Recall from the lecture the following variants of the **SAT**-problem:

- **LEX-MAXIMAL MODEL SAT**
- **WEIGHT-MAXIMAL MODEL SAT**
- **LogLEX-MAXIMAL MODEL SAT**
- **CARD-MAXIMAL MODEL SAT**
- **CARD-MINIMAL MODEL SAT**

And also recall the following problem reductions.

Reduction 1. From **LEX-MAXIMAL MODEL SAT** to **WEIGHT-MAXIMAL MODEL SAT**:

Consider an arbitrary instance $\varphi; (x_1, \ldots, x_n)$ of **LEX-MAXIMAL MODEL SAT**, where φ is a Boolean formula over variables X and $(x_1, \ldots, x_n)$ is an ordering of the variables in X.

We define the instance $\varphi; (x_1, \ldots, x_n); (w(x_1), \ldots, w(x_n)); z$ of **WEIGHT-MAXIMAL MODEL SAT** as follows:

- Formula φ (and, hence, also variable set X) is left unchanged.
- For every $i \in \{1, \ldots, n\}$, we define the weight $w(x_i) = 2^{n-i}$.
- We set $z = x_n$.

Reduction 2. From **LogLEX-MAXIMAL MODEL SAT** to **CARD-MAXIMAL MODEL SAT**:

Consider an arbitrary instance $\varphi; (x_1, \ldots, x_n)$ of **LogLEX-MAXIMAL MODEL SAT**, where φ is a Boolean formula and $(x_1, \ldots, x_n)$ is an ordering of logarithmically many variables in φ; moreover, let $Y = \{y_1, \ldots, y_m\}$ denote the remaining variables in φ.

We add the following fresh variables:

- "copies" of each variable x_i, i.e. for every $i \in \{1, \ldots, n\}$, we introduce $2^{n-i} - 1$ new variables $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(r_i)}$ with $r_i = 2^{n-i} - 1$.
- a primed copy of each variable in Y: $Y' = \{y'_1, \ldots, y'_m\}$.

Then we construct the instance $\psi; z$ of **CARD-MAXIMAL MODEL SAT** as follows: We set $z = x_n$ and we define ψ as

$$\psi = \varphi \wedge \bigwedge_{i=1}^{n} \left((x_i \leftrightarrow x_i^{(1)}) \wedge \cdots \wedge (x_i \leftrightarrow x_i^{(r_i)}) \right) \wedge \bigwedge_{i=1}^{m} y_i \leftrightarrow \neg y'_i$$

Reduction 3. From **CARD-MAXIMAL MODEL SAT** to **CARD-MINIMAL MODEL SAT**:

Consider an arbitrary instance $\varphi; x_i$ of **CARD-MAXIMAL MODEL SAT** where φ is a Boolean formula over the variables $X = \{x_1, \ldots, x_n\}$.

We add primed and double-primed copies of the variables, i.e. $X' = \{x'_1, \ldots, x'_n\}$ and $X'' = \{x''_1, \ldots, x''_n\}$.

Then we construct the instance $\psi; x_i$ of **CARD-MINIMAL MODEL SAT** as follows:

$$\psi = \varphi \wedge \bigwedge_{i=1}^{n} \big((x_i \leftrightarrow \neg x_i') \wedge (x_i \leftrightarrow \neg x_i'')\big)$$

Exercise 1 (4 credits)**.** Prove the correctness of the above reduction from **LEX-MAXIMAL MODEL SAT** to **WEIGHT-MAXIMAL MODEL SAT**.

Solution In a small abuse of notation let $w(x_1, \ldots, x_n) := (w(x_1), \ldots, w(x_n))$. Moreover, consider the following

Definition 1. *Let X be some set (or ordering) of variables, let $\mathcal{I}$ be an interpretation and let w be the weight function as introduced in Reduction 1 over the variables in X, then let $w^{\mathcal{I}}$ be defined as*

$$w^{\mathcal{I}}(x) := \begin{cases} w(x) & \text{if } \mathcal{I} \models x \\ 0 & \text{otw.} \end{cases}$$

Moreover, for some $X' \subseteq X$ let $\omega^{\mathcal{I}}(X') := \sum_{x \in X'} w^{\mathcal{I}}(x)$.

Lemma 2. *Let φ be a propositional formula over the set of variables X , let $\overline{x} := (x_1, \ldots, x_n)$ be a linear ordering of X. Then*

$(\varphi; \overline{x}) \in$ **LEX-MAXIMAL MODEL SAT** $\iff$ $\tau(\varphi; \overline{x}) \in$ **WEIGHT-MAXIMAL MODEL SAT**

where $\tau(\varphi; \overline{x}) = (\varphi; w(\overline{x}); x_n)$, i.e. the transformation as presented in Reduction 1.

Proof. Let $I := \{1, \ldots, n\}$. Each direction is shown separately.

- $\Rightarrow$ Assume that $(\varphi; \overline{x}) \in$ **LEX-MAXIMAL MODEL SAT**. Hence, there exists a lexicographical maximal model $\mathcal{I}$ of φ such that $\mathcal{I} \models x_n$. Hence, for any other model $\mathcal{J}$ of φ there must exist a $k \in I$ such that $\forall i \in \{1, \ldots, k-1\}$, $\mathcal{I}(x_i) = \mathcal{J}(x_i)$ and $\mathcal{I} \models x_k$ and $\mathcal{J} \not\models x_k$. If $\mathcal{J}$ does not exists, $\mathcal{I}$ is trivially weight maximal. Otherwise, it must be that

  $$\omega^{\mathcal{I}}(x_k, \ldots, x_n) \geq w^{\mathcal{I}}(x_k) = 2^{n-k} > \sum_{k+1}^{n} 2^{n-i} \geq \omega^{\mathcal{J}}(x_k, \ldots, x_n)$$

  That is, even if $\forall i \in \{k+1, \ldots, n\}$ $\mathcal{J} \models x_i$ one would have at most $\omega^{\mathcal{J}}(x_k, \ldots, x_n) = \sum_{k+1}^{n} 2^{n-i}$ and since both interpretations agree on all x_i smaller k it must be that$\omega^{\mathcal{I}}(\overline{x}) > \omega^{\mathcal{J}}(\overline{x})$. Therefore, from $\mathcal{J}$ being arbitrary, it follows that $\mathcal{I}$ is weight maximal. Furthermore, since $\mathcal{I} \models x_n$ one can conclude that x_n is true in a weight maximal model (w.r.t. $w(\overline{x})$). Hence, $\tau(\varphi; \overline{x}) \in$ **WEIGHT-MAXIMAL MODEL SAT**.

- $\Leftarrow$ Assume that $\tau(\varphi; \overline{x}) \in$ **WEIGHT-MAXIMAL MODEL SAT**. Hence, there exists a weight maximal model $\mathcal{I}$ of φ such that $\mathcal{I} \models x_n$. Let $\mathcal{J}$ be an arbitrary model of φ other than $\mathcal{I}$. If $\mathcal{J}$ does not exists, $\mathcal{I}$ is trivially lexicographically maximal. Consider the order imposed by the vector $\overline{x}$. Since $\mathcal{J}$ differs from $\mathcal{I}$ it must be that there exists a $k \in I$ where $\mathcal{I}(x_k) \neq \mathcal{J}(x_k)$ and where $\forall i \in \{1, \ldots, k-1\}$, $\mathcal{I}(x_i) = \mathcal{J}(x_i)$. Now there are two cases. If $\mathcal{I} \models x_k$ and $\mathcal{J} \not\models x_k$, then $\mathcal{I}$ is lexicographically greater than $\mathcal{J}$. If $\mathcal{I} \not\models x_k$ and $\mathcal{J} \models x_k$, then

  $$\omega^{\mathcal{J}}(x_k, \ldots, x_n) \geq w^{\mathcal{J}}(x_k) = 2^{n-k} > \sum_{k+1}^{n} 2^{n-i} \geq \omega^{\mathcal{I}}(x_k, \ldots, x_n)$$

  However, as above, this directly implies that $\omega^{\mathcal{I}}(\overline{x}) < \omega^{\mathcal{J}}(\overline{x})$ causing a contradiction. With $\mathcal{J}$ being arbitrary, it follows that $\mathcal{I}$ is the lexicographically maximal model. Moreover, since $\mathcal{I} \models x_n$ it follows that x_n is satisfied in the lexicographically maximal interpretation of φ, and thus $(\varphi; \overline{x}) \in$ **LEX-MAXIMAL MODEL SAT**.

$\square$

Exercise 2 (3 credits)**.** Prove the correctness of the above reduction from **LogLEX-MAXIMAL MODEL SAT** to **CARD-MAXIMAL MODEL SAT**.

Solution

Definition 3. *Let* $(\varphi; \overline{x}) \in$ **LogLEX-MAXIMAL MODEL SAT** *then* $\tau(\varphi; \overline{x})$ *is defined as* $\tau(\varphi; \overline{x}) := (\tau(\varphi), x_n)$*, with* $\tau(\varphi)$ *being the construction from Reduction 2. Moreover, for some* $k \in \{1, \dots, n\}$ *let* $\mathcal{X}_k := \{x_k, x_k^{(1)}, \dots, x_k^{(r_k)}\}$*, where* $r_k := 2^{n-k} - 1$ *(as defined in Reduction 2). Furthermore, let* $\mathcal{X}_{\leq k} := \bigcup_{i \in \{1,\dots,k\}} \mathcal{X}_k$*. Analogously for* $\mathcal{X}_{>k}$*.*

Definition 4. *Let* $\mathcal{I}$ *be an interpretation, then* $\mathfrak{S}(\mathcal{I}) := \{x \mid \mathcal{I}(x) = \textbf{true}\}$*. Moreover, if* $\mathcal{I}$ *is a model of the formula* φ *then one can construct the interpretation* $\tau(\mathcal{I})$ *such that* $\forall i \in \{1, \dots, m\}$ $\tau(\mathcal{I})(y_i) := \mathcal{I}(y_i) \wedge \tau(\mathcal{I})(y_i') := \neg\mathcal{I}(y_i)$*, as well as* $\forall i \in \{1, \dots, n\}$ $\tau(\mathcal{I})(x_i) := \mathcal{I}(x_i) \wedge \forall j \in \{1, \dots 2^{n-i} - 1\}$ $\tau(\mathcal{I})(x_i^{(j)}) := \mathcal{I}(x_i)$*.*

Observation 5. *For an arbitrary* $\mathcal{I}$ *such that* $\mathcal{I} \models \bigwedge_{i=1}^{n} \big((x_i \leftrightarrow x_i^{(1)}) \wedge \dots \wedge (x_i \leftrightarrow x_i^{(r_i)})\big)$ *then for any* $i \in \{1, \dots, n\}$ *one has* $\mathcal{I} \models x_i$ *if and only if* $\mathcal{I} \models x_i \wedge x_i^{(1)} \wedge \dots x_i^{(r_i)}$*.*

Observation 6. *Take an arbitrary model* $\mathcal{I}$ *of* $\tau(\varphi)$*. For any* $i \in \{1, \dots, n\}$ *one has* $\mathcal{I} \models y_i$ *if and only if* $\mathcal{I} \not\models y_i'$*. Hence, for any other model* $\mathcal{J}$ *of* $\tau(\varphi)$ *it must be that*

$$|\mathfrak{S}(\mathcal{J}) \cap (Y \cup Y')| = |\mathfrak{S}(\mathcal{I}) \cap (Y \cup Y')|$$

Lemma 7. *For the models* $\mathcal{I}$ *and* $\mathcal{J}$ *of* $\tau(\varphi)$*, consider the order* $\overline{x} := (x_1, \dots, x_n)$ *on the basis of which* $\tau(\varphi)$ *was constructed. Then it holds that, there exists* $k \in \{1, \dots, n\}$ *such that* $\forall i \in \{1, \dots, k-1\}$ *one has* $\mathcal{I}(x_i) = \mathcal{J}(x_i)$*, as well as* $\mathcal{J} \not\models x_k$ *and* $\mathcal{I} \models x_k$ *if and only if* $|\mathfrak{S}(\mathcal{J})| < |\mathfrak{S}(\mathcal{I})|$*.*

Proof. $\Rightarrow$ By Observation 5, this implies that $\mathcal{X}_k \subseteq \mathfrak{S}(\mathcal{I})$ and $\mathcal{X}_k \cap \mathfrak{S}(\mathcal{J}) = \emptyset$. Meaning that,

$$|\mathcal{X}_{\leq k} \cap \mathfrak{S}(\mathcal{I})| = |\mathcal{X}_{\leq k} \cap \mathfrak{S}(\mathcal{J})| + 2^{n-k}$$

However, by construction $|\mathcal{X}_{>k}| < 2^{n-k}$. Thus it follows that $|\mathfrak{S}(\mathcal{J}) \cap \mathcal{X}_{\leq n}| < |\mathfrak{S}(\mathcal{I}) \cap \mathcal{X}_{\leq n}|$. By Observation 6, this implies that $|\mathfrak{S}(\mathcal{J})| < |\mathfrak{S}(\mathcal{I})|$.

$\Leftarrow$ If $|\mathfrak{S}(\mathcal{J})| < |\mathfrak{S}(\mathcal{I})|$, by Observation 6, there must exists an $k \in \{1, \dots, n\}$ such that $\mathcal{I} \models x_k$ and $\mathcal{J} \not\models x_k$. With respect to the ordering $\overline{x}$ let this be the first of its occurrences. Assume that there exists a $l < k$ such that $\mathcal{I} \not\models x_l$ and $\mathcal{J} \models x_l$ If this is the case then this implies that $\mathcal{X}_l \subseteq \mathfrak{S}(\mathcal{J})$ and $\mathcal{X}_l \cap \mathfrak{S}(\mathcal{I}) = \emptyset$. Meaning that,

$$|\mathcal{X}_{\leq l} \cap \mathfrak{S}(\mathcal{J})| \geq |\mathcal{X}_{\leq l} \cap \mathfrak{S}(\mathcal{I})| + 2^{n-l}$$

However, by construction $|\mathcal{X}_{>l}| < 2^{n-l}$, thus even if k is satisfied by $\mathcal{I}$ this would not matter, as it would still be the case that $|\mathfrak{S}(\mathcal{I})| < |\mathfrak{S}(\mathcal{J})|$, which is a contradiction. Hence, l does not exist. Now with k being the first of its occurrences, it must be the case that $\forall i \in \{1, \dots, k-1\}$ one has $\mathcal{I}(x_i) = \mathcal{J}(x_i)$.

□

Observation 8. *For the models* $\mathcal{J}$ *and* $\mathcal{I}$ *of* $\tau(\varphi)$*. Then* $|\mathfrak{S}(\mathcal{I})| = |\mathfrak{S}(\mathcal{J})|$ *if and only if* $\mathfrak{S}(\mathcal{I}) \cap X = \mathfrak{S}(\mathcal{J}) \cap X$*.*

Proof. $\Rightarrow$ Assume that $|\mathfrak{S}(\mathcal{I})| = |\mathfrak{S}(\mathcal{J})|$ and $\mathfrak{S}(\mathcal{I}) \cap X \neq \mathfrak{S}(\mathcal{J}) \cap X$. Consider the order $\overline{x}$ on the basis of which $\tau(\varphi)$ was constructed. Given this order, and from Observation 6, there must be a $k \in \{1, \ldots, n\}$ such that $\forall i \in \{1, \ldots, k-1\}$ one has $\mathcal{I}(x_i) = \mathcal{J}(x_i)$, as well as $\mathcal{J}(x_k) \neq \mathcal{I}(x_k)$. W.l.o.g. $\mathcal{J} \not\models x_k$ and $\mathcal{I} \models x_k$, thus from Lemma 7 it follows that $|\mathfrak{S}(\mathcal{J})| < |\mathfrak{S}(\mathcal{I})|$, which is a contradiction.

$\Leftarrow$ This follows trivially from Observation 5 & 6.

□

Lemma 9. *Let φ be a propositional formula over the set of variables $X \cup Y$, let $\overline{x} := (x_1, \ldots, x_n)$ be a linear ordering of X and let $Y := \{y_1, \ldots, y_m\}$. Then*

$(\varphi; \overline{x}) \in$ **LogLEX-MAXIMAL MODEL SAT** $\iff$ $\tau(\varphi; \overline{x}) \in$ **CARD-MAXIMAL MODEL SAT**

Proof. Each direction is shown separately.

$\Rightarrow$ Assume that $(\varphi; \overline{x}) \in$ **LogLEX-MAXIMAL MODEL SAT**. Hence, there exists a lexicographically maximal (w.r.t. $\overline{x}$) model $\mathcal{I}$ of φ where x_n evaluates to true. Let $\mathcal{I}_\tau := \tau(\mathcal{I})$. By construction $\mathcal{I}_\tau \models \tau(\varphi)$. Hence, it remains to be shown that $\mathcal{I}_\tau$ is a cardinal maximal model. Take an arbitrary model $\mathcal{J}_\tau$ of $\tau(\varphi)$ other than $\mathcal{I}_\tau$. If none exists, then $\mathcal{I}_\tau$ is trivially maximal in cardinality. Firstly, by Observation 6 it must be that $|\mathfrak{S}(\mathcal{J}_\tau) \cap (Y \cup Y')| = |\mathfrak{S}(\mathcal{I}_\tau) \cap (Y \cup Y')|$. Secondly, two cases ought to be considered. On the one hand, if $\forall i \in \{1, \ldots, n\}$ $\mathcal{I}_\tau(x_i) = \mathcal{J}_\tau(x_i)$ it follows that $|\mathfrak{S}(\mathcal{I}_\tau)| = |\mathfrak{S}(\mathcal{J}_\tau)|$ by Observation 8. On the other hand, since $\mathcal{J}_\tau$ must model φ, one can restrict it to the variables in $X \cup Y$ to obtain $\mathcal{J}$ which still models φ. Since, $\mathcal{I}$ is lexicographically maximal with respect to $\overline{x}$, it follows that there must be a $k \in \{1, \ldots, n\}$ such that $\forall i \in \{1, \ldots, k-1\}$ one has $\mathcal{I}(x_i) = \mathcal{J}(x_i)$, as well as $\mathcal{J} \not\models x_k$ and $\mathcal{I} \models x_k$. However, since $\mathcal{I}_\tau$ agrees with $\mathcal{I}$ and since $\mathcal{J}_\tau$ agrees with $\mathcal{J}$ on all X's (and Y's), the same must hold for $\mathcal{I}_\tau$ and $\mathcal{J}_\tau$. Hence, from Lemma 7 one obtains $|\mathfrak{S}(\mathcal{J}_\tau)| < |\mathfrak{S}(\mathcal{I}_\tau)|$. Therefore, from the two cases one obtains $|\mathfrak{S}(\mathcal{J}_\tau)| \leq |\mathfrak{S}(\mathcal{I}_\tau)|$ for some arbitrary model of $\tau(\varphi)$, and thus $\mathcal{I}_\tau$ is cardinal maximal. Finally, since $\mathcal{I}_\tau \models x_n$ it follows that $(\tau(\varphi), x_n) \in$ **CARD-MAXIMAL MODEL SAT**.

$\Leftarrow$ Assume that $(\tau(\varphi), x_n) \in$ **CARD-MAXIMAL MODEL SAT**. Hence, there must exits a model $\mathcal{I}_\tau$ of $\tau(\varphi)$ that is maximal in its cardinality. Clearly, by restricting $\mathcal{I}_\tau$ to the variables in $X \cup Y$, one can construct an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \varphi$. Now, take any arbitrary model $\mathcal{J}$ of φ other than $\mathcal{I}$. Again, if $\mathcal{J}$ would not exists then $\mathcal{I}$ must be lexicographically maximal (w.r.t. $\overline{x}$). Then $\mathcal{J}_\tau := \tau(\mathcal{J})$ is a model of $\tau(\varphi)$. Moreover, by maximality of $\mathcal{I}_\tau$ it is known that $|\mathfrak{S}(\mathcal{J}_\tau)| \leq |\mathfrak{S}(\mathcal{I}_\tau)|$. Clearly, by Observation 6 it must be that $|\mathfrak{S}(\mathcal{J}_\tau) \cap (Y \cup Y')| = |\mathfrak{S}(\mathcal{I}_\tau) \cap (Y \cup Y')|$. Hence, by Observation 5, the only difference in cardinality can occur due to the respective evaluation of the variables in X. Thereby, inducing two cases. On the one hand, if $|\mathfrak{S}(\mathcal{J}_\tau)| = |\mathfrak{S}(\mathcal{I}_\tau)|$, then it follows by Observation 8, that $\mathfrak{S}(\mathcal{J}_\tau) \cap X = \mathfrak{S}(\mathcal{I}_\tau) \cap X$, implying that $\mathcal{I}$ is lexicographically equivalent to $\mathcal{J}$, i.e. they only differ on their assignments on the variables in Y. On the other hand, if $|\mathfrak{S}(\mathcal{J}_\tau)| < |\mathfrak{S}(\mathcal{I}_\tau)|$, then it follows from Lemma 7 that there exists a $k \in \{1, \ldots, n\}$ such that $\forall i \in \{1, \ldots, k-1\}$ one has $\mathcal{I}_\tau(x_i) = \mathcal{J}_\tau(x_i)$, as well as $\mathcal{J}_\tau \not\models x_k$ and $\mathcal{I}_\tau \models x_k$. However, this means that $\mathcal{I}_\tau$ is lexicographically greater than $\mathcal{J}_\tau$ w.r.t. the order $\overline{x}$. Since this carries over to $\mathcal{I}$ and $\mathcal{J}$ and since $\mathcal{I} \models x_n$ by assumption, it follows that $(\varphi, \overline{x}) \in$ **LogLEX-MAXIMAL MODEL SAT**.

□

Exercise 3 (3 credits)**.** Prove the correctness of the above reduction from **CARD-MAXIMAL MODEL SAT** to **CARD-MINIMAL MODEL SAT**.

Solution

Definition 10. *Let φ be a propositional formula over the set of variables $X := \{x_1, \ldots, x_n\}$, then $\tau(\varphi)$ the formula as defined in Reduction 3. Moreover, consider a model $\mathcal{I}$ of φ, then $\tau(\mathcal{I})$ is the interpretation $\forall i \in \{1, \ldots, n\}$ $\tau(\mathcal{I})(x_i) := \mathcal{I}(x_i) \;\wedge\; \tau(\mathcal{I})(x'_i) := \neg\mathcal{I}(x_i) \;\wedge\; \tau(\mathcal{I})(x''_i) := \neg\mathcal{I}(x_i)$.*

Observation 11. *Let φ be a propositional formula over the set of variables $X := \{x_1, \ldots, x_n\}$. Let $\mathcal{I}$ be a model of $\tau(\varphi)$. Then $|\mathfrak{S}(\mathcal{I}) \cap \tau(X)| = 2n - |\mathfrak{S}(\mathcal{I}) \cap X|$*

Proof. Clearly, $\forall i \in \{1, \ldots, n\}$ $x_i \notin \mathfrak{S}(\mathcal{I}) \iff x'_i \in \mathfrak{S}(\mathcal{I}) \wedge x''_i \in \mathfrak{S}(\mathcal{I})$. Hence,

$$|\mathfrak{S}(\mathcal{I})| = |\mathfrak{S}(\mathcal{I}) \cap \tau(X)| = |\mathfrak{S}(\mathcal{I}) \cap X| + 2(|X| - |\mathfrak{S}(\mathcal{I}) \cap X|) = 2n - |\mathfrak{S}(\mathcal{I}) \cap X|$$

where the first equality holds, if one considers only interpretations restricted to the variables in $\tau(X)$.[1] □

The following lemma demonstrates the correctness of Reduction 3

Lemma 12. *Let φ be a propositional formula over the set of variables $X := \{x_1, \ldots, x_n\}$. Then,*

$(\varphi; x_i) \in$ **CARD-MAXIMAL MODEL SAT** $\iff$ $(\tau(\varphi); x_i) \in$ **CARD-MINIMAL MODEL SAT**

Proof. Each direction is shown separately.

- $\Rightarrow$ Assume that $(\varphi; x_i) \in$ **CARD-MAXIMAL MODEL SAT**. Hence, x_i is satisfied in a cardinal maximal model $\mathcal{I}$ of φ. Using this, construct the interpretation $\mathcal{I}_\tau := \tau(\mathcal{I})$. By construction, $\mathcal{I}_\tau \models \tau(\varphi)$, thus it remains to be shown that it is cardinal minimal. Consider an arbitrary model $\mathcal{J}_\tau$ of $\tau(\varphi)$. Restrict $\mathcal{J}_\tau$ to the variables in X in order to create $\mathcal{J}$. Clearly, $\mathcal{J} \models \varphi$. By maximality of $\mathcal{I}$, any interpretation that satisfies φ must satisfy fewer or equally as many $x \in X$, i.e.

  $$r = |\mathfrak{S}(\mathcal{J}_\tau) \cap X| = |\mathfrak{S}(\mathcal{J})| \leq |\mathfrak{S}(\mathcal{I})| = |\mathfrak{S}(\mathcal{I}_\tau) \cap X| = o$$

  However, by Observation 11, this implies that

  $$|\mathfrak{S}(\mathcal{J}_\tau)| = |\mathfrak{S}(\mathcal{J}_\tau) \cap \tau(X)| = 2n - r \geq 2n - o = |\mathfrak{S}(\mathcal{I}_\tau) \cap \tau(X)| = |\mathfrak{S}(\mathcal{I}_\tau)|$$

  Now with $\mathcal{J}_\tau$ being arbitrary $\mathcal{I}_\tau$ is a cardinal minimal model of $\tau(\varphi)$. Moreover, by assumption x_i is satisfied by $\mathcal{I}$ and thus also by $\mathcal{I}_\tau$.

  Hence, $(\tau(\varphi); x_i) \in$ **CARD-MINIMAL MODEL SAT**.

- $\Leftarrow$ Assume that $(\tau(\varphi); x_i) \in$ **CARD-MINIMAL MODEL SAT**. Hence, x_i is satisfied in a cardinal minimal model $\mathcal{I}_\tau$ of $\tau(\varphi)$. Using this construct the interpretation $\mathcal{I}$ by restricting $\mathcal{I}_\tau$ to the set of variables in X. By construction, $\mathcal{I} \models \varphi$, thus it remains to be shown that $\mathcal{I}$ is cardinal maximal. By minimality of $\mathcal{I}_\tau$, any interpretation that satisfies $\tau(\varphi)$ must satisfy more or equally as many $x \in \tau(X)$. Now

[1] as is required to obtain any notion of maximality

consider an arbitrary model $\mathcal{J}$ of φ and construct $\mathcal{J}_\tau := \tau(\mathcal{J})$. Clearly, $\mathcal{J}_\tau \models \tau(\varphi)$ and thus

$$2n - |\mathfrak{S}(\mathcal{J}_\tau) \cap X| = |\mathfrak{S}(\mathcal{J}_\tau) \cap \tau(X)| = |\mathfrak{S}(\mathcal{J}_\tau)| \geq |\mathfrak{S}(\mathcal{I}_\tau)| = |\mathfrak{S}(\mathcal{I}_\tau) \cap \tau(X)| = 2n - |\mathfrak{S}(\mathcal{I}_\tau) \cap X|$$

which clearly is equivalent to $|\mathfrak{S}(\mathcal{J}_\tau) \cap X| \leq |\mathfrak{S}(\mathcal{I}_\tau) \cap X|$. However, by construction this is equivalent to $|\mathfrak{S}(\mathcal{J})| \leq |\mathfrak{S}(\mathcal{I})|$. Therefore, $\mathcal{I}$ is a cardinal maximal model of φ. Moreover, by assumption x_i is satisfied by $\mathcal{I}_\tau$ and thus also by $\mathcal{I}$. Hence, $(\varphi; x_i) \in$ **CARD-MAXIMAL MODEL SAT**.

$\square$