Chapter 1

Determinants and Inverse Matrices

1.1 Determinants

Checking the independence of multitudes of vectors may take an immense amount of time. What if you had a list of 5, 10, or even 100 vectors? The determinant of a matrix is a scalar value that also indicates whether the columns of a matrix are linearly independent. So, if you put all your vectors together in a matrix and take the determinant of that matrix, the result will tell you if all the vectors are independent or not. For a 2D matrix, the determinant is the area of the parallelogram defined by the column vectors. For a 3D matrix, the determinant is the volume of the parallelepiped (a six-dimensional figure formed by six parallelograms, such as a cube).[1]

Let's plot the parallelogram for this matrix (see figure **??**):

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

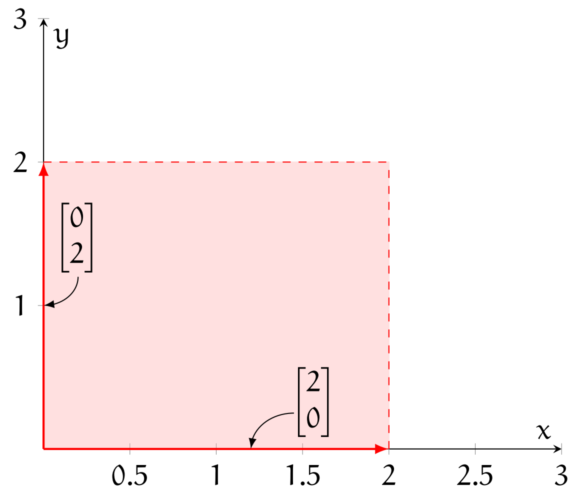


Figure 1.1: A parallelogram constructed from vectors $[2, 0]$ and $[0, 2]$

[1]Note that determinants can only be found for square, $n \times n$ matrices.

2 by 2 Determinant

The formal definition for calculating the determinant of a 2 by 2 matrix A is:

$$\det(A) = (a \cdot d) - (b \cdot c)$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For the matrix plotted above, the determinant is $(2 * 2) - (0 * 0)$. You can also see that 4.0 is the area, base (2) times height (2).

You can use the determinant to see what happens to a shape when it goes through a linear transformation. Let's scale the 2 by 2 matrix by 4:

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

Plot it (see figure **??**):

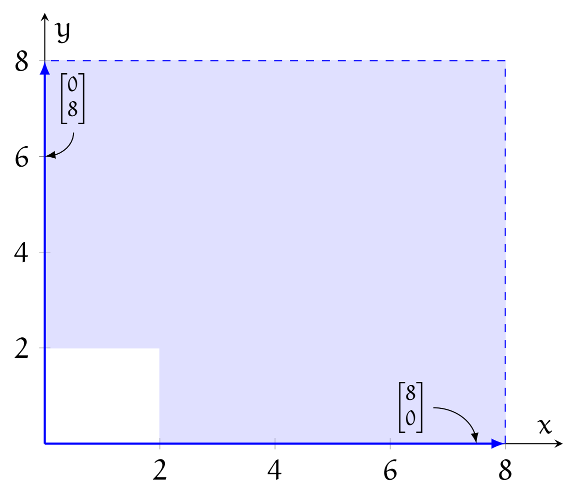


Figure 1.2: Scaling the matrix also scales the parallelogram.

Find the determinant using $(8 * 8) - (0 * 0) = 64$

You can see that scaling the matrix scaled the area by the scaling factor squared (see figure **??**).

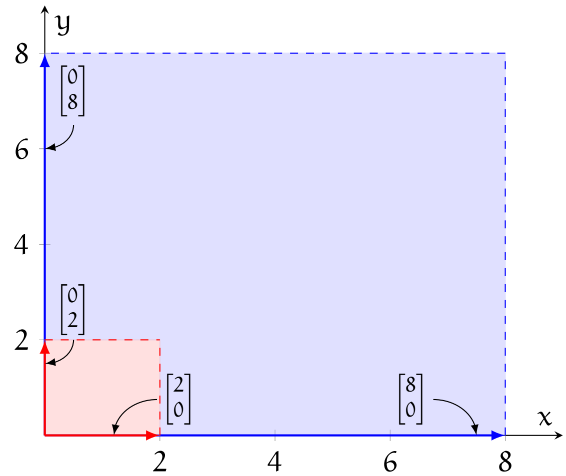


Figure 1.3: Scaling a matrix by a constant c increases the area of the parallelogram by a factor of c^2.

We can show why this is true mathematically. Suppose we have a 2 by 2 matrix A:

$$A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Then $\det(A) = wz - xy$. We can scale this matrix by a constant, c:

$$cA = c \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} cw & cx \\ cy & cz \end{bmatrix}$$

And we can take the determinant:

$$\det(cA) = \det\left(\begin{bmatrix} cw & cx \\ cy & cz \end{bmatrix}\right) = cw(cz) - cx(cy) = c^2(wz - xy) = c^2 \cdot \det(A)$$

Therefore, scaling a 2 by 2 matrix by a factor changes the determinant by that factor squared. What about higher dimensions? If each side of a cube were scaled by a factor of c, then the volume of the cube would change by a factor of c^3 (feel free to confirm this yourself). And if a tesseract (a four-dimensional cube) had each side scaled by a factor of c, then the hypervolume (four-dimensional volume) would be scaled by a factor of c^4. Do you notice a pattern?

In fact, scaling an $n \times n$ matrix by a constant factor, c, changes the determinant of that $n \times n$ matrix by a factor of c^n.

What happens if the columns of a matrix are not independent? Let's plot this matrix (see figure **??**):

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

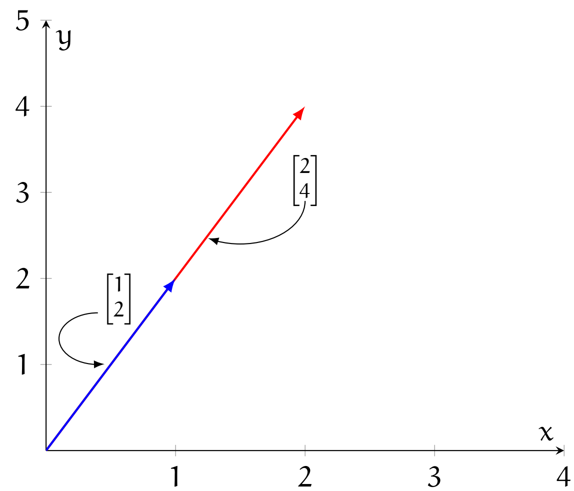


Figure 1.4: The vectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$ are co-linear, so there is no area between them and the determinant of $\begin{bmatrix}2 & 1\\4 & 2\end{bmatrix}$ is zero.

One vector overwrites the other. As you can see, the area is 0 because there is no space between the vectors. Therefore, the columns of the matrix are linearly dependent.

Exercise 1 Finding the Determinate

Plot the parallelogram represented by the columns of the matrix. What is the area of this parallelogram?

1. $\begin{bmatrix}1 & 4\\-3 & 1\end{bmatrix}$
2. $\begin{bmatrix}5 & -5\\5 & -1\end{bmatrix}$
3. $\begin{bmatrix}0 & -5\\-2 & 0\end{bmatrix}$

Working Space

Answer on Page ??

Calculating the determinant for a 2 by 2 matrix is easy. For a larger matrix, finding the determinant is more complex and requires breaking down the matrix into smaller matrices until you reach the 2x2 form. The process is called expansion by minors. For example,

3×3 **Determinant**

The determinant of a 3 by 3 matrix is found by

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

As you can see, this involves a recursive process of breaking a larger matrix into a smaller 2×2 matrix.

For our purposes, we simply want to first check to see if a matrix contains linearly independent rows and columns before using our Python code to solve.

1.2 Determinants in Python

Modify your code so that is uses the $np.linalg.det()$ function. If the determinant is not zero, then you can call the $np.linalg.solve()$ function. Your code should look like this:

```
if (np.linalg.det(D) != 0):
    j = np.linalg.solve(D,e)
    print(j)
else:
    print("Rows and columns are not independent.")
```

How does this work below the hood? Let's also write a recursive python function that finds our determinant:

There are two base cases:

- The matrix is of size 1×1
- The matrix is of size 2×2

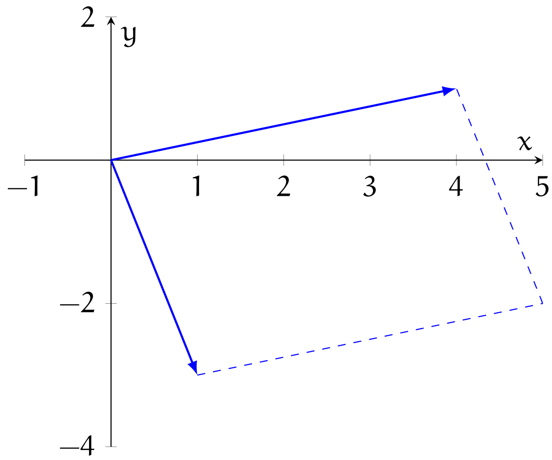
And further sizes can be simplified into one of the base cases:

Appendix A

Answers to Exercises

Answer to Exercise ?? (on page ??)

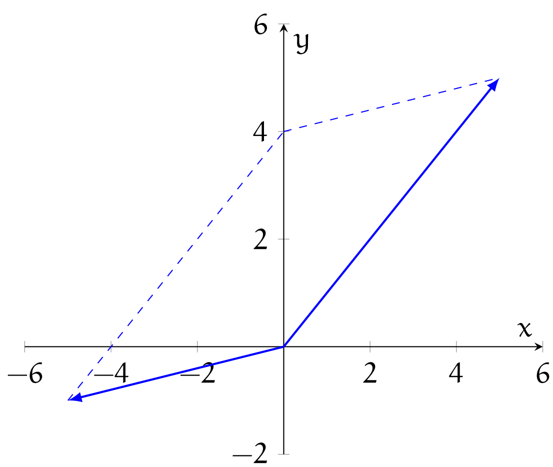
1. Our two vectors from the columns of the matrix are $[1, -3]$ and $[4, 1]$. Plotting:



The area of this parallelogram is the same as the determinant of the matrix:

$$\det\left(\begin{bmatrix} 1 & 4 \\ -3 & 1 \end{bmatrix}\right) = 1 \cdot 1 - (4 \cdot -3) = 1 + 12 = 13$$

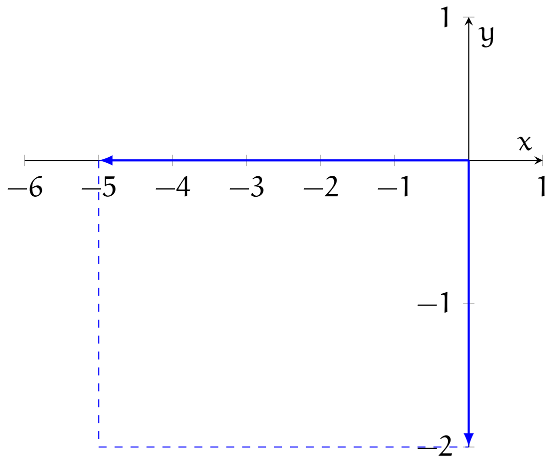
2. Our two vectors from the columns of the matrix are $[5, 5]$ and $[-5, -1]$. Plotting:



The area of this parallelogram is the same as the determinant of the matrix:

$$\det\left(\begin{bmatrix} 5 & -5 \\ 5 & -1 \end{bmatrix}\right) = 5 \cdot -1 - (-5 \cdot 5) = -5 + 25 = 20$$

3. Our two vectors from the columns of the matrix are $[0, -2]$ and $[-5, 0]$. Plotting:



This is a rectangle, and we can see the area is $5 \cdot 2 = 10$. However, the determinant is:

$$\det\left(\begin{bmatrix} 0 & -5 \\ -2 & 0 \end{bmatrix}\right) = 0 \cdot 0 - (-5 \cdot -2) = 0 - 10 = -10$$

We will discuss this unusual response in a future chapter.

Index