

Exercise 31

Axioms expressing relations between modalities:

$$\mathcal{F} \models \Box_2 A \supset \Box_1 A$$

whenever in $\mathcal{F} := \langle W, R_1, R_2 \rangle$ we have: R_1 is a sub-relation of R_2 (i.e. $R_1 \subseteq R_2$).

The inverse task is even more important: To find axioms that characterize given semantic connections between relations. R_1 is a sub-relation of R_2 , in a multi-relation frame \mathcal{F} , whenever

$$\mathcal{F} \models \Box_2 A \supset \Box_1 A$$

Let $\mathcal{F} := \langle W, R_1, R_2, \dots, R_n \rangle$ be a frame for $n \in \mathcal{N}$.

- $R_i \subseteq R_j$ implies $\mathcal{F} \models \Box_j \varphi \supset \Box_i \varphi$
Assume $R_i \subseteq R_j$, for the frame \mathcal{F} . Moreover, take an arbitrary model \mathcal{M} based on the frame \mathcal{F} . If it is the case that $\mathcal{M}, s \not\models \Box_j \varphi$, then clearly $\mathcal{M}, s \not\models \Box_j \varphi \supset \Box_i \varphi$. Otherwise, $\mathcal{M}, s \models \Box_j \varphi$, elevated to the meta-level is $\forall t(sR_j t \Rightarrow \mathcal{M}, t \models \varphi)$. By assumption we know that $R_i \subseteq R_j$. That is, every world accessible from s by a relation in R_i is also reachable from s by a relation in R_j . Since all worlds accessible from s through a relation in R_j satisfies φ , one clearly obtains $\forall t(sR_i t \Rightarrow \mathcal{M}, t \models \varphi)$, which returning to a language level results in $\mathcal{M}, s \models \Box_i \varphi$.
- $\mathcal{F} \models \Box_j \varphi \supset \Box_i \varphi$ implies $R_i \subseteq R_j$
Assume $\mathcal{F} \models \Box_j \varphi \supset \Box_i \varphi$. Moreover, consider that $\exists e \in R_i \setminus R_j$. Clearly, e must have a state of origin s and a state of destination t . Select a model \mathcal{M} based on \mathcal{F} such that $V(p) := W \setminus \{t\}$. Now, elevating $\mathcal{M}, s \models \Box_j p \supset \Box_i p$ to the meta level, results in $\forall t(sR_j t \Rightarrow \mathcal{M}, t \models p) \Rightarrow \forall t(sR_i t \Rightarrow \mathcal{M}, t \models p)$. Since by assumption $e \notin R_j$ the statement $\forall t(sR_j t \Rightarrow \mathcal{M}, t \models p)$ holds. However, $\forall t(sR_i t \Rightarrow \mathcal{M}, t \models p)$ can not, due to the fact that $sR_i t \wedge \mathcal{M}, t \not\models p$.

Exercise 32

Characterizing the relation between future and past time:

1. $\mathcal{F} \models A \supset [P] \langle F \rangle A$ iff $\forall s \forall t(sR_P t \Rightarrow tR_F s)$
2. $\mathcal{F} \models A \supset [F] \langle P \rangle A$ iff $\forall s \forall t(sR_F t \Rightarrow tR_P s)$

Assumptions. For some model \mathcal{M} and some present s . The statement

- $\mathcal{M}, s \models [P] \varphi$ elevated to the meta-level $\forall t(sR_P t \Rightarrow \mathcal{M}, s \models \varphi)$;

- $\mathcal{M}, s \models \langle P \rangle \varphi$ elevated to the meta-level $\exists t(sR_P t \text{ and } \mathcal{M}, s \models \varphi)$;
- $\mathcal{M}, s \models [F] \varphi$ elevated to the meta-level $\forall t(sR_F t \Rightarrow \mathcal{M}, s \models \varphi)$;
- $\mathcal{M}, s \models \langle F \rangle \varphi$ elevated to the meta-level $\exists t(sR_F t \text{ and } \mathcal{M}, s \models \varphi)$;

Hence, given the assumptions above it follows.

1. $\mathcal{F} \models \varphi \supset [P] \langle F \rangle \varphi$ iff $\forall s \forall t (sR_P t \Rightarrow tR_F s)$.

- " \Rightarrow ". This can be shown, by assuming that $\mathcal{F} \models \varphi \supset [P] \langle F \rangle \varphi$ and that $\exists s \exists t (sR_P t \text{ and } \neg tR_F s)$. Since by assumption the formula has to hold in every model based on the frame \mathcal{F} . Consider the model \mathcal{M} , where $V(p) := \{s\}$. Now consider the formula elevated to the meta-level at state s , $\mathcal{M}, s \models p \Rightarrow \forall u (sR_P u \Rightarrow \exists v (uR_F v \wedge \mathcal{M}, v \models p))$. By assumption, $\neg tR_F s$ and for all other states u , by definition, $\mathcal{M}, u \not\models p$. Therefore, $\exists v (uR_F v \wedge \mathcal{M}, v \models p)$ can not hold. However, by assumption $\mathcal{M}, s \models p$ and $\exists t sR_P t$. That is, the premise of the internal implication holds, but the consequence does not, thus the consequence of the external implication does not hold and thus the whole statement does not hold. Hence, moving back to the language level, one can conclude that $\mathcal{F} \not\models \varphi \supset [P] \langle F \rangle \varphi$. Thereby contradicting the assumption.
- " \Leftarrow ". Let \mathcal{F} be a frame that satisfies the property $\forall s \forall t (sR_P t \Rightarrow tR_F s)$. If $\mathcal{F} \not\models \varphi$, then $\mathcal{F} \models \varphi \supset [P] \langle F \rangle \varphi$ holds trivially. Otherwise, $\mathcal{F} \models \varphi$. Take an arbitrary model \mathcal{M} based on the frame \mathcal{F} and an arbitrary state s . By assumption $\mathcal{M}, s \models \varphi$. Now consider an arbitrary state t accessible from s through R_P . Due to the fact that $sR_P t$, given the assumptions $tR_F s$. That is, $sR_P t$ implies $tR_F s$ and $\mathcal{M}, s \models \varphi$. Therefore, from the assumption that $\mathcal{M} \models \varphi$, given an arbitrary state s , if a state t is accessible from s , there exists a state u , namely s , at which $\mathcal{M}, u \models \varphi$. Or more concise, $\forall t (sR_P t \Rightarrow \exists u (tR_F u \text{ and } \mathcal{M}, u \models \varphi))$, which is exactly the formula $\mathcal{F} \models \varphi \supset [P] \langle F \rangle \varphi$ elevated to the meta-level in the respective model and state.

2. $\mathcal{F} \models \varphi \supset [F] \langle P \rangle \varphi$ iff $\forall s \forall t (sR_F t \Rightarrow tR_P s)$.

This is the exact analogue to the previous statement, the only difference is the P and F are switched.

Exercise 33

Prove that (1) is equivalent to axiom (K) (with \Box replaced by \mathbf{K}_i).

$$(\mathbf{K}_i A \wedge \mathbf{K}_i (A \supset B)) \supset \mathbf{K}_i B \iff \mathbf{K}_i (A \supset B) \supset \mathbf{K}_i A \supset \mathbf{K}_i B$$

A nice way to show this fact is to establish that $((A \wedge B) \supset C) \supset (A \supset (B \supset C))$ and $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$ are CL tautologies. This is sufficient, as

$$(\mathbf{K}_i A \wedge \mathbf{K}_i(A \supset B)) \supset \mathbf{K}_i B \iff \mathbf{K}_i(A \supset B) \supset \mathbf{K}_i A \supset \mathbf{K}_i B$$

can be translated into

$$\begin{aligned} & (((p_{\mathbf{K}_i A} \wedge p_{\mathbf{K}_i(A \supset B)}) \supset p_{\mathbf{K}_i B}) \supset (p_{\mathbf{K}_i(A \supset B)} \supset p_{\mathbf{K}_i A} \supset p_{\mathbf{K}_i B})) \wedge \\ & ((p_{\mathbf{K}_i(A \supset B)} \supset p_{\mathbf{K}_i A} \supset p_{\mathbf{K}_i B}) \supset ((p_{\mathbf{K}_i A} \wedge p_{\mathbf{K}_i(A \supset B)}) \supset p_{\mathbf{K}_i B})) \end{aligned}$$

To take a different approach from the one below, those tautologies are shown by utilising equivalence transformations. That is,

$$\begin{array}{ll} A \supset (B \supset C) & \text{Definition of } \supset \\ \neg A \vee (\neg B \vee C) & \text{Commutativity of } \vee \\ (\neg A \vee \neg B) \vee C & \text{DeMorgan} \\ \neg(A \wedge B) \vee C & \text{Definition of } \supset \\ (A \wedge B) \supset C & \end{array}$$

At every step we have an equivalence transformation. Thus, both $((A \wedge B) \supset C) \supset (A \supset (B \supset C))$ and $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$ are shown.

Alternatively and more cumbersome.

- " \implies " Consider an arbitrary frame \mathcal{F} . Where $\mathcal{F} \models (\mathbf{K}_i \varphi \wedge \mathbf{K}_i(\varphi \supset \psi)) \supset \mathbf{K}_i \psi$. Now take an arbitrary models based on the frame \mathcal{F} and state s . Clearly, $\mathcal{M}, s \models (\mathbf{K}_i \varphi \wedge \mathbf{K}_i(\varphi \supset \psi)) \supset \mathbf{K}_i \psi$. Consider the following cases
 - **Case 1:** $\mathcal{M}, s \not\models (\mathbf{K}_i \varphi \wedge \mathbf{K}_i(\varphi \supset \psi))$
 - * **Case 1.1:** $\mathcal{M}, s \not\models \mathbf{K}_i \varphi$
If this is the case by semantics of " \supset ", $\mathcal{M}, s \models \mathbf{K}_i \varphi \supset \mathbf{K}_i \psi$ holds. Now again by the semantics of " \supset " it follows that $\mathcal{M}, s \models \mathbf{K}_i(\varphi \supset \psi) \supset \mathbf{K}_i \varphi \supset \mathbf{K}_i \psi$ regardless of the evaluation of $\mathbf{K}_i(\varphi \supset \psi)$.
 - * **Case 1.2:** $\mathcal{M}, s \not\models \mathbf{K}_i(\varphi \supset \psi)$
By the semantics of " \supset ", it clearly follows that $\mathcal{M}, s \models \mathbf{K}_i(\varphi \supset \psi) \supset \mathbf{K}_i \varphi \supset \mathbf{K}_i \psi$ regardless of the evaluation of $\mathbf{K}_i \varphi \supset \mathbf{K}_i \psi$.
 - **Case 2:** $\mathcal{M}, s \models (\mathbf{K}_i \varphi \wedge \mathbf{K}_i(\varphi \supset \psi))$
 $\mathcal{M}, s \models \mathbf{K}_i(\varphi \supset \psi)$ means that $\forall t(sR_i t \Rightarrow \mathcal{M}, t \models \varphi \supset \psi)$ and $\mathcal{M}, s \models \mathbf{K}_i \varphi$ means that $\forall t(sR_i t \Rightarrow \mathcal{M}, t \models \varphi)$. Hence, for all states t reachable from s through R_i we have $\mathcal{M}, t \models \varphi$ and $\mathcal{M}, t \models \varphi \supset \psi$. Thus, it follows that $\mathcal{M}, t \models \psi$, leading to $\mathcal{M}, s \models \mathbf{K}_i \psi$. Therefore, $\mathcal{M}, s \models \mathbf{K}_i(\varphi \supset \psi) \supset \mathbf{K}_i \varphi \supset \mathbf{K}_i \psi$.

- " \Leftarrow " Consider an arbitrary frame \mathcal{F} . Where $\mathcal{F} \models \mathbf{K}_i(\varphi \supset \psi) \supset \mathbf{K}_i\varphi \supset \mathbf{K}_i\psi$. Now take an arbitrary models based on the frame \mathcal{F} and state s . Clearly, $\mathcal{M}, s \models \mathbf{K}_i(\varphi \supset \psi) \supset \mathbf{K}_i\varphi \supset \mathbf{K}_i\psi$. Consider the following cases
 - **Case 1:** $\mathcal{M}, s \not\models \mathbf{K}_i(\varphi \supset \psi)$
 By semantics of " \wedge ", $\mathcal{M}, s \not\models (\mathbf{K}_i\varphi \wedge \mathbf{K}_i(\varphi \supset \psi))$ and thus by semantics of " \supset " it follows that $\mathcal{M}, s \models (\mathbf{K}_i\varphi \wedge \mathbf{K}_i(\varphi \supset \psi)) \supset \mathbf{K}_i\psi$.
 - **Case 2:** $\mathcal{M}, s \models \mathbf{K}_i(\varphi \supset \psi)$
 - * **Case 2.1:** $\mathcal{M}, s \not\models \mathbf{K}_i\varphi$
 By semantics of " \wedge ", $\mathcal{M}, s \not\models (\mathbf{K}_i\varphi \wedge \mathbf{K}_i(\varphi \supset \psi))$ and thus by semantics of " \supset " it follows that $\mathcal{M}, s \models (\mathbf{K}_i\varphi \wedge \mathbf{K}_i(\varphi \supset \psi)) \supset \mathbf{K}_i\psi$.
 - * **Case 2.2:** $\mathcal{M}, s \models \mathbf{K}_i\varphi$
 By semantics of " \wedge ", $\mathcal{M}, s \models (\mathbf{K}_i\varphi \wedge \mathbf{K}_i(\varphi \supset \psi))$ $\mathcal{M}, s \models \mathbf{K}_i(\varphi \supset \psi)$ means that $\forall t(sR_it \Rightarrow \mathcal{M}, t \models \varphi \supset \psi)$ and $\mathcal{M}, s \models \mathbf{K}_i\varphi$ means that $\forall t(sR_it \Rightarrow \mathcal{M}, t \models \varphi)$. Hence, for all states t reachable from s through R_i we have $\mathcal{M}, t \models \varphi$ and $\mathcal{M}, t \models \varphi \supset \psi$. Thus, it follows that $\mathcal{M}, t \models \psi$, leading to $\mathcal{M}, s \models \mathbf{K}_i\psi$. Therefore, $\mathcal{M}, s \models (\mathbf{K}_i\varphi \wedge \mathbf{K}_i(\varphi \supset \psi)) \supset \mathbf{K}_i\psi$.

Exercise 34

Prove: $\mathcal{F} \models A5$ iff $\mathcal{F} \models$ *negative introspection*.

By definition $\mathcal{F} \models A5$ is $\mathcal{F} \models \Diamond\varphi \supset \Box\Diamond\varphi$. Since epistemic logic operates over multiple modalities, consider $\mathcal{F} \models \Diamond_i\varphi \supset \Box_i\Diamond_i\varphi$ for some arbitrary i . Having only defined a correspondence between the epistemic modal operator and the box operator, the first step is to employ the definition of \Diamond , i.e. $\Diamond := \neg\Box\neg$. Resulting in $\mathcal{F} \models \neg\Box_i\neg\varphi \supset \Box_i\neg\Box_i\neg\varphi$. Now we can replace (previous exercise) the \Box_i with \mathbf{K}_i to obtain $\mathcal{F} \models \neg\mathbf{K}_i\neg\varphi \supset \mathbf{K}_i\neg\mathbf{K}_i\neg\varphi$, as the version of (A5) in the language of epistemic logic with multiple agents. Therefore, the it remains to show that,

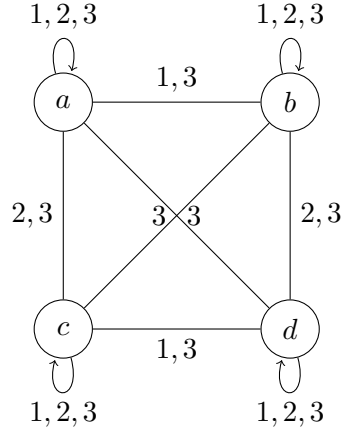
$$\forall \varphi \in \mathcal{L}_K (\mathcal{F} \models \neg\mathbf{K}_i\neg\varphi \supset \mathbf{K}_i\neg\mathbf{K}_i\neg\varphi) \iff \forall \varphi \in \mathcal{L}_K (\mathcal{F} \models \neg\mathbf{K}_i\varphi \supset \mathbf{K}_i\neg\mathbf{K}_i\varphi)$$

where \mathcal{L}_K is the language of epistemic logic with multiple agents. That is, (A5) is a schemata and therefore φ can be replaced by an arbitrary formula. Consider an arbitrary formula ψ and let $\varphi := \neg\psi$. By (A5) it follows that $\mathcal{F} \models \neg\mathbf{K}_i\neg\neg\psi \supset \mathbf{K}_i\neg\mathbf{K}_i\neg\neg\psi$, which by propositional reasoning is equivalent to $\mathcal{F} \models \neg\mathbf{K}_i\psi \supset \mathbf{K}_i\neg\mathbf{K}_i\psi$. Since, ψ is arbitrary, it follows $\forall \varphi \in \mathcal{L}_K \mathcal{F} \models \neg\mathbf{K}_i\varphi \supset \mathbf{K}_i\neg\mathbf{K}_i\varphi$. Similarly, if $\forall \varphi \in \mathcal{L}_K \mathcal{F} \models \neg\mathbf{K}_i\varphi \supset \mathbf{K}_i\neg\mathbf{K}_i\varphi$ then especially $\forall \neg\psi \in \mathcal{L}_K \mathcal{F} \models \neg\mathbf{K}_i\neg\psi \supset \mathbf{K}_i\neg\mathbf{K}_i\neg\psi$, and since ψ arbitrary $\forall \varphi \in \mathcal{L}_K \mathcal{F} \models \neg\mathbf{K}_i\neg\varphi \supset \mathbf{K}_i\neg\mathbf{K}_i\neg\varphi$.

Exercise 35

Augment this interpretation to represent an additional agent 3 who knows nothing about the weather in Hanoi and Manaus.

Note: Undirected edges (without arrows) represent two directed edges.



with V being defined as,

- $V(m) := \{a, c\}$
- $V(h) := \{a, b\}$

Lastly, the new edges have to be added, as not to violate transitivity required by the *positive introspection*.

Exercise 36

Construct an (infinite) satisfiable set of formulas over p , \mathbf{K}_1 , \mathbf{K}_2 and the classical connectives that has no finite Kripke model. (Proof!)

[I am terribly sorry for the ugly proof (assuming it is one).]

Firstly, some syntactic sugar. Let $\mathbf{P}_x\varphi := \neg K_x\neg\varphi$ for some agent x .

Secondly, defining the terminology for the construction of the set of formulas.

Consider the following. For all $k \in \mathbb{N}$ with $k \geq 2$,

$$\varphi_{K_a}(k) := \begin{cases} \mathbf{K}_a\neg p & k = 2 \\ \mathbf{K}_a\mathbf{K}_b\neg p & k = 3 \\ \mathbf{K}_a\mathbf{K}_b\varphi_{K_a}(k-2) & \text{otw.} \end{cases}$$

and

$$\varphi_{K_b}(k) := \begin{cases} \mathbf{K}_b \neg p & k = 2 \\ \mathbf{K}_b \mathbf{K}_a \neg p & k = 3 \\ \mathbf{K}_b \mathbf{K}_a \varphi_{K_b}(k-2) & \text{otw.} \end{cases}$$

where $\varphi_K(k) := \varphi_{K_a}(k) \wedge \varphi_{K_b}(k)$. Moreover,

$$\varphi_P(k) := \begin{cases} \mathbf{P}_a \mathbf{P}_b p & k = 2 \\ \mathbf{P}_a \mathbf{P}_b \mathbf{P}_a p & k = 3 \\ \mathbf{P}_a \mathbf{P}_b \varphi_P(k-2) & \text{otw.} \end{cases}$$

Now, let

$$\varphi(k) := \varphi_K(k) \wedge \varphi_P(k)$$

and

$$\varphi_*(k) := (\mathbf{P}_a \mathbf{P}_b)^k \varphi(k)$$

where $(\mathbf{P}_a \mathbf{P}_b)^k := \underbrace{\mathbf{P}_a \mathbf{P}_b \dots \mathbf{P}_a \mathbf{P}_b}_{k\text{-times}}$. Lastly,

$$\Gamma_k := \bigcup_{i=2}^k \{\varphi(i)\}$$

as well as,

$$\Gamma := \bigcup_{i=2}^{\infty} \{\varphi(i)\}$$

Before moving on, an intuitive explanation of the construction. Given a state s the formula $\varphi_K(k)$ ensures that any state reachable by an ab -path of length $k-1$, i.e. a path with $k-1$ steps using the accessibility relations R_a and R_b in alternating fashion, can not be in $V(p)$. While $\varphi_P(k)$ requires that there exists at least one state, accessible from s by an ab -path of length at most k where p does hold. However, since any world on the path less then k steps away can not be in $V(p)$, it is not at most k but exactly k . Hence, any model satisfying $\varphi(k)$ has to have at least k states. A natural choice for such a model would be the ab -path, where the last state will be the only one in $V(p)$. Now as Γ is an infinite union of formulas requiring models of increasing size, for any model of a fixed size, it will be possible to find a formula in Γ requiring a model of larger size. Moreover, to find a model for Γ one has to find a state, where all formulas hold. However, if one attaches all $\varphi(k)$ directly to single state, the requirements imposed by $\varphi_K(k)$ lead to unsatisfiability. Therefore, the sequence $(\mathbf{P}_a \mathbf{P}_b)^k$ allows for sufficient buffer, such that one can find a model, without engaging in the complications that arise due the possible interplay of formulas in Γ . e.g. $\varphi_K(k_1)$ for a large k_1 making $\varphi_P(k_2)$ for a small k_2 impossible.

The first task at hand is to show that Γ has no finite model. This shall be split in multiple parts.

1. For every path $s \rightsquigarrow t$ of length n in an epistemic Kripke model \mathcal{M} , with two accessibility relations R_a and R_b , there exists an ab -path $s \rightsquigarrow_{ab} t$ of length m such that $m \leq n$.

This statement follows directly from transitivity. That is, consider a path $s \rightsquigarrow t$ where for $x, y \in \{a, b\}$ and $x \neq y$ there exists a sequence

$$s_s R_y s_i, s_i R_x s_{i+1}, \dots, s_{j-1} R_x s_j, s_j R_y s_t$$

by transitivity, it follows that $s_i R_x s_j$. Hence,

$$s_s R_y s_i, s_i R_x s_j, s_j R_y s_t$$

By replacing the new sequence of steps with the original path one obtains a path of smaller (or equal) size. Moreover, since any sequence of steps in the same accessibility relation can be replaced by a single step, an exhaustive application of this replacement results in an ab -path.

(Note: The same holds for the special cases of path over a single accessibility relation. They simply collapse into a single step, which is a trivial ab -path)

2. For any model $\mathcal{M} \models \varphi(k)$ for $k \in \{k \mid k > 1\}$, it follows that $|W| \geq k$.

From the previous result follows that any arbitrary path between two states, can be replaced by a shorter ab -path, thus justifying the focus on ab -paths only.

Consider an arbitrary model \mathcal{M} such that there exists a state s_1 where $\mathcal{M}, s_1 \models \varphi(k)$, which is the same as $\mathcal{M}, s_1 \models \varphi_K(k) \wedge \varphi_P(k)$. Now, since $\mathcal{M}, s_1 \models \varphi_K(k)$ and $\mathcal{M}, s_1 \models \varphi_P(k)$, it is required that there exists a state s_{k+1} accessible from s_1 by an ab -path of length k , i.e. a path with k steps using the accessibility relations R_a and R_b in alternating fashion, such that $\mathcal{M}, s_{k+1} \models p$. From graph theory, it is known for a graph without weights that, if a vertex occurs twice in a path between two distinct vertices, it can not be the shortest path. Hence, any ab -path using a state twice, can be reached by a shorter ab -path, and thus no state in this path can satisfy p . Therefore, to satisfy $\varphi_P(k)$ one requires an ab -path with $k + 1$ unique states. Hence, $|W| > k$.

(Note: The two parts of φ_K , $\varphi_{K_a}(k)$ and $\varphi_{K_b}(k)$ are required to ensure that the number of alternations determine the truth value of p , regardless of which accessibility relation is used for the first step in the ab -path. That is, without $\varphi_{K_b}(k)$ it would be permitted that $s_1 R_a s_2$ such that $s_1 = s_2$.)

3. For any model $\mathcal{M} \models \varphi_*(k)$ for $k \in \{k \mid k > 1\}$, it follows that $|W| \geq k$.

There is some state reachable by an *ab*-path where $\varphi(k)$ holds. To satisfy $\varphi(k)$ at least k unique states are required. Hence, any model satisfying $\varphi_*(k)$ requires at least as many states.

4. Γ has no finite model.

Γ is an infinite union of formulas requiring models of increasing size, for any model of a fixed size, it will be possible to find a formula in Γ requiring a model of larger size.

The second task at hand requires the construction of a model $\mathcal{M} := \langle W, R_a, R_b, V \rangle$ such that there exists a state $o := (0, 0)$ that $\forall \psi \in \Gamma, \mathcal{M}, o \models \psi$.

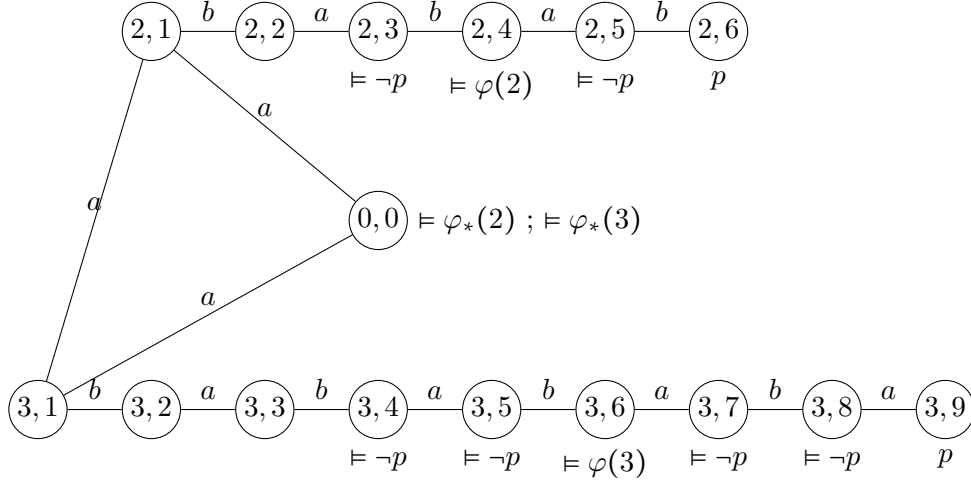
To that end consider the following model $W := \mathbb{N} \times \mathbb{N}$ and

- $R_a := R_{R_a} \cup O_{R_a} \cup E_{R_a} \cup B_{R_a}$
 - $R_{R_a} := \{(w, w) \mid \forall w \in W\}$
 - $O_{R_a} := \bigcup_{k>1} \{((0, 0), (k, 1)), ((k, 1), (0, 0))\}$
 - $E_{R_a} := \bigcup_{j,k>1} \{((j, 1), (k, 1))\}$
 - $B_{R_a} := \bigcup_{k>1} \bigcup_{1 < i = 2n < 3k} \{((k, i), (k, i+1)), ((k, i+1), (k, i))\}$
- $R_b := R_{R_b} \cup B_{R_b}$
 - $R_{R_b} := \{(w, w) \mid \forall w \in W\}$
 - $B_{R_b} := \bigcup_{k>1} \bigcup_{0 < i = 2n+1 < 3k} \{((k, i), (k, i+1)), ((k, i+1), (k, i))\}$

Lastly, $V(p) := \bigcup_{k>1} \{(k, 3k)\}$.

Again a brief explanation of the structure. R_{R_a} and R_{R_b} ensure reflexivity. O_{R_a} connects all paths to a single origin point, while E_{R_a} ensures that R_a remains an equivalence relation. That is, since all branches start with a step in the relation R_a , one has to symmetrically connect all worlds in the initial step (to ensure that R_a is euclidean). Lastly, B_{R_a} symmetrically connects each even state with the next one, while B_{R_b} symmetrically connects each odd state with the next one. Hence, no transitive relations are required.

Example for $k < 4$. (Reflexive edges are not depicted)



It is to show that for an arbitrary $k > 1$, $\mathcal{M}, o \models \varphi_*(k)$. Firstly, $\mathcal{M}, o \models (\mathbf{P}_a \mathbf{P}_b)^k \varphi(k)$. Consider $(k, 2k)$, which is by definition an ab -path of length $2k$ away. Hence, one has to test whether $\varphi(k)$ holds at $(k, 2k)$, to conclude that $\varphi_*(k)$ holds as o . As for $\varphi_P(k)$, given the evaluation function $\mathcal{M}, (k, 3k) \models p$, which is exactly k alternating steps away from $(k, 2k)$. Hence, $\mathcal{M}, (k, 2k) \models \varphi_P(k)$. Moreover, for $\varphi_K(k)$ to hold, all states in all ab -paths of length $k-1$ can not satisfy p . Given the construction, this means that $\forall s \in \{(k, k+1), \dots, (k, 3k-1)\}$ it must be that $\mathcal{M}, s \not\models p$, which is clearly the case. Hence, $\mathcal{M}, o \models \varphi_*(k)$. Since, this can be done for an arbitrary k , it follows that $\forall \psi \in \Gamma, \mathcal{M}, o \models \psi$.