

## CHAPTER 1

# Series

When writing a number with an infinite decimal, such as the Golden Ratio (also known as the Golden Number):

$$\phi = 1.618033988 \dots$$

The decimal system means we can rewrite the Golden Ratio (or any irrational number) as an infinite sum:

$$\phi = 1 + \frac{6}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{0}{10^4} + \frac{3}{10^5} + \dots$$

You might recall from the chapter on Riemann Sums that we can represent the addition of many (or infinite) with big sigma notation:

$$\sum_{i=1}^n a_i$$

where  $i$  is the index as discussed in Sequences and  $n$  is the number of terms. For infinite sums,  $n = \infty$ .

### 1.1 Partial Sums

Let's quickly define a *partial sum*. A partial sum is where we only look at the first  $n$  terms of a series. For the general series,  $\sum_{i=1}^n a_i$ , the partial sums are:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\dots$$

$$s_n = a_1 + a_2 + \dots + a_n = \sum_{i=1}^n a_i$$

**Example:** A series is given by  $\sum_{i=1}^{\infty} (-\frac{3}{4})^i$ . What is the value of the partial sum  $s_4$ ?

**Solution:**  $s_4$  is the sum of the first 4 terms:

$$\begin{aligned} & (-\frac{3}{4})^1 + (-\frac{3}{4})^2 + (-\frac{3}{4})^3 + (-\frac{3}{4})^4 \\ &= \frac{-3}{4} + \frac{9}{16} + \frac{-27}{64} + \frac{81}{256} = \frac{-75}{256} \end{aligned}$$

## 1.2 Reindexing

Sometimes it is necessary to re-index series. This means changing what  $n$  the series starts at. In general,

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+1}^{\infty} a_{n-1} \text{ and } \sum_{n=i}^{\infty} a_n = \sum_{n=i-1}^{\infty} a_{n+1}$$

In other words, to increase the index by 1, you need to replace  $n$  with  $(n - 1)$  and to decrease the index by 1, you need to replace  $n$  with  $(n + 1)$ . Let's visualize why this is true (see figure 1.1). Notice that for each series, the terms are the same. This is similar to shifting functions: to move the function to the left on the  $x$ -axis, you plot  $f(x + 1)$ , and to move it to the right,  $f(x - 1)$ .

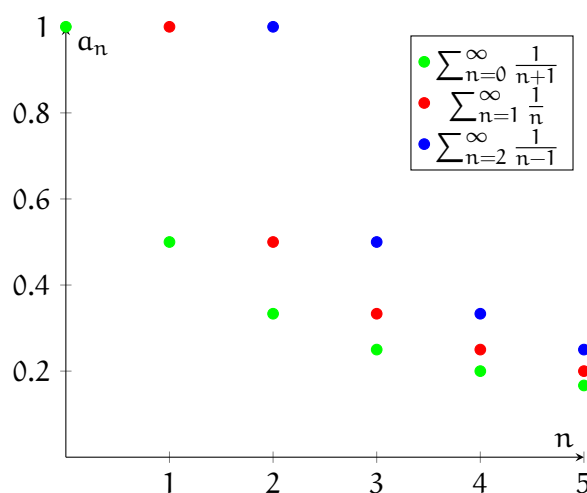


Figure 1.1:  $\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$

We can also prove each reindexing rule mathematically. Recall that

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

We also know that

$$\sum_{n=2}^{\infty} a_{n-1} = a_{2-1} + a_{3-1} + a_{4-1} + \dots = a_1 + a_2 + a_3 + \dots$$

Therefore,  $\sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} a_{n-1}$ .

Similarly,

$$\sum_{n=0}^{\infty} a_{n+1} = a_{0+1} + a_{1+1} + a_{2+1} + \dots = a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$$

**Example:** Reindex the series  $\sum_{n=3}^{\infty} \frac{n+1}{n^2-2}$  to begin with  $n = 1$ .

**Solution:** We are decreasing the index, so we will use  $\sum_{n=i-1}^{\infty} a_{n+1} = \sum_{n=i}^{\infty} a_n$ . We will apply this rule twice, to decrease the index from 3 to 1:

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(n+1)+1}{(n+1)^2-2} &= \sum_{n=2}^{\infty} \frac{n+2}{(n+1)^2-2} \\ \sum_{n=1}^{\infty} \frac{(n+1)+2}{[(n+1)+1]^2-2} &= \sum_{n=1}^{\infty} \frac{n+3}{(n+2)^2-2} \end{aligned}$$

It is easier and faster to be able to reindex a series by more than one step at a time. Using the example above, we can write an even more general rule for reindexing:

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+j}^{\infty} a_{n-j}$$

where  $i$  and  $j$  are integers. (Then, to decrease the index, you would choose a  $j$  such that  $j < 0$ .)

## 1.3 Convergent and Divergent Series

Just like sequences, series can also be convergent or divergent. Consider the series  $\sum_{i=1}^{\infty} i$ . Given what you already know about the meaning of "convergent" and "divergent", guess whether  $\sum_{i=1}^{\infty} i$  is convergent or divergent.

Let's determine the first few partial sums of the series (shown graphically in figure 1.2):

n	Terms	Partial Sum
1	1	1
2	1+2	3
3	1+2+3	6
4	1+2+3+4	10

As you can see, as  $n$  increases, the value of the partial sum increases without approaching a particular value. We can also see that the value of the first  $n$  terms summed together is  $\frac{n(n+1)}{2}$ . This means that as  $n$  approaches  $\infty$ , the sum also approaches  $\infty$  and the series is divergent.

Obviously, for a series to not become overly large, the values of the terms should decrease as  $i$  increases (that is, each subsequent term is smaller than the one before it). Take the series  $\sum_{i=1}^{\infty} \frac{1}{2^i}$ . As  $i$  increases,  $\frac{1}{2^i}$  decreases. Let's look at the first few partial sums of this series (shown graphically in figure 1.3):

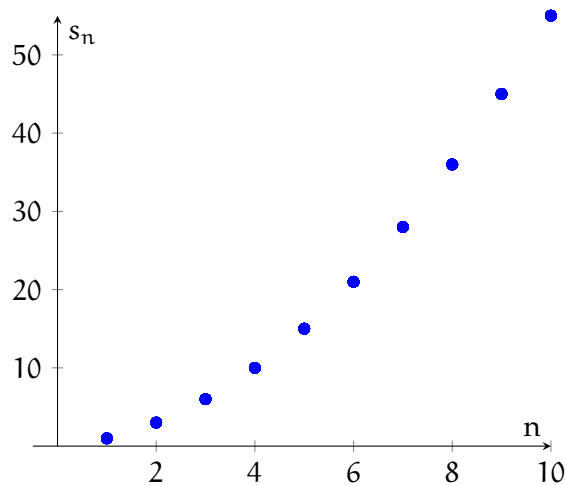


Figure 1.2: For the divergent series  $\sum_{i=1}^n i$ , the value of the partial sum increases to infinity as  $n$  increases

n	Terms	Partial Sum
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2} + \frac{1}{4}$	$\frac{3}{4}$
3	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$\frac{7}{8}$
4	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	$\frac{15}{16}$

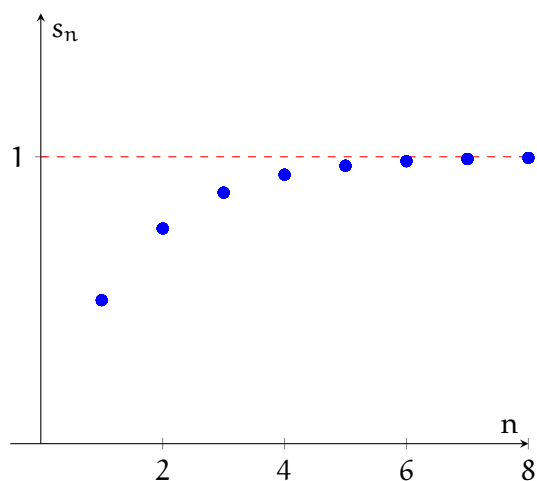


Figure 1.3: For the convergent series  $\sum_{i=1}^n \frac{1}{2^i}$ , the value of the partial sum approaches 1 as  $n$  increases

Do you see the pattern? The  $n^{\text{th}}$  partial sum is equal to  $\frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$ . And as  $n$  approaches  $\infty$ , the partial sum approaches 1. The series  $\sum_{i=1}^{\infty} \frac{1}{2^i}$  is convergent.

Let's define the sequence  $\{s_n\}$ , where  $s_n$  is the  $n^{\text{th}}$  partial sum of a series:

$$s_n = \sum_{i=1}^n a_i$$

If the sequence  $\{s_n\}$  is convergent and  $\lim_{n \rightarrow \infty} s_n$  exists, then the series  $\sum_{i=1}^{\infty} a_i$  is also convergent. And if the sequence  $\{s_n\}$  is divergent, then the series  $\sum_{i=1}^{\infty} a_i$  is also divergent.

**Example:** Is the harmonic series,  $\sum_{n=1}^{\infty} \frac{1}{n}$  convergent or divergent?

**Solution:** You may think that the series is convergent, since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . Let's see if we can confirm this. We begin by looking at the partial sums  $s_2$ ,  $s_4$ ,  $s_8$ , and  $s_{16}$ :

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{3}{4}$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{7}{8}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) > \\ 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) = 1 + \frac{15}{16}$$

Notice that, in general,  $s_{2^n} > 1 + \frac{n}{2}$  for  $n > 1$ . Taking the limit as  $n \rightarrow \infty$ , we see that  $\lim_{n \rightarrow \infty} s_{2^n} > \lim_{n \rightarrow \infty} 1 + \frac{n}{2} = \infty$ . Therefore,  $s_{2^n}$  also approaches  $\infty$  as  $n$  gets larger and the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

This example shows a very important point: A series whose terms decrease to zero as  $n$  gets large is not necessarily convergent. What we can say, though, is that if the limit as  $n$  approaches infinity of the terms of a series does not exist or is not zero, then the series is divergent (i.e., not convergent). This is called the **Test for Divergence**, and we will explore it further in the next chapter.

### 1.3.1 Properties of Convergent Series

We just saw that if  $\lim_{n \rightarrow \infty} a_n \neq 0$  then the series  $\sum_{n=1}^{\infty} a_n$  diverges. The contrapositive statement gives a property of convergent series:

$$\text{If the series } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n \rightarrow \infty} a_n = 0$$

If a series is made of other convergent series, it may be convergent. Recall, if a series is convergent, this means the  $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i = L$ . By the properties of limits, we can also say that the series multiplied by a constant is convergent:

$$\sum_{n=1}^{\infty} c a_n = c \cdot L = c \sum_{n=1}^{\infty} a_n$$

Suppose there is another convergent series such that  $\lim_{n \rightarrow \infty} \sum_{i=1}^n b_i = M$ . In this case, the sum of those series is also convergent. That is:

$$\sum_{n=1}^{\infty} (a_n + b_n) = L + M = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Similarly, the difference of the series is convergent:

$$\sum_{n=1}^{\infty} (a_n - b_n) = L - M = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

## 1.4 Geometric Series

A geometric series is the sum of a geometric sequence, and has the form:

$$\sum_{n=1}^{\infty} ar^n \text{ or } \sum_{n=1}^{\infty} ar^{n-1}$$

Where  $a$  is some constant and  $r$  is the common ratio. For  $\sum_{n=1}^{\infty} ar^{n-1}$ ,  $a$  is also the first term.

**Example:** Write the series  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$  in sigma notation.

**Solution:** We see that the first term is  $a = 1$  and the common ratio is  $\frac{1}{2}$ , so we can write the series:

$$\sum_{n=1}^{\infty} 1\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

When are geometric series convergent? First, let's consider the case where  $r = 1$ . If this is true, then  $s_n = a + a + a + \dots + a = na$ . As  $n$  approaches  $\infty$ , the sum will approach  $\pm\infty$  (depending on whether  $a$  is positive or negative), and the series is divergent.

When  $r \neq 1$ , we can write  $s_n$  and  $rs_n$ :

$$s_n = a + ar + ar^2 + \dots + ar^{n-1}$$

$$rs_n = ar + ar^2 + ar^3 + \dots + ar^n$$

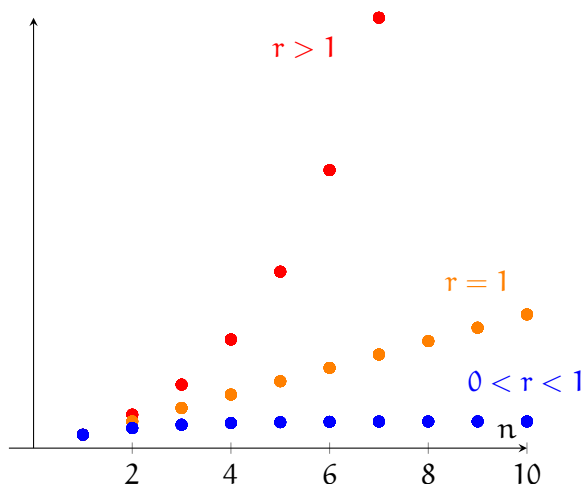


Figure 1.4: Geometric sequences are divergent if  $r \geq 1$

Subtracting  $rs_n$  from  $s_n$ , we get:

$$\begin{aligned} s_n - rs_n &= (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) \\ &= a - ar^n \end{aligned}$$

Solving for  $s_n$ , we find:

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

We take the limit as  $n \rightarrow \infty$  to determine for what values of  $r$  the series converges:

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} \\ &= \lim_{n \rightarrow \infty} \left[ \frac{a}{1 - r} - \frac{ar^n}{1 - r} \right] = \frac{a}{1 - r} - \left( \frac{a}{1 - r} \right) \lim_{n \rightarrow \infty} r^n \end{aligned}$$

This introduces the question: When is  $\lim_{n \rightarrow \infty} r^n$  convergent? From the sequences chapter, we know this limit converges if  $|r| < 1$  (that is,  $-1 < r < 1$ ). If this is true, then  $\lim_{n \rightarrow \infty} r^n = 0$  and

$$\lim_{n \rightarrow \infty} s_n = \frac{a}{1 - r}$$

(see figures 1.4 and 1.5 for a visual)

**Example:** Find the sum of the geometric series given by  $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \cdots$ .

**Solution:** The first term is  $a = 2$ , and each common ratio is  $r = -\frac{1}{3}$ . Since  $|r| < 1$ , we know that the series converges. We can calculate the value of the sum using the geometric series

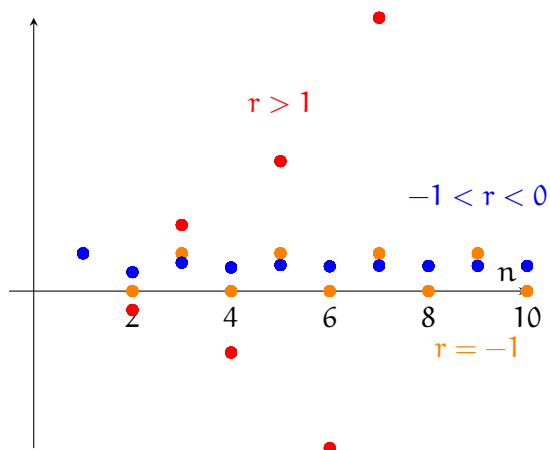


Figure 1.5: Geometric sequences are divergent if  $r \leq 1$ . Notice that for  $r = -1$ , the partial sums alternate between the initial term and zero.

formula:

$$\sum_{i=1}^{\infty} a(r)^{i-1} = \frac{a}{1-r}$$

$$\sum_{i=1}^{\infty} 2\left(\frac{-1}{3}\right)^{i-1} = \frac{2}{1-\frac{-1}{3}} = \frac{2}{\frac{4}{3}} = \frac{6}{4} = 1.5$$

We can confirm this graphically (see figure 1.6). You can also write out the first several partial sequences. You should find the sums approach 1.5 as  $n$  increases.

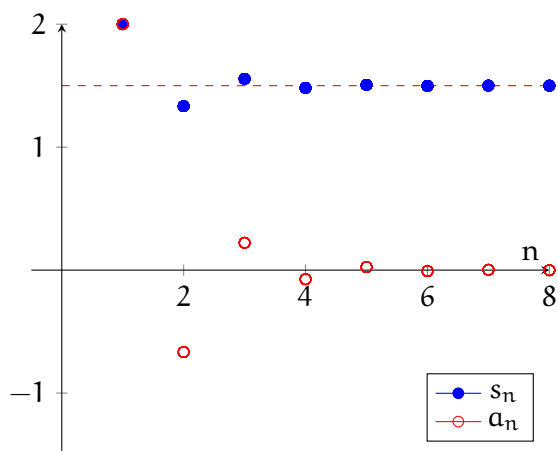


Figure 1.6: the  $n^{\text{th}}$  term and partial sums of  $\sum_{i=1}^n 2\left(\frac{-1}{3}\right)^{i-1}$

**Example:** What is the value of  $\sum_{n=1}^{\infty} 2^{2n}5^{1-n}$

**Solution:** The key here is to re-write the series in the form  $\sum_{n=1}^{\infty} ar^{n-1}$  so we can use the



fact that convergent geometric series sum to  $\frac{a}{1-r}$ .

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{2n} 5^{1-n} &= \sum_{n=1}^{\infty} (2^2)^n \left(\frac{1}{5}\right)^{n-1} \\ &= \sum_{n=1}^{\infty} 4 \cdot (4)^{n-1} \left(\frac{1}{5}\right)^{n-1} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^{n-1}\end{aligned}$$

Which is in the form  $\sum_{n=1}^{\infty} ar^{n-1}$  with  $a = 4$  and  $r = \frac{4}{5}$ . Since  $|r| < 1$ , the series converges to

$$\frac{a}{1-r} = \frac{4}{1-\frac{4}{5}} = \frac{4}{\frac{1}{5}} = 20$$

### Exercise 1

Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

1.  $3 - 4 + \frac{16}{3} - \frac{64}{9} + \dots$

2.  $2 + 0.5 + 0.125 + 0.03125 + \dots$

3.  $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

4.  $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

Working Space

Answer on Page 15

### Exercise 2

Find a value of  $c$  such that  $\sum_{n=0}^{\infty} (1 + c)^{-n} = \frac{5}{3}$ .

Working Space

Answer on Page 15

**Exercise 3**

For what values of  $p$  does the series  $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$  converge?

*Working Space*

*Answer on Page 15*

**1.5 p-series**

A  $p$ -series takes the form  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  and converges if  $p > 1$  and diverges if  $p \leq 1$ . We won't prove this here, since it requires the application of a test you will learn about in the next chapter.

**Example** Write the series  $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$ . Is it convergent or divergent?

**Solution:** We see that  $a_n = \frac{1}{\sqrt[3]{n}}$ , so the infinite series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

. We see that this is a  $p$ -series with  $p = \frac{1}{3}$ . Since  $p < 1$ , the series is divergent.

**Exercise 4**

Euler found that the exact sum of the p-series where  $p = 2$  is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

And that the exact sum of the p-series where  $p = 4$  is:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use this and the properties of convergent series to find the sum of each of the following series:

1.  $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$
2.  $\sum_{n=2}^{\infty} \frac{1}{n^2}$
3.  $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$
4.  $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$
5.  $\sum_{n=1}^{\infty} \left(\frac{4}{n^2} + \frac{3}{n^4}\right)$

*Working Space*

*Answer on Page 16*

**Exercise 5**

For what values of  $k$  does the series  $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$  converge?

*Working Space*

*Answer on Page 16*

**1.6 Alternating Series**

An alternating series is one in which the terms alternate between positive and negative. Here is an example:

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

Alternating series are generally of the form

$$a_n = (-1)^n b_n \text{ or } a_n = (-1)^{n-1} b_n$$

Where  $b_n$  is positive (and therefore,  $|a_n| = b_n$ ).

An alternating series is convergent if (i)  $b_{n+1} \leq b_n$  and (ii)  $\lim_{n \rightarrow \infty} b_n = 0$ . In other words, we say that if the absolute value of the terms of a series decrease towards zero, then the series converges. This is called the **Alternating Series Test**.

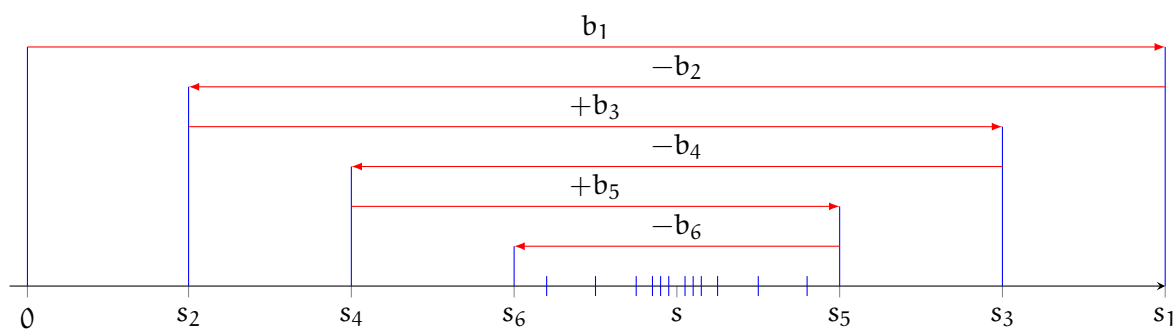


Figure 1.7: As  $n$  increases,  $s_n$  approaches  $s$

**Example:** Is the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  convergent?

**Solution:** The Alternating series test states that an alternating series is convergent if  $|a_{n+1}| < |a_n|$ :

$$\left| \frac{(-1)^{n-1+1}}{n+1} \right| < \left| \frac{(-1)^{n-1}}{n} \right|$$

$$\frac{1}{n+1} < \frac{1}{n}$$

Since  $|a_{n+1}| < |a_n|$  and the series is alternating,  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  is convergent.

### Exercise 6

Test the following alternating series for convergence:

*Working Space*

1.  $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$
2.  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$
3.  $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

*Answer on Page 16*



# Answers to Exercises

## Answer to Exercise 1 (on page 9)

1. We need to identify  $a$  and  $r$ . If we use the form  $\sum_{n=1}^{\infty} ar^{n-1}$ , then  $a = 3$ . To find the common ratio, we can evaluate  $\frac{a_{n+1}}{a_n} = \frac{-4}{3}$ . We can then write the series as  $\sum_{n=1}^{\infty} 3 \left(\frac{-4}{3}\right)^{n-1}$ . In this case,  $r = \frac{-4}{3}$  and  $|r| \geq 1$ , and therefore the series is divergent.
2. Following the process outlined above, we see that  $a = 2$  and  $r = \frac{1}{4}$ . Therefore, the series is  $\sum_{n=1}^{\infty} 2 \left(\frac{1}{4}\right)^{n-1}$ . Since  $|r| < 1$ , the series converges to  $\frac{a}{1-r} = \frac{2}{1-1/4} = \frac{2.4}{3} = \frac{8}{3}$ .
3. We need to rewrite the series into a standard form in order to identify  $a$  and  $r$ :

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4(4)^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{-3}{4}\right)^{n-1}$$

So  $r = \frac{-3}{4}$  and  $|r| < 1$ . Therefore, the series converges to  $\frac{1/4}{1-(-3/4)} = \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}$ .

4. We need to rewrite the series into a standard form in order to identify  $a$  and  $r$ :

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)^n}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)(e^2)^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} e^2 \left(\frac{e^2}{6}\right)^{n-1}$$

Therefore,  $r = \frac{e^2}{6} \approx 1.232$ . Since  $|r| > 1$ , the series diverges.

## Answer to Exercise 2 (on page 9)

We want to rewrite this as a geometric series of the form  $\sum_{n=i}^{\infty} ar^{n-1}$ , so we can use the fact that the sum of a convergent geometric series is  $\frac{a}{1-r}$ .  $\sum_{n=0}^{\infty} (1+c)^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{1+c}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{1+c}\right)^{n-1}$ . This is a geometric series with  $a = 1$  and  $r = \frac{1}{1+c}$ . So, the value of the series is  $\frac{1}{1-\frac{1}{1+c}} = \frac{1}{\frac{c}{1+c}} = \frac{1+c}{c}$ . Setting this equal to  $\frac{5}{3}$  and solving for  $c$ , we find that  $c = \frac{3}{2}$ .

## Answer to Exercise 3 (on page 10)

$-2 < p < 2$  Let's rewrite this geometric series into standard form:  $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n = \sum_{n=1}^{\infty} \frac{p}{2} \left(\frac{p}{2}\right)^{n-1}$  which means  $a = \frac{p}{2}$  and  $r = \frac{p}{2}$ . We know that geometric series converge if  $|r| < 1$ , so we

set up an inequality and solve for  $p$ :

$$\begin{aligned} \left| \frac{p}{2} \right| &< 1 \\ -1 &< \frac{p}{2} < 1 \\ -2 &< p < 2 \end{aligned}$$

### Answer to Exercise 4 (on page 11)

1. Separating the terms, we see that  $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4} = \sum_{n=1}^{\infty} \left( \frac{n^2}{n^4} + \frac{1}{n^4} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{6} + \frac{\pi^4}{90}$
2. Notice that this series starts at  $n = 2$ . By the properties of series, we know that  $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$ . Therefore,  $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \right) - \frac{1}{1^2} = \frac{\pi^2}{6} - 1$
3. We can begin by reindexing this series:  $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2}$ . Similar to the previous problem, we also know that  $\sum_{n=4}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \right) - \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \right) = \frac{\pi^2}{6} - \frac{49}{36}$
4. We can rewrite this series as  $\sum_{n=1}^{\infty} \left( \frac{3}{n} \right)^4 = \sum_{n=1}^{\infty} (3^4) \frac{1}{n^4} = 81 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{81\pi^4}{90} = \frac{9\pi^4}{10}$
5. We can re-write the series as  $\sum_{n=1}^{\infty} \left( \frac{4}{n^2} + \frac{3}{n^4} \right) = \sum_{n=1}^{\infty} \frac{4}{n^2} + \sum_{n=1}^{\infty} \frac{3}{n^4} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} + 3 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{4\pi^2}{6} + \frac{3\pi^4}{90} = \frac{2\pi^2}{3} + \frac{\pi^4}{30}$

### Answer to Exercise 5 (on page 12)

This is a  $p$ -series where  $p = 2k$ . We know that  $p$ -series converge for  $p > 1$ :  $2k > 1 \rightarrow k > \frac{1}{2}$ .

### Answer to Exercise 6 (on page 13)

1. The series is convergent if  $\left| \frac{(-1)^{n+1} 3(n+1)}{4(n+1)-1} \right| < \left| \frac{(-1)^n 3n}{4n-1} \right|$  if  $\frac{3n+3}{4n+4-1} < \frac{3n}{4n-1}$  and if  $\frac{3n+3}{4n+3} < \frac{3n}{4n-1}$  if  $(3n+3)(4n-1) < (3n)(4n+3)$  if  $12n^2 + 12n - 3n - 3 < 12n^2 + 9n$  if  $-3 < 0$  which is true. Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$  is convergent.
2. The series is convergent if  $\left| (-1)^{n+1} \frac{(n+1)^2}{(n+1)^3+1} \right| < \left| (-1)^{n+1} \frac{n^2}{n^3+1} \right|$ , which is true if  $\frac{(n+1)^2}{(n+1)^3+1} < \frac{n^2}{n^3+1}$  if  $(n+1)^2(n^3+1) < (n^2)((n+1)^3+1)$  if  $(n^2+2n+1)(n^3) <$



$(n^2)(n^3 + 3n^2 + 3n + 1 + 1)$  if  $n^5 + 2n^4 + n^3 < n^5 + 3n^4 + 3n^3 + 2n^2$ , which is true for all  $n \geq 1$ . Therefore,  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$  is convergent.

3. The series is convergent if  $|(-1)^{n-1+1} e^{2/(n+1)}| < |(-1)^{n-1} e^{2/n}|$ , which is true if  $e^{2/(n+1)} < e^{2/n}$ , which is true if  $\frac{2}{n+1} < \frac{2}{n}$  which is true for all  $n \geq 1$ . Therefore,  $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$  is convergent.





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