Computation of the mean of an inverse Gaussian distribution

Consider an inverse Gaussian distribution with location parameter μ and scale parameter λ . The pdf is given as

$$f(x; \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right).$$

We are interested in computing the mean, i.e.

$$E[x] = \int_0^\infty dx \, x \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right).$$

In order to do that consider t > 0 and define a function g

$$g(t) = \int_0^\infty \mathrm{d}x \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda (tx - \mu)^2}{2\mu^2 x}\right).$$

The integral can be solved by the substitution $tx \to z$:

$$g(t) = \int_0^\infty dz \frac{1}{t} \sqrt{\frac{\lambda t^3}{2\pi x^3}} \exp\left(\frac{\lambda t (z - \mu)^2}{2\mu^2 z}\right)$$
$$= \int_0^\infty dz \sqrt{\frac{\lambda t}{2\pi x^3}} \exp\left(-\frac{\lambda t (z - \mu)^2}{2\mu^2 z}\right)$$
$$= 1$$

In the last line we used that the integral is the normalization of an inverse Gaussian distribution with location parameter μ and scale parameter λt .

Now we compute

$$\frac{\partial g(t)}{\partial t} = \int_0^\infty dx \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(tx-\mu)^2}{2\mu^2 x}\right) \left(-\lambda \frac{tx-\mu}{2\mu^2 x}x\right)$$
$$= -\frac{\lambda}{2\mu^2} \left(tE[x] - \mu\right)$$

If we compare that with above, where g(t)=1 and thus $\frac{\partial g(t)}{\partial t}=0$, we find

$$tE[x] - \mu = 0.$$

By setting t = 1 we obtain

$$E[x] = \mu$$
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