Non-Monotonic Reasoning

Complexity Results for Non-Monotonic Logics

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Goal

What are we doing here?

Showing tight complexity bounds for a set of nonmonotonic logics

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Introduction

Core Concepts

Definition: Fixed Point

For a set Σ of premisses, $\Delta \subseteq \Sigma$ is stable under the operator Γ iff

$$\Gamma(\Delta) = \Delta$$

Definition: Consequence

For $\Delta \subseteq \mathcal{L}$ we have

$$cons(\Delta) := \{\phi \mid \Delta \models \phi\}$$

Definition: Notation

For $\Delta \subseteq \mathcal{L}$ and and an unary operator Θ:

$$\Theta(\Delta) := \{\Theta\phi \mid \phi \in \Delta\}$$
$$\overline{\Delta} := \mathcal{L} \setminus \Delta$$

Complexity Concepts: Definitions

Definition: Oracle

Let ϕ be an oracle (program) that solves all problems in Φ in unit-time. Then $p \in \Theta^{\phi}$ is a problem solvable in Θ given the oracle ϕ.

Definition: Polynomial Hierarchy

For $k = 0$:

$$\Delta_0^P = \Sigma_0^P = \Pi_0^P = P$$

For $k \geq 0$:

$$\Delta_{k+1}^P = P^{\Sigma_k^P}, \quad \Sigma_{k+1}^P = NP^{\Sigma_k^P}, \quad \Pi_{k+1}^P = co\Sigma_{k+1}^P = coNP^{\Sigma_k^P}$$

Examples: $SAT \in \Sigma_1^P$, $QBF_{2,\exists} \in \Sigma_2^P$

Complexity Concepts: Polynomial Hierarchy

$$
\begin{tikzcd}
{} & \cdots & {} \\
\Sigma_3^P = NP^{\Sigma_2^P} = NP^{NP^{NP}} \arrow[u, dash] \arrow[ur] & & \Pi_3^P = coNP^{\Sigma_2^P} = coNP^{NP^{NP}} \arrow[u, dash] \arrow[ul] \\
& \Delta_3^P = P^{\Sigma_2^P} = P^{NP^{NP}} \arrow[ul] \arrow[ur] & \\
\Sigma_2^P = NP^{\Sigma_1^P} = NP^{NP} \arrow[uu] \arrow[ur] & & \Pi_2^P = coNP^{\Sigma_1^P} = coNP^{NP} \arrow[uu] \arrow[ul] \\
& \Delta_2^P = P^{\Sigma_1^P} = P^{NP} \arrow[ul] \arrow[ur] & \\
\Sigma_1^P = NP^{\Sigma_0^P} = NP^{P} \arrow[uu] \arrow[ur] & & \Pi_1^P = coNP^{\Sigma_0^P} = coNP^{P} \arrow[uu] \arrow[ul] \\
& \Delta_0^P = \Sigma_0^P = \Pi_0^P = \Delta_1^P = P \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Complexity Concepts: $QBF_{2,\exists}$

Definition: $QBF_{2,\exists}$

For $Q \in QBF_{2,\exists}$ (QBF := Quantified Boolean Formulas)

$$Q := \exists p_1 \dots p_n \forall q_1 \dots \forall q_m \; E$$

where E is a propositional formula, $I := \{1, \dots, n\}$ and $(p_i)_{i \in I}, (q_i)_{i \in I}$ are families of mutually distinct propositional variables, i.e. $\nu(x)^{\mathcal{I}} \in \{\textbf{True}, \textbf{False}\}$ for x propositional variable.

Definition: $QBF_{2,\exists}$ - **Validity**

$Q \in QBF_{2,\exists}$ is valid $\iff$ $\exists$ variable assignment ν fixing $(p_i)_{i \in I}$ $\forall \sigma \supset \nu$ E is true.

Logics

- Default Logic (Reiter),
- Autoepistemic Logic (Moore),
- nonmonotonic logic N (Marek and Truszczyński) and
- nonmonotonic logic (McDermott and Doyle).

Definition: Three decision Problems

Let ϕ be a formula and Σ a set of premisses

existence: $\exists \Delta \supseteq \Sigma : \Delta$ *is a fixed-point*

brave/credulous reasoning: $\exists \Delta$ *stable-extension* $: \phi \in \Delta$

cautious/sceptical reasoning: $\forall \Delta$ *stable-extension* $: \phi \in \Delta$

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Spoilers

Complexity Results	existence	brave	cautious
Default Logic	Σ_2^P	Σ_2^P	Π_2^P
Autoepistemic Logic	Σ_2^P	Σ_2^P	Π_2^P
nonmonotonic logic N	?	?	?
nonmonotonic logic	Σ_2^P	Σ_2^P	Π_2^P

Spoilers

Complexity Results	existence	brave	cautious
Default Logic	Σ_2^P-*comp.*	Σ_2^P-*comp.*	Π_2^P-*comp.*
Autoepistemic Logic	Σ_2^P-*comp.*	Σ_2^P-*comp.*	Π_2^P-*comp.*
nonmonotonic logic N	Σ_2^P-*comp.*	Σ_2^P-*comp.*	Π_2^P-*comp.*
nonmonotonic logic	Σ_2^P-*comp.*	Σ_2^P-*comp.*	Π_2^P-*comp.*

Default Logic

Default Logic: Definitions

Definition: Default

A *default* is

$$\frac{\alpha : \beta_1, \beta_2, \ldots, \beta_n}{\omega}$$

(with $\alpha, \beta_1, \beta_2, \ldots, \beta_n, \omega$ propositional sentences) is satisfied by a deductively closed set of sentences Φ, if

$$\alpha \in \Phi \wedge \beta_1, \beta_2, \ldots, \beta_n \textit{ consistent with } \Phi \implies \omega \in \Phi$$

A default is called

- normal $:\iff \frac{\alpha:\omega}{\omega}$;
- semi-normal $:\iff \frac{\alpha:(\gamma\wedge\omega)}{\omega}$.

Definition: Propositional Default Theory

A *propositional default theory* is a pair $\langle W, D\rangle$ where W is a finite set of propositional sentences and D a set of defaults.

Default Logic: Definitions

Definition: Extension

Let $\langle W, D \rangle$ be a default theory, let S be a set of propositional formulas. Then $\Gamma(S)$ is the smallest set satisfying:

- $W \subseteq \Gamma(S)$,
- $\Gamma(S)$ deductively closed,
- $$\frac{\alpha : \beta_1, \beta_2, \ldots, \beta_n}{\omega} \land \alpha \in \Gamma(S) \land \neg\beta_1, \neg\beta_2, \ldots, \neg\beta_n \notin S \implies \omega \in \Gamma(S)$$
  .

Informally: A default extension of $\langle W, D \rangle$ is a grounded minimal deductively closed set of propositional formulas containing W and satisfying all defaults in D.

Default Logic: Finite Characterisation

Definition: Generating Defaults

Let E be an extension of the propositional default theory $\mathcal{T} = \langle W, D\rangle$. The set of generating defaults for E respect to $\mathcal{T}$ is

$$GD(E, \mathcal{T}) := \left\{ \frac{\alpha : \beta_1, \beta_2, \ldots, \beta_n}{\omega} \in D \,\middle|\, \alpha \in E \wedge \neg\beta_1, \neg\beta_2, \ldots, \neg\beta_n \notin E \right\}$$

Definition: Consequence

Let D be a set of default then

$$CONSEQUENTS(D) := \left\{ \omega \,\middle|\, \frac{\alpha : \beta_1, \beta_2, \ldots, \beta_n}{\omega} \in D \right\}$$

Proposition: Finite Characterisation of Extension

Let E be an extension of a default theory $\mathcal{T} = \langle W, D\rangle$. Then

$$E = cons(W \cup CONSEQUENTS(GD(E, \mathcal{T})))$$

Default Logic: Main Result

Theorem: Existence

Deciding whether a propositional default theory $\langle W, D\rangle$ has an extension is Σ_2^P-*complete*. (Note: the problem remains Σ_2^P-*complete* even if restricted to semi-normal default theories.)

Proof of Σ_2^P:

It can be shown that **existence** in default logic can be reduced to a Σ_2^P problem in nonmonotonic logic N

Proof of Σ_2^P-hard:

Proof by reduction to from $QBF_{2,\exists}$ to **existence** in default logic.
Let $Q := \exists p_1 \ldots p_n \forall q_1 \ldots \forall q_m\, E$ be transformed in polynomial time into the default theory $\langle W, D\rangle$ where $W := \varnothing$

$$D := \left\{ \frac{\top : p_1}{p_1}, \frac{\top : \neg p_1}{\neg p_1}, \ldots, \frac{\top : p_n}{p_n}, \frac{\top : \neg p_n}{\neg p_n}, \frac{\top : \neg E}{\bot} \right\}$$

Show

$$Q \text{ valid} \iff \langle W, D\rangle \text{ has an extension}$$

Assume $\langle W, D\rangle$ has an extension Δ.

- $\forall i \in I$ either $p_i \in \Delta$ or $\neg p_i \in \Delta$
- Show $\Delta \vDash E$.
  - W is consistent
  - thus, Δ must be consistent as
    - > from $\Delta = \mathcal{L}$
    - > we obtain $\Gamma(\Delta) = \Gamma(\mathcal{L}) = cons(W) \neq \Delta$.
  - Since $\bot \notin \Delta$ and $\frac{\top:\neg E}{\bot} \in D$ it must be that $\neg(\neg E) \in \Delta$.
- By combining $\Delta = cons(\{p_i \mid p_i \in \Delta\} \cup \{\neg p_i \mid \neg p_i \in \Delta\})$
- with $\Delta \vDash E$
- we obtain $\{p_i \mid p_i \in \Delta\} \cup \{\neg p_i \mid \neg p_i \in \Delta\} \vDash E$.
- Hence, Q is valid.

Default Logic: Main Result - Proof "$\Longrightarrow$"

Assume Q is valid.

- $\exists$ variable assignment ν fixing $(p_i)_{i \in I}$ s.t. $\forall \sigma \supset \nu$ E is true.
- Let $\Delta = cons(\{p_i \mid \nu(p_i) = \textbf{True}\} \cup \{\neg p_i \mid \nu(p_i)) = \textbf{False}\})$
- Hence, $\Delta \vDash E$,
- from which $E \in cons(\Delta)$ follows.
- $\Gamma(\Delta) \subseteq \Delta$ since
  - $\varnothing \subseteq \Delta$,
  - Δ is deductively closed and
  - $\forall d \in D : d$ satisfied implies $\omega \in \Delta$.
- $\Delta \subseteq \Gamma(\Delta)$ since
  - $p_i \in \Delta \iff p_i \in \Gamma(\Delta)$ and
  - $\neg p_i \in \Delta \iff \neg p_i \in \Gamma(\Delta)$.
- Obviously $\Gamma(\Delta) \subseteq \Delta$ and $\Delta \subseteq \Gamma(\Delta)$ implies $\Delta = \Gamma(\Delta)$.
- Therefore, Δ extension of $\langle W, D \rangle$.

Default Logic: Auxiliary Results - Brave Reasoning

Theorem: Brave Reasoning

Deciding whether a formula ϕ is an element of some extension of a propositional default theory $\langle W, D \rangle$ is Σ_2^P-*complete* (even for normal default theory)

Proof (Idea) of Σ_2^P-hard:

Let $Q := \exists p_1 \dots p_n \forall q_1 \dots \forall q_m\, E$ be transformed in polynomial time into a default theory $\langle W, D \rangle$ such that $W := \varnothing$

$$D := \left\{ \frac{\top : p_1}{p_1}, \frac{\top : \neg p_1}{\neg p_1}, \dots, \frac{\top : p_n}{p_n}, \frac{\top : \neg p_n}{\neg p_n} \right\}$$

- $\exists$ bijective mapping $f : \{\text{truth value assignments}\} \to \{\text{extensions of } \langle \varnothing, D \rangle\}$
- Hence, Q valid $\iff$ $\exists$ extension Δ of $\langle \varnothing, D \rangle$ such that $E \in \Delta$

Theorem: Cautious Reasoning

Deciding whether a formula ϕ is an element of all extensions of a propositional default theory $\langle W, D \rangle$ is Π_2^P-*complete* (even for normal default theory)

Proof (Idea) of Π_2^P-hard:

Let $Q := \exists p_1 \dots p_n \forall q_1 \dots \forall q_m \; E$ be transformed in polynomial time into a default theory $\langle W, D \rangle$ such that $W := \varnothing$

$$D := \left\{ \frac{\top : p_1}{p_1}, \frac{\top : \neg p_1}{\neg p_1}, \dots, \frac{\top : p_n}{p_n}, \frac{\top : \neg p_n}{\neg p_n}, \frac{\top : \neg E}{\neg E} \right\}$$

- Q not valid $\iff$ $\neg E$ belongs to each extension of $\langle \varnothing, D \rangle$.

Default Logic: Auxiliary Results - nonmonotonic logic N

Corollary: Reasoning in nonmonotonic logic N

Given a set of premisses Σ and $\phi \in \mathcal{L}$ (language of auto-epistemic logic)

- **existence** is Σ_2^P-*hard* (Σ_2^P-*complete*)
- **brave reasoning** for ϕ is Σ_2^P-*hard* (Σ_2^P-*complete*)
- **cautious reasoning** for ϕ is Π_2^P-*hard* (Π_2^P-*complete*)

Proof (Idea):

It can be shown that **existence** in default logic can be reduced to a Σ_2^P problem in nonmonotonic logic N. Hence, it is a fragment of nonmonotonic logic N, i.e. hardness carries over.

Autoepistemic Logic

Autoepistemic Logic: Definitions

Definition: Language $\mathcal{L}_{ae}$

The language of autoepistemic logic $\mathcal{L}_{ae}$ consists of the language of the classic propositional calculus $\mathcal{L}$ with the syntactic operators $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \bot, \top$ augmented with the "introspective" operator L (i.e. intuitively $L\phi$ means ϕ *is believed*).

Definition: Semantics

A propositional interpretation is extended by regarding $L\phi$ as atomic formula. Every non-atomic formula obtains its truth value by classic truth recursion.
The classical consequence relation on $\mathcal{L}$ is extended to $\mathcal{L}_{ae}$, such that for $\Sigma \subseteq \mathcal{L}_{ae}$ and $\phi \in \mathcal{L}_{ae}$

$$\Sigma \models \phi \iff \forall \mathcal{I} : \mathcal{I} \models \Sigma \Rightarrow \mathcal{I} \models \phi$$

Definition: Stable Expansion

Δ is a *stable expansion* of $\Sigma \iff \Delta = cons(\Sigma \cup L(\Delta) \cup \neg L(\overline{\Delta}))$

Autoepistemic Logic: Finite Characterisation

Definition: Lbase

An *Lbase* is the set $Lbase(\Sigma) := Sf^L(\Sigma) \cup \neg Sf^L(\Sigma)$ where $Sf^L(\Sigma)$ is the set of sub-formulas of each formula $\phi \in \Sigma$ of the form $L\phi$, i.e. $Sf^L(\Sigma) := \{L\phi \in Sf(\Sigma)\}$.

Definition: Σ-full

For a set of premises Σ a set $\Lambda \subseteq Lbase(\Sigma)$ is Σ-full iff $\forall L\phi \in Sf^L(\Sigma)$:

$$\Sigma \cup \Lambda \models \phi \iff L\phi \in \Lambda \quad \wedge \quad \Sigma \cup \Lambda \not\models \phi \iff \neg L\phi \in \Lambda$$

Proposition: Correspondence

For each set of premises Σ there is a one-to-one correspondence between the stable expansions of Σ and the Σ-full sets.

Definition: Kernel

For the expansion $E =: SE_\Sigma(\Lambda)$, with E corresponding the Σ-full set Λ we have

$$\Lambda = Lbase(\Sigma) \cap (\{L\phi \in E\} \cup \{\neg L\phi \notin E\})$$

With Λ being the kernel of $SE_\Sigma(\Lambda)$

Proposition: Membership

Let Σ be a set of premises, Λ is a Σ-*full* set and $\phi \in \mathcal{L}_{ae}$. Then
$\phi \in SE_\Sigma(\Lambda) \iff \Theta \vDash \phi$ where

$$\Theta := \Sigma \cup \Lambda \cup \{L\psi \mid L\psi \in Sf^q(\phi) \wedge \psi \in SE_\Sigma(\Lambda)\} \cup \{\neg L\psi \mid L\psi \in Sf^q(\phi) \wedge \psi \notin SE_\Sigma(\Lambda)\}$$

and Sf^q are all subformulas except that formulas of the form $L\phi$ do not have further subformulas.

Autoepistemic Logic: Main Result

Theorem: Existence

Deciding whether a set of premises Σ has a stable expansion is Σ_2^P complete.

Proof of Σ_2^P:

Was previously shown.

Proof of Σ_2^P-hard:

Proof by reduction to from $QBF_{2,\exists}$ to **existence** in autoepistemic logic.
Let $Q := \exists p_1 \dots p_n \forall q_1 \dots \forall q_m \, E$ be transformed in polynomial time into a set of autoepistemic formulas

$$\Sigma := \{p_1 \leftrightarrow Lp_1, \dots, p_n \leftrightarrow Lp_n, LE\}$$

Show

$$Q \text{ valid} \iff \Sigma \text{ has a stable expansion}$$

Autoepistemic Logic: Main Result - Proof "$\Longleftarrow$"

Assume Δ is a stable expansion of Σ.

- Firstly, check that Δ is consistent, i.e. $\Delta \neq \mathcal{L}_{ae}$.
  - Assume $\Delta = \mathcal{L}_{ae}$
  - thus, $\overline{\Delta} = \varnothing$
  - leading to $cons(\Sigma \cup L(\Delta) \cup \neg L(\varnothing)) = cons(\Sigma \cup L(\Delta))$
  - Consider $\mathcal{I}$ such that $\forall x \in atoms(\mathcal{L}_{ae}) : \nu^{\mathcal{I}}(x) = True$
    - > Σ is consistent,
    - > $L(\Delta)$ is consistent, leading to
    - > $\Sigma \cup L(\Delta)$ is consistent
  - Now since by definition $cons(\Sigma \cup L(\Delta)) = \{\phi \mid \Sigma \cup L(\Delta) \vDash \phi\}$ and $\Sigma \cup L(\Delta)$ it follows that
  - $cons(\Sigma \cup L(\Delta))$ is consistent.
  - $\lightning$

Autoepistemic Logic: Main Result - Proof "$\Longleftarrow$"

Assume Δ is a stable expansion of Σ.

- We have $\Sigma \subset \Delta$
- and $p_i \in \Delta$ or $\neg p_i \in \Delta$
  - we know either $Lp_i \in \Delta$ or $\neg Lp_i \in \Delta$
  - by $p_i \leftrightarrow Lp_i \in \Sigma \subset \Delta$ and by closure under consequence
  - $p_i \in \Delta$ or $\neg p_i \in \Delta$.
- We know $\Lambda = \{Lp_i \mid Lp_i \in \Delta\} \cup \{\neg Lp_i \mid \neg Lp_i \in \Delta\} \cup \{LE\}$
- From $LE \in \Delta$ we get $E \in \Delta$
- By Proposition "Membership" we get $\Sigma \cup \Lambda \models E$
- $q_i \notin \Sigma \cup \Lambda$
- Hence, truth value of Q sole depends on p_i's
- Therefore, Q is valid.

Autoepistemic Logic: Main Result - Proof "$\Longrightarrow$"

Assume Q is valid.

- $\exists$ variable assignment ν fixing $(p_i)_{i \in I}$ s.t. $\forall \sigma \supset \nu$ E is true.
- Consider $\Lambda = \{Lp_i \mid \nu(p_i) = \textbf{True}\} \cup \{\neg Lp_i \mid \nu(p_i) = \textbf{False}\} \cup \{LE\}$
- Claim Λ is Σ-*full*
  - $Sf^L(\Sigma) = \{Lp_i \mid \forall i \in I\} \cup \{LE\}$
  - $\forall i \in I$ $p_i \leftrightarrow Lp_i$ implies
    > $Lp_i \in \Lambda \iff \Sigma \cup \Lambda \models p_i$
    > $\neg Lp_i \in \Lambda \iff \Sigma \cup \Lambda \not\models p_i$
  - notice $\Sigma \cup \Lambda \models E$
  - thus, Λ is Σ-*full*.
- We have at least one Σ-*full* set.
- There must be at least one stable expansion of Σ.

Autoepistemic Logic: Auxiliary Results - Brave Reasoning

Theorem: Brave Reasoning

The problem of deciding whether a formula ϕ belongs to at least one stable expansion of a set of premises Σ is Σ_2^P-*complete*.

Proof (Idea) of Σ_2^P-*hard*:

- Any stable expansion Δ is closed under logical inference.
- Hence, $\top \in \Delta$
- Therefore, Σ has a stable expansion $\iff$ $\exists\Delta$ stable expansion of Σ $\top \in \Delta$
- Thus Σ has a stable expansion $\leq_P$ brave reasoning
- We obtain, Brave reasoning is Σ_2^P-*hard*

Theorem: Cautious Reasoning

The problem of deciding whether a formula ϕ belongs to at all stable expansion of a set of premises Σ is Π_2^P-*complete*.

Proof (Idea) for Π_2^P-*hard*:

- Σ has a stable expansion $\iff$ $\exists\Delta$ stable expansion of Σ $\top \in \Delta$
- Σ has a no stable expansion $\iff$ $\forall\Delta$ stable expansion of Σ $\bot \in \Delta$
- Σ has a no stable expansion $\leq_P$ cautious reasoning
- Σ has a no stable expansion in Π_2^P-complete (complement)
- cautious reasoning in Π_2^P-*hard*

Corollary: Consistent Stable Expansion

Deciding whether a set of premises Σ has a consistent stable expansion is Σ_2^P-complete .

Proof (Idea):

We made sure that $\Delta \neq \mathcal{L}_{ae}$

Theorem: Consistent Brave Reasoning

The problem of deciding whether a formula ϕ belongs to at least one consistent stable expansion of a set of premises Σ is Σ_2^P-*complete*.

Bye! Have a good night!

Thank you for your attention!