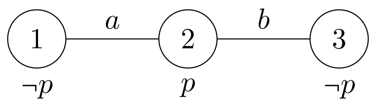
Exercise 44

(Analogously to exercise 38:) Separate $\mathbf{D}_G$ from $\mathbf{K}_i$, $\mathbf{S}_G$, and $\mathbf{S}_G\mathbf{S}_G$ in a single connected model, if possible.

Firstly, let $ab := G = \{a, b\}$. Consider the following epistemic model $\mathcal{M}$.



1. $(\mathbf{D}_G, \mathbf{K}_a)$:
   Consider state 2. That is, $\mathcal{M}, 2 \nvDash \mathbf{K}_a p$ due to 1, and $\mathcal{M}, 2 \vDash \mathbf{D}_G p$, because taking the intersection of R_a and R_b it follows that state 2 is isolated. Hence all states accessible through R_{D_G} satisfy p.

2. $(\mathbf{D}_G, \mathbf{K}_b)$:
   Consider state 2. That is, $\mathcal{M}, 2 \nvDash \mathbf{K}_b p$ due to 3, and $\mathcal{M}, 2 \vDash \mathbf{D}_G p$, because taking the intersection of R_a and R_b it follows that state 2 is isolated. Hence all states accessible through R_{D_G} satisfy p.

3. $(\mathbf{D}_G, \mathbf{S}_G)$:
   Consider state 2. That is, $\mathcal{M}, 2 \nvDash \mathbf{S}_G p$ due to $\mathcal{M}, 2 \vDash \mathbf{K}_a p$ and $\mathcal{M}, 2 \vDash \mathbf{K}_b p$. Moreover, $\mathcal{M}, 2 \vDash \mathbf{D}_G p$, because taking the intersection of R_a and R_b it follows that state 2 is isolated. Hence all states accessible through R_{D_G} satisfy p.

4. $(\mathbf{D}_G, \mathbf{S}_G\mathbf{S}_G)$:
   Consider state 2. That is, $\mathcal{M}, 2 \nvDash \mathbf{S}_G p$, due to the fact that, as established above, $\mathcal{M}, 2 \nvDash \mathbf{S}_G p$ there exists at least one state accessible form 2 via R_a and via R_b such that $\mathbf{S}_G p$ does not hold. Hence, $\mathcal{M}, 2 \nvDash \mathbf{K}_a \mathbf{S}_G p$ and $\mathcal{M}, 2 \nvDash \mathbf{K}_b \mathbf{S}_G p$. Moreover, $\mathcal{M}, 2 \vDash \mathbf{D}_G p$, because taking the intersection of R_a and R_b it follows that state 2 is isolated. Hence all states accessible through R_{D_G} satisfy p.

Exercise 45

Present a Kripke- and a Beth-countermodel for $\neg\neg A \supset A$.

Before moving forward, an overview of the required definitions.

Definition 1. *A model* $\mathcal{M} := \langle M, \leqslant, D, \Vdash \rangle$.

- $(M, \leqslant)$ *is a partially ordered set.*
- D *assigns a structure to* $\gamma \in M$*, s.t.* $\alpha, \beta \in M$ $\alpha \leqslant \beta \Rightarrow D(\alpha) \subseteq D(\beta)$*. (subset relation)*

- $\Vdash \subseteq M \times M$.

  1. $\alpha \Vdash p$, *if there is a bar* B *for* α, *s.t.* $\forall \beta \in B$, $D(\beta) \vDash p$;
  2. $\alpha \Vdash \varphi \wedge \psi$, *if* $\alpha \Vdash \varphi$ *and* $\alpha \Vdash \psi$;
  3. $\alpha \Vdash \varphi \vee \psi$, *if there is a bar* B *for* α, *s.t.* $\forall \beta \in B(\beta \Vdash \varphi$ *or* $\beta \Vdash \psi)$;
  4. $\alpha \Vdash \varphi \supset \psi$, *if* $\forall \beta \geqslant \alpha(\beta \Vdash \varphi \Rightarrow \beta \Vdash \psi)$;
  5. $\alpha \Vdash \forall x\, \varphi(x)$, *if* $\forall \beta \geqslant \alpha(\forall b \in |D(\beta)|\, \beta \Vdash \varphi(b))$;
  6. $\alpha \Vdash \exists x\, \varphi(x)$, *if there is a bar* B *for* α, *s.t.* $\forall \beta \in B(\exists b \in |D(\beta)|\, \beta \Vdash \varphi(b))$;
  7. $\alpha \Vdash \neg\varphi$, *if* $\forall \beta \geqslant \alpha(\beta \nVdash \varphi)$.

Moreover,

Definition 2. *A formula* φ *holds in a model* $\mathcal{M}$ *if* $\alpha \Vdash cl(\varphi)$ *for all* α, *where* $cl(\varphi)$ *is the universal closure of* φ.

Furthermore, a nice lemma was also presented.

Lemma 0.1. *The following statements hold:*

1. *For* $\alpha \leqslant \beta$, $\alpha \Vdash \varphi \Rightarrow \beta \Vdash \varphi$;
2. *For* $\alpha \nVdash \varphi \Leftrightarrow$ *there is a path* P *through* α *such that* $\forall \beta \in P(\beta \nVdash \varphi)$;
3. *For* $\alpha \Vdash \varphi \Leftrightarrow$ *there is a bar* B *for* α *such that* $\forall \beta \in B(\beta \Vdash \varphi)$;

Moreover, the definition for a Beth model

Definition 3. $\mathcal{M}$ *is a Beth model if* $|D(\alpha)|$ *is a fixed set* D *for all* α.

1. $\alpha \Vdash p$, *if there is a bar* B *for* α, *s.t.* $\forall \beta \in B$, $D(\beta) \vDash p$;
2. $\alpha \Vdash \varphi \wedge \psi$, *if* $\alpha \Vdash \varphi$ *and* $\alpha \Vdash \psi$;
3. $\alpha \Vdash \varphi \vee \psi$, *if there is a bar* B *for* α, *s.t.* $\forall \beta \in B(\beta \Vdash \varphi$ *or* $\beta \Vdash \psi)$;
4. $\alpha \Vdash \varphi \supset \psi$, *if* $\forall \beta \geqslant \alpha(\beta \Vdash \varphi \Rightarrow \beta \Vdash \psi)$;
5. $\alpha \Vdash \forall x\varphi(x) \Leftrightarrow \forall a \in D(\alpha \Vdash \varphi(a))$
6. $\alpha \Vdash \exists x\, \varphi(x)$, *if there is a bar* B *for* α, *s.t.* $\forall \beta \in B(\exists b \in |D(\beta)|\, \beta \Vdash \varphi(b))$;
7. $\alpha \Vdash \neg\varphi$, *if* $\forall \beta \geqslant \alpha(\beta \nVdash \varphi)$.

and a Kripke model is defined as

Definition 4. $\mathcal{M}$ *is a Kripke model if in (1), (3) and (6),* $B = \{a\}$, *i.e.*

1. $\alpha \Vdash p$, *if* $D(\alpha) \vDash p$;

2. $\alpha \Vdash \varphi \wedge \psi$, *if* $\alpha \Vdash \varphi$ *and* $\alpha \Vdash \psi$;

3. $\alpha \Vdash \varphi \vee \psi$, *if* $\alpha \Vdash \varphi$ *or* $\alpha \Vdash \psi$;

4. $\alpha \Vdash \varphi \supset \psi$, *if* $\forall \beta \geqslant \alpha(\beta \Vdash \varphi \Rightarrow \beta \Vdash \psi)$;

5. $\alpha \Vdash \forall x\, \varphi(x)$, *if* $\forall \beta \geqslant \alpha(\forall b \in |D(\beta)|\ \beta \Vdash \varphi(b))$;

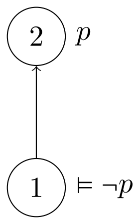
6. $\alpha \Vdash \exists x\, \varphi(x)$, *if* $\exists a \in |D(\alpha)|\ \alpha \Vdash \varphi(a)$;

7. $\alpha \Vdash \neg\varphi$, *if* $\forall \beta \geqslant \alpha(\beta \nVdash \varphi)$.

Starting with the semantic unravelling of the sentence $\neg\neg\varphi \supset \varphi$.

$$\begin{array}{ll}
\alpha \Vdash \neg\neg\varphi \supset \varphi & \Longleftrightarrow \\
\forall \beta \geqslant \alpha(\beta \Vdash \neg\neg\varphi \Rightarrow \beta \Vdash \varphi) & \Longleftrightarrow \\
\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\mathit{not}\ \gamma \Vdash \neg\varphi) \Rightarrow \beta \Vdash \varphi) & \Longleftrightarrow \\
\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\mathit{not}\ (\forall \delta \geqslant \gamma\ \mathit{not}\ \delta \Vdash \varphi)) \Rightarrow \beta \Vdash \varphi) & \Longleftrightarrow \\
\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\exists \delta \geqslant \gamma\ \delta \Vdash \varphi) \Rightarrow \beta \Vdash \varphi) & \Longleftrightarrow
\end{array}$$

Firstly, consider the following Kripke model.



where $|D(1)| = |D(2)| = \{a\}$ and p is a predicate of arity 0. First, one has to confirm that this model actually satisfies the required properties. Clearly, the set of worlds is a partial order (reflexive edges are not drawn). Since $1 \leqslant 2$ and $D(1) \nvDash p$ and $D(2) \vDash p$, it is the case that $D(1) \subseteq D(2)$. Hence, given

$$\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\exists \delta \geqslant \gamma\ \delta \Vdash p) \Rightarrow \beta \Vdash p)$$

and the fact that this is a Kripke model it follows

$$\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\exists \delta \geqslant \gamma\ D(\delta) \vDash p) \Rightarrow D(\beta) \vDash p)$$

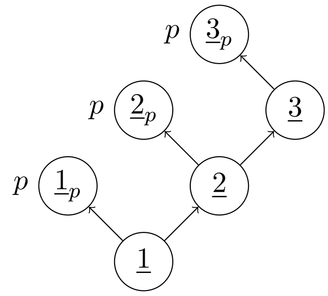
Now consider 1 as α and 1 as β, by reflexivity $1 \geqslant 1$, resulting in

$$\forall \gamma \geqslant 1(\exists \delta \geqslant \gamma\ D(\delta) \vDash p) \Rightarrow D(1) \vDash p$$

Clearly $D(1) \vDash p$ can not be the case. Hence, if the premise is correct $\mathcal{M}$ is a counter model. To establish exactly that two case distinctions are required.

- *Case 1:* For γ is 1, we have $2 \geqslant 1$ such that $D(2) \vDash p$.
- *Case 2:* For γ is 2, we have $2 \geqslant 2$ such that $D(2) \vDash p$.

Hence, the premise of the implication is satisfied by $\mathcal{M}$, thus a counter Kripke model is found.
Secondly, consider the following Beth model $\mathcal{M} := \langle M, \leqslant, D, \Vdash \rangle$. Where $M := M_o \cup M_p = \{\underline{1}, \underline{2}, \underline{3} \dots\} \cup \{\underline{1}_p, \underline{2}_p, \underline{3}_p, \dots\}$ and $\leqslant$ is the reflexive, anti-symmetric and transitive closure of $\{(\underline{1}, \underline{1}_p), (\underline{1}, \underline{2}), (\underline{2}, \underline{2}_p), (\underline{2}, \underline{3}), (\underline{3}, \underline{3}_p), \dots\}$, as well as $\forall \alpha \in M_o\ D(\alpha) \nvDash p$ and $\forall \alpha \in M_p\ D(\alpha) \vDash p$. The following is a visualisation for the first three steps.



First, one has to confirm that this model actually satisfies the required properties. Clearly, the set of worlds is a partial order (reflexive and transitive edges are not drawn). Moreover, $\forall \alpha |D(\alpha)| = \{\}$, since only propositional statements are considered. For any arbitrary $k > 0$ it follows that, $\underline{k} \leqslant \underline{k}_p$ and $D(\underline{k}) \nvDash p$ and $D(\underline{k}_p) \vDash p$, it is the case that $D(\underline{k}) \subseteq D(\underline{k}_p)$. Similarly, since $\underline{k} \leqslant \underline{k+1}$ and $D(\underline{k}) \nvDash p$ and $D(\underline{k+1}) \nvDash p$, it is the case that $D(\underline{k}) \subseteq D(\underline{k+1})$. Hence, given

$$\forall \beta \geqslant \alpha (\forall \gamma \geqslant \beta (\exists \delta \geqslant \gamma\ \delta \Vdash p) \Rightarrow \beta \Vdash p)$$

and the fact that this is a Beth model it follows

$$\forall \beta \geqslant \alpha (\forall \gamma \geqslant \beta (\exists \delta \geqslant \gamma\ \exists \mathcal{B}_\delta \forall \epsilon \in \mathcal{B}_\delta\ D(\epsilon) \vDash p) \Rightarrow \exists \mathcal{B}_\beta \forall \gamma \in \mathcal{B}_\beta\ D(\gamma) \vDash p)$$

Where $\exists \mathcal{B}_\alpha \forall \beta \in \mathcal{B}_\alpha\ D(\beta) \vDash p$ is a shorthand for "if there is a bar B for α, s.t. $\forall \beta \in B$, $D(\beta) \vDash p$".

Now consider $\underline{1}$ as α. The statement $\exists \mathcal{B}_\beta \forall \gamma \in \mathcal{B}_\beta\ D(\gamma) \vDash p$ can not hold due to the fact that it would require that at some point there exists a bar, such that for all states in the bar it follows that p holds. However, with M_o being infinite and $\underline{1} \leqslant \underline{k}$ for $\underline{k} \in M_o$ such bar can not exist. That is, at every point of the path $\underline{1}, \underline{2}, \dots, \underline{k}$ we know that p can not hold. Moreover, it is possible to find an path of arbitrary length of that kind. Hence, for any given bar, there exists a path of that kind that intersects with this bar. Thereby, invalidating the statement $\exists \mathcal{B}_\beta \forall \gamma \in \mathcal{B}_\beta\ D(\gamma) \vDash p$.

Hence, if the premise is correct $\mathcal{M}$ is a counter model. To establish exactly that, two case distinctions are required. Consider an arbitrary $k \geqslant 1$

- *Case 1:* For γ is $\underline{k}$, we have $\underline{k}_p$ as δ due to $\underline{k}_p \geqslant \underline{k}$ such that $D(\underline{k}_p) \vDash p$. In this case $\mathcal{B}_\delta = \mathcal{B}_{\underline{k}_b} = \{\underline{k}_b\}$.
- *Case 2:* For γ is $\underline{k}_p$, we have $\underline{k}_p$ as δ due to $\underline{k}_p \geqslant \underline{k}_p$ such that $D(\underline{k}_p) \vDash p$. In this case $\mathcal{B}_\delta = \mathcal{B}_{\underline{k}_b} = \{\underline{k}_b\}$.

Hence, the premise of the implication is satisfied by $\mathcal{M}$, thus a counter Beth model is found.

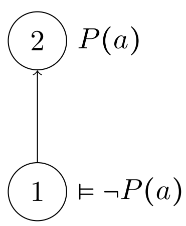
Exercise 46

Present a Kripke-countermodel for $\neg\forall x \neg P(x) \supset \exists x P(x)$.

Starting with the semantic unravelling of the sentence $\neg\forall x \neg P(x) \supset \exists x P(x)$ with respect to a Kripke model.

$$\neg\forall x \neg P(x) \supset \exists x P(x)$$
$$\forall \beta \geqslant \alpha(\beta \Vdash \neg\forall x \neg P(x) \Rightarrow \beta \Vdash \exists x P(x))$$
$$\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\mathit{not}\ \gamma \Vdash \forall x \neg P(x)) \Rightarrow \beta \Vdash \exists x P(x))$$
$$\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\mathit{not}\ \forall \delta \geqslant \gamma \forall x_\delta \in |D(\delta)|(\delta \Vdash \neg P(x_\delta))) \Rightarrow \beta \Vdash \exists x P(x))$$
$$\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\mathit{not}\ \forall \delta \geqslant \gamma \forall x_\delta \in |D(\delta)|(\forall \epsilon \geqslant \delta(\mathit{not}\ D(\epsilon) \vDash P(x_\delta)))) \Rightarrow \beta \Vdash \exists x P(x))$$
$$\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\mathit{not}\ \forall \delta \geqslant \gamma \forall x_\delta \in |D(\delta)|(\forall \epsilon \geqslant \delta(\mathit{not}\ D(\epsilon) \vDash P(x_\delta)))) \Rightarrow \exists x_\beta \in |D(\beta)|(D(\beta) \vDash P(x_\beta)))$$
$$\forall \beta \geqslant \alpha(\forall \gamma \geqslant \beta(\exists \delta \geqslant \gamma \exists x_\delta \in |D(\delta)|(\exists \epsilon \geqslant \delta(D(\epsilon) \vDash P(x_\delta)))) \Rightarrow \exists x_\beta \in |D(\beta)|(D(\beta) \vDash P(x_\beta)))$$

Consider the following Kripke model $\mathcal{M}$.



where $|D(1)| = |D(2)| = \{a\}$ and $D(1) \nvDash P(a)$ while $D(2) \vDash P(a)$.
First, one has to confirm that this model actually satisfies the required properties. Clearly, the set of worlds is a partial order (reflexive edges are not drawn). Since $1 \leqslant 2$ and $D(1) \nvDash P(a)$ and $D(2) \vDash P(a)$, it is the case that $D(1) \subseteq D(2)$.

Now consider 1 as α and 1 as β, by reflexivity $1 \geqslant 1$, resulting in

$$\forall \gamma \geqslant \beta (\exists \delta \geqslant \gamma \exists x_\delta \in |D(\delta)| (\exists \epsilon \geqslant \delta (D(\epsilon) \vDash P(x_\delta)))) \Rightarrow \exists x_\beta \in |D(\beta)| (D(\beta) \vDash P(x_\beta))$$

With a being the only element in the domain and with $D(1) \nvDash P(a)$ it follows that $\exists x_1 \in |D(1)|\ D(1) \vDash P(x_1)$ can not hold. Hence, if the premise is correct $\mathcal{M}$ is a counter model. To establish exactly that two case distinctions are required.

- *Case 1:* For γ is 1, we have $2 \geqslant 1$ for δ and $2 \geqslant 2$ for ϵ such that $D(2) \vDash P(a)$.
- *Case 2:* For γ is 2, we have $2 \geqslant 2$ for δ and $2 \geqslant 2$ for ϵ such that $D(2) \vDash P(a)$.

Hence, the premise of the implication is satisfied by $\mathcal{M}$, thus a counter Kripke model is found.

Exercise 47

> Consider the classical laws of distribution ($\vee$ over $\wedge$, $\wedge$ over $\vee$). Which parts of these laws (implications) hold and which fail for intuitionistic logic? Provide sequent or natural deduction proofs for the positive cases and Kripke and/or Beth counterexamples for the negative cases.

As far as I am aware the laws in question are:

1. $(P \wedge (Q \vee R)) \supset ((P \wedge Q) \vee (P \wedge R))$
2. $(P \vee (Q \wedge R)) \supset ((P \vee Q) \wedge (P \vee R))$
3. $((P \wedge Q) \vee (P \wedge R)) \supset (P \wedge (Q \vee R))$
4. $((P \vee Q) \wedge (P \vee R)) \supset (P \vee (Q \wedge R))$

Note that the inference

$$\frac{\Gamma, \psi, \chi \ \vdash \ \varphi}{\Gamma, \psi \wedge \chi \ \vdash \ \varphi}$$

is a short cut for

$$\cfrac{\cfrac{\cfrac{\Gamma, \psi, \chi \ \vdash \ \varphi}{\Gamma, \psi, \psi \wedge \chi \ \vdash \ \varphi}}{\Gamma, \psi \wedge \chi, \psi \wedge \chi \ \vdash \ \varphi}}{\Gamma, \psi \wedge \chi \ \vdash \ \varphi}$$

For $(P \wedge (Q \vee R)) \supset ((P \wedge Q) \vee (P \wedge R))$ the sequent proof is

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, Q \vdash P} \quad \dfrac{\dfrac{}{Q \vdash Q}}{P, Q \vdash Q}}{P, Q \vdash P \wedge Q}}{P, Q \vdash (P \wedge Q) \vee (P \wedge R)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, R \vdash P} \quad \dfrac{\dfrac{}{R \vdash R}}{P, R \vdash R}}{P, R \vdash P \wedge R}}{P, R \vdash (P \wedge Q) \vee (P \wedge R)}}{P, (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R)}}{P \wedge (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R)}}{\vdash (P \wedge (Q \vee R)) \supset ((P \wedge Q) \vee (P \wedge R))}
$$

For $(P \vee (Q \wedge R)) \supset ((P \vee Q) \wedge (P \vee R))$ the sequent proof is

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P \vdash P \vee Q} \quad \dfrac{\dfrac{}{P \vdash P}}{P \vdash P \vee R}}{P \vdash (P \vee Q) \wedge (P \vee R)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q \vdash Q}}{Q, R \vdash Q}}{Q, R \vdash P \vee Q} \quad \dfrac{\dfrac{\dfrac{}{R \vdash R}}{Q, R \vdash R}}{Q, R \vdash P \vee R}}{Q, R \vdash (P \vee Q) \wedge (P \vee R)}}{Q \wedge R \vdash (P \vee Q) \wedge (P \vee R)}}{P \vee (Q \wedge R) \vdash (P \vee Q) \wedge (P \vee R)}}{\vdash (P \vee (Q \wedge R)) \supset ((P \vee Q) \wedge (P \vee R))}
$$

For $((P \wedge Q) \vee (P \wedge R)) \supset (P \wedge (Q \vee R))$ the sequent proof is

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, Q \vdash P} \quad \dfrac{\dfrac{\dfrac{}{Q \vdash Q}}{P, Q \vdash Q}}{P, Q \vdash Q \vee R}}{P, Q \vdash P \wedge (Q \vee R)}}{P \wedge Q \vdash P \wedge (Q \vee R)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, R \vdash P} \quad \dfrac{\dfrac{\dfrac{}{R \vdash R}}{P, R \vdash R}}{P, R \vdash Q \vee R}}{P, R \vdash P \wedge (Q \vee R)}}{P \wedge R \vdash P \wedge (Q \vee R)}}{(P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R)}}{\vdash ((P \wedge Q) \vee (P \wedge R)) \supset (P \wedge (Q \vee R))}
$$

For $((P \vee Q) \wedge (P \vee R)) \supset (P \vee (Q \wedge R))$ the sequent proof is

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, P \vdash P}}{P, P \vdash P \vee (Q \wedge R)} \quad \dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, R \vdash P}}{P, R \vdash P \vee (Q \wedge R)}}{P, (P \vee R) \vdash P \vee (Q \wedge R)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{Q, P \vdash P}}{Q, P \vdash P \vee (Q \wedge R)} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Q \vdash Q}}{Q, R \vdash Q} \quad \dfrac{\dfrac{}{R \vdash R}}{Q, R \vdash R}}{Q, R \vdash (Q \wedge R)}}{Q, R \vdash P \vee (Q \wedge R)}}{Q, (P \vee R) \vdash P \vee (Q \wedge R)}}{(P \vee Q), (P \vee R) \vdash P \vee (Q \wedge R)}}{(P \vee Q) \wedge (P \vee R) \vdash P \vee (Q \wedge R)}}{\vdash ((P \vee Q) \wedge (P \vee R)) \supset (P \vee (Q \wedge R))}
$$