LEC2B: Matrix Algebra

1. Basic Concepts
2. Order of a Matrix
3. Addition of Matrices
4. Subtraction of Matrices
5. Multiplications of Matrices

Reading Assignment: The lecture ppt.

Why matrices?

Matrices were discovered and developed in the eighteenth and nineteenth centuries. They provide a theoretically and practically useful way of approaching many types of problems including:

- Solution of Systems of Linear Equations,
- Trend-surface analysis,
- Principal component analysis
- Cluster analysis
- Discriminant analysis

1. Basic concepts

Matrices, though they may appear weird objects at first, are a very important tool in expressing and discussing problems which arise from real life cases. Matrix algebra is not difficult!

Example

Consider two families A and B (though we may easily take more than two). Every month, the two families have expenses such as: utilities, health, entertainment, food, etc... Let us restrict ourselves to: food, utilities, and health. How would one represent the data collected? Many ways are available but one of them has an advantage of combining the data so that it is easy to manipulate them. Indeed, we will write the data as follows:

$$\text{Month} = \begin{pmatrix} \text{Family} & \text{Food} & \text{Utilities} & \text{Health} \\ A & a & b & c \\ B & d & e & f \end{pmatrix}$$

1. Basic concepts

If we have no problem confusing the names and what the expenses are, then we may write

$$\mathrm{Month} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

or

$$Month = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

This is what we call a Matrix. Individual numbers within a matrix are called the Element. The size of the matrix, as a block, is defined by the number of Rows and the number of Columns. In this case, the above matrix has 2 rows and 3 columns. You may easily come up with a matrix which has m rows and n columns. In this case, we say that the matrix is a (m x n) matrix (pronounce m-by-n matrix).

Keep in mind that:

m is the number of rows, and n is the number of columns.

e.g., our above matrix is a (2x3) matrix.

We usually designate a matrix symbolically by boldface capital letters such as **A** and its elements by subscripted italic lowercase letters such as a_{ij}, which represent the element in the ith row and jth column of matrix A.

Examples

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}$$

where $a_{11} = 2$, $a_{12} = 1$, and so on.

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 3 \end{pmatrix}$$

where $b_{11} = 3$, $b_{12} = 2$, and so on.

2. Order of a matrix

- The order of a matrix is an expression of its size. For example, the order of A is 2 x 3 and the order of B is 3 x 2. Sometime we write (most time we don't) the order of a matrix at the lower right of the matrix, e.g.,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}_{2\times 3}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3\times 2} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 3 \end{pmatrix}_{3\times 2}$$

A square matrix

When the numbers of rows and columns are equal, we call the matrix a square matrix. A square matrix of order n, is a (n x n) matrix.

Example of a square matrix of order 2 and 3:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 2 & 8 \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

Elementary Matrix Operations

Back to our example, let us assume that the matrices for the months of January, February, and March are

$$J = \begin{pmatrix} 600 & 250 & 350 \\ 550 & 180 & 400 \end{pmatrix}, \; F = \begin{pmatrix} 650 & 330 & 250 \\ 600 & 270 & 400 \end{pmatrix}, \text{ and } M = \begin{pmatrix} 580 & 270 & 350 \\ 625 & 350 & 410 \end{pmatrix}$$

To make sure that the reader knows what these numbers mean, you should be able to give the Health-expenses for family A and Food-expenses for family B during the month of February. The answers are ???. What is the matrix-expense for the two families for the first quarter? The idea is to add the three matrices above. It is easy to determine the total expenses for each family and each item, then the answer is

$$\text{First Quarter} = \begin{pmatrix} 1830 & 850 & 950 \\ 1775 & 800 & 1210 \end{pmatrix}$$

3. Addition of matrices

To add entries one by one.

For example, we have

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}=\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix} \qquad B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}=\begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \end{pmatrix}$$

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}+\begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \end{pmatrix}=\begin{pmatrix} 5 & 3 & 4 \\ 2 & 9 & 6 \end{pmatrix}$$

4. Subtraction of matrices

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}-\begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \end{pmatrix}=\begin{pmatrix} -1 & -1 & 2 \\ 0 & -1 & -2 \end{pmatrix}$$

Can you add or subtract the following two matrices ?

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 3 \end{pmatrix}$$

No, you can't because they are not in the same order or they don't have the same number of row and column!

Rule: The two matrices being added or subtracted must be of the same order.

Double a matrix

Clearly, if you want to double a matrix, it is enough to add the matrix to itself. So we have

$$A+A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}+\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}=\begin{pmatrix} 2a_{11} & 2a_{12} & 2a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \end{pmatrix}$$

This suggests the following rule

$$A+A=2A=2\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

In general, for any number c, we will have

$$cA=c\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}=\begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}$$

It is easy to see that for the number 0

$$0A = 0$$

$$0A = \begin{pmatrix} 0a_{11} & 0a_{12} & 0a_{13} \\ 0a_{21} & 0a_{22} & 0a_{23} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example: Given

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \end{pmatrix}$$

$$2A - B = 2\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 \\ 1 & 3 & 0 \end{pmatrix}$$

Let us summarize these two rules about matrices.

Addition of Matrices: In order to add two matrices, we add the entries one by one.

Note: Matrices involved in the addition operation must have the same size.

Multiplication of a Matrix by a Number: In order to multiply a matrix by a number, you multiply every entry by the given number.

The zero matrix

For the addition of matrices, one special matrix plays a role similar to the number zero. Indeed, if we consider the matrix with all its entries equal to 0, then it is easy to check that this matrix has behavior similar to the number zero. For example, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

or **A + 0 = A**

What about multiplying two matrices? Such operation exists but the calculations involved are complicated.

5. Multiplication of Matrices

For the following two **2×2 matrices**,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix};$$

Their multiplication is

$$A \times B = AB = C$$ or $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

where

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}; \qquad c_{12} = a_{11}b_{12} + a_{12}b_{22};$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21}; \qquad c_{22} = a_{21}b_{12} + a_{22}b_{22};$$

or

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Example

Please take a piece of paper and verify:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}; \qquad B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix};$$

Answer:

$$AB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2\times 1 + 1\times 2 & 2\times 3 + 1\times 4 \\ 3\times 1 + 2\times 2 & 3\times 3 + 2\times 4 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 7 & 17 \end{pmatrix}$$

For the following **3×3 matrices**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix};$$

Their multiplication is

$$AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

where

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}; \quad c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}; \quad c_{13} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}; \quad c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}; \quad c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$$

$$c_{31} = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}; \quad c_{32} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}; \quad c_{33} = a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}$$

Example

$$A=\begin{pmatrix}2&1&3\\1&4&2\\2&1&3\end{pmatrix}\qquad B=\begin{pmatrix}1&2&3\\3&2&2\\2&1&1\end{pmatrix}$$

$$c_{11}=a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}=11;\quad c_{12}=a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}=7;\quad c_{13}=a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33}=11$$
$$c_{21}=a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}=17;\quad c_{22}=a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}=12;\quad c_{23}=a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33}=13$$
$$c_{31}=a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}=11;\quad c_{32}=a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}=9;\quad c_{33}=a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33}=11$$

$$AB=\begin{pmatrix}2&1&3\\1&4&2\\2&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&2\\2&1&1\end{pmatrix}=\begin{pmatrix}11&7&11\\17&12&13\\11&9&11\end{pmatrix}$$

Note: In above examples we did multiplications for two square matrices and the two matrices are in the same order.

Can you multiply the following two matrices?

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 3 \end{pmatrix}$$

Yes, you can. Here is how:

$$AB = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 17 \\ 11 & 24 \end{pmatrix}$$

where

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 13; \qquad c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 17$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = 11; \qquad c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = 24$$

Multiplication of two matrices of different orders

Therefore, we do not need to have two matrices of the same size to multiply them (Remember: the two matrices being added or subtracted must be of the same order).

In fact, the general rule says that in order to perform the multiplication *AB*, where *A* is a (m x n) matrix and *B* a (k x l) matrix, then we must have ***n = k***, i.e., **the # of columns of A must equal to the # of rows of B.** The result will be a (m x l) matrix, i.e.,

$$A_{m\times n}B_{n\times l} = C_{m\times l}$$

Examples

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_{2\times 2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}_{2\times 2}$$

$$\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}_{2\times 2} \begin{pmatrix} I \\ S \end{pmatrix}_{2\times 1} = \begin{pmatrix} 0.6I + 0.3S \\ 0.4I + 0.7S \end{pmatrix}_{2\times 1}$$

$$A \times B = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}_{2\times 3} \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 3 \end{pmatrix}_{3\times 2} = \begin{pmatrix} 13 & 17 \\ 11 & 24 \end{pmatrix}_{2\times 2}$$

Can you do the following?

$$\begin{pmatrix} I \\ S \end{pmatrix}_{2\times 1} \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}_{2\times 2} = ???$$

$$\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}_{2x2} \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 2 \end{pmatrix}_{3x2} = ???$$

No, you can't. So we have to be very careful about multiplying matrices. Sentences like "multiply the two matrices A and B" may not make sense.

The matrix multiplication is not commutative

Now, another question:

AB = BA ?

Let's consider the two matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So,

AB $\neq$ B A

The matrix multiplication is not commutative

the order in which matrices are multiplied is important. In fact, this little setback is a major problem in playing around with matrices. This is something that you must always be careful with. Let us show you another setback. We have

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The product of two non-zero matrices may be equal to the zero-matrix!

Algebraic Properties of Matrix Operations

In this section, we give some general results about the three operations: addition, multiplication, and multiplication with numbers, called scalar multiplication.

Properties involving Addition

Let A, B, and C be m x n matrices. We have

1. **A+B = B+A**
2. **(A+B)+C = A + (B+C)**
3. **A + 0 = A**

where **0** is the **m x n zero-matrix** (all its entries are equal to 0);

Properties involving multiplication

1. Let A, B, and C be three matrices. If you can perform the products AB, (AB)C, BC, and A(BC), then we have

(AB)C = A (BC)

Note, for example, that if A is 2x3, B is 3x3, and C is 3x1, then the above products are possible (in this case, (AB)C is 2x1 matrix).

2. If α and β are numbers, and A is a matrix, then we have

$$\alpha(\beta A) = (\alpha\beta)A$$

3. If α is a number, and A and B are two matrices such that the product is possible, then we have

$$\alpha(AB) = (\alpha A)B = A(\alpha B)$$

4. If A is an n x m matrix and 0 the m x k zero-matrix, then

$$A\mathcal{O} = \mathcal{O}$$

Note the two zero-matrices may be different.

Properties involving Addition and Multiplication.

1. Let A, B, and C be three matrices. If you can perform the appropriate products, then we have

(A+B)C = AC + BC

and

A(B+C) = AB + AC

2. If and β are numbers, A and B are matrices, then we have

$$\alpha(A+B) = \alpha A + \alpha B$$

and

$$(\alpha+\beta)A = \alpha A + \beta A$$

Examples

Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & 1 & 5 \end{pmatrix}.$$

Evaluate (AB)C and A(BC). Check that you get the same matrix.

Answer. We have

$$AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

so

$$(AB)C = \begin{pmatrix} -1 \\ -2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -5 \\ 0 & -2 & -10 \end{pmatrix}.$$

On the other hand, we have

$$BC = \begin{pmatrix} 2 \\ -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 10 \\ 0 & -1 & -5 \end{pmatrix}$$

so

They are same!

$$A(BC) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 10 \\ 0 & -1 & -5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -5 \\ 0 & -2 & -10 \end{pmatrix}.$$

Unit Matrix

We have seen that matrix multiplication is different from normal multiplication (between numbers). Are there some similarities? For example, is there a matrix which plays a similar role as the number 1? The answer is yes.

Indeed, consider the n x n matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In particular, we have

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix In has similar behavior as the number 1. Indeed, for any n x n matrix A, we have

$$AI_n = I_n A = A$$

The matrix I_n is called the **Identity or Unit Matrix** of order n.

Example

Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}.$$

Then it is easy to check that

$$AB = I_2 \text{ and } BA = I_2.$$

The identity matrix behaves like the number 1 not only among the matrices of the form n x n. Indeed, for any n x m matrix A, we have

$$I_n A = A \text{ and } AI_m = A.$$

In particular, we have

$$I_4 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Thanks!