Computation of the mean of an inverse Gaussian distribution

Consider an inverse Gaussian distribution with location parameter mu and scale parameter λ. The pdf is given as

$$f(x; \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right).$$

We are interested in computing the mean, i.e.

$$\mathrm{E}[x] = \int_0^\infty \mathrm{d}x \, x \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right).$$

In order to do that consider $t > 0$ and define a function g

$$g(t) = \int_0^\infty \mathrm{d}x \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(\frac{\lambda(tx-\mu)^2}{2\mu^2 x}\right).$$

The integral can be solved by the substitution $tx \to z$:

$$\begin{aligned} g(t) &= \int_0^\infty \mathrm{d}z \frac{1}{t} \sqrt{\frac{\lambda t^3}{2\pi x^3}} \exp\left(\frac{\lambda t(z-\mu)^2}{2\mu^2 z}\right) \\ &= \int_0^\infty \mathrm{d}z \sqrt{\frac{\lambda t}{2\pi x^3}} \exp\left(\frac{\lambda t(z-\mu)^2}{2\mu^2 z}\right) \\ &= 1 \end{aligned}$$

In the last line we used that the integral is the normalization of an inverse Gaussian distribution with location parameter mu and scale parameter λt.

Now we compute

$$\begin{aligned} \frac{\partial g(t)}{\partial t} &= \int_0^\infty \mathrm{d}x \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(\frac{\lambda(tx-\mu)^2}{2\mu^2 x}\right) \left(-\lambda \frac{tx-\mu}{2\mu^2 x} x\right) \\ &= -\frac{\lambda}{2\mu^2} \left(t\mathrm{E}[x] - \mu\right) \end{aligned}$$

If we compare that with above, where $g(t) = 1$ and thus $\frac{\partial g(t)}{\partial t} = 0$, we find

$$t\mathrm{E}[x] - \mu = 0.$$

By setting $t = 1$ we obtain

$$\mathrm{E}[x] = \mu.$$