

# Partial Fractions

How can you add fractions with different denominators, like  $\frac{1}{x} + \frac{2}{x+3}$ ? You would need to make the denominators the same; after that, you could just add the numerators. You achieve this by multiplying the numerator and denominator of each fraction by the denominator of the other fraction:

$$\frac{1}{x} + \frac{2}{x+3} = \frac{1}{x} \left( \frac{x+3}{x+3} \right) + \frac{2}{x+3} \left( \frac{x}{x} \right)$$

Recall that when the numerator and denominator of a fraction are the same, the fraction is equal to one. So, we are not changing the *value* of each fraction, since we are just multiplying by one. Continuing, we can perform the multiplication and see that:

$$\begin{aligned} \frac{1}{x} + \frac{2}{x+3} &= \frac{x+3}{x(x+3)} + \frac{2x}{x(x+3)} \\ &= \frac{(x+3) + 2x}{x(x+3)} = \frac{3x+3}{x^2+3x} = \frac{3(x+1)}{x^2+3x} \end{aligned}$$

The inverse of this process is called **partial fraction decomposition** (or partial fraction expansion). This method has applications in many fields, but we will find it most useful as a tool to evaluate integrals in a later chapter.

Let  $g(x)$  be a rational function such that

$$g(x) = \frac{P(x)}{Q(x)}$$

Where  $P(x)$  and  $Q(x)$  are polynomials. If  $g(x)$  is proper (that is, the degree of  $P$  is less than the degree of  $Q(x)$ ) then we can express  $g(x)$  as the sum of simpler rational fractions. If  $g(x)$  is improper (that is, the degree of  $P$  is greater than or equal to the degree of  $Q$ ), then we must first perform long division to obtain a remainder,  $R(x)$ , where the degree of  $R$  is less than the degree of  $Q$ :

$$g(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

## 1.0.1 Improper fractions

What is  $\int \frac{x^3+x}{x-1} dx$ . Using long division, we see that:

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

(see figure 1.1 for an explanation). Then we can also say that:

$$\frac{x^3 + x}{x - 1} = x^2 + x + 2 + \frac{2}{x - 1}$$

$$\begin{array}{r}
 \phantom{x-1} \overline{x^2 + x + 2} \\
 x-1 \overline{) x^3 + 0x^2 + x} \\
 \underline{-(x^3 - x^2)} \phantom{+ x} \\
 \phantom{x-1} x^2 + x \\
 \underline{-(x^2 - x)} \\
 \phantom{x-1} 2x \\
 \underline{-(2x - 2)} \\
 \phantom{x-1} 2
 \end{array}$$

Figure 1.1: Evaluating  $(x^3 + x) \div (x - 1)$  with the long division method

When you start with an improper fraction, use long division to reduce it to a term plus a proper fraction, then use the methods outlined below to further manipulate the proper fraction.

### Exercise 1

Use long division to reduce the following improper rational functions to a term plus a proper rational fraction.

1.  $\frac{x^4 + x^3 + 2x^2 + 2x - 3}{x^2 - 3x + 2}$

2.  $\frac{2x^3 + 5}{x^3 - 3x^2 + 2x - 4}$

3.  $\frac{3x^4 - 2x^3 - x^2 + 1}{x^3 - 3x}$

*Working Space*

*Answer on Page 7*

## 1.0.2 Proper fractions

When the order of the numerator is less than or equal to the denominator, there are three further possibilities.

### No repeated linear factors

In the first case, the denominator,  $Q(x)$  is composed of distinct linear factors. In this case, we can say that  $Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$ , where no factor is repeated (including constant multiples). Then, there exists  $A, B, C, \dots$ , such that:

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x + b_1} + \frac{B}{a_2x + b_2} + \cdots$$

Let's see an example of this by decomposing  $\frac{4x^2 - 7x - 12}{x(x+2)(x-3)}$ . We start by defining  $A, B$ , and  $C$ , such that:

$$\frac{4x^2 - 7x - 12}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$

Multiplying both sides by  $x(x+2)(x-3)$  we get:

$$4x^2 - 7x - 12 = A(x+2)(x-3) + B(x)(x-3) + C(x)(x+2)$$

We have 3 unknowns and only one equation! Don't worry — remember this equation is true for all  $x$ , so we can choose a convenient value of  $x$  to isolate each unknown in turn. Starting, let  $x = 0$ . Then:

$$\begin{aligned} 4(0)^2 - 7(0) - 12 &= A(0+2)(0-3) + B(0)(0-3) + C(0)(0+2) \\ -12 &= A(2)(-3) + 0 + 0 \end{aligned}$$

Notice that the  $B$  and  $C$  disappear, and we can solve for  $A$ :

$$A = \frac{-12}{-6} = 2$$

We can solve for  $B$  by setting  $x = -2$  and for  $C$  by setting  $x = 3$  (notice, we've used all three zeroes of the denominator polynomial):

$$\begin{aligned} 4(-2)^2 - 7(-2) - 12 &= A(-2+2)(-2-3) + B(-2)(-2-3) + C(-2)(-2+2) \\ 4(4) + 14 - 12 &= 0 + B(-2)(-5) + 0 \\ 16 + 2 &= 10B \\ B &= \frac{9}{5} \end{aligned}$$

and

$$4(3)^2 - 7(3) - 12 = A(3+2)(3-3) + B(3)(3-3) + C(3)(3+2)$$

$$4(9) - 21 - 12 = 0 + 0 + C(3)(5)$$

$$36 - 33 = 15C$$

$$C = \frac{1}{5}$$

We can then decompose our original fraction:

$$\frac{4x^2 - 7x - 12}{x(x+2)(x-3)} = \frac{2}{x} + \frac{9}{5(x+2)} + \frac{1}{5(x-3)}$$

You can check your answer by cross-multiplying and adding. You should get the same rational function back.

### Repeated linear factors

The second case is if  $Q(x)$  has repeated factors (such as  $x^2 + 8x + 16 = (x+4)^2$ ). Suppose the first linear factor,  $(a_1x + b_1)$  is repeated  $r$  times (that is,  $Q(x)$  contains the factor  $(a_1x + b_1)^r$ ). Then, instead of  $\frac{A}{a_1x+b_1}$ , we should write:

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Let's look at a concrete example to see how this works:

**Example:** Decompose  $\frac{x^2+x+1}{(x+1)^2(x+2)}$

**Solution:** We start by defining:

$$\frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

Multiplying both sides by  $(x+1)^2(x+2)$ :

$$x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Since there are only 2 roots to  $(x+1)^2(x+2)$ , we will use another method called "equating the coefficients" to find  $A$ ,  $B$ , and  $C$ . We start by expanding the right side of the equation:

$$x^2 + x + 1 = A(x^2 + 3x + 2) + B(x+2) + C(x^2 + 2x + 1)$$

Distributing and combining, we find that:

$$x^2 + x + 1 = Ax^2 + 3Ax + 2A + Bx + 2B + Cx^2 + 2Cx + C$$

$$x^2 + x + 1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$

For this equation to be true, we know that:

$$A + C = 1$$

$$3A + B + 2C = 1$$

$$2A + 2B + C = 1$$

(That is, the coefficient for  $x^2$  on the left, 1, must be equal to the coefficient for  $x^2$  on the right,  $(A + C)$ , and so on.) We now have a system of 3 equations and 3 unknowns. When you solve for each, you should find that:

$$A = -2$$

$$B = 1$$

$$C = 3$$

Therefore,

$$\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{-2}{x + 1} + \frac{1}{(x + 1)^2} + \frac{3}{x + 2}$$

### Irreducible quadratic factors

Sometimes, we cannot express a polynomial as the product of two linear statements (that is, terms in the form  $ax + b$ ). Take  $x^2 + 1$ , which cannot be expressed as the product of real, linear terms. What do you do if something like  $x^2 + 1$  is in the denominator? In this case, when we write an expression for  $\frac{P(x)}{Q(x)}$ , we include a term in the form:

$$\frac{Ax + B}{ax^2 + bx + c}$$

For example, we can write:

$$\frac{x}{(x - 2)(x^2 + 1)(x^2 + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + 4}$$

**Example:** Decompose  $\frac{2x^2 - x + 4}{x^3 + 4x}$

**Solution:** We begin by factoring the denominator:

$$x^3 + 4x = x(x^2 + 4)$$

Which cannot be factored further. Therefore, we define:

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

Which implies that:

$$2 = A + B$$

$$C = -1$$

$$4A = 4$$

Therefore,  $A = 1$ ,  $B = 1$ , and  $C = -1$  and we can say that:

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

### Repeated irreducible quadratic factors

Lastly, the denominator might contain repeated irreducible quadratic factors. Similar to repeated linear factors, when setting up your partial fractions, instead of only writing

$$\frac{A}{ax^2 + bx + c}$$

For a quadratic factor that is repeated  $r$  times, your equation should include:

$$\frac{A_1}{ax^2 + bx + c} + \frac{A_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r}{(ax^2 + bx + c)^r}$$

### Exercise 2

Decompose the following proper fractions

*Working Space*

1.  $\frac{x-4}{x^2+5x-6}$
2.  $\frac{x^2+x+1}{(x^2+1)^2}$
3.  $\frac{x^2+x+1}{(x+1)^2(x+2)}$

*Answer on Page 7*

# Answers to Exercises

## Answer to Exercise 1 (on page 2)

1.  $x^2 + 4x + 12 \frac{30x-27}{x^2-3x+2}$

2.  $2 + \frac{6x^2-4x+13}{x^3-3x^2+2x-4}$

3.  $3x - 2 + \frac{8x^2-6x+1}{x^3-3x}$

## Answer to Exercise 2 (on page 6)

1.  $\frac{10}{7(x+6)} + \frac{-3}{7(x-1)}$

2.  $\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}$

3.  $\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$





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