Chapter 1

Antiderivatives

In your study of calculus, you have learned about derivatives, which allow us to find the rate of change of a function at any given point. Derivatives are powerful tools that help us analyze the behavior of functions. Now, we will explore another concept called antiderivatives, which are closely related to derivatives.

An antiderivative, also known as an integral or primitive, is the reverse process of differentiation. It involves finding a function whose derivative is equal to a given function. In simple terms, if you have a function and you want to find another function that, when differentiated, gives you the original function back, you are looking for its antiderivative. Consider the graph of $f(x)$ below. We can sketch a possible antiderivative of f by noting the slope of the antiderivative is equal to the value of f. We will refer to the antiderivative of f as $F(x)$ (that is, $F'(x) = f(x)$).

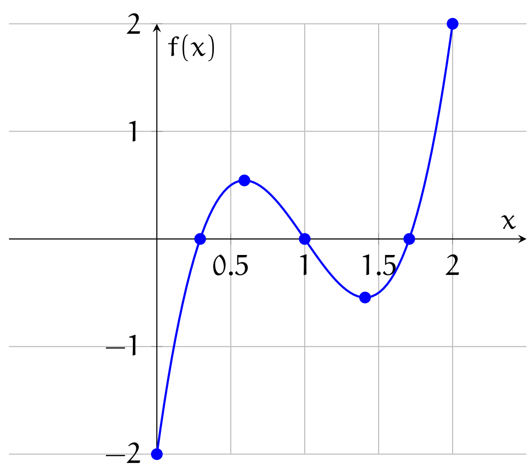

Figure 1.1: Plot of f with select points

If we are given a coordinate for $F(x)$, then we can use the graph of $f(x)$ to sketch $F(x)$. Suppose we know that $F(0) = 1$. From the graph of f, we also know that

x	$f(x) =$ slope of $F(x)$
0	-2
≈ 0.3	0
≈ 0.6	≈ 0.5
1	0
≈ 1.4	≈ -0.4
≈ 1.7	0
2	2

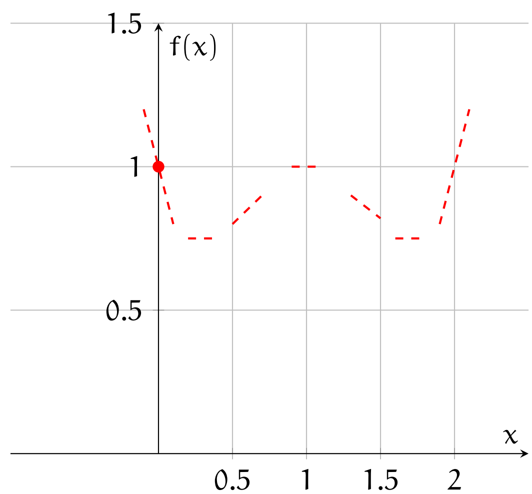

Figure 1.2: Beginning sketch of $F(x)$

Then we can connect these slopes to have an approximate sketch of $F(x)$:

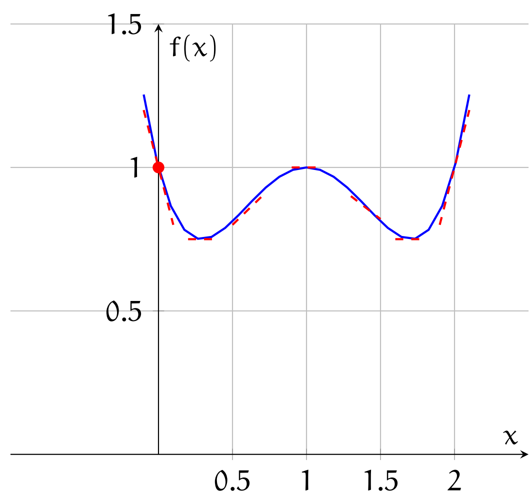

Figure 1.3: Sketch of $F(x)$

The symbol used to represent an antiderivative is $\int$. It is called the integral sign. For example, if $f(x)$ is a function, then the antiderivative of $f(x)$ with respect to x is denoted as $\int f(x)\,dx$. The dx at the end indicates that we are integrating with respect to x.

Another way to state this is that F is the antiderivative of f on an interval, I, if $F'(x) = f(x)$ over the interval. The relationship between f and F is discussed more in the chapter on the Fundamental Theorem of Calculus.

Finding antiderivatives requires using specific techniques and rules. Some common antiderivative rules include:

- The power rule: If $f(x) = x^n$, where n is any real number except -1, then the antiderivative of $f(x)$ is given by $\int f(x)\,dx = \frac{1}{n+1}x^{n+1} + C$, where C is the constant of integration.
- The constant rule: The antiderivative of a constant function is equal to the constant times x. For example, if $f(x) = 5$, then $\int f(x)\,dx = 5x + C$.
- The sum and difference rule: If $f(x)$ and $g(x)$ are functions, then $\int(f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$. Similarly, $\int(f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$.

Antiderivatives have various applications in mathematics and science. They allow us to calculate the total accumulation of a quantity over a given interval, compute areas under curves, and solve differential equations, among other things.

1.1 General Antiderivatives

It is important to note that an antiderivative is not a unique function. Since the derivative of a constant is zero, any constant added to an antiderivative will still be an antiderivative of the original function. This is why we include the constant of integration, denoted by C, in the antiderivative expression.

Stated formally, if F is an antiderivative of f on interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

A concrete example of this is $f(x) = x^2$. Let us define $F(x)$ such that $F'(x) = f(x)$. That is, there is some function F such that the derivative of F is x^2. One possible solution for F is $F(x) = \frac{1}{3}x^3$. You can check using the power rule that $\frac{d}{dx}F(x) = f(x)$. What if we added or subtracted a constant from F? Let us define $G(x) = \frac{1}{3}x^3 + 2$. Well, $G'(x) = f(x)$ also! Same for $H(x) = \frac{1}{3}x^3 - 7$. Several possible antiderivatives of $f(x) = x^2$ are shown in figure 1.4.

Since taking a derivative "erases" any constant, you must always add back in the unknown constant, C, when finding the general antiderivative.

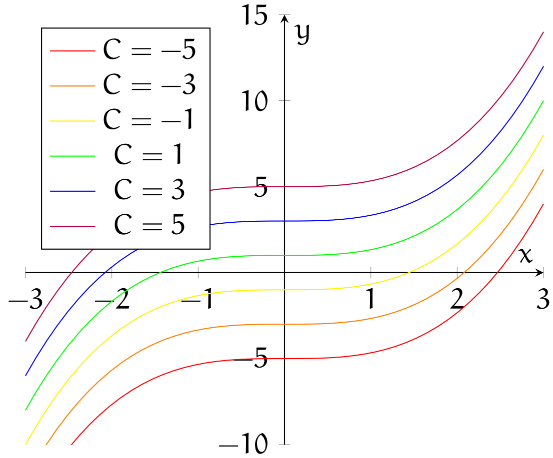

Figure 1.4: If $F'(x) = x^2$, then the general solution is $F(x) = \frac{1}{3}x^3 + C$

1.2 Specific Antiderivatives

If you are given a condition, you can often solve for C and find a specific antiderivative. For example, suppose that in addition to knowing that $F'(x) = x^2$, we also know that $F(3) = 2$. We can use the fact that F passes through $(3, 2)$ to find the value of C:

$$F(x) = \frac{1}{3}x^3 + C$$
$$F(3) = \frac{1}{3}(3)^2 + C = 2$$
$$39 + C = 2$$
$$C = -7$$

Therefore, the specific solution to $F'(x) = x^2$ with the condition that $F(3) = 2$ is $F(x) = \frac{1}{3}x^3 - 7$.

1.3 Antiderivatives of Trig Functions

We already know that $\frac{d}{dx} \sin x = \cos x$. Taking $\sin x$ to be $F(x)$ and $\cos x$ to be $f(x)$, we see that $F'(x) = f(x)$ and therefore $\sin x$ is the antiderivative of $\cos x$.

Exercise 1

What is the antiderivative of $\sin x$? Explain your answer.

Working Space

Answer on Page 11

You should have found that the antiderivative of $\sin x$ is $-\cos x$. Other general antiderivatives of trigonometric functions are presented in the table below.

Function	Antiderivative
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
$-\csc^2 x$	$\cot x + C$
$-\csc x \cot x$	$\csc x + C$

Notice this is the flipped version of the derivatives of trigonometric functions presented in the Trigonometric Functions chapter. This hints at the relationship between derivatives and integrals: they are opposite processes.

1.4 Other Important Antidervatives

The Power Rule only applies when $n \neq -1$. Then what is the antiderivative of $f(x) = \frac{1}{x}$? Recall from the chapters on derivatives that $\frac{d}{dx} \ln x = \frac{1}{x}$ (see figure 1.5). Therefore, the general antiderivative of $\frac{1}{x}$ is $\ln |x| + C$. We have to take the absolute value because of the domain restrictions of $\ln x$. Notice that for $x < 0$, the slope of $\ln |x|$ is negative and decreasing (becoming more negative) and the value of $\frac{1}{x}$ is also negative and decreasing. Similarly, for $x > 0$, the slope of $\ln |x|$ is positive and decreasing (becoming less positive) and the value of $\frac{1}{x}$ is also positive and decreasing.

Since the derivative of e^x is e^x, it follows that the general antiderivative of e^x is $e^x + C$. What if there is a multiplying factor in the exponent, such as e^{kx}? Recall that $\frac{d}{dx} e^{kx} = ke^{kx}$. It follows that $\frac{d}{dx} \frac{1}{k} e^{kx} = e^{kx}$. Therefore, the general antiderivative of e^{kx} is $\frac{1}{k} e^{kx} + C$. (See figure 1.6 for an example where $k = 2$.)

Often, the base of an exponential function isn't e. We can also find the general antiderivative of b^x, where $b \neq e$. Recall that $\frac{d}{dx} b^x = \ln b b^x$. Therefore $\frac{d}{dx} \frac{1}{\ln b} b^x = b^x$, and the general

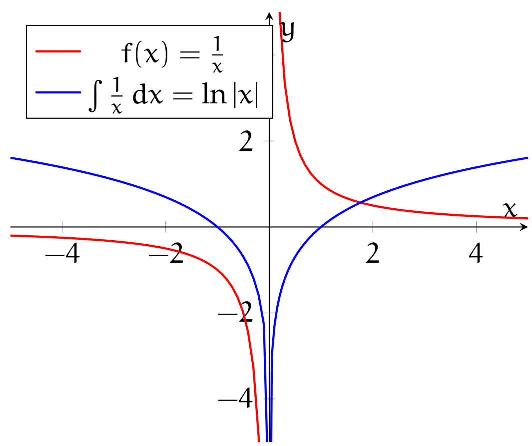

Figure 1.5: $\frac{1}{x}$ and its antiderivative, $\ln|x|$

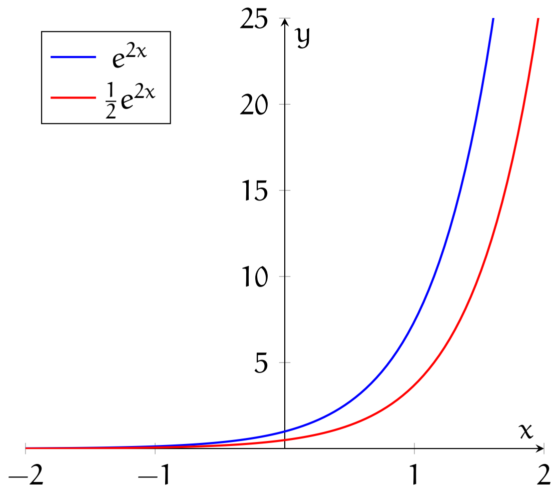

Figure 1.6: e^{2x} and its antiderivative $\frac{1}{2}e^{2x}$

antiderivative of b^x is $\frac{b^x}{\ln b}$.

1.5 Higher order antiderivatives

What if we are given the second order derivative, or a higher order? Take this example: $f''(x) = 2x + 3e^x$. The antiderivative of f'' is f'. Applying the Power Rule and knowing the antiderivative of e^x is e^x, we find that $f'(x) = x^2 + 3e^x + C_1$. We designate the constant as C_1 because we'll have to determine the antiderivative a second time and we don't want to confuse our constants with each other. To find f, we apply the Power Rule again, and we find that $f(x) = \frac{1}{3}x^3 + 3e^x + C_1x + C_2$. You can check if this is correct by taking the derivative of $f(x)$ twice, which should yield the $f''(x)$ originally given.

In summary, antiderivatives are the reverse process of differentiation. They help us find functions whose derivatives match a given function. Understanding antiderivatives is crucial for various advanced calculus concepts and real-world applications.

Now, let's explore different techniques and methods for finding antiderivatives and discover how they can be applied in solving problems.

1.6 Additional Practice

Exercise 2

A particle moving in a straight line has an acceleration given by $a(t) = 6t+4$ (in units of $\frac{cm}{s^s}$). If its initial velocity is $-6\frac{cm}{s}$ and its initial position is $9cm$, what is the function $s(t)$ that describes the particle's position in cm?

Working Space

Answer on Page 11

Exercise 3

Let $f'(x) = 2\sin x$. If $f(\pi) = 1$, write an expression for $f(x)$.

Working Space

Answer on Page 11

Exercise 4

Find the general antiderivatives of the following functions:

1. $f(x) = x^2 + 2x - 4$
2. $g(x) = \sqrt[3]{x^2} + x\sqrt{x}$
3. $h(x) = \frac{1}{5} - \frac{2}{x}$
4. $r(\theta) = 2\sin\theta - \sec^2\theta$

Working Space

Answer on Page 12

Exercise 5

Find the f that satisfies the given conditions:

1. $f'(\theta) = \sin\theta + \cos\theta$, $f(\pi) = 2$
2. $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$ and $f(1) = 1$

Working Space

Answer on Page 12

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

Since we are finding the antiderivative of $\sin x$, we will define $f(x) = \sin x$. We are looking for a F such that $F'(x) = \sin x$. The derivative of $\cos x$ is $-\sin x \neq f(x)$. But the derivative of $-\cos x = \sin x = f(x)$. Since $\frac{d}{dx}[-\cos x] = \sin x$, the antiderivative of $\sin x$ is $-\cos x$.

Answer to Exercise 2 (on page 7)

First, we will find $v(t)$ by taking the antiderivative of $a(t)$ and using the initial condition $v(0) = -6$

$$\int 6t + 4\,, dt = 3t^2 + 4t + C = v(t)$$

$$v(0) = 3(0)^2 + 4(0) + C = -6$$

$$C = -6$$

Therefore, the velocity function is $v(t) = 3t^2 + 4t - 6$. Now we repeat the process to find $s(t)$:

$$\int 3t^2 + 4t - 6\,, dt = t^3 + 2t^2 - 6t + C = s(t)$$

$$s(0) = (0)^3 + 2(0)^2 - 6(0) + C = 9$$

$$C = 9$$

Therefore, the position function is $s(t) = t^3 + 2t^2 - 6t + 9$.

Answer to Exercise 3 (on page 8)

The antiderivative of $\sin x$ is $-\cos x$ and therefore the general solution is $f(x) = -2\cos x + C$. We use the given condition, $f(\pi) = 1$ to find C:

$$f(\pi) = -2\cos\pi + C = 1$$

$$C = 1 + 2\cos\pi = 1 + 2(-1) = -1$$

Therefore, the specific solution is $f(x) = -2\cos x - 1$

Answer to Exercise 4 (on page 8)

1. By the power rule, the antiderivative of x^2 is $\frac{1}{3}x^3$, the antiderivative of $2x$ is x_2, and the antiderivative of 4 is $4x$. So the general antiderivative of $f(x)$ is $\frac{1}{3}x^3 + x^2 - 4x + C$
2. We can rewrite $g(x)$ to more clearly see the powers of x. $g(x) = x^{\frac{2}{3}} + x^{\frac{3}{2}}$. Applying the Power rule, we find the general antiderivative of $g(x)$ is $\frac{3}{5}x^{\frac{5}{3}} + \frac{2}{5}x^{\frac{5}{2}} + C$.
3. Recalling that the antiderivative of $\frac{1}{x}$ is $\ln|x|$, the general antiderivative of $h(x)$ is $\frac{1}{5}x - 2\ln|x| + C$
4. The antiderivative of $\sin\theta$ is $-\cos\theta$ and the antiderivative of $\sec^2\theta$ is $\tan x$. Therefore, the general antiderivative of $r(\theta)$ is $-2\cos\theta - \tan\theta + C$

Answer to Exercise 5 (on page 8)

1. The antiderivative of $\sin\theta$ is $-\cos\theta$ and the antiderivative of $\cos\theta$ is $\sin\theta$. The general form of f is $f(\theta) = -\cos\theta + \sin\theta + C$. Substituting $\theta = \pi$, we find that $f(\pi) = -\cos\pi + \sin\pi + C = 1 + 0 + C = 2$, which implies $C = 1$. Therefore $f(\theta) = -\cos\theta + \sin\theta + 1$.
2. The general antiderivative of f'' is $f'(x) = 4x^3 + 3x^2 - 4x + C_1$. We don't have a condition for f', so we continue to find f. The antiderivative of f' is $f(x) = x^4 + x^3 - 2x^2 + C_1x + C_2$. We can find C_2 with the condition $f(0) = 4$. $f(0) = C_2 = 4$, so we know $f(x) = x^4 + x^3 - 2x^2 + C_1x + 4$. Using the condition $f(1) = 1$, we find that $C_1 = 3$. Therefore, the specific solution is $f(x) = x^4 + x^3 - 2x^2 + 3x - 4$.

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