Exercise 1 (5 credits)**.** Recall the following characterizations of the complexity classes Σ_i^{P} and Π_i^{P} for $i \geq 1$.

Theorem 2. *Let L be a language and $i \geq 1$.*

- *Then $L \in \Sigma_i^{\mathsf{P}}$ iff there is a polynomially balanced relation R such that the language $\{x\#y \mid (x, y) \in R\}$ is in Π_{i-1}^{P} and*

$$L = \{x \mid \text{there exists a } y \text{ with } |y| \leq |x|^k \text{ s.t. } (x, y) \in R\}$$

- *Then $L \in \Pi_i^{\mathsf{P}}$ iff there is a polynomially balanced relation R such that the language $\{x\#y \mid (x, y) \in R\}$ is in $\Sigma_{i-1}^{\mathsf{P}}$ and*

$$L = \{x \mid \text{for all } y \text{ with } |y| \leq |x|^k, (x, y) \in R\}$$

Corollary 3. *Let L be a language and $i \geq 1$.*

- *Then $L \in \Sigma_i^{\mathsf{P}}$ iff there is a polynomially balanced, polynomial-time decidable $(i+1)$-ary relation R such that*

$$L = \{x \mid \exists y_1 \forall y_2 \exists y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R\}$$

  where Q is $\forall$ if i is even and $\exists$ if i is odd.

- *Then $L \in \Pi_i^{\mathsf{P}}$ iff there is a polynomially balanced, polynomial-time decidable $(i+1)$-ary relation R such that*

$$L = \{x \mid \forall y_1 \exists y_2 \forall y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R\}$$

  where Q is $\exists$ if i is even and $\forall$ if i is odd.

Give a rigorous proof of this corollary.

Hint. Use the above theorem and proceed by induction on i. It suffices to prove the correctness of the characterization of Σ_i^{P}. You may use the characterization of Π_i^{P} in the induction step.

Solution To make the proof Corollary of more concise consider the following.

Definition 4. *For some n let $\mathcal{R}^n$ represent the set of all n-ary relations. Moreover, if $R \in \mathcal{R}^2$, then let $\mathcal{L}(R) := \{x\#y \mid (x, y) \in R\}$. Furthermore, as an abbreviation let p.b. stand for polynomially balanced and let p.d. stand for polynomial-time decidable*

Before delving into the proof of Corollary consider the following remark.

Remark 5. *As proposed in the slides. It is possible to omit the condition $|y| \leq |x|^k$, due to fact that R is a polynomially balanced relation.*

Moreover, some preliminary results.

Lemma 6. *For some $R \in \mathcal{R}^2$ with R p.b. it follows that $(x, y) \in R =_{\mathsf{P}} x\#y \in \mathcal{L}(R)$.*

Proof. Let τ be the bijection $\tau(x, y) = x\#y$ (and its inverse $\tau^{-1}(x\#y) = (x, y)$).

Firstly, τ can be computed in polynomial time with respect to $|x|$[1]. That is, given the input (x, y) one can easily create the string $x\#y$ by copying x adding a # and copying y, and since R is p.b. it must be that $|y| \leq |x|^k$ for some $k > 0$. Hence, it follows that this transformation can be done in linear time with respect to $|x|$.

Secondly, τ^{-1} can be computed in polynomial time with respect to $|x|$. That is, given the input $x\#y$ one can easily create the tuple (x, y) iterating over $x\#y$ until # is reached. Then one merely copies everything before the separator into the first position of the tuple and everything after into the second position. Since R is p.b. it must be that $|y| \leq |x|^k$ for some $k > 0$. Hence, it follows that this transformation can be done in linear time with respect to $|x|$.

Thirdly, it must be established that

$$(x, y) \in R \iff \tau(x, y) \in \mathcal{L}(R) \wedge \tau^{-1}(x\#y) \iff x\#y \in \mathcal{L}(R)$$

However, since $\tau(x, y) = x\#y$ and $\tau^{-1}(x\#y) = (x, y)$ this follows by construction of $\mathcal{L}$. $\square$

Finally, allowing the demonstration of the following corollary.

Corollary 7. *Let L be a language and $i \geq 1$.*

- *$L \in \Sigma_i^{\mathsf{P}}$ iff there is a polynomially balanced, polynomial-time decidable $(i+1)$-ary relation R such that*

  $$L = \{x \mid \exists y_1 \forall y_2 \exists y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R\}$$

  where Q is $\forall$ if i is even and $\exists$ if i is odd.

- *$L \in \Pi_i^{\mathsf{P}}$ iff there is a polynomially balanced, polynomial-time decidable $(i+1)$-ary relation R such that*

  $$L = \{x \mid \forall y_1 \exists y_2 \forall y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R\}$$

  where Q is $\exists$ if i is even and $\forall$ if i is odd.

Proof. Firstly, in the subsequent proof a p.b. relation $R \in \mathcal{R}^2$ will be obtained by applying Theorem 2 w.l.o.g. assume that $\forall (x, y) \in R\ |x| > 1$. As for any polynomially balanced relation R, where there exists an $(x, y) \in R$ such that $x \leq 1$ it is possible to construct the relation $R' := \{(x\boxtimes, y) \mid (x, y) \in R\}$ (where $\boxtimes$ is some new character) that still is polynomially balanced, i.e. increasing the size of x can not invalidate the condition for being polynomially balanced. Moreover, this construction can clearly be done in polynomial time. Moreover, using R' one can just as easily reconstruct R. Hence, $\mathcal{L}(R')$ will live at the same level of the polynomial hierarchy as $\mathcal{L}(R)$.

Secondly, the claim can be demonstrated by induction on i.

- **IH:** For a fixed $i > 0$.

$$\begin{aligned} L \in \Sigma_i^{\mathsf{P}} \iff & \exists R \in \mathcal{R}^{i+1}\ R \ p.b. \wedge R \ p.d. \wedge \\ & L = \{x \mid \exists y_1 \forall y_2 \exists y_3 \ldots Q y_i\ (x, y_1, \ldots, y_i) \in R\} \\ & (i \text{ even} \Rightarrow Q = \forall) \wedge (i \text{ odd} \Rightarrow Q = \exists) \end{aligned}$$

[1] This is stronger than required.

and

$$\begin{aligned} L \in \Pi_i^{\mathsf{P}} \iff & \exists R \in \mathcal{R}^{i+1}\ R\ p.b. \land R\ p.d. \land \\ & L = \{x \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_i\ (x, y_1, \ldots, y_i) \in R\} \land \\ & (i \text{ even} \Rightarrow Q = \exists) \land (i \text{ odd} \Rightarrow Q = \forall) \end{aligned}$$

- **IB:** For $i = 1$. Firstly, starting from $L \in \Sigma_1^{\mathsf{P}}$, by Theorem 2, on obtains

$$L \in \Sigma_1^{\mathsf{P}} \iff \exists R \in \mathcal{R}^2\ R\ p.b. \land \mathcal{L}(R) \in \Pi_0^{\mathsf{P}} \land L = \{x \mid \exists y\ (x, y) \in R\}$$

Since, $\Pi_0^{\mathsf{P}} = \mathsf{P}$ this is equivalent to

$$L \in \Sigma_1^{\mathsf{P}} \iff \exists R \in \mathcal{R}^2\ R\ p.b. \land \mathcal{L}(R) \in \mathsf{P} \land L = \{x \mid \exists y\ (x, y) \in R\}$$

By Lemma 6, since $\mathcal{L}(R) \in \mathsf{P}$, it follows that $(x, y) \in R$ can be decided in polynomial time. Hence, the previous equality is equivalent to

$$L \in \Sigma_1^{\mathsf{P}} \iff \exists R \in \mathcal{R}^2\ R\ p.b. \land R\ p.d. \land L = \{x \mid \exists y\ (x, y) \in R\}$$

which is precisely what was desired.

Secondly, starting from $L \in \Pi_1^{\mathsf{P}}$ this is done completely in analogue. That is, by Theorem 2, on obtains

$$L \in \Pi_1^{\mathsf{P}} \iff \exists R \in \mathcal{R}^2\ R\ p.b. \land \mathcal{L}(R) \in \Sigma_0^{\mathsf{P}} \land L = \{x \mid \forall y\ (x, y) \in R\}$$

Since, $\Sigma_0^{\mathsf{P}} = \mathsf{P}$ this is equivalent to

$$L \in \Pi_1^{\mathsf{P}} \iff \exists R \in \mathcal{R}^2\ R\ p.b. \land \mathcal{L}(R) \in \mathsf{P} \land L = \{x \mid \forall y\ (x, y) \in R\}$$

By Lemma 6, since $\mathcal{L}(R) \in \mathsf{P}$, it follows that $(x, y) \in R$ can be decided in polynomial time. Hence, the previous equality is equivalent to

$$L \in \Pi_1^{\mathsf{P}} \iff \exists R \in \mathcal{R}^2\ R\ p.b. \land R\ p.d. \land L = \{x \mid \forall y\ (x, y) \in R\}$$

which is precisely what was desired.

- **IS:** Let $i = n + 1$. Observe the following

$$
\begin{aligned}
& L \in \Sigma^{\mathsf{P}}_{n+1} \\
\overset{\text{(i)}}{\iff} & \exists R_2 \in \mathcal{R}^2 \ R_2 \ p.b. \wedge \mathcal{L}(R_2) \in \Pi^{\mathsf{P}}_n \wedge L = \{x \mid \exists y \ (x, y) \in R_2\} \\
\overset{\text{(ii)}}{\iff} & \exists R_2 \in \mathcal{R}^2 \ R_2 \ p.b. \wedge \exists R_{n+1} \in \mathcal{R}^{n+1} \ R_{n+1} \ p.b. \wedge R_{n+1} \ p.d. \wedge \\
& \mathcal{L}(R_2) = \{x\#y \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ (x\#y, y_1, \ldots, y_n) \in R_{n+1}\} \wedge \\
& L = \{x \mid \exists y \ (x, y) \in R_2\} \wedge \\
& (n \text{ even} \Rightarrow Q = \exists) \wedge (n \text{ odd} \Rightarrow Q = \forall) \\
\overset{\text{(iii)}}{\iff} & \exists R_2 \in \mathcal{R}^2 \ R_2 \ p.b. \wedge \exists R_{n+2} \in \mathcal{R}^{n+2} \ R_{n+2} \ p.b. \wedge R_{n+2} \ p.d. \wedge \\
& \mathcal{L}(R_2) = \{x\#y \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ (x, y, y_1, \ldots, y_n) \in R_{n+2}\} \wedge \\
& L = \{x \mid \exists y \ (x, y) \in R_2\} \wedge \\
& (n \text{ even} \Rightarrow Q = \exists) \wedge (n \text{ odd} \Rightarrow Q = \forall) \\
\overset{\text{(iv)}}{\iff} & \exists R_2 \in \mathcal{R}^2 \ R_2 \ p.b. \wedge \exists R_{n+2} \in \mathcal{R}^{n+2} \ R_{n+2} \ p.b. \wedge R_{n+2} \ p.d. \wedge \\
& R_2 = \{(x, y) \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ (x, y, y_1, \ldots, y_n) \in R_{n+2}\} \wedge \\
& L = \{x \mid \exists y \ (x, y) \in \{(x, y) \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ (x, y, y_1, \ldots, y_n) \in R_{n+2}\}\} \wedge \\
& (n \text{ even} \Rightarrow Q = \exists) \wedge (n \text{ odd} \Rightarrow Q = \forall) \\
\overset{\text{(v)}}{\iff} & \exists R_{n+2} \in \mathcal{R}^{n+2} \ R_{n+2} \ p.b. \wedge R_{n+2} \ p.d. \wedge \\
& L = \{x \mid \exists y \ (x, y) \in \{(x, y) \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ (x, y, y_1, \ldots, y_n) \in R_{n+2}\}\} \wedge \\
& (n \text{ even} \Rightarrow Q = \exists) \wedge (n \text{ odd} \Rightarrow Q = \forall) \\
\overset{\text{(vi)}}{\iff} & \exists R_{n+2} \in \mathcal{R}^{n+2} \ R_{n+2} \ p.b. \wedge R_{n+2} \ p.d. \wedge \\
& L = \{x \mid \exists y \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ (x, y, y_1, \ldots, y_n) \in R_{n+2}\} \wedge \\
& (n \text{ even} \Rightarrow Q = \exists) \wedge (n \text{ odd} \Rightarrow Q = \forall) \\
\overset{\text{(vii)}}{\iff} & \exists R_{n+2} \in \mathcal{R}^{n+2} \ R_{n+2} \ p.b. \wedge R_{n+2} \ p.d. \wedge \\
& L = \{x \mid \exists y_1 \forall y_2 \exists y_3 \ldots Q y_{n+1} \ (x, y, y_1, \ldots, y_n) \in R_{n+2}\} \wedge \\
& (n + 1 \text{ even} \Rightarrow Q = \forall) \wedge (n + 1 \text{ odd} \Rightarrow Q = \exists)
\end{aligned}
$$

(i) Here Theorem 2 was applied.

(ii) Here the **IH** was applied, i.e.

$$
\begin{aligned}
\mathcal{L}(R_2) \in \Pi^{\mathsf{P}}_n \iff & \exists R_{n+1} \in \mathcal{R}^{n+1} \ R_{n+1} \ p.b. \wedge R_{n+1} \ p.d. \wedge \\
& \mathcal{L}(R_2) = \{x\#y \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ (x\#y, y_1, \ldots, y_n) \in R_{n+1}\} \wedge \\
& (n \text{ even} \Rightarrow Q = \exists) \wedge (n \text{ odd} \Rightarrow Q = \forall)
\end{aligned}
$$

(iii) Firstly, $\Rightarrow$. Starting from

$$
\begin{aligned}
& \exists R_2 \in \mathcal{R}^2 \ R_2 \ p.b. \wedge \exists R_{n+1} \in \mathcal{R}^{n+1} \ R_{n+1} \ p.b. \wedge R_{n+1} \ p.d. \wedge \\
& \mathcal{L}(R_2) = \{x\#y \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ (x\#y, y_1, \ldots, y_n) \in R_{n+1}\}
\end{aligned}
$$

Taking the relation R_{n+1} one can construct the relation $R_{n+2} \in \mathcal{R}^{n+2}$ such that

$$R_{n+2} = \{(x, y, y_1, \ldots, y_n) \mid (x\#y, y_1, \ldots, y_n) \in R_{n+1}\}$$

To do so one merely has to split the first entry in $(x\#y, y_1, \ldots, y_n) \in R_{n+1}$ into two, which can be done in polynomial time (similar argument as in Lemma 6). Moreover, by construction it clearly holds that

$$(x, y, y_1, \ldots, y_n) \in R_{n+2} \iff (x\#y, y_1, \ldots, y_n) \in R_{n+1}$$

Since by assumption R_2 is polynomially balanced it follows that there exists a k such that for any $(x, y) \in R_2$ one has $|y| \leq |x|^k$. Furthermore, it is known that R_{n+1} is p.b., thus there exists a k' such that for any $1 \leq i \leq n$ one has $|y_i| \leq |x\#y|^{k'} \leq |x| + 1 + |x|^k$. By assumption, i.e. $|x| > 1$, it follows that there exists a $k^* \geq k$ such that $|y_i| \leq |x|^{k*}$ and $|y| \leq |x|^{k*}$. Hence, R_{n+2} is polynomially balanced. Additionally, one knows that R_{n+1} is p.d., thus R_{n+2} can be decided by concatenating the first two entries and querying R_{n+1}. Both operations can be done in polynomial time, thus R_{n+2} is p.d.. Hence, one obtains

$$\begin{aligned}&\exists R_2 \in \mathcal{R}^2\ R_2\ p.b. \wedge \exists R_{n+2} \in \mathcal{R}^{n+2}\ R_{n+2}\ p.b. \wedge R_{n+2}\ p.d. \wedge\\ &\mathcal{L}(R_2) = \{x\#y \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n\ (x, y, y_1, \ldots, y_n) \in R_{n+2}\}\end{aligned}$$

Secondly, $\Leftarrow$. This argument is essentially the same as the previous one, but in reverse (and with slight alterations in the complexity arguments). That is, starting from

$$\begin{aligned}&\exists R_2 \in \mathcal{R}^2\ R_2\ p.b. \wedge \exists R_{n+2} \in \mathcal{R}^{n+2}\ R_{n+2}\ p.b. \wedge R_{n+2}\ p.d. \wedge\\ &\mathcal{L}(R_2) = \{x\#y \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n\ (x, y, y_1, \ldots, y_n) \in R_{n+2}\}\end{aligned}$$

Taking the relation R_{n+2} one can construct the relation $R_{n+1} \in \mathcal{R}^{n+1}$ such that

$$R_{n+1} = \{(x\#y, y_1, \ldots, y_n) \mid (x, y, y_1, \ldots, y_n) \in R_{n+2}\}$$

To do so one merely has to concatenate the first two entries in $(x, y, y_1, \ldots, y_n) \in R_{n+1}$ using the separator #, which can be done in polynomial time (similar argument as in Lemma 6). Moreover, it clearly holds that

$$(x\#y, y_1, \ldots, y_n) \in R_{n+1} \iff (x, y, y_1, \ldots, y_n) \in R_{n+2}$$

It is known that R_{n+2} is p.b., thus there exists a k such that for $1 \leq i \leq n$, $|y_i| \leq |x|^k$ and $|y| \leq |x|^k$. Now since $|x| < |x\#y|$ it must be that R_{n+1} is p.b. as well. Additionally, one knows that R_{n+2} is p.d., thus R_{n+1} can be decided by splitting the first entry on # and querying R_{n+2}. Both operations can be done in polynomial time, thus R_{n+1} is p.d.. Hence, one obtains

$$\begin{aligned}&\exists R_2 \in \mathcal{R}^2\ R_2\ p.b. \wedge \exists R_{n+1} \in \mathcal{R}^{n+1}\ R_{n+1}\ p.b. \wedge R_{n+1}\ p.d. \wedge\\ &\mathcal{L}(R_2) = \{x\#y \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n\ (x\#y, y_1, \ldots, y_n) \in R_{n+1}\}\end{aligned}$$

(iv) This equality is guaranteed by the following. Take an arbitrary relation R. Clearly, $(x, y) \in R \iff x\#y \in \mathcal{L}(R)$. Hence, in this particular case one has $(x, y) \in R_2 \iff x\#y \in \mathcal{L}(R_2)$. Now starting from

$$\begin{aligned}&\mathcal{L}(R_2) = \{x\#y \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n\ (x, y, y_1, \ldots, y_n) \in R_{n+2}\} \wedge\\ &\quad L = \{x \mid \exists y\ (x, y) \in R_2\}\end{aligned}$$

due to

$$\begin{aligned}(\alpha,\beta)\in R_2 &\iff \alpha\#\beta\in\mathcal{L}(R_2)\\ &\iff \alpha\#\beta\in\{x\#y \mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\\ &\iff (\alpha,\beta)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\end{aligned}$$

one obtains the equivalent statement

$$\begin{aligned}R_2 &= \{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\wedge\\ L &= \{x\mid \exists y\ (x,y)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

(v) Firstly, $\Rightarrow$. Starting from

$$\begin{aligned}&\exists R_2\in\mathcal{R}^2\ R_2\ p.b.\wedge\exists R_{n+2}\in\mathcal{R}^{n+2}\ R_{n+2}\ p.b.\wedge R_{n+2}\ p.d.\wedge\\ &R_2=\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\wedge\\ &L=\{x\mid \exists y\ (x,y)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

one can simply use weakening to obtain the part of the statement, where R_2 does not occur.

$$\begin{aligned}&\exists R_{n+2}\in\mathcal{R}^{n+2}\ R_{n+2}\ p.b.\wedge R_{n+2}\ p.d.\wedge\\ &L=\{x\mid \exists y\ (x,y)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

Thereby, eradicating all references of R_2.

Secondly, $\Leftarrow$. Starting from

$$\begin{aligned}&\exists R_{n+2}\in\mathcal{R}^{n+2}\ R_{n+2}\ p.b.\wedge R_{n+2}\ p.d.\wedge\\ &L=\{x\mid \exists y\ (x,y)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

One can define the relation $R_2 := \{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}$. Since it is known that R_{n+2} is p.b. this implies that there exists a k such that $|y|\le|x|^k$, thus implying that R_2 is p.b.. Allowing one to conclude that

$$\begin{aligned}&\exists R_2\in\mathcal{R}^2\ R_2\ p.b.\wedge\exists R_{n+2}\in\mathcal{R}^{n+2}\ R_{n+2}\ p.b.\wedge R_{n+2}\ p.d.\wedge\\ &R_2=\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\wedge\\ &L=\{x\mid \exists y\ (x,y)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

(vi) Starting from $\{x\mid \exists y\ (x,y)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}$. Notice that

$$\begin{aligned}&(\alpha,\beta)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\\ &\iff \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (\alpha,\beta,y_1,\ldots,y_n)\in R_{n+2}\end{aligned}$$

From this it follows that

$$\begin{aligned}&\exists y\ (\alpha,y)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\\ &\iff \exists y\forall y_1\exists y_2\forall y_3\ldots Qy_n\ (\alpha,y,y_1,\ldots,y_n)\in R_{n+2}\end{aligned}$$

and therefore

$$\begin{aligned}&\{x\mid \exists y\ (x,y)\in\{(x,y)\mid \forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\\ &=\{x\mid \exists y\forall y_1\exists y_2\forall y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\end{aligned}$$

(vii) This particular renaming of bound variables is clearly an equivalence transformation. Hence,

$$n \text{ even} \Rightarrow Q = \exists \wedge n \text{ odd} \Rightarrow Q = \forall \iff n+1 \text{ even} \Rightarrow Q = \forall \wedge n+1 \text{ odd} \Rightarrow Q = \exists$$

remains to be established. However, this follows directly from the fact that n is even if and only if $n+1$ is odd. That is, if n was even, one has $Q = \exists$. However, this implies that for $Q = \exists$ for $n+1$ being odd and if $n+1$ is odd then $Q = \exists$, meaning that $Q = \exists$ for n is even. Analogous for the other case.

The other case, i.e. for $L \in \Pi^{\mathsf{P}}_{n+1}$ is done completely analogously (see Appendix). □

Exercise 8 (5 credits)**.** Recall the Σ_2^{P}-hardness proof of **MINIMAL MODEL SAT** by reduction from the QSAT_2-problem: Let an arbitrary instance of QSAT_2 be given by the QBF

$$\psi = (\exists x_1, \ldots, x_k)(\forall y_1, \ldots, y_\ell)\varphi$$

Now let $\{x_1', \ldots, x_k', z\}$ be fresh propositional variables. Then we construct an instance of **MINIMAL MODEL SAT** by the *variable* z and the *formula*

$$\chi = \big(\bigwedge_{i=1}^{k}(\neg x_i \leftrightarrow x_i')\big) \wedge \big(\neg\varphi \vee (y_1 \wedge \cdots \wedge y_\ell \wedge z)\big)$$

Recall from the lecture that we have already proved the following implication:
ψ is **true** (in every interpretation) $\Rightarrow$ z is **true** in a minimal model of χ.

Give a rigorous proof also of the opposite direction, i.e.:
z is **true** in a minimal model of χ $\Rightarrow$ ψ is **true** (in every interpretation).

Hint. Let $\mathcal{J}$ be a minimal model of χ and let z be **true** in $\mathcal{J}$.

- First show that then $\mathcal{J}(y_j) = \textbf{true}$ for every j.
- Second, let $\mathcal{I}$ be the truth assignment obtained by restricting $\mathcal{J}$ to the variables $\{x_1, \ldots, x_k\}$. Show that (by the minimality of $\mathcal{J}$) $\mathcal{I}$ is indeed a partial assigment on $\{x_1, \ldots, x_k\}$ s.t. for any values assigned to $\{y_1, \ldots, y_\ell\}$, the formula φ is **true**.

Solution Firstly, a restatement of the reduction, to unify with the notation used in the solution.

Definition 9. *Let $\varphi := (\exists x_1, \ldots, x_k)(\forall y_1, \ldots, y_\ell)\psi$ be a $\mathbf{QBF}_{2,\exists}$-formula, then let*

$$\chi(\varphi) := \big(\bigwedge_{i=1}^{k}(\neg x_i \leftrightarrow x_i')\big) \wedge \big(\neg\psi \vee (y_1 \wedge \cdots \wedge y_\ell \wedge z)\big)$$

Moreover, $\mathcal{Y}(\varphi) := \{y_1, \ldots, y_\ell\}$, $\mathcal{X}(\varphi) := \{x_1, \ldots, x_k\}$ and $\mathcal{X}'(\varphi) := \{x_1', \ldots, x_k'\}$. Lastly, let $\tau : \mathcal{X}(\varphi) \to \mathcal{X}'(\varphi)$ a bijection such that $x_i \mapsto \tau(x_i) = x_i'$.

Remark 10. *The function τ is thus merely a function that given $x \in \mathcal{X}(\varphi)$ allows one to access the corresponding $x' \in \mathcal{X}'(\varphi)$. Moreover, corresponding in this case merely means they occur in the sub-formula $\neg x_i \leftrightarrow x_i'$ of $\chi(\varphi)$. Lastly, by construction of the sub-formula $\bigwedge_{i=1}^{k}(\neg x_i \leftrightarrow x_i')$ one can be sure that τ is actually bijective.*

Secondly, in this proof the notion of subset minimality is required. To wield the usual notion of subset minimality, it is necessary to conceptualise an interpretation as a set of atoms, where an atom is true under this interpretation if and only if it is part of the set. Here, a marginally different approach shall be chosen.

Definition 11. *Let $\mathcal{I}$ be an interpretation over the set of atoms $A_{\mathcal{I}}$. Then $\mathfrak{S}(\mathcal{I}) := \{x \mid \forall x \in A_{\mathcal{I}}\ \mathcal{I}(x) = \textbf{true}\}$. Moreover, an interpretation $\mathcal{J}$ then $\mathcal{I} \subseteq \mathcal{J}$ if and only if $\mathfrak{S}(\mathcal{I}) \subseteq \mathfrak{S}(\mathcal{J})$ and $\mathcal{I} \subset \mathcal{J}$ if and only if $\mathfrak{S}(\mathcal{I}) \subset \mathfrak{S}(\mathcal{J})$ Moreover, $\mathcal{I}$ is a subset minimal if and only if $\nexists \mathcal{I}'\ \mathcal{I}' \subset \mathcal{I}$. Similarly, $\mathcal{I}$ is a subset minimal interpretation of a formula φ if and only if $\mathcal{I} \models \varphi \wedge \nexists \mathcal{I}'\ \mathcal{I}' \subset \mathcal{I} \wedge \mathcal{I}' \models \varphi$.*

Remark 12. *Notice that if $\mathcal{I}$ is a subset minimal interpretation of the formula φ, then $\mathfrak{S}(\mathcal{I}) \subseteq Var(\varphi)$. That is, if there would exists an $x \in \mathfrak{S}(\mathcal{I})$ such that $x \notin Var(\varphi)$, $\mathcal{I}$ would not be subset minimal.*

Thirdly, the notion of extension is required.

Definition 13. *Let $\mathcal{I}$ be an interpretation. Then an extension of $\mathcal{I}$ by the atoms X, is any interpretation $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$ and $\forall x \in X$ $\mathcal{J}(x) = \mathbf{true} \vee \mathcal{J}(x) = \mathbf{false}$.*

As suggested in the given hint, to demonstrate Lemma XX, two precursory results are demonstrated.

Lemma 14. *Let $\varphi := (\exists x_1, \dots, x_k)(\forall y_1, \dots, y_\ell)\psi$ be a $\mathbf{QBF}_{2,\exists}$-formula, such that there exists a minimal model $\mathcal{J}$ of $\chi(\varphi)$, where $\mathcal{J} \models z$. Then $\forall y \in \mathcal{Y}(\varphi)$ $\mathcal{J} \models y$.*

Proof. Assume that $\mathcal{J}$ exists, thus it is known that $\mathcal{J} \models z$ and that $\mathcal{J}$ is a subset minimal interpretation of $\chi(\varphi)$. Assume that there exists a $y \in \mathcal{Y}(\varphi)$ such that $\mathcal{J} \not\models y$. If this is the case, then clearly $\mathcal{J} \not\models (y_1 \wedge \cdots \wedge y_\ell \wedge z)$. However, since $\mathcal{J} \models \chi(\varphi)$, it must be that $\mathcal{J} \models \neg\psi$. By construction it is known that $z \notin Var(\varphi)$. Therefore, the only occurrence of z in $\chi(\varphi)$ is in the sub-formula $(y_1 \wedge \cdots \wedge y_\ell \wedge z)$. Now, with the one y evaluating to **false** under $\mathcal{J}$, the truth value of z is immaterial in the evaluation of $\chi(\varphi)$. Hence, one can construct the interpretation $\mathcal{J}'$ such that $\forall x \neq z$ $\mathcal{J}'(x) := \mathcal{J}(x)$ and $\mathcal{J}'(z) := \mathbf{false}$ that satisfies $\chi(\varphi)$. Hence, by definition one obtains $\mathcal{J}' \subset \mathcal{J}$, which clearly violates the assumed subset minimality of $\mathcal{J}$. Therefore, one can conclude that $\forall y \in \mathcal{Y}(\varphi)$ $\mathcal{J} \models y$. $\square$

Guided by the hint, the second relevant lemma.

Lemma 15. *Let $\varphi := (\exists x_1, \dots, x_k)(\forall y_1, \dots, y_\ell)\psi$ be a $\mathbf{QBF}_{2,\exists}$-formula, such that there exists a minimal model $\mathcal{J}$ of $\chi(\varphi)$, where $\mathcal{J} \models z$. Let $\mathcal{I}_{|X}$ be the interpretation $\forall x \in \mathcal{X}(\varphi)$ $\mathcal{I}_{|X}(x) = \mathcal{J}(x) \wedge \mathfrak{S}(\mathcal{I}_{|X}) \subseteq \mathcal{X}(\varphi)$, i.e. it is $\mathcal{J}$ restricted to the variables in $\mathcal{X}(\varphi)$. Then it holds that for any arbitrary extension $\mathcal{I}_{|X\cup Y}$ of $\mathcal{I}_{|X}$ by the variables in $\mathcal{Y}(\varphi)$, it must be that $\mathcal{I}_{|X\cup Y} \models \psi$.*

Proof. Towards a contradiction, assume that there exists an extension $\mathcal{I}_{|X\cup Y}$ of $\mathcal{I}_{|X}$ by the variables in $\mathcal{Y}(\varphi)$ such that $\mathcal{I}_{|X\cup Y} \not\models \psi$. Hence, by semantics this implies that $\mathcal{I}_{|X\cup Y} \models \neg\psi$. Now using $\mathcal{I}_{|X\cup Y}$ an interpretation $\mathcal{J}'$ will be constructed such that $\mathcal{J}' \subset \mathcal{J}$ and $\mathcal{J}' \models \chi(\varphi)$. The sought after interpretation is defined such that

- $\forall x \in \mathcal{X}(\varphi)$ $\mathcal{J}'(x) := \mathcal{I}_{|X\cup Y}(x)$;
- $\forall x \in \mathcal{X}(\varphi)$ $\mathcal{J}'(\tau(x)) := \neg\mathcal{I}_{|X\cup Y}(x)$;
- $\forall y \in \mathcal{Y}(\varphi)$ $\mathcal{J}'(y) := \mathcal{I}_{|X\cup Y}(y)$;
- $\mathcal{J}'(z) := \mathbf{false}$.

Notice that $\mathcal{J}'$ was constructed using $\mathcal{I}_{|X\cup Y}$, which is an extension of $\mathcal{I}_{|X}$, which itself is merely a restriction of $\mathcal{J}$ to the variables in $\mathcal{X}(\varphi)$. Hence, it follows that $\forall x \in \mathcal{X}(\varphi)$ $\mathcal{J}'(x) = \mathcal{J}(x)$. Moreover, together with the fact that $\mathcal{J} \models \bigwedge_{i=1}^{k}(\neg x_i \leftrightarrow x_i')$ (and the construction of τ) it follows that $\forall x' \in \mathcal{X}'(\varphi)$ $\mathcal{J}'(x') = \mathcal{J}(x')$. Therefore, $\mathcal{J}$ and $\mathcal{J}'$ agree on the variables in $\mathcal{X}(\varphi) \cup \mathcal{X}'(\varphi)$, i.e. $\mathfrak{S}(\mathcal{J}') \cap (\mathcal{X}(\varphi) \cup \mathcal{X}'(\varphi)) = \mathfrak{S}(\mathcal{J}) \cap (\mathcal{X}(\varphi) \cup \mathcal{X}'(\varphi))$. Now, by Lemma 14, it is known that for any $y \in \mathcal{Y}(\varphi)$ it must be that $\mathcal{J} \models y$, i.e. $\mathfrak{S}(\mathcal{J}) \cap \mathcal{Y}(\varphi) = \mathcal{Y}(\varphi)$. Hence, $\mathfrak{S}(\mathcal{J}') \cap \mathcal{Y}(\varphi) \subseteq \mathfrak{S}(\mathcal{J}) \cap \mathcal{Y}(\varphi)$. Furthermore, by assumption it is known that $\mathcal{J} \models z$ while $\mathcal{J}'$ does not, thus $\mathfrak{S}(\mathcal{J}') \cap \{z\} \subset \mathfrak{S}(\mathcal{J}) \cap \{z\}$ holds. To summarise,

(i) $\mathfrak{S}(\mathcal{J}') \cap (\mathcal{X}(\varphi) \cup \mathcal{X}'(\varphi)) = \mathfrak{S}(\mathcal{J}) \cap (\mathcal{X}(\varphi) \cup \mathcal{X}'(\varphi))$,

(ii) $\mathfrak{S}(\mathcal{J}') \cap \mathcal{Y}(\varphi) \subseteq \mathfrak{S}(\mathcal{J}) \cap \mathcal{Y}(\varphi)$ and

(iii) $\mathfrak{S}(\mathcal{J}') \cap \{z\} \subset \mathfrak{S}(\mathcal{J}) \cap \{z\}$.

As this covers all variables assigned in $\mathcal{J}$ by subset minimality and all variables assigned in $\mathcal{J}'$ by construction, one can conclude that $\mathfrak{S}(\mathcal{J}') \subset \mathfrak{S}(\mathcal{J})$ which by definition implies that $\mathcal{J}' \subset \mathcal{J}$.

What remains to be shown is that $\mathcal{J}' \models \chi(\varphi)$. From (i) and the fact that $\mathcal{J} \models \chi(\varphi)$ one obtains $\mathcal{J}' \models \bigwedge_{i=1}^{k}(\neg x_i \leftrightarrow x_i')$. Furthermore, by assumption it is known that $\mathcal{I}_{|X \cup Y} \models \neg\varphi$, by construction one knows that $\forall x \in \mathcal{X}(\varphi) \cup \mathcal{Y}(\varphi)\ \mathcal{J}'(x) = \mathcal{I}_{|X \cup Y}(x)$, as well as $\mathcal{X}(\varphi) \cup \mathcal{Y}(\varphi) = \mathit{Var}(\varphi) = \mathit{Var}(\psi)$. Hence, one can conclude that $\mathcal{J}' \models \neg\psi$. Which thereby, forces that $\mathcal{J}' \models \chi(\varphi)$, thus clearly contradicting the subset minimality of $\mathcal{J}$.

□

Finally, allowing the proof of the main result.

Lemma 16. *Let* $\varphi := (\exists x_1, \ldots, x_k)(\forall y_1, \ldots, y_\ell)\psi$ *be a* $\mathbf{QBF}_{2,\exists}$*-formula.* z *is* **true** *in a minimal model of* $\chi(\varphi) \implies \varphi$ *is* **true**.

Proof. If the antecedent is not satisfied, the statement holds vacuously. Hence, to demonstrate this claim, under the assumption that z is **true** in a minimal model of $\chi(\varphi)$, the formula φ is **true**. This is precisely the case if there exists a partial assignment $\mathcal{I}$ of the variables $\mathcal{X}(\varphi)$ such that any extension $\mathcal{I}'$ by the variables $\mathcal{Y}(\varphi)$ satisfies ψ, i.e. $\mathcal{I}' \models \psi$. However, this is precisely what Lemma 15 provides. That is, using this lemma it is possible to construct such a partial truth assignment for the variables in $\mathcal{X}(\varphi)$. □

Appendix Let $i = n+1$. Observe the following

$$
\begin{aligned}
& L \in \Pi^{\mathsf{P}}_{n+1} \\
\overset{\text{(i)}}{\Longleftrightarrow}\ & \exists R_2 \in \mathcal{R}^2\ R_2\ p.b. \wedge \mathcal{L}(R_2) \in \Sigma^{\mathsf{P}}_n \wedge L = \{x \mid \forall y\ (x,y) \in R_2\} \\
\overset{\text{(ii)}}{\Longleftrightarrow}\ & \exists R_2 \in \mathcal{R}^2\ R_2\ p.b. \wedge \exists R_{n+1} \in \mathcal{R}^{n+1}\ R_{n+1}\ p.b. \wedge R_{n+1}\ p.d. \wedge \\
& \mathcal{L}(R_2) = \{x\#y \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n\ (x\#y, y_1, \dots, y_n) \in R_{n+1}\} \wedge \\
& L = \{x \mid \forall y\ (x,y) \in R_2\} \wedge \\
& (n \text{ even} \Rightarrow Q = \forall) \wedge (n \text{ odd} \Rightarrow Q = \exists) \\
\overset{\text{(iii)}}{\Longleftrightarrow}\ & \exists R_2 \in \mathcal{R}^2\ R_2\ p.b. \wedge \exists R_{n+2} \in \mathcal{R}^{n+2}\ R_{n+2}\ p.b. \wedge R_{n+2}\ p.d. \wedge \\
& \mathcal{L}(R_2) = \{x\#y \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n\ (x, y, y_1, \dots, y_n) \in R_{n+2}\} \wedge \\
& L = \{x \mid \forall y\ (x,y) \in R_2\} \wedge \\
& (n \text{ even} \Rightarrow Q = \forall) \wedge (n \text{ odd} \Rightarrow Q = \exists) \\
\overset{\text{(iv)}}{\Longleftrightarrow}\ & \exists R_2 \in \mathcal{R}^2\ R_2\ p.b. \wedge \exists R_{n+2} \in \mathcal{R}^{n+2}\ R_{n+2}\ p.b. \wedge R_{n+2}\ p.d. \wedge \\
& R_2 = \{(x,y) \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n\ (x, y, y_1, \dots, y_n) \in R_{n+2}\} \wedge \\
& L = \{x \mid \forall y\ (x,y) \in \{(x,y) \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n\ (x, y, y_1, \dots, y_n) \in R_{n+2}\}\} \wedge \\
& (n \text{ even} \Rightarrow Q = \forall) \wedge (n \text{ odd} \Rightarrow Q = \exists) \\
\overset{\text{(v)}}{\Longleftrightarrow}\ & \exists R_{n+2} \in \mathcal{R}^{n+2}\ R_{n+2}\ p.b. \wedge R_{n+2}\ p.d. \wedge \\
& L = \{x \mid \forall y\ (x,y) \in \{(x,y) \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n\ (x, y, y_1, \dots, y_n) \in R_{n+2}\}\} \wedge \\
& (n \text{ even} \Rightarrow Q = \forall) \wedge (n \text{ odd} \Rightarrow Q = \exists) \\
\overset{\text{(vi)}}{\Longleftrightarrow}\ & \exists R_{n+2} \in \mathcal{R}^{n+2}\ R_{n+2}\ p.b. \wedge R_{n+2}\ p.d. \wedge \\
& L = \{x \mid \forall y \exists y_1 \forall y_2 \exists y_3 \dots Q y_n\ (x, y, y_1, \dots, y_n) \in R_{n+2}\} \wedge \\
& (n \text{ even} \Rightarrow Q = \forall) \wedge (n \text{ odd} \Rightarrow Q = \exists) \\
\overset{\text{(vii)}}{\Longleftrightarrow}\ & \exists R_{n+2} \in \mathcal{R}^{n+2}\ R_{n+2}\ p.b. \wedge R_{n+2}\ p.d. \wedge \\
& L = \{x \mid \forall y_1 \exists y_2 \forall y_3 \dots Q y_{n+1}\ (x, y, y_1, \dots, y_n) \in R_{n+2}\} \wedge \\
& (n+1 \text{ even} \Rightarrow Q = \exists) \wedge (n+1 \text{ odd} \Rightarrow Q = \forall)
\end{aligned}
$$

(i) Here Theorem 2 was applied.

(ii) Here the **IH** was applied, i.e.

$$
\begin{aligned}
\mathcal{L}(R_2) \in \Sigma^{\mathsf{P}}_n \iff & \exists R_{n+1} \in \mathcal{R}^{n+1}\ R_{n+1}\ p.b. \wedge R_{n+1}\ p.d. \wedge \\
& \mathcal{L}(R_2) = \{x\#y \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n\ (x\#y, y_1, \dots, y_n) \in R_{n+1}\} \wedge \\
& (n \text{ even} \Rightarrow Q = \forall) \wedge (n \text{ odd} \Rightarrow Q = \exists)
\end{aligned}
$$

(iii) Firstly, $\Rightarrow$. Starting from

$$
\begin{aligned}
& \exists R_2 \in \mathcal{R}^2\ R_2\ p.b. \wedge \exists R_{n+1} \in \mathcal{R}^{n+1}\ R_{n+1}\ p.b. \wedge R_{n+1}\ p.d. \wedge \\
& \mathcal{L}(R_2) = \{x\#y \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n\ (x\#y, y_1, \dots, y_n) \in R_{n+1}\}
\end{aligned}
$$

Taking the relation R_{n+1} one can construct the relation $R_{n+2} \in \mathcal{R}^{n+2}$ such that

$$R_{n+2} = \{(x, y, y_1, \dots, y_n) \mid (x\#y, y_1, \dots, y_n) \in R_{n+1}\}$$

To do so one merely has to split the first entry in $(x\#y, y_1, \dots, y_n) \in R_{n+1}$ into two, which can be done in polynomial time (similar argument as in Lemma 6). Moreover, by construction it clearly holds that

$$(x, y, y_1, \dots, y_n) \in R_{n+2} \iff (x\#y, y_1, \dots, y_n) \in R_{n+1}$$

Since by assumption R_2 is polynomially balanced it follows that there exists a k such that for any $(x, y) \in R_2$ one has $|y| \leq |x|^k$. Furthermore, it is known that R_{n+1} is p.b., thus there exists a k' such that for any $1 \leq i \leq n$ one has $|y_i| \leq |x\#y|^{k'} \leq |x| + 1 + |x|^k$. By assumption, i.e. $|x| > 1$, it follows that there exists a $k^* \geq k$ such that $|y_i| \leq |x|^{k*}$ and $|y| \leq |x|^{k*}$. Hence, R_{n+2} is polynomially balanced. Additionally, one knows that R_{n+1} is p.d., thus R_{n+2} can be decided by concatenating the first two entries and querying R_{n+1}. Both operations can be done in polynomial time, thus R_{n+2} is p.d.. Hence, one obtains

$$\begin{aligned}
&\exists R_2 \in \mathcal{R}^2 \; R_2 \; p.b. \wedge \exists R_{n+2} \in \mathcal{R}^{n+2} \; R_{n+2} \; p.b. \wedge R_{n+2} \; p.d. \wedge \\
&\mathcal{L}(R_2) = \{x\#y \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n \; (x, y, y_1, \dots, y_n) \in R_{n+2}\}
\end{aligned}$$

Secondly, $\Leftarrow$. This argument is essentially the same as the previous one, but in reverse (and with slight alterations in the complexity arguments). That is, starting from

$$\begin{aligned}
&\exists R_2 \in \mathcal{R}^2 \; R_2 \; p.b. \wedge \exists R_{n+2} \in \mathcal{R}^{n+2} \; R_{n+2} \; p.b. \wedge R_{n+2} \; p.d. \wedge \\
&\mathcal{L}(R_2) = \{x\#y \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n \; (x, y, y_1, \dots, y_n) \in R_{n+2}\}
\end{aligned}$$

Taking the relation R_{n+2} one can construct the relation $R_{n+1} \in \mathcal{R}^{n+1}$ such that

$$R_{n+1} = \{(x\#y, y_1, \dots, y_n) \mid (x, y, y_1, \dots, y_n) \in R_{n+2}\}$$

To do so one merely has to concatenate the first two entries in $(x, y, y_1, \dots, y_n) \in R_{n+1}$ using the separator #, which can be done in polynomial time (similar argument as in Lemma 6). Moreover, it clearly holds that

$$(x\#y, y_1, \dots, y_n) \in R_{n+1} \iff (x, y, y_1, \dots, y_n) \in R_{n+2}$$

It is known that R_{n+2} is p.b., thus there exists a k such that for $1 \leq i \leq n$, $|y_i| \leq |x|^k$ and $|y| \leq |x|^k$. Now since $|x| < |x\#y|$ it must be that R_{n+1} is p.b. as well. Additionally, one knows that R_{n+2} is p.d., thus R_{n+1} can be decided by splitting the first entry on # and querying R_{n+2}. Both operations can be done in polynomial time, thus R_{n+1} is p.d.. Hence, one obtains

$$\begin{aligned}
&\exists R_2 \in \mathcal{R}^2 \; R_2 \; p.b. \wedge \exists R_{n+1} \in \mathcal{R}^{n+1} \; R_{n+1} \; p.b. \wedge R_{n+1} \; p.d. \wedge \\
&\mathcal{L}(R_2) = \{x\#y \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n \; (x\#y, y_1, \dots, y_n) \in R_{n+1}\}
\end{aligned}$$

(iv) This equality is guaranteed by the following. Take an arbitrary relation R. Clearly, $(x, y) \in R \iff x\#y \in \mathcal{L}(R)$. Hence, in this particular case one has $(x, y) \in R_2 \iff x\#y \in \mathcal{L}(R_2)$. Now starting from

$$\begin{aligned}
\mathcal{L}(R_2) &= \{x\#y \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n \; (x, y, y_1, \dots, y_n) \in R_{n+2}\} \wedge \\
L &= \{x \mid \forall y \; (x, y) \in R_2\}
\end{aligned}$$

due to

$$\begin{aligned}(\alpha,\beta)\in R_2 &\iff \alpha\#\beta\in\mathcal{L}(R_2)\\ &\iff \alpha\#\beta\in\{x\#y \mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\\ &\iff (\alpha,\beta)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\end{aligned}$$

one obtains the equivalent statement

$$\begin{aligned}R_2 &= \{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\wedge\\ L &= \{x\mid \forall y\ (x,y)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

(v) Firstly, $\Rightarrow$. Starting from

$$\begin{aligned}&\exists R_2\in\mathcal{R}^2\ R_2\ p.b.\wedge\exists R_{n+2}\in\mathcal{R}^{n+2}\ R_{n+2}\ p.b.\wedge R_{n+2}\ p.d.\wedge\\ &R_2=\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\wedge\\ &L=\{x\mid \forall y\ (x,y)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

one can simply use weakening to obtain the part of the statement, where R_2 does not occur.

$$\begin{aligned}&\exists R_{n+2}\in\mathcal{R}^{n+2}\ R_{n+2}\ p.b.\wedge R_{n+2}\ p.d.\wedge\\ &L=\{x\mid \forall y\ (x,y)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

Thereby, eradicating all references of R_2.

Secondly, $\Leftarrow$. Starting from

$$\begin{aligned}&\exists R_{n+2}\in\mathcal{R}^{n+2}\ R_{n+2}\ p.b.\wedge R_{n+2}\ p.d.\wedge\\ &L=\{x\mid \forall y\ (x,y)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

One can define the relation $R_2 := \{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}$. Since it is known that R_{n+2} is p.b. this implies that there exists a k such that $|y|\le|x|^k$, thus implying that R_2 is p.b.. Allowing one to conclude that

$$\begin{aligned}&\exists R_2\in\mathcal{R}^2\ R_2\ p.b.\wedge\exists R_{n+2}\in\mathcal{R}^{n+2}\ R_{n+2}\ p.b.\wedge R_{n+2}\ p.d.\wedge\\ &R_2=\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\wedge\\ &L=\{x\mid \forall y\ (x,y)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\end{aligned}$$

(vi) Starting from $\{x\mid \forall y\ (x,y)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}$. Notice that

$$\begin{aligned}&(\alpha,\beta)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\\ &\iff \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (\alpha,\beta,y_1,\ldots,y_n)\in R_{n+2}\end{aligned}$$

From this it follows that

$$\begin{aligned}&\forall y\ (\alpha,y)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\\ &\iff \forall y\exists y_1\forall y_2\exists y_3\ldots Qy_n\ (\alpha,y,y_1,\ldots,y_n)\in R_{n+2}\end{aligned}$$

and therefore

$$\begin{aligned}&\{x\mid \forall y\ (x,y)\in\{(x,y)\mid \exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\}\\ &=\{x\mid \forall y\exists y_1\forall y_2\exists y_3\ldots Qy_n\ (x,y,y_1,\ldots,y_n)\in R_{n+2}\}\end{aligned}$$

(vii) This particular renaming of bound variables is clearly an equivalence transformation. Hence,

$$n \text{ even} \Rightarrow Q = \forall \wedge n \text{ odd} \Rightarrow Q = \exists \iff n+1 \text{ even} \Rightarrow Q = \exists \wedge n+1 \text{ odd} \Rightarrow Q = \forall$$

remains to be established. However, this follows directly from the fact that n is even if and only if $n+1$ is odd. That is, if n was even, one has $Q = \forall$. However, this implies that for $Q = \forall$ for $n+1$ being odd and if $n+1$ is odd then $Q = \forall$, meaning that $Q = \forall$ for n is even. Analogous for the other case.