Chapter 1

Polar Coordinates

We have already seen how to plot a function with (x, y) coordinates. For every x that we put into a function, it returns a y. These pairs of coordinates tell us where on the xy-plane to graph the function. This coordinate system, where x and y are oriented horizontally and vertically, is called the *Cartesian* coordinate system. It can be used to describe 2D space, but it is not the only way.

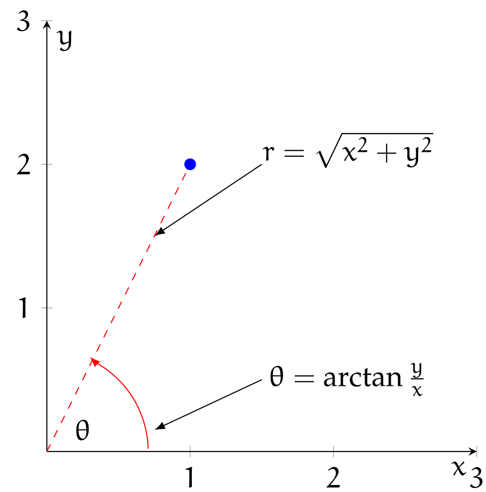


Figure 1.1: The point $(1, 2)$ is $\sqrt{5}$ units from the origin and approximately 1.107 radians counterclockwise from horizontal

Instead of thinking about the horizontal and vertical position, we could think about distance from the origin and rotation about the origin. Take the Cartesian coordinate point $(1, 2)$ (see figure 1.1). How far is $(1, 2)$ from the origin, $(0, 0)$? We can create a right triangle, where the legs are parallel to the x and y axes. This means the leg lengths are 1 and 2, and we can use the Pythagorean theorem to find the length of the hypotenuse (which is the distance from the origin to the point):

$$c^2 = a^2 + b^2$$

$$c^2 = 1^2 + 2^2 = 1 + 4 = 5$$

$$c = \sqrt{5}$$

Therefore, the Cartesian point $(1, 2)$ is $\sqrt{5}$ units from the origin. This is not enough to find our point: there are infinite points that are $\sqrt{5}$ from the origin (see 1.2). To identify a particular point that is a distance of $\sqrt{5}$ from the origin, we also need an *angle of rotation*. By convention, angles are measured from the positive x- axis. This means points on the positive x-axis have an angle of $\theta = 0$, points on the positive y-axis have an angle of $\theta = \frac{\pi}{2}$, and so on.

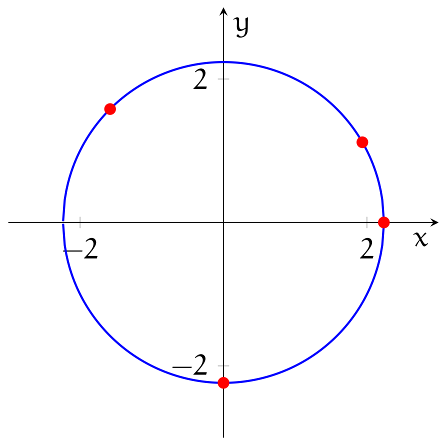


Figure 1.2: There are infinite points $\sqrt{5}$ from the origin, represented by the circle with a radius of $\sqrt{5}$ centered about the origin

We can use trigonometry to find the appropriate angle of rotation for our Cartesian point. There are many ways to do this, but using arctan is the most straightforward. Recall that:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

That is, for a given angle in a right triangle, the tangent of that angle is given by the length of the opposite leg divided by the adjacent leg. In our case, the opposite leg is the vertical distance (y-value of the Cartesian point) and the adjacent leg is the horizontal distance (x-value of the Cartesian point), which means:

$$\tan\theta = \frac{2}{1}$$

$$\theta = \arctan 2 \approx 1.107 \text{ radians}$$

1.1 Plotting Polar Coordinate Points

How do we plot polar coordinate points? Begin by locating the angle given by the second coordinate (remember, the angle is measured counterclockwise from the horizontal). Your point will lie somewhere on this line. Next, move outwards along the angle by the radius given by the first coordinate.

Example: Plot the polar coordinate point $(2, \frac{\pi}{3})$.

Solution: Begin by locating $\theta = \frac{\pi}{3}$ (see figure 1.3)

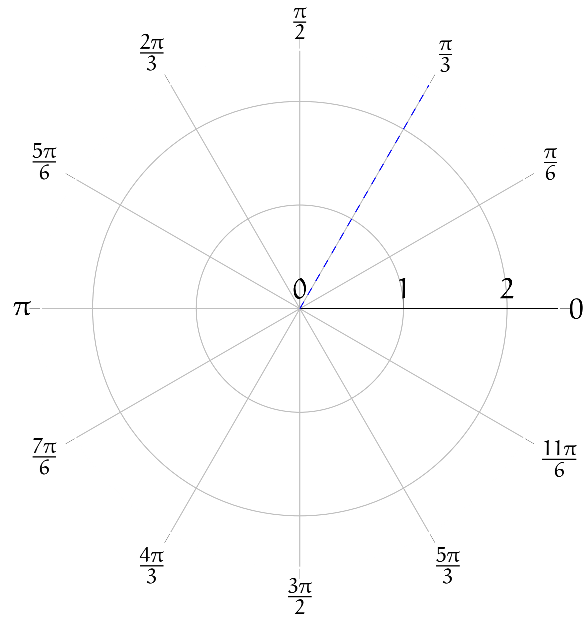


Figure 1.3: $\theta = \frac{\pi}{3}$

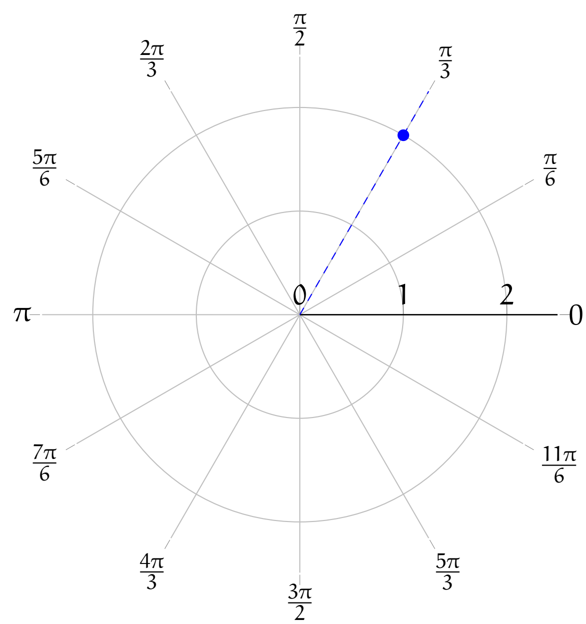


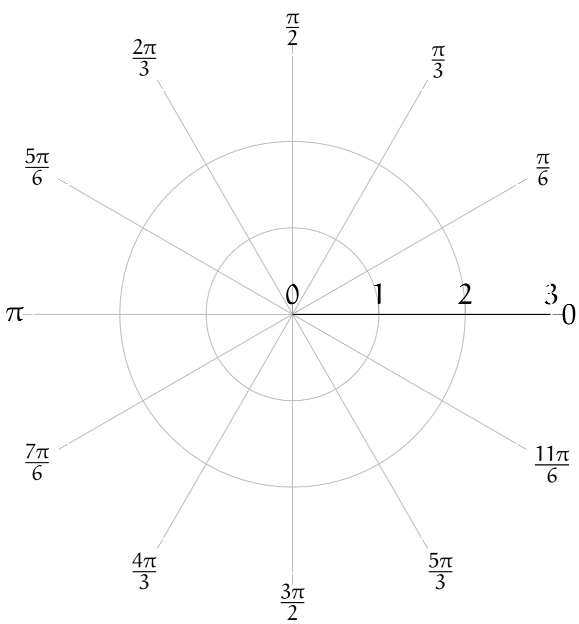
Figure 1.4: $(2, \frac{\pi}{3})$

Then, move your finger or pencil along the line $\theta = \frac{\pi}{3}$ until you reach $r = 2$ (see figure 1.4).

Exercise 1

Plot the following polar coordinate points on the provided polar axis (hint: negative angles are taken counterclockwise):

1. $(1, \pi)$
2. $(1.5, \frac{\pi}{2})$
3. $(1.5, -\frac{\pi}{6})$
4. $(2, \frac{3\pi}{4})$



Working Space

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1.2 Equivalent Points

Unlike the Cartesian coordinate system, two different coordinates may lie at the same location. Consider the points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ (see figure 1.5). When a radius is negative, you move *backwards* back over the origin, like a mirror image.

1.3 Changing coordinate systems

1.3.1 Cartesian to Polar

From the example above, you should see that a given Cartesian coordinate, (x, y), can also be expressed as a polar coordinate, (r, θ), where r is the distance from the origin and θ is the angle of rotation from the horizontal. (Note: Polar functions are generally given as r defined in terms of θ, which means the *dependent* variable is listed first in the coordinate pair, unlike Cartesian coodinates.) Additionally,

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

Example: Express the Cartesian point $(-3, 4)$ in polar coordinates.

Solution: Taking $x = -3$ and $y = 4$, we find that:

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

We follow the convention of only taking the positive solution to the square root. Finding θ:

$$\theta = \arctan \frac{4}{-3}$$

When you evaluate the arctan with a calculator, you are likely to get back $\theta = -0.928$. Recall that $tan\theta = \tan\theta \pm n\pi$, where n is an integer. We know our Cartesian point, $(-3, 4)$, is in the II quadrant, while the angle -0.928 radians would fall in the IV quadrant. So, clearly, -0.928 radians is not correct. Most calculators restrict the output of arctan to angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because there are actually multiple angles where $\tan\theta = -\frac{4}{3}$. Since $\tan\theta = \tan\theta \pm n\pi$, we also know that:

$$\arctan -\frac{4}{3} = -0.928 \pm n\pi$$

Another possible θ is $-0.928 + \pi \approx 2.214$, which does fall in the appropriate quadrant. This means the polar coordinates $(5, 2.214)$ are the same as the Cartesian coordinates $(-3, 4)$. *Note:* It is standard practice to express angles in radians, and not degrees, when using polar coordinates.

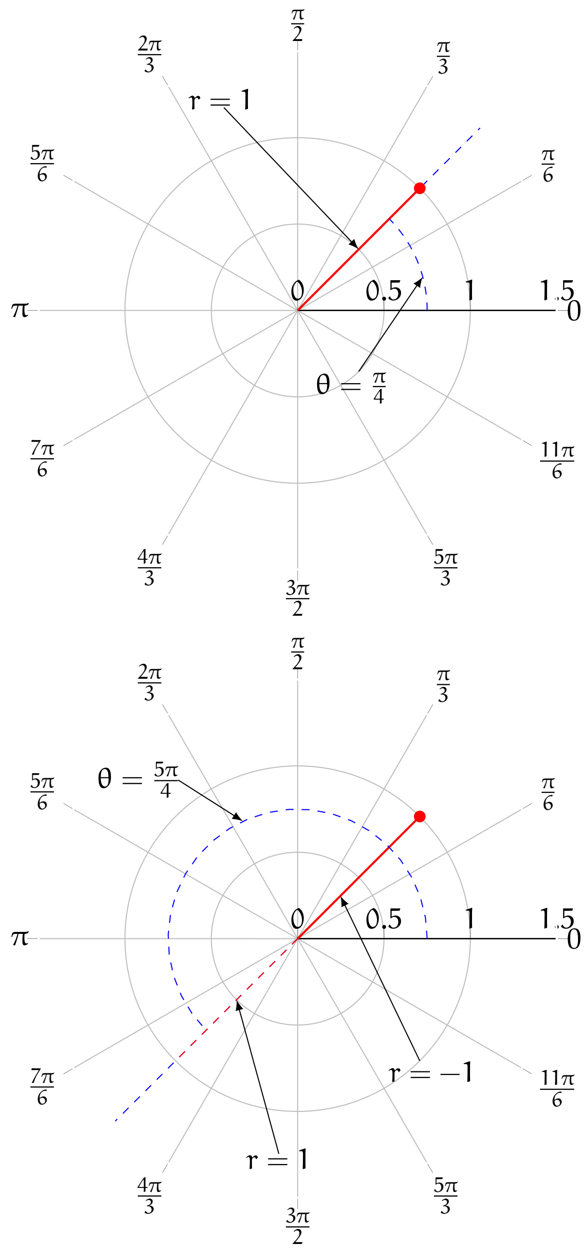

Figure 1.5: The polar coordinates points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ are the same location on a polar axis

1.3.2 Polar to Cartesian

We can also leverage our knowledge of right triangles to convert polar coordinates to Cartesian coordinates. Take the polar coordinate $(2, \frac{\pi}{4})$ (see figure 1.6). We can draw a right triangle with legs parallel to the x and y axes (not shown in the figure) and a hypotenuse that goes from the origin to the polar coordinate $(2, \frac{\pi}{4})$.

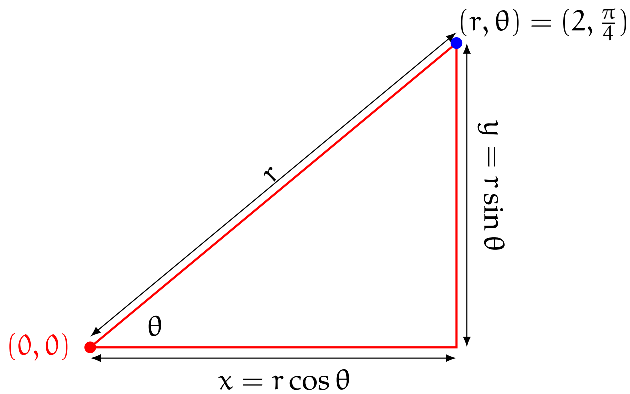

Figure 1.6: To convert from polar to Cartesian coordinates, use the identities $x = r\cos\theta$ and $y = r\sin\theta$

Recall from trigonometry that:

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

We know that the hypotenuse of this triangle has a length of r. The opposite leg is vertical and is the same length as the distance of the polar coordinate from the x-axis. Therefore, the length of the vertical leg represents the y value of that same polar coordinate if it were expressed in Cartesian coordinates. So, we can say that:

$$\sin\theta = \frac{y}{r}$$

And therefore:

$$y = r\sin\theta$$

By a similar process, we also see that:

$$x = r\cos\theta$$

This is easy to visualize and understand for $0 \leq \theta \leq \frac{\pi}{2}$, but it also holds for other values of θ.

Example: Express the polar coordinate $(\frac{3}{2}, \frac{2\pi}{3})$ in Cartesian coordinates.

Solution: From the polar coordinate, we see that $\theta = \frac{2\pi}{3}$ and $r = \frac{3}{2}$. Therefore:

$$x = r\cos\theta = \frac{3}{2} \cdot \cos\frac{2\pi}{3} = \frac{3}{2} \cdot -\frac{1}{2} = -\frac{3}{4}$$

$$y = r\sin\theta = \frac{3}{2} \cdot \sin\frac{2\pi}{3} = \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

The Cartesian coordinate $(-\frac{3}{4}, \frac{3\sqrt{3}}{4})$ has the same location as the given polar coordinate.

Exercise 2

Convert the following polar coordinates to Cartesian coordinates:

1. $(2, \frac{3\pi}{2})$
2. $(\sqrt{2}, \frac{3\pi}{4})$
3. $(3, -\frac{\pi}{4})$
4. $(-3, -\frac{\pi}{3})$
5. $(2, -\frac{\pi}{2})$

Working Space

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Exercise 3

Convert the following Cartesian coordinates to polar coordinates. Restrict θ to $0 \leq \theta < 2\pi$.

1. $(-4, 4)$
2. $(3, 3\sqrt{3})$
3. $(\sqrt{3}, -1)$
4. $(-6, 0)$
5. $(-2, -2)$

Working Space

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1.4 Circles in Polar Coordinates

Many conic sections, including circles, are simpler to express as polar functions than as Cartesian functions. Consider a circle with a radius of 2 centered about the origin. The polar function for this is $r = 2$ for all θ. Let's write a Cartesian function for the same circle.

We know that for every point on the circle, the distance to the origin is 2. This means that, by the Pythagorean theorem,

$$r^2 = x^2 + y^2$$

.

(see figure 1.7)

We can solve this equation for y, given that $r = 2$ (in this case):

$$y = \pm\sqrt{2^2 - x^2}$$

Notice that this is really two equations: $y = \sqrt{2^2 - x^2}$ and $y = -\sqrt{2^2 - x^2}$. This is more complex than the polar equation, $r = 2$.

As seen above, the equation of a circle with radius R centered on the origin is simply $r = R$ in polar coordinates. What if we want a circle centered somewhere else? Polar coordinates

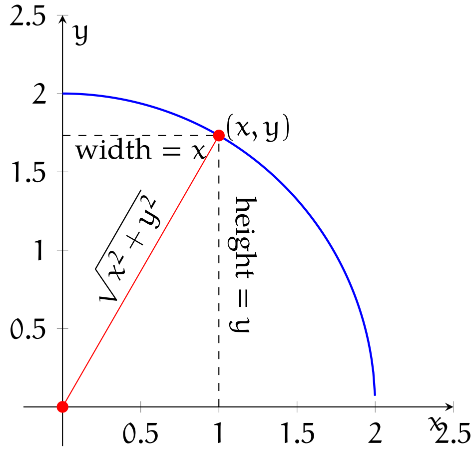


Figure 1.7: All (x, y) pairs on the circle are the same distance from the origin

are best when a circle is bisected by the x or y axis. Consider the polar equation $r = 3\sin\theta$. Let's use a table to find some points and plot the function:

θ	$r = 3\sin\theta$
0	0
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{\pi}{2}$	3
$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{3}{2}$
π	0

Here is how those points look plotted (see figures 1.8 and 1.9):

So, the polar equation $r = 3\sin\theta$ gives a circle with radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$.

Example: Describe the graph of $r = \cos\theta$. Feel free to make a rough plot on the blank polar axis below:

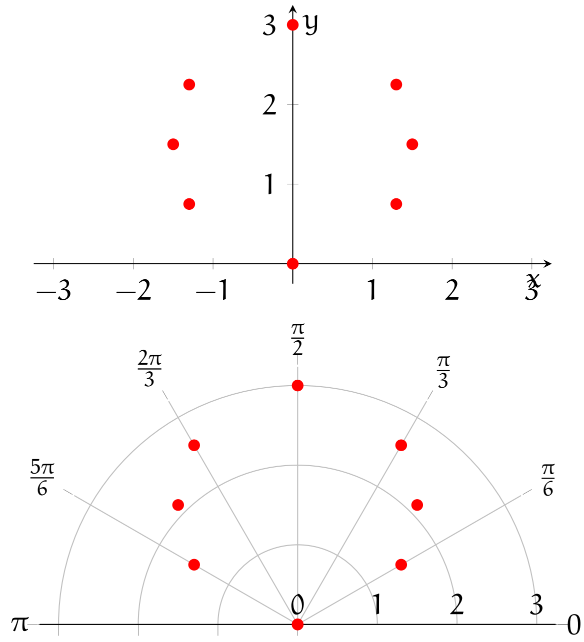


Figure 1.8: Several points for $r = 3\sin\theta$ plotted on Cartesian and polar coordinate systems

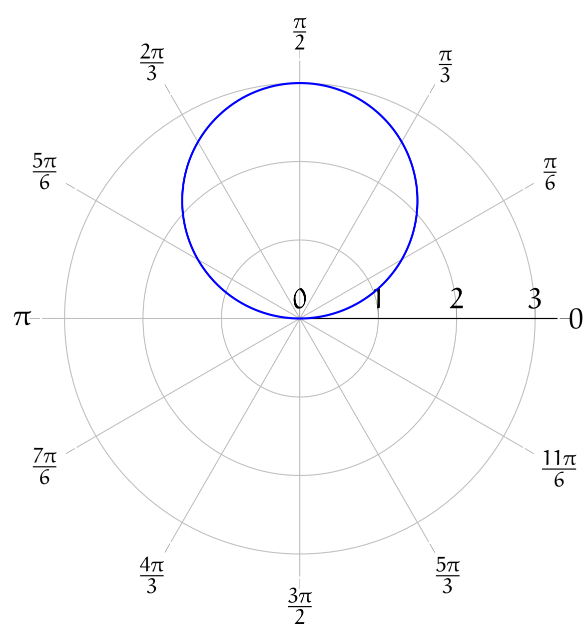
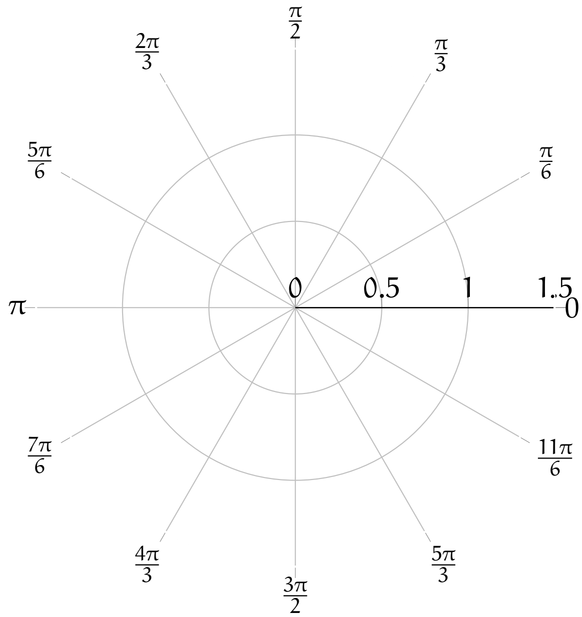
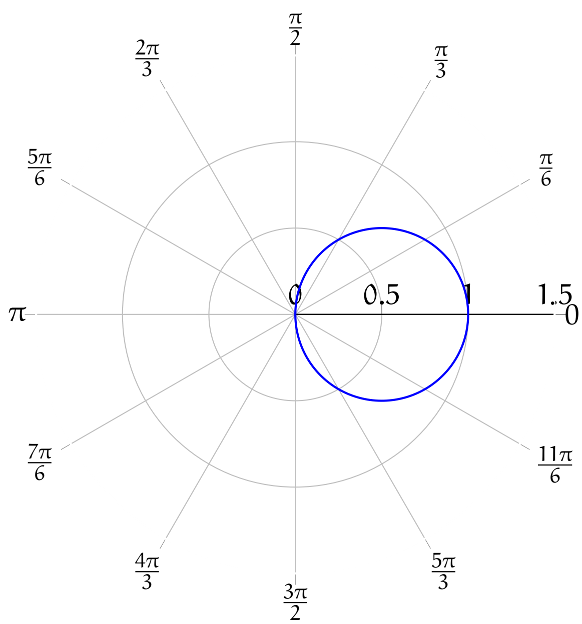


Figure 1.9: $r = 3\sin\theta$ plotted on a polar coordinate system



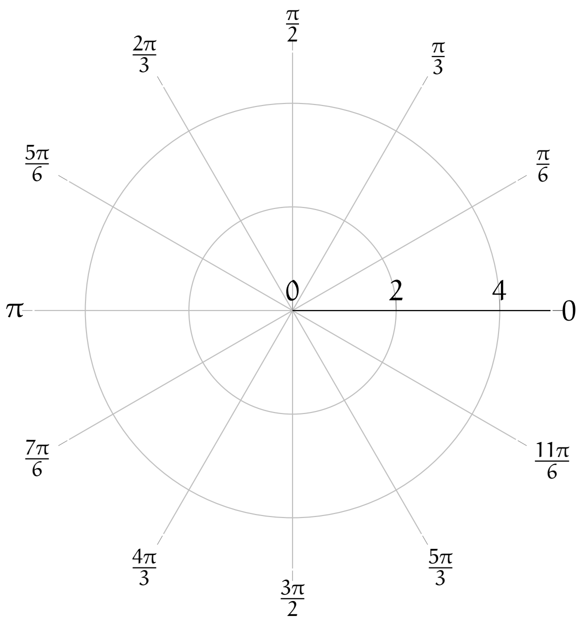
Solution: This plot will look like a circle of radius 0.5 centered at $(0.5, 0)$ (in polar coordinates).



Exercise 4

Sketch the following polar functions on the provided polar axis for $0 \leq \theta < 2\pi$:

1. $r = 3$
2. $\theta = \pi$
3. $r = 2\cos\frac{\theta}{2}$
4. $r = -4\sin\theta$
5. $r = \theta$



Working Space

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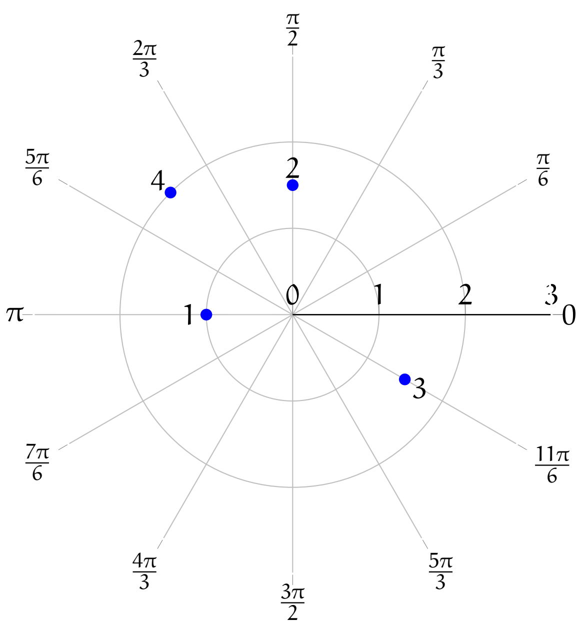
Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)



Answer to Exercise 2 (on page 8)

1. $(0, -2)$. $x = 2 \cdot \cos \frac{3\pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin \frac{3\pi}{2} = 2 \cdot -1 = -2$.
2. $(-1, 1)$. $x = \sqrt{2} \cdot \cos \frac{3\pi}{4} = \sqrt{2} \cdot -\frac{\sqrt{2}}{2} = \frac{2}{2} = -1$ and $y = \sqrt{2} \cdot \sin \frac{3\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{2} = 1$.
3. $(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2})$. $x = 3 \cdot \cos -\frac{\pi}{4} = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$ and $y = 3 \cdot \sin -\frac{\pi}{4} = 3 \cdot -\frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}$.
4. $(-\frac{3}{2}, -\frac{3\sqrt{3}}{2})$. $x = (-3) \cdot \cos \frac{\pi}{3} = (-3) \cdot \frac{1}{2} = -\frac{3}{2}$ and $y = (-3) \cdot \sin \frac{\pi}{3} = (-3) \cdot \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$.
5. $(0, -2)$. $x = 2 \cdot \cos -\frac{pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin -\frac{\pi}{2} = 2 \cdot -1 = -2$.

Answer to Exercise 3 (on page 9)

1. $(4\sqrt{2}, \frac{3\pi}{4})$. $r = \sqrt{x^2+y^2} = \sqrt{32} = 4\sqrt{2}$. $\arctan\frac{y}{x} = \arctan\frac{4}{-4} = \arctan -1 = -\frac{\pi}{4} + n\pi$. We take $\theta = \frac{3\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.

2. $(6, \frac{\pi}{3})$. $r = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9+27} = \sqrt{36} = 6$. $\arctan\frac{3\sqrt{3}}{3} = \arctan\sqrt{3} = \frac{\pi}{3} + n\pi$. We take $\theta = \frac{\pi}{3}$ to satisfy the domain restriction and be in the correct quadrant.

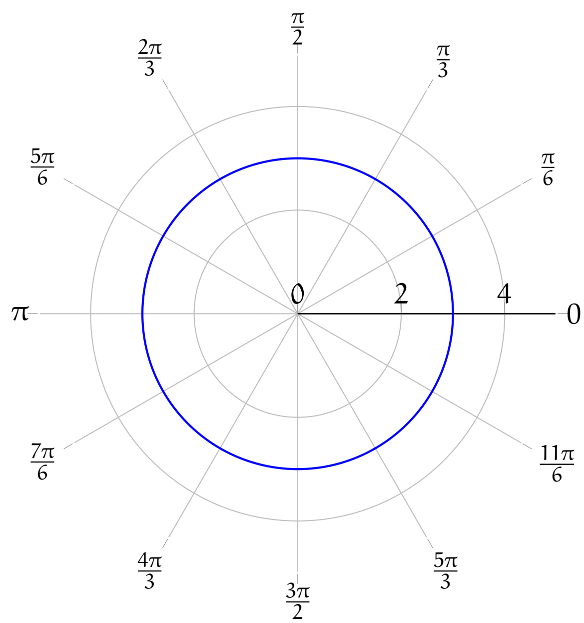
3. $(2, \frac{11\pi}{6})$. $r = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2$. $\arctan\frac{-1}{\sqrt{3}} = -\frac{\pi}{6} + n\pi$. We take $\theta = \frac{11\pi}{6}$ to satisfy the domain restriction and have the point in the correct quadrant.

4. $(6, \pi)$. $r = \sqrt{(-6)^2 + 0^2} = 6$. $\arctan\frac{0}{-6} = \pi + n\pi$. We take $\theta = \pi$ to satisfy the domain restriction.

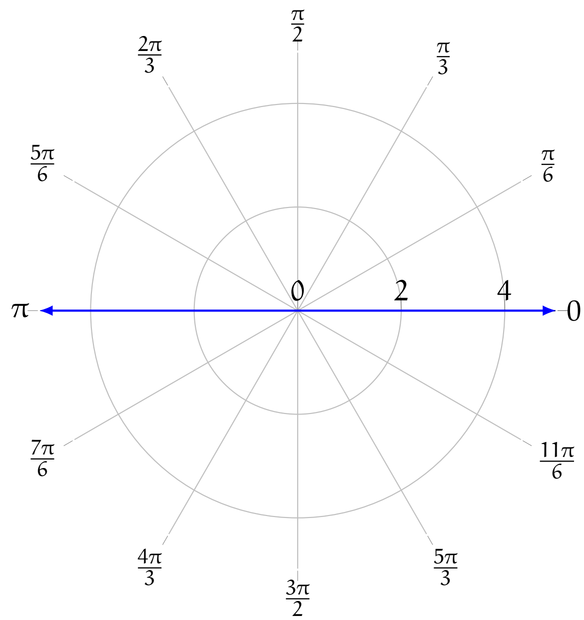
5. $(2\sqrt{2}, \frac{5\pi}{4})$. $r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$. $\arctan\frac{-2}{-2} = \arctan 1 = \frac{\pi}{4} + n\pi$. We take $\theta = \frac{5\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.

Answer to Exercise ?? (on page 13)

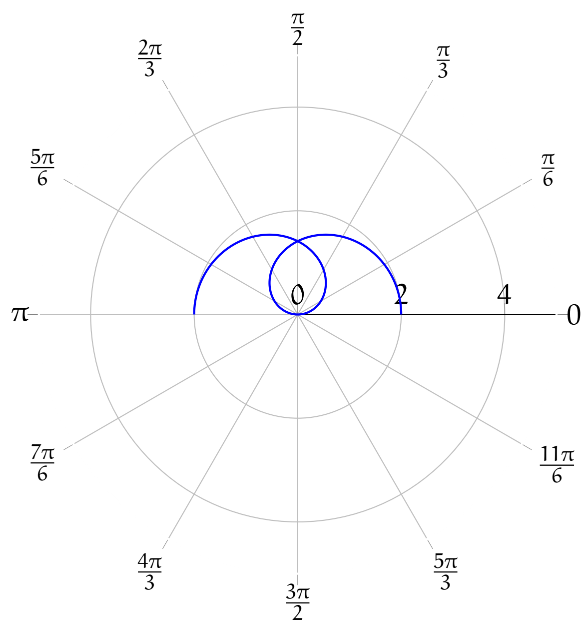
1. $r = 3$



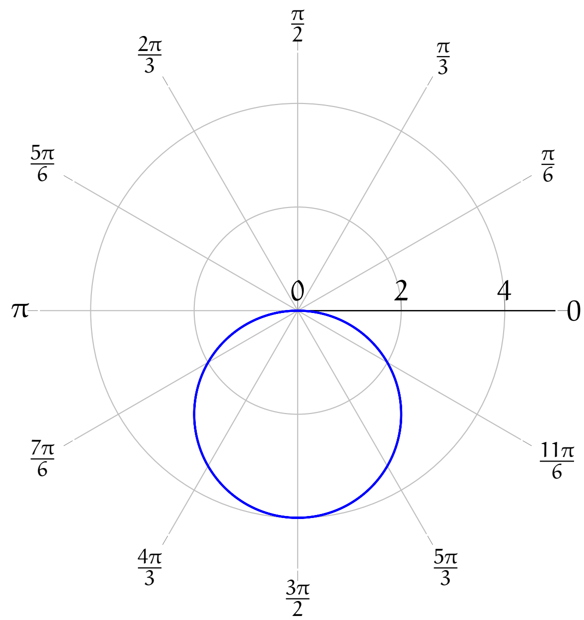
2. $\theta = \pi$ Because r includes all real numbers, negative r is possible and the line $\theta = \pi$ extends in both directions



3. $r = 2\cos\frac{\theta}{2}$



4. $r = -4\sin\theta$



5. $r = \theta$ (The spiral continues, but is beyond the boundary of the graph)