

## CHAPTER 1

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# Projectile Motion

A projectile is an object that, once thrown or dropped, continues to move only under the influence of gravity. Throwing a baseball, shooting a cannon, and diving off a high diving board are all examples. NASA flight planners use projectile motion to plan flight paths for space vehicles, such as sending rovers to Mars. You've already learned how to describe and model one-dimensional projectile motion in the Falling Bodies chapter. Now, we consider projectiles that also have horizontal motion, and therefore are moving in two dimensions.

First, we will compare the motion of projectiles that are dropped versus horizontally launched from the same height. This will frame our discussion of the important concept of independence of motion: the vertical and horizontal motions of a projectile can be considered and described independent from each other. This will allow you to predict how far horizontally launched objects will travel before hitting the ground. Next, you'll learn to describe the motion of projectiles launched at an angle (like some heavy ground artillery). Finally, you'll use what you've learned to create a model of any projectile motion.

### 1.1 Comparing Projectiles

This video was mentioned at the end of the kinematics chapter: <https://www.youtube.com/watch?v=zMF4CD7i3hg>. From the video, we can see that the addition of horizontal motion does not effect how fast an object is acted upon by gravity. Both objects hit the ground at the same time, regardless of whether horizontal motion was added or not. This is because of a concept called *independence of motion*.

### 1.2 Independence of Motion

In projectile motion, such as a ball being thrown off of a cliff, the horizontal and vertical components of motion are independent of each other. This means that horizontal motion (and forces in the horizontal direction) do not affect vertical motion (and forces in the vertical direction), and vice versa. This is because gravity only acts in the vertical direction.<sup>1</sup>

If you recall the falling bodies chapter, you know that the vertical motion of an object in

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<sup>1</sup>This holds true for objects on sloped surfaces. Gravitational motion is just separated into components parallel and perpendicular to the slope. This will be covered in a future chapter.

Horizontal Motion	Vertical Motion
$\Delta x = v_{0x} t$	$\Delta y = v_{0y} t + \frac{1}{2}(-g)t^2$
$v_x = v_{0x}$ (constant!)	$v_y = v_{0y} + (-g)t$
$a_x = 0$	$a_y = -g$

Figure 1.1: Projectile motion equations from kinematics equations.

free fall is described by the equations of uniformly accelerated motion, with a constant acceleration of  $9.8\frac{\text{m}}{\text{s}^2}$  downward (this may simplified to  $10\frac{\text{m}}{\text{s}^2}$  for simplicity in many calculations).

In the horizontal direction, if we ignore air resistance (a common practice for elementary physics), there are no forces acting on the object. This means that the object will continue to move at a constant velocity in the horizontal direction. Note that from launch to landing, the time must be the same for both components. Take a minute to think about why this is if you are unsure.

Because the horizontal and vertical motions are independent of each other, we can use different equations to describe the motion in each direction (See Figure ??). Note that the velocity in the horizontal direction is *constant*, while the velocity in the vertical direction is *not constant* due to the acceleration of gravity ( $9.8\frac{\text{m}}{\text{s}^2}$  downward).

Let's take a look at some simple graphs comparing x motions and y motions, shown in Figures ?? and ???. Note that these are a general representation of projectile motion, and the scales or values may not be accurate.

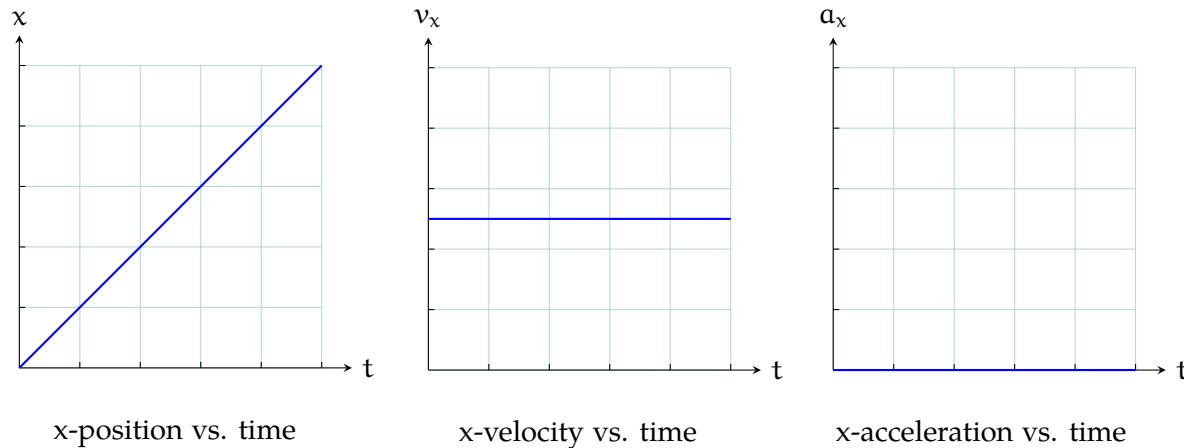


Figure 1.2: Vertical motion graphs

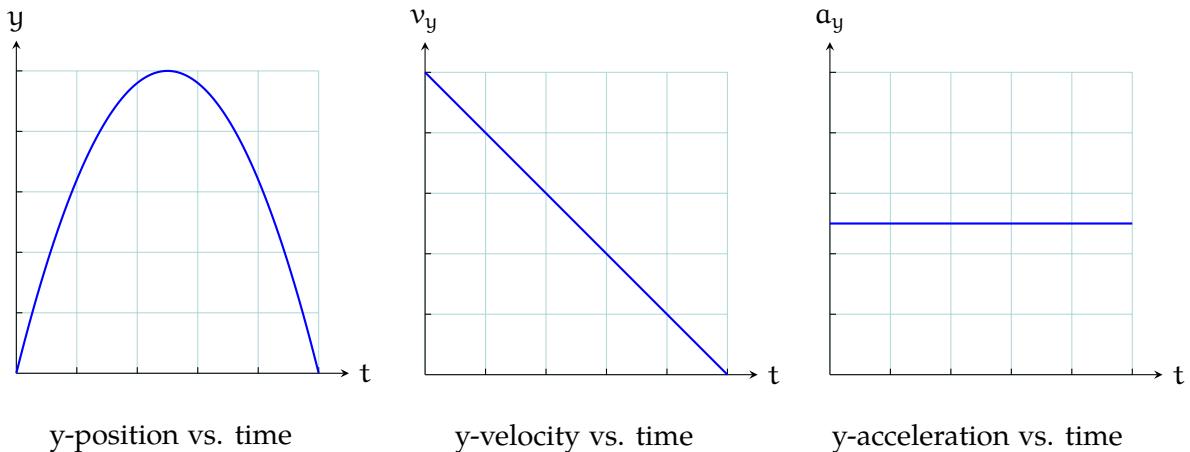


Figure 1.3: Vertical motion graphs

### 1.3 Horizontally-launched Projectiles

Imagine a ball is thrown horizontally off of a cliff. The ball has an initial horizontal velocity of 20 m/s, but no initial vertical velocity. The ball will continue to move horizontally at a constant velocity, while simultaneously accelerating downward due to gravity. The ball falls for 5 seconds before hitting the ground. We will use  $-10 \frac{\text{m}}{\text{s}^2}$  for the acceleration due to gravity to make calculations easier. See figure ??.

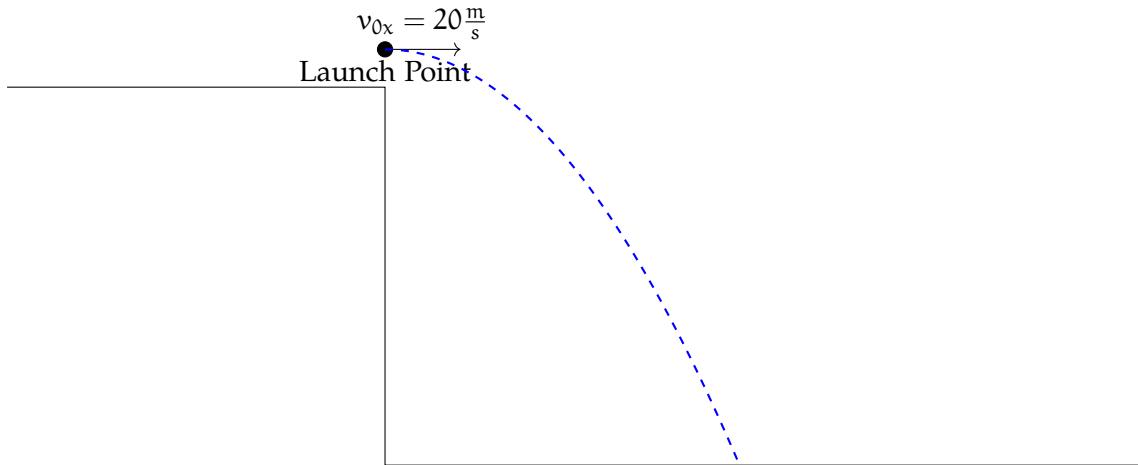


Figure 1.4: Diagram of a ball being thrown horizontally off of a cliff. Note: figure is not to scale.

Let's calculate the change in  $y$ .

The vertical motion can be described by the equation:

$$\Delta y = v_{0y}t + \frac{1}{2}(-g)t^2$$

Since the initial vertical velocity  $v_{0y} = 0$ , this simplifies to:

$$\Delta y = \frac{1}{2}(-g)t^2$$

Substituting in the values for  $g$  and  $t$ :

$$\Delta y = \frac{1}{2}(-10 \frac{\text{m}}{\text{s}^2})(5\text{s})^2$$

Calculating this gives:

$$\Delta y = -125\text{m}$$

So the ball falls a vertical distance of 122.5 m from its origin before hitting the ground.

Now let's calculate the change in  $x$ . The horizontal motion can be described by the equation:

$$\Delta x = v_{0x}t$$

Substituting in the values for  $v_{0x}$  and  $t$ :

$$\Delta x = (20 \frac{\text{m}}{\text{s}})(5\text{s}) = 100\text{m}$$

**Exercise 1**    **Horizontally-launched Projectile**

A rock is thrown horizontally off of the edge of a cliff with an initial velocity of 8 m/s. If the rock falls for 3 seconds before hitting the ground, (a) how far from the base of the cliff does the rock land? (b) How high is the cliff?

Use  $-10 \frac{\text{m}}{\text{s}^2}$  for the acceleration due to gravity.

Working Space

Answer on Page ??

### 1.3.1 Newton's Cannon and Escape Velocity

A real world application of horizontally launched projectiles is a thought experiment known as Newton's cannon (sometimes called Newton's cannonball). Newton theorized that a cannon on the top of a mountain above Earth's atmosphere could launch a cannonball at some high velocity could do one of three things:

- The cannonball would not have enough horizontal velocity, and fall back to the Earth.
- The cannonball would get shot at enough velocity to be in continuous orbit around the Earth. (This is how satellites stay in orbit around the Earth!).
- The cannonball would get shot at such a high velocity that it would escape the Earth's gravitational pull, and fly off into space. This is known as *escape velocity*.

Take a look at this simulation of Newton's cannon: <https://physics.weber.edu/schroeder/software/NewtonCannon.html>. Then, watch this video: <https://www.youtube.com/watch?v=ALRdYPMpqQs>.

At anywhere below 7000 m/s, the cannonball falls back to Earth. In a range of 7,000 m/s to 8,000 m/s (the simulation doesn't go above 8,000 m/s), the cannonball enters an orbit around the Earth. For Earth, the escape velocity is  $\approx 11,200$  m/s.

For any given planet, the escape velocity can be calculated using the formula:

$$v_e = \sqrt{\frac{2GM}{d}}$$

where  $G$  is the gravitational constant,  $M$  is the mass of the planet, and  $d$  is the distance from the center of the planet to the object. We will cover this in more detail in the orbits chapter.

## 1.4 Projectiles launched at an Angle

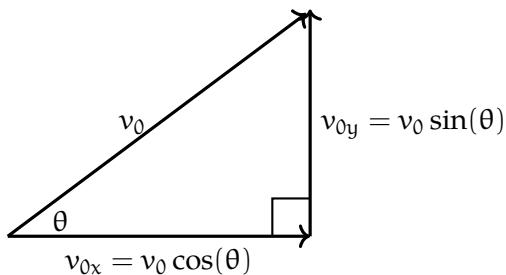
### 1.4.1 From the Ground

Now, let's imagine a ball is launched from the ground at an angle. The angle of launch influences both horizontal and vertical components of the initial velocity. We can use trigonometry to separate the initial velocity into its horizontal and vertical components.

$$v_{0x} = v_0 \cos(\theta) \quad v_{0y} = v_0 \sin(\theta)$$

In the previous section, when an object is thrown directly horizontally,  $\theta = 0^\circ$ , so  $v_{0x} = v_0$ . However, we now have  $v_0$  being separated into two components, so the  $v_{0x}$  will be less than  $v_0$ , as  $\cos(\theta)$  and  $\sin(\theta)$  are always less than or equal to 1.

The vertical component of  $v_0$  is the initial velocity,  $v_{0y}$ . This component, unlike  $v_{0x}$ , is constantly changing due to gravity, at a rate of  $-9.8 \frac{\text{m}}{\text{s}^2}$ . The peak of an object being launched at some angle is when  $v_{0y}$  is equal to 0.

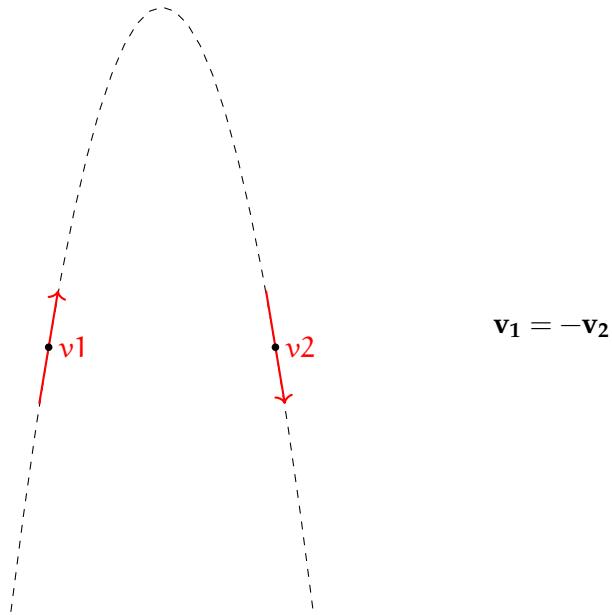


For example, take a watermelon launched at  $20 \frac{\text{m}}{\text{s}}$  at an angle of  $30^\circ$  above the horizontal. What would the horizontal and vertical components of the initial velocity be?

Well, we can use the equations above to find out:

$$v_{0x} = (20) \cos(30^\circ) \approx 17.32 \text{ m/s}, \quad v_{0y} = (20) \sin(30^\circ) = 10 \text{ m/s}$$

It is important to note that the upwards velocity and downwards velocity are equal in magnitude at any given height (given the same vertical initial and final height), but *opposite in direction*, due to the symmetric nature of projectile motion.



One last formula of note is the range formula, describing the horizontal range of any object launched at an angle on *level ground*:

**Range Formula**

$$R = \frac{v_0^2 \sin(2\theta)}{g} \quad (1.1)$$

where the object starts and lands at the same vertical height.

**Exercise 2**    **Projectile Motion at an angle**

A pumpkin is launched at 25 m/s at an angle of  $40^\circ$  above the horizontal. Find (a) the peak height of the pumpkin, and (b) the horizontal distance the pumpkin travels before hitting the ground. Use  $-10 \frac{\text{m}}{\text{s}^2}$  for the acceleration due to gravity. You may assume the ground is level the entire time.

*Working Space*

*Answer on Page ??*

**Exercise 3**    **Launch Angle for Maximum Range**

(a) At what angle should you launch for an object to go the furthest given a maximum launch velocity? (b) How can you get the same horizontal distance with two different launch angles?

*Working Space*

*Answer on Page ??*

**Exercise 4**    **Projectile Motion at an angle**

There is a target 100 meters away. I must shoot a bow at 14 meters per second. At what angle will I be able to hit my target? If I cannot hit the target, calculate the velocity needed to reach the target. Use  $-10 \frac{\text{m}}{\text{s}^2}$  for the acceleration due to gravity. You may assume the ground is level the entire time.

Working Space

Answer on Page ??

**1.4.2 From a Height Above the Ground**

Now we can imagine a ball being launched at some angle from a height above the ground. This is similar to the horizontally launched projectile, except now we have an initial vertical velocity component.

Let's say a ball is launched from a height of 50 m at an angle of  $30^\circ$  above the horizontal with an initial velocity of 20 m/s. We can find the horizontal and vertical components of the initial velocity as follows:

$$v_{0x} = (20) \cos(30^\circ) \approx 17.32 \text{ m/s}, \quad v_{0y} = (20) \sin(30^\circ) = 10 \text{ m/s.}$$

The vertical motion can be described by the equation:

$$\Delta y = y_0 + v_{0y}t + \frac{1}{2}(-g)t^2 = 50 + 10t - 5t^2$$

Likewise, the horizontal motion of the projectile can be described by the equation:

$$\Delta x = v_{0x}t = 17.32t$$

To find the peak of the projectile, we can set  $v_y = 0$  and solve for t:

$$v_y = v_{0y} + (-g)t$$

$$0 = 10 - 5t^2$$

$$t_{\text{peak}} = \sqrt{2} \text{ s} \approx 1.414 \text{ s}$$

Substituting this time into the vertical and horizontal equations, we can find the peak height and horizontal distance at the peak:

$$\Delta y = 50 + 10(\sqrt{2}) - 5(\sqrt{2})^2 = 50 + 10\sqrt{2} - 10 = 40 + 10\sqrt{2} \approx 54.14 \text{ m}$$

$$\Delta x = 17.32(\sqrt{2}) \approx 24.49 \text{ m}$$

And to find the total time of flight, we can set  $\Delta y = 0$  and solve for t:

$$\begin{aligned} 0 &= 50 + 10t - 5t^2 \\ 0 &= -5t^2 + 10t + 50 \\ 0 &= t^2 - 2t - 10 \end{aligned}$$

Using the quadratic formula, we find:

$$\begin{aligned} t &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-10)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4+40}}{2} \\ &= \frac{2 \pm \sqrt{44}}{2} \\ &= 1 \pm \sqrt{11} \approx 4.32 \text{ s (taking the positive root)} \end{aligned}$$

## 1.5 Simulating Projectile Motion

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*This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.*

## APPENDIX A

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# Answers to Exercises

### Answer to Exercise ?? (on page ??)

(a) To find how far from the base of the cliff the rock lands, we can use the horizontal motion equation:

$$\begin{aligned}\Delta x &= v_{0x}t \\ \Delta x &= (8 \frac{\text{m}}{\text{s}})(3\text{s}) = 24\text{m}\end{aligned}$$

(b) To find the height of the cliff, we can use the vertical motion equation. Since  $v_{0y} = 0$ , and the initial vertical position is at the top of the cliff (we can call this  $y_0 = 0$ ), we have:

$$\begin{aligned}y_f &= v_{0y}t + \frac{1}{2}(-g)t^2 \\ y_f &= 0 + \frac{1}{2}(-10)(3)^2 \\ y_f &= -45\text{m}\end{aligned}$$

We get a negative value for  $y_f$  because the rock is falling downward from its initial position, the top of the cliff which we selected as the origin. Therefore, the height of the cliff is 45 meters.

So the rock lands 24 meters from the base of the cliff, and the cliff is 45 meters high.

### Answer to Exercise ?? (on page ??)

Let's first find the horizontal and vertical components of the initial velocity:

$$\begin{aligned}v_{0x} &= (25) \cos(40^\circ) \approx 19.15 \text{ m/s}, \\ v_{0y} &= (25) \sin(40^\circ) \approx 16.07 \text{ m/s}.\end{aligned}$$

(a) To find the peak height, we can use the vertical motion equations. Taking it in two

steps, let's first find the time to reach the peak height, where the vertical velocity  $v_y = 0$ :

$$\begin{aligned} v_y &= v_{0y} + (-g)t \\ 0 &= 16.07 - 10t \\ t_{\text{peak}} &= \frac{16.07}{10} \approx 1.607 \text{ s} \end{aligned}$$

Now, we can use this time to find the peak height using the vertical displacement equation:

$$\begin{aligned} \Delta y &= v_{0y}t + \frac{1}{2}(-g)t^2 \\ \Delta y &= (16.07)(1.607) + \frac{1}{2}(-10)(1.607)^2 \\ \Delta y &\approx 12.91 \text{ m} \end{aligned}$$

(b) To find the total horizontal distance traveled, we first need the total time of flight. Since the motion is symmetric, the total time will be twice the time to reach the peak height:

$$t_{\text{total}} = 2t_{\text{peak}} \approx 2(1.607) \approx 3.214 \text{ s} = 3.21 \text{ s}$$

Since there is no horizontal acceleration, we can use the horizontal motion equation to find the horizontal distance:

$$\Delta x = v_{0x}t_{\text{total}} \approx (19.15)(3.21) \approx 61.5 \text{ m}$$

So, the peak height of the pumpkin is approximately 12.91 m, and the horizontal distance it travels before hitting the ground is approximately 61.5 m.

## Answer to Exercise ?? (on page ??)

To maximize the horizontal distance of a projectile being launched, the vertical and horizontal components must be proportionally equal. This is only possible at a launch angle of  $45^\circ$ . Let's prove this mathematically.

Let the initial launch speed be  $v$  and the launch angle be  $\theta$ . The horizontal and vertical velocity components are

$$v_x = v \cos \theta, \quad v_y = v \sin \theta.$$

The vertical motion kinematics equation can be solved for:

$$0 = t \left( v \sin \theta - \frac{1}{2}gt \right)$$

This has two solutions:  $t = 0$  (the time at launch) and

$$t = \frac{2v \sin \theta}{g}.$$

Thus, the horizontal range is

$$R_x = v_x \cdot t = v \cos \theta \cdot \frac{2v \sin \theta}{g} = \frac{v^2}{g} \sin(2\theta).$$

The function  $\sin(2\theta)$  reaches its maximum value of 1 when  $2\theta = 90^\circ$ , or  $\theta = 45^\circ$ . Therefore, the optimal launch angle for maximum horizontal distance is  $45^\circ$ .

The range equation above can result in two different values:  $\theta$  and  $90^\circ - \theta$ . There are two ways to get a certain horizontal launch distance, one at an angle less than  $45^\circ$  and one at an angle greater than  $45^\circ$ . For example, a projectile launched at  $30^\circ$  will travel the same horizontal distance as one launched at  $60^\circ$ , assuming the same initial velocity. However, the projectile launched at  $30^\circ$  will spend less time in the air and have a lower peak height than the one launched at  $60^\circ$ .

## Answer to Exercise ?? (on page ??)

Using the horizontal distance equation,  $R$ , with  $v_0 = 14$  m/s and  $R = 100$  m, we have:

$$\begin{aligned} R &= \frac{v^2 \sin(2\theta)}{g} \\ \sin(2\theta) &= \frac{Rg}{v^2} \\ \sin(2\theta) &= \frac{(100)(10)}{14^2} \approx 5.10 \end{aligned}$$

Since the sine of an angle cannot be greater than 1, it is impossible to hit the target with a launch speed of 14 m/s.

Instead, we can calculate the minimum launch speed needed to hit the target at a  $45^\circ$  angle, while satisfying the 100 m requirement. Assuming the  $\sin(2\theta) = 1$  for a  $45^\circ$  launch angle, we can rearrange the range equation to solve for  $v$ :

$$\begin{aligned} v &= \sqrt{\frac{Rg}{\sin(2\theta)}} \\ v &= \sqrt{\frac{(100)(10)}{1}} = \sqrt{1000} \approx 31.62 \text{ m/s} \end{aligned}$$

So at an assumed launch angle of  $45^\circ$ , a minimum launch speed of approximately 31.62 m/s is needed to hit the target 100 meters away.





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