Chapter 1

Rules for Finding Derivatives

Derivatives play a key role in calculus, providing us with a means of calculating rates of change and the slopes of curves. Here, we present some common rules used to calculate derivatives.

1.1 Constant Rule

The derivative of a constant is zero. If c is a constant and x is a variable, then:

$$\frac{d}{dx}c = 0 \tag{1.1}$$

1.2 Power Rule

For any real number n, the derivative of x^n is:

$$\frac{d}{dx}x^n = nx^{n-1} \tag{1.2}$$

1.3 Product Rule

The derivative of the product of two functions is:

$$\frac{d}{dx}(fg) = f'g + fg' \tag{1.3}$$

where f' and g' denote the derivatives of f and g, respectively.

1.4 Quotient Rule

The derivative of the quotient of two functions is:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \tag{1.4}$$

1.5 Chain Rule

The derivative of a composition of functions is:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \tag{1.5}$$

1.6 Practice

Exercise 1

If f is the function given, find f'.

1. $f(x) = x \sin x$
2. $f(x) = (x^3 - \cos x)^5$
3. $f(x) = \sin^3 x$

Working Space

Answer on Page 7

Exercise 2

Let $f(x) = 7x - 3 + \ln x$. Find $f'(x)$ and $f'(1)$

Working Space

Answer on Page 7

Exercise 3

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] The position of a particle in the xy-plane is given by the parametric equations $x(t) = t^3 - 3t^2$ and $y(t) = 12t - 3t^2$. State a coordinate point (x, y) at which the particle is at rest.

Working Space

Answer on Page 7

Exercise 4

Let $f(x) = \sqrt{x^2 - 4}$ and $g(x) = 3x - 2$. Find the derivative of $f(g(x))$ at $x = 3$.

Working Space

Answer on Page 7

Exercise 5

The a particle's position on the x-axis is given by $x(t) = (t - a)(t - b)$, where a and b are constants and $a \neq b$. At what time(s) is the particle at rest?

Working Space

Answer on Page 8

Exercise 6

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] Let $f(x) = \frac{x}{x+2}$. At what values of x does f have the property that the line tangent to f has a slope of $\frac{1}{2}$?

Working Space

Answer on Page 8

Exercise 7

For $t \geq 0$, the position of a particle moving along the x-axis is given by $x(t) = \sin t - \cos t$. (a) When does the velocity first equal 0? (b) What is the acceleration at the time when the velocity first equals 0?

Working Space

Answer on Page 8

Exercise 8

The graph of $y = e^(\tan x) - 2$ crosses the x-axis at one point on the interval $[0, 1]$. What is the slope of the graph at this point?

Working Space

Answer on Page 9

Exercise 9

The function f is define by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.
(a) Find $f'(x)$.
(b) Write an equation for the line tangent to the graph at $x = -3$.

Working Space

Answer on Page 9

Exercise 10

For $0 \leq t \leq 12$, a particle moves along the x-axis. The velocity of the particle at a time t is given by $v(t) = \cos \frac{\pi}{6} t$. What is the acceleration of the particle at time $t = 4$?

Working Space

Answer on Page 10

Exercise 11

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let f and g be the functions given by $f(x) = e^x$ and $g(x) = x^4$. On what intervals is the rate of change of $f(x)$ greater than the rate of change of $g(x)$?

Working Space

Answer on Page 10

1.7 Conclusion

These rules form the basis for calculating derivatives in calculus. Many more complex rules and techniques are built upon these fundamental rules.

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

1. $\frac{dy}{dx} = \frac{d}{dx}[x \sin x] = x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x = x(-\cos x) + \sin x(1) = \sin x - x\cos x$
2. By the chain rule, $f'(x) = 5(x^3 - \cos x)^4 \cdot \frac{d}{dx}(x^3 - \cos x) = 5(x^3 - \cos x)^4 \cdot (3x^2 + \sin x)$
3. By the chain rule, $f'(x) = \frac{d}{d(\sin x)}[\sin^3 x] \times \frac{d}{dx}\sin x = 3\sin^2 x \cdot \cos x$

Answer to Exercise 2 (on page 2)

$f'(x) = \frac{d}{dx}(7x) - \frac{d}{dx}(3) + \frac{d}{dx}(\ln x) = 7 - 0 + \frac{1}{x} = 7 - \frac{1}{x}$ and $f'(1) = 7 - \frac{1}{1} = 6$

Answer to Exercise 3 (on page 3)

The particle is at rest when $x'(t) = y'(t) = 0$. First, we find each of the derivatives:

$$x'(t) = 3t^2 - 6t$$
$$y'(t) = 12 - 6t$$

We can solve $y' = 0$ for t and find that the y-velocity is 0 when $t = 2$. Substituting $t = 2$ into our expression for x', we find $x'(2) = 3(2)^2 - 6(2) = 0$. Therefore, the particle is at rest when $t = 0$. to find the xy-coordinate, we substitute $t = 2$ into $x(t)$ and $y(t)$:

$$x(2) = (2)^3 - 3(2)^2 = 8 - 12 = -4$$
$$y(2) = 12(2) - 6(2) = 24 - 12 = 12$$

Therefore, the particle is at rest when it is located at $(-4, 12)$.

Answer to Exercise 4 (on page 3)

$f(g(x)) = \sqrt{(3x-2)^2 - 4} = \sqrt{9x^2 - 12x}$ and $\frac{d}{dx}f(g(x)) = \frac{18x-12}{2\sqrt{9x^2-12x}}$. Substituting $x = 3$, we find $f'(g(x)) = \frac{18(3)-12}{2\sqrt{9(3)^2-12(3)}} = \frac{42}{2\sqrt{45}} = \frac{21}{3\sqrt{5}} = \frac{7}{\sqrt{5}}$

Answer to Exercise 5 (on page 3)

First, recall that the velocity of a particle is the derivative of its position function. Therefore, $v(t) = x'(t) = \frac{d}{dt}[(t-a)(t-b)]$. Applying the Product Rule for derivatives, we see that $v(t) = (t-a)(1) + (t-b)(1) = 2t - a - b$. To find the time(s) when the particle is at rest, we set $v(t) = 0$ and solve for t.

$$0 = 2t - a - b$$
$$2t = a + b$$
$$t = \frac{a+b}{2}$$

Answer to Exercise 6 (on page 4)

The question is asking when the derivative of f is $\frac{1}{2}$. We will take the derivative and set it equal to $\frac{1}{2}$.

$$f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{2}{(x+2)^2}$$
$$\frac{2}{(x+2)^2} = \frac{1}{2}$$
$$4 = (x+2)^2$$
$$\pm 2 = x + 2$$
$$x = 2 - 2 = 0 \text{ and } x = -2 - 2 = -4$$

Answer to Exercise 7 (on page 4)

(a) Let t_0 be the time at which the particle is first at rest. The velocity of the particle is given by $v(t) = x'(t) = \cos t + \sin t$. Setting $v(t) = 0$, we find:

$$\cos t = -\sin t$$

which is true for $t = \frac{3\pi + 4n}{4}$, where n is an integer. Therefore, the first time the velocity is 0 is $t_0 = \frac{3\pi}{4}$.
(b) To find the acceleration at $t = \frac{3\pi}{4}$, we take the derivative of the velocity function to yield the acceleration function.

$$a(t) = v'(t) = -\sin t + \cos t$$

. Substituting $t = \frac{3\pi}{4}$, we find the acceleration is $-\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

Answer to Exercise 8 (on page 4)

First, we find the x such that $y = 0$

$$0 = e^{\tan x} - 2$$

$$2 = e^{\tan x}$$

$$\ln 2 = \tan x$$

$$x = \arctan(\ln 2) = \arctan 0.693 \approx 0.606$$

Then, we find the slope of the function at $x = 0.606$ by finding $y'(0.606)$

$$y' = e^{\tan x}(\sec x)^2 = \frac{e^{\tan x}}{(\cos x)^2}$$

$$y'(0.606) = \frac{e^{\tan 0.606}}{(\cos 0.606)^2} = 2.961$$

Answer to Exercise 9 (on page 5)

(a) Apply the chain rule to find $f'(x)$

$$f'(x) = \frac{1}{2\sqrt{25 - x^2}} \cdot (-2x) = \frac{-x}{\sqrt{25 - x^2}}$$

.

(b) First, substitute $x = -3$ into $f'(x)$

$$f'(-3) = \frac{-(-3)}{\sqrt{25 - (-3)^2}} = \frac{3}{\sqrt{16}} = \frac{3}{4}$$

This is the slope of the line. To complete an equation for the tangent line, we need a point. We know the tangent line touches $f(x)$ at $x = -3$, so the tangent line must pass through the point $(-3, f(-3))$.

$$f(-3) = \sqrt{25 - (-3)^2} = 4$$

We use $m = \frac{3}{4}$ and the coordinate point $(x_1, y_1) = (-3, 16)$ to complete the equation $y - y_1 = m(x - x_1)$

$$y - 16 = \frac{3}{4}(x + 3)$$

Answer to Exercise 10 (on page 5)

$$a(t) = v'(t) = -\frac{\pi}{6}\sin\frac{\pi}{6}t$$

$$a(4) = -\frac{\pi}{6}\sin\frac{2\pi}{3} = -\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{12}$$

Answer to Exercise 11 (on page 5)

Recall that the rate of change of a function is given by the derivative of that function. Therefore, we are looking for the interval(s) where $f'(x) > g'(x)$. First, we find each derivative:

$$f'(x) = e^x$$

$$g'(x) = 4x^3$$

We are looking for x-values such that $e^x > 4x^3$. This inequality can be restated as $e^x - 4x^3 > 0$. Using a calculator, you should find that $e^x - 4x^3 = 0$ when $x \approx 0.831$ and $x \approx 7.384$. We will check values on either side of and in the interval $x \in (0.831, 7.384)$ to determine the sign value of $e^x - 4x^3$. We know that when $x = 0$, $e^x - 4x^3 > 0$, when $x = 5$, $e^x - 4x^3 < 0$, and when $x = 10$, $e^x - 4x^3 > 0$. Therefore, $f'(x)$ is greater than $g'(x)$ on the open intervals $x \in (-\infty, 0.831) \cup (7.384, \infty)$.

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