Chapter 1

Projections

The word ”projection” has two main meanings in everyday life. One is a projection as a forecast or estimate of something in the future based on the current situation; another is the result of shining a light to cast a shadow or show a movie. Both these definitions apply to mathematical projection.

Projections are used in many fields, such as science, math, engineering, and finance. Here are a few examples:

- Investors evaluate risk and return of a portfolio by projecting an asset’s return onto a reference portfolio.
- Astronomers analyze the motion of stellar objects by projecting the object’s true motion onto the plane of the sky.
- Robotics engineers use projections to prevent robots from running into obstacles by projecting the robot’s position onto the optimal path.

Mathematically, a projection describes the relationship of one vector to another in terms of direction and orthogonality. Given two vectors, $\mathbf{u}$ and $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ separates $\mathbf{u}$ into two components. The first component signifies how much $\mathbf{u}$ lies in the direction of $\mathbf{v}$. The second signifies the component of $\mathbf{u}$ that is orthogonal (perpendicular) to $\mathbf{v}$.

The figure 1.1 depicts a projection. The perpendicular line dropped from the end of $\mathbf{u}$ is the *orthogonal* component. The portion of $\mathbf{u}$ that lies in the direction of $\mathbf{v}$ is the blue segment.

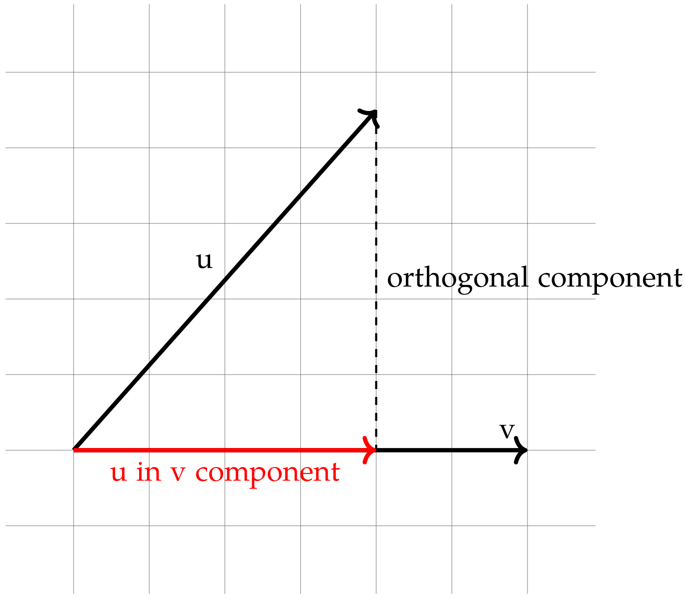

Figure 1.1: The projection of u onto v

You can also think of a projection as the shadow cast by one vector onto the other by an overhead light. See Figure 1.2

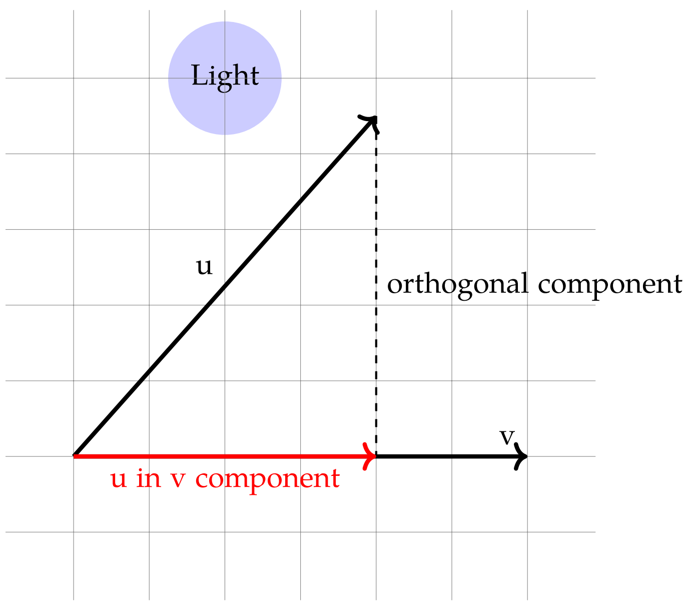

Figure 1.2: Projection of u onto v with a light included to simulate a shadow.

The projected vector can be in any direction and its length can extend beyond the vector onto which it is projecting. See Figure 1.3.

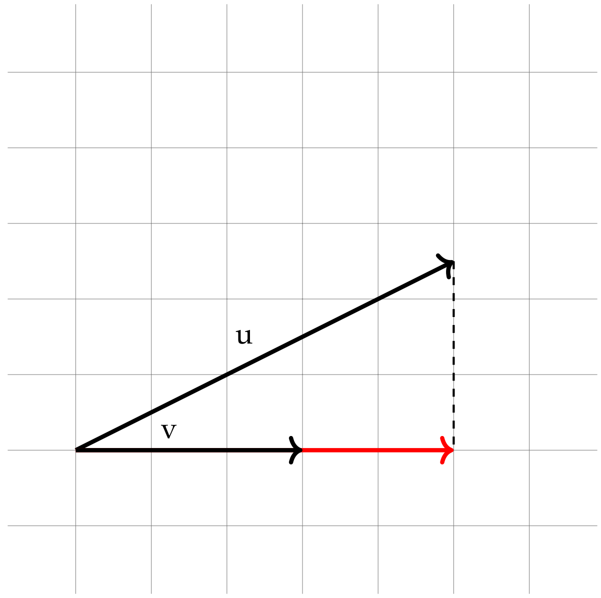

Figure 1.3: Projection vector extended beyond v.

To calculate the projection of **v** onto **u**, use this formula:

$$\text{Projection of } \mathbf{u} \text{ onto } \mathbf{v}: \quad \text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$$

where $\mathbf{u} \cdot \mathbf{v}$ is the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Note that either form of the equation works, and the **v** being multiplied by the dot product quotient cannot be cancelled because it is a vector, not a scalar.

Note that the denominator is the magnitude squared of vector **v**.

$$\left(\sqrt{a_1^2 + a_2^2 + ... + a_n^2}\right)^2$$

You learned previously that this is the same as the dot product of a vector with itself.

$$v \cdot v$$

In the examples that follow, we will simplify to the dot product notation.

Let's look at a specific example:

$$u = (1, 4, 6)$$
$$v = (-2, 6, 2)$$

$$\mathbf{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{u \cdot v}{\| v \|^2} v$$

$$\mathbf{proj_v}(\mathbf{u}) = \frac{(1,4,6)\cdot(-2,6,2)}{(-2,6,2)\cdot(-2,6,2)}(-2,6,2)$$
$$\mathbf{proj_v}(\mathbf{u}) = (\frac{34}{44}(-2,6,2)$$
$$\mathbf{proj_v}(\mathbf{u}) = (-1.545, 4.64, 1.545)$$

As you work your way through this course, you will have a chance to apply the calculations you learn in this chapter to a variety of problems. Specifically, the next chapter shows how to transform a set of linearly independent vectors into a set of orthogonal ones. Projections are essential to that transformation.

Exercise 1 Projections

Find the projection of **a** on **b** where:

$$a = (1,3)$$
$$b = (-4,6)$$

Working Space

Answer on Page 7

1.1 Projections in Python

Create a file called `projections.py` and enter this code:

```
import numpy as np

# define two vectors
a = np.array([1, 4, 6])
b = np.array([-2, 6, 2])

# use np.dot() to calculate the dot product
projection_a_on_b = (np.dot(a, b)/np.dot(b, b))*b

print("The projection of vector a on vector b is:", projection_a_on_b)
```

1.2 Where to Learn More

Watch this Introduction to Projections from Khan Academy from your digital resources: https://rb.gy/yf0i3

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

Compute dot product of **a** and **b**:

$$1 * -4 + 3 * 6 = -4 + 18 = 14$$

Compute the dot product of **b** and **b**

$$16 + 36 = 52$$

$$14/52 * (-4, 6) = (-1.076, 1.61)$$

Index