Exercise 3.1

> Define the syntax and standard semantics for third-order logic. *For simplicity, throughout this assignment you may restrict your attention to a language with no function symbols and function variables, and higher-order quantification restricted to sets, and sets of sets (i.e. unary relations).*

In the lecture simple type theory was given as a vehicle for thinking about higher-order logics. Moreover, in van Benthemthere exists a characterisation of how to "generate" a k-order logic, $\mathcal{L}_k$, by using simply type theory. However, given the restriction of the language (relational and monadic) used throughout this exercise, a less general definition will be used to obtain the required third-order logic.

Definition 1. *Let $\mathbb{T}_n^n$ for be defined inductively*

- $o \in \mathbb{T}_0^0$ *the type boolean;*
- $\iota \in \mathbb{T}_1^1$ *the type individual;*
- $\tau \in \mathbb{T}_n^n$ *then* $\tau := \sigma \to o$ *for* $\sigma \in \mathbb{T}_{n-1}^{n-1}$

Moreover, let

$$\mathbb{T}_m^n := \bigcup_{i=m}^{n} \mathbb{T}_i^i$$

and let

$$\mathbb{T} := \bigcup_{i=m}^{\infty} \mathbb{T}_i^i$$

If $m = 0$ the subscript will be dropped.

Remark This restriction is possible since in the languages considered here, no function symbols and function variables are present, thus implying that for all types of the form $\tau \to \sigma$ it must follow that $\sigma = o$. Moreover, since all predicates and all predicate symbols will be monadic, types of the form $\tau_1 \to \cdots \to \tau_n \to \sigma$ for $\tau_1, \ldots, \tau_n, \sigma \in \mathbb{T}$ and $n > 1$ can be excluded. Notice, $\mathbb{T}$ is a fragment of the relational variant of the simple type theory.

Moreover, this leads to the definition of order.

Definition 2. *Let $\tau \in \mathbb{T}$ be a type, its order $|\tau|$ is defined as $|\tau| = n$ iff $\tau \in \mathbb{T}_n^n$.*

Consider the following observation.

Lemma 0.1. *For every k, $|\{\tau \mid \forall \tau \in \mathbb{T}\ |\tau| = k\}| = 1$.*

Proof. For $k < 3$ this is clearly the case, i.e. o for $k = 0$, ι for $k = 1$ and $\iota \to o$ for $k = 2$. Moreover, by induction, let τ such that $|\tau| = k + 1 > 2$. Then $\tau = \sigma \to o$, by IH, $|\sigma| = k$ and thus unique. Hence, τ is unique. $\square$

This allows to uniquely specify the type by giving its order.

Using this definition of order, it is here where the signature used in most of what follows will be introduced.

Definition 3. *Let $L := \langle CS, FS, PS \rangle$ be a signature. If*

- $\forall c^\tau \in CS \ |\tau| = 1$, *i.e. all constant symbols represent individuals;*
- $FS = \varnothing$ *and*
- $\forall P^\tau \in PS \ |\tau| = 2 \wedge arity(P) = 1$, *i.e. all predicate symbols represent sets of individuals;*

then L is called an e-signature.

Notice that this requires, the signature to be typed.

k-Order Logic

This is simply a more general version of the following sub-section. This definition draws upon the definition of the syntax of L_ω as presented in van Benthem. For a given language L, a L_k-term is defined as follows:

Definition 4. *Let L be an e-signature. Then the set of L_k-terms is defined as*

$$\mathrm{Term}(L_k) = \bigcup_{1 \le i \le k} \mathrm{Term}_i(L_k)$$

where

- $\mathrm{Term}_1(L_k)$ *is defined as:*
  - *constant symbols,* $c^\iota \in \mathrm{Term}_1(L_k)$.
  - *variable symbols,* $x^\iota \in \mathrm{Term}_1(L_k)$*;*
- $\mathrm{Term}_i(L_k)$ *for* $1 \le i \le k$ *is defined as:*
  - *variable symbols* $x^\tau \in \mathrm{Term}_i(L_k)$, *if* $|\tau| = i$*;*

If apparent from the context the type indicator of a symbol will be dropped, i.e. x instead of x^τ for some type τ.

From terms to formulas

Definition 5. *Let L be an e-signature and $k > 0$. Then the set of L_k-formulas, or short $\mathcal{L}_k$, is defined as*

- $\bot \in \mathcal{L}_k$;
- *If* $t \in \mathrm{Term}_1(L_k)$ *and* R *is a relation symbol then* $R(t) \in \mathcal{L}_k$;
- *If* $t \in \mathrm{Term}_i(L_k)$ *and* X^τ *is a variable s.t.* $|\tau| = i+1 \leq k$, *then* $X(t) \in \mathcal{L}_k$;
- *If* $\varphi, \psi \in \mathcal{L}_k$, *then* $\varphi \circ \psi \in \mathcal{L}_k$ *for* $\circ \in \{\wedge, \vee, \rightarrow\}$ *and* $\neg\varphi \in \mathcal{L}_k$;
- *If* $\varphi \in \mathcal{L}_k$ *and for* $\tau \in \mathbb{T}_1^k$, *then* $\forall X^\tau \varphi \in \mathcal{L}_k$ *and* $\exists X^\tau \varphi \in \mathcal{L}_k$.

Moving on to the semantics, as with the syntactic definition a restricted form of the semantics presented in van Benthemwill be given.

Definition 6. *Let L be an e-signature. Let D be a non-empty domain. Given D and $\tau \in \mathbb{T}$ let D_τ be inductively defined as*

- *if* $\tau = o$ *then* $D_\tau := \{true, false\}$;
- *if* $\tau = \iota$ *then* $D_\tau := D$;
- *else* $\tau := \sigma \rightarrow o$ *then* $D_\tau := \wp(D_\sigma)$.

Definition 7. *Let L be an e-signature and let $\varphi \in \mathcal{L}_k$. Let $\mathcal{I} := \langle D, I \rangle$ be a L_k-structure. Let $\mathcal{I}$ interpret φ. The truth value of φ wrt. $\mathcal{I}$ is defined inductively.*

- *for* P *being a predicate symbol and* $t \in \mathrm{Term}_1(L_k)$, $\mathcal{I} \models P(t)$ *if and only if* $I(t) \in I(P)$;
- *for* X^τ *being a variable with* $|\tau| = n > 1$ *and* $t \in \mathrm{Term}_{n-1}(L_k)$, $\mathcal{I} \models X(t)$ *if and only if* $I(t) \in I(X)$ *and* $I(X) \in D_\tau$;
- *for* ψ *being a formula,* $\mathcal{I} \models \forall x^\tau\ \psi$ *if and only if for all* x^τ*-variants* $\mathcal{I}'$, *i.e.* $\mathcal{I}' \in \{\mathcal{I} \cup \{x^\tau \mapsto c\} \mid \forall c \in D_\tau\}$, $\mathcal{I}' \models \psi$;
- *for* ψ *being a formula,* $\mathcal{I} \models \forall x^\tau\ \psi$ *if and only if for some* x^τ*-variants* $\mathcal{I}'$, *i.e.* $\mathcal{I}' \in \{\mathcal{I} \cup \{x^\tau \mapsto c\} \mid \forall c \in D_\tau\}$, $\mathcal{I}' \models \psi$;
- $\wedge$, $\vee$, $\rightarrow$, $\neg$ *are interpreted as usual.*

Third-Order Logic

Now the specific case for third-order logic. Clearly, the set of permissible types is $\mathbb{T}^3 := \{o, \iota, \iota \rightarrow o, (\iota \rightarrow o) \rightarrow o\}$

Definition 8. *Let L be an e-signature. Then the set of L_3-terms is defined as* $\mathrm{Term}(L_3) := \mathrm{Term}_1(L_3) \cup \mathrm{Term}_2(L_3) \cup \mathrm{Term}_3(L_3)$ *such that*

- *constant symbols,* $c^\iota \in \mathrm{Term}_1(L_3)$*;*
- *variable symbols,* $x^\iota \in \mathrm{Term}_1(L_3)$*;*
- *variable symbols,* $X^{\iota\to o} \in \mathrm{Term}_2(L_3)$*;*
- *variable symbols,* $\mathbf{X}^{(\iota\to o)\to o} \in \mathrm{Term}_3(L_3)$*;*

Notice, the stylistic separation based on the type of variable.

For a given e-signature L a L_3-formula is defined as follows:

Definition 9. *Let L be an e-signature. Then the set of $\mathcal{L}_3$-formulas, or short $\mathcal{L}_3$, is defined as*

- $\bot \in \mathcal{L}_k$*;*
- *If* $t \in \mathrm{Term}_1(L_3)$ *and R is a relation symbol then* $R(t) \in \mathcal{L}_3$*;*
- *If* $t \in \mathrm{Term}_1(L_3)$ *and* $X^{\iota\to o}$ *is a variable, then* $X(t) \in \mathcal{L}_3$*;*
- *If* $T \in \mathrm{Term}_2(L_3)$ *and* $\mathbf{X}^{(\iota\to o)\to o}$ *is a variable, then* $\mathbf{X}(T) \in \mathcal{L}_3$*;*
- *If* $\varphi, \psi \in \mathcal{L}_3$*, then* $\varphi \circ \psi \in \mathcal{L}_3$ *for* $\circ \in \{\wedge, \vee, \to\}$ *and* $\neg\varphi \in \mathcal{L}_3$*;*
- *If* $\varphi \in \mathcal{L}_3$*, then*
  - $\forall x^\iota \varphi \in \mathcal{L}_3$ *and* $\exists x^\iota \varphi \in \mathcal{L}_3$*;*
  - $\forall X^{\iota\to o} \varphi \in \mathcal{L}_3$ *and* $\exists X^{\iota\to o} \varphi \in \mathcal{L}_3$*;*
  - $\forall \mathbf{X}^{(\iota\to o)\to o} \varphi \in \mathcal{L}_3$ *and* $\exists \mathbf{X}^{(\iota\to o)\to o} \varphi \in \mathcal{L}_3$*;*

Moving on to the semantics.

Definition 10. *Let L be an e-signature. Let D is a non-empty domain, then let*

- $D_o \coloneqq \{\mathit{true}, \mathit{false}\}$*;*
- $D_\iota \coloneqq D$*;*
- $D_{\iota\to o} \coloneqq \wp(D_\iota)$*;*
- $D_{(\iota\to o)\to o} \coloneqq \wp(D_{\iota\to o})$*.*

Definition 11. *Let L be an e-signature and let $\varphi \in \mathcal{L}_3$. Let $\mathcal{I} \coloneqq \langle D, I\rangle$ be a L_3-structure. Let $\mathcal{I}$ interpret φ. The truth value of φ wrt. $\mathcal{I}$ is defined inductively.*

- *for predicate symbol P and* $t \in \mathrm{Term}_1(L_3)$, $\mathcal{I} \vDash P(t) \iff I(t) \in I(P)$*;*
- *for* $X^{\iota\to o}$ *and* $t \in \mathrm{Term}_1(L_3)$, $\mathcal{I} \vDash X(t) \iff I(t) \in I(X) \subseteq D_\iota$*;*

- *for* $\mathbf{X}^{(\iota\to o)\to o}$ *and* $T \in \mathrm{Term}_2(L_3)$, $\mathcal{I} \models X(t) \iff I(T) \in I(\mathbf{X}) \subseteq D_{\iota\to o}$;
- *for* ψ *being a formula,* $\mathcal{I} \models \forall x^{\iota}\ \psi$ *if and only if for all* x*-variants* $\mathcal{I}'$, $\mathcal{I}' \models \psi$;
- *for* ψ *being a formula,* $\mathcal{I} \models \forall X^{\iota\to o}\ \psi$ *if and only if for all* X*-variants* $\mathcal{I}'$, $\mathcal{I}' \models \psi$;
- *for* ψ *being a formula,* $\mathcal{I} \models \forall \mathbf{X}^{(\iota\to o)\to o}\ \psi$ *if and only if for all* $\mathbf{X}$*-variants* $\mathcal{I}'$, $\mathcal{I}' \models \psi$;
- $\exists x^{\iota}\ \psi$, $\exists X^{\iota\to o}\ \psi$ *and* $\exists \mathbf{X}^{(\iota\to o)\to o}\ \psi$, *analogue as above (replace "all" with "some");*
- $\wedge$, $\vee$, $\to$, $\neg$ *are interpreted as usual.*

Exercise 3.2

> Show how third-order logic can be re-interpreted as first-order logic, by defining a Henkin semantics and restricting attention to the third- order formulae that are valid under those semantics. Deduce that this set of formulae is recursively enumerable.

In a standard model the type structure is build by making the next level to be equal to the power set over previous level. This can be generalised, thereby obtaining the notion of a Henkin-L_k-prestructure .

Definition 12. *Let* L *be an e-signature. Then* $\mathcal{H} := \langle D, I\rangle_{\mathfrak{H}}$ *is a Henkin-*L_3*-prestructure if*

- $I(o) = D_o^{\mathfrak{H}} := \{\text{true}, \text{false}\}$;
- $I(\iota) = D_\iota^{\mathfrak{H}} := D$;
- $I(\iota \to o) = D_{\iota\to o}^{\mathfrak{H}} :\subseteq \wp(D_\iota)$;
- $I((\iota \to o) \to o) = D_{(\iota\to o)\to o}^{\mathfrak{H}} :\subseteq \wp(D_{\iota\to o})$.
- I *interprets the constant symbols in the signature* L*, i.e.*
  - *if* c *is a constant symbol then* $I(c) \in D_\iota^{\mathfrak{H}}$;
  - *if* R *is a relation symbol then* $I(R) \in D_{\iota\to o}^{\mathfrak{H}}$ *(or equally* $I(R) \subseteq D_\iota^{\mathfrak{H}}$*).*

What follows adapts the definition of standard semantics by the generalisations introduced above. Essentially, the only thing that changed is changing the domain at each type level.

Definition 13. *Let L be an e-signature and let $\varphi \in \mathcal{L}_3$. Let $\mathcal{H} := \langle D, I \rangle_{\mathfrak{H}}$ be a Henkin-L_3-prestructure. Let $\mathcal{H}$ interpret φ. The truth value of φ wrt. $\mathcal{I}$ is defined inductively.*

- *for predicate symbol P and $t \in \mathrm{Term}_1(L_3)$, $\mathcal{H} \vDash P(t) \iff I(t) \in I(P)$;*
- *for $X^{\iota \to o}$ and $t \in \mathrm{Term}_1(L_3)$, $\mathcal{H} \vDash X(t) \iff I(t) \in I(X) \subseteq D^{\mathfrak{H}}_{\iota}$;*
- *for $\mathbf{X}^{(\iota \to o) \to o}$ and $T \in \mathrm{Term}_2(L_3)$, $\mathcal{H} \vDash X(t) \iff I(T) \in I(\mathbf{X}) \subseteq D^{\mathfrak{H}}_{\iota \to o}$;*
- *for ψ being a formula, $\mathcal{H} \vDash \forall x^{\iota}\ \psi$ if and only if for all x-variants $\mathcal{H}'$, i.e. $\mathcal{H}' \in \{\mathcal{H} \cup \{x \mapsto c\} \mid \forall c \in D^{\mathfrak{H}}_{\iota}\}$, $\mathcal{H}' \vDash \psi$;*
- *for ψ being a formula, $\mathcal{H} \vDash \forall X^{\iota \to o}\ \psi$ if and only if for all X-variants $\mathcal{H}'$, i.e. $\mathcal{H}' \in \{\mathcal{H} \cup \{X \mapsto S\} \mid \forall S \in D^{\mathfrak{H}}_{\iota \to o}\}$, $\mathcal{H}' \vDash \psi$;*
- *for ψ being a formula, $\mathcal{H} \vDash \forall \mathbf{X}^{(\iota \to o) \to o}\ \psi$ if and only if for all $\mathbf{X}$-variants $\mathcal{H}'$, i.e. $\mathcal{H}' \in \{\mathcal{H} \cup \{\mathbf{X} \mapsto \mathbf{S}\} \mid \forall \mathbf{S} \in D^{\mathfrak{H}}_{(\iota \to o) \to o}\}$, $\mathcal{H}' \vDash \psi$;*
- *for ψ being a formula, $\mathcal{H} \vDash \exists x^{\iota}\ \psi$ if and only if for some x-variants $\mathcal{H}'$, $\mathcal{H}' \vDash \psi$;*
- *for ψ being a formula, $\mathcal{H} \vDash \exists X^{\iota \to o}\ \psi$ if and only if for some X-variants $\mathcal{H}'$, $\mathcal{H}' \vDash \psi$;*
- *for ψ being a formula, $\mathcal{H} \vDash \exists \mathbf{X}^{(\iota \to o) \to o}\ \psi$ if and only if for some $\mathbf{X}$-variants $\mathcal{H}'$, $\mathcal{H}' \vDash \psi$;*
- *$\wedge$, $\vee$, $\to$, $\neg$ are interpreted as usual.*

The notion of a Henkin-prestructure can be strengthened to the notion of a Henkin-structure.

Definition 14. *Let L be an e-signature. Let $\mathcal{H}$ be a Henkin-L_3-prestructure. $\mathcal{H}$ is a Henkin-L_3-structure if it is closed under definability*[1]*, i.e. for each $\varphi \in \mathcal{L}_3$ and $\tau = (\sigma \to o) \in \mathbb{T}^3_1$*

$$\{x \mid \forall x \in D_{\sigma}, \mathcal{H}'\ x^{\sigma}\text{-variant},\ \mathcal{H}' \vDash \varphi\} \in D^{\mathfrak{H}}_{\tau}.$$

Finally, leading to the definition of Henkin-validity.

Definition 15. *A formula $\varphi \in \mathcal{L}_3$ is called Henkin-valid, if it is true in all Henkin structures.*

The first step towards, providing a first order semantics for third-order logic under Henkin-semantics, i.e. showing that Henkin-validity can be reduced to first-order logical consequence, is to define a translation of $\mathcal{L}_3$ to a set of first-order formulas.

[1] alternative characterisation $\forall t^{\tau} \in \mathrm{Term}(L_3)\ I(t) \in D^{\mathfrak{H}}_{\tau}$ for $\tau \in \mathbb{T}^3_1$.

Definition 16. *Let L be an e-signature, then*

$$L_3^{\hookleftarrow} := L \cup \{E_{\iota\to o}/2, E_{(\iota\to o)\to o}/2\} \cup \{T_\iota/1, T_{\iota\to o}/1, T_{(\iota\to o)\to o}/1\}$$

Building on this consider the following syntactic translation.

Definition 17. *Let L be a e-signature and let $\varphi \in \mathcal{L}_3$ then $\cdot^{\hookleftarrow}$ is defined recursively*

- *If $\varphi = X^{\iota\to o}(t)$ then $\varphi^{\hookleftarrow} := E_{\iota\to o}(t, x)$;*
- *If $\varphi = \mathbf{X}^{(\iota\to o)\to o}(t)$ then $\varphi^{\hookleftarrow} := E_{(\iota\to o)\to o}(x, t)$;*
- *If $\varphi = \psi \circ \chi$ then $\varphi^{\hookleftarrow} := \psi^{\hookleftarrow} \circ \chi^{\hookleftarrow}$ for all $\circ \in \{\wedge, \vee, \to\}$;*
- *If $\varphi = \neg\psi$ then $\varphi^{\hookleftarrow} := \neg\psi^{\hookleftarrow}$;*
- *If $\varphi = \forall X^{\iota\to o}\psi$ then $\varphi^{\hookleftarrow} := \forall x\, T_{\iota\to o}(x) \to \psi^{\hookleftarrow}$ where x is fresh;*
- *If $\varphi = \forall \mathbf{X}^{(\iota\to o)\to o}\psi$ then $\varphi^{\hookleftarrow} := \forall x\, T_{(\iota\to o)\to o}(x) \to \psi^{\hookleftarrow}$ where x is fresh;*
- *If $\varphi = \exists X^{\iota\to o}\psi$ then $\varphi^{\hookleftarrow} := \exists x\, T_{\iota\to o}(x) \wedge \psi^{\hookleftarrow}$ where x is fresh;*
- *If $\varphi = \exists \mathbf{X}^{(\iota\to o)\to o}\psi$ then $\varphi^{\hookleftarrow} := \exists x\, T_{(\iota\to o)\to o}(x) \wedge \psi^{\hookleftarrow}$ where x is fresh;*

Moreover, let $\mathcal{L}_3^{\hookleftarrow} := \{\varphi^{\hookleftarrow} \mid \forall\varphi \in \mathcal{L}_3\}$ be the set of formulas over $\mathcal{L}_3^{\hookleftarrow}$.

It is easy to see that $\mathcal{L}_3^{\hookleftarrow} \subseteq \mathcal{L}_1$. Furthermore, this translation has a semantic counterpart.

Definition 18. *Let L be a e-signature and let $\mathcal{H} := \langle D, I\rangle_{\mathfrak{H}}$ be a Henkin-L_3-prestructure. Then one obtains the following standard first-order $L_3^{\hookleftarrow}$-structure $\mathcal{H}^{\hookleftarrow} = \langle D^{\hookleftarrow}, I^{\hookleftarrow}\rangle$.*

- *$D^{\hookleftarrow} := D_\iota^{\mathfrak{H}} \cup D_{\iota\to o}^{\mathfrak{H}} \cup D_{(\iota\to o)\to o}^{\mathfrak{H}}$;*
- *$I^{\hookleftarrow}(T_\tau) := D_\tau^{\mathfrak{H}}$ for $\tau \in \mathbb{T}_1^3$;*
- *$I^{\hookleftarrow}(E_\tau) :=\in$ for $\tau \in \mathbb{T}_2^3$, i.e. for $\tau := \sigma \to o$, $I^{\hookleftarrow}(E_\tau)(x, y)$ if and only if $x \in D_\sigma^{\mathfrak{H}}$, $y \in D_\tau^{\mathfrak{H}}$ and $x \in y$;*
- *the interpretation of L_3-symbols is the same as in $\mathcal{H}$.*

Now consider the following lemma

Lemma 0.2. *Let $\mathcal{H}$ be a Henkin-L_3-prestructure and $\tau \in \mathbb{T}_1^3$, then $\mathcal{H}'$, x^τ-variants of $\mathcal{H}$ if and only if $\mathcal{H}'^{\hookleftarrow}$, x-variants of $\mathcal{H}^{\hookleftarrow}$ if restricted to $I^{\hookleftarrow}(x) \in I^{\hookleftarrow}(T_\tau)$. And where x is the variable replacing x^τ after the transformation.*

Proof. Since $\mathcal{H}' = \mathcal{H} \cup \{x^\tau \mapsto m\}$ for some $m \in D_\tau^{\mathfrak{H}}$, thus $(\mathcal{H}')^{\hookleftarrow} = \mathcal{H}^{\hookleftarrow} \cup \{x \mapsto m\}$. Hence, $(\mathcal{H}')^{\hookleftarrow}$ differs from $\mathcal{H}^{\hookleftarrow}$ by only x and where $m \in D_\tau^{\mathfrak{H}}$ implies that $I(x) \in I^{\hookleftarrow}(T_\tau)$. Similar in the other direction. □

Lemma 0.3. *Let $\mathcal{H}$ be a Henkin-L_3-prestructure and $\varphi \in \mathcal{L}_3$. Then $\mathcal{H} \vDash \varphi$ if and only if $\mathcal{H}^{\hookleftarrow} \vDash \varphi^{\hookleftarrow}$.*

Proof. This will be done by induction on the structure on φ.

Induction Basis:

- $\varphi = P(t)$ for P predicate symbol. Trivial.
- $\mathcal{H} \vDash X^{\iota\to o}(t)$ for some term $t \in \mathrm{Term}_1(L_3)$. Meaning that $I(t) \in I(X)$, $I(X) \in D^{\mathfrak{H}}_{\iota\to o}$ and that $I(t) \in D^{\mathfrak{H}}_{\iota}$. Hence, by construction of $I^{\hookleftarrow}$, this is equivalent to $I^{\hookleftarrow}(E)(I(t), I(X))$, since $I(t) = I^{\hookleftarrow}(t)$ and $I(X) = I^{\hookleftarrow}(x)$, it this is equivalent to $\mathcal{H}^{\hookleftarrow} \vDash E(t, x)$.
- $\mathcal{H} \vDash \mathbf{X}^{(\iota\to o)\to o}(T)$ for some term $T \in \mathrm{Term}_2(L_3)$. Analogue to above.

Thus all atoms are covered.

Induction Step:

- $\varphi = \psi \circ \chi$ for $\circ \in \{\wedge, \vee, \to\}$. Follows by semantics as usual.
- $\varphi = \neg\psi$. Follows by semantics as usual.
- $\varphi = \forall x\, \psi$. Hence, $\mathcal{H}' \vDash \psi$ for all x^{ι}-variants of $\mathcal{H}$. Moreover, $(\forall x\, \psi)^{\hookleftarrow} = \forall x\,(T_{\iota}(x) \to \psi^{\hookleftarrow})$. Take an arbitrary x-variant $(\mathcal{H}^{\hookleftarrow})'$ of $\mathcal{H}^{\hookleftarrow}$, such that x is mapped to m. If $m \notin D^{\mathfrak{H}}_{\iota}$, then $(\mathcal{H}^{\hookleftarrow})' \nvDash T_{\iota}(x)$ and therefore $(\mathcal{H}^{\hookleftarrow})' \vDash T_{\iota}(x) \to \psi^{\hookleftarrow}$. Otherwise, $(\mathcal{H}^{\hookleftarrow})'$ is a restricted x-variant. Thus there exists an x^{ι}-variant $\mathcal{H}'$ of $\mathcal{H}$ such that $(\mathcal{H}^{\hookleftarrow})' = (\mathcal{H}')^{\hookleftarrow}$ and then by IH $(\mathcal{H}^{\hookleftarrow})' \vDash \psi^{\hookleftarrow}$. Hence, $T_{\iota}(x) \to \psi^{\hookleftarrow}$ for all x-variants of $\mathcal{H}^{\hookleftarrow}$, thus resulting in $\mathcal{H}^{\hookleftarrow} \vDash \forall x\, \psi^{\hookleftarrow}$.

  In the other direction, $(\mathcal{H}^{\hookleftarrow})' \vDash \forall x(T_{\iota}(x) \to \psi^{\hookleftarrow})$ for all x-variants of $\mathcal{H}^{\hookleftarrow}$. Hence, for all variants where x maps to an element $m \in D^{\mathfrak{H}}_{\iota}$, $(\mathcal{H}^{\hookleftarrow})' \vDash \psi^{\hookleftarrow}$. Since, for each x^{ι}-variant of $\mathcal{H}$ there exists such an restricted variant and by IH it follows that, for all x^{ι}-variants $\mathcal{H}'$ of $\mathcal{H}$, $\mathcal{H}' \vDash \psi$. Therefore, resulting in $\mathcal{H} \vDash \forall x\, \psi$.
- $\varphi = \forall X^{\iota\to o}\, \psi$ and $\varphi = \forall \mathbf{X}\, \psi$ analogue to above.
- $\varphi = \exists x^{\iota}\, \psi$. Hence, $\mathcal{H}' \vDash \psi$ for some x^{ι}-variants of $\mathcal{H}$, let x be mapped to m. By Lemma **??**, there exists an x-variant $(\mathcal{H}^{\hookleftarrow})'$ of $\mathcal{H}^{\hookleftarrow}$ such that $(\mathcal{H}')^{\hookleftarrow} = (\mathcal{H}^{\hookleftarrow})'$. By IH $(\mathcal{H}')^{\hookleftarrow} \vDash \psi^{\hookleftarrow}$ and since $m \in D^{\mathfrak{H}}_{\iota}$ by construction $(\mathcal{H}^{\hookleftarrow})'$ is a restricted variant such that $m \in I^{(\mathcal{H}^{\hookleftarrow})'}(T_{\iota})$. Therefore, $(\mathcal{H}^{\hookleftarrow})' \vDash T_{\iota}(x) \wedge \psi^{\hookleftarrow}$. Resulting in $\mathcal{H}^{\hookleftarrow} \vDash \exists x T_{\iota}(x) \wedge \psi^{\hookleftarrow}$.

  Starting from $(\mathcal{H}^{\hookleftarrow})' \vDash T_{\iota}(x) \wedge \psi^{\hookleftarrow}$ for some x-variants of $\mathcal{H}^{\hookleftarrow}$. Clearly, this variant must be restricted to those values in $I^{(\mathcal{H}^{\hookleftarrow})'}(T_{\iota})$. Hence, by Lemma **??**, there exists an x^{ι}-variant $(\mathcal{H}')^{\hookleftarrow}$ of $\mathcal{H}$ such that $(\mathcal{H}')^{\hookleftarrow} = (\mathcal{H}^{\hookleftarrow})'$. Hence, resulting in $\mathcal{H} \vDash \exists x^{\iota}\, \psi$.
- $\varphi = \exists X\, \psi$ and $\varphi = \exists \mathbf{X}\, \psi$, analogue to above.

$\square$

Having the next step is to force arbitrary standard structures to behave as the translated Henkin-prestructure. This can be done syntactically.

Definition 19. *For the signature $L_3^{\hookleftarrow}$ with the underlying e-signature L, let Γ contain*

1. *Non-emptiness:* $\exists x\, T_\iota(x)$;
2. *L_3-correctness:*
   (a) $T_\iota(c)$ *for all constant symbols c in L;*
   (b) $\forall x(P(x) \to T_\iota(x))$ *for all predicate symbols P in L;*
3. *disjointness:* $\forall \tau, \sigma \in \mathbb{T}_1^3$ *s.t.* $\tau \neq \sigma$, $\forall x(T_\tau(x) \to \neg T_\sigma(x))$;
4. *inclusion:* $\forall x\, (T_\iota(x) \vee T_{\iota\to o}(x) \vee T_{(\iota\to o)\to o}(x))$;
5. *elementhood:*
   (a) $\forall x \forall y E_{\iota\to o}(y,x) \to T_{\iota\to o}(x) \wedge T_\iota(y)$ *and*
   (b) $\forall x \forall y E_{(\iota\to o)\to o}(y,x) \to T_{(\iota\to o)\to o}(x) \wedge T_{\iota\to o}(y)$;
6. *extensionality:*

$$\forall x \forall y \big(T_{\iota\to o}(x) \wedge T_{\iota\to o}(y) \wedge \forall z(E_{\iota\to o}(z,x) \leftrightarrow E_{\iota\to o}(z,y)) \to x = y\big)$$

   and

$$\forall x \forall y \big(T_{(\iota\to o)\to o}(x) \wedge T_{(\iota\to o)\to o}(y) \wedge \forall z(E_{(\iota\to o)\to o}(z,x) \leftrightarrow E_{(\iota\to o)\to o}(z,y)) \to x = y\big).$$

Next, one has to check if those formulas are actually valid in every Henkin-prestructure.

Lemma 0.4. *For any Henkin-L_3-prestructure $\mathcal{H}$ it holds that $\mathcal{H}^{\hookleftarrow} \vDash \Gamma$.*

Proof. Firstly, Γ is clearly consistent. Now, consider an arbitrary Henkin-L_3-prestructure $\mathcal{H}$. Since $D_\iota^{\mathfrak{H}}$ is not empty, $I^{\hookleftarrow}(T_\iota)$ is not empty as well, thus (1) is satisfied. Moreover, by definition (2) is also satisfied. Since all $D_\tau^{\mathfrak{H}}$ for $\tau \in \mathbb{T}_1^3$ are disjoint, (3) follows from the translation or the $D_\tau^{\mathfrak{H}}$ to $I^{\hookleftarrow}(T_\tau)$. Similarly, (4) follows from construction of the domain and the respective interpretations of the T's. (5) follows again directly from the type hierarchy and the definition of the respective E's. Lastly, from set theory it is known that two sets are equal if they have the same members. Hence, by construction of the interpretations of the E's and T's under $\mathcal{H}^{\hookleftarrow}$, (6) is satisfied. □

Now consider the following extension of Γ.

Definition 20. *For the signature $L_3^{\hookleftarrow}$, let Γ_T be defined by adding all comprehension axioms of the form*

$$\left(\exists Y^{\iota\to o}\forall x^{\iota}(Y(x)\leftrightarrow\varphi)\right)^{\hookleftarrow}$$

and

$$\left(\exists \mathbf{Y}^{(\iota\to o)\to o}\forall X^{\iota\to o}(\mathbf{Y}(X)\leftrightarrow\varphi)\right)^{\hookleftarrow}$$

where $\varphi\in\mathcal{L}_3$ such that Y not free.

Those axioms are satisfied not by all Henkin-prestructures but only Henkin-structures.

Lemma 0.5. *For any Henkin-L_3-structure $\mathcal{H}$ it holds that $\mathcal{H}^{\hookleftarrow}\vDash\Gamma_T$.*

Proof. Firstly, Γ is satisfied since $\mathcal{H}$ is a special case of a Henkin-L_3-prestructure. Consider $\varphi := \left(\exists Y^{\tau}\forall x^{\sigma}(Y(x)\leftrightarrow\psi)\right)^{\hookleftarrow}$, where $\psi\in\mathcal{L}_3$, $\tau := (\sigma\to o)\in\mathbb{T}_1^3$ and such that Y not free. By Lemma **??** it suffices to show that $\mathcal{H}\vDash \exists Y^{\tau}\forall x^{\sigma}(Y(x)\leftrightarrow\psi)$. Let $I(Y) := \{x \mid \forall x\in D_\sigma, \mathcal{H}'\ x^{\sigma}\text{-}variant,\ \mathcal{H}'\vDash\psi\}\in D_\tau^{\mathfrak{H}}$, the existence of which is guaranteed by closure wrt. definability. $\square$

Additionally, the following lemma is needed.

Lemma 0.6. *For any Henkin-L_3-prestructure $\mathcal{H}$, if $\mathcal{H}\vDash\Gamma_T\smallsetminus\Gamma$ then $\mathcal{H}$ is a Henkin-L_3-structure.*

Proof. Suppose $\mathcal{H}$ satisfies all comprehension axioms. Consider $\varphi := \exists Y^{\tau}\forall x^{\sigma}(Y(x)\leftrightarrow\psi)^{\hookleftarrow}$, where $\psi\in\mathcal{L}_3$, $\tau := (\sigma\to o)\in\mathbb{T}_1^3$ and such that Y not free. Hence, for an arbitrary ψ there must be $\{x \mid \forall x\in D_\sigma, \mathcal{H}'\ x^{\sigma}\text{-}variant,\ \mathcal{H}'\vDash\psi\}\in D_\tau^{\mathfrak{H}}$. Otherwise, there would not be an element in $D_\tau^{\mathfrak{H}}$ that satisfies the requirements of demanded by syntactic claim of the existence of Y. From this it follows that $\mathcal{H}$ is closed wrt. definability. $\square$

Finally, allowing the first correspondence between third-order logic and first-order logic.

Lemma 0.7. *Let $\mathcal{I}$ be an $L_3^{\hookleftarrow}$-structure, then $\mathcal{I}\vDash\Gamma$ if and only if there exists a Henkin-L_3-prestructure $\mathcal{H}$ such that $\mathcal{H}^{\hookleftarrow}\cong\mathcal{I}$.*

Proof. Since, by Lemma **??** it is known that $\mathcal{H}^{\hookleftarrow}\vDash\Gamma^{\hookleftarrow}$ and from the assumption that $\mathcal{H}^{\hookleftarrow}\cong\mathcal{I}$ it follows that $\mathcal{I}\vDash\Gamma$ (see Homework 1).

As for the other direction. First the construction of $\mathcal{H}$. From $\mathcal{I}$ the following domains can be extracted:

- Let $D_\iota^{\mathfrak{H}} := I^{\mathcal{I}}(T_\iota)$, which is possible since (1) requires $I^{\mathcal{I}}(T_\iota)\neq\varnothing$.
- Let $D_{\iota\to o}^{\mathfrak{H}} := \{\mathcal{S}(y)\mid\forall y\in I^{\mathcal{I}}(T_{\iota\to o})\}$, where $\mathcal{S}(y) := \{x\mid\forall x\ I^{\mathcal{I}}(E_{\iota\to o})(x,y)\}$.

- Let $D^{\mathfrak{H}}_{(\iota\to o)\to o} := \{\mathcal{T}(z) \mid \forall z \in I^{\mathcal{I}}(T_{(\iota\to o)\to o})\}$, where $\mathcal{T}(z) := \{\mathcal{S}(y) \mid \forall y\, I^{\mathcal{I}}(E_{(\iota\to o)\to o})(y,z)\}$.

Now due to (2) one can simply re-use the interpretation of the constant and predicate symbols from $\mathcal{I}$ in $\mathcal{H}$. Now with an appropriate type hierarchy, i.e. $D^{\mathfrak{H}}_{\iota} = D^{\mathcal{H}} \neq \emptyset$, $D^{\mathfrak{H}}_{\iota\to o} \subseteq \wp(D^{\mathcal{H}})$ as well as $D^{\mathfrak{H}}_{(\iota\to o)\to o} \subseteq \wp^2(D^{\mathcal{H}}) := \wp(\wp(D^{\mathcal{H}}))$, and a complete assignment of predicate and constant symbols, one can conclude that $\mathcal{H}$ is a Henkin-L_3-prestructure.

What remains to show is the isomorphism. Normally, this can be constructed by induction. However, with only three levels, constructing the isomorphism $\pi : D^{\mathcal{I}} \to D^{\mathcal{H}^{\hookleftarrow}}$ can be done by hand.

$$\pi(x) := \begin{cases} x & x \in I^{\mathcal{I}}(T_{\iota}) \\ \mathcal{S}(x) & x \in I^{\mathcal{I}}(T_{\iota\to o}) \\ \mathcal{T}(x) & x \in I^{\mathcal{I}}(T_{(\iota\to o)\to o}) \end{cases}$$

Firstly, observe that through (4) π covers the whole domain, and by (3) every element is mapped deterministically. Hence, π is a well-defined function. By definition of $\mathcal{H}^{\hookleftarrow}$ and given the construction of $\mathcal{H}$, it is known that

$$D^{\mathcal{H}^{\hookleftarrow}} = I^{\mathcal{I}}(T_{\iota}) \cup \{\mathcal{S}(y) \mid \forall y \in I^{\mathcal{I}}(T_{\iota\to o})\} \cup \{\mathcal{T}(z) \mid \forall z \in I^{\mathcal{I}}(T_{(\iota\to o)\to o})\}$$

and since π is well defined, one obtains surjectivity. Moreover, if there would be two $x, y \in D^{\mathcal{I}}$ mapping to the same element $z \in D^{\mathcal{H}^{\hookleftarrow}}$. Then this would mean that they are in the same $I^{\mathcal{I}}(T_{\tau})$ and that they always agree on $I^{\mathcal{I}}(E_{\tau})$. However, by (6) this implies that $x = y$, thus π is injective. Hence, π is bijective. Now, by (2) constant and predicate symbols assignment of the original L is preserved. By construction, if $x \in I^{\mathcal{I}}(T_{\tau})$ holds then $h(x) \in D^{\mathfrak{H}}_{\tau}$ and thus, by $\cdot^{\hookleftarrow}$, $h(x) \in I^{\mathcal{H}^{\hookleftarrow}}(T_{\tau})$. By (5) the appropriate "type structure" is enforced such that by construction, one can conclude that $I^{\mathcal{I}}(E_{\tau})(x,y)$ then $I^{\mathcal{H}^{\hookleftarrow}}(E_{\tau})(h(x), h(y))$. Thereby establishing that π is isomorphic. □

This can be strengthened to Henkin-structures, leading to the following lemma

Lemma 0.8. *Let $\mathcal{I}$ be a $L_3^{\hookleftarrow}$-structure. Then $\mathcal{I} \models \Gamma_T$ iff there is a Henkin-L_3-structure $\mathcal{H}$ such that $\mathcal{H}^{\hookleftarrow} \cong \mathcal{I}$.*

Proof. If $\mathcal{H}^{\hookleftarrow} \cong \mathcal{I}$ for some Henkin-L_3-structure $\mathcal{H}$, then by isomorphism and by Lemma **??**, $\mathcal{I} \models \Gamma_T$.
It is known that if $\mathcal{I} \models \Gamma$ then $\mathcal{I} \cong \mathcal{H}^{\hookleftarrow}$ for $\mathcal{H}$ being a Henkin-L_3-prestructure. Additionally, assume that $\mathcal{I} \models \Gamma_T$ since isomorphism preserves truth, it therefore follows that $\mathcal{H}^{\hookleftarrow}$ satisfies all comprehension axioms. Therefore, one can conclude, by Lemma **??** and Lemma **??**, that $\mathcal{H}$ is a Henkin-L_3-structure. □

Which then culminates in

Lemma 0.9. *The set of Henkin-valid formulas is equal to the set of logical first-order consequences from* Γ_T. *That is, let* $\mathbb{H}$ *be the set of all Henkin-*L_3*-structures and likewise let* $\mathbb{I}$ *be the set of all* $L_3^{\hookleftarrow}$*-structures* $\mathcal{I}$ *such that* $\mathcal{I} \vDash \Gamma_T$. *For every* $\varphi \in \mathcal{L}_3$

$$\forall \mathcal{H} \in \mathbb{H}\, \mathcal{H} \vDash \varphi \iff \forall \mathcal{I} \in \mathbb{I}\, \mathcal{I} \vDash \varphi^{\hookleftarrow}$$

Proof. By Lemma **??** one knows $\mathcal{H} \vDash \varphi$ for all $\mathcal{H} \in \mathbb{H}$, if and only if $\mathcal{H}^{\hookleftarrow} \vDash \varphi^{\hookleftarrow}$ for all $\mathcal{H} \in \mathbb{H}$.
By Lemma **??** one knows $\mathcal{H}^{\hookleftarrow} \vDash \varphi^{\hookleftarrow}$ for all $\mathcal{H} \in \mathbb{H}$, if and only if $\mathcal{I} \vDash \varphi^{\hookleftarrow}$ for all $\mathcal{I} \in \mathbb{I}$ such that $\mathcal{I} \vDash \Gamma_T$. □

Thereby, reducing Henkin-validity to first-order consequence.

The last remaining thing is to argue why the set of Henkin-valid formulas is recursively enumerable.

Lemma 0.10. *The set of Henkin-valid formulas is is recursively enumerable.*

Proof. By Lemma **??**, one knows that that this is the same as the set of formulas $Cl(\Gamma_T) := \{\varphi \mid \forall \varphi \in \mathcal{L}_3\ \Gamma_T \vDash \varphi\}$. If Γ_T were to be recursive then $Cl(\Gamma_T)$ would be recursively enumerable. Clearly, Γ being finite, is recursive. Consider $\varphi := \left(\exists Y^{\iota\to o} \forall x^{\iota}(Y(x) \leftrightarrow \psi)\right)^{\hookleftarrow}$. Clearly, checking if a third-order formula is well formed, i.e. checking if $\psi \in \mathcal{L}_3$, is decidable. Moreover, constructing φ from ψ is computable, and $\cdot^{\hookleftarrow}$ is also computable. Hence, there exists an algorithm that decides whether φ is in the set $\Gamma_T \setminus \Gamma$ or whether it is not. Therefore, one can conclude that Γ_T is recursive. Thereby, it follows that $Cl(\Gamma_T)$ is axiomatisable, and thus recursively enumerable. □

Exercise 3.3

> Finally, show how third-order logic can be reduced to second-order logic, by effectively translating third-order formulae that are valid in the standard semantics to second-order formulae that are valid in the standard semantics.

Consider the following extension of Γ.

Definition 21. *For the signature* $L_3^{\hookleftarrow}$, *let* Γ_R *be defined over the extended signature* $L_3^{2\hookleftarrow}$ *by adding the representability axioms*

$$\forall R^{\iota\to o} \exists x\big(T_{\iota\to o}(x) \land \forall y(T_{\iota}(y) \to (R(y) \leftrightarrow E_{\iota\to o}(y,x))\big)$$

and

$$\forall R^{\iota\to o}\exists x\big(T_{(\iota\to o)\to o}(x) \land \forall y(T_{\iota\to o}(y) \to (R(y) \leftrightarrow E_{(\iota\to o)\to o}(y,x))\big)$$

Moreover, let $\mathcal{L}_3^{2\hookleftarrow}$ be the first-order language extended to a second-order one by allowing for monadic second order variables only.

Clearly, both representability axioms are part of $\mathcal{L}_3^{2\hookleftarrow}$.

Lemma 0.11. *For every full Henkin-prestructure $\mathcal{H}$, $\mathcal{H}^{\hookleftarrow} \vDash \Gamma_R$*

Proof. Let $\mathcal{H}$ be full Henkin-L_3-prestructure. By Lemma **??**, $\mathcal{H}^{\hookleftarrow} \vDash \Gamma$. Since $\mathcal{H}$ full, $D^{\mathfrak{H}}_{\iota\to o} = \wp(D)$ and $D^{\mathfrak{H}}_{(\iota\to o)\to o} = \wp^2(D)$. That is, all possible sets and sets of sets over $D^{\mathcal{H}}$ are in the domains of the type hierarchy. Now given the translation, those sets are translated into elements in $D^{\mathcal{H}^{\hookleftarrow}}$. That is, let $\tau = (\sigma \to o) \in \mathbb{T}^3_1$. Take an arbitrary subset of $X \subseteq D^{\mathcal{H}^{\hookleftarrow}}$. Let, $X_\tau := I^{\hookleftarrow}(T_\sigma) \cap X$ clearly, since $I^{\hookleftarrow}(T_\sigma) = D^{\mathfrak{H}}_\sigma = D_\sigma$, $X_\tau \in D_\tau$. By $\mathcal{H}$ being full, this is the same as $X_\tau \in D^{\mathfrak{H}}_\tau$ and therefore there must be a corresponding $x \in D^{\mathcal{H}^{\hookleftarrow}}$ such that $x \in I^{\hookleftarrow}(T_\tau)$ and therefore by construction if follows that for all $y \in D^{\mathcal{H}^{\hookleftarrow}}$ such that $y \in I^{\hookleftarrow}(T_\sigma)$ then $I^{\hookleftarrow}(E_\tau)(y,x)$ if and only if $y \in X_\tau$. □

Notice that a standard interpretation is the same as a full Henkin-prestructure.

Lemma 0.12. *Let $\mathcal{I}$ be an $L_3^{2\hookleftarrow}$-structure, then $\mathcal{I} \vDash \Gamma_R$ if and only if there exists a full Henkin-L_3-prestructure $\mathcal{H}$ such that $\mathcal{H}^{\hookleftarrow} \cong \mathcal{I}$.*

Proof. Again, if there exists a full Henkin-L_3-prestructure $\mathcal{H}$ such that $\mathcal{H}^{\hookleftarrow} \cong \mathcal{I}$. By Lemma **??**, it is known that $\mathcal{H}^{\hookleftarrow} \vDash \Gamma_R$ and since $\mathcal{H}^{\hookleftarrow} \cong \mathcal{I}$, one can conclude that $\mathcal{I} \vDash \Gamma_R$.

On the other hand, if $\mathcal{I} \vDash \Gamma_R$, then $\mathcal{I} \vDash \Gamma$. Moreover, one can safely restrict $\mathcal{I}$ to a $L_3^{\hookleftarrow}$-structure, as those languages are essentially the same, i.e. same constant, function and predicate symbols. By Lemma **??**, it thus follows that there exists a Henkin-L_3-prestructure $\mathcal{H}$ such that $\mathcal{I} \cong \mathcal{H}^{\hookleftarrow}$. Now the representability axioms force that for any $\tau = (\sigma \to o) \in \mathbb{T}^3_1$, and any arbitrary subset $X \subseteq I^{\mathcal{I}}(T_\sigma)$, there must be an $x \in D^{\mathcal{I}} \cap I^{\mathcal{I}}(T_\tau)$ such that $\forall y \in D^{\mathcal{I}} \cap I^{\mathcal{I}}(T_\sigma)$ it must be that $E_\tau(y,x)$ iff $y \in X$. That is, for any subset of $I^{\mathcal{I}}(T_\sigma)$ there must be an element in $I^{\mathcal{I}}(T_\tau)$ that represent this set. Hence, through the isomorphism the same must hold for $\mathcal{H}^{\hookleftarrow}$ and by the definition of $\cdot^{\hookleftarrow}$ it therefore follows that $\mathcal{H}$ is full and therefore a standard model, i.e. since each set in the type hierarchy has a representative in $\mathcal{H}^{\hookleftarrow}$ the set must be present in $\mathcal{H}$. □

Finally, one obtains the following

Lemma 0.13. *Let $\mathbb{H}$ be the set of all full Henkin-L_3-prestructures (i.e. all standard structures) and likewise let $\mathbb{I}$ be the set of all $L_3^{2\hookleftarrow}$-structures $\mathcal{I}$ such that $\mathcal{I} \vDash \Gamma_R$. For every $\varphi \in \mathcal{L}_3$*

$$\forall \mathcal{H} \in \mathbb{H}\ \mathcal{H} \vDash \varphi \iff \forall \mathcal{I} \in \mathbb{I}\ \mathcal{I} \vDash \varphi^{\hookleftarrow}$$

Proof. By Lemma **??** one knows $\mathcal{H} \vDash \varphi$ for all $\mathcal{H} \in \mathbb{H}$, if and only if $\mathcal{H}^{\hookleftarrow} \vDash \varphi^{\hookleftarrow}$ for all $\mathcal{H} \in \mathbb{H}$.
By Lemma **??** one knows $\mathcal{H}^{\hookleftarrow} \vDash \varphi^{\hookleftarrow}$ for all $\mathcal{H} \in \mathbb{H}$, if and only if $\mathcal{I} \vDash \varphi^{\hookleftarrow}$ for all $\mathcal{I} \in \mathbb{I}$ such that $\mathcal{I} \vDash \Gamma_R$. $\square$