Exercise 1

Exercise 1.7 (page 6)
Two structures (D_1, I_1) and (D_2, I_2) in the same language $\mathcal{L}$ are called isomorphic, often written as $(D_1, I_1) \simeq (D_2, I_2)$, if there is a bijection $\psi : D_1 \to D_2$ s.t.

1. $\psi(I_1(c)) = I_2(c)$,
2. $\psi(I_1(f)(m_1, \ldots, m_n)) = I_2(f)(\psi(m_1), \ldots, \psi(m_n)))$ for all $m_1, \ldots, m_n \in D_1$, and
3. $(m_1, \ldots, m_n) \in I_1(P)$ iff $(\psi(m_1), \ldots, \psi(m_n)) \in I_2(P)$ for all $m_1, \ldots, m_n \in D_1$.

The theory of a structure $\mathcal{S}$ is defined as $Th(\mathcal{S}) = \{A \textit{ sentence} \mid \mathcal{S} \models A\}$. S_1 and S_2 are called elementarily equivalent if $Th(\mathcal{S}_1) = Th(\mathcal{S}_2)$.

(a) Show that two isomorphic structures are elementarily equivalent. Hint: first show $\psi(I_1(t)) = I_2(t)$ by induction on the term structure of t and then (a) by induction on formula structure.

The language of arithmetic is $\mathcal{L}_{\mathbb{N}} = \{0/0, s/1, +/2, \cdot/2, < /2\}$. The standard model of arithmetic is the $\mathcal{L}_{\mathbb{N}}$-structure $\mathcal{N} = (\mathbb{N}, I)$ where I is the obvious standard-interpretation of the symbols in $\mathcal{L}_{\mathbb{N}}$.

(b) Show that there is a structure which is elementary equivalent but not isomorphic to $\mathcal{N}$. Hint: Add a new constant symbol c to $\mathcal{L}_{\mathbb{N}}$, successively force c to be larger than each natural number and apply the compactness theorem.

A structure as in (b) is called non-standard model of arithmetic.

Firstly, some clarifications $\overline{x} = (x_i)_{i \in \{0,\ldots,n\}}$ is the notation for a family of variable symbols. In analogue for terms $\overline{t}$ and elements of an domain $\overline{m}$. Moreover, given the definition in the script, there is not distinct variable assignment. That is, for a structure $\mathcal{M} := \langle M, \mathcal{I} \rangle$ the interpretation function can be understood as $\mathcal{I} := I \cup \Delta$ where $\Delta := \{\overline{x} \mapsto \overline{m}\}$ is responsible for the assignment of domain elements to variable symbols. That is, this is merely used to make the free variable assignment explicit. Moreover, for a term t and a formula φ, the notation $t[\overline{x}]$ and $\varphi[\overline{x}]$ are used to make the free variables explicit. Lastly, let $(\tau \circ \Delta)$ be $\{\overline{x} \mapsto \overline{\tau(m)}\}$.

Starting by an induction on the structure of terms.

IH: Let $t[\overline{x}]$ be a term, let $\mathcal{M}_1 := \langle M_1, I_1 \rangle$ and let $\mathcal{M}_2 := \langle M_2, I_2 \rangle$ such that there exists an isomorphic function $\tau : M_1 \to M_2$ and let $\Delta_1 := \{\overline{x} \mapsto \overline{m_1}\}$ be an extension of I_1. Then

$$\tau((I_1 \cup \Delta_1)(t[\overline{x}])) = (I_2 \cup (\tau \circ \Delta))(t[\overline{x}])$$

IB:

- $t[\overline{x}] = c$. Starting from $\tau((I_1 \cup \Delta_1)(c)$, by definition of I_1 the constant symbol c will be mapped to an element $m_1 \in M_1$ resulting in $\tau(m_1) = m_2$. By definition of τ one obtains $\tau(m_1) = m_2 = I_2(c)$. Since, c is a constant symbol it follows that $I_2(c) = (I_2 \cup (\tau \circ \Delta_1))(c)$.

- $t[\overline{x}] = x$. Starting from $\tau((I_1 \cup \Delta_1)(x))$, by definition of $\mathcal{I}_1$ the variable symbol x will be mapped to an element $m_1 \in M_1$ by an assignment in Δ_1 (otherwise $\mathcal{I}_1$ would not be an interpretation). That is, $\{x \mapsto m_1\} \subseteq \Delta_1$. Resulting in $\tau((I_1 \cup \Delta_1)(x)) = \tau((I_1 \cup \{x \mapsto m_1\})(x)) = \tau(m_1) = m_2 = (I_2 \cup \{x \mapsto m_2\})(x) = (I_2 \cup \{x \mapsto \tau(m_1)\})(x) = (I_2 \cup (\tau \circ \Delta_1))(x)$.

IS:

- $t[\overline{x}] = f(\overline{t})[\overline{x}]$. Starting from $\tau((I_1 \cup \Delta_1)(f(\overline{t})[\overline{x}]))$ which is just a short-hand for

  $$\tau((I_1 \cup \Delta_1)(f(t_1[\overline{x}_1], \ldots, t_n[\overline{x}_n])).$$

  By definition of I_1 one obtains

  $$\tau(I_1(f)(\mathcal{I}_1(t_1[\overline{x}_1]), \ldots, \mathcal{I}_1(t_n[\overline{x}_n]))).$$

  By definition of τ one obtains

  $$I_2(f)(\tau(\mathcal{I}_1(t_1[\overline{x}_1])), \ldots, \tau(\mathcal{I}_1(t_n[\overline{x}_n])))).$$

  By IH it follows that $\forall i \in \{1, \ldots, n\}$,

  $$\tau(\mathcal{I}_1(t_i[\overline{x}_i])) = \mathcal{I}_2(t_i[\overline{x}_i])) = (I_2 \cup (\tau \circ \Delta_1))(t_i[\overline{x}_i])).$$

  Moreover, since the interpretation of f is not influenced by the part of the interpretation responsible for free variable assignment it follows that

  $$\mathcal{I}_2(f)(\mathcal{I}_2(t_1[\overline{x}_1]), \ldots, \mathcal{I}_2(t_n[\overline{x}_n]))),$$

  which by definition of the interpretation is

  $$\mathcal{I}_2(f(t_1[\overline{x}_1], \ldots, t_n[\overline{x}_n])) = (I_2 \cup (\tau \circ \Delta))(f(t_1, \ldots, t_n)[\overline{x}]).$$

From here an induction over the structure of formulas in warranted.

IH: Let $\varphi[\overline{x}]$ be a second-order formula, let $\mathcal{M}_1 := \langle M_1, I_1 \rangle$ and let $\mathcal{M}_2 := \langle M_2, I_2 \rangle$ such that there exists an isomorphic function $\tau : M_1 \to M_2$ and let $\Delta_1 := \{\overline{x} \mapsto \overline{m_1}\}$ be an arbitrary extension of I_1. Then

$$\langle M_1, (I_1 \cup \Delta_1) \rangle \vDash \varphi[\overline{x}] \iff \langle M_2, (I_2 \cup (\tau \circ \Delta)) \rangle \vDash \varphi[\overline{x}]$$

IB: Note $\overline{X}$ is here clearly empty.

- $\varphi[\overline{x}] = P(\overline{t})[\overline{x}]$. Starting from $\langle M_1, \mathcal{I}_1 \rangle \vDash P(\overline{t})[\overline{x}]$, which by semantics is $(I_1 \cup \Delta_1)(P(\overline{t})[\overline{x}])$. This is equivalent to

$$(I_1 \cup \Delta_1)(P(t_1[\overline{x}_1], \ldots, t_n[\overline{x}_n]).$$

By the definition of the interpretation function this is equivalent to

$$(I_1 \cup \Delta_1)(P)(\mathcal{I}_1(t_1[\overline{x}_1]), \ldots, \mathcal{I}_1(t_n[\overline{x}_n])).$$

Due to the fact that for the assignment of the predicate symbol P is independent of the assignments in Δ_1 one obtains,

$$I_1(P)(\mathcal{I}_1(t_1[\overline{x}_1]), \ldots, \mathcal{I}_1(t_n[\overline{x}_n]))),$$

which is simply another depiction of the statement

$$(\mathcal{I}_1(t_1[\overline{x}_1]), \ldots, \mathcal{I}_1(t_n[\overline{x}_n])) \in (I_1)(P).$$

Now by the definition of τ one obtains

$$(\tau(\mathcal{I}_1(t_1[\overline{x}_1])), \ldots, \tau(\mathcal{I}_1(t_n[\overline{x}_n]))) \in (I_2)(P).$$

Now, given the fact that $t_1, \ldots, t_n$ are terms and by the observation above, i.e. $\forall i \in \{1, \ldots n\}\ \mathcal{I}_1(t_i[\overline{x}_i])) = (I_2 \cup (\tau \circ \Delta_1))(t_i[\overline{x}_i])) = \mathcal{I}_2(t_i[\overline{x}_i])$, this is the same as

$$(\mathcal{I}_2(t_1[\overline{x}_1]), \ldots, \mathcal{I}_2(t_n[\overline{x}_n])) \in (I_2)(P).$$

Again by the fact that the interpretation of the symbol P is not influenced by Δ_1 and by rewriting the term one obtains

$$(\mathcal{I}_2)(P)(\mathcal{I}_2(t_1[\overline{x}_1]), \ldots, \mathcal{I}_2(t_n[\overline{x}_n]))).$$

By the definition of the interpretation it thus follows

$$\mathcal{I}_2(P(t_1[\overline{x}_1], \ldots, t_n[\overline{x}_n]))$$

which finally leads to $\langle M_2, I_2 \cup (\tau \circ \Delta_1) \rangle \vDash P(\overline{t})[\overline{x}]$.

- $\varphi[\overline{x}] = \bot$. Starting from $\langle M_1, \mathcal{I}_1 \rangle \nvDash \bot$. In the set notation $(I_1 \cup \Delta_1)(\bot) = I_1(\bot) = \{\}$, which means given the definition of τ that $\tau(I_1(\bot)) = I_2(\bot) = \{\}$, which leads to $\langle M_2, I_2 \cup (\tau \circ \Delta_1) \rangle \nvDash \bot$, as no element can be in the empty set.

- $\varphi[\overline{x}] = (t_1[\overline{x}_1] = t_2[\overline{x}_2])$. Starting from $\langle M_1, \mathcal{I}_1 \rangle \vDash t_1[\overline{x}_1] = t_2[\overline{x}_2]$, which by semantics is $(I_1 \cup \Delta_1)(t_1[\overline{x}_1]) = (I_1 \cup \Delta_1)(t_2[\overline{x}_2])$. Since those are terms, it follows by the observation above that $(I_2 \cup (\tau \circ \Delta_1))(t_1[\overline{x}_1]) = (I_2 \cup (\tau \circ \Delta_1))(t_2[\overline{x}_2])$, which by semantics is $\langle M_2, I_2 \cup (\tau \circ \Delta_1) \rangle \vDash t_1[\overline{x}_1] = t_2[\overline{x}_2]$.

IS:

- $\varphi[\overline{x}] = \neg\psi[\overline{x}]$: Starting from $\langle M_1, \mathcal{I}_1 \rangle \vDash \neg\psi[\overline{x}]$, which by semantics is $\langle M_1, \mathcal{I}_1 \rangle \nvDash \psi[\overline{x}]$. By IH it follows that $\langle M_2, (\tau \circ \Delta_1) \rangle \nvDash \psi[\overline{x}]$ and by semantics one obtains $\langle M_2, (\tau \circ \Delta_1) \rangle \vDash \neg\psi[\overline{x}]$.

- $\varphi[\overline{x}] = \psi[\overline{x}_\psi] \wedge \chi[\overline{x}_\chi]$: Starting from

  $$\langle M_1, \mathcal{I}_1 \rangle \vDash \psi[\overline{x}_\psi] \wedge \chi[\overline{x}_\chi],$$

  which by semantics is

  $$\langle M_1, \mathcal{I}_1 \rangle \vDash \psi[\overline{x}_\psi] \textit{ and } \langle M_1, \mathcal{I}_1 \rangle \vDash \chi[\overline{x}_\chi].$$

  By IH it follows that

  $$\langle M_2, (\tau \circ \Delta_1) \rangle \vDash \psi[\overline{x}_\psi] \textit{ and } \langle M_2, (\tau \circ \Delta_1) \rangle \vDash \chi[\overline{x}_\chi]$$

  and by semantics one obtains

  $$\langle M_2, (\tau \circ \Delta_1) \rangle \vDash \psi[\overline{x}_\psi] \wedge \chi[\overline{x}_\chi].$$

- $\varphi[\overline{x}] = \psi[\overline{x}_\psi] \vee \chi[\overline{x}_\chi]$: Starting from

  $$\langle M_1, \mathcal{I}_1 \rangle \vDash \psi[\overline{x}_\psi] \vee \chi[\overline{x}_\chi],$$

  which by semantics is

  $$\langle M_1, \mathcal{I}_1 \rangle \vDash \psi[\overline{x}_\psi] \textit{ or } \langle M_1, \mathcal{I}_1 \rangle \vDash \chi[\overline{x}_\chi].$$

  By IH it follows that

  $$\langle M_2, (\tau \circ \Delta_1) \rangle \vDash \psi[\overline{x}_\psi] \textit{ or } \langle M_2, (\tau \circ \Delta_1) \rangle \vDash \chi[\overline{x}_\chi]$$

and by semantics one obtains

$$\langle M_2, (\tau \circ \Delta_1)\rangle \vDash \psi[\overline{x}_\psi] \vee \chi[\overline{x}_\chi].$$

- $\varphi[\overline{x}] = \psi[\overline{x}_\psi] \to \chi[\overline{x}_\chi]$: Starting from

$$\langle M_1, \mathcal{I}_1\rangle \vDash \psi[\overline{x}_\psi] \to \chi[\overline{x}_\chi],$$

which by semantics is

$$\langle M_1, \mathcal{I}_1\rangle \vDash \psi[\overline{x}_\psi] \Rightarrow \langle M_1, \mathcal{I}_1\rangle \vDash \chi[\overline{x}_\chi].$$

By IH it follows that

$$\langle M_2, (\tau \circ \Delta_1)\rangle \vDash \psi[\overline{x}_\psi] \Rightarrow \langle M_2, (\tau \circ \Delta_1)\rangle \vDash \chi[\overline{x}_\chi]$$

and by semantics one obtains

$$\langle M_2, (\tau \circ \Delta_1)\rangle \vDash \psi[\overline{x}_\psi] \to \chi[\overline{x}_\chi].$$

- $\varphi[\overline{x}] = (\forall x\, \psi)[\overline{x}]$: Starting from $\langle M_1, \mathcal{I}_1\rangle \vDash (\forall x\, \psi)[\overline{x}]$. By semantics

$$\forall m \in M_1\ \langle M_1, ((I_1 \cup \Delta_1) \cup \{x \mapsto m\})\rangle \vDash \psi[\overline{x}, x]$$

which is clearly the same as

$$\forall m_1 \in M_1\ \langle M_1, I_1 \cup (\Delta_1 \cup \{x \mapsto m_1\})\rangle \vDash \psi[\overline{x}, x].$$

Consider the fact that the IH was formulated for arbitrary extension assigning free variables it follows,

$$\forall m_1 \in M_1\ \langle M_2, I_2 \cup (\tau \circ (\Delta_1 \cup \{x \mapsto m_1\}))\rangle \vDash \psi[\overline{x}, x].$$

Moreover, with $\{x \mapsto m_1\}$ mapping a variable symbol to an element in M_1 and with τ mapping from M_1 to M_2, the composition of those assignments results in

$$\forall m_1 \in M_1\ \langle M_2, I_2 \cup ((\tau \circ \Delta_1) \cup \{x \mapsto \tau(m_1)\})\rangle \vDash \psi[\overline{x}, x].$$

Due to the fact that τ is bijective this is equal to

$$\forall m_2 \in M_2\ \langle M_2, (I_2 \cup (\tau \circ \Delta_1)) \cup \{x \mapsto m_2\}\rangle \vDash \psi[\overline{x}, x],$$

which by semantics is equal to $\langle M_2, I_2 \cup (\tau \circ \Delta_1)\rangle \vDash \forall x\, \psi[\overline{x}]$.

- $\varphi[\overline{x}] = (\exists x\, \psi)[\overline{x}]$: Starting from $\langle M_1, \mathcal{I}_1 \rangle \vDash (\exists x\, \psi)[\overline{x}]$. By semantics

  $$\exists m \in M_1 \ \langle M_1, ((I_1 \cup \Delta_1) \cup \{x \mapsto m\}) \rangle \vDash \psi[\overline{x}, x]$$

  which is clearly the same as

  $$\exists m_1 \in M_1 \ \langle M_1, I_1 \cup (\Delta_1 \cup \{x \mapsto m_1\}) \rangle \vDash \psi[\overline{x}, x].$$

  Consider the fact that the IH was formulated for arbitrary extension assigning free variables it follows,

  $$\exists m_1 \in M_1 \ \langle M_2, I_2 \cup (\tau \circ (\Delta_1 \cup \{x \mapsto m_1\})) \rangle \vDash \psi[\overline{x}, x].$$

  Moreover, with $\{x \mapsto m_1\}$ mapping a variable symbol to an element in M_1 and with τ mapping from M_1 to M_2, the composition of those assignments results in

  $$\exists m_1 \in M_1 \ \langle M_2, I_2 \cup ((\tau \circ \Delta_1) \cup \{x \mapsto \tau(m_1)\}) \rangle \vDash \psi[\overline{x}, x].$$

  Due to the fact that τ is bijective this is equal to

  $$\exists m_2 \in M_2 \ \langle M_2, (I_2 \cup (\tau \circ \Delta_1)) \cup \{x \mapsto m_2\} \rangle \vDash \psi[\overline{x}, x],$$

  which by semantics is equal to $\langle M_2, I_2 \cup (\tau \circ \Delta_1) \rangle \vDash \exists x\, \psi[\overline{x}]$.

Clearly, the case for sentence is merely a special case of the previous proposition. That is, let φ be a sentence, let $\mathcal{M}_1 := \langle M_1, I_1 \rangle$ and let $\mathcal{M}_2 := \langle M_2, I_2 \rangle$ such that there exists an isomorphic function $\tau : M_1 \to M_2$ and let $\Delta_1 := \{\}$ be an arbitrary extension of I_1 assigning free variables to elements of the domain. Then by the proposition above

$$\langle M_1, (I_1 \cup \Delta_1) \rangle \vDash \varphi \iff \langle M_2, (I_2 \cup (\tau \circ \Delta_1)) \rangle \vDash \varphi$$

which is the same as

$$\langle M_1, I_1 \rangle \vDash \varphi \iff \langle M_2, I_2 \rangle \vDash \varphi$$

Finally, given the fact that for all sentences $\varphi \in \mathcal{L}$ $\mathcal{M}_1 \vDash \varphi \Leftrightarrow \mathcal{M}_2 \vDash \varphi$. Therefore, clearly $Th(\mathcal{M}_1) = \{\psi \mid \mathcal{M}_1 \vDash \psi\} = \{\psi \mid \mathcal{M}_2 \vDash \psi\} = Th(\mathcal{M}_2)$.

(b) Show that there is a structure which is elementary equivalent but not isomorphic to $\mathcal{N}$. As suggested in the hint, we start by adding an additional constant $\underline{\omega}$ to the language $\mathcal{L}_{\mathbb{N}}$ to obtain $\mathcal{L}_\omega$. Moreover, let Q be the theory of minimal arithmetic and let $\Gamma := Th(\mathcal{N})$, clearly $Q \subseteq \Gamma$. From there we construct the theory Γ_ω as follows

$$\Gamma_\omega := \Gamma \cup \{s^k(0) < \underline{\omega} \mid 0 \leq k\}$$

Building upon the fact that $\mathcal{N} \models \Gamma$, we know that any finite subset of Γ can be satisfied by $\mathcal{N}$. Hence, Moreover, take any $T \subseteq_{fin} \Gamma_\omega$, clearly T has the form $T = \Gamma' \cup \{s^k(0) < \underline{\omega} \mid k \in \{k_1, \ldots, k_n\} \subseteq_{fin} \mathbb{N}\}$ for $\Gamma' \subseteq_{fin} \Gamma$. As already established Γ' can be satisfied. Moreover, lets $\mathcal{N}_m$ be the model where $\underline{\omega}$ is interpreted such that $I(\underline{\omega}) := I(s^{m+1}(0))$, with $m = \max(k_1, \ldots, k_n)$. Therefore, both Γ' and $\{s^k(0) < \underline{\omega} \mid k \in \{k_1, \ldots, k_n\} \subseteq_{fin} \mathbb{N}\}$ are satisfied by $\mathcal{N}_m$. That is, for an arbitrary m such a model can be found. Now, given the compactness theorem it follows that Γ_ω has a model. Let the model be called $\mathcal{N}_\omega$, and let $\omega := I_\omega(\underline{\omega})$. Moreover, due $\mathcal{N}_\omega \models \Gamma_\omega$ and $\Gamma \subset \Gamma_\omega$, one obtains $\mathcal{N}_\omega \models \Gamma$. Furthermore, by removing the mapping $\{\underline{\omega} \mapsto \omega\}$ from I_ω, the language can be restricted to $\mathcal{L}_\mathbb{N}$. Note that ω and its successors still remain in N_ω.

We show that $\mathcal{N} \not\cong \mathcal{N}_\omega$. Let $\tau : N \to N_\omega$ be an isomorphism between $\mathcal{N}$ and $\mathcal{N}_\omega$. Since τ is an isomorphism, it must conform with the predicate $<_\mathcal{N} := I(<)$. Clearly, it must be the case that $\tau(I(0)) = I_\omega(0)$. Moreover, this forces $\tau(I(s^k(0))) = I_\omega(s^k(0))$ (to be precise an this can be shown by a straight forward induction). Now, there must be an $n \in M$ such that $\tau(n) = \omega$. By definition $n = I(s^n(0))$ and by the observation above $\tau(I(s^n(0))) = I_\omega(s^n(0))$ implying that $\omega = I_\omega(s^n(0))$. which due to the fact that $(I_\omega(s^n(0)), I_\omega(s^{n+1}(0))) \in I_\omega(<)$ would imply that $\mathcal{N}_\omega$ can not be a model of Γ_ω. Hence, we conclude that this isomorphism can not exists.

To conclude the proof of the statement (b) it remains to show that $Th(\mathcal{N}) = Th(\mathcal{N}_\omega)$. It is known that $Th(\mathcal{N})$ is consistent and complete. Now if there would be a sentence φ in the language $\mathcal{L}_\mathbb{N}$ such that $\varphi \in Th(\mathcal{N}_\omega)$ and $\varphi \notin Th(\mathcal{N})$ this would mean that $\mathcal{N} \models \neg\varphi$ due to completeness. However, together with consistency this contradicts the fact that it was already shown that $Th(\mathcal{N}_\omega) \models Th(\mathcal{N})$. Thus one obtains $Th(\mathcal{N}) = Th(\mathcal{N}_\omega)$.

Exercise 2

> Exercise 1.8 (page 6)
> A theory T is called countably categorical if, whenever $\mathcal{S}_1$ and $\mathcal{S}_2$ are countably infinite models of T, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic. A theory T is called complete if, for every sentence A either $T \vdash A$ or $T \vdash \neg A$. Show that a countably categorical theory without finite models is complete.

Let T be a countably categorical theory without finite models and let $\overline{\mathcal{L}}$ be the set of sentences.

If T is not consistent, then T bust be complete, i.e. every sentence $\varphi \in \overline{\mathcal{L}}$ can be derived from $\perp$. Hence, from now on only consistent theories are considered.

Assume T is not complete. Hence, $\exists\varphi \in \overline{\mathcal{L}}\ T \nvdash \varphi$ *and* $T \nvdash \neg\varphi$. Therefore, both $T' := T \cup \{\varphi\}$ and $T'' := T \cup \{\neg\varphi\}$ remain consistent.[1] If they are consistent, then clearly there exists $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \vDash T'$ and $\mathcal{M}_2 \vDash T''$. Moreover, as every theory that has a model, must have a countable model one obtains $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable. Furthermore, any models satisfying T' and T'' must satisfy T, forcing this model to be infinite. Therefore, $\mathcal{M}_1$ and $\mathcal{M}_2$ are both countably infinite models satisfying T. Now given the knowledge that T is countably categorical, the two countably infinite models $\mathcal{M}_1$ and $\mathcal{M}_2$ must be isomorphic, i.e. $\mathcal{M}_1 \simeq \mathcal{M}_2$. Hence, by the previous exercise it must therefore be the case that $Th(\mathcal{M}_1) = Th(\mathcal{M}_2)$. However, by construction $\varphi \in Th(\mathcal{M}_1)$ and $\neg\varphi \in Th(\mathcal{M}_2)$, which is clearly a contradiction.

Exercise 3

> Exercise 2.4 (page 14)
> Show that $\forall X \forall Y(X \subseteq Y \to X \leq Y)$ is valid by giving a proof in **NK2**. Show that $\forall X \forall Y(X \leq Y \to X \subseteq Y)$ is not valid by specifying a counterexample.

Starting with the counter example for $\forall X \forall Y(X \leq Y \to X \subseteq Y)$. That is, consider the following standard structure $\mathcal{M} := \langle\{a, b\}, I\rangle$ and move the sentence to a semantic level. However, before this can be done the formula has to be expanded. Hence, from $\forall X \forall Y(X \leq Y \to X \subseteq Y)$ given

$$\begin{aligned} X \leq Y &\equiv \exists u \forall x \in X \exists y \in Y(u(x) = y) \equiv \exists u \forall x \exists y(x \in X \to (y \in Y \wedge (u(x) = y))) \\ &\equiv \exists u \forall x \exists y(X(x) \to (Y(y) \wedge (u(x) = y))) \end{aligned}$$

and

$$X \subseteq Y \equiv \forall x(x \in X \to x \in Y) \equiv \forall x(X(x) \to Y(x))$$

one obtains

$$\forall X \forall Y(\exists u \forall x \exists y(X(x) \to (Y(y) \wedge (u(x) = y))) \to \forall x(X(x) \to Y(x))).$$

Now one obtains

$\langle M, I\rangle \vDash \forall X \forall Y(\exists u \forall x \exists y(X(x) \to (Y(y) \wedge (u(x) = y))) \to \forall x(X(x) \to Y(x)))$

$\forall P \subseteq M^{ar(X)}\ \langle M, I \cup \{X \mapsto P\}\rangle \vDash \forall Y(\exists u \forall x \exists y(X(x) \to (Y(y) \wedge (u(x) = y))) \to \forall x(X(x) \to Y(x)))$

$\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)}\ \langle M, I \cup \{X \mapsto P, Y \mapsto Q\}\rangle \vDash \exists u \forall x \exists y(X(x) \to (Y(y) \wedge (u(x) = y))) \to \forall x(X(x) \to Y(x))$

[1] Assume that $\psi \in T$ and $\neg\psi \in T'$. Hence, it must be the case that $T \cup \{\varphi\} \vdash \neg\psi$. By the deduction theorem it thus follows that $T \vdash \varphi \to \neg\psi$. However, since $\varphi \to \neg\psi = \neg\varphi \vee \neg\psi = \neg\psi \vee \neg\varphi = \psi \to \neg\varphi$. This contradicts the assumption that $\psi \in T$ and $\neg\psi \in T$ hold at the same time. The same argument holds for T''

First the antecedent

$$
\begin{aligned}
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \\
&\quad \langle M, I \cup \{X \mapsto P, Y \mapsto Q\}\rangle \vDash \exists u \forall x \exists y (X(x) \to (Y(y) \wedge (u(x) = y))) \\
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \exists f : M^{ar(f)} \to M \\
&\quad \langle M, I \cup \{X \mapsto P, Y \mapsto Q, u \mapsto f\}\rangle \vDash \forall x \exists y (X(x) \to (Y(y) \wedge (u(x) = y))) \\
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \exists f : M^{ar(f)} \to M \forall m_1 \in M \\
&\quad \langle M, I \cup \{x \mapsto m_1, X \mapsto P, Y \mapsto Q, u \mapsto f\}\rangle \vDash \exists y (X(x) \to (Y(y) \wedge (u(x) = y))) \\
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \exists f : M^{ar(f)} \to M \forall m_1 \in M \exists m_2 \in M \\
&\quad \langle M, I \cup \{x \mapsto m_1, y \mapsto m_2, X \mapsto P, Y \mapsto Q, u \mapsto f\}\rangle \vDash X(x) \to (Y(y) \wedge (u(x) = y)) \\
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \exists f : M^{ar(f)} \to M \forall m_1 \in M \exists m_2 \in M \\
&\quad \langle M, I \cup \Delta_1\rangle \vDash X(x) \Rightarrow \langle M, I \cup \Delta_1\rangle \vDash (Y(y) \wedge (u(x) = y) \\
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \exists f : M^{ar(f)} \to M \forall m_1 \in M \exists m_2 \in M \\
&\quad \langle M, I \cup \Delta_1\rangle \vDash X(x) \Rightarrow (\langle M, I \cup \Delta_1\rangle \vDash Y(y) \; and \; \langle M, I \cup \Delta_1\rangle \vDash u(x) = y) \\
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \exists f : M^{ar(f)} \to M \forall m_1 \in M \exists m_2 \in M \\
&\quad \langle M, I \cup \Delta_1\rangle \vDash X(x) \Rightarrow (\langle M, I \cup \Delta_1\rangle \vDash Y(y) \; and \; (I \cup \Delta_1)(u)((I \cup \Delta_1)(x)) = (I \cup \Delta_1)(y)
\end{aligned}
$$

Secondly the consequence

$$
\begin{aligned}
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \\
&\quad \langle M, I \cup \{X \mapsto P, Y \mapsto Q\}\rangle \vDash \forall x (X(x) \to Y(x)) \\
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \forall m_3 \in M \\
&\quad \langle M, I \cup \{X \mapsto P, Y \mapsto Q, x \mapsto m_3\}\rangle \vDash X(x) \to Y(x) \\
&\forall P \subseteq M^{ar(X)} \forall Q \subseteq M^{ar(Y)} \forall m_3 \in M \\
&\quad \langle M, I \cup \Delta_2\rangle \vDash X(x) \Rightarrow \langle M, I \cup \Delta_2\rangle \vDash Y(x)
\end{aligned}
$$

Consider $P := \{a\}$ and $Q := \{b\}$. For the antecedent. Clearly there exists a function $f := \{(a, b)\}$ such that

- for $m_1 = a$ one can find b as m_2 in order to satisfy $f(a) = b$, i.e. $(P(a) \Rightarrow (Q(b) \; and \; (f(a) = b)))$
- for $m_1 = b$ the statement $(P(a) \Rightarrow (Q(b) \; and \; (f(a) = b)))$ holds trivially by the semantics of $\to$.

Hence, for P and Q the antecedent holds. However, for the consequence consider $m_3 = a$. Clearly, $P(a)$ holds however, $Q(a)$ does not.

Moving on towards the natural deduction proof.

$$
\begin{prooftree}
\AxiomC{$[X_0(a)]^2$}
\AxiomC{$[\forall x(X_0(x) \to Y_0(x))]^1$}
\RightLabel{$(\forall_E)$}
\UnaryInfC{$X_0(a) \to Y_0(a)$}
\RightLabel{$(\to_E)$}
\BinaryInfC{$Y_0(a)$}
\AxiomC{$[a = a]$}
\dashedLine
\UnaryInfC{$(\lambda z.z)(a) = a)$}
\RightLabel{$(\wedge_I)$}
\BinaryInfC{$Y_0(a) \wedge (\lambda z.z)(a) = a$}
\RightLabel{$(\exists_I)$}
\UnaryInfC{$\exists y(Y_0(y) \wedge (\lambda z.z)(y) = a)$}
\RightLabel{$(\to_I^2)$}
\UnaryInfC{$X_0(a) \to (\exists y(Y_0(y) \wedge (\lambda z.z)(y) = a))$}
\RightLabel{$(\forall_I)$}
\UnaryInfC{$\forall x(X_0(x) \to (\exists y(Y_0(y) \wedge (\lambda z.z)(y) = x)))$}
\RightLabel{$(\exists_I)$}
\UnaryInfC{$\exists u \forall x(X_0(x) \to (\exists y(Y_0(y) \wedge u(y) = x)))$}
\RightLabel{$(\to_I^1)$}
\UnaryInfC{$\forall x(X_0(x) \to Y_0(x)) \to \exists u \forall x(X_0(x) \to (\exists y(Y_0(y) \wedge u(y) = x)))$}
\RightLabel{$(\forall_I)$}
\UnaryInfC{$\forall Y(\forall x(X_0(x) \to Y(x)) \to \exists u \forall x(X_0(x) \to (\exists y(Y(y) \wedge u(y) = x))))$}
\RightLabel{$(\forall_I)$}
\UnaryInfC{$\forall X \forall Y(\forall x(X(x) \to Y(x)) \to \exists u \forall x(X(x) \to (\exists y(Y(y) \wedge u(y) = x))))$}
\end{prooftree}
$$

Exercise 4

> Give the proof of Proposition 2.3 (page 16).
> if $\mathcal{M}_1 \simeq \mathcal{M}_2$ and A is a second-order sentence, then $\mathcal{M}_1 \vDash A$ iff $\mathcal{M}_2 \vDash A$.

Firstly, some clarifications $\overline{x} = (x_i)_{i \in \{0,\dots,n\}}$ is the notation for a family of variable symbols. In analogue for terms $\overline{t}$ and elements of an domain $\overline{m}$. Moreover, given the definition in the script, there is not distinct variable assignment. That is, for a structure $\mathcal{M} := \langle M, \mathcal{I} \rangle$ the interpretation function can be understood as $\mathcal{I} := I \cup \Delta$ where $\Delta := \{\overline{x} \mapsto \overline{m}, \overline{X} \mapsto \overline{P}, \overline{u} \mapsto \overline{f}\}$ is responsible for the assignment of domain elements (as well as sets and functions) to variable symbols. That is, this is merely used to make the free variable assignment explicit. Moreover, for a term t and a formula φ, the notation $t[\overline{x}, \overline{X}, \overline{u}]$ and $\varphi[\overline{x}, \overline{X}, \overline{u}]$ are used to make the free variables explicit. Moreover, in the subsequent proof it will be shown that given an interpretation $\mathcal{I}_1$ with an arbitrary assignment of variable symbols Δ_1, it is possible to find a suitable assignment of variables Δ_2 for an to I_1 isomorphic structure I_2. And vice versa. This variable assignment will be constructed by using the isomorphism $\tau : M_1 \to M_2$. Firstly, for any subset $X_1 \subseteq M_1$ one can find a subset $X_2 \subseteq M_2$ such that $X_2 = \{\tau(x) \mid \forall x \in X_1\}$. The same holds for subsets of M_1^n. Moreover, with τ being bijective the same holds for τ^{-1} in the other direction as well. Furthermore, for a set X, let $g := X^f$ indicate that X is just the set definition of the function g (if permitted by the underlying set). Similarly let X^P indicate that X is just the set definition of the predicate Q. Meaning for every function and every predicate over the initial domain there exists a copy created by τ over the other domain. Hence, allowing for the syntactic sugar

$$
\begin{aligned}
\tau(f) &:= \{(\tau(m_1), \dots, \tau(m_n), \tau(m)) \mid f(m_1, \dots, m_n) = m\}^f \\
\tau(P) &:= \{(\tau(m_1), \dots, \tau(m_n)) \mid (m_1, \dots, m_n) \in P\}^P
\end{aligned}
$$

Lastly, the notation $(\tau \circ \Delta)$ will be similar as in the previous exercise the variable extension of the interpretation I_2 and is defined as $(\tau \circ \Delta) := \{\overline{x} \mapsto \overline{\tau(m)}, \overline{X} \mapsto \overline{\tau(P)}, \overline{u} \mapsto \overline{\tau(f)}\}$, where $\overline{m}, \overline{P}$ and $\overline{f}$ live over M_1.

Starting by an induction on the structure of terms.

IH: Let $t[\overline{x}, \overline{u}]$ be a term, let $\mathcal{M}_1 := \langle M_1, I_1 \rangle$ and let $\mathcal{M}_2 := \langle M_2, I_2 \rangle$ such that there exists an isomorphic function $\tau : M_1 \to M_2$ and let $\Delta_1 := \{\overline{x} \mapsto \overline{m_1}, \overline{X} \mapsto \overline{R}, \overline{u} \mapsto \overline{f_1}\}$ be an extension of I_1. Then

$$\tau((I_1 \cup \Delta_1)(t[\overline{x}, \overline{u}])) = (I_2 \cup (\tau \circ \Delta))(t[\overline{x}, \overline{u}])$$

IB:

- $t[\overline{x}, \overline{u}] = c$. Starting from $\tau((I_1 \cup \Delta_1)(c)$, by definition of I_1 the constant symbol c will be mapped to an element $m_1 \in M_1$ resulting in $\tau(m_1) = m_2$. By definition of τ one obtains $\tau(m_1) = m_2 = I_2(c)$. Since, c is a constant symbol it follows that $I_2(c) = (I_2 \cup (\tau \circ \Delta_1))(c)$.

- $t[\overline{x}, \overline{u}] = x$. Starting from $\tau((I_1 \cup \Delta_1)(x))$, by definition of $\mathcal{I}_1$ the variable symbol x will be mapped to an element $m_1 \in M_1$ by an assignment in Δ_1 (otherwise $\mathcal{I}_1$ would not be an interpretation). That is, $\{x \mapsto m_1\} \subseteq \Delta_1$. Resulting in $\tau((I_1 \cup \Delta_1)(x)) = \tau((I_1 \cup \{x \mapsto m_1\})(x)) = \tau(m_1) = m_2 = (I_2 \cup \{x \mapsto m_2\})(x) = (I_2 \cup \{x \mapsto \tau(m_1)\})(x) = (I_2 \cup (\tau \circ \Delta_1))(x)$.

IS:

- $t[\overline{x}, \overline{u}] = f(\overline{t})[\overline{x}, \overline{u}]$. Starting from $\tau((I_1 \cup \Delta_1)(f(\overline{t})[\overline{x}, \overline{u}]))$ which is just a shorthand for

  $$\tau((I_1 \cup \Delta_1)(f(t_1[\overline{x}_1, \overline{u}_1], \ldots, t_n[\overline{x}_n, \overline{u}_n])).$$

  By definition of I_1 one obtains

  $$\tau(I_1(f)(\mathcal{I}_1(t_1[\overline{x}_1, \overline{u}_1]), \ldots, \mathcal{I}_1(t_n[\overline{x}_n, \overline{u}_n]))).$$

  By definition of τ one obtains

  $$I_2(f)(\tau(\mathcal{I}_1(t_1[\overline{x}_1, \overline{u}_1])), \ldots, \tau(\mathcal{I}_1(t_n[\overline{x}_n, \overline{u}_n])))).$$

  By IH it follows that $\forall i \in \{1, \ldots, n\}$,

  $$\tau(\mathcal{I}_1(t_i[\overline{x}_i, \overline{u}_i])) = \mathcal{I}_2(t_i[\overline{x}_i, \overline{u}_i])) = (I_2 \cup (\tau \circ \Delta_1))(t_i[\overline{x}_i, \overline{u}_i])).$$

  Moreover, since the interpretation of f is not influenced by the part of the interpretation responsible for free variable assignment it follows that

  $$\mathcal{I}_2(f)(\mathcal{I}_2(t_1[\overline{x}_1, \overline{u}_1]), \ldots, \mathcal{I}_2(t_n[\overline{x}_n, \overline{u}_n]))),$$

which by definition of the interpretation is

$$\mathcal{I}_2(f(t_1[\overline{x}_1,\overline{u}_1],\ldots,t_n[\overline{x}_n,\overline{u}_n])) = (I_2 \cup (\tau \circ \Delta))(f(t_1,\ldots,t_n)[\overline{x},\overline{u}]).$$

- $t[\overline{x},\overline{u}] = u(\bar{t})[\overline{x},\overline{u}]$. Starting from $\tau((I_1 \cup \Delta_1)(u(\bar{t})[\overline{x},\overline{u}])$ which is simply

  $$\tau((I_1 \cup \Delta_1)(u(t_1[\overline{x}_1,\overline{u}_1],\ldots,t_n[\overline{x}_n,\overline{u}_n])).$$

  By the definition of an interpretation one obtains

  $$\tau((I_1 \cup \Delta_1)(u)(\mathcal{I}_1(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_1(t_n[\overline{x}_n,\overline{u}_n]))).$$

  Since this formula is interpreted it must be the case that $\{u \mapsto f_1\} \subseteq \Delta_1$. Hence, one obtains

  $$\tau((I_1 \cup \{u \mapsto f_1\})(u)(\mathcal{I}_1(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_1(t_n[\overline{x}_n,\overline{u}_n])))$$

  and subsequently

  $$\tau(f_1(\mathcal{I}_1(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_1(t_n[\overline{x}_n,\overline{u}_n]))) = \tau(f_1(m_{1_1},\ldots,m_{1_n})) = \tau(m_1).$$

  Hence, $\tau \circ f_1 : M_1^n \to M_2$. However, since τ is isomorphic therefore it can be used to find a copy of f_1 in M_2. Hence, on has to find a function f_2 that given the copies of $m_{1_1},\ldots,m_{1_n}$ in M_2 the function maps to the copy of m_1, i.e. $f_2(\tau(m_{1_1}),\ldots,\tau(m_{1_n})) = \tau(m_1)$. Since for every subset in $X \subseteq M_1^n$ one has $\tau(X) \subseteq M_2^n$ and for every subset in $X \subseteq M_2^n$ one has $\tau^{-1}(X) \subseteq M_1^n$ it is the function $\tau(f_1)$ that satisfies those requirements. As a reminder

  $$\begin{aligned} f_2 = \tau(f_1) &:= \tau(\{(m_{1_1},\ldots,m_{1_n},m_1) \mid f_1(m_{1_1},\ldots,m_{1_n}) = m_1\}^f) \\ &= \{(\tau(m_{1_1}),\ldots,\tau(m_{1_n}),\tau(m_1)) \mid f_1(m_{1_1},\ldots,m_{1_n}) = m_1\}^f. \end{aligned}$$

  Resulting in $\tau(f_1(m_{1_1},\ldots,m_{1_n})) = f_2(\tau(m_{1_1}),\ldots,\tau(m_{1_n})) = \tau(m_1)$. From $f_2(\tau(\mathcal{I}_1(t_1[\overline{x}_1,\overline{u}_1])),\ldots,\tau(\mathcal{I}_1(t_n[\overline{x}_n,\overline{u}_n])))$ and by IH this amounts to

  $$f_2(\mathcal{I}_2(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_2(t_n[\overline{x}_n,\overline{u}_n])).$$

  Furthermore, one obtains

  $$\begin{aligned} &(I_2 \cup \{u \mapsto f_2\})(u)(\mathcal{I}_2(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_2(t_n[\overline{x}_n,\overline{u}_n])) \\ &\qquad = (I_2 \cup \{u \mapsto \tau(f_1)\})(u)(\mathcal{I}_2(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_2(t_n[\overline{x}_n,\overline{u}_n])) \end{aligned}$$

  Given the definition of $(\tau \circ \Delta)$ this results in

  $$(I_2 \cup (\tau \circ \Delta))(u)(\mathcal{I}_2(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_2(t_n[\overline{x}_n,\overline{u}_n]))$$

  which by semantics is $\mathcal{I}_2(u(\bar{t})[\overline{x},\overline{u}])$.

From here an induction over the structure of formulas in warranted.

IH: Let $\varphi[\overline{x},\overline{X},\overline{u}]$ be a second-order formula, let $\mathcal{M}_1 := \langle M_1, I_1\rangle$ and let $\mathcal{M}_2 := \langle M_2, I_2\rangle$ such that there exists an isomorphic function $\tau : M_1 \to M_2$ and let $\Delta_1 := \{\overline{x} \mapsto \overline{m_1}, \overline{X} \mapsto \overline{P_1}, \overline{u} \mapsto \overline{f_1}\}$ be an arbitrary extension of I_1. Then

$$\langle M_1, (I_1 \cup \Delta_1)\rangle \vDash \varphi[\overline{x},\overline{X},\overline{u}] \iff \langle M_2, (I_2 \cup (\tau \circ \Delta))\rangle \vDash \varphi[\overline{x},\overline{X},\overline{u}]$$

IB: Note $\overline{X}$ is here clearly empty.

- $\varphi[\overline{x},\overline{X},\overline{u}] = P(\bar{t})[\overline{x},\overline{X},\overline{u}]$. Starting from $\langle M_1, \mathcal{I}_1\rangle \vDash P(\bar{t})[\overline{x},\overline{u}]$, which by semantics is $(I_1 \cup \Delta_1)(P(\bar{t})[\overline{x},\overline{u}])$. This is equivalent to

  $$(I_1 \cup \Delta_1)(P(t_1[\overline{x}_1,\overline{u}_1],\ldots,t_n[\overline{x}_n,\overline{u}_n])).$$

  By the definition of the interpretation function this is equivalent to

  $$(I_1 \cup \Delta_1)(P)(\mathcal{I}_1(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_1(t_n[\overline{x}_n,\overline{u}_n])).$$

  Due to the fact that for the assignment of the predicate symbol P is independent of the assignments in Δ_1 one obtains,

  $$I_1(P)(\mathcal{I}_1(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_1(t_n[\overline{x}_n,\overline{u}_n]))),$$

  which is simply another depiction of the statement

  $$(\mathcal{I}_1(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_1(t_n[\overline{x}_n,\overline{u}_n])) \in (I_1)(P).$$

  Now by the definition of τ one obtains

  $$(\tau(\mathcal{I}_1(t_1[\overline{x}_1,\overline{u}_1])),\ldots,\tau(\mathcal{I}_1(t_n[\overline{x}_n,\overline{u}_n]))) \in (I_2)(P).$$

  Now, given the fact that $t_1,\ldots,t_n$ are terms and by the observation above, i.e. $\forall i \in \{1,\ldots n\}\ \mathcal{I}_1(t_i[\overline{x}_i,\overline{u}_i])) = (I_2 \cup (\tau \circ \Delta_1))(t_i[\overline{x}_i,\overline{u}_i])) = \mathcal{I}_2(t_i[\overline{x}_i,\overline{X}_i,\overline{u}_i])$, this is the same as

  $$(\mathcal{I}_2(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_2(t_n[\overline{x}_n,\overline{u}_n])) \in (I_2)(P).$$

  Again by the fact that the interpretation of the symbol P is not influenced by Δ_1 and by rewriting the term one obtains

  $$(\mathcal{I}_2)(P)(\mathcal{I}_2(t_1[\overline{x}_1,\overline{u}_1]),\ldots,\mathcal{I}_2(t_n[\overline{x}_n,\overline{u}_n]))).$$

  By the definition of the interpretation it thus follows

  $$\mathcal{I}_2(P(t_1[\overline{x}_1,\overline{u}_1],\ldots,t_n[\overline{x}_n,\overline{u}_2]))$$

  which finally leads to $\langle M_2, I_2 \cup (\tau \circ \Delta_1)\rangle \vDash P(\bar{t})[\overline{x},\overline{u}]$.

- $\varphi[\overline{x}, \overline{X}, \overline{u}] = \bot$. Starting from $\langle M_1, \mathcal{I}_1 \rangle \not\models \bot$. In the set notation $(I_1 \cup \Delta_1)(\bot) = I_1(\bot) = \{\}$, which means given the definition of τ that $\tau(I_1(\bot)) = I_2(\bot) = \{\}$, which leads to $\langle M_2, I_2 \cup (\tau \circ \Delta_1) \rangle \not\models \bot$, as no element can be in the empty set.

- $\varphi[\overline{x}, \overline{X}, \overline{u}] = (t_1[\overline{x}_1, \overline{u}_1] = t_2[\overline{x}_2, \overline{u}_2])$. Starting from $\langle M_1, \mathcal{I}_1 \rangle \models t_1[\overline{x}_1, \overline{u}_1] = t_2[\overline{x}_2, \overline{u}_2]$, which by semantics is $(I_1 \cup \Delta_1)(t_1[\overline{x}_1, \overline{u}_1]) = (I_1 \cup \Delta_1)(t_2[\overline{x}_2, \overline{u}_2])$. Since those are terms, it follows by the observation above that $(I_2 \cup (\tau \circ \Delta_1))(t_1[\overline{x}_1, \overline{u}_1]) = (I_2 \cup (\tau \circ \Delta_1))(t_2[\overline{x}_2, \overline{u}_2])$, which by semantics is $\langle M_2, I_2 \cup (\tau \circ \Delta_1) \rangle \models t_1[\overline{x}_1, \overline{u}_1] = t_2[\overline{x}_2, \overline{u}_2]$.

- $\varphi[\overline{x}, \overline{u}] = X(\bar{t})[\overline{x}, \overline{u}]$. Starting from $\langle M_1, \mathcal{I}_1 \rangle \models X(\bar{t})[\overline{x}, \overline{u}]$, which by semantics is $(I_1 \cup \Delta_1)(P(\bar{t})[\overline{x}, \overline{u}])$. This is equivalent to

  $$(I_1 \cup \Delta_1)(X(t_1[\overline{x}_1, \overline{u}_1], \ldots, t_n[\overline{x}_n, \overline{u}_n]).)$$

  By the definition of the interpretation function this is equivalent to

  $$(I_1 \cup \Delta_1)(X)(\mathcal{I}_1(t_1[\overline{x}_1, \overline{u}_1]), \ldots, \mathcal{I}_1(t_n[\overline{x}_n, \overline{u}_n])).$$

  Due to the fact that for the assignment of the predicate variable symbol X it must be the case that $\{X \mapsto P\} \subseteq \Delta_1$ one obtains,

  $$(I_1 \cup \{X \mapsto P\})(X)(\mathcal{I}_1(t_1[\overline{x}_1, \overline{u}_1], \ldots, \mathcal{I}_1(t_n[\overline{x}_n, \overline{u}_n])).$$

  which is simply another depiction of the statement

  $$(\mathcal{I}_1(t_1[\overline{x}_1, \overline{u}_1]), \ldots, \mathcal{I}_1(t_n[\overline{x}_n, \overline{u}_n])) \in (I_1 \cup \{X \mapsto P\})(X)$$

  and subsequently of

  $$(\mathcal{I}_1(t_1[\overline{x}_1, \overline{u}_1]), \ldots, \mathcal{I}_1(t_n[\overline{x}_n, \overline{u}_n])) \in P.$$

  Now, given the fact that $t_1, \ldots, t_n$ are terms and by the observation above, i.e. $\forall i \in \{1, \ldots n\}\ \mathcal{I}_1(t_i[\overline{x}_i, \overline{u}_i])) = (I_2 \cup (\tau \circ \Delta_1))(t_i[\overline{x}_i, \overline{X}_i, \overline{u}_i])) = \mathcal{I}_2(t_i[\overline{x}_i, \overline{u}_i])$. Moreover, since for every subset in $X \subseteq M_1^n$ one has $\tau(X) \subseteq M_2^n$ and for every subset in $X \subseteq M_2^n$ one has $\tau^{-1}(X) \subseteq M_1^n$ it is the predicate $\tau(P)$ that fulfils the desired requirements. Hence, one obtains

  $$(\tau(\mathcal{I}_1(t_1[\overline{x}_1, \overline{u}_1])), \ldots, \tau(\mathcal{I}_1(t_n[\overline{x}_n, \overline{u}_n]))) \in \tau(P)$$

  which is logically equivalent to the previous one. Leading to

  $$(\mathcal{I}_2(t_1[\overline{x}_1, \overline{u}_1])), \ldots, \mathcal{I}_2(t_n[\overline{x}_n, \overline{u}_n])) \in (I_2 \cup \{X \to \tau(P)\})(X)$$

  and finally by the definition of $(\tau \circ \Delta)$ to

  $$(\mathcal{I}_2(t_1[\overline{x}_1, \overline{u}_1])), \ldots, \mathcal{I}_2(t_n[\overline{x}_n, \overline{u}_n])) \in (I_2 \cup (\tau \circ \Delta))(X)$$

By the definition of the interpretation it thus follows

$$\mathcal{I}_2(P(t_1[\overline{x}_1, \overline{u}_1], \ldots, t_n[\overline{x}_n, \overline{u}_n]))$$

which finally leads to $\langle M_2, I_2 \cup (\tau \circ \Delta_1)\rangle \models P(\bar{t})[\overline{x}, \overline{u}]$.

IS:

- $\varphi[\overline{x}, \overline{X}, \overline{u}] = \neg\psi[\overline{x}, \overline{X}, \overline{u}]$: Starting from $\langle M_1, \mathcal{I}_1\rangle \models \neg\psi[\overline{x}, \overline{X}, \overline{u}]$, which by semantics is $\langle M_1, \mathcal{I}_1\rangle \not\models \psi[\overline{x}, \overline{X}, \overline{u}]$. By IH it follows that $\langle M_2, (\tau \circ \Delta_1)\rangle \not\models \psi[\overline{x}, \overline{X}, \overline{u}]$ and by semantics one obtains $\langle M_2, (\tau \circ \Delta_1)\rangle \models \neg\psi[\overline{x}, \overline{X}, \overline{u}]$.

- $\varphi[\overline{x}, \overline{X}, \overline{u}] = \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \wedge \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi]$: Starting from

$$\langle M_1, \mathcal{I}_1\rangle \models \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \wedge \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi],$$

which by semantics is

$$\langle M_1, \mathcal{I}_1\rangle \models \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \textit{ and } \langle M_1, \mathcal{I}_1\rangle \models \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi].$$

By IH it follows that

$$\langle M_2, (\tau \circ \Delta_1)\rangle \models \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \textit{ and } \langle M_2, (\tau \circ \Delta_1)\rangle \models \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi]$$

and by semantics one obtains

$$\langle M_2, (\tau \circ \Delta_1)\rangle \models \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \wedge \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi].$$

- $\varphi[\overline{x}, \overline{X}, \overline{u}] = \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \vee \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi]$: Starting from

$$\langle M_1, \mathcal{I}_1\rangle \models \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \vee \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi],$$

which by semantics is

$$\langle M_1, \mathcal{I}_1\rangle \models \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \textit{ or } \langle M_1, \mathcal{I}_1\rangle \models \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi].$$

By IH it follows that

$$\langle M_2, (\tau \circ \Delta_1)\rangle \models \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \textit{ or } \langle M_2, (\tau \circ \Delta_1)\rangle \models \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi]$$

and by semantics one obtains

$$\langle M_2, (\tau \circ \Delta_1)\rangle \models \psi[\overline{x}_\psi, \overline{X}_\psi, \overline{u}_\psi] \vee \chi[\overline{x}_\chi, \overline{X}_\chi, \overline{u}_\chi].$$

- $\varphi[\overline{x},\overline{X},\overline{u}] = \psi[\overline{x}_\psi,\overline{X}_\psi,\overline{u}_\psi] \to \chi[\overline{x}_\chi,\overline{X}_\chi,\overline{u}_\chi]$: Starting from

  $$\langle M_1,\mathcal{I}_1\rangle \vDash \psi[\overline{x}_\psi,\overline{X}_\psi,\overline{u}_\psi] \to \chi[\overline{x}_\chi,\overline{X}_\chi,\overline{u}_\chi],$$

  which by semantics is

  $$\langle M_1,\mathcal{I}_1\rangle \vDash \psi[\overline{x}_\psi,\overline{X}_\psi,\overline{u}_\psi] \Rightarrow \langle M_1,\mathcal{I}_1\rangle \vDash \chi[\overline{x}_\chi,\overline{X}_\chi,\overline{u}_\chi].$$

  By IH it follows that

  $$\langle M_2,(\tau\circ\Delta_1)\rangle \vDash \psi[\overline{x}_\psi,\overline{X}_\psi,\overline{u}_\psi] \Rightarrow \langle M_2,(\tau\circ\Delta_1)\rangle \vDash \chi[\overline{x}_\chi,\overline{X}_\chi,\overline{u}_\chi]$$

  and by semantics one obtains

  $$\langle M_2,(\tau\circ\Delta_1)\rangle \vDash \psi[\overline{x}_\psi,\overline{X}_\psi,\overline{u}_\psi] \Rightarrow \chi[\overline{x}_\chi,\overline{X}_\chi,\overline{u}_\chi].$$

- $\varphi[\overline{x},\overline{X},\overline{u}] = (\forall x\,\psi)[\overline{x},\overline{X},\overline{u}]$: Starting from $\langle M_1,\mathcal{I}_1\rangle \vDash (\forall x\,\psi)[\overline{x},\overline{X},\overline{u}]$. By semantics

  $$\forall m \in M_1\ \langle M_1,((I_1\cup\Delta_1)\cup\{x\mapsto m\})\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}]$$

  which is clearly the same as

  $$\forall m_1 \in M_1\ \langle M_1,I_1\cup(\Delta_1\cup\{x\mapsto m_1\})\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}].$$

  Consider the fact that the IH was formulated for arbitrary extension assigning free variables it follows,

  $$\forall m_1 \in M_1\ \langle M_2,I_2\cup(\tau\circ(\Delta_1\cup\{x\mapsto m_1\}))\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}].$$

  Moreover, with $\{x\mapsto m_1\}$ mapping a variable symbol to an element in M_1 and with τ mapping from M_1 to M_2, the composition of those assignments results in

  $$\forall m_1 \in M_1\ \langle M_2,I_2\cup((\tau\circ\Delta_1)\cup\{x\mapsto\tau(m_1)\})\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}].$$

  Due to the fact that τ is bijective this is equal to

  $$\forall m_2 \in M_2\ \langle M_2,(I_2\cup(\tau\circ\Delta_1))\cup\{x\mapsto m_2\}\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}],$$

  which by semantics is equal to $\langle M_2,I_2\cup(\tau\circ\Delta_1)\rangle \vDash \forall x\,\psi[\overline{x},\overline{X},\overline{u}]$.

- $\varphi[\overline{x},\overline{X},\overline{u}] = (\exists x\,\psi)[\overline{x},\overline{X},\overline{u}]$: Starting from $\langle M_1,\mathcal{I}_1\rangle \vDash (\exists x\,\psi)[\overline{x},\overline{X},\overline{u}]$. By semantics

$$\exists m \in M_1\ \langle M_1,((I_1 \cup \Delta_1) \cup \{x \mapsto m\})\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}]$$

which is clearly the same as

$$\exists m_1 \in M_1\ \langle M_1, I_1 \cup (\Delta_1 \cup \{x \mapsto m_1\})\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}].$$

Consider the fact that the IH was formulated for arbitrary extension assigning free variables it follows,

$$\exists m_1 \in M_1\ \langle M_2, I_2 \cup (\tau \circ (\Delta_1 \cup \{x \mapsto m_1\}))\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}].$$

Moreover, with $\{x \mapsto m_1\}$ mapping a variable symbol to an element in M_1 and with τ mapping from M_1 to M_2, the composition of those assignments results in

$$\exists m_1 \in M_1\ \langle M_2, I_2 \cup ((\tau \circ \Delta_1) \cup \{x \mapsto \tau(m_1)\})\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}].$$

Due to the fact that τ is bijective this is equal to

$$\exists m_2 \in M_2\ \langle M_2,(I_2 \cup (\tau \circ \Delta_1)) \cup \{x \mapsto m_2\}\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u}],$$

which by semantics is equal to $\langle M_2, I_2 \cup (\tau \circ \Delta_1)\rangle \vDash \exists x\,\psi[\overline{x},\overline{X},\overline{u}]$.

- $\varphi[\overline{x},\overline{X},\overline{u}] = (\forall X\,\psi)[\overline{x},\overline{X},\overline{u}]$: Starting from $\langle M_1,\mathcal{I}_1\rangle \vDash (\forall X\,\psi)[\overline{x},\overline{X},\overline{u}]$. By semantics

$$\forall P_1 \subseteq M_1^{ar(X)}\ \langle M_1,((I_1 \cup \Delta_1) \cup \{X \mapsto P_1\})\rangle \vDash \psi[\overline{x},\overline{X},X,\overline{u}]$$

which is clearly the same as

$$\forall P_1 \subseteq M_1^{ar(X)}\ \langle M_1, I_1 \cup (\Delta_1 \cup \{X \mapsto P_1\})\rangle \vDash \psi[\overline{x},x,\overline{X},X,\overline{u}].$$

Consider the fact that the IH was formulated for arbitrary extension assigning free variables it follows,

$$\forall P_1 \subseteq M_1^{ar(X)}\ \langle M_2, I_2 \cup (\tau \circ (\Delta_1 \cup \{X \mapsto P_1\}))\rangle \vDash \psi[\overline{x},\overline{X},X,\overline{u}].$$

This is the same as

$$\forall P_1 \subseteq M_1^{ar(X)}\ \langle M_2, I_2 \cup ((\tau \circ \Delta_1) \cup \{X \mapsto \tau(P_1)\})\rangle \vDash \psi[\overline{x},\overline{X},X,\overline{u}].$$

Due to the fact a predicate can be understood as a set and the fact that for every subset in $X \subseteq M_1^n$ one has $\tau(X) \subseteq M_2^n$ and for every subset in $X \subseteq M_2^n$ one has $\tau^{-1}(X) \subseteq M_1^n$

$$\forall P_2 \subseteq M_2^{ar(X)}\ \langle M_2,(I_2 \cup (\tau \circ \Delta_1)) \cup \{X \mapsto P_2\}\rangle \vDash \psi[\overline{x},\overline{X},X,\overline{u}],$$

which by semantics is equal to $\langle M_2, I_2 \cup (\tau \circ \Delta_1)\rangle \vDash \forall X\,\psi[\overline{x},\overline{X},\overline{u}]$.

- $\varphi[\overline{x},\overline{X},\overline{u}] = (\exists X\,\psi)[\overline{x},\overline{X},\overline{u}]$: Starting from $\langle M_1,\mathcal{I}_1\rangle \vDash (\exists X\,\psi)[\overline{x},\overline{X},\overline{u}]$. By semantics

$$\exists P_1 \subseteq M_1^{ar(X)}\ \langle M_1,((I_1 \cup \Delta_1) \cup \{X \mapsto P_1\})\rangle \vDash \psi[\overline{x},\overline{X},X,\overline{u}]$$

which is clearly the same as

$$\exists P_1 \subseteq M_1^{ar(X)}\ \langle M_1, I_1 \cup (\Delta_1 \cup \{X \mapsto P_1\})\rangle \vDash \psi[\overline{x},x,\overline{X},X,\overline{u}].$$

Consider the fact that the IH was formulated for arbitrary extension assigning free variables it follows,

$$\exists P_1 \subseteq M_1^{ar(X)}\ \langle M_2, I_2 \cup (\tau \circ (\Delta_1 \cup \{X \mapsto P_1\}))\rangle \vDash \psi[\overline{x},\overline{X},X,\overline{u}].$$

This is the same as

$$\exists P_1 \subseteq M_1^{ar(X)}\ \langle M_2, I_2 \cup ((\tau \circ \Delta_1) \cup \{X \mapsto \tau(P_1)\})\rangle \vDash \psi[\overline{x},\overline{X},X,\overline{u}].$$

Due to the fact a predicate can be understood as a set and the fact that for every subset in $X \subseteq M_1^n$ one has $\tau(X) \subseteq M_2^n$ and for every subset in $X \subseteq M_2^n$ one has $\tau^{-1}(X) \subseteq M_1^n$

$$\exists P_2 \subseteq M_2^{ar(X)}\ \langle M_2,(I_2 \cup (\tau \circ \Delta_1)) \cup \{X \mapsto P_2\}\rangle \vDash \psi[\overline{x},\overline{X},X,\overline{u}],$$

which by semantics is equal to $\langle M_2, I_2 \cup (\tau \circ \Delta_1)\rangle \vDash \exists X\,\psi[\overline{x},\overline{X},\overline{u}]$.

- $\varphi[\overline{x},\overline{X},\overline{u}] = (\forall u\,\psi)[\overline{x},\overline{X},\overline{u}]$: Starting from $\langle M_1,\mathcal{I}_1\rangle \vDash (\forall u\,\psi)[\overline{x},\overline{X},\overline{u}]$. By semantics

$$\forall f_1 : M_1^{ar(u)} \to M_1\ \langle M_1,((I_1 \cup \Delta_1) \cup \{u \mapsto f_1\})\rangle \vDash \psi[\overline{x},\overline{X},\overline{u},u]$$

which is clearly the same as

$$\forall f_1 : M_1^{ar(u)} \to M_1\ \langle M_1, I_1 \cup (\Delta_1 \cup \{u \mapsto f_1\})\rangle \vDash \psi[\overline{x},x,\overline{X},\overline{u},u].$$

Consider the fact that the IH was formulated for arbitrary extension assigning free variables it follows,

$$\forall f_1 : M_1^{ar(u)} \to M_1\ \langle M_2, I_2 \cup (\tau \circ (\Delta_1 \cup \{u \mapsto f_1\}))\rangle \vDash \psi[\overline{x},\overline{X},\overline{u},u].$$

This is the same as

$$\forall f_1 : M_1^{ar(u)} \to M_1\ \langle M_2, I_2 \cup ((\tau \circ \Delta_1) \cup \{u \mapsto \tau(f_1)\})\rangle \vDash \psi[\overline{x},\overline{X},\overline{u},u].$$

Due to the observation that $\tau((I_1 \cup \Delta_1)(u)) = (I_2 \cup (\tau \circ \Delta))(u) = \tau(f_1) = f_2$ and due to the fact that τ is bijective one obtains [2]

$$\forall f_2 : M_2^{ar(u)} \to M_2\ \langle M_2,(I_2 \cup (\tau \circ \Delta_1)) \cup \{u \mapsto f_2\}\rangle \vDash \psi[\overline{x},\overline{X},\overline{u},u],$$

[2] That is, for every function over M_1 one can find through τ a function in M_2 that behaves virtually the same and the other way round. Alternatively think of the fact a function can be understood as a predicate and the fact that for every subset in $X \subseteq M_1^n$ one has $\tau(X) \subseteq M_2^n$ and for every subset in $X \subseteq M_2^n$ one has $\tau^{-1}(X) \subseteq M_1^n$

By semantics is equal to $\langle M_2, I_2 \cup (\tau \circ \Delta_1)\rangle \vDash \forall u\, \psi[\overline{x}, \overline{X}, \overline{u}]$.

- $\varphi[\overline{x}, \overline{X}, \overline{u}] = (\exists u\, \psi)[\overline{x}, \overline{X}, \overline{u}]$: Starting from $\langle M_1, \mathcal{I}_1\rangle \vDash (\exists u\, \psi)[\overline{x}, \overline{X}, \overline{u}]$. By semantics

$$\exists f_1 : M_1^{ar(u)} \to M_1\ \langle M_1, ((I_1 \cup \Delta_1) \cup \{u \mapsto f_1\})\rangle \vDash \psi[\overline{x}, \overline{X}, \overline{u}, u]$$

which is clearly the same as

$$\exists f_1 : M_1^{ar(u)} \to M_1\ \langle M_1, I_1 \cup (\Delta_1 \cup \{u \mapsto f_1\})\rangle \vDash \psi[\overline{x}, x, \overline{X}, \overline{u}, u].$$

Consider the fact that the IH was formulated for arbitrary extension assigning free variables it follows,

$$\exists f_1 : M_1^{ar(u)} \to M_1\ \langle M_2, I_2 \cup (\tau \circ (\Delta_1 \cup \{u \mapsto f_1\}))\rangle \vDash \psi[\overline{x}, \overline{X}, \overline{u}, u].$$

This is the same as

$$\exists f_1 : M_1^{ar(u)} \to M_1\ \langle M_2, I_2 \cup ((\tau \circ \Delta_1) \cup \{u \mapsto \tau(f_1)\})\rangle \vDash \psi[\overline{x}, \overline{X}, \overline{u}, u].$$

Due to the observation that $\tau((I_1 \cup \Delta_1)(u)) = (I_2 \cup (\tau \circ \Delta))(u) = \tau(f_1) = f_2$ and due to the fact that τ is bijective one obtains

$$\exists f_2 : M_2^{ar(u)} \to M_2\ \langle M_2, (I_2 \cup (\tau \circ \Delta_1)) \cup \{u \mapsto f_2\}\rangle \vDash \psi[\overline{x}, \overline{X}, \overline{u}, u],$$

which by semantics is equal to $\langle M_2, I_2 \cup (\tau \circ \Delta_1)\rangle \vDash \exists u\, \psi[\overline{x}, \overline{X}, \overline{u}]$.

Clearly, the case for sentence is merely a special case of the previous proposition. That is, let φ be a second order sentence, let $\mathcal{M}_1 := \langle M_1, I_1\rangle$ and let $\mathcal{M}_2 := \langle M_2, I_2\rangle$ such that there exists an isomorphic function $\tau : M_1 \to M_2$ and let $\Delta_1 := \{\}$ be an arbitrary extension of I_1 assigning free variables to elements of the domain. Then by the proposition above

$$\langle M_1, (I_1 \cup \Delta_1)\rangle \vDash \varphi \iff \langle M_2, (I_2 \cup (\tau \circ \Delta_1))\rangle \vDash \varphi$$

which is the same as

$$\langle M_1, I_1\rangle \vDash \varphi \iff \langle M_2, I_2\rangle \vDash \varphi$$

Exercise 5

Let $\vdash$ **NK2** denote deducibility in the natural deduction calculus **NK2** in the course notes. For every rule r with a *variable/term side condition*, let $r^!$ denote the rule minus that condition. For each $r^!$, deduce $A_1, \ldots, A_n \vdash_{NK2\cup r^!} B$ where each A_i is a sentence that clearly is true in the standard model of arithmetic, and B is a sentence that clearly is not true (you should not justify that each A_i/B has this property, but it should be easy to see). In each deduction, aim to use as few proof rules as possible. Consider all the relevant introduction and elimination rules for first and second order universal and existential quantifiers. The answers for two rules are given below. Present your answer for the remaining rules in the same format, using the notation conventions from the course notes

$$\dfrac{\dfrac{[\forall x \exists y(y > x)]}{\exists y(y > x)[x/y]}\,(\forall^!_E)}{\exists y(y > y)} \qquad \dfrac{\dfrac{\dfrac{[\forall u \exists v \forall x(v(x) > u(x))]}{\exists v \forall x(v(x) > u(x))[u/\lambda x.t]}\,(\forall^!_E v \equiv \lambda x.t)}{\exists v \forall x(v(x) > (\lambda x.t)(x))}}{\exists v \forall x(v(x) > v(x))}$$

Starting with the first order cases.

1. The rule $\forall_E$, i.e. $\frac{\forall x A}{A[x/t]}$ where t does not contain a variable bound in A. Is already solved in the exercise description.

2. The rule $\exists_I$, i.e. $\frac{A[x/t]}{\exists x A}$ where t does not contain a variable bound in A. Consider the case

$$\dfrac{\dfrac{[\forall y(y < s(y))]}{\forall y(y < x)[x/s(y)]}}{\exists x \forall y(y < x)}\,(\exists^!_I)$$

3. The rule $\forall_I$, i.e. $\frac{\overset{\pi}{A[x/\alpha]}}{\forall x A}$ where α does not occur in A and not in π. Consider the case

$$\dfrac{\dfrac{[\forall y(y = y)]}{\forall y(x = y)[x/y]}}{\forall x \forall y(x = y)}\,(\forall^!_I)$$

4. The rule $\exists^i_E$, i.e. $\frac{\exists x B \quad \begin{matrix}[B[x/\alpha]]^i \\ \pi \\ A\end{matrix}}{A}$ where α does not occur in A and does not occur in B and not in the open assumptions π on the right side of the proof. Consider the case

$$\dfrac{[\exists x \exists y(x \neq y)] \qquad \dfrac{[\exists y(x \neq y)[x/y]]^1}{\exists y(y \neq y)}}{\exists y(y \neq y)}\,(\exists^{!\,1}_E)$$

Note that here the false sentence is a discharged hypothesis.

5. The rule $\forall_E$, i.e. $\frac{\forall XA}{A[X/\lambda\overline{x}.B]}$ where B does not contain a variable that is bound in A. Consider the case

$$\dfrac{\dfrac{\dfrac{[\forall X\exists Y\forall x((X(x)\to\neg Y(x))\wedge(\neg X(x)\to Y(x)))]}{\exists Y\forall x((X(x)\to\neg Y(x))\wedge(\neg X(x)\to Y(x)))[X/\lambda z.Y(z)]}(\forall_E^!)}{\exists Y\forall x(((\lambda z.Y(z))(x)\to\neg Y(x))\wedge(\neg(\lambda z.Y(z))(x)\to Y(x)))}}{\exists Y\forall x((Y(x)\to\neg Y(x))\wedge(\neg Y(x)\to Y(x)))}$$

6. The rule $\forall_E$, i.e. $\frac{\forall uA}{A[u/\lambda\overline{x}.t]}$ where t does not contain a variable that is bound in A. Is already solved in the exercise description.

7. The rule $\forall_I$, i.e. $\frac{\overset{\pi}{A[X/X_0]}}{\forall XA}$ where X_0 does not occur in A and not in π. Consider the case

$$\dfrac{\dfrac{[\forall Y\forall x(Y(x)\to Y(x))]}{\forall Y\forall x(X(x)\to Y(x))[X/Y]}}{\forall X\forall Y\forall x(X(x)\to Y(x))}(\forall_I^!)$$

8. The rule $\forall_I$, i.e. $\frac{\overset{\pi}{A[u/u_0]}}{\forall uA}$ where u_0 does not occur in A and not in π. Consider the case

$$\dfrac{\dfrac{[\forall v\forall x(v(x)=v(x))]}{\forall v\forall x(u(x)=v(x))[u/v]}}{\forall u\forall v\forall x(u(x)=v(x))}(\forall_I^!)$$

9. The rule $\exists_I$, i.e. $\frac{A[X/\lambda\overline{x}.B]}{\exists XA}$ where B does not contain a variable bound in A. Consider the case

$$\dfrac{\dfrac{\dfrac{[\forall Y\forall x((\neg Y(x)\to\neg Y(x))\wedge(\neg\neg Y(x)\to Y(x)))]}{\forall Y\forall x(((\lambda z.\neg Y(z))(x)\to\neg Y(x))\wedge(\neg(\lambda z.\neg Y(z))(x)\to Y(x)))}}{\forall Y\forall x((X(x)\to\neg Y(x))\wedge(\neg X(x)\to Y(x)))[X/\lambda z.\neg Y(z)]}}{\exists X\forall Y\forall x((X(x)\to\neg Y(x))\wedge(\neg X(x)\to Y(x)))}(\exists_I^!)$$

10. The rule $\exists_I$, i.e. $\frac{A[u/\lambda\overline{x}.t]}{\exists uA}$ where t does not contain a variable bound in A. Consider the case

$$\dfrac{\dfrac{\dfrac{[\exists x\forall y(y<s(y))]}{\exists x\forall y(y<(\lambda z.s(y))(x))}}{\exists x\forall y(y<u(x))[u/\lambda z.s(y)]}}{\exists u\exists x\forall y(y<u(x))}(\exists_I^!)$$

11. The rule $\exists^i_E$, i.e. $\dfrac{\exists X B \qquad \begin{matrix}[B[X/X_0]]^i \\ \pi \\ A\end{matrix}}{A}$ where X_0 does not occur in A and not in the open assumptions π on the right side of the proof. Consider the case

$$\dfrac{[\exists X \exists Y \forall x((X(x) \to \neg Y(x)) \wedge (\neg X(x) \to Y(x)))] \qquad \dfrac{[\exists Y \forall x((X(x) \to \neg Y(x)) \wedge (\neg X(x) \to Y(x)))[X/Y]]^1}{\exists Y \forall x((Y(x) \to \neg Y(x)) \wedge (\neg Y(x) \to Y(x)))}}{\exists Y \forall x((Y(x) \to \neg Y(x)) \wedge (\neg Y(x) \to Y(x)))} \, (\exists!^1_E)$$

Note that here the false sentence is a discharged hypothesis.

12. The rule $\exists^i_E$, i.e. $\dfrac{\exists u B \qquad \begin{matrix}[B[u/u_0]]^i \\ \pi \\ A\end{matrix}}{A}$ where u_0 does not occur in A and not in the open assumptions π on the right side of the proof. Consider the case

$$\dfrac{[\exists u \exists v \forall x(u(x) \neq v(x))] \qquad \dfrac{[\exists v \forall x(u(x) \neq v(x))[u/v]]^1}{\exists v \forall x(v(x) \neq v(x))}}{\exists v \forall x(v(x) \neq v(x))} \, (\exists!^1_E)$$

Note that here the false sentence is a discharged hypothesis.