

## CHAPTER 1

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# Implicit Differentiation

Implicit differentiation is a technique in calculus for finding the derivative of a relation defined implicitly (that is, a relation between variables  $x$  and  $y$  that is not explicitly solved for one variable in terms of the other).

## 1.1 Implicit Differentiation Procedure

Consider an equation that defines a relationship between  $x$  and  $y$ :

$$F(x, y) = 0$$

To find the derivative of  $y$  with respect to  $x$ , we differentiate both sides of this equation with respect to  $x$ , treating  $y$  as an implicit function of  $x$ :<sup>1</sup>

$$\frac{d}{dx} F(x, y) = \frac{d}{dx} 0$$

Applying the chain rule during the differentiation on the left side of the equation gives:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

Finally, we solve for  $\frac{dy}{dx}$  to find the derivative of  $y$  with respect to  $x$ :

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

This result is obtained using the implicit differentiation method.

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<sup>1</sup>This  $\frac{d}{dx}$  form of the derivative is the same as  $y'$  said as taking the derivative of  $y$  with respect to  $x$ .

## 1.2 Example

Consider the equation of a circle with radius  $r$ :

$$x^2 + y^2 = r^2$$

First, we will find  $\frac{dy}{dx}$  without implicit differentiation. Next, we will apply implicit differentiation to get the same result.

### 1.2.1 Without Implicit Differentiation

First, we need to rearrange the equation to solve for  $y$ :

$$\begin{aligned}y^2 &= r^2 - x^2 \\y &= \pm\sqrt{r^2 - x^2}\end{aligned}$$

We take the derivative of  $y$  by applying the Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2 \pm \sqrt{r^2 - x^2}} \cdot (-2x) = \frac{-x}{\pm\sqrt{r^2 - x^2}}$$

Notice the denominator of this fraction is the same as the solution we found for  $y$ ,  $y = \pm\sqrt{r^2 - x^2}$ . So, we can also represent this as:

$$\frac{dy}{dx} = \frac{-x}{y}$$

### 1.2.2 With Implicit Differentiation

With implicit differentiation, we assume  $y$  is a function of  $x$  and apply the Chain Rule.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[r^2]$$

For  $x^2$  and  $r^2$ , we take the derivative as we normally would.<sup>2</sup> For  $y^2$ , we apply the Chain Rule, as outlined above.<sup>3</sup>

$$2x + 2y \frac{dy}{dx} = 0$$

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<sup>2</sup>The  $\frac{d}{dx}$  part disappears when taking the derivative of  $x$ , as the derivative of  $x$  with respect to  $x$  is just regular differentiation.

<sup>3</sup>Applying the chain rule is only allowed as  $y$  is not the variable we were taking the derivative with respect to.

Solving for  $\frac{dy}{dx}$ , we find

$$\frac{dy}{dx} = \frac{-x}{y}$$

, which is the same result as we found without implicit differentiation.

## 1.3 Folium of Descartes

It was relatively easy to rearrange the equation for a circle to solve for  $y$ , but that is not always the case. Consider the equation for the folium of Descartes (yes, that Descartes!):

$$x^3 + y^3 = 3xy$$

It is much more difficult to isolate  $y$  in this equation. In fact, were we to do so, we would need three separate equations to completely describe the original equation.

### 1.3.1 Example: Tangent to Folium of Descartes

In this example, we will use implicit differentiation to easily find the tangent line at a point on the folium.

(a) Find  $\frac{dy}{dx}$  if  $x^3 + y^3 = 6xy$

(b) Find the tangent to the folium  $x^3 + y^3 = 3xy$  at the point  $(2, 2)$

(c) Is there any place in the first quadrant where the tangent line is horizontal? If so, state the point(s).

Solution:

(a)  $\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y$$

$$x^2 + y^2 \frac{dy}{dx} = x \frac{dy}{dx} + y$$

Rearranging to solve for  $\frac{dy}{dx}$ :

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

(b) We already have the coordinate point,  $(2, 2)$ , so to write an equation for the tangent line, all we need is the slope. Substituting  $x = 2$  and  $y = 2$  into our result from part (a):

$$\frac{dy}{dx} = \frac{2 - 2^2}{2^2 - 2} = \frac{-2}{2} = -1$$

This is the slope,  $m$ . Using the point-slope form of a line, our tangent line is  $y - 2 = -(x - 2)$ .

(c) Recall that in the first quadrant,  $x > 0$  and  $y > 0$ . We will set our solution for  $\frac{dy}{dx}$  equal to 0:

$$\frac{y - x^2}{y + 2 - x} = 0$$

which implies that

$$y - x^2 = 0$$

Substituting  $y = x^2$  into the original equation:

$$x^3 + (x^2)^3 = 3(x)(x^2)$$

$$x^3 + x^6 = 3x^3$$

Which simplifies to

$$x^6 = 2x^3$$

Since we have excluded  $x = 0$  by restricting our search to the first quadrant, we can divide both sides by  $x^3$ :

$$x^3 = 2$$

$$x = \sqrt[3]{2} \approx 1.26$$

Substituting  $x \approx 1.26$  into our equation for  $y$ :

$$y \approx 1.26^2 = 1.59$$

Therefore, the folium has a horizontal tangent line at the point  $(1.26, 1.59)$ .

## 1.4 Practice

**Exercise 1**

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.]  
If  $\arcsin x = \ln y$ , what is  $\frac{dy}{dx}$ ?

**Working Space****Answer on Page 7****Exercise 2**

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.]  
The points  $(-1, -1)$  and  $(1, -5)$  are on the graph of a function  $y = f(x)$  that satisfies the differential equation  $\frac{dy}{dx} = x^2 + y$ . Use implicit differentiation to find  $\frac{d^2y}{dx^2}$ . Determine if each point is a local minimum, local maximum, or inflection point by substituting the  $x$  and  $y$  values of the coordinates into  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$ .

**Working Space****Answer on Page 7**



## APPENDIX A

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# Answers to Exercises

### Answer to Exercise 1 (on page 5)

Using implicit differentiation, we see that:

$$\frac{d}{dx} \arcsin x = \frac{d}{dx} \ln y$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{y} \frac{dy}{dx}$$

Multiplying both sides by  $y$  to isolate  $\frac{dy}{dx}$ , we find that:

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$$

### Answer to Exercise 2 (on page 5)

First, we need to find  $\frac{dy}{dx^2}$ :

$$\begin{aligned}\frac{d}{dx} \frac{dy}{dx} &= \frac{d}{dx} x^2 + \frac{d}{dx} y \\ &= 2x + \frac{dy}{dx} = 2x + x^2 + y\end{aligned}$$

At  $(-1, -1)$ ,  $\frac{dy}{dx} = (-1)^2 + (-1) = 0$  and  $\frac{d^2y}{dx^2} = 2(-1) + (-1)^2 + (-1) = -2 < 0$ . Since the slope of  $y$  is zero and the graph of  $y$  is concave down,  $(-1, -1)$  is a local maximum. At  $(1, -5)$ ,  $\frac{dy}{dx} = 1^2 + -5 = -4 \neq 0$  and  $\frac{d^2y}{dx^2} = 2(1) + 1^2 + (-5) = -2 \neq 0$ . Since neither the first nor second derivative of  $y$  are zero,  $(1, -5)$  is neither a local extrema nor an inflection point.





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