Chapter 1

Slope Fields

While separable differential equations are solvable, most differential equations are not separable. And, in fact, it's impossible to obtain an explicit formula as a solution to most differential equations. How do computers solve these, then? They start with a given quantity (usually initial conditions) and perform many small calculations to estimate the behavior of the solution. We can do this graphically with slope fields (also called direction fields), which allow us to visualize the family of solutions to the differential equation.

1.1 Drawing Slope Fields

When a differential equation is in the form

$$y' = f(x, y)$$

we can use the coordinates (x, y) to determine the slope of a solution to the differential equation at that coordinate. Take $y' = x + y$ as an example. According to this differential equation, a solution that passes through the point $(1, 1)$ would have a slope of 2. We can represent this with a small tick of slope 2 at the $(1, 1)$ (see figure 1.1).

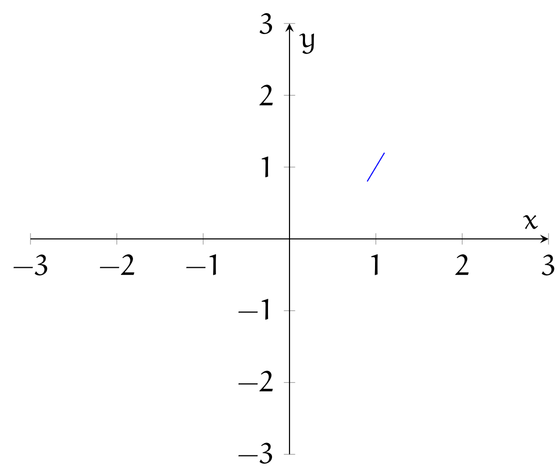


Figure 1.1: A solution to $y' = x + y$ that passes through $(1, 1)$ will have a slope of 2 at that point

Continuing, we want to choose coordinates that are easy to determine the slope. Notice that $y' = 0$ when $-x = y$, so let's go ahead and fill those ticks in (see figure 1.2):

We can repeat this process for all the coordinates shown, resulting in a slope field (see figure 1.3).

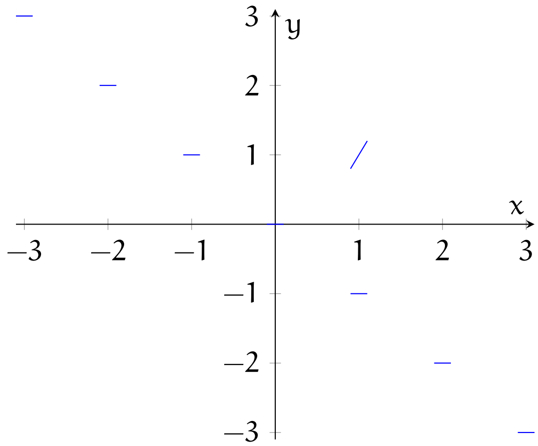


Figure 1.2: Solutions to $y' = x + y$ that lie on the line $y = -x$ will have a slope of 0.

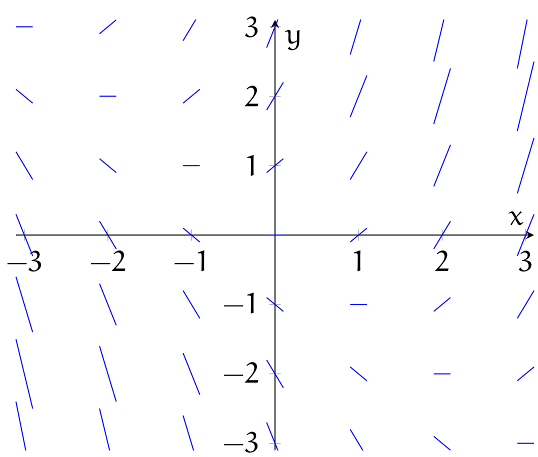


Figure 1.3: Slope field of $y' = x + y$

1.2 Sketching solutions on slope fields

If you are given an initial condition or a known point in the solution to the differential equation, you can begin sketching a curve on the slope field. Start at the given point and draw parallel to the nearby slopes. For example, suppose we know that particular solution to $y' = x + y$ passes through the point $(1, 0)$. Begin by extending the dash at $(1, 0)$ (see figure 1.4), changing the slope of your sketched solution to be approximately parallel to the nearby slopes (see figure 1.5).

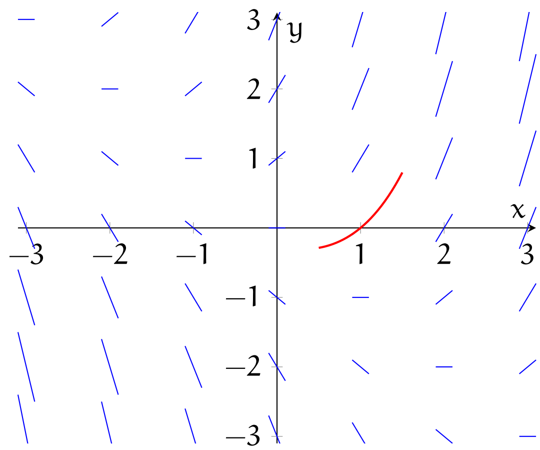


Figure 1.4: To begin sketching a solution to the differential equation, start at the point given as part of the solution

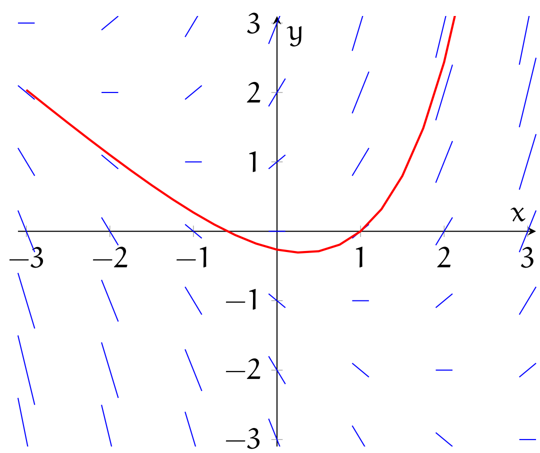


Figure 1.5: To sketch a solution to the differential equation, draw a function parallel to the nearby slopes that passes through the given point in the particular solution

While this method doesn't yield an exact, formulaic solution to the differential equation, it does allow us to visualize solutions and generally describe the behavior of any solutions. Sketching solutions in this way is logically similar to Euler's method for finding numerical approximations of solutions to differential equations, which we will discuss more in the next chapter.

1.3 Example: Application of Differential Equations to Electronics

Think back to the chapter on DC circuits. You learned that Ohm's Law relates voltage (electromotive force), current, and resistance for simple DC circuits:

$$V = IR$$

Simple resistors have a constant resistance, so once the voltage source (battery) is connected, the current is constant. There are other electronic components, such as inductors and capacitors, that behave differently. When current changes in an inductor, a voltage drop is induced across the inductor. This is described by the differential equation:

$$V = -L\frac{dI}{dt}$$

Where L is inductance, measured in henries (H), of the inductor. Consider, then, a circuit consisting of a constant-voltage battery, a fixed resistor, and an inductor (shown in figure 1.6). Since Kirchoff's Law states that the sum of the voltage drops across each component must equal the voltage supplied by the battery, we can write a differential equation to describe the circuit:

$$V = L\frac{dI}{dt} + RI$$

Where the current, I, is a function of time, t.

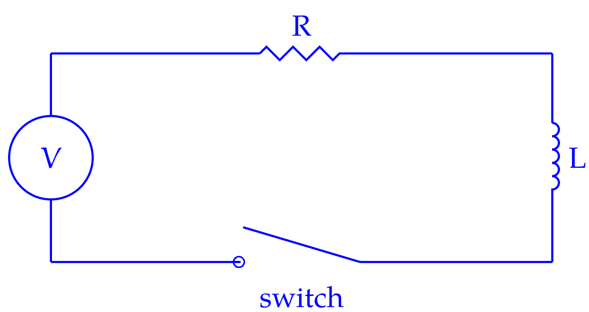


Figure 1.6: A simple circuit with a battery, resistor, inductor, and switch

Example: If the resistor is 12Ω, the inductance is $4H$, and the battery supplies a constant voltage of $60V$:

1. Draw a slope field for the differential equation describing the current in the circuit.
2. Describe the expected behavior of the current over a long period of time.
3. Identify any equilibrium solutions.
4. If the initial current at $t = 0$ is $I(0) = 0$, sketch the particular solution to the differential equation on the slope field.

Solution: Substituting the given values into the differential equation and rearranging to isolate $\frac{dI}{dt}$, we get $\frac{dI}{dt} = 15 - 3I$. Notice that the current is not dependent on time. When the slope is only dependent on the value of the function (as in this case), we call this an **autonomous differential equation**. This means that the slope will be the same of all values of t for a given I. The slope field is shown in figure 1.7.

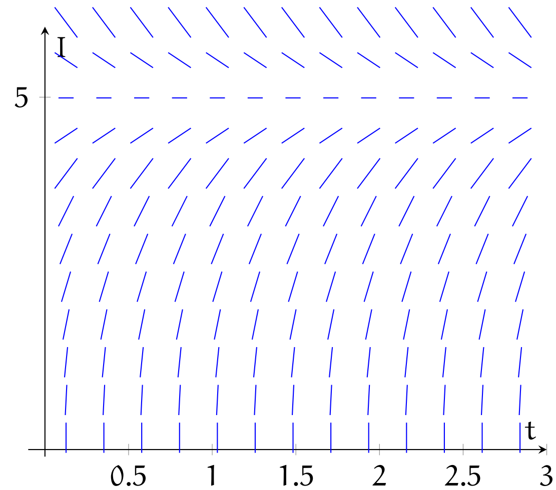


Figure 1.7: Slope field for the differential equation $\frac{dI}{dt} = 15 - 3I$

Examining the slope field, we see that the solutions tend towards $I(t) = 5$, which suggests that over an extended period of time, the current will approach 5 amperes. Similarly, if the initial current were 5 amperes, then the current would be constant at 5 amperes. Therefore, $I(t) = 5$ is an equilibrium solution. A sketch of the solution with $I(0) = 0$ is shown in figure 1.8.

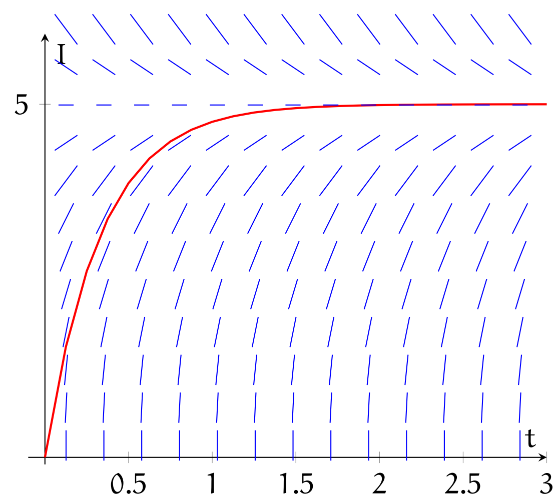


Figure 1.8: Slope field for the differential equation $\frac{dI}{dt} = 15 - 3I$

1.4 Practice

Exercise 1

Sketch the slope field for the differential equation $y' = x + y^2$. Use your slope field to sketch a solution that passes through the point $(0, 0)$.

Working Space

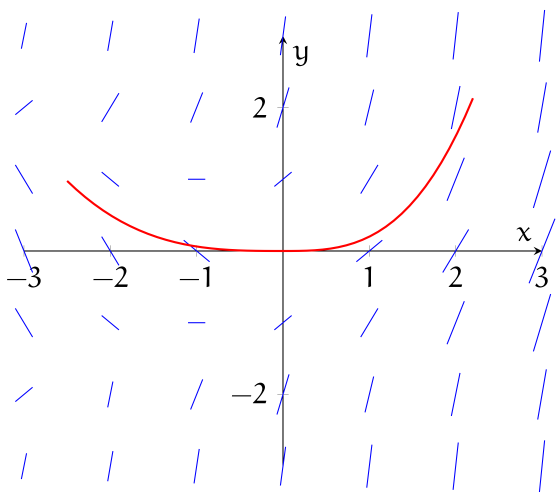
Answer on Page 7

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)



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