

# Volumes with Integrals

Suppose we wanted to know the volume of a theoretical irregular shape (we stipulate theoretical because, if you had this object and a large enough container, you could use displacement to determine the volume of the object). [fixme better intro]

## 1.1 Volume of a Sphere

Below, we will prove the volume of a sphere is given by  $\frac{4}{3}\pi r^3$  using the integral method. Suppose we have a sphere of radius  $r$  centered at the origin (see figure 1.1).

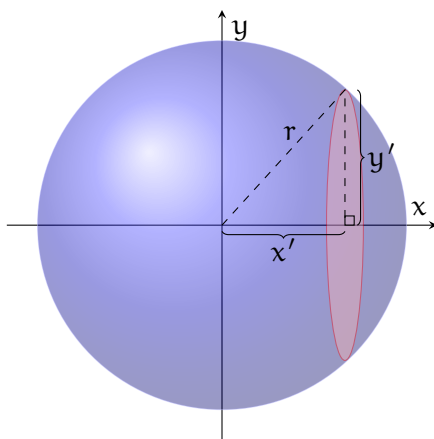


Figure 1.1: A vertical cross-section of a sphere

We begin by taking very thin vertical cross-sections. The radius of the cross-section is the height,  $y$ , of the sphere at the horizontal position,  $x$ . Since the edges of the cross-section lie on the sphere, we know the edge of the cross-section is distance  $r$  from the origin. Applying the Pythagorean theorem, we see that  $r^2 = x^2 + y^2$ , which implies that  $y = \sqrt{r^2 - x^2}$ . So, the area of the cross-section is given by  $\pi y^2 = \pi(r^2 - x^2)$ . If we imagine each cross section as having a width,  $dx$ , and taking the sum of all the cross sections from  $x = -r$  to  $x = r$ , we can write an integral equal to the volume of the sphere:

$$V_{\text{sphere}} = \int_{-r}^r \pi(r^2 - x^2) dx$$

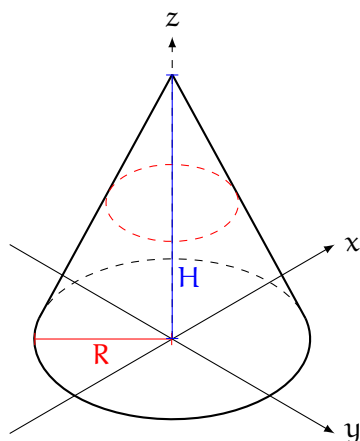
We can then evaluate that integral:

$$V_{\text{sphere}} = \pi \int_{-r}^r r^2 dx - \pi \int_{-r}^r x^2 dx$$

$$\begin{aligned}
 V_{\text{sphere}} &= \pi \left[ r^2 x \right]_{x=-r}^{x=r} - \frac{\pi}{3} \left[ x^3 \right]_{x=-r}^{x=r} \\
 V_{\text{sphere}} &= \pi \left[ r^3 - (-r^3) \right] - \frac{\pi}{3} \left[ r^3 - (-r^3) \right] \\
 V_{\text{sphere}} &= 2\pi r^3 - \frac{2\pi}{3} r^3 = \frac{4}{3} \pi r^3
 \end{aligned}$$

**Exercise 1**

Prove the volume of a regular cone is  $\frac{\pi}{3} R^2 H$ , where  $R$  is the radius of the base and  $H$  is the height of the cone. (Hint: A cone is a series of decreasing circles stacked on top of each other; see figure below.)



*Working Space*

*Answer on Page 13*

## 1.2 Volumes of Solids of Revolution

We can also find the volume of solids made by revolving a graph about the  $x$  or  $y$ -axis. Suppose the graph  $y = \sin x$  from  $x = 0$  to  $x = \frac{\pi}{2}$  were rotated vertically about the  $x$ -axis to form a solid. How could we find the volume of that solid? Well, we can imagine a rectangle of width  $dx$  and height  $y$  (see figure 1.2)

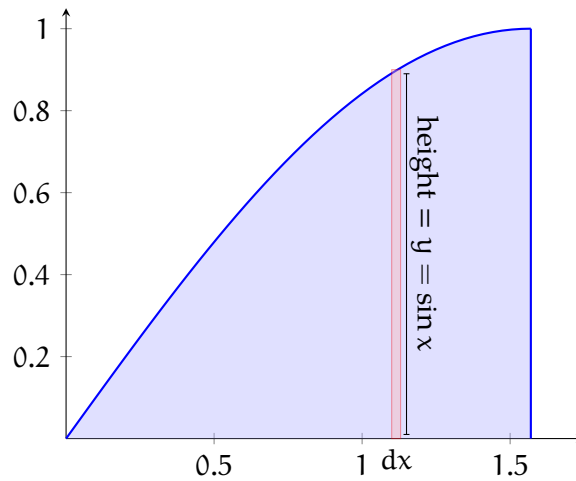


Figure 1.2: A cross section has width  $dx$  and height  $y = \sin x$

If we rotate the plot vertically about the  $x$ -axis, the rectangle becomes a cylinder with radius  $y = \sin x$  and height  $dx$  (see figure ??). Therefore, the volume of each cylindrical slice is  $V_{\text{slice}} = \pi r^2 dx = \pi \cdot \sin^2 x dx$ .

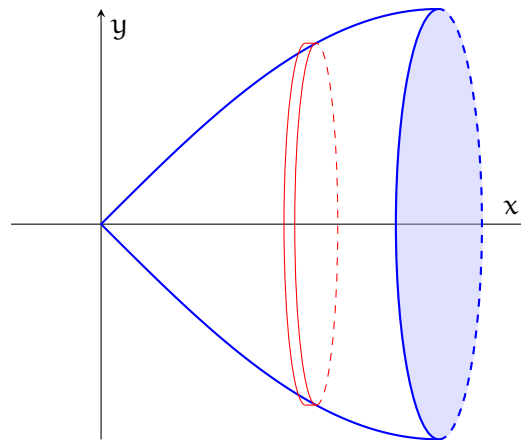


Figure 1.3: When rotated, the cross-section becomes a cylinder with radius  $\sin x$  and width  $dx$ , which has a total volume of  $\pi \sin^2 x dx$

We can find the total volume by integrating from 0 to  $\pi/2$ :

$$V = \pi \int_0^{\pi/2} \sin^2 x \, dx$$

Recall the half angle formula,  $\sin^2 x = \frac{1}{2} (1 - \cos 2x)$ . Substituting, we see that:

$$V = \frac{\pi}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$$

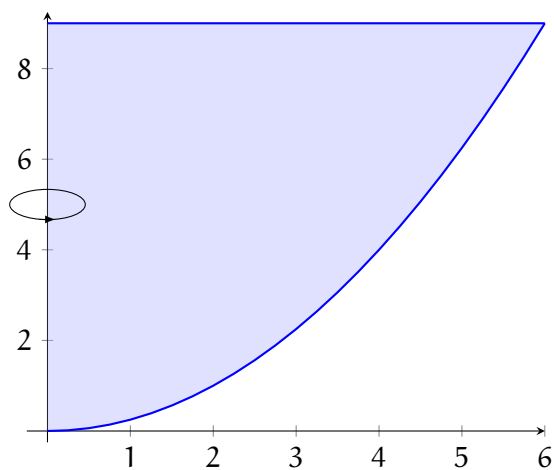
$$V = \frac{\pi}{2} \left( x - \frac{1}{2} \sin 2x \right) \Big|_{x=0}^{x=\pi/2}$$

$$V = \frac{\pi}{2} \left[ \left( \pi/2 - \frac{1}{2} \sin \pi \right) - \left( 0 - \frac{1}{2} \sin 0 \right) \right]$$

$$V = \frac{\pi}{2} [\pi/2 - 0 - 0 + 0] = \frac{\pi^2}{4}$$

**Exercise 2**

Find the volume of a solid created by rotating the region bounded by  $x = 2\sqrt{y}$ ,  $x = 0$ , and  $y = 9$  about the  $y$ -axis. A graph is shown below.

*Working Space**Answer on Page 14*

**Exercise 3**

Let  $f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$ . Bird's eggs of various sizes can be modeled by rotating  $f(x)$  about the  $x$ -axis, with different values of  $a$ ,  $b$ ,  $c$ , and  $d$  defining different sizes and shapes of eggs. For a domestic chicken,  $a = -0.02$ ,  $b = 0.03$ ,  $c = 0.12$ , and  $d = 0.454$ . For a mallard duck,  $a = -0.06$ ,  $b = 0.04$ ,  $c = 0.1$ , and  $d = 0.54$ . Use a calculator, such as a TI-89 or Wolfram Alpha, to determine which species lays a bigger egg.

*Working Space*

*Answer on Page 14*

**1.2.1 Using donuts for solids of revolution**

Sometimes there is space between the region we are rotating and the line we are rotating it about. Consider the region bounded between  $y = 2x$  and  $y = x^2$  (see figure 1.4):

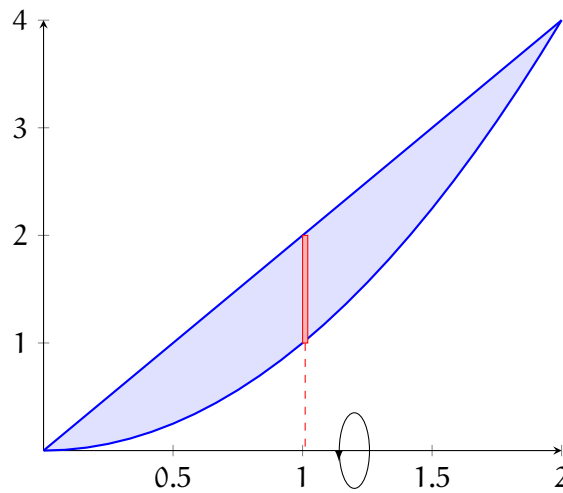


Figure 1.4: When rotated, the slices will become donuts with outer radius  $2x$  and inner radius  $x^2$

When rotated, the slices will take the form of donuts (or washers), the volume of which is  $\pi (R^2 - r^2) dx$ , where  $R$  is the outer radius and  $r$  is the inner radius. Therefore, in this case, the total volume of the rotated region is given by:

$$\begin{aligned}
 V &= \int_0^2 \pi \left[ (2x)^2 - (x^2)^2 \right] dx \\
 V &= \pi \int_0^2 4x^2 - x^4 dx = \pi \left[ \frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{x=0}^{x=2} \\
 V &= \pi \left[ \frac{4}{3}2^3 - \frac{1}{5}2^5 \right] = \pi \left[ \frac{32}{3} - \frac{32}{5} \right] \\
 V &= \frac{64\pi}{15}
 \end{aligned}$$

**Exercise 4**

What is the volume of the region bounded by  $y = x^2$  and  $y = 2\sqrt{x}$  when rotated about the  $y$ -axis?

*Working Space*

*Answer on Page 15*

**1.3 Volumes of Other Solids**

You can also model a solid as a base defined by a function with cross-sections of specific shapes. Consider the function  $y = x^2$  from  $x = 0$  to  $x = 2$  ( see figure 1.5). Suppose the area between the curve, the  $y$ -axis, and the line  $y = 4$  defines a base and each vertical cross-section is a square. So, the width of the each cross section is  $dx$ , the length is  $4 - x^2$ , and (because they are squares) the height in the  $z$ -plane is also  $4 - x^2$ . The volume of each cross-section is  $V_{\text{slice}} = (4 - x^2)^2 dx$  and the total volume of the solid is:

$$V = \int_0^2 (4 - x^2)^2 dx$$

$$V = \int_0^2 (16 - 8x^2 + x^4) dx$$

$$V = \left[ 16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_{x=0}^{x=2}$$



$$V = 16(2) - \frac{8}{3}(2)^3 + \frac{1}{5}(2)^5 = \frac{256}{15} \approx 17.067$$

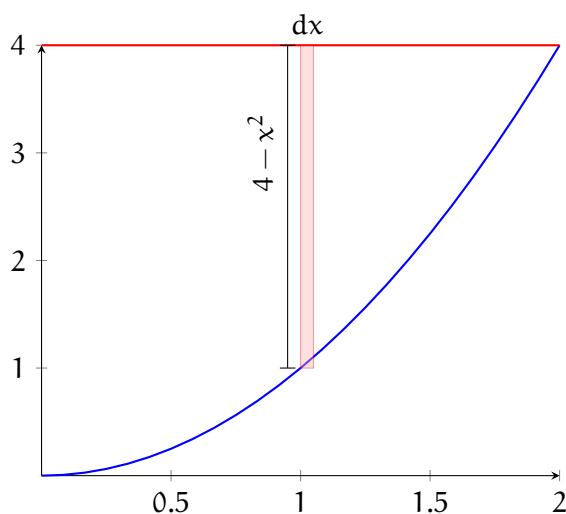


Figure 1.5:  $y = x^2$  with a vertical cross-section

You can use a similar method for triangular, semi-circular, or any other shape cross-section. The trick is writing everything in terms of  $x$  (when your cross sections are vertical and have width  $dx$ ) or  $y$  (when your cross section are horizontal and have length  $dy$ ).

**Exercise 5**

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let  $R$  be the region in the first quadrant bounded above by the graph  $y = \ln(3 - x)$ , for  $0 \leq x \leq 2$ .  $R$  is the base of a solid for which each cross section perpendicular to the  $x$ -axis is square. What is the volume of the solid? Give your answer to 3 decimal places.

*Working Space*

*Answer on Page 16*

**Exercise 6**

Find the volume of a solid whose base is defined by the ellipse  $9x^2 + 16y^2 = 25$  and is made up of isosceles-triangular cross-sections perpendicular to the  $x$ -axis (with the hypotenuse in the base of the solid).

*Working Space*

*Answer on Page 16*

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*This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.*

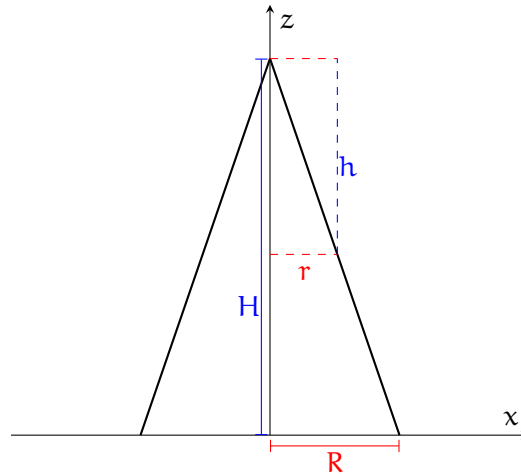


# Answers to Exercises

## Answer to Exercise 1 (on page 2)

Imagine a side view of the cone (see figure below), an isosceles triangle with height  $H$  and base  $2R$ . If we take horizontal cross-sections, then each cross-section is a circle  $h$  from the top with a radius  $r$ . Because the triangles are similar (FIXME: better wording/explanation here), we also know that  $\frac{H}{h} = \frac{R}{r}$ . Therefore, we can define  $r$  in terms of  $h$ :  $r = \frac{hR}{H}$  and the volume of each subsequent cross-section is  $\pi r^2 dh = \pi \frac{h^2 R^2}{H^2} dh$ . We start with  $h = 0$  and end with  $h = H$ :

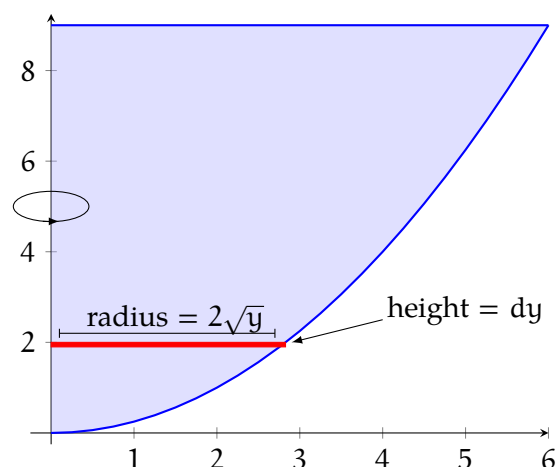
$$\begin{aligned} V_{\text{cone}} &= \int_0^H \pi \frac{h^2 R^2}{H^2} dh = \pi \frac{R^2}{H^2} \int_0^H h^2 dh \\ &= \pi \frac{R^2}{H^2} \left[ \frac{1}{3} h^3 \right]_{h=0}^{h=H} = \pi \frac{R^2}{3H^2} [H^3 - 0^3] \\ &= \pi \frac{R^2}{3H^2} H^3 = \frac{\pi}{3} R^2 H \end{aligned}$$



## Answer to Exercise 2 (on page 5)

If we are rotating about the  $y$  axis, we should make our slices horizontal, so their width is  $dy$  (see graph below). Then, the volume of each cylinder is given by  $V = \pi r^2 dy$  and the total volume is given by:

$$\begin{aligned} V &= \int_0^9 \pi [2\sqrt{y}]^2 dy \\ V &= 4\pi \int_0^9 y dy = 2\pi y^2 \Big|_{y=0}^{y=9} \\ V &= 2\pi (9)^2 = 162\pi \end{aligned}$$



## Answer to Exercise 3 (on page 6)

Since the graph is rotated around the  $x$ -axis, we will take vertical slices with width  $dx$ , and rotate them to make cylinders with radius  $f(x)$  and height  $dx$ . The volume of each egg is given by:

$$\int_{-1}^1 \pi [f(x)]^2 dx$$

To determine our limits of integration, we note that  $\sqrt{1-x^2} = 0$  (and therefore,  $f(x) = 0$ ) when  $x = \pm 1$ .

For the chicken:

$$V_{\text{chickenegg}} = \pi \int_{-1}^1 \left[ (-0.02x^3 + 0.03x^2 + 0.12x + 0.454) \sqrt{1-x^2} \right]^2 dx$$

For the mallard duck:

$$V_{\text{duckegg}} = \pi \int_{-1}^1 \left[ \left( -0.06x^3 + 0.04x^2 + 0.1x + 0.54 \right) \sqrt{1-x^2} \right]^2 dx$$

Using a calculator, we find that  $V_{\text{chickenegg}} \approx 0.897$  and  $V_{\text{duckegg}} \approx 1.263$ . Therefore, mallard ducks lay larger eggs than chickens do.

## Answer to Exercise ?? (on page 8)

First, since we are revolving around the  $y$ -axis, we know our slices will have width  $dy$ . We will rewrite the functions as  $x$  in terms of  $y$ :

$$x = \sqrt{y}$$

$$x = \frac{y^2}{4}$$

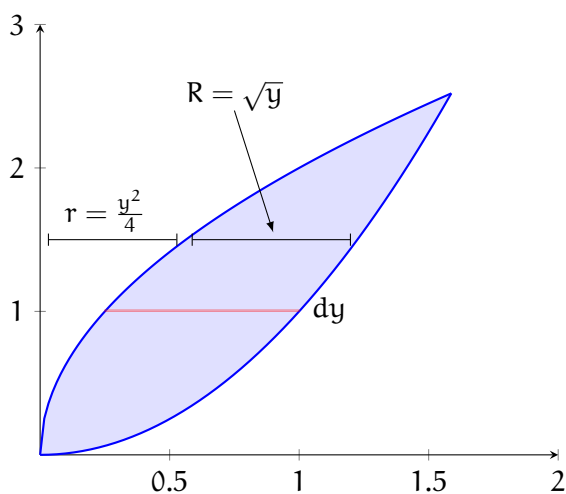
Setting them equal to each other to find the  $y$ -value at which they intercept:

$$\sqrt{y} = \frac{y^2}{4}$$

$$4 = \frac{y^2}{\sqrt{y}} = y^{3/2}$$

$$y = \sqrt[3]{4^2} = 2\sqrt[3]{2}$$

Examining a graph (shown below), we see that the outer radius is  $x = \sqrt{y}$  and the inner radius is  $x = \frac{y^2}{4}$ .



So, the total volume of the solid of revolution is given by:

$$V = \pi \int_0^{2\sqrt[3]{2}} (\sqrt{y})^2 - \left(\frac{y^2}{4}\right)^2 dy$$

$$V = \pi \int_0^{2\sqrt[3]{2}} \left[ y - \frac{y^4}{16} \right] dy$$

$$V = \pi \left[ \frac{1}{2}y^2 - \frac{1}{80}y^5 \right]_{y=0}^{y=2\sqrt[3]{2}}$$

$$V = \pi \left[ \frac{6}{5}2^{2/3} \right] \approx 5.9844$$

### Answer to Exercise 5 (on page 10)

If each cross section is a square, then the volume of each cross section is given by  $s^2 dx$ , where  $s$  is the side length of the square. Since the side length is equal to the distance between the graph of  $y$  and the  $x$ -axis, we can see that  $s = y = \ln(3 - x)$ . And, therefore, the total volume of all the cross sections is given by  $\int_0^2 [\ln(3 - x)]^2 dx$ . Using a calculator, this integral evaluates to  $\approx 1.029$ .

### Answer to Exercise 6 (on page 11)

Since the cross-sections are perpendicular to the  $x$ -axis, they will have width  $dx$  and we will integrate across the domain of the ellipse. Setting  $y = 0$  to find the domain of the ellipse:

$$9x^2 = 25 \rightarrow x^2 = \frac{25}{9} \rightarrow x = \pm \frac{5}{3}$$

A right isosceles triangle with hypotenuse  $h$  has area  $\frac{1}{4}h^2$ . In this case, each triangle's hypotenuse is given by the distance between the top and bottom of the ellipse. The top of the ellipse is defined by  $y = \frac{1}{4}\sqrt{25 - 9x^2}$  and the bottom by  $y = -\frac{1}{4}\sqrt{25 - 9x^2}$ . Therefore, the length of each hypotenuse is  $\frac{1}{2}\sqrt{25 - 9x^2}$ .

Then, each cross-section has a total volume of  $\frac{1}{4}h^2 dx = \frac{1}{4} \left( \frac{1}{2}\sqrt{25 - 9x^2} \right)^2 dx$  and the volume of the solid is:

$$\begin{aligned} V_{\text{solid}} &= \int_{-5/3}^{5/3} \frac{1}{4} \left( \frac{1}{2}\sqrt{25 - 9x^2} \right)^2 dx \\ &= \frac{1}{4} \int_{-5/3}^{5/3} \frac{1}{4} (25 - 9x^2) dx \end{aligned}$$



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$$\begin{aligned} &= \frac{1}{6} \int_{-5/3}^{5/3} (25 - 9x^2) \, dx = \frac{1}{16} \left[ 25x - 3x^3 \right]_{x=-5/3}^{x=5/3} \\ &= \frac{1}{16} \left[ \left( 25 \left( \frac{5}{3} \right) - 25 \left( \frac{-5}{3} \right) \right) - \left( 3 \left( \frac{5}{3} \right)^3 - 3 \left( \frac{-5}{3} \right)^3 \right) \right] \\ &= \frac{1}{16} \left[ \frac{250}{3} - \left( \frac{375}{27} + \frac{375}{27} \right) \right] = \frac{1}{16} \left[ \frac{250}{3} - \frac{250}{9} \right] = \frac{1}{16} \left[ \frac{750}{9} - \frac{250}{9} \right] = \frac{1}{16} \left[ \frac{500}{9} \right] = \frac{125}{36} \end{aligned}$$





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