Chapter 1

Calculus with Polar Coordinates

We've been working in Cartesian coordinates, which are rectangular, with x representing the horizontal position and y representing the vertical position. Another way to represent a position in 2D space is with **polar coordinates**. In this coordinate system, the first number and independent variable is Θ and represents the degrees of rotation from the the x axis. The second number is r and represents how far the point is from the origin (see figure ??).

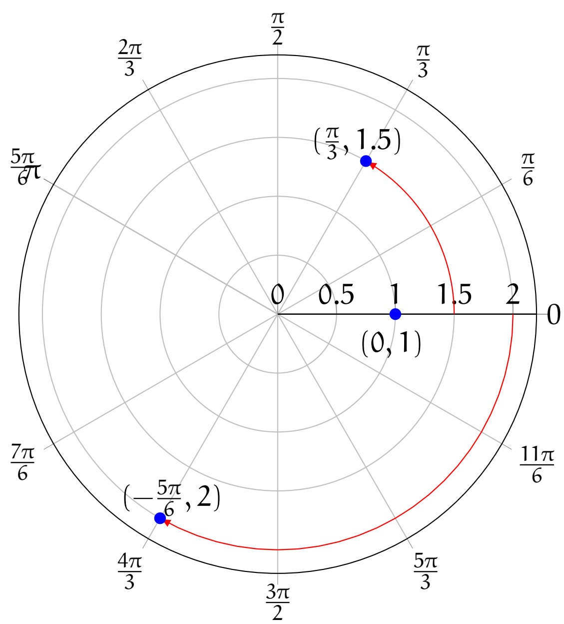

Figure 1.1: Polar coordinates give a degree of rotation, θ, and a distance from the origin, r

1.1 Derivatives of Polar Functions

Consider the cardioid $r = 2 + \sin\theta$ (see figure ??). What is the slope of the line tangent to the curve at $\theta = \frac{\pi}{2}$?

From a visual inspection, we can guess that the slope of the tangent line is zero. Let's prove this mathematically:

First, recall that to convert polar coordinates to Cartesian coordinates, we can use the

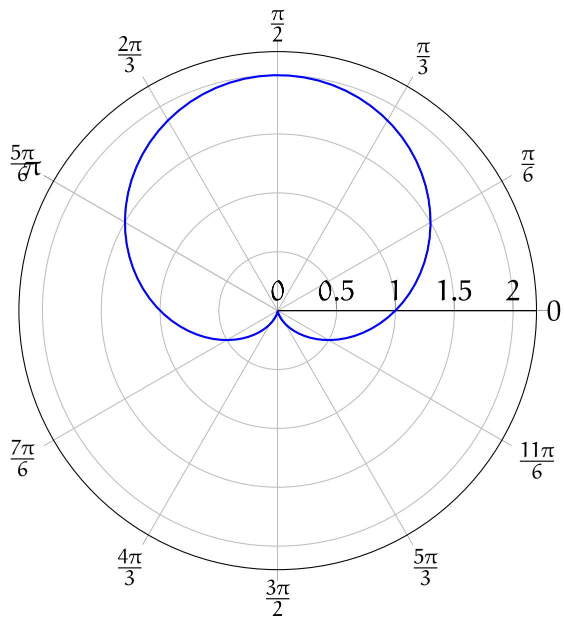

Figure 1.2: $r = 2 + \sin\theta$

trigonometric identities:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

So we can write the parametric equation:

$$x = [2 + \sin\theta]\cos\theta$$
$$y = [2 + \sin\theta]\sin\theta$$

Recall from parametric equations that we can use implicit differentiation to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(2\sin\theta + \sin^2\theta\right) = 2\cos\theta + 2\sin\theta\cos\theta$$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(2\cos\theta + \sin\theta\cos\theta\right) = \cos^2\theta - \sin^2\theta - 2\sin\theta$$

Substituting $\theta = \frac{pi}{2}$, we find that:

$$\frac{dy}{d\theta} = 2(0) + 2(1)(0) = 0$$

$$\frac{dx}{d\theta} = (0)^2 - (1)^2 - 2(1) = -3$$

And therefore,

$$\frac{dy}{dx} = \frac{0}{-3} = 0$$

Which is the result we expected from examining the graph of $r = 2 + \sin\theta$.

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] What is the slope of the line tangent to the polar curve $r = 1 + 2\sin\theta$ at $\theta = 0$?

Working Space

Answer on Page 5

Exercise 2

The figure below shows the graphs of polar curves $r = 2\cos 3\theta$ and $r = 2$. What is the sum of the areas of the shaded regions? [fix me graph]

Working Space

Answer on Page 5

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

Recall that for a polar function, $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$. At $\theta = 0$, $r = 1 + 2\sin 0 = 1$ and $\frac{dr}{d\theta} = 2\cos 0 = 2$. Substituting, we find that $\frac{dy}{dx} = \frac{2\sin 0 + 1\cos 0}{2\cos 0 - 1\sin 0} = \frac{0+1}{2-0} = \frac{1}{2}$.

Answer to Exercise 2 (on page 3)

We know the area of the circle is $\pi r^2 = \pi(2)^2 = 4\pi$. To find the area of the shaded regions, we need to subtract the area of the trefoil from the area of the circle. The area of the trefoil is given by $\frac{1}{2}\int_0^{\pi} [2\cos 3\theta]^2 \, d\theta = [\text{fixme finish solution}]$

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