Chapter 1

Power Series

Consider the function $f(x) = \frac{1}{1-x}$. This looks similar to the value of convergent geometric series, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If we let $a = 1$ and $r = x$, then we see that $\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$. We can reindex to begin at $n = 0$ and see that:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is a **power series**. We use power series in place of functions for many applications, such as integrals where an explicit antiderivative don't exist, solving differential equations, and computer scientists representing functions on computers. Consider $f(x) = \frac{1}{1-x^2}$. What is $\int f(x)\,dx$? We can't directly use u-substitution, and this is not a derivative of any inverse trigonometric function. [You may have realized we could integrate this explicitly by using partial fractions, but this is not true for other functions, and we are using this as a demonstration anyway.] One way to evaluate this integral would be to represent $f(x)$ as a power series, then integrate the series. This is easier, since we know how to take the integral of any polynomial ($\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$). First, we discuss what power series are further.

1.1 Power Series

Power series are series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 = c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

for some fixed x. Depending on x, the series may converge or diverge. For example, the power series $\sum_{n=0}^{\infty} x^n$ converges for $-1 < x < 1$ and diverges for all other values of x. This is because $\sum_{n=0}^{\infty} x^n$ is essentially a geometric series with $r = x$, which we already know converges for $|r| < 1$.

The form given above is for a power series centered on 0, but a power series can be centered on any value, a. In that case, it looks like this:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

Which we say is a *power series in* $(x-a)$, or a *power series centered at* a, or a *power series about* a.

Example: Find a power series representation for $f(x) = \frac{2x-4}{x^2-4x+3}$.

Solution: Since we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we will use partial fractions to decompose the function into two fractions. (The process is left as an exercise for the student.) We find that:

$$\frac{2x-4}{x^2-4x+3} = \frac{1}{x-1} + \frac{1}{x-3}$$

Noting that $\frac{1}{x-1} = (-1)\cdot\frac{1}{1-x}$, we can say that:

$$\frac{1}{x-1} = (-1)\cdot\sum_{n=0}^{\infty} x^n$$

Now, let's look at $\frac{1}{x-3}$. We can show that:

$$\frac{1}{x-3} = \frac{\frac{1}{3}}{\frac{x}{3}-1} = \frac{1}{3}\frac{1}{\frac{x}{3}-1} = \frac{-1}{3}\frac{1}{1-\frac{x}{3}}$$

Substituting $\frac{x}{3}$ for x into $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we see that:

$$\frac{1}{1-\frac{x}{3}} = \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

Therefore:

$$\frac{1}{x-3} = \left(\frac{-1}{3}\right)\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

Adding the terms, we see that:

$$\frac{1}{x-1} + \frac{1}{x-3} = (-1)\sum_{n=0}^{\infty} x^n + \left(\frac{-1}{3}\right)\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

1.2 Power Series Convergence

Sometimes, you will be asked to find one or more values of x for which a power series converges. To do this, choose a test to apply, then find x such that the test is passed.

Example: For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution: We will apply the Ratio Test (since there is a factorial in the series) and find x such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)n!x \cdot x^n}{n!x^n} = \lim_{n\to\infty} \frac{(n+1)\cdot x}{1}$$

which converges to 0 when $x = 0$ and diverges for all other values of x. Therefore, $\sum_{n=0}^{\infty} n!x^n$ converges if $x = 0$.

Example: For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n}$ converge?

Solution: We will use the Ratio Test again. We are looking for an x such that:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{\frac{(x-4)^{n+1}}{2(n+1)}}{\frac{(x-4)^n}{2n}} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{(x-4)(x-4)^n}{2n+2} \cdot \frac{2n}{(x-4)^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{(x-4)(x-4)^n(2n)}{(x-4)^n(2n+2)} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{(x-4)(2n)}{2n+2} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{2(x-4)}{2+\frac{2}{n}} \right| < 1$$

$$2 \cdot \lim_{n\to\infty} \left| \frac{x-4}{1+\frac{1}{n}} \right| < 1$$

$$2 \cdot |x-4| < 1$$

$$|x-4| < \frac{1}{2}$$

Which is true when

$$-\frac{1}{2} < x-4 < \frac{1}{2}$$

$$3.5 < x < 4.5$$

We are not done yet, though! We know the series converges for $3.5 < x < 4.5$ and diverges for $x < 3.5$ and $x > 4.5$. What about when $x = 3.5$ and $x = 4.5$? (These are the cases where the Ratio Test is indeterminate, because $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.) We need to test each case. Substituting $x = 3.5$ into the series yields:

$$\sum_{n=1}^{\infty} \frac{(3.5-4)^n}{2n} = \sum_{n=1}^{\infty} \frac{\left(\frac{-1}{2}\right)^n}{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^{n+1}}$$

This is an alternating series, so we apply the alternating series test. First, we check that $|a_{n+1}| < |a_n|$:

$$\frac{1}{(n+1) \cdot 2^{n+2}} < \frac{1}{n \cdot 2^{n+1}}$$

Which is true for all $n > 0$. Next, we check if $\lim_{n\to\infty} |a_n| = 0$:

$$\lim_{n\to\infty} \frac{1}{n \cdot 2^{n+1}} = \frac{1}{\infty} = 0$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^{n+1}}$ is convergent and $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n}$ is convergent for $x = 3.5$. Next, we test $x = 4.5$ for convergence:

$$\sum_{n=1}^{\infty} \frac{(4.5-4)^n}{2n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot \frac{1}{2n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+1} \frac{1}{n}$$

This series is less than the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ for all n. We know the harmonic series diverges, therefore, by the direct comparison test, $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+1} \frac{1}{n}$ must also diverge. So, our final answer to the original question is that the series is convergent fort $3.5 \leq x < 4.5$.

1.2.1 Radius of Convergence

There are three possible outcomes when testing a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ for convergence:

1. The series only converges for $x = a$
2. The series converges for all x
3. The series converges if $|x - a| < R$ and diverges for $|x - a| > R$, where R is some positive number

We call R the **radius of convergence** . If we rearrange $|x - a| < R$, we can see why this is called a radius (see figure 1.1):

$$a - R < x < a + R$$

When $x = a \pm R$, the series could be convergent or divergent. You will need to test the endpoints of the windown of convergence to determine if the interval is open or closed. Thus, there are four possiblities for the interval of convergence:

1. $(a - R, a + R)$

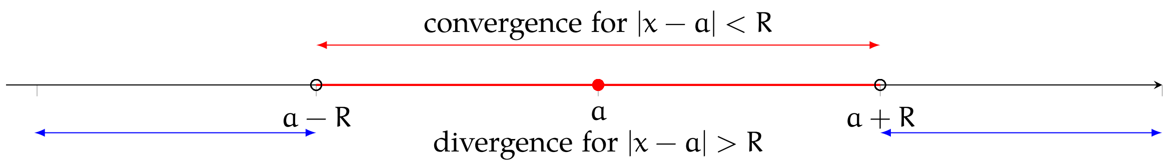

Figure 1.1: R is called the radius of convergence because it is half the width of the window of convergence

2. $[a-R, a+R)$
3. $(a-R, a+R]$
4. $[a-R, a+r]$

In the example of $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n}$ (shown above), $a = 4$ and $R = 0.5$, and we found that the power series is convergent for $x \in [3.5, 4.5)$.

Example: For what values of x is the Bessel function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ convergent?

Solution: Because there is a factorial, we will apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}}{frac(-1)^n x^{2n} 2^{2n}(n!)^2} \right| < 1$$

$$\lim_{n\to\infty} \frac{x^{2n}x^2 2^{2n} n!n!}{2^{2n}2^2(n=1)!(n+1)!x^{2n}} < 1$$

$$\lim_{n\to\infty} \frac{x^2 n!n!}{2^2(n+1)n!(n+1)n!} < 1$$

$$\lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1$$

Because $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ for all x, the Bessel function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ is convergent for all real values of x, and the interval of convergence is $(-\infty, \infty)$.

Example: Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Solution: Again, we apply the ratio test to find values of x such that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$:

$$\lim_{n\to\infty} \left| \frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+1+1}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| < 1$$

$$\lim_{n\to\infty}\left|\frac{(-3)x}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{1}\right|<1$$
$$\lim_{n\to\infty}\left|(-3)x\sqrt{\frac{n+2}{n+1}}\right|<1$$
$$3|x|\lim_{n\to\infty}\sqrt{\frac{n+2}{n+1}}<1$$
$$3|x|\lim_{n\to\infty}\sqrt{\frac{1+2/n}{1+1/n}}=3|x|(1)<1$$
$$3|x|<1$$
$$|x|<\frac{1}{3}$$

Therefore, the radius of convergence is $\frac{1}{3}$. We need to test the endpoints, $x=\frac{-1}{3}$ and $x=\frac{1}{3}$, to determine the interval of convergence. First, we will test if $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$ when $x=\frac{-1}{3}$:

$$\sum_{n=0}^{\infty}\frac{(-3)^n\left(\frac{-1}{3}\right)^n}{\sqrt{n+1}}=\sum_{n=0}^{\infty}\frac{1^n}{\sqrt{n+1}}=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$$

This is a p-series such that $p<1$, so it is divergent, and our original series does not converge for $x=\frac{-1}{3}$. Next, we test $x=\frac{1}{3}$:

$$\sum_{n=0}^{\infty}\frac{(-3)^n\left(\frac{1}{3}\right)^n}{\sqrt{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$$

which is an alternating series that converges by the alternating series test. Therefore, the interval of convergence for $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$ is $x\in\left(\frac{-1}{3},\frac{1}{3}\right]$.

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] What is the radius of convergence of the series $\sum_{n=0}^{\infty}\frac{(x-4)^{2n}}{3^n}$?

Working Space

Answer on Page 11

Exercise 2

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] A power series is given by $\frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \cdots$. Write the series in sigma notation and use the Ratio Test to determine the interval of convergence.

Working Space

Answer on Page 11

1.3 Calculus with Power Series

You can integrate and differentiate power series. Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. Recall that $f(x)$ is just a very long polynomial and that the derivative of a polynomial x^n is $n \cdot x^{n-1}$. We can then state that:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots + c_n(x-a)^n\right]$$

$$f'(x) = 0 + c_1 + 2c_2(x-a)^1 + \cdots + nc_n(x-a)^{n-1}$$

$$f'(x) = \sum_{n=1}^{\infty} c_n(x-a)^{n-1}$$

which is true when x is in the interval of convergence for the series.

Similarly, we know $\int x^n \, dx = \frac{1}{n+1}x^{n+1}$. We can then say that:

$$\int f(x)\,dx = \int \left[c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots + c_n(x-a)^n\right], dx$$

$$\int f(x)\,dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots \frac{c_n}{n+1}(x-a)^{n+1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

Where C is the integration constant. Again, this is true when x is in the interval of convergence for the series.

Example: Express $\frac{1}{(1-x)^2}$ as a power series by differentiating $\frac{1}{1-x}$.

Solution: Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$.

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

Differentiating both sides:

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$(-1)\cdot\frac{1}{(1-x)^2}\cdot\frac{d}{dx}(1-x) = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Reindexing to begin at $n = 0$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$$

Because $\sum_{n=0}^{\infty}$ has a radius of convergence of 1, so does $\sum_{n=0}^{\infty}(n+1)x^n$. We can confirm our series makes sense by plotting the partials sums for $n = 3, 5, and 7$ with the original function (see figure 1.2).

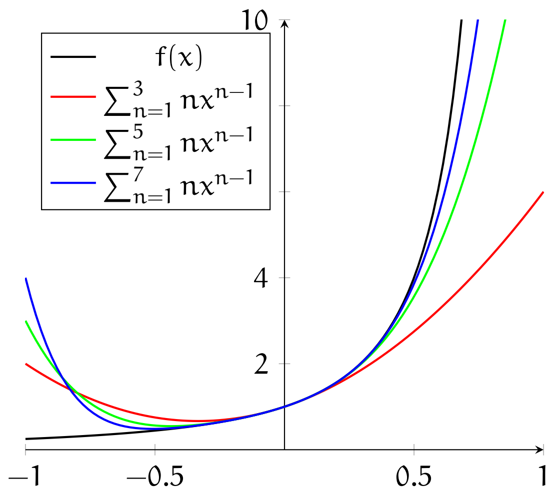

Figure 1.2: The function $f(x) = \frac{1}{(1-x)^2}$ is equal to the power series $\sum_{n=1}^{\infty} nx^{n-1}$

Example: Find a power series representing $\ln(1+x)$.

Solution: We know that $\frac{d}{dx}\frac{1}{1-x} = \ln(1-x)$. Replacing x with $-x$, we see that:

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$$

which converges for $|x| < 1$. We can then integrate both sides:

$$\int \frac{1}{1+x}\,dx = \int \left[\sum_{n=0}^{\infty}(-x)^n\right] dx$$

$$\ln(1+x) = \int \left(1 - x + x^2 - x^3 + \cdots\right) dx$$

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + C$$

when $|x| < 1$. To find C, substitute $x = 0$ and solve:

$$\ln(1+0) = C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{0^n}{n} = C + 0$$

$$C = \ln 1 = 0$$

So, our final answer is $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$.

Exercise 3

Find a power series representation for $f(x) = \arctan x$.

Working Space

Answer on Page 12

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

Since this sum has terms to the n^{th} power, we will apply the Root Test, which states a series is convergent if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-4)^{2n}}{3^n}\right|} < 1$$

$$\lim_{n\to\infty} \left|\frac{(x-4)^{2n/n}}{3^{n/n}}\right| < 1$$

$$\lim_{n\to\infty} \frac{\left|(x-4)^2\right|}{3} < 1$$

$$(x-4)^2 < 3$$

$$|x-4| < \sqrt{3}$$

Therefore, the radius of convergence is $\sqrt{3}$.

Answer to Exercise 2 (on page 7)

We see that the series is alternating, so we know it involves $(-1)^n$ (we will begin indexing at $n = 0$). The powers of x are given by x^{2n+1} and the denominators are given by $2n+3$. Therefore, the sum in sigma notation is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3}$. Applying the ratio test, the series is convergent when:

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{2(n+1)+3}\frac{2n+3}{(-1)^n x^{2n+1}}\right| < 1$$

$$\lim_{n\to\infty} \left|\frac{x^{2n+3}}{2n+5}\frac{2n+3}{x^{2n+1}}\right| < 1$$

$$\left|x^2\right| \lim_{n\to\infty} \frac{2n+3}{2n+5} < 1$$

$$\left|x^2\right| < 1$$

$$|x| < 1$$

So, we know that the series is convergent on the open interval $x \in (-1, 1)$. We check the endpoints, $x = -1, 1$ for convergence.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+3} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+3}$$

When $x = -1$, the series is an alternating series such that $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$. Therefore, the series converges for $x = -1$.

$$\sum_{n=1}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3}$$

which is also an alternating series such that $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$. Therefore, the series converges for $x = 1$ and the interval of convergence is $x \in [-1, 1]$, which can also be written as $-1 \le x \le 1$.

Answer to Exercise 3 (on page 9)

Recall that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$. Replacing x with $-x^2$, we see that $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left[-x^2\right]^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$. Then we can also say that $\arctan x = \int \frac{1}{1+x^2}\, dx = \int \left[sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx$. Evaluating the integral, $\int \left[sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Knowing that $\arctan 0 = 0$, we find that $C = 0$ and $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

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