Pendulum

Reference: W. Rubinowicz, W. Królikowski, Mechanika teoretyczna (Theoretical mechanics), Państwowe Wydawnictwo Naukowe, Warszawa, 1998, pp. 91-99.

Analysis: Explicit dynamics, bilaterally constrained motion.

Purpose: Examine the accuracy of an analysis involving rigid rod constraint.

Summary: A mathematical pendulum composed of a mass point and a weightless rod swings with a large amplitude. Pendulum period, energy conservation, constraint satisfaction and convergence are examined.

The period of an oscillatory mathematical pendulum reads

$$T = 2\pi\sqrt{\frac{l}{g_3}}\left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right) k^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right) k^2 + ...\right) \tag{1}$$

where

$$k = \sin\left(\frac{\theta_{max}}{2}\right) \tag{2}$$

and l is the length of the pendulum, g_3 is the vertical component of the gravity acceleration and θ_{max} is the maximal tilt angle of the pendulum. Let us assume the initial velocity of the pendulum to be zero. Thus $\theta_{max} = \theta(0)$. Taking the rest configuration position of the mass point $\bar{\mathbf{x}} = [0, 0, 0]$ and considering the swing in the $x - z$ plane, the initial position of the pendulum reads

$$\bar{\mathbf{x}}(0) = \left[\begin{array}{c} l\sin(\theta_{max}) \\ 0 \\ l(1 - \cos(\theta_{max})) \end{array}\right] \tag{3}$$

Without the initial kinetic energy ($E_k(0) = 0$), the energy conservation requires that

$$E_k(t) + E_p(t) = E_p(0) \tag{4}$$

where

$$E_p(0) = m g_3 \bar{x}_3(0) \tag{5}$$

and m is the scalar mass.

Input parameters

Length (m)	$l = 1$
Mass (kg)	$m = 1$
Initial angle $\theta(0) = \theta_{max}$ (rad)	$\theta_{max} = \pi/2$
Gravity acceleration (m/s^2)	$\mathbf{g} = [0, 0, -\pi^2]$

The gravity acceleration g_3 has been chosen so that for $\theta_{max} = 0\,\text{deg}$ there holds $T = 2s$.

Results

The table below summarizes the results for the time step $h = 0.001$. The solution is accurate and stable after 1 and 10 swings. Figure 1 illustrates the energy balance over one period of the pendulum. The potential and kinetic energies sum up to π^2.

	Target	*Solfec*	Ratio
Pendulum period - 1 swing (s)	2.360	2.360	1.000
Total energy - 1 swing (J)	π^2	9.86960	1.000
Pendulum period - 10 swings (s)	23.61	23.60	1.000
Total energy - 10 swings (J)	π^2	9.86960	1.000

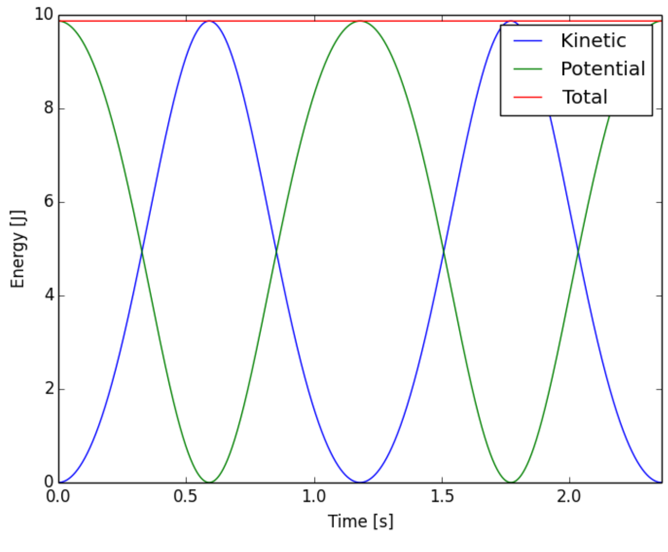

Figure 1: Energy balance over one period of the pendulum.