Chapter 1

Span

1.1 Spans of Vectors

Knowing whether two vectors are linearly dependent or independent allows us to accurately describe the span of those two vectors (this expands to include any number of vectors). In the previous chapter, we saw that linear combinations of two linearly dependent vectors can only make vectors that lie on the same line as the two starting vectors. We saw this in 2D, but it also applies to 3D vectors. Consider the two vectors $\mathbf{u} = [2, 4, 3]$ and $\mathbf{v} = [4, 8, 6]$, shown in figure 1.1.

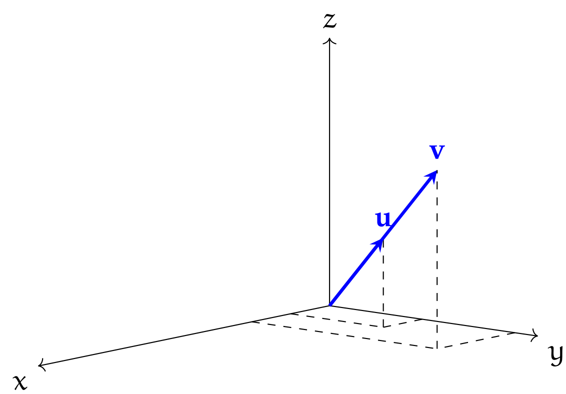

Figure 1.1: 3-dimensional vectors, $\mathbf{u}$ and $\mathbf{v}$

Notice that these two vectors are colinear (that is, they are on the same line), therefore they are linearly dependent and any combination of $\mathbf{u}$ and $\mathbf{v}$ will lie on the same line as $\mathbf{u}$ and $\mathbf{v}$. Therefore, we say the *the span of* $\mathbf{u}$ *and v is a line*. In fact, for any size list of linearly dependent vectors (whether it's one vector or one hundred), the span of that list is a line.

Now that you have a sense of what a span is, it is time for the formal mathematical definition. A vector span is the collection of vectors obtained by scaling and combining the original set of vectors in all possible proportions. Formally, if the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ contains vectors from a vector space V, then the span of S is given by:

$$\text{Span}(S) = \{a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n} : a_1, a_2, ..., a_n \in \mathbb{R}\} \tag{1.1}$$

This means that any vector in the Span(S) can be written as a linear combination of the vectors in S.

1.1.1 Spans of Independent Vectors

What if our list of vectors aren't all linearly dependent on each other? We've seen in 2 dimensions that any two independent vectors can be linearly combined to create any vector in $\mathbb{R}^2$. So, the span is described as a *plane* (in fact, it is the entire xy-plane, which we also call $\mathbb{R}^2$). How does this expand to 3-dimensional vectors?

Let's again consider two 3-dimensional vectors: $\mathbf{u} = [2, 4, 3]$ and $\mathbf{v} = [2, 1, 0]$, as shown in figure 1.2.

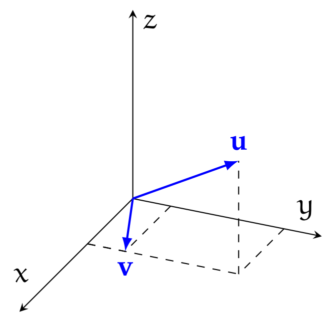

Figure 1.2: Linearly independent 3-dimensional vectors, **u** and **v**

Just like in two dimensions, any two independent vectors in $\mathbb{R}^3$ define a plane (see figure 1.3). This also applies to higher dimensions: the span of any two linearly independent vectors is a plane.

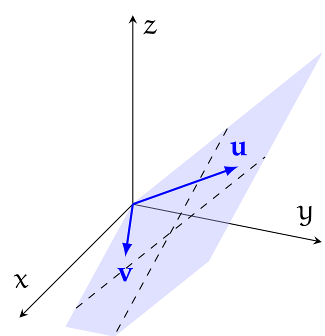

Figure 1.3: Linearly independent 3-dimensional vectors, **u** and **v**, define a plane.

If we have 3 independent vectors, then we can define a *3-dimensional space*. To understand this, first imagine a plane formed by two independent 3-dimensional vectors like in figure 1.3). If a third independent vector is introduced, it must not lie on the plane: if it did, it would be a linear combination of the first two and therefore not independent. This third vector allows us to move off the plane, and therefore all three independent vectors span $\mathbb{R}^3$. In review, 1 vector or set of dependent vectors span a *line*, 2 vectors or sets of dependent vectors span a *plane*, and 3 vectors or sets of dependent vectors span $\mathbb{R}^3$.

Example: Do the vectors $\mathbf{r} = [5, 4, -6]$, $\mathbf{s} = [0, -5, -10]$, and $\mathbf{t} = [0, 2, 4,]$ span a line, plane, or $\mathbb{R}^3$?

Solution: We need to determine the number of *independent vectors*. First, we'll check if **r** and **s** are independent. They are independent if the only solution to the equation below is $a_1 = a_2 = 0$:

$$a_1\,[5, 4, -6] + a_2\,[0, -5, -10] = [0, 0, 0]$$

Which we can write as a system of equations:

$$5a_1 + 0a_2 = 0$$
$$4a_1 - 5a_2 = 0$$
$$-6a_2 - 10a_2 = 0$$

From the first equation, we see that $5a_1 = 0$ which implies that $a_1 = 0$. Substituting that into the second equation:

$$4(0) - 5a_2 = 0$$
$$-5a_2 = 0$$
$$a_2 = 0$$

Therefore, vectors **r** and **s** are independent. Now let's check **r** and **t**:

$$a_1\,[5, 4, -6] + a_2\,[0, 2, 4,] = [0, 0, 0]$$

Which we can re-write as a system of equations:

$$5a_1 + 0a_2 = 0$$
$$4a_1 + 2a_2 = 0$$
$$-6a_1 + 4a_2 = 0$$

Again, from the first equation, we see that $a_1 = 0$. Substituting into the second:

$$4(0) + 2a_2 = 0$$
$$2a_2 = 0$$
$$a_2 = 0$$

Therefore, **r** and **t** are also independent. Last, we'll check **s** and **t** for independence:

$$a_1\,[0, -5, -10] + a_2\,[0, 1, 2,] = [0, 0, 0]$$

The system of equations:

$$0a_1 + 0a_2 = 0$$
$$-5a_1 + a_2 = 0$$
$$-10a_1 + 2a_2 = 0$$

The first equation doesn't tell us anything, since it would be true no matter what a_1 and a_2 are. We can solve the second equation for a_2 and substitute into the third equation:

$$a_2 = 5a_1$$
$$-10a_1 + 2(5a_1) = 0$$
$$-10a_1 + 10a_1 = 0$$

Which is also true for all a_1. In fact, there are many solutions to $a_1 [0, -5, -10] + a_2 [0, 1, 2,] = [0, 0, 0]$, $a_1 = 1$ and $a_2 = 5$ is an example. Therefore, $\mathbf{s}$ and $\mathbf{t}$ are *dependent*. So, we really have 2 independent vectors in the list, and therefore $\text{span}(\mathbf{r}, \mathbf{s}, \mathbf{t})$ is a plane.

Exercise 1 Determining Span

Geometrically describe (line, plane, or $\mathbb{R}^3$) the span of the list of vectors.

1. $[1, 2, 4]$ and $[-2, -4, -8]$
2. $[2, 0, 0]$ and $[0, 1, 3]$
3. $[3, 0, 0]$ and $[0, 3, 3]$ and $[3, 3, 2]$

Working Space

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1.2 Determinants

Checking all these vectors by hand takes a long time. What if you had a list of 5, 10, or even 100 vectors? The determinant of a matrix is a scalar value that indicates whether the columns of a matrix are linearly independent. So, if you put all your vectors together in a matrix and take the determinant of that matrix, the result will tell you if all the vectors are independent or not. For a 2D matrix, the determinant is the area of the parallelogram defined by the column vectors. For a 3D matrix, the determinant is the volume of the parallelepiped (a six-dimensional figure formed by six parallelograms, such as a cube).

Let's plot the parallelogram for this matrix (see figure 1.4):

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

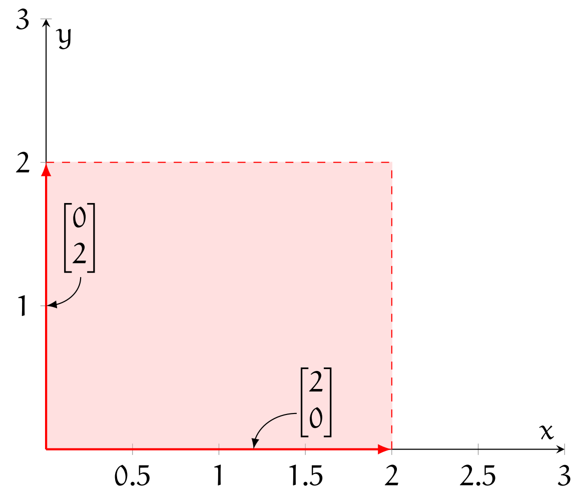

Figure 1.4: A parallelogram constructed from vectors $[2, 0]$ and $[0, 2]$

The formal definition for calculating the determinant of a 2 by 2 matrix is:

$$\det(A) = (a * d) - (b * c)$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For the matrix plotted above, the determinant is $(2 * 2) - (0 * 0)$. You can also see that 4.0 is the area, base (2) times height (2).

You can use the determinant to see what happens to a shape when it goes through a linear transformation. Let's scale the 2 by 2 matrix by 4:

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

Plot it (see figure 1.5):

Find the determinant.

$$(8 * 8) - (0 * 0) = 64$$

You can see that scaling the matrix scaled the area by the scaling factor squared (see figure 1.6).

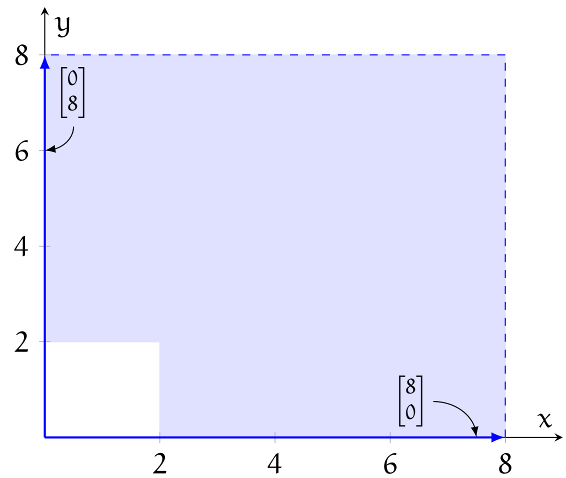

Figure 1.5: Scaling the matrix also scales the parallelogram.

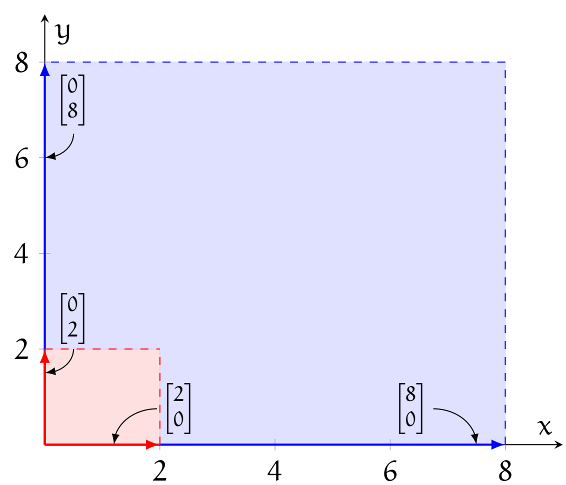

Figure 1.6: Scaling a matrix by a constant c increases the area of the parallelogram by a factor of c^2.

We can show why this is true mathematically. Suppose we have a 2 by 2 matrix A:

$$A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Then $\det(A) = wz - xy$. We can scale this matrix by a constant, c:

$$cA = c \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} cw & cx \\ cy & cz \end{bmatrix}$$

And we can take the determinant:

$$\det(cA) = \det\left(\begin{bmatrix} cw & cx \\ cy & cz \end{bmatrix}\right) = cw(cz) - cx(cy) = c^2(wz - xy) = c^2 \cdot \det(A)$$

Therefore, scaling a 2 by 2 matrix by a factor changes the determinant by that factor squared. What about higher dimensions? If each side of a cube were scaled by a factor of c, then the volume of the cube would change by a factor of c^3 (feel free to confirm this yourself). And if a tesseract (a four-dimensional cube) had each side scaled by a factor of c, then the hypervolume (four-dimensional volume) would be scaled by a factor of c^4. Do you notice a pattern?

In fact, scaling an $n \times n$ matrix by a constant factor, c, changes the determinant of that $n \times n$ matrix by a factor of c^n.

What happens if the columns of a matrix are not independent? Let's plot this matrix (see figure 1.7):

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

One vector overwrites the other. As you can see, the area is 0 because there is no space between the vectors. Therefore, the columns of the matrix are linearly dependent.

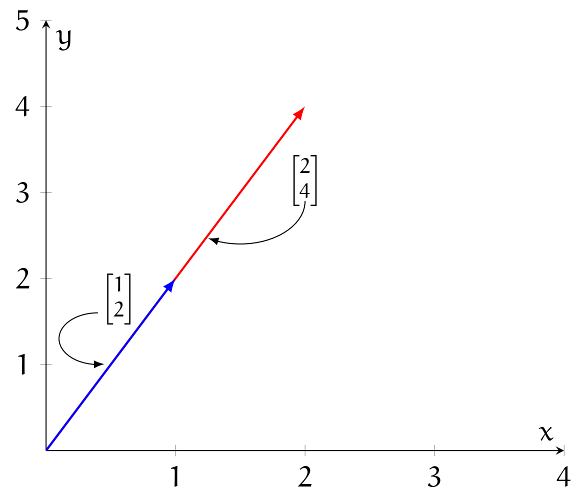

Figure 1.7: The vectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$ are co-linear, so there is no area between them and the determinant of $\begin{bmatrix}2 & 1\\4 & 2\end{bmatrix}$ is zero.

Exercise 2 Finding the Determinate

Plot the parallelogram represented by the columns of the matrix. What is the area of this parallelogram?

1. $\begin{bmatrix}1 & 4\\-3 & 1\end{bmatrix}$
2. $\begin{bmatrix}5 & -5\\5 & -1\end{bmatrix}$
3. $\begin{bmatrix}0 & -5\\-2 & 0\end{bmatrix}$

Working Space

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Calculating the determinant for a 2 by 2 matrix is easy. For a larger matrix, finding the determinant is more complex and requires breaking down the matrix into smaller matrices until you reach the 2x2 form. The process is called expansion by minors. For our purposes, we simply want to first check to see if a matrix contains linearly independent rows and columns before using our Python code to solve.

Modify your code so that is uses the $np.linalg.det()$ function. If the determinant is not zero, then you can call the $np.linalg.solve()$ function. Your code should look like this:

```
if (np.linalg.det(D) != 0):
    j = np.linalg.solve(D,e)
    print(j)
else:
     print("Rows and columns are not independent.")
```

1.3 Where to Learn More

Watch this video on *Linear Combinations and Vector Spans from Khan Academy*: http://rb.gy/g1snk

The Wolfram Demonstrations website has a fun, interactive demo where you can enter values for 2D and 3D matrices and see how the area or volume changes. https://demonstrations.wolfram.com/DeterminantsSeenGeometrically/#more

If you are curious about the *Expansion of Minors*, see: https://mathworld.wolfram.com/DeterminantExpansionbyMinors.html

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This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Appendix A

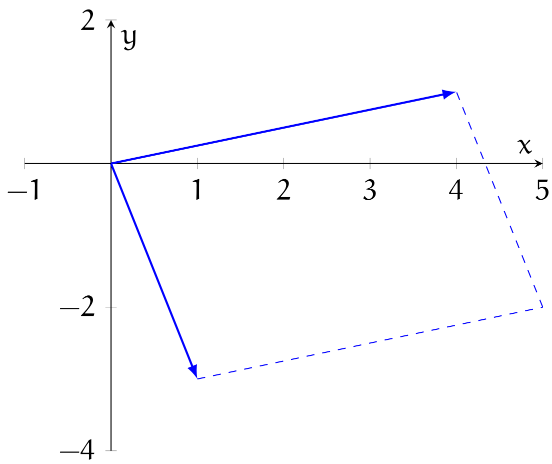
Answers to Exercises

Answer to Exercise 1 (on page 4)

1. Since the second vector is a scalar multiple of the first, the span of $S = \{[1, 2, 4], [-2, -4, -8]\}$ is a *line*.
2. Since the second vector is not a scalar multiple of the first, the span of $S = \{[2, 0, 0,], [0, 1, 3]\}$ is a *plane*.
3. None of the three vectors are scalar multiples or linear combinations of the other two. Therefore, the span of $S = \{[3, 0, 0], [0, 3, 3], [3, 3, 2]\}$ is $\mathbb{R}^3$.

Answer to Exercise 2 (on page 8)

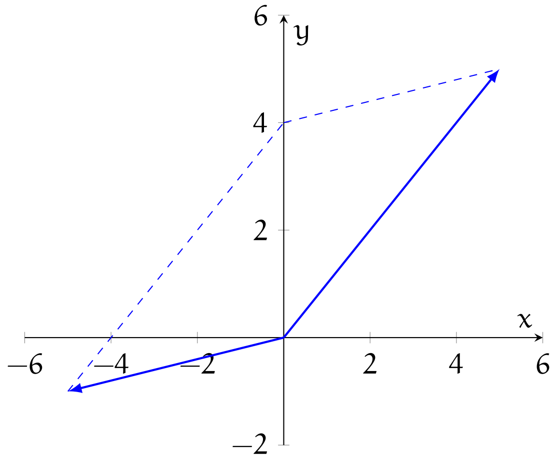
1. Our two vectors from the columns of the matrix are $[1, -3]$ and $[4, 1]$. Plotting:



The area of this parallelogram is the same as the determinant of the matrix:

$$\det\left(\begin{bmatrix} 1 & 4 \\ -3 & 1 \end{bmatrix}\right) = 1 \cdot 1 - (4 \cdot -3) = 1 + 12 = 13$$

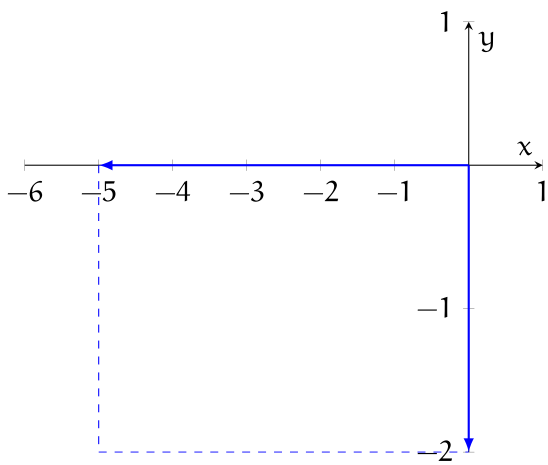
2. Our two vectors from the columns of the matrix are $[5, 5]$ and $[-5, -1]$. Plotting:



The area of this parallelogram is the same as the determinant of the matrix:

$$\det\left(\begin{bmatrix} 5 & -5 \\ 5 & -1 \end{bmatrix}\right) = 5 \cdot -1 - (-5 \cdot 5) = -5 + 25 = 20$$

3. Our two vectors from the columns of the matrix are $[0, -2]$ and $[-5, 0]$. Plotting:



This is a rectangle, and we can see the area is $5 \cdot 2 = 10$. However, the determinant is:

$$\det\left(\begin{bmatrix} 0 & -5 \\ -2 & 0 \end{bmatrix}\right) = 0 \cdot 0 - (-5 \cdot -2) = 0 - 10 = -10$$

We will discuss this unusual response in a future chapter.