Riemann Sums

1.1 The Meaning of the Area Under a Function

Let's look at the example of a hammer tossed in the air from a previous chapter. As you may recall, if a hammer is tossed up from the ground at 5 m/s, its velocity can be described as $v(t) = 5 - 9.8t$ (on Earth, where the acceleration due to gravity is approximately -9.8 $\frac{m}{s^2}$). The velocity function of our hammer from when it is tossed ($t = 0$) to when it hits the ground $t \approx 1.02$) is shown in figure 1.1.
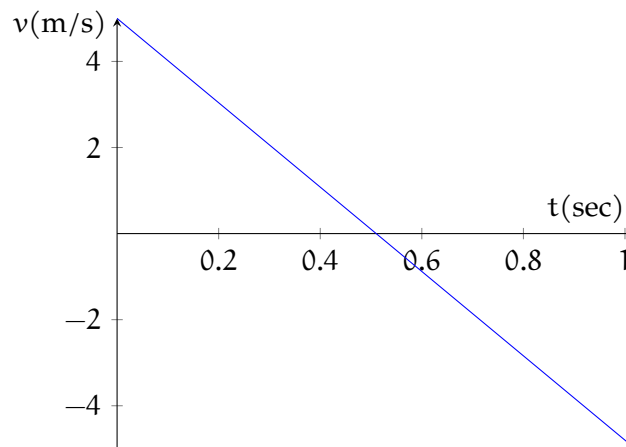

Figure 1.1: Velocity of a hammer thrown upwards at 5 m/s

Now, suppose we only have this velocity function, and we want to know how high above its initial position the hammer is tossed. Examine the graph: At approximately what time does the hammer reach its peak height? (Hint: what should the hammer's velocity be when it reaches its peak?). At the highest point of its flight, the hammer's velocity will be 0 $\frac{m}{s}$, which occurs at approximately $t = 0.5s$ (it's actually $t = 0.5102s$ but we don't need to be that precise for this example).

Now that we know *when* the hammer reaches its peak, how can we determine *how high* that peak is? Recall that velocity is the slope of the position-time graph. Since slope is change in position divided by change in time (in this case, as time is on the x-axis and position on the y-axis), then the slope must have units of [position]/[time] which could be $\frac{m}{s}$, $\frac{miles}{hr}$, and so on. These are units of velocity!

In figure 1.1, you can see that the units on the x-axis are seconds, and on the y-axis, the units are $\frac{m}{s}$. If we are looking for a *displacement* (that is, how far from its initial position the hammer has traveled), we are looking for a solution with units of meters. To yield

an answer with those units, we wouldn't use the slope of the graph; this would yield an answer with units $\frac{m}{s^s}$, the units for acceleration. Instead, we need to *multiply*! The area between the velocity function and the x-axis (see figure 1.2) can be found this way:

$$Area = \frac{1}{2}bh$$

where b is the base of the triangle and h is the height.

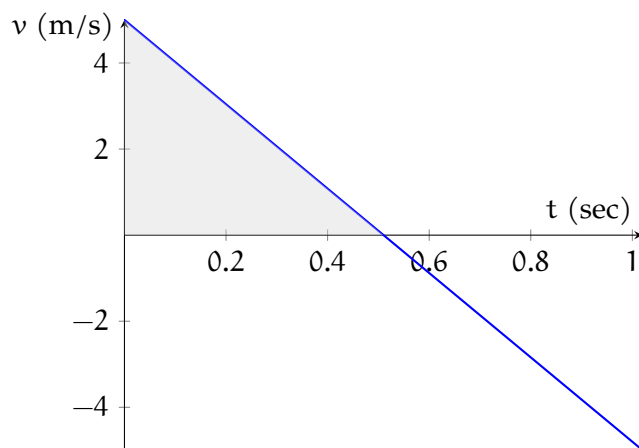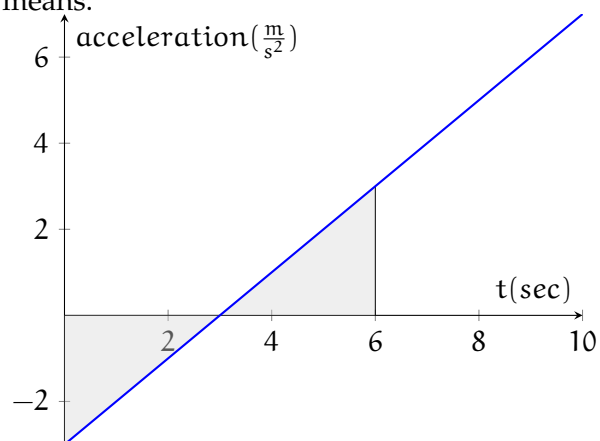$$Area = \frac{1}{2}(0.5s)(5\frac{m}{s})$$

$$Area = 1.25m$$

Figure 1.2: The area under $v(t)$ from $x = 0$ to $x = 0.5$ is equal to the displacement of the hammer

Notice that when multiplying the change in time (0.5 s) by the change in velocity (1.25 $\frac{m}{s}$), the seconds units cancel, yielding a result with units of meters. Therefore, the hammer reaches a peak height of ≈ 1.25 m, which you can confirm by examining the graph originally presented for the hammer toss in the chapter on graph shape.

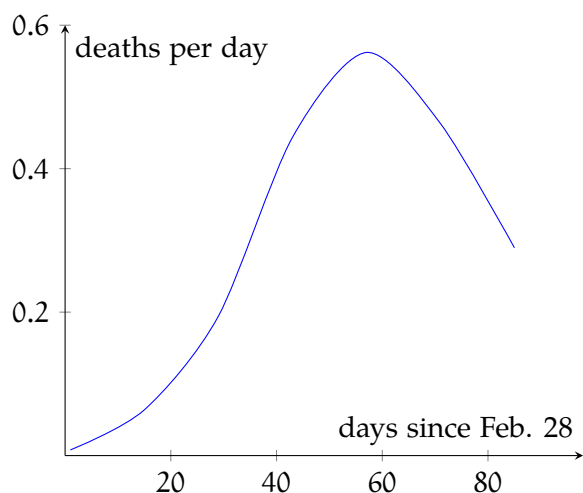### 1.1.1   Determining the Meaning of the Area with Units

Exercise 1

What units will the area shown in the graph have? Based on your answer, does the area represent a displacement, a net change in velocity, or a net change in acceleration? Calculate the shaded area [hint: areas below the x-axis are negative]. Write a sentence in plain English explaining what the are you calculated means.

Exercise 2

The graph below shows historical data of the number of deaths due to SARS in Singapore over several months in 2003. What would the area under the curve represent?

Exercise 3

Oil leaked from a tank at a rate of $r(t)$ liters per hour. A site engineer recorded the leak rate over a period of 10 hours, shown in the table. Plot the data. How could you estimate the total volume of oil lost?

$t(h)$	0	2	4	6	8	10
$r(t)(L/h)$	8.7	7.6	6.8	6.2	5.7	5.3

1.2   Estimating the area under functions

In the hammer example above, it was easy to determine the area under the function, since the area took the shape of a triangle. However, what about finding the area under a more complex function, such as $f(x) = \sin x + x$ (shown in figure 1.3)?
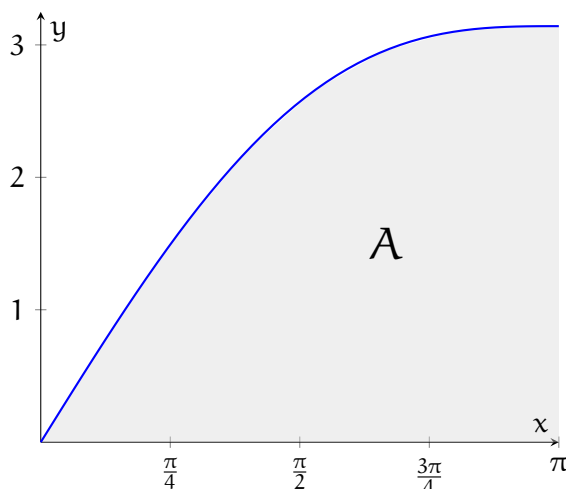

Figure 1.3: $f(x) = \sin x + x$

How can we determine the area under $f(x) = \sin x + x$ from $x = 0$ to $x = \pi$? We can *estimate* the area of that region by dividing the region into rectangles, finding the areas of the rectangles, and adding the areas. As an example, we will divide the region under $f(x) = \sin x + x$ into 4 intervals, shown in figure 1.4.
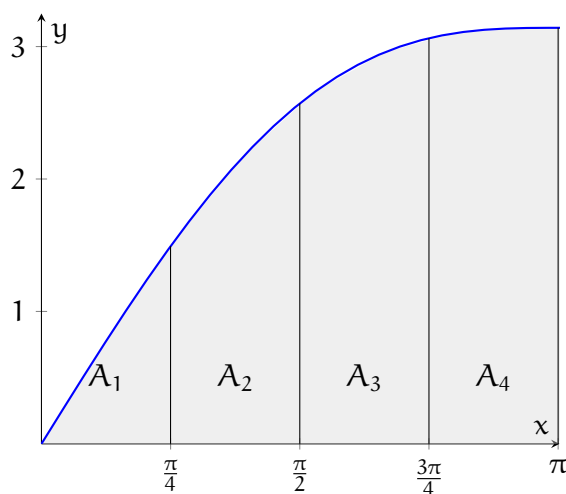

Figure 1.4: $f(x) = \sin x + x$ divided into 4 regions

As you can see in figure 1.4, each rectangle will have a width of $\frac{\pi}{4}$. But what about the height? One way is to use the value of the function at the rightmost value of each rectangle, as shown in figure 1.5.
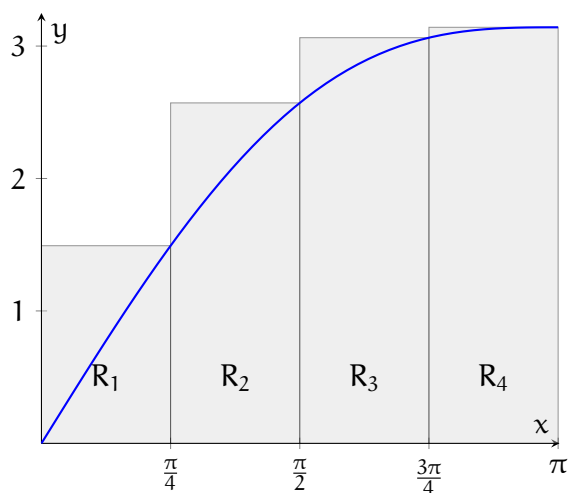
Figure 1.5: Four rectangle sections with heights determined by rightmost value of $f(x)$ on each interval

We can easily calculate the areas of each of these rectangles:

$$\frac{\pi}{4} \times f(\frac{\pi}{4}) + \frac{\pi}{4} \times f(\frac{\pi}{2}) + \frac{\pi}{4} \times f(\frac{3\pi}{4}) + \frac{\pi}{4} \times f(\pi)$$

$$\approx \frac{\pi}{4} \times (1.4925 + 2.5708 + 3.0633 + 3.1416) = 8.0646$$

Based on figure 1.5, will the calculated area be an overestimate or an underestimate? Each of the rectangles overshoots the function, so this will be an overestimate. What about using the leftmost value of $f(x)$ of each interval to determine the height of the rectangles? This is shown in figure 1.6.
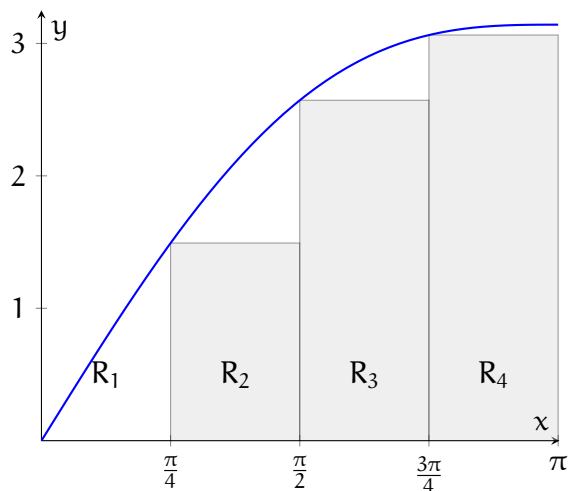

Figure 1.6: Four rectangle sections with heights determined by leftmost value of $f(x)$ on each interval
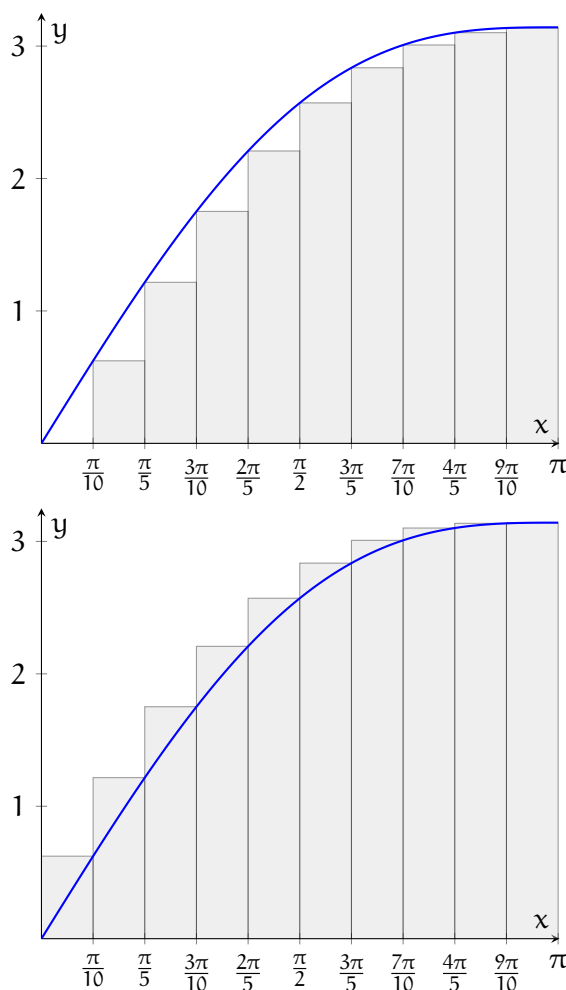
Figure 1.7: $\sin x + x$ broken into 10 intervals using either the left or right value to determine the height.

Notice that because $f(0) = 0$, the height of the first rectangle is zero, so we don't see it on the graph. To find the area of these rectangles:

$$\frac{\pi}{4} \times f(0) + \frac{\pi}{4} \times f(\frac{\pi}{4}) + \frac{\pi}{4} \times f(\frac{\pi}{2}) + \frac{\pi}{4} \times f(\frac{3\pi}{4})$$

$$\approx \frac{\pi}{4} \times (0 + 1.4925 + 2.5708 + 3.0633) = 5.5972$$

This is an underestimate. Therefore, the true value of the area under $f(x) = \sin x + x$ is between 5.5972 and 8.0646. This is an awfully wide window! We can narrow our estimate by increasing the number of intervals. Graphs of $f(x)$ with 10 intervals are shown in figure 1.7.

The total area for the left-determined rectangles is ≈ 6.4248, and for the right-determined, it is ≈ 7.4118. Therefore, we have narrowed the range for the true area under the curve to

$6.4248 < A < 7.4118$. In general, as you increase the number of intervals, you get closer to the true area.

For a strictly increasing function, the right sum will be an overestimate and the left sum will be an underestimate of the true area under the curve. In the exercise below, you will examine a strictly decreasing function:

Exercise 4

Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$ using four rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an overestimate or an underestimate? Repeat using left endpoints.

Working Space

Answer on Page 22

You should have found that for the strictly decreasing function $f(x) = \frac{1}{x}$, the right-determined sum is an *underestimate*, while the left-determined sum is an *overestimate*.

1.3   The Riemann Sum

In the previous section, we estimated the area under functions by dividing the area into approximating rectangles. This method is called a *Riemann Sum*. We will use a general example to formally define the Riemann sum. Consider a generic function divided into strips of equal width (shown in figure 1.8). The width of each strip is

$$\Delta x = \frac{b-a}{n}$$

where a is the left endpoint of the interval, b is the right endpoint of the interval, and n is the number of strips. So, the right endpoints of the sections are

$$x_1 = a + \Delta x$$

.

$$x_2 = a + 2\Delta x$$

$$\ldots \; x_n = a + n\Delta x$$

As above, we can use the value of the function to determine the height of a rectangle whose area approximates the area of the section. (E.g. for the ith strip, the width is Δx and the height is $f(x_i)$, see figure 1.9). So, the total area approximated by the rectangles is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

This is the formal definition of the right Riemann sum. You can also take a left Riemann sum or a midpoint Riemann sum, as discussed below.

1.3.1   Right Riemann Sums

As seen above, a right Riemann sum uses the right-most value of $f(x)$ to determine the height of the rectangle (an example is shown in figure 1.10). We will refer to the right Riemann sum as R_n, where n is the number of intervals.

1.3.2   Left Riemann Sums

When taking a left Riemann sum, the height of the rectangle is determined by the value of the function at the lower (left-most) x-value. See figure 1.11. We will refer to left Riemann sums as L_n, where n is the number of intervals. So, the total area approximated by a Left Riemann sum is

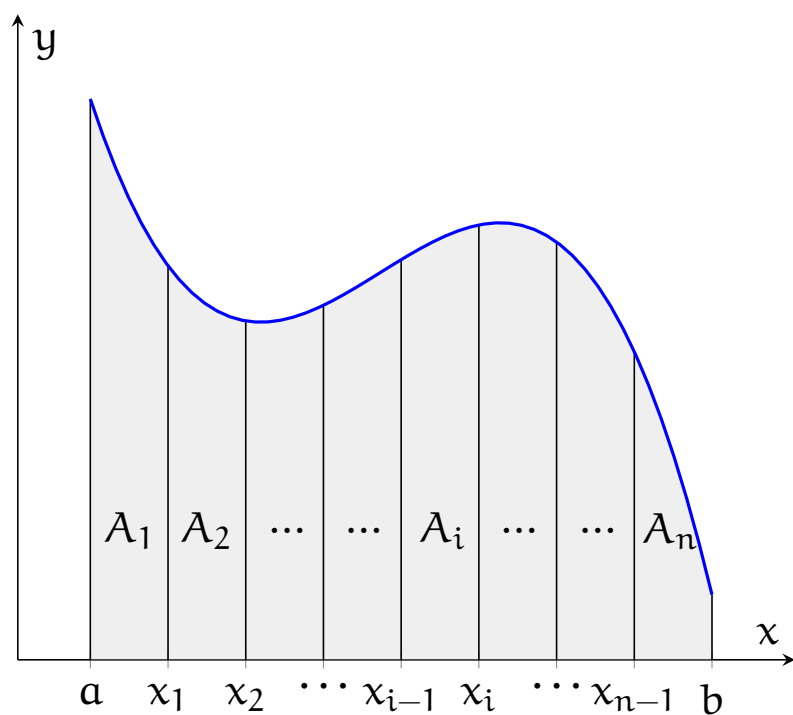$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

Figure 1.8: A representative function divided into n strips of equal width
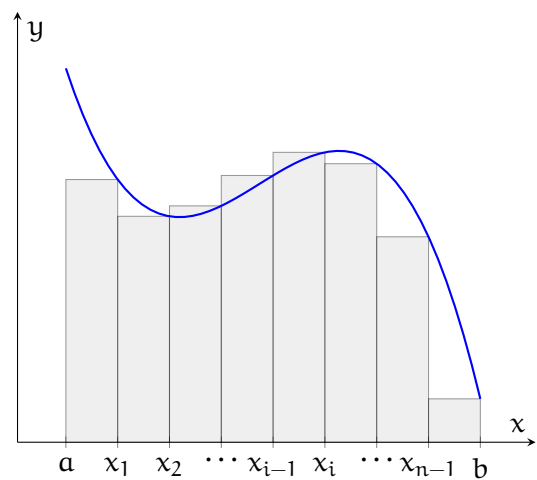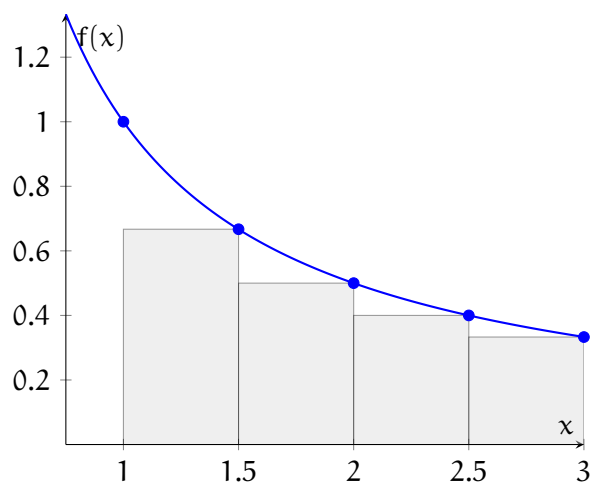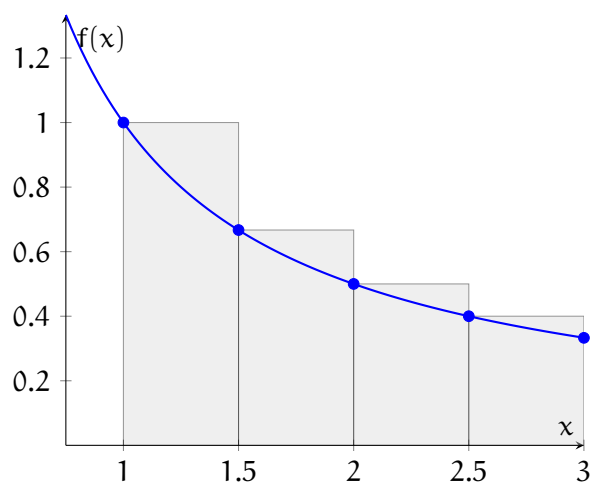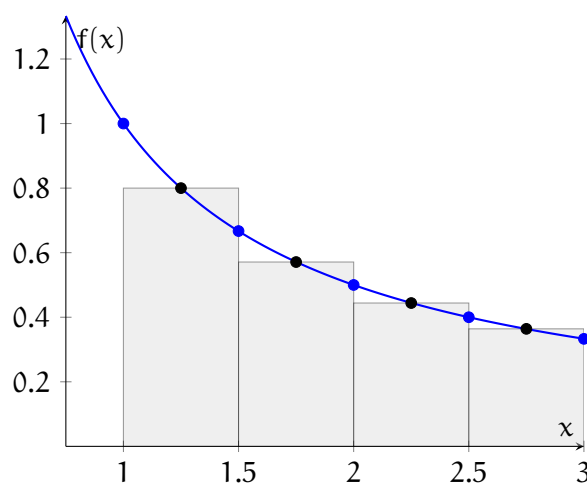

Figure 1.9: A representative function divided into n rectangles of equal width, with rectangle height determined by the right endpoint of the subinterval

Figure 1.10: R_4 for $f(x) = \frac{1}{x}$

Figure 1.11: L_4 for $f(x) = \frac{1}{x}$

Figure 1.12: M_4 of $f(x) = \frac{1}{x}$

1.3.3   Midpoint Riemann Sums

A midpoint Riemann sum uses the value of $f(x)$ at the midpoint of the division to determine the height of the rectangle, as shown in figure 1.12. We will refer to the midpoint Riemann sum as M_n, where n is the number of intervals. So, the total area approximated by the rectangles is

$$M_n = f(\frac{x_0 + x_1}{2})\Delta x + f(\frac{x_1 + x_2}{x})\Delta x + \ldots + f(\frac{x_{n-1} + x_n}{2})\Delta x$$

1.3.4   Riemann sum sigma notation

As you may recall, mathematicians use sigma notation to concisely express sums, such as Riemann sums. We can rewrite the definition of a right Riemann sum in sigma notation:

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

where n is the number of subintervals. So, the actual area under the curve is the limit as n approaches ∞ of the above sum. Let's apply this by writing a sum that represents the area, A, of the region that lies between the x-axis and the function $f(x) = e^{-x}$ from $x = 0$ to $x = 2$.

First, we find an expression for Δx:

$$\Delta x = \frac{2 - 0}{n} = \frac{2}{n}$$

Recall that $x_i = a + i\Delta x$. Since a (the beginning of the interval) $= 0$, then the general expression for x_i in this case is $0 + i \times \frac{2}{n} = \frac{2i}{n}$. Substituting our expressions for Δx and x_i into the sum formula, we see that:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} e^{\frac{-2i}{n}} \frac{2}{n}$$

We can also interpret a sum as the area under a specific function. Take the expression:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin \frac{i\pi}{n}$$

There are two expressions in the sum: $\frac{\pi}{n}$ and $\sin \frac{i\pi}{n}$. It makes sense that $\Delta x = \frac{\pi}{n}$ and $f(x_i) = \sin \frac{i\pi}{n}$. Because $\Delta x = \frac{b-a}{n} = \frac{\pi}{n}$, it follows that the interval of the area has a width of π. We will need to examine the other expression, $\sin \frac{i\pi}{n}$, to determine an exact window.

Since $f(x_i) = \sin \frac{i\pi}{n}$, it follows that the function we are looking for is a sine function. Further, the expression for $x_i = \frac{i\pi}{n}$. Recall that $x_i = a + i\Delta x$, where a is the left-most boundary of the interval. Substituting what we have found already, we see that:

$$x_i = a + i\frac{\pi}{n} = \frac{i\pi}{n}$$

which implies that $a = 0$. Since we have established the interval is π wide, we can infer that $b = \pi$. Therefore, the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin \frac{i\pi}{n}$ is equal to the area under $f(x) = \sin x$ from $x = 0$ to $x = \pi$.

Exercise 5

Use the formal definition of a Right Riemann sum to write a limit of a sum that is equal to the total area under the graph of f on the specified interval. Do not evaluate the limit.

1. $f(x) = \frac{2x}{x^2+1}$, $1 \leq x \leq 3$

2. $f(x) = x^2 + \sqrt{1 + 2x}$, $4 \leq x \leq 7$

3. $f(x) = \sqrt{\sin x}$, $0 \leq x \leq \pi$

Working Space

Answer on Page 22

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

Figure 1.13: Speed of *Endeavour* from launch to booster separation

Exercise 6

Use the formal definition of a Right Rie-
mann sum to find a region on a graph
whose are is equal to the given limit. Do
not evaluate the limit.

1. $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$

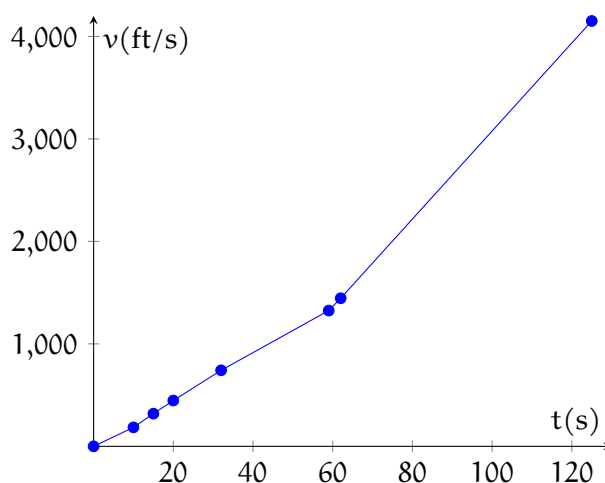2. $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

Working Space

1.3.5   Real-world Riemann Sums

Sometimes we are working from real data, and the intervals aren't evenly spaced. That's
ok! We can still use Riemann sums to make an estimate. Consider the velocity data from
the 1992 launch of the space shuttle *Endeavour*, shown in tabular form in figure 1.13:

We can use a Riemann sum to estimate how far the space shuttle traveled in the first 62
seconds of flight. First, let's visualize our data (see figure 1.14). There are 7 time intervals
from the data, but we only need the first 6. We can find a reasonable range for the distance
the space shuttle travels by finding the left and right Riemann sums. Remember: Becayse
this data is strictly increasing, the left sum will be our lower bound and the right sum
will be our upper bound.

First, we'll find L_6. The width of the first interval is 10 seconds ($10-0 = 10$) and the height

Figure 1.14: Plot of time, velocity data for the *Endeavour*

of the rectangle will be $v(0) = 0$. Calculations for the additional intervals are shown in the table:

Interval	Width(s)	Height(ft/s)	Area(ft)
1	10	0	0
2	5	185	925
3	5	319	1595
4	12	447	5364
5	27	742	20034
6	3	1325	3975

Adding the areas, we find the lower limit for the distance traveled is 31,893 feet. We can determine the upper bound, R_6, in a similar manner:

Interval	Width(s)	Height(ft/s)	Area(ft)
1	10	185	1850
2	5	319	1595
3	5	447	2235
4	12	742	8904
5	27	1325	35775
6	3	1445	4335

Adding the areas, we find the upper limit for the distance traveled is 54,694 feet. Therefore, the *Endeavour* traveled between 31,893 and 54,694 feet during the first 62 seconds of this flight.

1.4   Code for a Riemann Sum

You can create a program that automatically calculates a Riemann sum. Create a file called `riemann.py` and type the following into it:

```python
import matplotlib.pyplot as plt
import sys
import math

from matplotlib.table import Rectangle

# Did the user supply two arguments?
if len(sys.argv) != 3:
    print(f"Usage: {sys.argv[0]} <stop> <divisions>")
    print(f"Numerically integrates 1/x from 1 to <stop>.")
    print(f"Calculates the value of 1/x at <divisions> spots in the range.")
    exit(1)

# Check to make sure the number of divisions is greater than zero?
divisions = int(sys.argv[2])
if divisions <= 0:
    print("ERROR: Divisions must be at least 1.")
    exit(1)

# Is the stopping point after 1.0?
stop = float(sys.argv[1])
if stop <= 1.0:
    print("ERROR: Stopping point must be greater than 1.0")
    exit(1)

start = 1.0
step_size = (stop - start)/divisions

print(f"Step size is {step_size:.5f}.")
x_values = []
y_values = []
sum = 0.0
for i in range(divisions):
    current_x = start + i * step_size
    current_y = 1.0/current_x
    area = current_y * step_size
    print(f"{i}: 1 / {current_x:.3f} = {current_y:4f}, area of rect = {area:8f} ")

    x_values.append(current_x)
    y_values.append(current_y)
```

```python
    sum += area
    print(f"\tCumulative={sum:.3f}, ln({current_x:.3f})={math.log(current_x):.3f}")

print(f"Numerical integration of 1/x from 1.0 to {stop:.4f} is {sum:.4f}")
print(f"The natural log of {stop:.4f} is {math.log(stop):.4f}")

# Create data for the smooth 1/x line
SMOOTH_DIVISIONS = 200
smooth_start = start - 0.15
smooth_stop = stop + 1.0
smooth_step = (smooth_stop - smooth_start)/SMOOTH_DIVISIONS
smooth_x_values = []
smooth_y_values = []
for i in range(SMOOTH_DIVISIONS):
    current_x = smooth_start + i * smooth_step
    current_y = 1.0/current_x
    smooth_x_values.append(current_x)
    smooth_y_values.append(current_y)

# Put it on a plot
fig, ax = plt.subplots()
ax.set_xlim((smooth_x_values[0], smooth_x_values[-1]))
ax.set_ylim((0, smooth_y_values[0]))
ax.set_title("Riemann Sums for 1/x")

# Make the Riemann rects
for i in range(divisions):
    current_x = x_values[i]
    next_x = current_x + step_size
    current_y = y_values[i]
    rect = Rectangle((current_x, 0), step_size, current_y, edgecolor="green", facecolor=
    ax.add_patch(rect)

# Make the true 1/x curve
ax.plot(smooth_x_values, smooth_y_values, c="k", label="1/x")

# Show the user
plt.show()
```

This program will calculate and display a graph of the left Riemann sum of $\frac{1}{x}$ from 1 to the provided stop value with the indicated number of subintervals. When you run it, you will see a graph in a new window and something like this in the terminal:

```
Step size is 0.40000.
```

```
0: 1 / 1.000 = 1.000000, area of rect = 0.400000
        Cumulative=0.400, ln(1.000)=0.000
1: 1 / 1.400 = 0.714286, area of rect = 0.285714
        Cumulative=0.686, ln(1.400)=0.336
2: 1 / 1.800 = 0.555556, area of rect = 0.222222
        Cumulative=0.908, ln(1.800)=0.588
3: 1 / 2.200 = 0.454545, area of rect = 0.181818
        Cumulative=1.090, ln(2.200)=0.788
4: 1 / 2.600 = 0.384615, area of rect = 0.153846
        Cumulative=1.244, ln(2.600)=0.956
5: 1 / 3.000 = 0.333333, area of rect = 0.133333
        Cumulative=1.377, ln(3.000)=1.099
6: 1 / 3.400 = 0.294118, area of rect = 0.117647
        Cumulative=1.495, ln(3.400)=1.224
7: 1 / 3.800 = 0.263158, area of rect = 0.105263
        Cumulative=1.600, ln(3.800)=1.335
8: 1 / 4.200 = 0.238095, area of rect = 0.095238
        Cumulative=1.695, ln(4.200)=1.435
9: 1 / 4.600 = 0.217391, area of rect = 0.086957
        Cumulative=1.782, ln(4.600)=1.526
Numerical integration of 1/x from 1.0 to 5.0000 is 1.7820
The natural log of 5.0000 is 1.6094
```

Exercise 7

Use the Python program you created to find L_{10}, L_{50}, L_{100}, L_{500}, L_{1000}, and L_{5000} for the function $\frac{1}{x}$ from $x = 1$ to $x = 5$. What do you notice about the results?

Working Space

Answer on Page 23

1.5   Riemann Sum Practice

Exercise 8

t (hours)	4	7	12	15
R(t) (L/hr)	6.5	6.2	5.9	5.6

A tank contains 50 liters of water after 4 hours of filling. Water is being added to the tank at rate R(t). The value of R(t) at select times is shown in the table. Using a right Riemann sum, estimate the amount of water in the tank after 15 hours of filling.
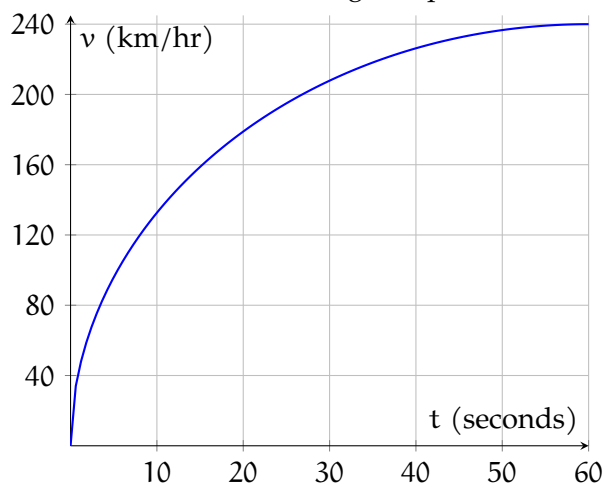
Working Space

Answer on Page 23

Exercise 9

Let $f(x) = x - 2\ln x$. Estimate the area under f from $x = 1$ to $x = 5$ using four rectangles and the value of f at the midpoint of each interval. Sketch the curve and your approximating rectangles.

Working Space

Answer on Page 24

Exercise 10

A graph of a car's velocity over a period of 60 seconds is shown. Estimate the distance traveled during this period.

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua. org/) for more details.
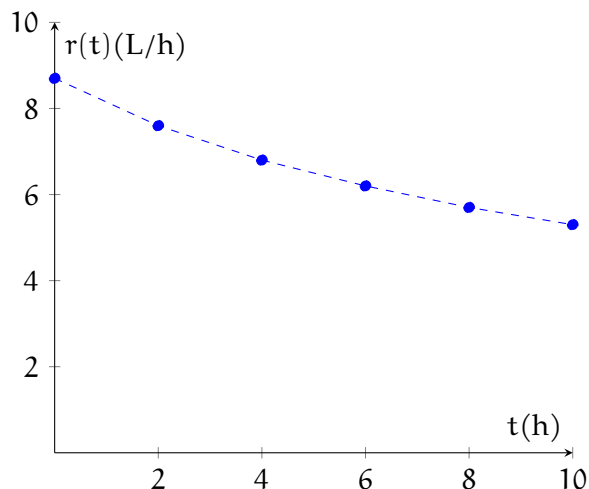
Answers to Exercises

Answer to Exercise 1 (on page 3)

The units on the x-axis are s, and the units on the y-axis are $\frac{m}{s^2}$. The area would then have units of $s \times \frac{m}{s^2} = \frac{m}{s}$. Based on the units, the area represents a net change in velocity. The area above and below the axis are equal $(4.5\frac{m}{s})$; therefore, the total area is 0. This means the object's starting and ending velocity are the same.

Answer to Exercise 2 (on page 4)

The units of the area will be $days \times \frac{deaths}{day} = deaths$. The area under the curve represents the total number of people who died of SARS in Singapore during the time period represented [from March 1 to May 24 (if you took the time to do the math for the dates)].
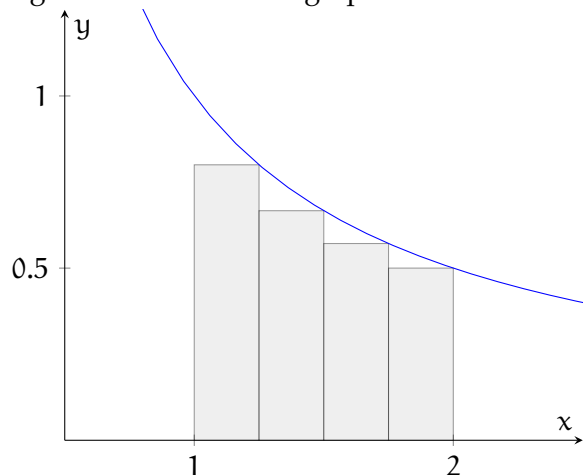
Answer to Exercise 3 (on page 4)

Based on the units, the area under the data would represent the total oil lost. One way to estimate this area would be to create rectangles, but there are other valid methods.
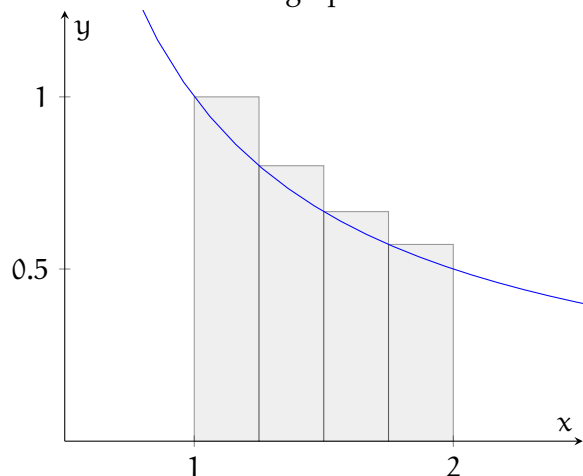
Answer to Exercise 4 (on page 8)

Right-determined sum graph:

The area of the right-determined sum is $0.25 \times (0.8 + 0.6667 + 0.5714 + 0.5) = 0.4202$. This is an underestimate of the actual area.

Left-determined sum graph:

The area of the left-determined sum is $0.25 \times (1 + 0.8 + 0.6667 + 0.5714) = 0.7595$. This is an overestimate of the actual area.

Answer to Exercise 5 (on page 13)

1. $\Delta x = \frac{3-1}{n} = \frac{2}{n}$ and $x_i = 1 + i\frac{2}{n} = 1 + \frac{2i}{n}$. Substituting, we get $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{2(1 + \frac{2i}{n})}{(1 + \frac{2i}{n})^2 + 1} \cdot \frac{2}{n}$

2. $\Delta x = \frac{7-4}{n} = \frac{3}{n}$ and $x_i = 4 + \frac{3i}{n}$. Substituting, we get $\lim_{n \to \infty} \sum_{i=1}^{n} [(4 + \frac{3i}{n})^2 + \sqrt{1 + 2(4 + \frac{3i}{n})}]\frac{3i}{n}$

3. $\Delta x = \frac{\pi - 0}{n} = \frac{\pi}{n}$ and $x_i = \frac{i\pi}{n}$. Substituting, we get $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin \frac{i\pi}{n}} \frac{\pi}{n}$

Answer to Exercise 6 (on page 14)

1. $\Delta x = \frac{3}{n}$, which implies $b - a = 3$. We could interpret $\sqrt{1 + \frac{3i}{n}}$ two ways: either $f(x) = \sqrt{1+x}$ and $x_i = \frac{3i}{n}$ or $f(x) = \sqrt{x}$ and $x_i = 1 + \frac{3i}{n}$. In the first case, we can find that $a = 0$ and $b = 3$, so the limit of the sum represents the area under $f(x) = \sqrt{1+x}$ from $x = 0$ to $x = 3$. For the second case, we can find that $a = 1$ and $b = 4$, so the limit of the sum represents the area under $f(x) = \sqrt{x}$ from $x = 1$ to $x = 4$.

2. $\Delta x = \frac{\pi}{4n}$, which implies $b - a = \frac{\pi}{4}$. We can see that $x_i = \frac{i\pi}{4n}$, which implies $a = 0$ and therefore also that $b = \frac{\pi}{4}$. Therefore, the limit of the sum represents the total area under $f(x) = \tan x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Answer to Exercise 7 (on page 18)

Number of Intervals	Calculated Area
10	1.7820
50	1.6419
100	1.6256
500	1.6126
1000	1.6110
5000	1.6098

The area approaches the natural log of the endpoint, $\ln 5 \approx 1.6094$.

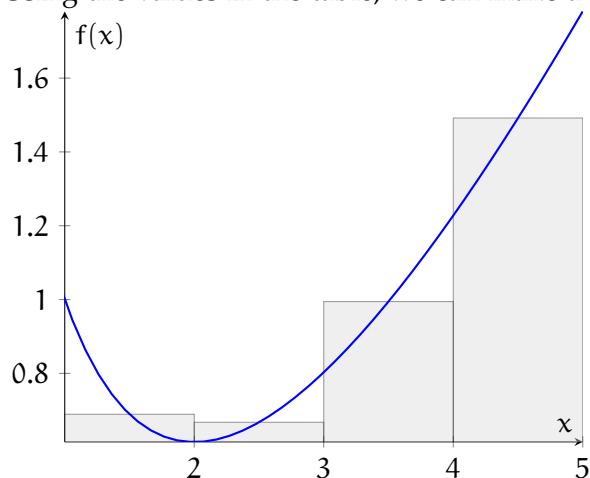## Answer to Exercise 8 (on page 19)

The volume of water will be the amount of water at 4 hours (50 liters) plus the area under the graph of $R(t)$ from $t = 4$ to $t = 15$. We will estimate this area with a right Riemann sum. The approximate volume added from $t = 4$ to $t = 7$ is $(7 - 4) * (6.2) = 18.6$ liters. The approximate volume added from $t = 7$ to $t = 12$ is $(12 - 7) * (5.9) = 29.5$ liters. The approximate volume added from $t = 12$ to $t = 15$ is $(15 - 12) * (5.6) = 16.8$ liters. Therefore, the approximate total volume of water in the tank at $t = 15$ is $50 + 18.6 + 29.5 + 16.8 = 114.9$ liters.

Answer to Exercise 9 (on page 19)

We will divide the area from $x = 1$ to $x = 5$ into four intervals at $x = 2$, $x = 3$, and $x = 4$. Next, we will find the value of $f(x)$ at the midpoint of each interval:

Interval	Midpoint	Value of $f(x)$ at midpoint
1	1.5	≈ 0.68907
2	2.5	≈ 0.66742
3	3.5	≈ 0.99447
4	4.5	≈ 1.49185

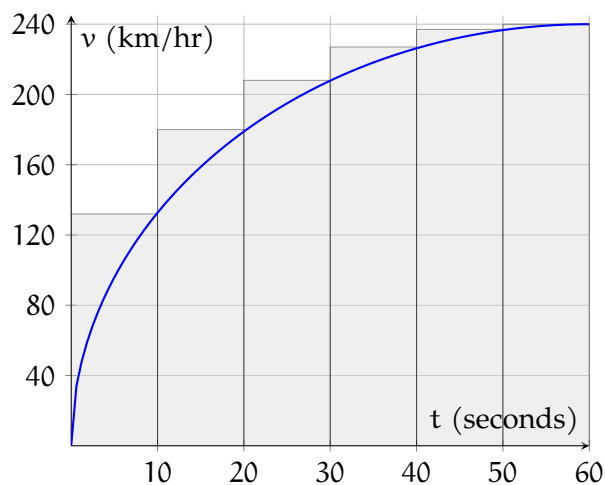Using the values in the table, we can make a possible sketch of $f(x)$:

And we calculate the total area in the rectangles:

$$1 \times (0.68907 + 0.66742 + 0.99447 + 1.49185 = 3.84281$$

Answer to Exercise 10 (on page 20)

The question allows the student to choose the type of sum (left, right, or midpoint) and the number of intervals. A possible solution is given, but there are many ways to answer the question.

The tricky part here is noticing the units! In order to have a solution in kilometers, we will need to convert km/hr to m/s when we calculate the areas. A possible solution is to divide the graph into 6 intervals (one every 10 seconds) and use a right Riemann sum.

We can use the graph to *estimate* the height of each rectangle. Some reasonable estimates are $f(10) = 130\frac{km}{hr} \approx 36.1\frac{ms}{s}$, $f(20) = 180\frac{km}{hr} = 50\frac{m}{s}$, $f(30) = 210\frac{km}{hr} \approx 58.3\frac{m}{s}$, $f(40) = 230\frac{km}{hr} \approx 63.9\frac{m}{s}$, $f(50) = 235\frac{km}{hr} \approx 65.3\frac{m}{s}$, and $f(60) = 240\frac{km}{hr} \approx 66.7\frac{m}{s}$. [Any values within ±5 of the listed values are reasonable.] Noting that each interval is 10 sec wide and usingß the estimates of $f(x)$ listed, we can estimate that the distance traveled is $10~sec \times (36.1\frac{m}{s} + 50\frac{m}{s} + 58.3\frac{m}{s} + 63.9\frac{m}{s} + 65.3\frac{m}{s} + 66.7\frac{m}{s}) = 3403$ meters.

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