Chapter 1

Double Integrals Over Non-Rectangular Regions

Now that we've seen how to evaluate double integrals over rectangular regions, let's consider non-rectangular regions. Suppose we are interested in the integral of a function, $f(x, y)$, over a region, D, which exists such that it can be bounded by inside a rectangular region, R (see figure 1.1). We can then define a new function:

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

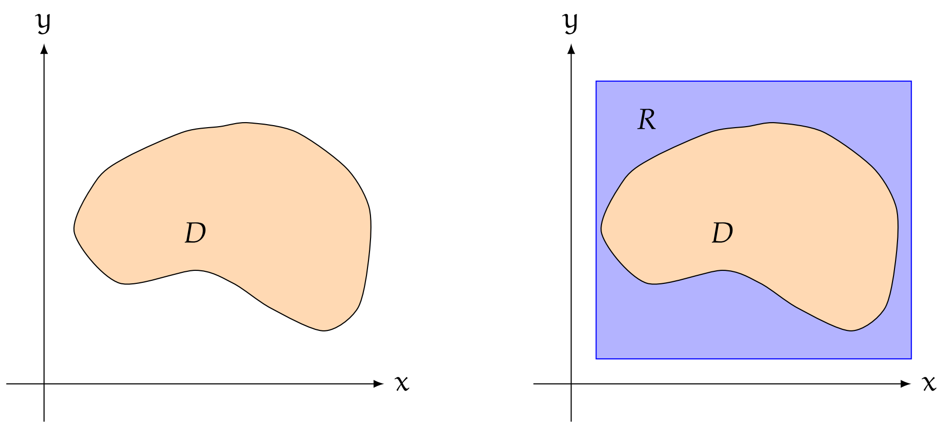


Figure 1.1: We can find a rectangular region, R, that completely encloses D

Then, we can see that:

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA$$

This makes sense intuitively, since integrating over F outside of D doesn't contribute anything to the integral, and the integral of F inside D is equal to the integral of f inside D. In general, there are two types of regions for D. A region is **type I** if it lies between two continuous functions of x and can be defined thusly:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

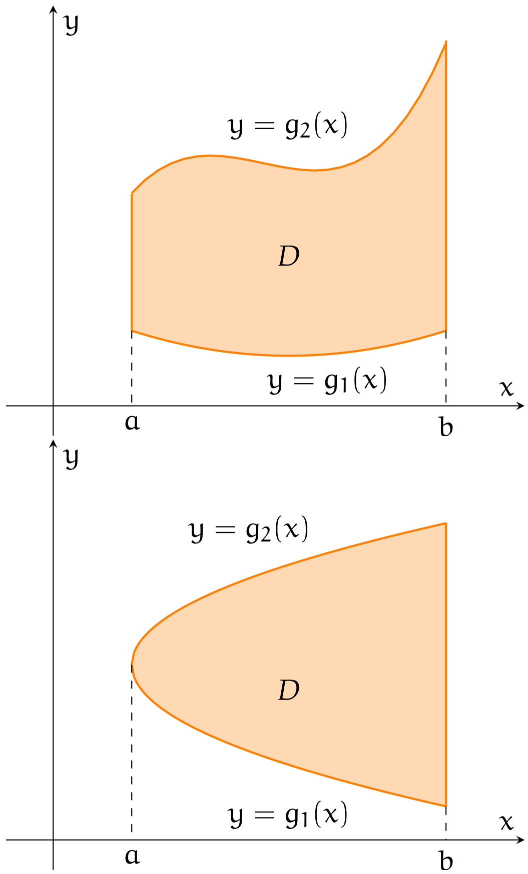


Figure 1.2: Two examples of type I domains

Some type I regions are shown in figure 1.2. To evaluate $\iint_D f(x,y)\,dA$, we begin by choosing a rectangle $R = [a,b] \times [c,d]$ such that D is completely contained in R. We again define $F(x,y)$ such that $F(x,y) = f(x,y)$ on D and $F = 0$ outside of D. Then, by Fubini's theorem:

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

Since $F(x,y) = 0$ when $y \le g_1(x)$ or $y \ge g_2(x)$, we know that:

$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

Substituting this into the iterated integral above, we see that for a type I region $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Another way to visualize the double integral over a type I region is shown in figure 1.3. For any value of $x \in [a, b]$, we know that $g_1(x) \le y \le g_2(x)$. The inner integral represents moving along one blue line from $y = g_1(x)$ to $y = g_2(x)$ and integrating with respect to y. Next, for the outer integral, we integrate with respect to x, which is represented by moving the line from $x = a$ to $x = b$.

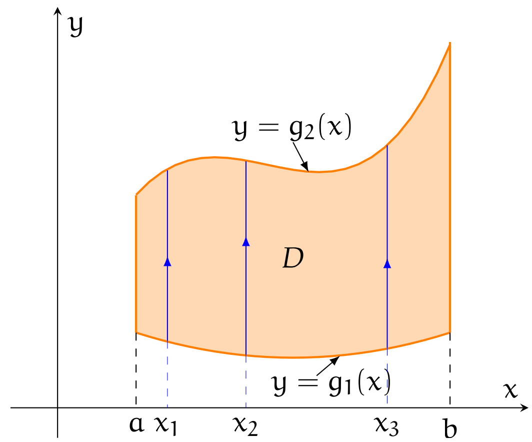


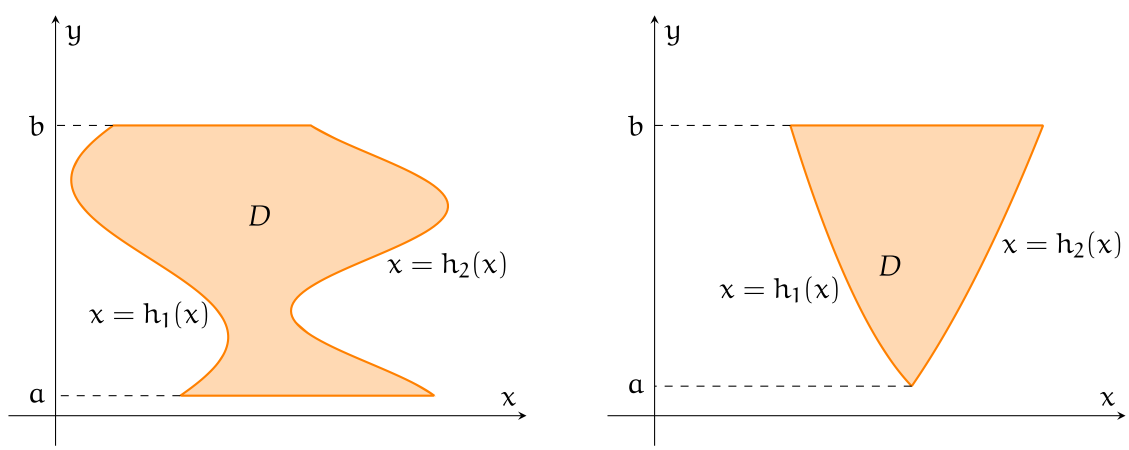
Figure 1.3: On type I domains, for a given value of x, $g_1(x) \le y \le g_2(x)$

A **type II** region is a region such that we can define the limits of x in terms of y (see figure 1.4). In other words, a type II region can be defined as:

$$D = \{(x, y) \mid c \le y \le d, h_1(y0 \le x \le h_2(y))\}$$

In a similar manner to above, we can show that:

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$



(a) Example 1

(b) Example 2

Figure 1.4: Two examples of type II domains

You can annotate type II regions with horizontal lines to show that, for a given y values, all x values in the region are contained in $h_1(y) \le x \le h_2(y)$ (see figure 1.5).

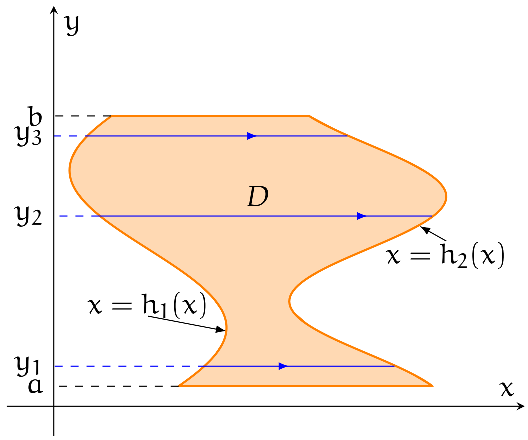


Figure 1.5: On type II domains, for a given value of y, $h_1(y) \le x \le h_2(y)$

1.1 Determining Region Type

Many regions can be described as either type I or type II. Consider the region between the curves $y = \frac{3}{2}(x-1)$ and $y = \frac{1}{2}(x-1)^2$ (see figure 1.6).

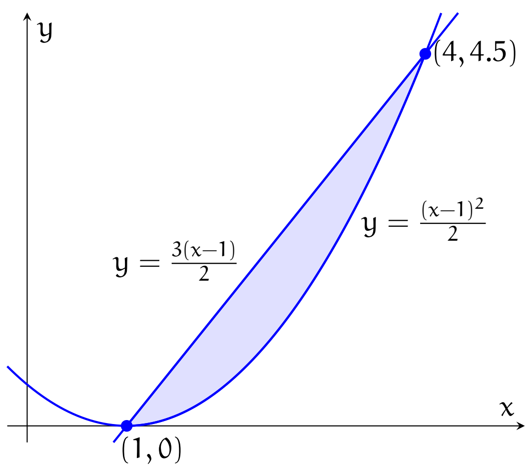
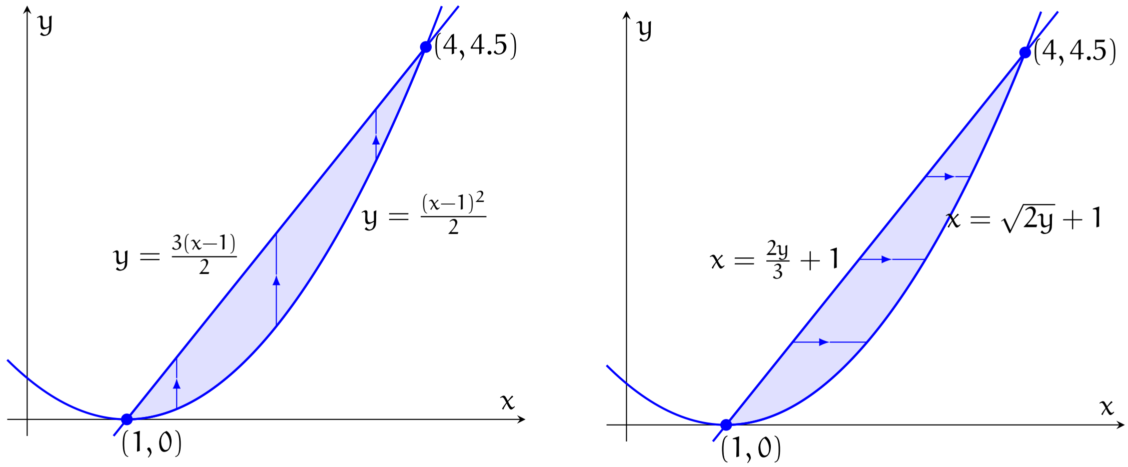


Figure 1.6: The region that lies between $y = \frac{(x-1)^2}{2}$ and $y = \frac{3(x-1)}{2}$ can be classified as type I or type II

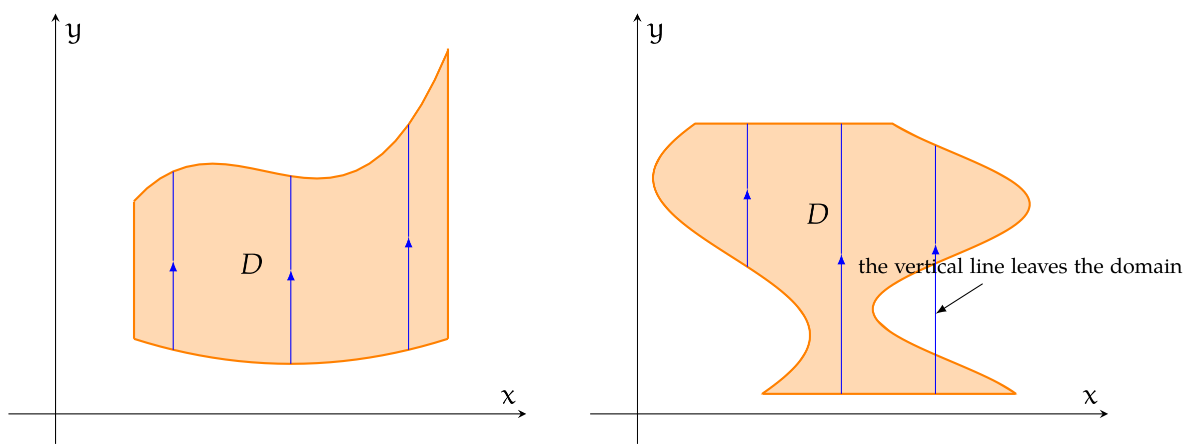
We could classify this as a type I (see figure 1.7a) or a type II domain (see figure 1.7b).



(a) Type I region: $\frac{(x-1)^2}{2} \leq y \leq \frac{3(x-1)}{2}$ (b) Type II region: $\frac{2y}{3} + 1 \leq x \leq \sqrt{2y} + 1$

Figure 1.7: Comparison of Type I and Type II descriptions of the same region.

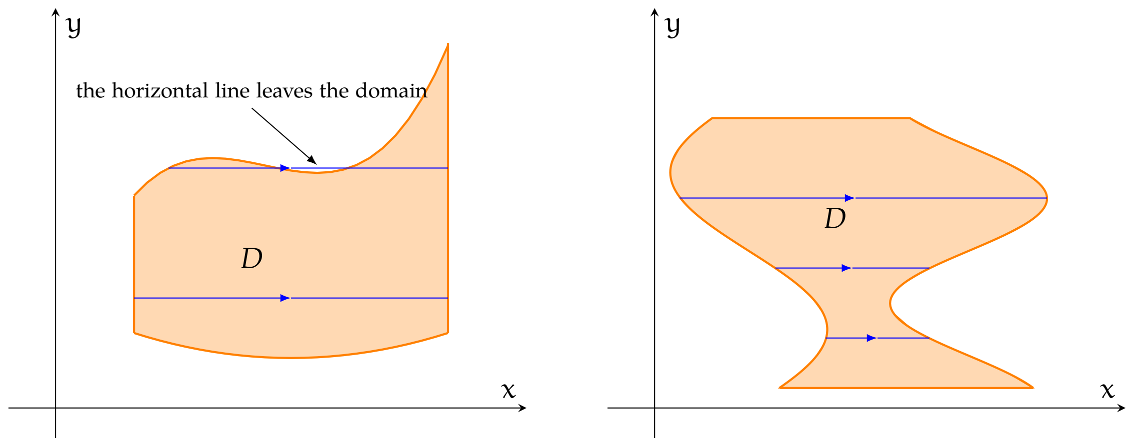
However, not all domains can be classified as both type I or type II. A region can be classified as type I if you can take a vertical line ($x = c$, where c is some number in the domain of the region) and move it across the region without any gaps. Consider the two regions shown in figure **??**. The top is type I, and if you put vertical lines, the line traverses the entire region without leaving it. On the other hand, the lower region is not type I. There are vertical lines where the line exits the lower region before traversing the entire region.



(a) A domain that *is* Type I (b) A domain that is *not* Type I

Figure 1.8: Comparing two regions: one that satisfies the Type I condition and one that does not.

To determine if a region can be classified as type II, you can use a horizontal line. If you can move a horizontal line ($y = c$, where c is in the domain of the region) across the region and the line always traverses the entire region without leaving and re-entering it, then the region is type II (see figure **??**).



(a) A domain that is *not* Type II

(b) A domain that *is* Type II

Figure 1.9: Comparing two regions: one that fails the Type II condition and one that satisfies it.

Example: Evaluate $\iint_D (2x + y)\, dA$, where D is the region bounded by the parabolas $y = 3x^2$ and $y = 2 + x^2$. Region D is shown in figure 1.10.

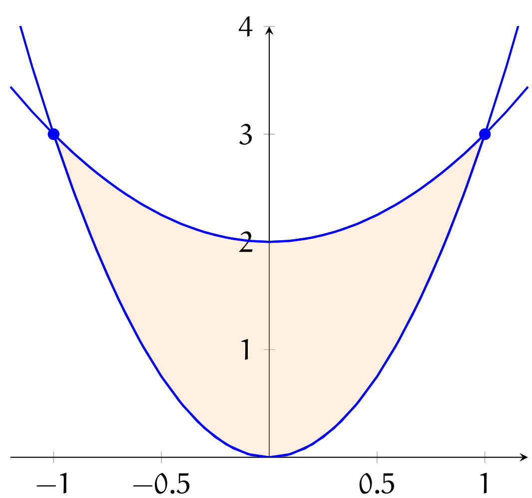


Figure 1.10: Region D is bounded above by $y = 2 + x^2$ and below by $y = 3x^2$

Solution:This is a type I region, since for a given x, $y \in \left[3x^2, 2 + x^2\right]$. We can define region D as $D = \{(x, y) \mid -1 \leq x \leq 1, 3x^2 \leq y \leq 2 + x^2\}$. Therefore,

$$\iint_D (2x + y)\, dA = \int_{-1}^{1} \int_{3x^2}^{2+x^2} (2x + y)\ dy\, dx$$

$$= \int_{-1}^{1} \left[\int_{3x^2}^{2+x^2} 2x \, dy + \int_{3x^2}^{2+x^2} y \, dy \right] dx$$

$$= \int_{-1}^{1} \left[2xy|_{y=3x^2}^{y=2+x^2} + \frac{1}{2} y^2|_{y=3x^2}^{y=2+x^2} \right] dx$$

$$= \int_{-1}^{1} \left[2x \left(2 + x^2 - 3x^2 \right) + \frac{1}{2} \left((2+x^2)^2 - (3x^2)^2 \right) \right] dx$$

$$= \int_{-1}^{1} \left[2 + 4x + 2x^2 - 4x^3 - 4x^4 \right] dx$$

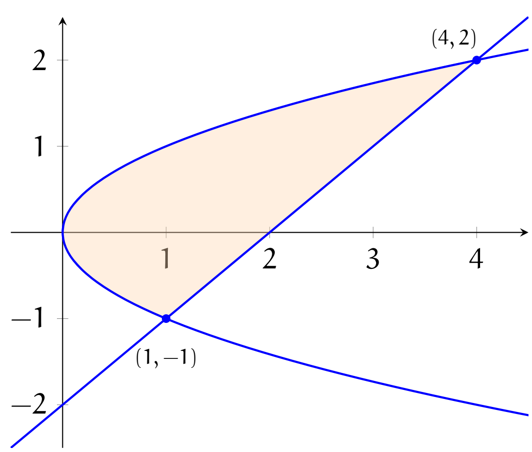
$$= \left[2x + 2x^2 + \frac{2}{3}x^3 - x^4 - \frac{4}{5}x^5 \right]_{x=-1}^{x=1}$$

$$= \left(2 + 2 + \frac{2}{3} - 1 - \frac{4}{5} \right) - \left(-2 + 2 - \frac{2}{3} - 1 + \frac{4}{5} \right)$$

$$= 4 + \frac{4}{3} - \frac{8}{5} = \frac{56}{15}$$

Example: Set up integrals to evaluate $\iint_D xy \, dA$ if D is the region bounded by $y = x - 2$ and $x = y^2$ as both a type I and type II region. Which method will be easier to evaluate? Evaluate the easier double integral.

Solution: Let's begin by visualizing D:



This region could be classified as type I and type II. To set up an integral as if D were a type I region, we need to describe y in terms of x. However, we run into a little problem. For $0 \le x \le 1$, the region is bounded by the parabola $x = y^2$, and for $1 \le x \le 4$, the region is bounded by both the parabola and the line. So, we will have to split the integral into to parts: $0 \le x \le 1$ and $1 \le x \le 4$:

$$\iint_D xy \, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} xy \, dy \, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} xy \, dy \, dx$$

To set up an integral as if D were a type II region, we need to describe x in terms of y. This time, we will not need to split the integral, because $y+2 \leq x \leq y^2$ over the entire region:

$$\iint_D xy\, dA = \int_{-1}^{2} \int_{y+2}^{y^2} xy\, dx\, dy$$

It is easier to evaluate the integral with D as a type II region, since we do not need to split the integral. Evaluating:

$$\int_{-1}^{2} \int_{y+2}^{y^2} xy\, dx\, dy = \int_{-1}^{2} \frac{y}{2} \left[x^2\right]_{x=y+2}^{x=y^2}\, dy$$

$$= \int_{-1}^{2} \frac{y}{2} \left[\left(y^2\right)^2 - (y+2)^2\right]\, dy = \frac{1}{2} \int_{-1}^{2} y \left[y^4 - \left(y^2 + 4y + 4\right)\right]\, dy$$

$$= \frac{1}{2} \int_{-1}^{2} y^5 - y^3 - 4y^2 - 4y\, dy = \frac{1}{2} \left[\frac{1}{6}y^6 - \frac{1}{4}y^4 - \frac{4}{3}y^3 - \frac{4}{2}y^2\right]_{y=-1}^{y=2}$$

$$= \frac{1}{2} \left[\frac{1}{6}(2)^6 - \frac{1}{4}(2)^4 - \frac{4}{3}(2)^3 - 2(2)^2 - \left(\frac{1}{6}(-1)^6 - \frac{1}{4}(-1)^4 - \frac{4}{3}(-1)^3 - 2(-1)^2\right)\right]$$

$$= \frac{1}{2} \left[\frac{32}{3} - 4 - \frac{32}{3} - 8 - \frac{1}{6} + \frac{1}{4} - \frac{4}{3} + 2\right] = -\frac{45}{8}$$

Exercise 1 Double Integrals over Non-Rectangular Regions

Working Space

Evaluate the double integral.

1. $\iint_D e^{-y^2}\, dA$, $D = \{(x, y) \mid 0 \le y \le 3, 0 \le x \le 2y\}$.
2. $\iint_D x \sin y\, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 2$.
3. $\iint_D (2y - x)\, dA$, D is bounded by the circle with center at the origin and radius 3.

Answer on Page 17

1.2 Double Integrals in Other Coordinate Systems

Consider a region composed of a semi-circular ring (see figure **??**). Describing the region in Cartesian coordinates is complicated; you would have to split it into three regions (see figure ...). However, in polar coordinates, we can describe the whole region in one statement:

$$D = \{(r, \theta) \mid 1 \le r \le 4,\ 0 \le \theta \le \pi\}$$

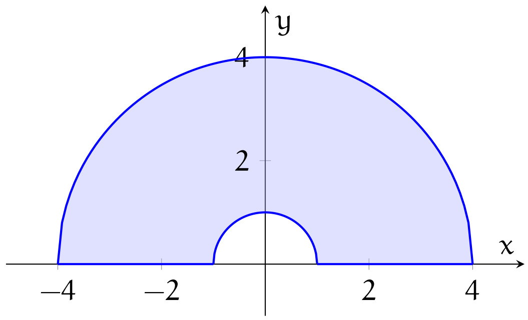


Figure 1.11: A semi-circular ring

There are many instances where a region is simpler to describe in polar coordinates, so how do we take double integrals in polar coordinates? Suppose we want to integrate some function, $f(x, y)$, over a polar rectangle described by $D = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$ (see figure 1.12). Similar to Cartesian coordinates, we can divide this region into many smaller polar rectangles, with each subrectangle defined by $D_{ij} = \{(r, \theta) \mid r_{i-1} \leq r \leq r_i,\ \theta_{i-1} \leq \theta \leq \theta_i\}$. The center of each subrectangle has polar coordinates (r_i^*, θ_j^*), where:

$$r_i^* - \frac{1}{2}\left(r_{i-1} + r_i\right)$$

$$\theta_j^* = \frac{1}{2}\left(\theta_{j-1} + \theta_j\right)$$

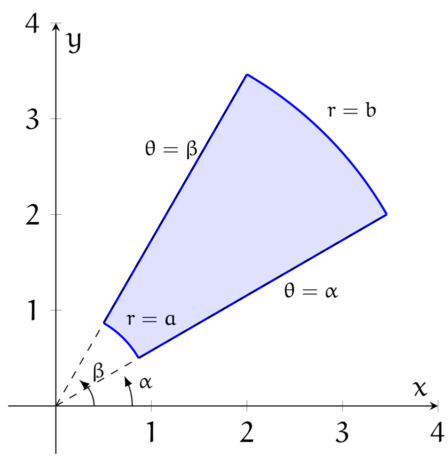


Figure 1.12: A polar rectangle described by $D = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$

Each subrectangle is a larger radius sector minus a smaller radius sector, each with the same central angle, $\Delta\theta = \theta_j - \theta_{j-1}$. The total area of each subrectangle is given by:

$$\Delta A_i = \frac{1}{2}\left(r_i\right)^2 \delta\theta - \frac{1}{2}\left(r_{i-1}\right)^2 \Delta\theta = \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right) \Delta\theta$$

Substituting $\left(r_i^2 - r_{i-1}^2\right) = (r_i + r_{i-1})(r_i - r_{i-1})$, we see that:

$$\Delta A_i = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta$$

Recall that we have defined $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$. Additionally, $\Delta r = r_i - r_{i-1}$. Substituting this, we find a simplified expression for the area of each subrectangle:

$$\Delta A_i = r_i^* \Delta r \Delta\theta$$

Therefore, the Riemann sum of $f(x, y)$ over the region is:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_i$$

(Recall that to convert from Cartesian to polar coordinates, we use $x = r\cos\theta$ and $y = r\sin\theta$). Substituting for δA_i:

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta\theta$$

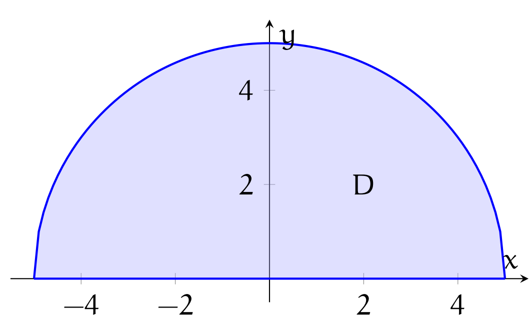
Taking the limit as $n \to \infty$, the Riemann sum becomes the double integral:

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

Therefore, if f is continuous on the polar rectangle $a \le r \le b$, $\alpha \le \theta \le \beta$, then:

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

Example: Evaluate $\iint_D x^2 y \, dA$, where D is the semi-circle shown below.



Solution: Since the region is a semi-circle with radius 5, we can describe D as $D = \{(r,\theta) \mid 0 \le r \le 5,\ 0 \le \theta \le \pi\}$. Therefore,

$$\begin{aligned}
\iint_D x^2 y \, dA &= \int_0^{\pi} \int_0^{5} (r\cos\theta)^2 (r\sin\theta)\, r \, d\theta \, dr \\
&= \int_0^{\pi} \int_0^{5} r^4 \cos^2\theta \sin\theta \, dr \, d\theta \\
&= \int_0^{\pi} \cos^2\theta \sin\theta \left[\frac{1}{5} r^5\right]_{r=0}^{r=5} d\theta \\
&= \int_0^{\pi} \cos^2\theta \sin\theta \frac{5^5}{5} \, d\theta = 625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta
\end{aligned}$$

Using u-substitution, let $u = \cos\theta$. Then $-du = \sin\theta d\theta$, therefore:

$$\begin{aligned}
625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta &= 625 \int_{\theta=0}^{\theta=\pi} -u^2 \, du \\
&= -625 \frac{1}{3} u^3 |_{\theta=0}^{\theta=\pi} = -625 \frac{1}{3} \left(\cos^3\theta\right) |_{\theta=0}^{\theta=\pi} \\
&= -\frac{625}{3}\left[(-1)^3 - (1)^3\right] = -\frac{625}{3}(-2) = \frac{1250}{3}
\end{aligned}$$

Exercise 2 Changing to Polar Coordinates

Working Space

Evaluate the following iterated integrals by converting to polar coordinates:

1. $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$
2. $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\,dx\,dy$
3. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

Answer on Page 17

Exercise 3 Using Polar Coordinates in Multiple Integration

Find the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy-plane.

Working Space

Answer on Page 19

Exercise 4 The volume of a pool

A circular swimming pool has a 40-ft diameter. The depth of the pool is constant along the north-south axis and increases from 3 feet at the west end to 10 feet at the east end. What is the total volume of water in the pool?

Working Space

Answer on Page 20

APPENDIX A

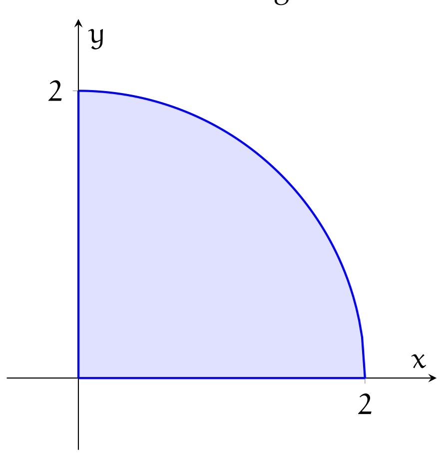
Answers to Exercises

Answer to Exercise 1 (on page 9)

1. $\iint_D e^{-y^2}\, dA = \int_0^3 \int_0^{2y} e^{-y^2}\, dx\, dy = \int_0^3 \left[e^{-y^2} x|_{x=0}^{x=2y} \right] dy = \int_0^3 2ye^{-y^2}\, dy = -e^{-y^2}|_{y=0}^{y=3} = 1 - e^{-9} \approx 0.9999$
2. $\iint_D x \sin y\, dA = \int_0^2 \int_0^{x^2} x \sin y\, dy\, dx = \int_0^2 x \int_0^{x^2} \sin y\, dy\, dx = \int_0^2 x\, [-\cos y]_{y=0}^{y=x^2}$
$= \int_0^2 x \left(\cos 0 - \cos x^2\right) dx = \int_0^2 \left(x - x\cos x^2\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{2}\sin x^2\right]_{x=0}^{x=2} = \frac{1}{2}(2)^2 - \frac{1}{2}\left(\sin 2^2 - \sin 0\right)$
$= 2 - \frac{1}{2}(\sin 4 - 0) = 2 - \frac{\sin 4}{2} \approx 2.378$
3. We can describe the region as $D = \{(x, y) \mid -3 \le x \le -3, -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}\}$.
Therefore, $\iint_D (2y - x)\, dA = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (2x - y)\, dy\, dx = \int_{-3}^{3} \left[2xy - \frac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$
$= \int_{-3}^{3} \left[2x\left(\sqrt{9-x^2} + \sqrt{9-x^2}\right) - \frac{1}{2}\left(9 - x^2 - \left(9 - x^2\right)\right)\right] dx = \int_{-3}^{3} 4x\sqrt{9-x^2}\, dx$. Let $u = 9 - x^2$, then $du = -2x$ and $4x = -2du$. Substituting, $\int_{-3}^{3} 4x\sqrt{9-x^2}\, dx = \int_{x=-3}^{x=3} -2\sqrt{u}\, du = -2 \cdot \frac{2}{3}u^{3/2}|_{x=-3}^{x=3} = -\frac{4}{3}\left[\left(9 - x^2\right)\right]_{x=-3}^{x=3} = 0$

Answer to Exercise 2 (on page 13)

1. Let's visualize the region in the xy-plane:



The region is a quarter-circle that can be described with $D = \{(r, \theta) \mid 0 \le r \le 2,\ 0 \le$

$\theta \leq \pi/2\}$. We can then rewrite the integral in polar coordinates:

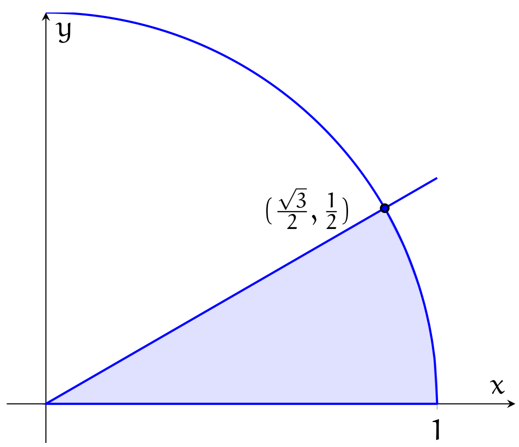
$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^2 re^{-r^2}\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}e^{-r^2} \right]_{r=0}^{r=2} d\theta = \int_0^{\pi/2} \left(-\frac{1}{2}\right) \left[e^{-4} - 1\right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 1 - e^{-4}\, d\theta = \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \int_0^{\pi/2} 1\, d\theta$$

$$= \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \theta\Big|_{\theta=0}^{\theta=\pi/2} = \frac{\pi}{4}\left(1 - \frac{1}{e^4}\right)$$

2. The region is bounded by the x-axis, the line $y = x/\sqrt{3}$, and the circle $x^2 + y^2 = 1$:



We see that the region defined in polar coordinates is $D = \{(r, \theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/6\}$. And therefore:

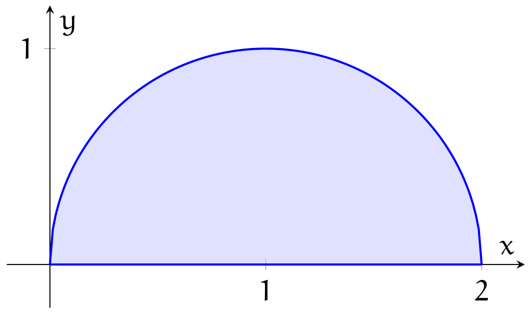
$$\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy = \int_0^{\pi/6} \int_0^1 r\,(r\cos\theta)\,(r\sin\theta)^2\, dr\, d\theta$$

$$= \int_0^{\pi/6} \left[\cos\theta \sin^2\theta\right] d\theta \cdot \int_0^1 r^4\, dr$$

$$= \left(\frac{1}{3}\sin^3\theta\Big|_{\theta=0}^{\theta=\pi/6}\right) \cdot \left(\frac{1}{5}r^5\Big|_{r=0}^{r=1}\right)$$

$$= \frac{1}{15} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{120}$$

3. Visualizing the region:

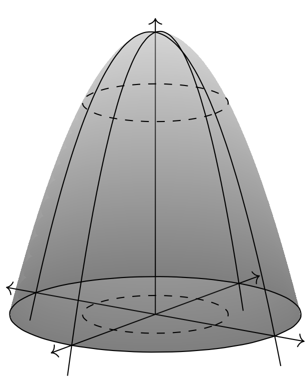


We see that the region is the top half of a circle of radius 1 centered at (1, 0). In polar coordinates, this region is $D = \{(r, \theta) \mid 0 \le r \le 2\cos\theta, 0 \le \theta \le \pi/2\}$. And therefore:

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r\sqrt{r^2}\, dr\, d\theta$$
$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{3}\left[r^3\right]_{r=0}^{r=2\cos\theta} d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{8}{3}\int_0^{\pi/2} \cos\theta\left(1-\sin^2\theta\right) d\theta$$
$$= \frac{8}{3}\left[\int_0^{\pi/2} \cos\theta\, d\theta - \int_0^{\pi/2} \cos\theta\sin^2\theta\, d\theta\right]$$
$$= \frac{8}{3}\left[(\sin\theta)_{\theta=0}^{\theta=\pi/2} - \left(\frac{1}{3}\sin^3\theta\right)_{\theta=0}^{\theta=\pi/2}\right]$$
$$= \frac{8}{3}\left[(1-0) - \frac{1}{3}\left(1^3-0^3\right)\right] = \frac{8}{3}\cdot\frac{2}{3} = \frac{16}{9}$$

Answer to Exercise 3 (on page 14)

We are finding the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy-plane.



We can use polar coordinates to simplify the double integral. In polar coordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so $x^2 + y^2 = r^2$. The volume under the surface and above the xy-plane is given by

$$V = \int\int (4 - r^2) r \, dr \, d\theta, \tag{1.1}$$

where r ranges from 0 to 2 (since $4 - r^2 \geq 0$ if $0 \leq r \leq 2$) and θ ranges from 0 to 2π.

Hence,

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{1}{4} r^4 \right]_0^2 d\theta \\
&= \int_0^{2\pi} (8 - 4) \, d\theta \\
&= \int_0^{2\pi} 4 \, d\theta \\
&= [4\theta]_0^{2\pi} \\
&= 8\pi.
\end{aligned}$$

So, the volume of the solid is 8π cubic units.

Answer to Exercise 4 (on page 15)

Let's describe the footprint of the pool as a 20-foot radius circle centered at the origin (that is, a region $D = \{(r, \theta) \mid 0 \leq r \leq 20, \ 0 \leq \theta \leq 2\pi\}$). Further, let's take north-south as parallel to the y-axis and east-west as parallel to the x-axis. So, the depth of water is then given by $z = f(x, y) = \frac{7}{40}x + \frac{13}{2}$ over the footprint of the pool. The total volume of water is given by:

$$\begin{aligned}
&\int_0^{2\pi} \int_0^{20} r \left(\frac{7}{40} r \cos\theta + \frac{13}{2} \right) dr \, d\theta \\
&= \int_0^{2\pi} \int_0^{20} \left[\frac{7}{40} r^2 \cos\theta + \frac{13}{2} r \right] dr \, d\theta \\
&= \int_0^{2\pi} \left[\frac{7\cos\theta}{40} \int_0^{20} r^2 \, dr + \frac{13}{2} \int_0^{20} r \, dr \right] d\theta \\
&= \int_0^{2\pi} \left[\frac{7\cos\theta}{40} \left(\frac{1}{3} r^3 \right)_{r=0}^{r=20} + \frac{13}{2} \left(\frac{1}{2} r^2 \right)_{r=0}^{r=20} \right] d\theta
\end{aligned}$$

$$= \int_0^{2\pi} \left[\frac{1400}{3} \cos\theta + 1300 \right] \, d\theta = \left[\frac{1400}{3} \sin\theta + 1300\theta \right]_{\theta=0}^{\theta=2\pi}$$

$$= 2600\pi \text{ cubic feet}$$

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