Chapter 1

Linear Combinations of Vectors

In the introductory linear algebra chapter, you learned that vectors and matrices can be rotated, inverted, and added. In this chapter, we will explore linear combinations of vectors and the span of group of vectors. The **span** of a group of vectors is the set of vectors that can be made with linear combinations of the original group of vectors. We will offer mathematical and visual explanations later in the chapter. First, let's examine linear combinations.

A *linear combination* is simply the addition of vectors with leading scalar multipliers. In algebra, we can express polynomials similar to the form $5x + -4y$. In linear algebra, we do this with a vectors: $3\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Another way to say this is:

Linear Combination of Vectors

A linear combination of a list of n vectors, $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ takes the form:

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n}$$

where $a_1, a_2, \cdots, a_n \in \mathbb{R}$. You may see the list of vectors as $\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_n}$, both are acceptable variables.

Example: Find a linear combination of $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ that gives the vector $\begin{bmatrix} 17 \\ -4 \\ 2 \end{bmatrix}$.

Solution: We are looking for a_1 and a_2 such that:

$$a_1\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ -4 \\ 2 \end{bmatrix}$$

Looking at each dimension separately, we get the system of equations:

$$2a_1 + 1a_2 = 17$$

$$1a_1 - 2a_2 = -4$$

$$-3a_1 + 4a_2 = 2$$

If we can solve this system of equations, we will find a_1 and a_2. Let's multiply the first equation by 2 and add it to the second equation:

$$2(2a_1 + a_2) + (a_1 - 2a_2) = 2(17) + (-4)$$

$$4a_1 + 2a_2 + a_1 - 2a_2 = 34 - 4$$

$$5a_1 = 30$$

$$a_1 = 6$$

Now we can take a_1 and substitute it back into any equation in our system to find a_2. Let's use the third equation:

$$-3(6) + 4a_2 = 2$$

$$-18 + 4a_2 = 2$$

$$4a_2 = 20$$

$$a_2 = 5$$

Since we used all 3 equations, we know $a_1 = 6$ and $a_2 = 5$ are solutions to all 3 equations. If we had only used the first two equations to find a_1 and a_2, we would want to substitute our values back into the third equation to make sure our solution holds for that equation also.

Therefore,

$$6\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ -4 \\ 2 \end{bmatrix}.$$

Exercise 1 Linear Combinations

Find a linear combination of the first two vectors that yields the third vector.

1. $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}, \begin{bmatrix}4\\5\end{bmatrix}$
2. $\begin{bmatrix}9\\4\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-5\\3\end{bmatrix}$
3. $\begin{bmatrix}7\\-2\end{bmatrix}, \begin{bmatrix}-8\\4\end{bmatrix}, \begin{bmatrix}6\\-2\end{bmatrix}$

Working Space

Answer on Page ??

Sometimes, a set of vectors cannot be combined to make a specific vector. Take the pair of vectors we have looked at before: $\begin{bmatrix}2\\1\\-3\end{bmatrix}$ and $\begin{bmatrix}1\\-2\\4\end{bmatrix}$. Can we find a combination to make vector $\begin{bmatrix}17\\-4\\5\end{bmatrix}$? Let's try. We define a_1 and a_2 such that:

$$a_1 \begin{bmatrix}2\\1\\-3\end{bmatrix} + a_2 \begin{bmatrix}1\\-2\\4\end{bmatrix} = \begin{bmatrix}17\\-4\\5\end{bmatrix}$$

Which creates the system of equations:

$$2a_1 + a_2 = 17$$

$$a_1 - 2a_2 = -4$$

$$-3a_1 + 4a_2 = 5$$

We have two variables (a_1 and a_2) and three equations. Let's use the first two to find a_1 and a_2, then check our answers by substituting our solutions into the third equation. First, we'll multiply the second equation by -2 and add that to the first equation:

$$2a_1 + a_2 + (-2)(a_1 - 2a_2) = 17 + (-2)(-4)$$

$$2a_1 + a_2 - 2a_1 + 4a_2 = 17 + 8$$
$$5a_2 = 25$$
$$a_2 = 5$$

Substituting for a_2 back into the first equation and solving for a_1:

$$2a_1 + 5 = 17$$

$$2a_1 = 12$$
$$a_1 = 6$$

Now, let's check if $a_1 = 6$, $a_2 = 5$ is a solution to the third equation:

$$-3(6) + 4(5) = 5$$

$$-18 + 20 = 2 \neq 5$$

Therefore, there is no linear combination of the vectors $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ that yields $\begin{bmatrix} 17 \\ -4 \\ 5 \end{bmatrix}$.

Linear Combinations as systems.

We are aiming to know if $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{w}$. We can rewrite this as a matrix multiplication:

$$[v_1 \ v_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{w}$$

Then, to check whether w is a linear combination of v_1 and v_2, solve the system of equations. If a solution exists, then yes; if not, then no. Note that the a_n values are only scalar numbers, while the v_n's remain vectors.

We will talk more about systems of equations in Chapter **??**.

1.1 Visualizing Linear Combinations

First, let's look at what vectors can be made from linear combinations of the 2-dimensional unit vectors $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose we are looking for a linear combination of $\mathbf{i}$ and $\mathbf{j}$ to create the vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$. We can find such a linear combination:

$$3\mathbf{i} + (-4)\,\mathbf{j} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

In fact, with $\mathbf{i}$ and $\mathbf{j}$, we can create any 2-dimensional vector. To prove this, consider a generic vector, $\mathbf{z} = \begin{bmatrix} a \\ b \end{bmatrix}$, where $a, b \in \mathbb{R}$. We are looking for a linear combination of $\mathbf{i}$ and $\mathbf{j}$ such that:

$$c_1\mathbf{i} + c_2\mathbf{j} = \begin{bmatrix} a \\ b \end{bmatrix}$$

The above equation yields the system of equations:

$$c_1\,(1) + c_2\,(0) = a$$

$$c_1\,(0) + c_2\,(1) = b$$

And the solution to this system of equations is:

$$c_1 = a$$

$$c_2 = b$$

Therefore, using $\mathbf{i}$ and $\mathbf{j}$, we can construct any vector in $\mathbb{R}^2$ (that is, any vector in the xy-plane). What about combinations of other vectors?

Let's consider linear combinations of two vectors: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. The vectors are shown in figure **??**.

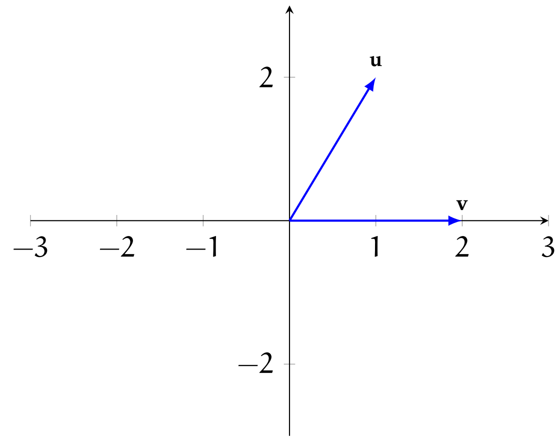
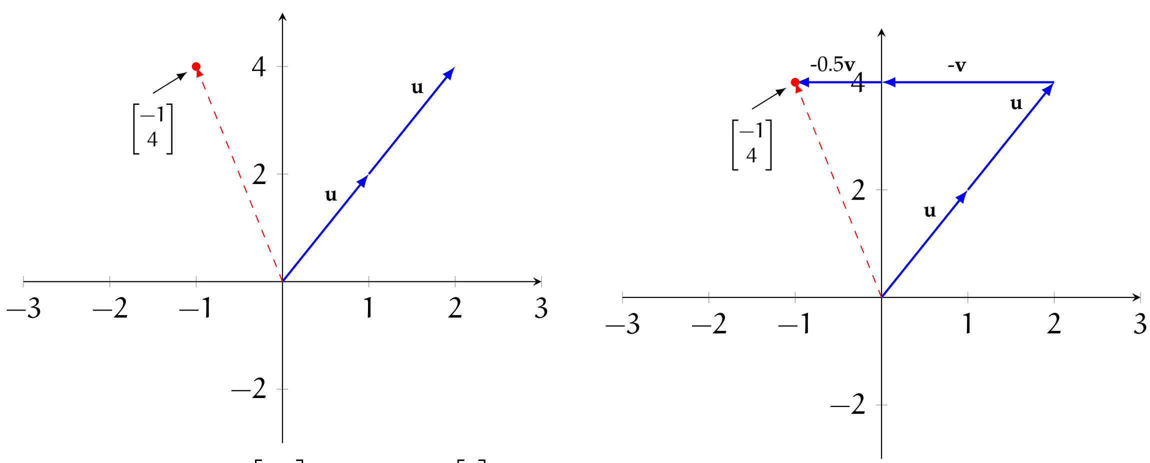


Figure 1.1: The vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Suppose we want to construct the vector $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Since only $\mathbf{u}$ has value in the y-dimension, we can start by adding $\mathbf{u}$ vectors to reach $y = 4$ (see figure **??**). Next, we can use $\mathbf{v}$ vectors to reach $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (see figure **??**).



(a) To create vector $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ with $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, we begin by adding two $\mathbf{u}$ vectors to reach a y-value of 4.

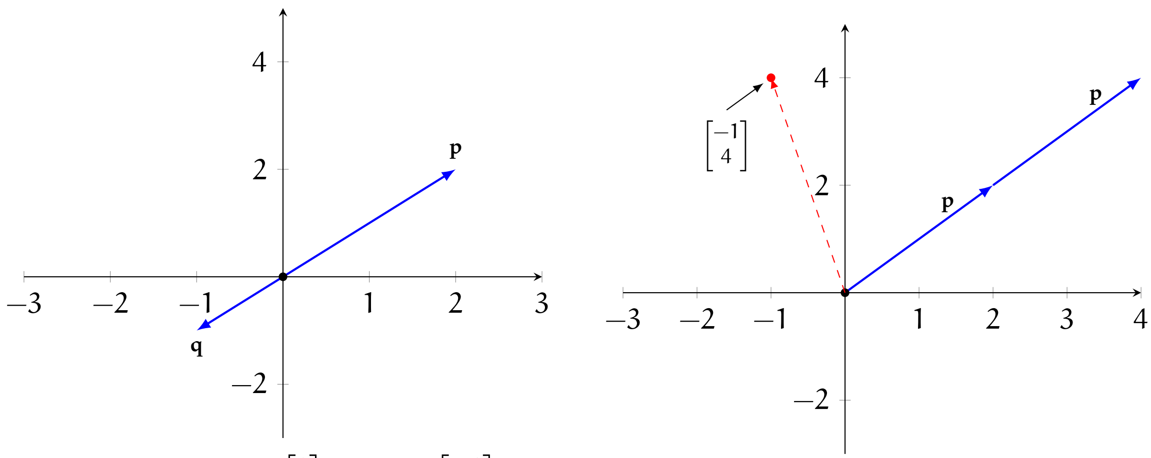
(b) If $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, then $2\mathbf{u} - 1.5\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Figure 1.2: Using linear combinations of $\mathbf{u}$ and $\mathbf{v}$ to construct $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Using this method, we can imagine reaching any point in $\mathbb{R}^2$: we add or subtract as many $\mathbf{u}$ vectors as needed to reach the appropriate y-value, then add or subtract as many $\mathbf{v}$

vectors to reach the appropriate x-value. The vectors are *not multiples* of each other, so we can say that they span all of $\mathbb{R}^2$.

Let's look at another pair of vectors: $\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ (see figure **??**). Again, let's try to use $\mathbf{p}$ and $\mathbf{q}$ to construct the vector $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. We begin by using $\mathbf{p}$ to reach the y-value of 4 (see figure **??**).



(a) The vectors $\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(b) Adding two $\mathbf{p}$ vectors to reach $y = 4$.

Figure 1.3: Visualizing combinations of $\mathbf{p}$ and $\mathbf{q}$.

But now we run into a problem: no matter how many multiples of $\mathbf{q}$ vectors we add or subtract, we just move along the $\mathbf{p}$ vector and never reach our goal of $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (see figure **??**). Notice that $\mathbf{p}$ and $\mathbf{q}$ lie on the same line, (for a better visualization, refer back to **??**). In other words, the vectors are *scalar multiples* of each other: $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = -2 \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. When two vectors lie on the same line or a scalar multiples of each other, we call them *linearly dependent*.

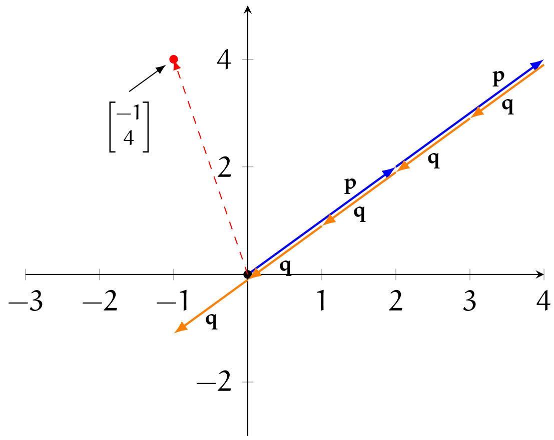


Figure 1.4: There is no linear combination of $\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ that yields the vector $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

FIXME add summary

In the next chapter, we will be looking more at linear combinations, span, and independence in the form of systems of equations. Future chapters will look even further at these concepts, linear dependence and independence, and their applications to various subspaces.

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`) for more details.

Appendix A

Answers to Exercises

Answer to Exercise ?? (on page ??)

1. We are looking for a_1 and a_2 such that:

$$a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Which creates the system of equations:

$$a_1 - 3a_2 = 4$$

$$2a_1 + a_2 = 5$$

We can multiply the first equation by -2 and add it to the second to solve for a_2:

$$-2\left(a_1 - 3a_2\right) + 2a_1 + a_2 = -2\left(4\right) + 5$$

$$6a_2 + a_2 = -8 + 5$$
$$7a_2 = -3$$
$$a_2 = -\frac{3}{7}$$

Substituting a_2 back into an equation and solving for a_1:

$$a_1 - 3\left(-\frac{3}{7}\right) = 4$$

$$a_1 + \frac{9}{7} = 4$$
$$a_1 = \frac{19}{7}$$

Therefore, $\frac{19}{7}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{7}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

2. We are looking for a_1 and a_2 such that:

$$a_1 \begin{bmatrix} 9 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Which creates the system of equations:

$$9a_1 = -5$$

$$4a_1 + a_2 = 3$$

We can find a_1 from the first equation:

$$a_1 = -\frac{5}{9}$$

Substituting for a_1 back into the second equation and solving for a_2:

$$4\left(-\frac{5}{9}\right) + a_2 = 3$$

$$a_2 - \frac{20}{9} = 3$$

$$a_2 = \frac{47}{9}$$

Therefore, $-\frac{5}{9}\begin{bmatrix}9\\4\end{bmatrix} + \frac{47}{9}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-5\\3\end{bmatrix}$.

3. We are looking for a_1 and a_2 such that:

$$a_1\begin{bmatrix}7\\-2\end{bmatrix} + a_2\begin{bmatrix}-8\\4\end{bmatrix} = \begin{bmatrix}6\\-2\end{bmatrix}$$

Which yields the system of equations:

$$7a_1 - 8a_2 = 6$$

$$-2a_1 + 4a_2 = -2$$

Doubling the second equation and adding it to the first:

$$7a_1 - 8a_2 + 2(-2a_1 + 4a_2) = 6 + 2(-2)$$

$$7a_1 - 8a_2 - 4a_1 + 8a_2 = 6 - 4$$

$$3a_1 = 2$$

$$a_1 = \frac{2}{3}$$

Substituting for a_1 back into the second equation and solving for a_2:

$$-2\left(\frac{2}{3}\right) + 4a_2 = -2$$

$$-\frac{4}{3} + 4a_2 = -2$$

$$4a_2 = -\frac{2}{3}$$
$$a_2 = -\frac{1}{6}$$

Therefore, $\frac{2}{3}\begin{bmatrix} 7 \\ -2 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

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