Chapter 1

Volumes with Integrals

Suppose we wanted to know the volume of a theoretical irregular shape (we stipulate theoretical because, if you had this object and a large enough container, you could use displacement to determine the volume of the object). [fixme better intro]

1.1 Volume of a Sphere

Below, we will prove the volume of a sphere is given by $\frac{4}{3}\pi r^3$ using the integral method. Suppose we have a sphere of radius r centered at the origin (see figure 1.1).

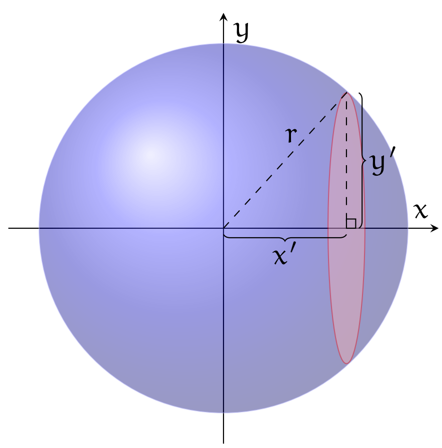


Figure 1.1: A vertical cross-section of a sphere

We begin by taking very thin vertical cross-sections. The radius of the cross-section is the height, y, of the sphere at the horizontal position, x. Since the edges of the cross-section lie on the sphere, we know the edge of the cross-section is distance r from the origin. Applying the Pythagorean theorem, we see that $r^2 = x^2 + y^2$, which implies that $y = \sqrt{r^2 - x^2}$. So, the area of the cross-section is given by $\pi y^2 = \pi(r^2 - x^2)$. If we imagine each cross section as having a width, dx, and taking the sum of all the cross sections from $x = -r$ to $x = r$, we can write an integral equal to the volume of the sphere:

$$V_{\text{sphere}} = \int_{-r}^{r} \pi(r^2 - x^2)\, dx$$

We can then evaluate that integral:

$$V_{\text{sphere}} = \pi \int_{-r}^{r} r^2\, dx - \pi \int_{-r}^{r} x^2\, dx$$

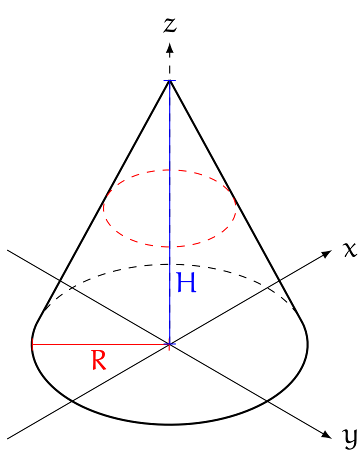
$$V_{\text{sphere}} = \pi \left[r^2 x\right]_{x=-r}^{x=r} - \frac{\pi}{3}\left[x^3\right]_{x=-r}^{x=r}$$

$$V_{\text{sphere}} = \pi \left[r^3 - (-r^3)\right] - \frac{\pi}{3}\left[r^3 - (-r^3)\right]$$

$$V_{\text{sphere}} = 2\pi r^3 - \frac{2\pi}{3} r^3 = \frac{4}{3}\pi r^3$$

Exercise 1

Prove the volume of a regular cone is $\frac{\pi}{3}R^2H$, where R is the radius of the base and H is the height of the cone. (Hint: A cone is a series of decreasing circles stacked on top of each other; see figure below.)



Working Space

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1.2 Volumes of Solids of Revolution

We can also find the volume of solids made by revolving a graph about the x or y-axis. Suppose the graph $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ were rotated vertically about the x-axis to form a solid. How could we find the volume of that solid? Well, we can imagine a rectangle of width dx and height y (see figure 1.2)

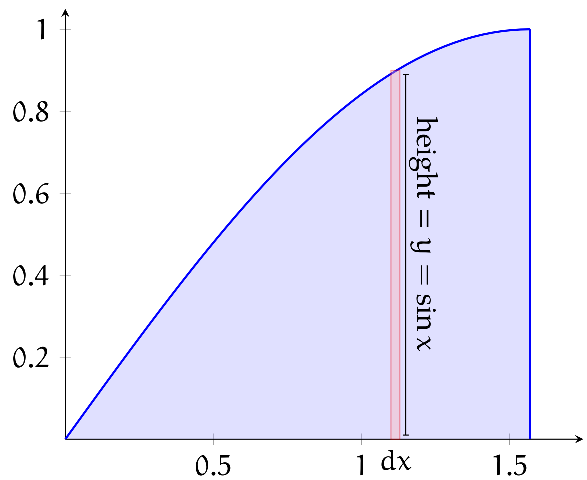


Figure 1.2: A cross section has width dx and height $y = \sin x$

If we rotate the plot vertically about the x-axis, the rectangle becomes a cylinder with radius $y = \sin x$ and height dx (see figure **??**). Therefore, the volume of each cylindrical slice is $V_{slice} = \pi r^2 dx = \pi \cdot \sin^2 x dx$.

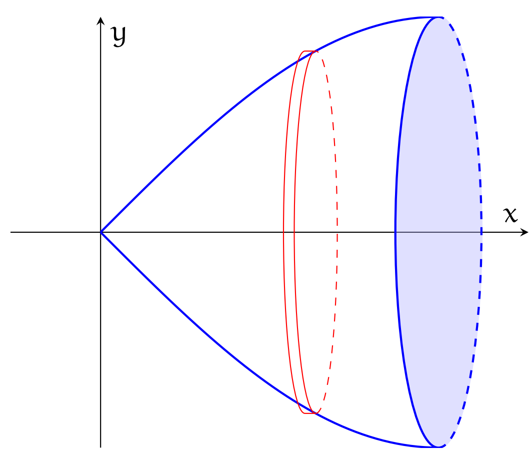


Figure 1.3: When rotated, the cross-section becomes a cylinder with radius $\sin x$ and width dx, which has a total volume of $\pi \sin^2 x dx$

We can find the total volume by integrating from 0 to $\pi/2$:

$$V = \pi \int_0^{\pi/2} \sin^2 x \, dx$$

Recall the half angle formula, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Substituting, we see that:

$$V = \frac{\pi}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$$

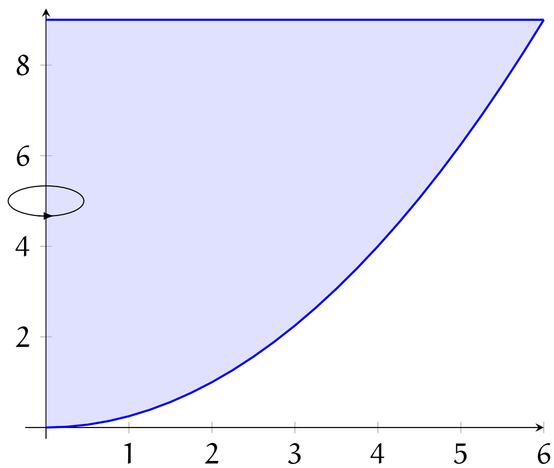
$$V = \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right) \Big|_{x=0}^{x=\pi/2}$$

$$V = \frac{\pi}{2} \left[\left(\pi/2 - \frac{1}{2} \sin \pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right]$$

$$V = \frac{\pi}{2} [\pi/2 - 0 - 0 + 0] = \frac{\pi^2}{4}$$

Exercise 2

Find the volume of a solid created by rotating the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$ about the y-axis. A graph is shown below.



Working Space

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Exercise 3

Let $f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$. Bird's eggs of various sizes can be modeled by rotating $f(x)$ about the x-axis, with different values of a, b, c, and d defining different sizes and shapes of eggs. For a domestic chicken, $a = -0.02$, $b = 0.03$, $c = 0.12$, and $d = 0.454$. For a mallard duck, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Use a calculator, such as a TI-89 or Wolfram Alpha, to determine which species lays a bigger egg.

Working Space

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1.2.1 Using donuts for solids of revolution

Sometimes there is space between the region we are rotating and the line we are rotating it about. Consider the region bounded between $y = 2x$ and $y = x^2$ (see figure 1.4):

When rotated, the slices will take the form of donuts (or washers), the volume of which is $\pi\left(R^2 - r^2\right) dx$, where R is the outer radius and r is the inner radius. Therefore, in this case, the total volume of the rotated region is given by:

$$V = \int_0^2 \pi \left[(2x)^2 - \left(x^2\right)^2\right] dx$$

$$V = \pi \int_0^2 4x^2 - x^4 \, dx = \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{x=0}^{x=2}$$

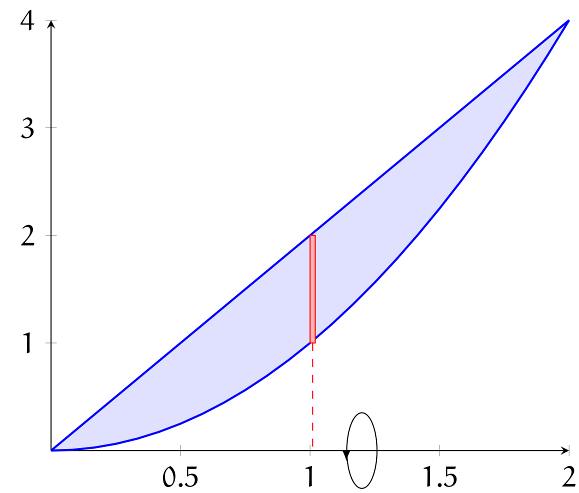

Figure 1.4: When rotated, the slices will become donuts with outer radius $2x$ and inner radius x^2

$$V = \pi\left[\frac{4}{3}2^3 - \frac{1}{5}2^5\right] = \pi\left[\frac{32}{3} - \frac{32}{5}\right]$$

$$V = \frac{64\pi}{15}$$

Exercise 4

What is the volume of the region bounded by $y = x^2$ and $y = 2\sqrt{x}$ when rotated about the y-axis?

Working Space

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1.3 Volumes of Other Solids

You can also model a solid as a base defined by a function with cross-sections of specific shapes. Consider the function $y = x^2$ from $x = 0$ to $x = 2$ (see figure 1.5). Suppose the area between the curve, the y-axis, and the line $y = 4$ defines a base and each vertical cross-section is a square. So, the width of the each cross section is dx, the length is $4-x^2$, and (because they are squares) the height in the z-plane is also $4 - x^2$. The volume of each cross-section is $V_{slice} = \left(4 - x^2\right)^2 dx$ and the total volume of the solid is:

$$V = \int_0^2 \left(4 - x^2\right)^2 \, dx$$

$$V = \int_0^2 \left(16 - 8x^2 + x^4\right) \, dx$$

$$V = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{x=0}^{x=2}$$

$$V = 16(2) - \frac{8}{3}(2)^3 + \frac{1}{5}(2)^5 = \frac{256}{15} \approx 17.067$$

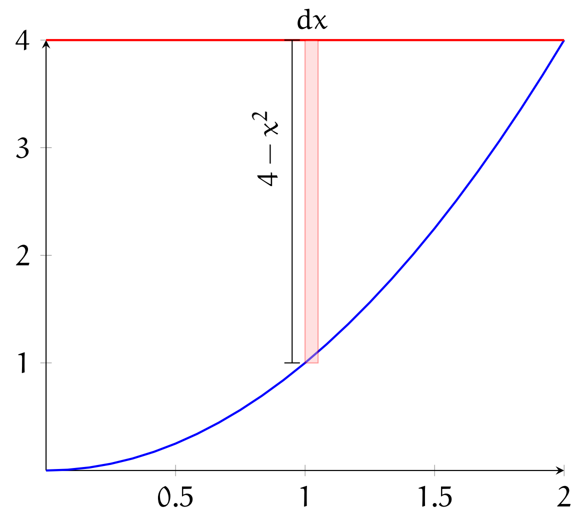

Figure 1.5: $y = x^2$ with a vertical cross-section

You can use a similar method for triangular, semi-circular, or any other shape cross-section. The trick is writing everything in terms of x (when you cross sections are vertical and have width dx) or y (when your cross section are horizontal and have length dy).

Exercise 5

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let R be the region in the first quadrant bounded above by the graph $y = \ln(3 - x)$, for $0 \leq x \leq 2$. R is the base of a solid for which each cross section perpendicular to the x-axis is square. What is the volume of the solid? Give your answer to 3 decimal places.

Working Space

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Exercise 6

Find the volume of a solid whose base is defined by the ellipse $9x^2 + 16y^2 = 25$ and is made up of isosceles-triangular cross-sections perpendicular to the x-axis (with the hypotenuse in the base of the solid).

Working Space

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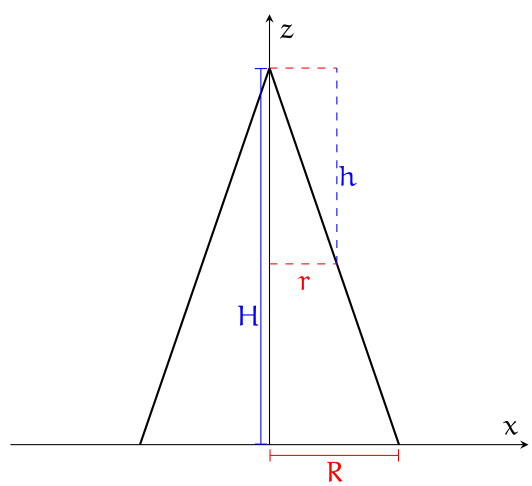
Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

Imagine a side view of the cone (see figure below), an isosceles triangle with height H and base $2R$. If we take horizontal cross-sections, then each cross-section is a circle h from the top with a radius r. Because the triangles are similar (FIXME: better wording/explanation here), we also know that $\frac{H}{h} = \frac{R}{r}$. Therefore, we can define r in terms of h: $r = \frac{hR}{H}$ and the volume of each subsequent cross-section is $\pi r^2 dh = \pi \frac{h^2 R^2}{H^2} dh$. We start with $h = 0$ and end with $h = H$:

$$
\begin{aligned}
V_{cone} &= \int_0^H \pi \frac{h^2 R^2}{H^2}\, dh = \pi \frac{R^2}{H^2} \int_0^H h^2\, dh \\
&= \pi \frac{R^2}{H^2} \left[\frac{1}{3} h^3\right]_{h=0}^{h=H} = \pi \frac{R^2}{3H^2} \left[H^3 - 0^3\right] \\
&= \pi \frac{R^2}{3H^2} H^3 = \frac{\pi}{3} R^2 H
\end{aligned}
$$



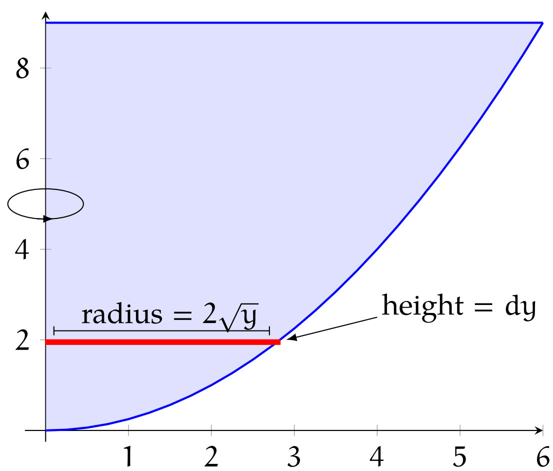
Answer to Exercise 2 (on page 5)

If we are rotating about the y axis, we should make our slices horizontal, so their width is dy (see graph below). Then, the volume of each cylinder is given by $V = \pi r^2 dy$ and the total volume is given by:

$$V = \int_0^9 \pi \left[2\sqrt{y}\right]^2 \, dy$$

$$V = 4\pi \int_0^9 y \, dy = 2\pi y^2 \Big|_{y=0}^{y=9}$$

$$V = 2\pi \left(9\right)^2 = 162\pi$$



Answer to Exercise 3 (on page 6)

Since the graph is rotated around the x-axis, we will take vertical slices with width dx, and rotate them to make cylinders with radius $f(x)$ and height dx. The volume of each egg is given by:

$$\int_{-1}^{1} \pi \left[f(x)\right]^2 \, dx$$

To determine our limits of integration, we note that $\sqrt{1-x^2} = 0$ (and therefore, $f(x) = 0$) when $x = \pm 1$.

For the chicken:

$$V_{chickenegg} = \pi \int_{-1}^{1} \left[\left(-0.02x^3 + 0.03x^2 + 0.12x + 0.454\right)\sqrt{1-x^2}\right]^2 \, dx$$

For the mallard duck:

$$V_{duckegg} = \pi \int_{-1}^{1} \left[\left(-0.06x^3 + 0.04x^2 + 0.1x + 0.54 \right) \sqrt{1-x^2} \right]^2 dx$$

Using a calculator, we find that $V_{chickenegg} \approx 0.897$ and $V_{duckegg} \approx 1.263$. Therefore, mallard ducks lay larger eggs than chickens do.

Answer to Exercise ?? (on page 8)

First, since we are revolving around the y-axis, we know our slices will have width dy. We will rewrite the functions as x in terms of y:

$$x = \sqrt{y}$$

$$x = \frac{y^2}{4}$$

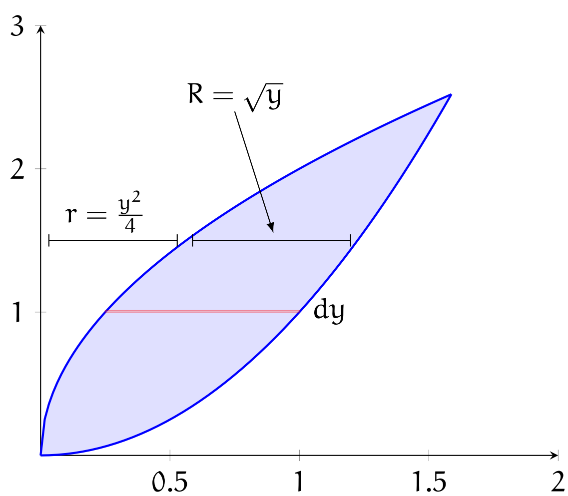
Setting them equal to each other to find the y-value at which they intercept:

$$\sqrt{y} = \frac{y^2}{4}$$

$$4 = \frac{y^2}{\sqrt{y}} = y^{3/2}$$

$$y = \sqrt[3]{4^2} = 2\sqrt[3]{2}$$

Examining a graph (shown below), we see that the outer radius is $x = \sqrt{y}$ and the inner radius is $x = \frac{y^2}{4}$.



So, the total volume of the solid of revolution is given by:

$$V = \pi \int_0^{2\sqrt[3]{2}} (\sqrt{y})^2 - \left(\frac{y^2}{4}\right)^2 \, dy$$

$$V = \pi \int_0^{2\sqrt[3]{2}} \left[y - \frac{y^4}{16}\right] \, dy$$

$$V = \pi \left[\frac{1}{2}y^2 - \frac{1}{80}y^5\right]_{y=0}^{y=2\sqrt[3]{2}}$$

$$V = \pi \left[\frac{6}{5}2^{2/3}\right] \approx 5.9844$$

Answer to Exercise 5 (on page 10)

If each cross section is a square, then the volume of each cross section is given by $s^2 dx$, where s is the side length of the square. Since the side length is equal to the distance between the graph of y and the x-axis, we can see that $s = y = \ln(3-x)$. And, therefore, the total volume of all the cross sections is given by $\int_0^2 [\ln(3-x)]^2 \, dx$. Using a calculator, this integral evaluates to ≈ 1.029.

Answer to Exercise 6 (on page 11)

Since the cross-sections are perpendicular to the x-axis, they will have width dx and we will integrate across the domain of the ellipse. Setting $y = 0$ to find the domain of the ellipse:

$$9x^2 = 25 \to x^2 = \frac{25}{9} \to x = \pm\frac{5}{3}$$

A right isosceles triangle with hypotenuse h has area $\frac{1}{4}h^2$. In this case, each triangle's hypotenuse is given by the distance between the top and bottom of the ellipse. The top of the ellipse if defined by $y = \frac{1}{4}\sqrt{25-9x^2}$ and the bottom by $y = -\frac{1}{4}\sqrt{25-9x^2}$. Therefore, the length of each hypotenuse is $\frac{1}{2}\sqrt{25-9x^2}$.

Then, each cross-section has a total volume of $\frac{1}{4}h^2 dx = \frac{1}{4}\left(\frac{1}{2}\sqrt{25-9x^2}\right)^2 dx$ and the volume of the solid is:

$$V_{\text{solid}} = \int_{-5/3}^{5/3} \frac{1}{4}\left(\frac{1}{2}\sqrt{25-9x^2}\right)^2 \, dx$$

$$= \frac{1}{4}\int_{-5/3}^{5/3} \frac{1}{4}\left(25-9x^2\right) \, dx$$

$$= \frac{1}{6}\int_{-5/3}^{5/3} \left(25 - 9x^2\right)\ dx = \frac{1}{16}\left[25x - 3x^3\right]_{x=-5/3}^{x=5/3}$$

$$= \frac{1}{16}\left[\left(25\left(\frac{5}{3}\right) - 25\left(\frac{-5}{3}\right)\right) - \left(3\left(\frac{5}{3}\right)^3 - 3\left(\frac{-5}{3}\right)^3\right)\right]$$

$$= \frac{1}{16}\left[\frac{250}{3} - \left(\frac{375}{27} + \frac{375}{27}\right)\right] = \frac{1}{16}\left[\frac{250}{3} - \frac{250}{9}\right] = \frac{1}{16}\left[\frac{750}{9} - \frac{250}{9}\right] = \frac{1}{16}\left[\frac{500}{9}\right] = \frac{125}{36}$$