

Vectors and Matrices

The last chapter provided an overview of linear algebra, using several image examples. In this chapter, we will focus primarily on vector-matrix multiplications. First, we will show how matrices can be used to represent a set of linear equations. Then, we will provide you with a general definition of vector-matrix multiplication, followed by a few examples. You will have an opportunity to solve a problem manually, then by using Python. In this chapter, we will use two-dimensional matrices for simplicity, but a matrix can have any number of dimensions.

1.1 Matrices

We've been looking at vectors, which are usually represented as a column of numbers. For example, while we may write $\mathbf{v} = [1, 2, 3]$ in line, the vector is really:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

A matrix can be made of many columns, like the 3×2 matrix shown below:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{bmatrix}$$

We describe the size and shape of matrices by saying *an $m \times n$ matrix*, where m is the number of rows and n is the number of columns. A vector is simply a one-column matrix. For example, the vector \mathbf{v} above is 3×1 . Matrices aren't restricted to 2 dimensions: a matrix can be 3, 4, or any number of dimensions. For example, a $3 \times 2 \times 4$ matrix would be made of 4 stacked 3×2 matrices.

Exercise 1 Matrix Dimensions 1

Write the dimensions of the following matrices:

1.
$$\begin{bmatrix} -3 & 0 & 4 & -2 & -4 \\ -1 & 5 & 3 & 4 & -2 \\ -3 & 2 & 3 & -5 & 1 \end{bmatrix}$$

2.
$$\begin{bmatrix} -3 & 1 \end{bmatrix}$$

3.
$$\begin{bmatrix} -3 & 2 & -3 \\ 4 & 0 & -3 \\ -5 & -4 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

Working Space

Answer on Page 13

Exercise 2 Matrix Dimensions 2

Create a matrix with the indicated dimensions.

1. 1×3

2. 2×4

3. 4×3

Working Space

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1.1.1 Zero Matrices

Recall that we can represent a generic zero vector as $\mathbf{0}$, which indicates a vector of any number of dimensions filled with zeros. Just like vectors, there are *zero matrices*, which can be of any number of dimensions, all filled with zeros. In two dimensions, zero matrices

are denoted as $O_{m \times n}$, where the subscript is the dimension of the matrix. The subscript can be expanded to denote any number of dimensions.

1.2 Matrix Arithmetic

1.2.1 Adding and Subtracting Matrices

Matrices that are the same dimension can be added and subtracted. Just like vectors, to add matrices you add the elements in the same position:

$$\begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -2+5 & -1+2 \\ 2+(-1) & 4+(-4) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

And to subtract matrices, you subtract the elements in the same position:

$$\begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -2-5 & -1-2 \\ 2-(-1) & 4-(-4) \end{bmatrix} = \begin{bmatrix} -7 & -3 \\ 3 & 8 \end{bmatrix}$$

Formally, for 2-dimensional matrices, we can say that:

Adding and Subtracting Matrices

For two $m \times n$ matrices, the sum of the matrices is the matrix of the sums of the elements in analogous positions:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_{m1} & y_{m2} & y_{m3} & \cdots & y_{mn} \end{bmatrix} = \begin{bmatrix} x_{11} + y_{11} & x_{12} + y_{12} & x_{13} + y_{13} & \cdots & x_{1n} + y_{1n} \\ x_{21} + y_{21} & x_{22} + y_{22} & x_{23} + y_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{m1} + y_{m1} & x_{m2} + y_{m2} & x_{m3} + y_{m3} & \cdots & x_{mn} + y_{mn} \end{bmatrix}$$

To subtract matrices, simply add the negative of the second matrix (that is, $A - B = A + -B$). Additionally, matrix addition is commutative ($A + B = B + A$).

Exercise 3 Adding and Subtracting MatricesFind $A + B$, $A - B$, and $B - A$.

Working Space

1. $A = \begin{bmatrix} 0 & 4 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & -2 & 5 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & -4 & -2 \\ 1 & -3 & 5 \\ -5 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & -1 \\ -5 & -3 & -2 \\ -5 & 3 & -4 \end{bmatrix}$.

3. $A = \begin{bmatrix} -2 & -1 & -5 & -1 \\ 5 & -4 & 4 & 3 \\ -5 & -2 & 3 & -5 \\ 0 & 5 & -4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -2 & 3 & -5 \\ 0 & 5 & -4 & -3 \end{bmatrix}$.

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1.2.2 Multiplying Matrices

Surprisingly (it may be to you), matrix multiplication has dimension limits. We cannot multiply any two matrices: the first matrix must have the same number of columns as the second has number of rows. Let's examine the origin of the dimension limits on matrix multiplication. We begin with a review of the vector dot product.

Recall that in order to find the dot product of two vectors, they must be the same length (that is, the same number of dimensions). The result is always a scalar: one number. You can review finding the dot product of vectors and practice the dimension limits on the vector dot product in the next exercise.

Exercise 4 **Vector Dot Product Review**

Find all possible pairs of vectors that can be used to find a dot product, then find the dot products.

Working Space

1. $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. $\mathbf{b} = \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$

3. $\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

4. $\mathbf{d} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

5. $\mathbf{e} = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 1 \end{bmatrix}$

6. $\mathbf{f} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

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To multiply two matrices, it is helpful to think of the rows of the first matrix and the columns of the second matrix as vectors. Let's see how this shakes out for two 2×2 matrices:

Let's look at this more concretely. For two-dimensional matrices, it can be helpful to move your left index finger across the row and right index finger down the column, as shown in figure 1.2.

Since each entry in the product matrix is the dot product between a row of the first matrix and a column of the second matrix, the first matrix must have the same number of elements in each row as the second has in each column. Another way to say this is that

$$\begin{array}{c}
 \mathbf{a}_1 \rightarrow \\
 \mathbf{a}_2 \rightarrow
 \end{array}
 \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}
 \cdot
 \begin{array}{c}
 \mathbf{b}_1 \quad \mathbf{b}_2 \\
 \downarrow \quad \downarrow \\
 \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix}
 \end{array}
 =
 \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \end{bmatrix}$$

$$\mathbf{A} \quad \cdot \quad \mathbf{B}$$

Figure 1.1: Each entry in \mathbf{C} , c_{ij} , is the dot product of the i^{th} row of \mathbf{A} , \mathbf{a}_i , and the j^{th} column of \mathbf{B} , \mathbf{b}_j .

$$\begin{array}{c}
 \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} 5 \times -1 + 4 \times -4 \\ -5 \times -1 + 1 \times -4 \end{bmatrix} \\
 \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -21 & 5 \times -2 + 4 \times -4 \\ -5 \times -1 + 1 \times -5 \end{bmatrix} \\
 \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -21 & -26 \\ 5 \times -1 + 1 \times -5 \end{bmatrix} \\
 \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -21 & -26 \\ 0 & -5 \times -2 + 1 \times -4 \end{bmatrix} \\
 \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -21 & -26 \\ 0 & 6 \end{bmatrix}
 \end{array}$$

Figure 1.2: You can use your fingers to trace across matrix \mathbf{A} and down matrix \mathbf{B} to find $\mathbf{A} \cdot \mathbf{B}$.

the number of columns of the first matrix must match the number of rows in the second matrix.

Matrix Multiplication

For two-dimensional matrices, the inner dimensions must match in order to carry out matrix multiplication. That is, if we want to find $\mathbf{A} \cdot \mathbf{B}$, and \mathbf{A} has dimensions $m \times n$, then \mathbf{B} must have dimensions $n \times p$, where m , n , and p are integers. The resulting matrix will have dimensions $m \times p$ (m and p may be equal or unequal).

Exercise 5 Multiplying Matrices 1

Multiply the matrices.

Working Space

$$1. \begin{bmatrix} -5 & -2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 & -5 \\ 3 & 0 & 3 \\ 4 & -1 & -4 \\ -1 & -4 & 2 \end{bmatrix}$$

$$2. \begin{bmatrix} 1 \\ 5 \\ -5 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 5 & 1 \end{bmatrix}$$

$$3. \begin{bmatrix} -1 & 4 & -4 \\ 5 & -3 & 5 \\ -1 & -4 & 4 \\ -4 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -3 & 5 & 1 \\ -3 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

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Exercise 6 **Multiplying Matrices 2**Find $A \cdot B$ and $B \cdot A$.*Working Space*

1. $A = \begin{bmatrix} -2 \\ 2 \\ 1 \\ -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 & -5 & -2 \end{bmatrix}$

2. $A = \begin{bmatrix} -4 & -2 \\ 2 & 5 \\ -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 & -4 \\ 1 & -4 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ -4 & 0 & -5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ -4 & -1 \\ -2 & 3 \\ -5 & 1 \end{bmatrix}$

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What have you noticed about the results of $A \cdot B$ as compared to $B \cdot A$? You should have noticed that the product matrices are *different dimensions*. This leads us to the next unusual property of matrix multiplication: it is *non-commutative*. That is, the *order* in which you multiply matrices affects the result. This is very different from scalar values!

As you saw in the second matrix multiplication exercise, A is a 2×4 matrix and B is a 4×2 matrix, then AB is a 2×2 matrix, while BA is a 4×4 matrix. It is obvious, then, that $A \cdot B \neq B \cdot A$. What if A and B are square matrices?

Example: Find $A \cdot B$ and $B \cdot A$ if $A = \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$.

Solution:

$$A \cdot B = \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} -3(-1) + 5(4) & -3(1) + 5(-3) \\ -1(-1) + 0(4) & -1(1) + 0(-3) \end{bmatrix} = \begin{bmatrix} 23 & -18 \\ 1 & -1 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1(-3) + 1(-1) & -1(5) + 1(0) \\ 4(-3) + -3(-1) & 4(5) + -3(0) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -9 & 20 \end{bmatrix}$$

As you can see, even if A and B are square, matrix multiplication is still not commutative.

Non-Commutation of Matrix Multiplication

For two matrices A and B , where neither is an identity matrix or a zero matrix:

$$A \cdot B \neq B \cdot A$$

Properties of the Zero Matrix

Just like the number 0, the zero matrix, O has unique mathematical properties:

Properties of the Zero Matrix

For a matrix, A , and a zero matrix, O

1. $A + O = A$
2. $A + -A = O$
3. $0 \cdot A = O$

The Identity Matrix

There is another special matrix, called the *identity matrix*, usually denoted with I . An identity matrix is all zeroes except for a diagonal line of ones. A 3×3 identity matrix is shown below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All identity matrices are square (that is, they have the same number of rows as they do columns). The identity matrix has the special property that whenever a vector or matrix is multiplied by I , it doesn't change. Let's look at some examples:

Example: If $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, what is $I\mathbf{x}$? (Take I to be a 2×2 identity matrix.)

Solution:

$$I\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \cdot (2) + 0 \cdot (-3) \\ 0 \cdot (2) + 1 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Example: If $B = \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$, what is $I \cdot B$?

Solution:

$$I \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 0 \cdot (5) & 1 \cdot (5) + 0 \cdot (-4) \\ 0 \cdot (-2) + 1 \cdot (5) & 0 \cdot (5) + 1 \cdot (-4) \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$$

Properties of the Identity Matrix

An $n \times n$ identity matrix, I , does not change any vectors or matrices it multiplies. That is:

$$1. \ I \cdot \mathbf{x} = \mathbf{x}$$

$$2. \ I \cdot B = B$$

where \mathbf{x} is an $n \times 1$ vector and B is an $n \times p$ matrix (p may be, but is not necessarily, equal to n).

1.2.3 Can We Divide Matrices?

Matrices cannot be divided. Suppose we have a matrix, A , a vector \mathbf{x} , and another vector \mathbf{b} such that:

$$A \cdot \mathbf{x} = \mathbf{b}$$

Now, if we know A and \mathbf{x} , it is easy to find \mathbf{b} . What if, on the other hand, we know A and \mathbf{b} and want to find \mathbf{x} ? We might be tempted to do something like this:

$$\mathbf{x} = \frac{\mathbf{b}}{A}$$

While this would be correct if \mathbf{x} , \mathbf{b} , and A were scalars, but it is not for matrices. However, there is an analogy we can make. Instead of trying to divide by A , we can multiply by its *inverse*:

Inverse Matrices

Given a matrix A , and vectors \mathbf{b} and \mathbf{x} , if

$$A \cdot \mathbf{x} = \mathbf{b}$$

Then,

$$\mathbf{x} = A^{-1} \cdot \mathbf{b}$$

A^{-1} is called the *inverse matrix*. We will explore inverse matrices and how to find them in the next chapter.

This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.

Answers to Exercises

Answer to Exercise 1 (on page 2)

1. 3×5
2. 1×2
3. 4×3

Answer to Exercise 2 (on page 2)

1. The matrix should have 1 row and 3 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

2. The matrix should have 2 rows and 4 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

3. The matrix should have 4 rows and 3 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Answer to Exercise 3 (on page 4)

1. $A + B = \begin{bmatrix} -2 & 7 & -2 & 10 \end{bmatrix}$. $A - B = \begin{bmatrix} 2 & 1 & 2 & 0 \end{bmatrix}$. $B - A = \begin{bmatrix} -2 & -1 & -2 & 0 \end{bmatrix}$

2. $A + B = \begin{bmatrix} 9 & -4 & -3 \\ -4 & -6 & 3 \\ -10 & 6 & -4 \end{bmatrix}$. $A - B = \begin{bmatrix} -1 & -4 & -1 \\ 6 & 0 & 7 \\ 0 & 0 & 4 \end{bmatrix}$. $B - A = \begin{bmatrix} 1 & 4 & 1 \\ -6 & 0 & -7 \\ 0 & 0 & -4 \end{bmatrix}$.

$$3. \mathbf{A} + \mathbf{B} = \begin{bmatrix} -7 & -3 & -2 & -6 \\ 5 & 1 & 0 & 0 \end{bmatrix}, \mathbf{A} - \mathbf{B} = \begin{bmatrix} 3 & 1 & -8 & 4 \\ 5 & -9 & 8 & 6 \end{bmatrix}, \mathbf{B} - \mathbf{A} = \begin{bmatrix} -3 & -1 & 8 & -4 \\ -5 & 9 & -8 & -6 \end{bmatrix}.$$

Answer to Exercise 4 (on page 5)

It is possible to compute $\mathbf{a} \cdot \mathbf{d}$, $\mathbf{b} \cdot \mathbf{e}$, and $\mathbf{c} \cdot \mathbf{f}$:

$$1. \mathbf{a} \cdot \mathbf{d} = 1(-5) + 2(-1) = -5 + (-2) = -7$$

$$2. \mathbf{b} \cdot \mathbf{e} = -3(1) + 3(-5) + 5(3) + -5(1) = -3 + (-15) + 15 - 5 = -8$$

$$3. \mathbf{c} \cdot \mathbf{f} = 1(4) + 2(1) + -1(-3) = 4 + 2 + 3 = 9$$

Answer to Exercise 5 (on page 7)

$$1. \begin{bmatrix} -2 & -1 & 5 \end{bmatrix}$$

$$2. \begin{bmatrix} 0 & 5 & 1 \\ 0 & 25 & 5 \\ 0 & -25 & -5 \\ 0 & 20 & 4 \\ 0 & 5 & 1 \end{bmatrix}$$

$$3. \begin{bmatrix} -9 & -17 & 11 \\ -6 & 40 & -4 \\ 15 & 7 & -13 \\ 9 & -8 & -1 \end{bmatrix}$$

Answer to Exercise 6 (on page 8)

$$1. \mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 8 & -6 & 10 & 4 \\ -8 & 6 & -10 & -4 \\ -4 & 3 & -5 & -2 \\ 8 & -6 & 10 & 4 \end{bmatrix} \text{ and } \mathbf{B} \cdot \mathbf{A} = \begin{bmatrix} 13 \end{bmatrix}$$

$$2. \mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} -2 & 16 & 16 \\ 5 & -24 & -8 \\ -4 & 22 & 12 \end{bmatrix} \text{ and } \mathbf{B} \cdot \mathbf{A} = \begin{bmatrix} 8 & 6 \\ -12 & -22 \end{bmatrix}$$

3. $A \cdot B = \begin{bmatrix} -22 & 1 \\ 15 & -4 \end{bmatrix}$ and $B \cdot A = \begin{bmatrix} 12 & 0 & 15 & 3 \\ -4 & 0 & 1 & -15 \\ -16 & - & -17 & -11 \\ -14 & 0 & -10 & -21 \end{bmatrix}$



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