Differentiation

We have done some differentiation, but you haven't been given the real definition yet, because it is based on limits.

The idea is that we can find the slope between two points on the graph a and b like this:
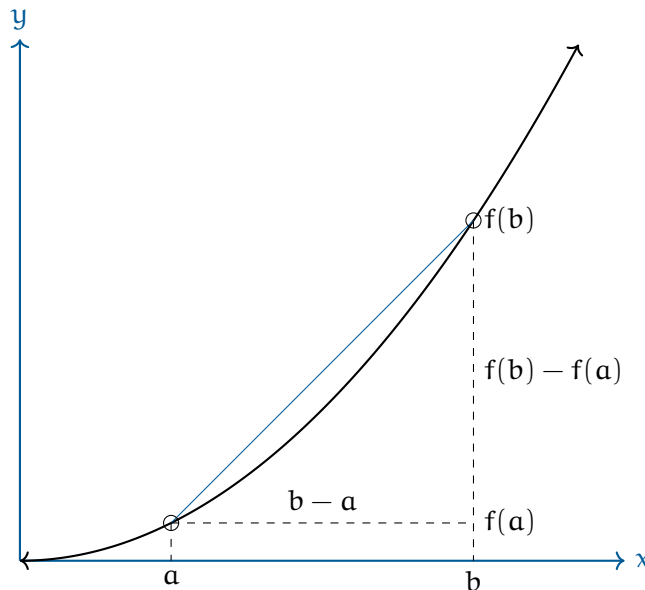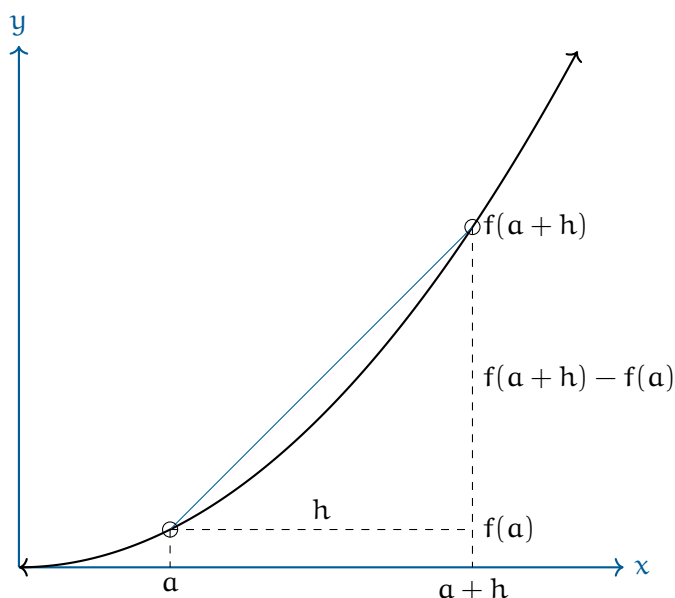
$$m = \frac{f(b) - f(a)}{b - a}$$

Figure 1.1: The slope of point a using points a and b.

If we want to find the slope at a, we take the limit of this as the b goes to a:

$$f'(a) = \lim_{b \to b} \frac{f(b) - f(a)}{b - a}$$

This idea is usually expressed using h as the difference between b and a:

The formula then becomes:

Figure 1.2: Limit of a point using h difference.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

As h decreases to very close to zero, almost infinitesimal, what we are finding is the sloe of the tangent line to point a.

1.1 Differentiability

Warning: Not every function is differentiable everywhere. For example, if $f(x) = |x|$, you get a corner at zero.

To the left of zero, the slope is -1. To the right of zero, the slope is 1. At zero? The derivative is not defined because the slopes are two different numbers.

If a function has a derivative everywhere, it is said to be *differentiable*. Generally, you can think of differentiable functions as smooth — their graphs have no corners or sharp turns. Another place where a function is not differentiable is at a vertical tangent, as the slope reaches $\pm\infty$.

It is important to note: if f is differentiable at a, it *must* also be continuous at a.

Likewise, if f is not continuous at a it is not differentiable. An example of this is a jump discontinuity. FIXME diagram here of jump discontinuity.
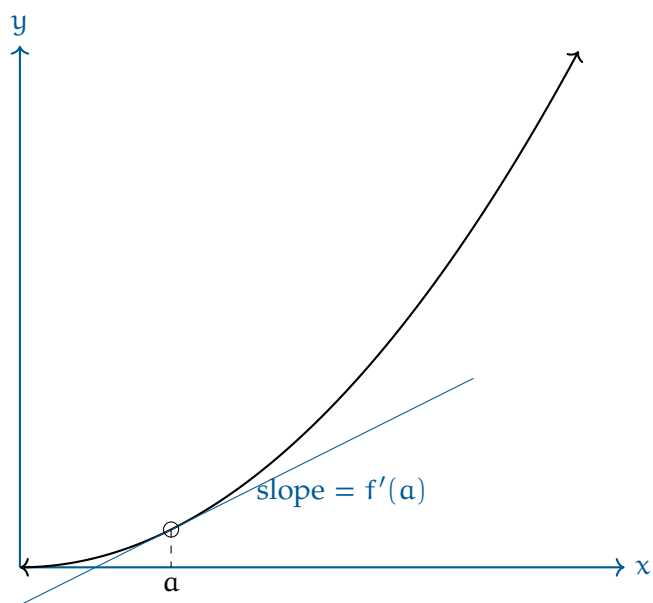
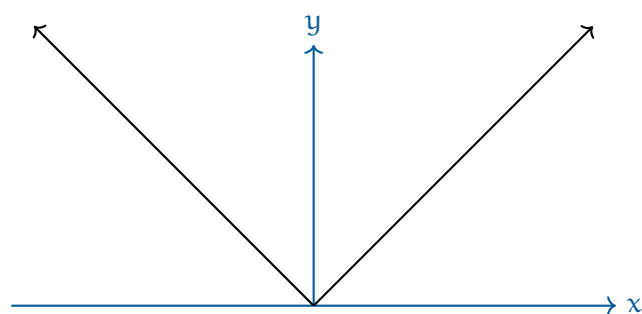Figure 1.3: Slope at a as h approaches 0.

Figure 1.4: Absolute value function.

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] Let f be the function defined by $f(x) = \sqrt{|x - 2|}$ for all x. Classify each of the following statements as true or false.

1. f is continuous at $x = 2$.

2. f is differentiable at $x = 2$.

3. $\lim_{x \to 2} f(x) = 0$.

4. $x = 2$ is a vertical asymptote of the graph of $f(x)$.

Working Space

Answer on Page 5

1.2   Using the definition of derivative

Let's say that you want to know the slope of $f(x) = -3x^2$ at $x = 2$. Using the definition of the derivative, that would be:

$$f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{-3(2 + h)^2 - \left(-3(2)^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{-12 - 12h + -3(h)^2 + 12}{h} = -12$$

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Answers to Exercises

Answer to Exercise 1 (on page 4)

1. True. $f(2)$ exists and $\lim_{x\to2^+} f(x) = \lim_{x\to2^-} f(x) = f(2) = 0$.

2. False. Because of the absolute value, there is a corner in the graph of f at $x = 2$. $\lim_{x\to2^+} f'(x) < 0$ and $\lim_{x\to2^-} f'(x) < 0$. Therefore there is a discontinuity in $f'(x)$ at $x = 2$ and $f(x)$ is not differentiable at $x = 2$.

3. True. $\sqrt{|2-2|} = \sqrt{0} = 0$.

4. False. $f(2)$ is defined at $x = 2$.