

- Resource Usage Constraint

$$\hat{u}u_{1p} \leq uen_{1p} + ul_p + uv_{1p} \quad \forall p \quad (12)$$

$$\hat{u}u_{tp} \leq uen_{tp} + uv_{tp} \quad \forall t \in \{2..T\}p \quad (13)$$

- Custom bounds on when reschedules are allowed

$$rsc_{tt'mdc} = 0 \quad \forall t \in \{2..T\}, t' \in \{2..T\}mdc \quad (14)$$

$$rsc_{tt'mdc} = 0 \quad \forall tt'mdc, \text{ where } t = t' = 1 \quad (15)$$

- Bounds on Schedules/Reschedules

$$\sum_{t'} rsc_{tt'mdc} \leq pst_{mdc} \quad \forall tmdc \quad (16)$$

$$\sum_t sc_{tmdc} \leq pw_{mdc} \quad \forall mdc \quad (17)$$

- Bounds on states

$$ul_p \leq uen_p * 3 \quad \forall p \quad (18)$$

$$pw_{mdc} \leq pea_{dc} * 20 \quad \forall mdc \quad (19)$$

$$pst_{mdc} \leq pea_{dc} * 4 \quad \forall tmdc \quad (20)$$

## 3.4 Transition Probabilities

### 3.4.1 Uncertainty Sources

1. Number of patients arriving this period -  $pw_{0dc}$

- $pea_{dc}$  - represents the random number of patients arriving. Follows a poisson distribution.

2. Transition between patient complexities within the wait list -  $pw_{m \geq 1, dc}$

- $pwt_{mdc}$  represents the random number of patients of complexity  $d$ , CPU  $c$ , waiting for  $m$  period, who became more complex. Follows binomial distribution.
- $ptp_{mdc}$  represents transition probability to a higher complexity

3. Transition between patient complexities within the scheduled list -  $pst_{mdc}$

- $pst_{mdc}$  represents the random number of patients of complexity  $d$ , CPU  $c$ , waiting for  $m$  period, scheduled into  $t$  period, who became more complex. Follows binomial distribution.

4. Amount of expected units of PPE resource for the next time period -  $ue_{1p}$

- $ued_p$  represents the random deviation of resource  $p$  from the expectation. Follows uniform distribution.
- $uen_p$  expected units of resource  $p$  per period

### 3.4.2 Transition Constraints

1. Transition from  $u\vec{e}$  to  $u\vec{e}'$  - Resource Carry Over

$$ul'_p = \hat{ul}_p + ued_p \quad \forall p \in \{\text{Carry Over}\} \quad (21)$$

$$ul'_p = ued_p \quad \forall p \in \{\text{Non Carry Over}\} \quad (22)$$

2. Transition from  $p\vec{w}$  to  $p\vec{w}'$  - Flow of patients on waitlist

$$pw'_{0dc} = pea_{dc} \quad \forall dc \quad (23)$$

$$pw'_{mdc} = \hat{pw}_{m-1,dc} + \overbrace{pwt_{m-1,d-1,c} - pwt_{m-1,dc}}^{\text{change in complexities}} \quad \forall m \in \{1 \dots M-1\}dc \quad (24)$$

$$pw'_{Mdc} = \sum_{M-1}^M (\hat{pw}_{mdc} + \overbrace{pwt_{md-1,c} - pwt_{mdc}}^{\text{change in complexities}}) \quad \forall dc \quad (25)$$

3. Transition from  $p\vec{s}$  to  $p\vec{s}'$  - Flow of patients scheduled

$$ps'_{t0dc} = 0 \quad \forall tdc \quad (26)$$

$$ps'_{Tmdc} = 0 \quad \forall mdc \quad (27)$$

$$ps'_{tmdc} = \hat{ps}_{t+1,m-1,dc} + \overbrace{pst_{t+1,m-1,d-1,c} - pst_{t+1,m-1,dc}}^{\text{change in complexities}} \quad \forall t \in \{1 \dots T-1\}m \in \{1 \dots M-1\}dc \quad (28)$$

$$ps'_{tMdc} = \sum_{M-1}^M (\hat{ps}_{t+1mdc} + \overbrace{pst_{t+1,md-1,c} - pst_{t+1,mdc}}^{\text{change in complexities}}) \quad \forall t \in \{1 \dots T-1\}dc \quad (29)$$

### 3.5 Costs

Cost will come from 4 sources:

- waiting ( $cw$ )
- rescheduling ( $cc$ )
- scheduling ( $cs_t = \sum_t cw^t$ )
- resource violation ( $M$ )

$$\begin{aligned} C = c(p\vec{w}, p\vec{s}, r\vec{sc}, u\vec{v}) &= \overbrace{\sum_{mdc} cw(pw_{mdc})}^{\text{cost of waiting}} + \overbrace{\sum_{tmdc} cs_t(sc_{tmdc})}^{\text{Prefer earlier appointments}} \\ &+ \overbrace{\sum_{\substack{tt'mdc \\ t' > t}} (cs_{t'-t} + cc) * (rsc_{tt'mdc})}^{\text{Bad Reschedule}} - \overbrace{\sum_{\substack{tt'mdc \\ t' < t}} (cs_{t-t'} - cc) * (rsc_{tt'mdc})}^{\text{Good Reschedule}} + M \sum_{tp} uv_{tp} \end{aligned} \quad (30)$$

## 4 LP Model

### 4.1 Full LP

Given a full MDP model, the equivalent LP would look as follows:

$$\max_{\vec{v}} \sum \alpha(\vec{S}) v(\vec{S}) \quad (31)$$

subject to

$$c(p\vec{w}, p\vec{s}, r\vec{sc}, u\vec{v}) + \gamma \sum_{\vec{p}} p(p\vec{ea}, p\vec{wt}, p\vec{st}, u\vec{ed}) v(\vec{S}' | \vec{S}, \vec{A}, p\vec{ea}, p\vec{wt}, p\vec{st}, u\vec{ed}) \geq v(\vec{S}) \quad \forall \vec{S}, \vec{A} \quad (32)$$

## 5 Aproximate Dynamic Programming Model

### 5.1 Full ADP Model

Let's convert it into ADP. We do that by changing  $v(\vec{S})$  to an approximation as follows:

$$v(\vec{ul}, \vec{pw}, \vec{ps}) = \beta^0 + \sum_p \beta_p^{ul} ul_p + \sum_{mdc} \beta_{mdc}^{pw} pw_{mdc} + \sum_{tmdc} \beta_{tmdc}^{ps} ps_{tmdc} \quad (33)$$

This gives the following LP

$$\max_{\vec{\beta}} \sum \alpha(\vec{ul}, \vec{pw}, \vec{ps}) v(\vec{ul}, \vec{pw}, \vec{ps}) \quad (34)$$

Subject to:

$$\begin{aligned} & c(\vec{pw}, \vec{ps}, r\vec{sc}, u\vec{v}) + \gamma \sum_{\vec{p}} p \left( \beta^0 + \left( \beta_{p \in \text{co}}^{ul} (\hat{ul}_p + uen_d) \right) + \left( \beta_{p \in \text{nco}}^{ul} (uen_d) \right) + \right. \\ & \left( \sum_{dc} \beta_{0dc}^{pw} pea_{dc} \right) + \left( \sum_{m=1, dc}^{M-1} \beta_{mdc}^{pw} (\hat{pw}_{m-1dc} + pwt_{m-1d-1c} - pwt_{m-1dc}) \right) + \\ & \left( \sum_{dc} \beta_{Mdc}^{pw} \left( \sum_{M-1}^M \hat{pw}_{mdc} + pwt_{md-1c} - pwt_{mdc} \right) \right) + \end{aligned} \quad (35)$$

$$\begin{aligned} & \left( \sum_{tdc} \beta_{t0dc}^{ps} 0 \right) + \left( \sum_{mdc} \beta_{Tmdc}^{ps} 0 \right) + \left( \sum_{t=1m=1dc}^{T-1M-1} \beta_{tmdc}^{ps} (\hat{ps}_{t+1m-1dc} + pst_{t+1m-1d-1c} - pst_{t+1m-1dc}) \right) + \\ & \left( \sum_{t=1Mdc}^{T-1} \beta_{tmdc}^{ps} \left( \sum_{M-1}^M (\hat{ps}_{t+1mdc} + pst_{t+1md-1c} - pst_{t+1mdc}) \right) \right) \geq v(\vec{ul}, \vec{pw}, \vec{ps}) \quad \forall \vec{S} \vec{A} \\ & \vec{\beta} \geq 0 \quad \forall \beta \end{aligned} \quad (36)$$

And all auxiliary constraints in "Auxiliary Variable Definition"

### 5.2 Expectation ADP Model

Steps for rearranging and converting the constraint (35) into expectation:

- $E[ued_p] = 0$
- $E[pwt_{mdc}] = ptp_{mdc} * \hat{pw}_{mdc}$
- $E[pst_{tmdc}] = ptp_{mdc} * \hat{ps}_{tmdc}$

$$\beta^0 : \quad (1 - \gamma) \beta^0 \quad (37)$$

$$\beta_p^{ul} : \quad \sum_{p \in \text{co}} \beta_p^{ul} (ul_p - \gamma(\hat{ul}_p)) + \sum_{p \in \text{nco}} \beta_p^{ul} (ul_p) \quad (38)$$

$$\beta_{mdc}^{pw} : \quad \sum_{dc} \beta_{0dc}^{pw} (pw_{0dc} - \gamma(pea_{dc})) + \sum_{m=1dc}^{M-1} \beta_{mdc}^{pw} (pw_{mdc} - \gamma(\hat{pw}_{m-1dc} + E[pwt_{m-1d-1c}] - E[pwt_{m-1dc}])) + \quad (39)$$

$$\begin{aligned} & \sum_{dc} \beta_{Mdc}^{pw} (pw_{Mdc} - \gamma \sum_{M-1}^M (\hat{pw}_{mdc} + E[pwt_{md-1c}] - E[pwt_{mdc}])) \\ \beta_{tmdc}^{ps} : & \sum_{tdc} \beta_{t0dc}^{ps} (ps_{t0dc}) + \sum_{mdc} \beta_{Tmdc}^{ps} (ps_{Tmdc}) + \\ & \sum_{t=1m=1dc}^{T-1M-1} \beta_{tmdc}^{ps} (ps_{tmdc} - \gamma(\hat{ps}_{t+1m-1dc} + E[pst_{t+1m-1d-1c}] - E[pst_{t+1m-1dc}])) + \\ & \sum_{t=1dc}^{T-1} \beta_{tmdc}^{ps} (ps_{tmdc} - \gamma \sum_{M-1}^M (\hat{ps}_{t+1mdc} + E[pst_{t+1md-1c}] - E[pst_{t+1mdc}])) \end{aligned} \quad (40)$$

Let's say  $E[V]$  is the addition of all parts above. Then the ADP model converted to expectation would look as follows:

$$\max_{\vec{\beta}} \left( \sum \beta^0 + \sum \beta^{ut} E[ut] + \sum \beta^{pw} E[pw] + \sum \beta^{ps} E[ps] \right) \quad (41)$$

Subject to:

$$E[V] \leq c(p\vec{w}, \vec{s}\vec{c}, r\vec{s}\vec{c}, \vec{u}\vec{v}) \quad \forall \vec{S}\vec{A} \quad (42)$$

### 5.3 ADP Master Problem

Converting to Dual

$$\min_w \sum_{\vec{S}\vec{A}} w(\vec{S}\vec{A}) c(\vec{S}\vec{A}) \quad (43)$$

Subject To:

$$\beta^0 : \sum_{\vec{S}\vec{A}} \vec{w}(1 - \gamma) = 1 \quad (44)$$

$$\beta^{ul} : \sum_{\vec{S}\vec{A}} \vec{w}(ul_p - \gamma(\hat{u}l_p)) \geq E[ul_p] \quad \forall p \in \{\text{carry over}\} \quad (45)$$

$$\sum_{\vec{S}\vec{A}} \vec{w}(ul_p) \geq E[ul_p] \quad \forall p \in \{\text{non carry over}\} \quad (46)$$

$$\beta^{pw} : \sum_{\vec{S}\vec{A}} \vec{w}(pw_{0dc} - \gamma(pea_{dc})) \geq E[ps_{0dc}] \quad \forall dc \quad (47)$$

$$\sum_{\vec{S}\vec{A}} \vec{w}(pw_{mdc} - \gamma(\hat{p}w_{m-1dc} + E[pwt_{m-1d-1c}] - E[pwt_{m-1dc}])) \geq E[pw_{mdc}] \quad \forall m \in \{1 \dots M-1\} dc \quad (48)$$

$$\sum_{\vec{S}\vec{A}} \vec{w}(pw_{Mdc} - \gamma \sum_{M-1}^M (\hat{p}w_{mdc} + E[pwt_{md-1c}] - E[pwt_{mdc}])) \geq E[pw_{Mdc}] \quad \forall dc \quad (49)$$

$$\beta^{ps} : \sum_{\vec{S}\vec{A}} \vec{w}(ps_{t0dc}) \geq E[ps_{t0dc}] \quad \forall tdc \quad (50)$$

$$\sum_{\vec{S}\vec{A}} \vec{w}(ps_{Tmdc}) \geq E[ps_{Tmdc}] \quad \forall mdc \quad (51)$$

$$\sum_{\vec{S}\vec{A}} \vec{w}(ps_{tmdc} - \gamma(\hat{p}s_{t+1m-1dc} + E[pst_{t+1m-1d-1c}] - E[pst_d])) \geq E[ps_{tmdc}] \quad \forall \{1 \dots T-1\} \{1 \dots M-1\} dc \quad (52)$$

$$\sum_{\vec{S}\vec{A}} \vec{w}(ps_{tMdc} - \gamma \sum_{M-1}^M (\hat{p}s_{t+1mdc} + E[pst_{t+1md-1c}] - E[pst_d])) \geq E[ps_{tMdc}] \quad \forall \{1 \dots T-1\} dc \quad (53)$$

$$\text{State Action Bounds:} \quad w \geq 0 \quad \forall w \quad (54)$$

### 5.4 ADP Pricing Problem

$$\min_{(\vec{u}\vec{e}, \vec{u}\vec{u}, \vec{p}\vec{w}, \vec{p}\vec{e}, \vec{p}\vec{s}) \in S, (\vec{s}\vec{c}, r\vec{s}\vec{c}) \in A} c(\vec{p}\vec{w}, \vec{s}\vec{c}, r\vec{s}\vec{c}, \vec{u}\vec{v}) - E[V] \quad (55)$$

Subject to: constraints in sections "State-Action Constraints" and "Auxiliary Variable Definition"

## 6 Solution Explanation

### 6.1 Algorithm for solving

To get  $\vec{\beta}$  values, which will be used to generate an action follow steps below:

1. Perform a monte-carlo simulation (following some arbitrary policy) to get  $E[ue], E[uu], E[pw], E[pw]$

2. Create an initial feasible set of state-action pairs -  $\vec{w}$
3. Solve model in section "ADP Master Problem" where each state-action pairs in  $\vec{w}$  corresponds to a variable and parameters for all the constraints for a specific action.
4. Solve model in section "ADP Pricing Problem", where duals from problem in step 3 correspond to  $\vec{\beta}$  values.
  - If objective function is less than 0, add solution as a single state-action pair to  $\vec{w}$  and go to step 3
  - If objective function is greater than 0, continue to next step
5. Duals from problem in step 3 correspond to final  $\vec{\beta}$  values

## 6.2 Generating an Action

Once  $\vec{\beta}$  values have been approximated, you may use the model below to generate a recommended action for a specific state.

$$\begin{aligned}
& \min_{\vec{A}} c(\vec{S}, \vec{A}) + \gamma v(S') \\
v(S') = & \left( \beta^0 + \left( \sum_p \beta_{1p}^{ue} (uen_d + ue_{1p} - \hat{u}u_{1p} + uv_{1p}) \right) + \left( \sum_{t=2,p}^T \beta_{tp}^{ue} uen_d \right) + \right. \\
& \left( \sum_{tp}^{T-1} \beta_{tp}^{uu} (\hat{u}u_{t+1,p} + \sum_{mdc} E[pst_{tmdc}](U_{pd+1c} - U_{pdc})) \right) + \\
& \left( \sum_{dc} \beta_{0dc}^{pw} pea_{dc} \right) + \left( \sum_{m=1,dc}^{M-1} \beta_{mdc}^{pw} (\hat{p}w_{m-1dc} + E[pwt_{m-1d-1c}] - E[pwt_{m-1dc}]) \right) + \\
& \left( \sum_{dc} \beta_{Mdc}^{pw} \left( \sum_{M-1}^M \hat{p}w_{mdc} + E[pwt_{md-1c}] - E[pwt_{mdc}] \right) \right) + \\
& \left( \sum_{t=1m=1dc}^{T-1M-1} \beta_{tmdc}^{ps} (\hat{p}s_{t+1m-1dc} + E[pst_{t+1m-1d-1c}] - E[pst_{t+1m-1dc}]) \right) + \\
& \left. \left( \sum_{t=1Mdc}^{T-1} \beta_{tmdc}^{ps} \left( \sum_{M-1}^M (\hat{p}s_{t+1mdc} + E[pst_{t+1md-1c}] - E[pst_{t+1mdc}]) \right) \right) \right)
\end{aligned} \tag{56}$$

Subject to: constraints in section "State-Action Constraints"