Chapter 1

Calculus with Polar Coordinates

We have been working in Cartesian coordinates, which are rectangular, with x representing the horizontal position and y representing the vertical position. Another way to represent a position in 2D space is with **polar coordinates**. In this coordinate system, the first number and dependent variable is r, which represents how far the point is from the origin. The second number is θ, which represents the degrees of rotation from the the x axis (see figure **??**).

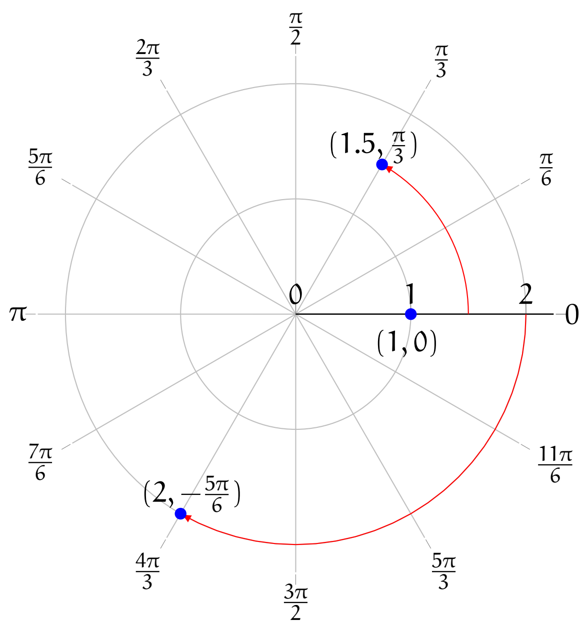


Figure 1.1: Polar coordinates give a degree of rotation, θ, and a distance from the origin, r, in the form of (r, θ)

1.1 Derivatives of Polar Functions

Consider the cardioid $r = 2 + \sin\theta$ (see figure **??**). What is the slope of the line tangent to the curve at $\theta = \frac{\pi}{2}$?

From a visual inspection, we can guess that the slope of the tangent line is zero. Let's prove this mathematically:

First, recall that to convert polar coordinates to Cartesian coordinates, we can use the

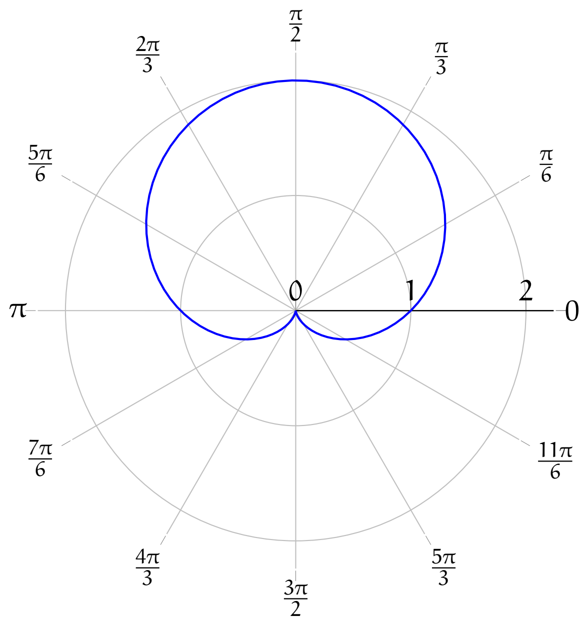


Figure 1.2: $r = 2 + \sin\theta$

trigonometric identities:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

So, we can write the parametric equation:

$$x = [2 + \sin\theta]\cos\theta$$
$$y = [2 + \sin\theta]\sin\theta$$

Recall from parametric equations that we can use implicit differentiation to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(2\sin\theta + \sin^2\theta\right) = 2\cos\theta + 2\sin\theta\cos\theta$$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(2\cos\theta + \sin\theta\cos\theta\right) = \cos^2\theta - \sin^2\theta - 2\sin\theta$$

Substituting $\theta = \frac{\pi}{2}$, we find that:

$$\frac{dy}{d\theta} = 2(0) + 2(1)(0) = 0$$

$$\frac{dx}{d\theta} = (0)^2 - (1)^2 - 2(1) = -3$$

Therefore,

$$\frac{dy}{dx} = \frac{0}{-3} = 0$$

Which is the result we expected from examining the graph of $r = 2 + \sin\theta$.

So, in general for polar equations,

Tangent to a Polar Function

For a polar function, $r = f(\theta)$, the slope of a tangent line is given by:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Where $y = r \cdot \sin\theta$ and $x = r \cdot \cos\theta$

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] What is the slope of the line tangent to the polar curve $r = 1 + 2\sin\theta$ at $\theta = 0$?

Working Space

Answer on Page 15

Exercise 2

Find the slope of the tangent line to the given polar curve at the value of θ specified. Use this to write an equation for the tangent line in Cartesian coordinates.

1. $r = \frac{2}{3}\cos\theta$, $\theta = \frac{\pi}{6}$
2. $r = \frac{1}{2\theta}$, $\theta = \frac{\pi}{2}$
3. $r = 2 + 3\cos\theta$, $\theta = \frac{2\pi}{3}$

Working Space

Answer on Page 15

1.2 Integrals of Polar Functions

Similar to Cartesian functions, an integral of a polar function tells us the area within the function. We say "within" as opposed to "under" because a polar function describes how far from the origin the graph is based on the angle. Consider the graph of $r = 2\sin\theta$ (figure 1.3). Geometrically, we expect the area inside the curve to be $\pi r^2 = \pi$. However, this is not the result we get from directly integrating the function (we only integrate from $\theta = 0$ to $\theta = \pi$ because the circle is complete when θ reaches π):

$$\int_0^{\pi} 2\sin\theta\, d\theta = -2\cos\theta\Big|_{\theta=0}^{\theta=\pi}$$

$$= -2\left[\cos\pi - \cos 0\right] = -2\left[-1-1\right] = 4 \neq \pi$$

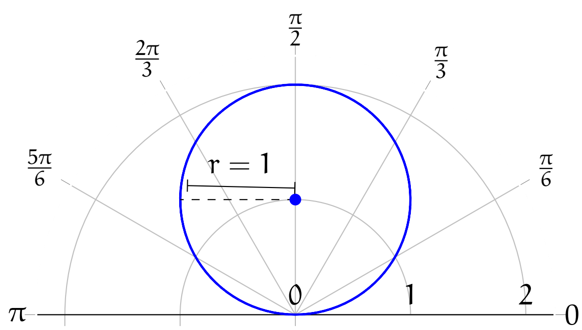


Figure 1.3: The graph of $r = 2\sin\theta$ is a circle of radius 1 centered at $(1, \frac{\pi}{2})$

Clearlym something else is happening here. We can just take the integral of a Cartesian function because the area of a rectangle is the base times the height. When integrating Cartesian functions, the base is given by the dx and the height by the function, $f(x)$. In polar coordinates, the integral sweeps across a θ interval, making a wedge, not a rectangle.

Let us consider a generic polar function, shown in figure 1.4

Suppose we are interested in a specific region, bounded by $a \leq \theta \leq b$ (see figure 1.5).

We can divide the region into many small sectors. Then, each small sector has a central angle $\Delta\theta$ and a radius $r(\theta_i^*)$, where $\theta_{i-1} < \theta_i^* < \theta_i$ (see figure 1.6).

What is the area of the i^{th} sector? Recall from the chapter on circles that the area of a sector with angle θ and length r is $A = \frac{1}{2}r^2\theta$. Substituting, we see the area of the i^{th} sector

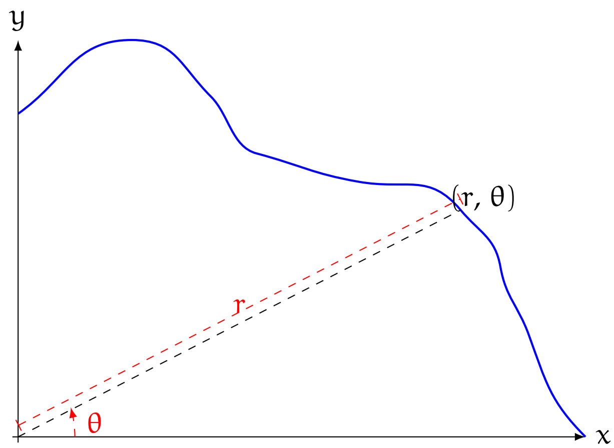


Figure 1.4: A generic polar function

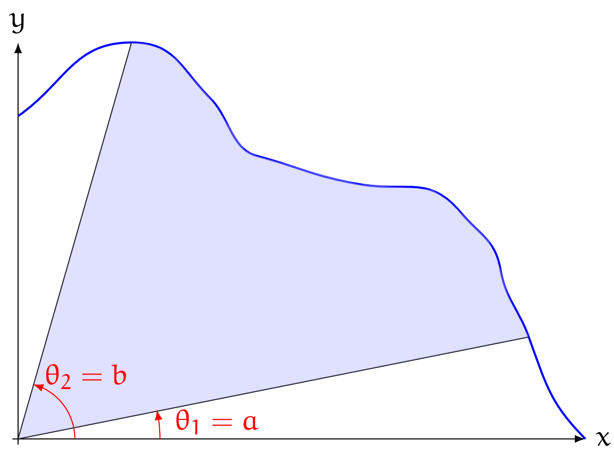


Figure 1.5: A generic polar with a region from $\theta = a$ to $\theta = b$ highlighted

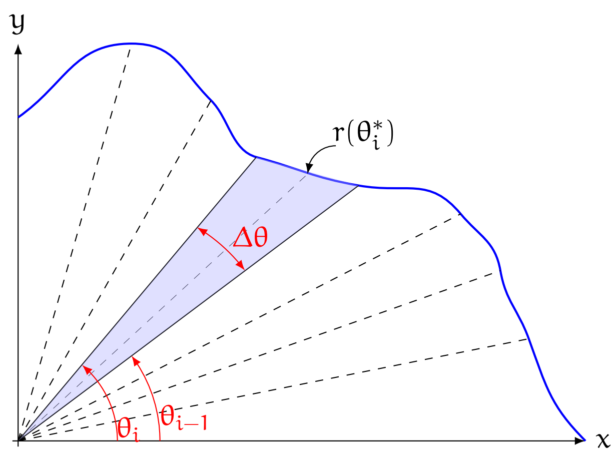


Figure 1.6: A single sector from θ_{i-1} to θ_i

is:

$$A_i = \frac{1}{2}\left[r(\theta_i^*)\right]^2 d\theta$$

Therefore, the total area of the whole sector from $\theta = a$ to $\theta = b$ is the limit as the number of sectors approaches infinity of sum of the areas of all the small sectors:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}\left[r(\theta_i^*)\right]^2 \Delta\theta$$

Does this look familiar? It is the definition of an integral!

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}\left[r(\theta_i^*)\right]^2 \Delta\theta = \int_a^b \frac{1}{2}\left[r(\theta)\right]^2\, d\theta$$

Area of a Polar Function

The area of a polar function is given by the integral

$$\int_a^b \frac{1}{2} r^2\, d\theta$$

Where r is a function of θ.

We can check this with the example from the beginning of the section. Recall that the polar function $r = 2\sin\theta$ graphs a circle with a radius of 1. Therefore, we expect the area enclosed by the graph of $r = 2\sin\theta$ from $\theta = 0$ to $\theta = \pi$ to be π:

$$A = \frac{1}{2}\int_0^{\pi}\left[2\sin\theta\right]^2\, d\theta$$

$$A = 2\int_0^{\pi}\sin^2\theta\, d\theta = 2\int_0^{\pi}\left[\frac{1-\cos 2\theta}{2}\right]\, d\theta$$

$$A = \int_0^{\pi}\left[1-\cos 2\theta\right]\, d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\pi}$$

$$A = [\pi - 0] - [0 - 0] = \pi$$

Which is the expected result, confirming our formula for the area within a polar function.

Example: The graph of $r = 3\sin 2\theta$ is shown below. What is the total area enclosed by the graph?

Solution: Since each lobe is symmetric to the others, we can find the area of one lobe and multiply it by four. To find the area of one lobe, we need to determine an interval for θ

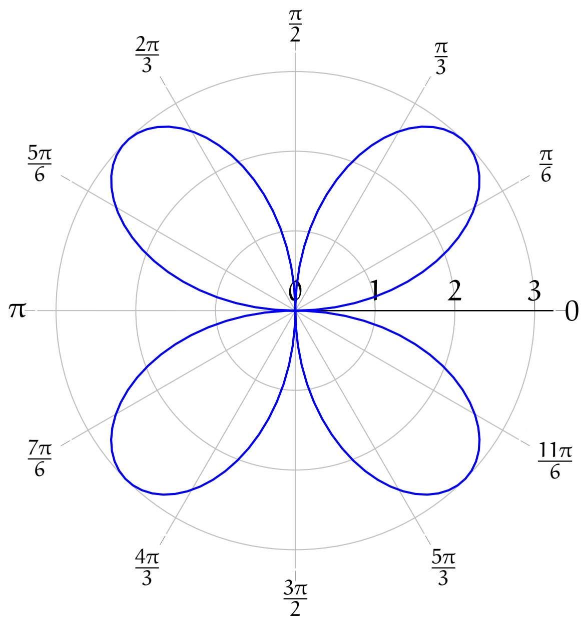


Figure 1.7: $r = 3\sin 2\theta$

that defines one lobe. You can imagine each lobe being draw out from the center and then back in. So, we will find where $r = 0$:

$$0 = 3\sin 2\theta$$

$$\sin 2\theta = 0$$

$$2\theta = n\pi$$

$$\theta = \frac{n\pi}{2}$$

Taking the first two solutions, $\theta = 0$ and $\theta = \frac{\pi}{2}$, as our limits of integration, we see that the area of one lobe is:

$$A_{\text{lobe}} = \frac{1}{2}\int_0^{\pi/2} [3\sin 2\theta]^2 \; d\theta$$

$$A_{\text{lobe}} = \frac{9}{2}\int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

Applying the half-angle formula $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, we see that:

$$A_{\text{lobe}} = \frac{9}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} \, d\theta = \frac{9}{4}\int_0^{\pi/2} 1 - \cos 4\theta \, d\theta$$

$$= \frac{9}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\theta=0}^{\theta=\pi/2} = \frac{9}{4}\left(\frac{\pi}{2} - 0\right) - \frac{9}{4}\left(\frac{1}{4}\right)(\sin 2\pi - \sin 0)$$

$$= \frac{9\pi}{8} - \frac{9}{16}(0) = \frac{9\pi}{8}$$

Since the area of one lobe is $\frac{9\pi}{8}$, the area of all four lobes is $\frac{9\pi}{2}$.

1.2.1 Area between polar curves

Consider the circle $r = 6\sin\theta$ and the cardioid $r = 2 + 2\sin\theta$. How can we find the area that lies inside the circle, but outside the cardioid (see figure 1.8)? First, let's find where these curves intersect. This will determine the limits of any integrals we take.

$$6\sin\theta = 2 + 2\sin\theta$$

$$3\sin\theta = 1 = \sin\theta$$

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

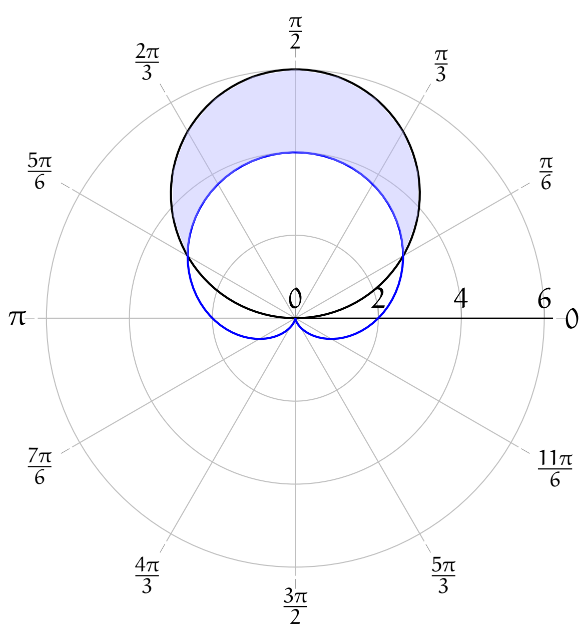


Figure 1.8: The area inside $r = 6\sin\theta$ and outside of $r = 2 + 2\sin\theta$ is highlighted

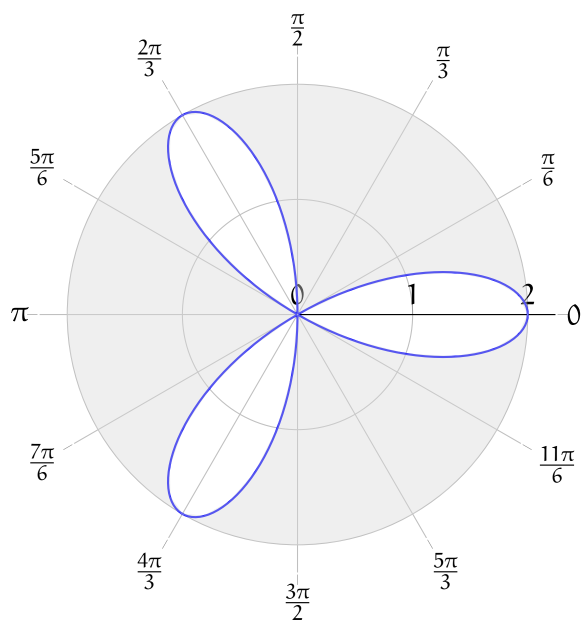
Recall that for Cartesian functions, to find the area between two curves, we subtract the area under the lower curve from the total area under the higher curve. In polar coordinates, we want to subtract the area in the inner curve from the total area in the outer curve. In this case, the outer curve is $r = 6\sin\theta$ and the inner curve is $r = 2 + 2\sin\theta$. We have already found our limits of integration ($\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$), so we set up and evaluate our integral:

$$
\begin{aligned}
A_{\text{between}} &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} [4\sin\theta]^2 \, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} [2+2\sin\theta]^2 \, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[16 sin^2\theta - 4 - 8\sin\theta - 4\sin^2\theta\right] \, d\theta \\
&= \int_{\pi/6}^{5\pi/6} \left[6\sin^2\theta - 4\sin\theta - 2\right] \, d\theta \\
&= \int_{\pi/6}^{5\pi/6} [3\,(1-\cos 2\theta) - 4\sin\theta - 2] \, d\theta \\
&= \int_{\pi/6}^{5\pi/6} [1 - 3\cos 2\theta - 4\sin\theta] \, d\theta \\
&= \left[\theta - \frac{3}{2}\sin 2\theta + 4\cos\theta\right]_{\theta=\pi/6}^{\theta=5\pi/6} \\
&= \left[\frac{5\pi}{6} - \frac{\pi}{6}\right] - \left[\frac{3}{2}\sin\left(2\cdot\frac{5\pi}{6}\right) - \frac{3}{2}\sin\left(2\cdot\frac{\pi}{6}\right)\right] + \left[4\cos\frac{5\pi}{6} - 4\cos\frac{\pi}{6}\right] \\
&= \frac{4\pi}{6} - \left[\frac{3}{2}\cdot -\frac{\sqrt{3}}{2} - \frac{3}{2}\cdot\frac{\sqrt{3}}{2}\right] + \left[4\cdot -\frac{\sqrt{3}}{2} - 4\cdot\frac{\sqrt{3}}{2}\right] \\
&= \frac{2\pi}{3} + \frac{3\sqrt{3}}{2} - 4\sqrt{3} = \frac{2\pi}{3} + \frac{3\sqrt{3} - 8\sqrt{3}}{2} = \frac{2\pi}{3} - \frac{5\sqrt{3}}{2}
\end{aligned}
$$

(*Note*: Because these polar functions are symmetric about the y-axis, we could have also taken the integral from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$ and doubled the result. We leave it as an exercise for the student to show this works.)

Exercise 3

[This question was originally presented as a multiple-choice, calculator- allowed problem on the 2012 AP Calculus BC exam.] The figure below shows the graphs of polar curves $r = 2\cos 3\theta$ and $r = 2$. What is the sum of the areas of the shaded regions to three decimal places?



Working Space

Answer on Page 17

Exercise 4

Find the area of the region bounded by the given curve and angles.

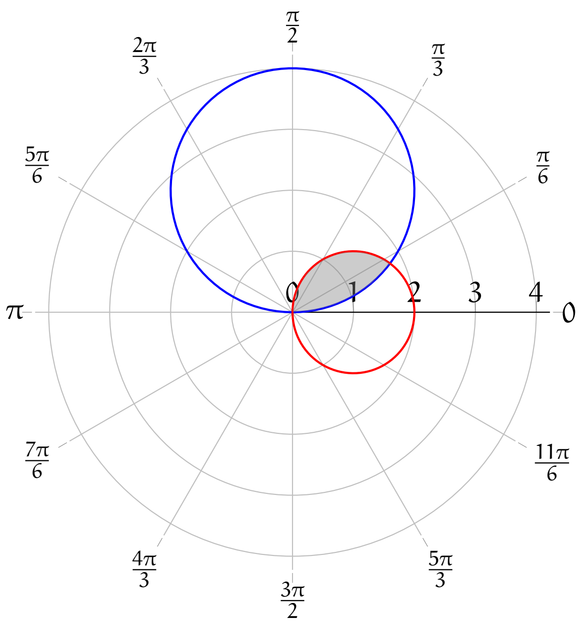
1. $r = e^{\theta/2}$, $\pi/4 \leq \theta \leq \pi/2$
2. $r = 2\sin\theta + \cos 2\theta$, $0 \leq \theta \leq \pi$
3. $r = 4 + 3\sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$

Working Space

Answer on Page 17

Exercise 5

Find the area of the region that lies between the curves $r = 4\sin\theta$ and $r = 2\cos\theta$. A graph is shown below.



Working Space

Answer on Page 18

`org/`) *for more details.*

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

Recall that for a polar function, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. We also know that $x = r\cos\theta$, which equals $[1 + 2\sin\theta] \cdot \cos\theta = \cos\theta + 2\sin\theta\cos\theta$ in this case. We also know that $y = r \cdot \sin\theta$, which equals $[1 + 2\sin\theta] \cdot \sin\theta = \sin\theta + 2\sin^2\theta$ in this case. Taking the derivative with respect to θ:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\sin\theta + 2\sin^2\theta\right]$$
$$\frac{dy}{d\theta} = \cos\theta + 4\sin\theta\cos\theta$$

And

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[\cos\theta + 2\sin\theta\cos\theta\right]$$
$$\frac{dx}{d\theta} = -\sin\theta - 2\sin^2\theta + 2\cos^2\theta$$

Evaluating each at $\theta = 0$:

$$\frac{dy}{d\theta} = \cos 0 + 4\sin 0\cos 0 = 1 + 0 = 1$$

$$\frac{dx}{d\theta} = -\sin 0 - 2\sin^2 0 + 2\cos^2 0 = 0 - 0 + 2 = 2$$

Therefore, $\frac{dr}{d\theta} = \frac{dy/d\theta}{dx/d\theta} = \frac{1}{2}$

Answer to Exercise 2 (on page 4)

1. Answer: slope $= -\frac{\sqrt{3}}{3}$ and an equation for the tangent line is $y - \frac{\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$.

   Explanation: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta} r\cdot\sin\theta}{\frac{d}{d\theta} r\cdot\cos\theta} = \frac{\frac{d}{d\theta}\left(\frac{2}{3}\cos\theta\sin\theta\right)}{\frac{d}{d\theta}\left(\frac{2}{3}\cos^2\theta\right)} = \frac{\frac{2}{3}(\cos^2\theta - \sin^2\theta)}{\frac{2}{3}(-2\cos\theta\sin\theta)} = \frac{\sin^2\theta - \cos^2\theta}{2\cos\theta\sin\theta}$

Substituting $\theta = \frac{\pi}{6}$: $\frac{dy}{dx} = \frac{\sin^2 \pi/6 - \cos^2 \pi/6}{2\cos \pi/6 \sin \pi/6} = \frac{(1/2)^2 - \left(\sqrt{3}/2\right)^2}{2\left(\sqrt{3}/2\right)(1/2)} = \frac{1/4 - 3/4}{\sqrt{3}/2} = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

To write an equation for a line, we need a Cartesian point. First, we find r at $\theta = \frac{\pi}{6}$: $r = \frac{2}{3}\cos\left(\frac{\pi}{6}\right) = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}$. So the point the tangent passes through is the polar coordinate $\left(\frac{\sqrt{3}}{3}, \frac{\pi}{6}\right)$. We convert this to Cartesian coordinates: $x = r\cos\theta = \frac{\sqrt{3}}{3} \cdot \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{2} = \frac{3}{6} = \frac{1}{2}$ And $y = r\sin\theta = \frac{\sqrt{3}}{3} \cdot \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{6}$

So, an equation for a line with slope $-\frac{\sqrt{3}}{3}$ that passes through Cartesian coordinate $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$ is: $y - \frac{\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$

2. Answer: slope $= \frac{2}{\pi}$ and an equation for the tangent line is $y - \frac{1}{\pi} = \frac{2}{\pi}x$

   Explanation: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}\left(\frac{\sin\theta}{2\theta}\right)}{\frac{d}{d\theta}\left(\frac{\cos\theta}{2\theta}\right)} = \frac{\frac{\theta\cos\theta - \sin\theta}{2\theta^2}}{-\frac{\theta\sin\theta + \cos\theta}{2\theta^2}} = \frac{\sin\theta - \theta\cos\theta}{\theta\sin\theta + \cos\theta}$.

   Substituting $\theta = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{\sin\frac{\pi}{2} - \left(\frac{\pi}{2}\right)\cos\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + \cos\frac{\pi}{2}} = \frac{1 - \left(\frac{\pi}{2}\right)\cdot 0}{\left(\frac{\pi}{2}\right)\cdot 1 + 0} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

   To write an equation for a line, we need a Cartesian point. First, we find r at $\theta = \frac{\pi}{2}$: $r = \frac{1}{2\theta} = \frac{1}{2\frac{\pi}{2}} = \frac{1}{\pi}$. So the tangent line passes through the point with polar coordinates $\left(\frac{1}{\pi}, \frac{\pi}{2}\right)$. We convert this to Cartesian coordinates: $x = r \cdot \cos\theta = \frac{1}{\pi} \cdot \cos\frac{\pi}{2} = \frac{1}{\pi} \cdot 0 = 0$ and $y = r \cdot \sin\theta = \frac{1}{\pi} \cdot \sin\frac{\pi}{2} = \frac{1}{\pi} \cdot 1 = \frac{1}{\pi}$.

   So, an equation for a line with slope $\frac{2}{\pi}$ that passes through Cartesian coordinate $\left(0, \frac{1}{\pi}\right)$ is $y - \frac{1}{\pi} = \frac{2}{\pi}x$

3. Answer: slope $= -\frac{5}{\sqrt{3}}$ and an equation for the tangent line is $y - \frac{\sqrt{3}}{4} = \left(-\frac{5}{\sqrt{3}}\right)\left(x + \frac{1}{4}\right)$.

   Explanation: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[(2+3\cos\theta)\cdot\sin\theta]}{\frac{d}{d\theta}[(2+3\cos\theta)\cdot\cos\theta]} = \frac{\cos\theta(2+3\cos\theta) - 3\sin^2\theta}{(2+3\cos\theta)\cdot(-\sin\theta) + \cos\theta(-3\sin\theta)} = \frac{\cos\theta(2+3\cos\theta) - 3\sin^2\theta}{-2\sin\theta(1+3\cos\theta)}$.

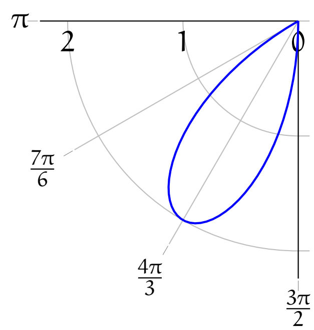
   Substituting $\theta = \frac{2\pi}{3}$: $\frac{dy}{dx} = \frac{\cos\frac{2\pi}{3}\left(2 + 3\cos\frac{2\pi}{3}\right) - 3\sin^2\frac{2\pi}{3}}{-2\sin\frac{2\pi}{3}\left(1 + 3\cos\frac{2\pi}{3}\right)} = \frac{\left(-\frac{1}{2}\right)\left(2 + 3\left(-\frac{1}{2}\right)\right) - 3\left(\frac{\sqrt{3}}{2}\right)^2}{-2\left(\frac{\sqrt{3}}{2}\right)\left(1 + 3\left(-\frac{1}{2}\right)\right)} = \frac{\left(-\frac{1}{2}\right)\left(2 - \frac{3}{2}\right) - 3\left(\frac{3}{4}\right)}{-\sqrt{3}\left(1 - \frac{3}{2}\right)} = \frac{\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \frac{9}{4}}{-\sqrt{3}\left(-\frac{1}{2}\right)} = \frac{-\frac{1}{4} - \frac{9}{4}}{\frac{\sqrt{3}}{2}} = -\frac{\frac{10}{4}}{\frac{\sqrt{3}}{2}} = -\frac{10 \cdot 2}{4 \cdot \sqrt{3}} = -\frac{5}{\sqrt{3}}$

   To write an equation for a tangent line, we need a Cartesian point. First, we find r at $\theta = \frac{2\pi}{3}$: $r = 2 + 3\cos\frac{2\pi}{3} = 2 + 3\left(\frac{-1}{2}\right) = 2 - \frac{3}{2} = \frac{1}{2}$. So the tangent line passes through polar coordinate $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$. We convert this to Cartesian coordinates: $x = r\cos\theta = \frac{1}{2}\cos\frac{2\pi}{3} = \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}$ and $y = r\sin\theta = \frac{1}{2}\sin\frac{2\pi}{3} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$.

   So, an equation with slope $-\frac{\sqrt{5}}{3}$ that passes through the Cartesian coordinate $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ is: $y - \frac{\sqrt{3}}{4} = \left(-\frac{5}{\sqrt{3}}\right)\left(x + \frac{1}{4}\right)$

Answer to Exercise 3 (on page 11)

We know the area of the circle is $\pi r^2 = \pi(2)^2 = 4\pi$. To find the area of the shaded regions, we need to subtract the area of the trefoil from the area of the circle. The trefoil has three equal areas. We can find the area of the leaf that is formed on the interval $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$ (see figure below).



The area of one leaf of the trefoil is given by $\frac{1}{2}\int_{\pi/6}^{\pi/2} [2\cos 3\theta]^2 \, d\theta$. Using a calculator, the area of one leaf is ≈ 1.0472. The area of the circle is given by $\pi r^2 = \pi(2)^2 \approx 12.5664$. The area of the shaded region is the area of the circle minus three times the area of a single leaf: $12.5664 - 3 \cdot 1.0472 = 9.4248 \approx 9.425$.

Answer to Exercise 4 (on page 12)

1. Answer: $A = e^{\pi/8}\left(e^{\pi/8} - 1\right)$

   Explanation: $A = \frac{1}{2}\int_{\pi/4}^{\pi/2} \left[e^{\theta/2}\right] , d\theta = \frac{1}{2} \cdot 2 \left[e^{\theta/2}\right]_{\theta=\pi/4}^{\theta=\pi/2} = e^{\pi/4} - e^{\pi/8} = e^{\pi/8}\left(e^{\pi/8} - 1\right) \approx 0.712$

2. Answer: The area is $\frac{1}{2}\left[e^{\pi/2} - e^{\pi/4}\right] \approx 1.309$

   Explanation: $A = \frac{1}{2}\int_{\pi/4}^{\pi/2} \left[e^{\theta/2}\right]^2 d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} e^{\theta} \, d\theta = \frac{1}{2} e^{\theta}\Big|_{\theta=\pi/4}^{\theta=\pi/2} = \frac{1}{2}\left[e^{\pi/2} - e^{\pi/4}\right] \approx 1.309$

3. Answer: $A = \frac{41}{4}\pi \approx 32.201$

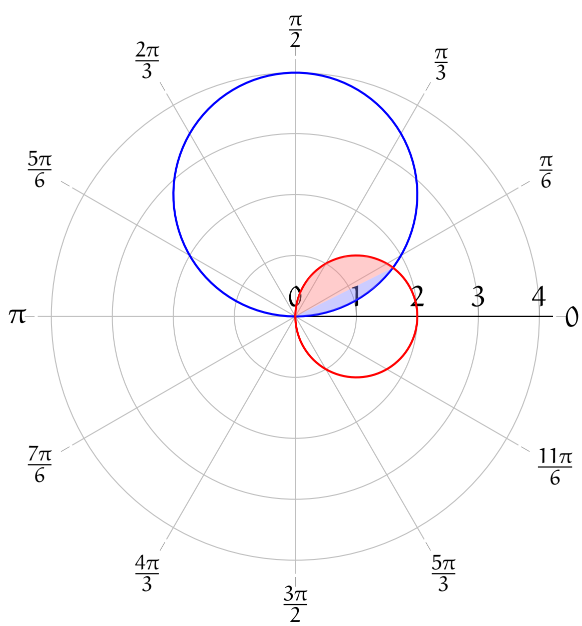
   Explanation: $A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} [4 + 3\sin\theta]^2 \, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[16 + 24\sin\theta + 9\sin^2\theta\right] \, d\theta =$

$\int_{-\pi/2}^{\pi/2} 8\,d\theta + 12\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta + \frac{9}{2}\int_{-\pi/2}^{\pi/2} \sin^2\theta\,d\theta = [8\theta]_{\theta=-\pi/2}^{\theta=\pi/2} + 12\,[-\cos\theta]_{\theta=-\pi/2}^{\theta=\pi/2} + \frac{9}{2}\int_{-\pi/2}^{\pi/2} \frac{1-\cos 2\theta}{2}\,d\theta = 8\left[\left(\frac{\pi}{2}\right) - \left(\frac{-\pi}{2}\right)\right] + 12\left[\left(-\cos\frac{\pi}{2}\right) - \left(-\cos\frac{-\pi}{2}\right)\right] + \frac{9}{4}\int_{-\pi/2}^{\pi/2} 1\,d\theta - \frac{9}{4}\int_{-\pi/2}^{\pi/2} \cos 2\theta\,d\theta = 8\pi + 12\,(0-0) + \frac{9}{4}\,[\theta]_{\theta=-\pi/2}^{\theta=\pi/2} - \frac{9}{4}\left[\frac{1}{2}\sin 2\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2} = 8\pi + \frac{9}{4}\left[\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\right] - \frac{9}{8}\left[\sin\left(2\cdot\frac{\pi}{2}\right) - \sin\left(2\cdot -\frac{\pi}{2}\right)\right] = 8\pi + \frac{9}{4}\pi - \frac{9}{8}\,[\sin(\pi) - \sin(-\pi)] = \frac{41}{4}\pi - \frac{9}{8}\,[0 - (-0)] = \frac{41}{4}\pi \approx 32.201$

Answer to Exercise 5 (on page 13)

Answer: The area between the circles is approximately 0.96174.

Explanation: Examining the graph, we see that the region we are interested in is the area within $r = 4\sin\theta$ from $\theta = 0$ to $\theta = \theta_i$ plus the area within $r = 2\cos\theta$ from $\theta = \theta_i$ to $\theta = \frac{\pi}{2}$, where θ_i is the angle where the two curves intersect. Examine the graph below to see why this is true.



Setting the equations equal to each other to find θ_i:

$$4\sin\theta_i = 2\cos\theta_i$$

$$\frac{\sin\theta_i}{\cos\theta_i} = \tan\theta_i = \frac{2}{4}$$

$$\theta_i = \arctan 1/2 \approx 0.464$$

So, the total area between the circles is:

$$\begin{aligned}
&\frac{1}{2}\int_0^{\theta_i} [4\sin\theta]^2 \, d\theta + \frac{1}{2}\int_{\theta_i}^{\pi/2} [2\cos\theta]^2 \, d\theta \\
&= 8\int_0^{\theta_i} \sin^2\theta \, d\theta + 2\int_{\theta_i}^{\pi/2} \cos^2\theta \, d\theta \\
&= 8\int_0^{\theta_i} \frac{1}{2}(1-\cos 2\theta) \, d\theta + 2\int_{\theta_i}^{\pi/2} \frac{1}{2}(1+\cos 2\theta) \, d\theta \\
&= 4\int_0^{\theta_i} (1-\cos 2\theta) \, d\theta + \int_{\theta_i}^{\pi/2} (1+\cos 2\theta) \, d\theta \\
&= 4\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\theta_i} + \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=\theta_i}^{\theta=\pi/2} \\
&= 4\left[(\theta_i - 0) - \frac{1}{2}(\sin 2\theta_i - \sin 0)\right] + \left[\left(\frac{\pi}{2} - \theta_i\right) + \frac{1}{2}(\sin\pi - \sin 2\theta_i)\right] \\
&= 4\left[\theta_i - \frac{1}{2}\sin 2\theta_i\right] + \frac{\pi}{2} - \theta_i - \frac{1}{2}\sin 2\theta_i \\
&= 4\theta_i - 2\sin 2\theta_i + \frac{\pi}{2} - \theta_i - \frac{1}{2}\sin 2\theta_i = 3\theta_i - \frac{5}{2}\sin 2\theta_i + \frac{\pi}{2}
\end{aligned}$$

Substituting $\theta_i = \arctan 1/2 \approx 0.464$:

$$= 3(0.464) - \frac{5}{2}\sin 0.927 + \frac{\pi}{2} \approx 0.96174$$

Index