Exercise 1 (5 credits)**.** Recall the $\Sigma_2\mathsf{P}$-hardness proof of the Abduction Solvability problem by reduction from QSAT_2: Let an arbitrary instance of the QSAT_2 problem be given by the formula $\varphi = (\exists X)(\forall Y)\psi(X,Y)$ with $X = \{x_1,\ldots,x_k\}$ and $Y = \{y_1,\ldots,y_l\}$. Moreover, let $X' = \{x'_1,\ldots,x'_k\}$, $R = \{r_1,\ldots,r_k\}$, and t be fresh variables. Then we define an instance of Solvability as $\mathcal{P} = \langle V,H,M,T\rangle$ with

$$\begin{aligned}
V &= X \cup Y \cup X' \cup R \cup \{t\}\\
H &= X \cup X'\\
M &= R \cup \{t\}\\
T &= \{\psi(X,Y) \rightarrow t\} \cup \{\neg x_i \vee \neg x'_i, x_i \rightarrow r_i, x'_i \rightarrow r_i \mid 1 \leq i \leq k\}
\end{aligned}$$

Give a rigorous correctness proof of this problem reduction, i.e., $\varphi \equiv \textbf{true} \Leftrightarrow \mathit{Sol}(\mathcal{P}) \neq \emptyset$.

Hint. As usual, prove both directions of the equivalence separately. It is convenient to use the notation from the lecture: For $A \subseteq X$, let A' denote the set $\{x' \mid x \in A\}$.

- For the "$\Rightarrow$"-direction, you start off with a partial assignment I on X. Let $A = I^{-1}(\textbf{true})$. Then it can be shown that $S = A \cup (X \setminus A)'$ is a solution of $\mathcal{P}$. In order to show that S is indeed a solution, you must prove carefully the two conditions that (1) $T \cup S$ is satisfiable and (2) $T \cup S \models M$.
- For the "$\Leftarrow$"-direction, first show that a solution S of $\mathcal{P}$ contains exactly one of $\{x_i, x'_i\}$. Why? Hence, S must be of the form $S = A \cup (X \setminus A)'$ for some $A \subseteq X$. It remains to show that for the assignment I on X with $I^{-1}(\textbf{true}) = A$, every extension J of I to the variables Y satisfies the formula $\psi(X,Y)$.

Solution

Lemma 2. *Let $\varphi := \exists X \forall Y\ \psi(X,Y)$ be a $\mathrm{QBF}_{\exists,2}$-formula, with $X = \{x_1,\ldots,x_k\}$ and $Y = \{y_1,\ldots,y_l\}$. Moreover, let $\tau(\varphi)$ the abduction instance presented in the reduction. Then $\varphi \equiv$ **true** $\iff$ $\mathit{Sol}(\mathcal{P}) \neq \emptyset$.*

Proof. $\Rightarrow$ Assume $\varphi \equiv \textbf{true}$ then there exists a partial assignment $\mathcal{I}_{|X}$ for the variables in X such that for any extension $\mathcal{I}_{|X\cup Y}$ by the variables in Y one has $\mathcal{I}_{|X\cup Y} \models \psi(X,Y)$. Using the partial assignment $\mathcal{I}_{|X}$ one can build the set $A := \{x \mid \mathcal{I}_{|X} \models x\}$. Now consider the set $S := A \cup (X \setminus A)'$. As suggested, the claim to be demonstrated is that S is a solution for $\tau(\varphi)$. To that end consider $T \cup S$, take an arbitrary model $\mathcal{J}$ satisfying $T \cup S$. Notice that currently it is not known that such a model actually exist. Firstly, since $\mathcal{J} \models S$ it must be that by construction one has $\forall x \in A\ \mathcal{J} \models x$ and $\forall x \in X \setminus A\ \mathcal{J} \models x'$. Since $x \in S$ if and only if $x' \notin S$, and the fact that for some $i \in \{1,\ldots k\}$ $\mathcal{J} \models \neg x_i \vee \neg x'_i$, one has $\mathcal{J} \models x$ if and only if $\mathcal{J} \not\models x'$. Therefore, for every $i \in \{1,\ldots,k\}$ $\mathcal{J}$ models either x_i or x'_i, and thus it follows that $\mathcal{J} \models r_i$. Secondly, by virtue of A being constructed from $\mathcal{I}_{|X}$ and by the fact that $\mathcal{J} \models A$, every x_i satisfied by $\mathcal{I}_{|X}$ must be satisfied by $\mathcal{J}$. Moreover, if there would exist an x_i such that $\mathcal{J} \models x_i$ but $\mathcal{I}_{|X} \not\models x_i$, then $x'_i \in S$ and thus $\mathcal{J} \models x'_i$. Thereby, violating the fact that $\mathcal{J} \models \neg x_i \vee \neg x'_i$. Hence, it is known that J agrees on $I_{|X}$ on the variables X. Therefore, it follows $\mathcal{J} \models \psi(X,Y)$ regardless of the truth values of the variables in Y under $\mathcal{J}$. Hence, it must be that $\mathcal{J} \models t$, which was the last piece required to establish that $\mathcal{J} \models M$. Thus one can conclude that $T \cup S \models M$. What remains is to verify that such a model actually exists, that is let $\mathcal{J}_e$ be the interpretation

- $\forall x \in A\ \mathcal{J}_e(x) := \mathbf{true}$;
- $\forall x \in X \setminus A\ \mathcal{J}_e(x') := \mathbf{true}$;
- $\forall r \in R\ \mathcal{J}_e(r) := \mathbf{true}$;
- $\forall y \in Y\ \mathcal{J}_e(y) := \mathbf{true}$;
- $\mathcal{J}_e(t) := \mathbf{true}$.

Since, $\mathcal{J}_e$ agrees with $\mathcal{I}_{|X}$ on all variables on X, the assignment of $y \in Y$ is irrelevant and thus it follows that $\mathcal{J}_e \models \psi(X, Y)$. With $\mathcal{J}_e \models \psi(X, Y) \land t$, it must be that $\mathcal{J}_e \models \psi(X, Y) \to t$. By construction, for an arbitrary $i \in \{1, \ldots, k\}$, $\mathcal{J}_e \models x_i$ iff $\mathcal{J}_e \not\models x'_i$ and thus $\mathcal{J}_e \models \neg x_i \lor \neg x'_i$. Lastly, since for any $i \in i \in \{1, \ldots, k\}$ one has, $\mathcal{J}_e \models x_i \lor x'_i$ it follows that $x_i \to r$ and $x'_i \to r$ are satisfied. Hence, $\mathcal{J}_e \models T \cup S$. Thereby, establishing that S is indeed a solution.

$\Leftarrow$ Assume that there exists a solution S for $\tau(\varphi)$. For S to be a solution it must that $S \subseteq H$, $T \cup S$ is satisfiable and that $T \cup S \models M$. Hence, it is known that there exists an interpretation $\mathcal{I}$ that $\mathcal{I} \models T \cup S$ and $\mathcal{I} \models M$. Now restrict $\mathcal{I}$ by removing the assignments concerning the variables in Y to create $\mathcal{J}$. Furthermore, assume that $\mathcal{J}'$ is an extension of $\mathcal{J}$ by the variables in Y such that $\mathcal{J}' \not\models \psi(X, Y)$. Since $\mathcal{J}'$ agrees with $\mathcal{I}$ on all variables in R, X and X', $\mathcal{J}' \models S \cup \{\neg x_i \lor \neg x'_i, x_i \to r_i, x'_i \to r_i \mid 1 \leq i \leq k\}$. However, since by assumption, $\mathcal{J}' \not\models \psi(X, Y)$, it must be that $\mathcal{J}' \models \psi(X, Y) \to t$ holds. Therefore, it follows that $\mathcal{J}' \models T \cup S$. Moreover, with $\psi(X, Y) \to t$ being vacuously true under $\mathcal{J}'$, there must exists an interpretation where t is not satisfied. W.l.o.g. let $\mathcal{J}'$ be this interpretation. Therefore, $\mathcal{J}' \models T \cup S$ but $\mathcal{J}' \not\models M$, which is clearly a contradiction. Hence, in every extension of $\mathcal{J}$ to the variables in Y, it must be that $\psi(X, Y)$ is satisfied. Now, by restricting $\mathcal{J}$ to the variables in X one has found a partial assignments for the variables in X where every extension by the variables Y satisfies the formula $\psi(X, Y)$. Meaning that $\varphi \equiv \mathbf{true}$.

□

Exercise 3 (5 credits)**.** Recall the Σ_2P-hardness proof of the Abduction Relevance problem by reduction from the Solvability problem: Let an arbitrary instance of the Solvability problem be given by the PAP $\mathcal{P} = \langle V, H, M, T \rangle$. W.l.o.g., let T consist of a single formula φ and let h, h', m' be fresh variables. Then we define an instance of the Relevance (resp. the Necessity) problem with the following PAP $\mathcal{P}' = \langle V', H', M', T' \rangle$:

$$\begin{aligned} V' &= V \cup \{h, h', m'\} \\ H' &= H \cup \{h, h'\} \\ M' &= M \cup \{m'\} \\ T' &= \{\neg h \vee \varphi\} \cup \{h' \to m \mid m \in M\} \cup \{\neg h \vee \neg h', h \to m', h' \to m'\} \end{aligned}$$

This reduction fulfills the following equivalences:

$\mathcal{P}$ has at least one solution iff h is relevant in $\mathcal{P}'$ iff h' is not necessary in $\mathcal{P}'$.

Give a rigorous proof of these equivalences.

Hint. The second equivalence is easy to show. For Then first equivalence, show both directions separately:

- For the "$\Rightarrow$"-direction, you start off with a solution S of $\mathcal{P}$ and construct a solution S' of $\mathcal{P}'$ with $h \in S'$. Prove carefully that S' is indeed a solution of $\mathcal{P}'$, i.e. (1) $T' \cup S'$ is satisfiable and (2) $T' \cup S' \models M'$.
- For the "$\Leftarrow$"-direction, you start off with a solution S' of $\mathcal{P}'$, s.t. $h \in S'$ and construct a solution S of $\mathcal{P}$. Prove carefully that S is indeed a solution of $\mathcal{P}$, i.e. (1) $T \cup S$ is satisfiable and (2) $T \cup S \models M$.

Solution

Observation 4. *Take an arbitrary PAP $\mathcal{P} = \langle V, H, M, T \rangle$. Let $\mathcal{P}_\tau$ be the PAP as constructed in the reduction. Then for any solution $S \in Sol(\mathcal{P}_\tau)$ it must be the case that $h \in S$ if and only if $h' \notin S$.*

Proof. For any solution $S \in Sol(\mathcal{P}_\tau)$ it must hold that $T \cup S \models m'$. However, this requires that either $h \in S$ or $h' \in S$. Assume that $h \in S$. Hence, any interpretation $\mathcal{I}$ satisfying $T \cup S$ must satisfy h. Moreover, it must also satisfy $\neg h \vee \neg h'$, which can only be the case if $\mathcal{I} \not\models h'$. Now assume that $h' \in S$ if this is the case $\mathcal{I} \models h'$, which is impossible. That is, there can not be an interpretation modelling S. However, since S is a solution, it must be that there exists at least one model of S. Thus the only possible conclusion is that $h' \notin S$. Due to symmetry the other direction can be done in analogue. $\square$

Lemma 5. *Take an arbitrary PAP $\mathcal{P} = \langle V, H, M, T \rangle$. Let $\mathcal{P}_\tau$ be the PAP as constructed in the reduction. Then h is relevant in $\mathcal{P}_\tau$ iff h' is not necessary in $\mathcal{P}_\tau$.*

Proof. By definition, h is relevant in $\mathcal{P}_\tau$ there exists a solution $S \in Sol(\mathcal{P}_\tau)$ such that $h \in S$. By Observation 4, this is equivalent to there exists a solution $S \in Sol(\mathcal{P}_\tau)$ such that $h' \notin S$. By definition, this is equivalent to h' is not necessary in $\mathcal{P}_\tau$. $\square$

Lemma 6. *Take an arbitrary PAP $\mathcal{P} = \langle V, H, M, T \rangle$. Let $\mathcal{P}_\tau$ be the PAP as constructed in the reduction. Then $\mathcal{P}$ has at least one solution iff h is relevant in $\mathcal{P}_\tau$.*

Proof. $\Rightarrow$ Let $S \in Sol(\mathcal{P})$. That is, $S \subseteq H$, $T \cup S \models M$ and that there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models T \cup S$. Consider $S_\tau := S \cup \{h\}$. The claim is that S_τ is a solution of $\mathcal{P}'$ where $h \in S$. Firstly, $S_\tau \subseteq H \cup \{h, h'\}$ and $h \in S$ by construction. Secondly, take some model $\mathcal{I}$ of $T \cup S$, extend it to the interpretation $\mathcal{I}_\tau$ by the variables $\{h, h', m'\}$ such that $\mathcal{I}_\tau \models h \wedge m'$ and $\mathcal{I}_\tau \not\models h'$. The claim is that $\mathcal{I}_\tau \models T_\tau \cup S_\tau$. Since $\mathcal{I} \models \varphi$ and $\mathcal{I}_\tau$ agrees with $\mathcal{I}$ on all variables in φ one has $\mathcal{I}_\tau \models \neg h \vee \varphi$. Due to $\mathcal{I}_\tau \not\models h'$ the implications $h' \rightarrow m$ for all $m \in M$ hold vacuously. By construction one has $\mathcal{I}_\tau \models \neg h \vee \neg h'$. From $\mathcal{I}_\tau \models m'$ it follows that $h \rightarrow m'$ and $h' \rightarrow m'$ hold. Since $\mathcal{I} \models S$ and $\mathcal{I}_\tau$ agrees with $\mathcal{I}$ on all variables in S and since $\mathcal{I}_\tau \models h$ it follows that $\mathcal{I}_\tau \models S_\tau$. Hence, one can conclude that $\mathcal{I}_\tau \models T_\tau \cup S_\tau$. Thirdly, consider an arbitrary model $\mathcal{J}$ of $T_\tau \cup S_\tau$. By construction, it is known that $h \in S$. Hence, $\mathcal{J} \models h$ and by extension $\mathcal{J} \models \varphi$. Moreover, since $T = \varphi$ and $S_\tau = S \cup \{h\}$, it must be that $\mathcal{J} \models T \cup S$. However, it is known that $T \cup S \models M$. Hence, it follows that $\mathcal{J} \models M$. Furthermore, due to the fact that $\mathcal{J} \models h$ it must be the case that $\mathcal{J} \models m'$. Hence, $\mathcal{J} \models M_\tau$. Having checked all conditions, one can conclude that S_τ is a solution of $\mathcal{P}'$ with $h \in S_\tau$.

$\Leftarrow$ Let $S_\tau \in Sol(\mathcal{P}')$ such that $h \in S_\tau$. Moreover, it is known that $S_\tau \subseteq H_\tau$, $T_\tau \cup S_\tau \models M_\tau$ and that there exists an interpretation $\mathcal{I}_\tau$ such that $\mathcal{I}_\tau \models T_\tau \cup S_\tau$. Consider $S := S_\tau \cap H$. Restrict $\mathcal{I}_\tau$ by removing the assignments of the variables h, h' and m', thereby creating $\mathcal{I}$. Claim $\mathcal{I} \models T \cup S$. Since $S \subset S_\tau$ it follows that $\mathcal{I}_\tau \models S$, and thus by construction $\mathcal{I} \models S$. From the fact that $\mathcal{I}_\tau \models T_\tau$ and the fact that $h \in S_\tau$ it must be that $\mathcal{I}_\tau \models \varphi$. Again with $\mathcal{I}$ and $\mathcal{I}_\tau$ agreeing on all variables in φ it follows that $\mathcal{I} \models \varphi$. Hence, $\mathcal{I} \models T \cup S$. Now one has to check whether $T \cup S \models M$. To that end, take an arbitrary $\mathcal{J} \models T \cup S$. Extend it by the variables h, h', m' to obtain $\mathcal{J}_\tau$ such that $\mathcal{J}_\tau \models h \wedge m'$ and $\mathcal{J}_\tau \not\models h'$. Clearly, $\mathcal{J}_\tau \models T_\tau \cup S_\tau$ (see above). However, this implies that $\mathcal{J}_\tau \models M_\tau$ and especially $\mathcal{J}_\tau \models M$. However, since $\mathcal{J}$ and $\mathcal{J}_\tau$ agree on all variables in M it follows that $\mathcal{J} \models M$ and therefore, $T \cup S \models M$. Having checked all conditions, one can conclude that S is a solution of $\mathcal{P}$.

$\square$