Systems of Equations as Matrices

1.1 Systems of Equations, REF, and RREF

We have talked a lot about matrices and vectors. Now we can use matrices and vectors to solve systems of equations. A system of linear equations can be written compactly using matrices and vectors. This matrix representation allows us to use algebraic tools such as row operations and matrix multiplication to analyze and solve the system more efficiently. Let's look at a very simple example from a previous section.

$$\begin{cases} 1x_1 + 2x_2 = -1 \\ 2x_1 + 0x_2 = 4 \end{cases}$$

We can write this in the form of a matrix! This may come as a surprise to some of you. How can we do this? Let's rewrite the the coefficients as a matrix and the x_n's as column vector, and the constant vector, or right-hand-side vector, called $\vec{b}$. Here is the coefficient matrix which we will call A:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

Then we can write the column vector of x, which we will terminalize as $\vec{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And we can write the $\vec{b}$ as the following column vector:

$$\vec{b} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Now, if we multiply A and $\vec{x}$,

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 \\ 2x_1 + 0x_2 \end{bmatrix}$$

We get exactly the left-hand sides of our original system of equations! Therefore, when we set this equal to the right-hand-side vector $\vec{b}$:

$$\vec{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

we get the fundamental system of equation formula represented as a matrix Equation

$$A\vec{x} = \vec{b} \tag{1.1}$$

This is a very important equation for linear algebra as a whole. Notice that the size of matrix has to respect the matrix multiplication rules:

$$\underset{m \times n}{A} \cdot \underset{n \times 1}{\vec{b}} = \underset{m \times 1}{\vec{p}}$$

Exercise 1   **Matrix Equation**

Rewrite the systems of equations in matrix form and identify A, $\vec{x}$, and $\vec{b}$.

$$\begin{cases} -4x_1 + 9x_2 - 8x_3 = 5 \\ -1x_1 + 0x_2 + 6x_3 = 7 \end{cases}$$

Working Space

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1.2   Row-Echelon Form

This brings up the question, how can we solve for the $\vec{x}$? There are multiple ways, but most commonly we form an augmented matrix and solve for Row-Echelon Form.

1.2.1   The Augmented Matrix

What is the augmented matrix? It is a way of represented the A matrix and $\vec{b}$ vector. We write the A matrix, seperated by a vertical line, and then writing the $\vec{b}$ vector:

$$[A \mid \vec{b}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{bmatrix}$$

The left block contains all the coefficients of the system of equations, and the right block contains only the solutions. Then, to solve for $\vec{x}$, we perform *Elementary Row Operations* to find the *Row-Echelon Form*, mathematically stated as $\mathrm{rref}(A|\vec{b})$. Let's continue with the matrix vector set we were experimenting with:

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 \\ 2x_1 + 0x_2 \end{bmatrix}$$

This can be written as an augmented matrix:

$$\left[A|\vec{b} \right] = \begin{bmatrix} 1 & 2 & | & -1 \\ 2 & 0 & | & 4 \end{bmatrix}$$

1.2.2   Elementary Row Operations

The Row-Echelon Form of a matrix is an *equivalent* form of a matrix obtained through Gaussian and Gauss-Jordan elimination. There are lots of videos on this process online, so multiple videos included in your digital resources to walk you through the process. The rules of Gaussian Elimination are simple, only 3 different elementary row operations, or ERO's can be performed:

- Swapping two rows

- multiply a row by a non-zero scalar

- dd a multiple of one row to another row

Why do these operations preserve the given linear system? Swapping two rows only changes the order in which the equations are written and therefore has no effect on the solution set. Multiplying a row by a nonzero scalar corresponds to multiplying both sides of an equation by the same scalar, which does not alter its solutions. Finally, if two equations are satisfied by the same solution, then any linear combination of those equations is also satisfied by that solution; in particular, adding a multiple of one equation

to another preserves the solution set. Before and after each operation, the augmented matrix contains exactly the same information as the system of equations and therefore represents the same solution set.

With these ERO's, we are trying to achieve a matrix with the following properties:

- Leading entries (not necessarily 1) move to the right as you go down

- Rows of zeros, if exist, are at the bottom

- Each pivot has zeros in all positions below it in the same column.

Lets walk through the steps to simplify $\text{ref}(A|\vec{b})$:

$$\begin{bmatrix} 1 & 2 & | & -1 \\ 2 & 0 & | & 4 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1 \implies \begin{bmatrix} 1 & 2 & | & -1 \\ 0 & -4 & | & 6 \end{bmatrix}$$

As you can see, we set Row 2 to be Row 2 minus 2 times Row 1. This gives us a zero in the first column of Row 2. Technically, this matrix is now in Row-Echelon Form (REF). However, we can continue applying elementary row operations to obtain an even more structured form called the *Reduced Row-Echelon Form*.

1.2.3   Reduced Row Echelon Form

A matrix is in *Reduced Row-Echelon Form* (RREF) if it satisfies all of the conditions for Row-Echelon Form, and in addition:

- Each leading entry is equal to 1

- Each leading entry is the only nonzero entry in its column

In other words, not only do pivots move to the right as we move down the rows, but each pivot column contains zeros both above and below the pivot, and each pivot is normalized to 1. These additional conditions make RREF especially useful, since the solutions to the system can often be read directly from the matrix.

Continuing our example, we start with the matrix in REF:

$$\begin{bmatrix} 1 & 2 & | & -1 \\ 0 & -4 & | & 6 \end{bmatrix}$$

First, make the leading entry in Row 2 equal to 1 by dividing the row by -4:

$$R_2 \rightarrow -\frac{1}{4}R_2 \implies \begin{bmatrix} 1 & 2 & | & -1 \\ 0 & 1 & | & -\frac{3}{2} \end{bmatrix}$$

Next, eliminate the entry above the pivot in column 2:

$$R_1 \rightarrow R_1 - 2R_2 \implies \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -\frac{3}{2} \end{bmatrix}$$

The matrix is now in Reduced Row-Echelon Form.

1.2.4   Writing the Solution Vector

Recall that the augmented matrix represents a system of linear equations. After reducing the matrix to Reduced Row-Echelon Form, we obtain

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -\frac{3}{2} \end{bmatrix}$$

Each row now corresponds directly to an equation in the variables x_1 and x_2:

$$x_1 = 2$$
$$x_2 = -\frac{3}{2}$$

Thus, the solution to the system can be written as the vector

$$\vec{x} = \begin{bmatrix} 2 \\ -\frac{3}{2} \end{bmatrix}$$

Since every variable corresponds to a pivot column, the system has a *unique solution*.

We have sucessfully solved the system of equations using matrix methods:

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

This is a fundamental technique in linear algebra, and you will use it often in this module and beyond!

1.3 Larger RREF Example

Let's solve a larger system by finding $\vec{x}$ in the matrix equation

$$A\vec{x} = \vec{b}.$$

1.3.1 Write the System and Identify A, $\vec{x}$, and $\vec{b}$

Consider the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 4 \\ 2x_1 + x_2 + 3x_3 + x_4 = 9 \\ x_1 + 3x_2 + 2x_3 + 2x_4 = 10 \\ 3x_1 + x_2 + 2x_3 + 4x_4 = 13 \end{cases}$$

The coefficient matrix, variable vector, and constant vector are:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 3 & 2 & 2 \\ 3 & 1 & 2 & 4 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 4 \\ 9 \\ 10 \\ 13 \end{bmatrix}.$$

So the matrix equation is:

$$A\vec{x} = \vec{b}.$$

1.3.2 Form the Augmented Matrix

$$\left[A | \vec{b} \right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 2 & 1 & 3 & 1 & 9 \\ 1 & 3 & 2 & 2 & 10 \\ 3 & 1 & 2 & 4 & 13 \end{array} \right].$$

1.3.3 Row-Reduce to REF

Use the pivot in Row 1 to clear entries below it in column 1:

$$R_2 \rightarrow R_2 - 2R_1$$
$$R_3 \rightarrow R_3 - R_1$$
$$R_4 \rightarrow R_4 - 3R_1$$

$$\implies \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -1 & | & 1 \\ 0 & 2 & 1 & 1 & | & 6 \\ 0 & -2 & -1 & 1 & | & 1 \end{bmatrix}.$$

Now use the pivot in Row 2 (column 2) to clear entries below it in column 2:

$$R_3 \to R_3 + 2R_2$$
$$R_4 \to R_4 - 2R_2$$

$$\implies \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -1 & | & 1 \\ 0 & 0 & 3 & -1 & | & 8 \\ 0 & 0 & -3 & 3 & | & -1 \end{bmatrix}.$$

Next use the pivot in Row 3 (column 3) to clear the entry below it in column 3:

$$R_4 \to R_4 + R_3$$

$$\implies \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -1 & | & 1 \\ 0 & 0 & 3 & -1 & | & 8 \\ 0 & 0 & 0 & 2 & | & 7 \end{bmatrix}.$$

At this point, the matrix is in Row-Echelon Form (REF).

1.3.4   Continue to RREF

Make the pivot in Row 4 equal to 1:

$$R_4 \to \frac{1}{2}R_4 \implies \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -1 & | & 1 \\ 0 & 0 & 3 & -1 & | & 8 \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Clear the entries above the pivot in column 4:

$$R_1 \to R_1 - R_4$$
$$R_2 \to R_2 + R_4$$
$$R_3 \to R_3 + R_4$$

$$\implies \begin{bmatrix} 1 & 1 & 1 & 0 & | & \frac{1}{2} \\ 0 & -1 & 1 & 0 & | & \frac{9}{2} \\ 0 & 0 & 3 & 0 & | & \frac{23}{2} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Make the pivot in Row 3 equal to 1:

$$R_3 \to \frac{1}{3}R_3 \implies \begin{bmatrix} 1 & 1 & 1 & 0 & | & \frac{1}{2} \\ 0 & -1 & 1 & 0 & | & \frac{9}{2} \\ 0 & 0 & 1 & 0 & | & \frac{23}{6} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Clear the entries above the pivot in column 3:

$$R_1 \to R_1 - R_3$$
$$R_2 \to R_2 - R_3$$

$$\implies \begin{bmatrix} 1 & 1 & 0 & 0 & | & -\frac{10}{3} \\ 0 & -1 & 0 & 0 & | & \frac{2}{3} \\ 0 & 0 & 1 & 0 & | & \frac{23}{6} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Make the pivot in Row 2 equal to 1:

$$R_2 \to -R_2 \implies \begin{bmatrix} 1 & 1 & 0 & 0 & | & -\frac{10}{3} \\ 0 & 1 & 0 & 0 & | & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & | & \frac{23}{6} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Clear the entry above the pivot in column 2:

$$R_1 \to R_1 - R_2$$

$$\implies \begin{bmatrix} 1 & 0 & 0 & 0 & | & -\frac{8}{3} \\ 0 & 1 & 0 & 0 & | & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & | & \frac{23}{6} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

This matrix is now in Reduced Row-Echelon Form (RREF).

1.3.5   Write the Solution Vector

Reading each row as an equation gives:

$$x_1 = -\frac{8}{3}$$
$$x_2 = -\frac{2}{3}$$
$$x_3 = \frac{23}{6}$$
$$x_4 = \frac{7}{2}$$

Thus the solution vector is:

$$\vec{x} = \begin{bmatrix} -\frac{8}{3} \\ -\frac{2}{3} \\ \frac{23}{6} \\ \frac{7}{2} \end{bmatrix}.$$

You can verify the solution by confirming that $A\vec{x} = \vec{b}$ using matrix multiplication.

Note that solution sets must be one of the following:

- No Solution (ie. parallel lines),

- Exactly One Unique Solution (ie. lines or planes intersecting at one point),

- or Infinitely Many Solutions (ie. same line or plane is redundantly in the set)

1.4   Free Variables

In the reduced row-echelon form (RREF) of a matrix, pivots determine which variables are constrained by the system.

1.4.1   Zero rows and redundant equations

A row of all zeros in RREF represents a redundant equation. That row corresponds to the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = 0,$$

which is always true and therefore imposes no restriction on the variables. Such a row arises when one of the original equations is a linear combination of the others.

Every row of zeros is a row without a pivot. Consequently, if a matrix has at least one zero row in RREF:

- The number of pivots is less than the number of rows,

- The rank of the matrix is less than the number of rows,

- The rows of the matrix are linearly dependent.

1.4.2   Free variables

Pivots are associated with *columns*, not rows. A variable is called a *free variable* if its column does not contain a pivot.

Thus:

- A *row* without a pivot (a zero row) indicates a redundant equation,

- A *column* without a pivot corresponds to a free variable.

Free variables can take arbitrary values and lead to infinitely many solutions when solving a homogeneous system.

1.4.3   Infinitely Many Solutions

Example: Consider the augmented matrix in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

Labeling the variable columns (and ignoring the constants c column), we write:

$$\begin{bmatrix} x_1 & x_2 & x_3 & c \\ 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

The pivot positions are in the columns corresponding to x_1 and x_3. Since the column corresponding to x_2 contains no pivot, x_2 is a free variable.

The system of equations represented by this matrix is:

$$x_1 = 8,$$
$$x_3 = -6.$$

The variable x_2 does not appear in any equation and may take any real value. Since x_1 and x_3 are bound, we can say the solution is :

$$x_1 = 8, \qquad x_3 = -6, \qquad x_2 \in \mathbb{R}$$

writen as a vector this is: $\vec{x} = \begin{bmatrix} 8 \\ x_2 \\ -6 \end{bmatrix}$ or in set builder notation: $\left\{ \begin{pmatrix} 8 \\ x_2 \\ -6 \end{pmatrix} : x_2 \in \mathbb{R} \right\}$

Thus, the presence of a column without a pivot leads to a free variable and infinitely many solutions.

1.4.4 Inconsistent sets

Let's look at another example: **Example** The augmented matrix

$$\begin{bmatrix} 1 & 2 & 4 & 12 \\ 12 & 24 & 48 & 9 \\ 3 & 6 & 12 & -4 \end{bmatrix}$$

simplifies to

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can interperite this As

$$x_1 + 2x_2 + 4x_3 = 0,$$
$$0x_1 + 0x_2 + 0x_3 = 1.$$

Note that the equation $0 = 1$ is never true, so the system is *inconsistent*, and there is *no solution*.

The presence of an all zero row signifies an inconsistent system.

FIXME exercises

1.5   Basis, Dimension, Rank

Let's define a few terms. These definitions are very basic, and will come back in the future:

Rank

Rank is the number of pivots in the RREF state of a matrix of size $m \times n$. In other words, the number of nonzero rows in any row-echelon form of a matrix A is denoted $\text{rank}(A)$. This has the constraints of:

- $\text{rank}(A) \leq m$

- $\text{rank}(A) \leq n$

Dimension

The **dimension** of the solution set of a system is the number of free variables. We calculate this with

$$\texttt{dimension} = \texttt{number of variables} - \texttt{rank}$$

or equivalently, for matrix A of size $m \times n$

$$\texttt{dimension} = n - \text{rank}(A)$$

Basis

A **basis** is a minimal set of direction vectors that describes all solutions. Specifically, this consists of $n - \text{rank}(A)$ vectors, one for each free variable

---

In summary,

$$
\begin{aligned}
\text{rank} \quad &= \quad \text{number of pivots} \\
\text{number of free variables} \quad &= \quad n - \text{rank} \\
\text{dimension of solution set} \quad &= \quad n - \text{rank} \\
\text{number of basis vectors} \quad &= \quad n - \text{rank}
\end{aligned}
$$

Exercises

Exercise 2   ## Augmented Matrix 1

Simplify the matrix:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

to RREF state and determine the type of
solution set.

Working Space

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Exercise 3   ## Augmented Matrices 2

Put the augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

into RREF and determine its solution.

Working Space

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Exercise 4   ## Augmented Matrices 3

Determine the solution for the following
set:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Working Space

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Exercise 5   **Rank 1**

Find the rank of

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Exercise 6   **Rank 2**

Let A be a 4×6 matrix with exactly 3 pivots in RREF.

- What is rank(A)?

- How many free variables are there?

Exercise 7   **Free Variables**

The RREF of a system has the form:

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Identify the pivot variables

2. Identify the free variables

3. State the dimension of the solution set

1.6   Creating Matrices in Python

Recall that matrices can be represented by 2D arrays in Python. Let's explore this idea further with the module `numpy`. Create a file called `matrices_creation.py` and enter this code:

```python
# import the python module that supports matrices
import numpy as np
# Use the function np.array to define a matrix that
# contains specific values that you supply.
A = np.array([[ 5, 1, 3],
              [ 1, -1, 8],
              [ 6, 2, 1]])
# The transpose function returns
A.transpose()
```

When you run it, you should see:

```
array([[ 5, 1, 6],
       [ 1, -1, 2],
       [ 3, 8, 1]])
```

As you can see, $A \neq A^{\mathsf{T}}$, so A is not symmetric. Try another:

```
# create a matrix, B
B = np.array([[ 5,  1,  6],
              [ 1, -1,  2],
              [ 6,  2,  1]])
B.transpose()
```

When you run it, you should see:

```
array([[ 5,  1,  6],
       [ 1, -1,  2],
       [ 6,  2,  1]])
```

B is symmetric. You can actually transpose any matrix using this function, but a matrix cannot be symmetric unless it is square.

Try this code to see what happens when you transpose a rectangular matrix.

```
# create a matrix, J
J = np.array([[ 5,  1,  3,  0],
              [ 1, -1,  8, 11],
              [ 6,  2,  1, -7]])
J.transpose()
```

Note that transposing a rectangular matrix changes its dimension from 3 by 4 to 4 by 3. You should see a transposed matrix, but it's not symmetric.

```
array([[ 5,  1,  6],
       [ 1, -1,  2],
       [ 3,  8,  1],
       [ 0, 11, -7]])
```

1.6.1   Creating Special Matrices in Python

Use the same file to add this code for creating a zero matrix.

```
# create an 8 by 10 Zero matrix.
 C = np.zeros((8, 10))
 C
```

When you run it, you should see:

```
array([[0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.]])
```

Add the following code to create an 8 by 8 Identity matrix.

```
# create an 8 by 8 Identity matrix
D = np.eye(8)
D
```

When you run it, you should see:

```
array([[1., 0., 0., 0., 0., 0., 0., 0.],
       [0., 1., 0., 0., 0., 0., 0., 0.],
       [0., 0., 1., 0., 0., 0., 0., 0.],
       [0., 0., 0., 1., 0., 0., 0., 0.],
       [0., 0., 0., 0., 1., 0., 0., 0.],
       [0., 0., 0., 0., 0., 1., 0., 0.],
       [0., 0., 0., 0., 0., 0., 1., 0.],
       [0., 0., 0., 0., 0., 0., 0., 1.]])
```

As you progress in your studies, you will learn the importance of diagonal matrices and of extracting the diagonal of a matrix. Let's see how to extract a diagonal, then create a diagonal matrix.

```
# create a  matrix
W = np.array([[1, 2, 3, 4],
              [5, 6, 7, 8],
              [-8, -7, -6, -5],
              [-4, -3, -2, -1]])
```

Extract the main diagonal using np.diag(<array>,<diagonal to extract>). Passing 0 as the second parameter specifies the main diagonal. A positive value extracts a diagonal from the upper part. A negative value extracts a diagonal from the lower part. Run this code then experiment passing other values to see what you get.

```
print(np.diag(W,0))
```

When you run it, you should see:

```
array([ 1,  6, -6, -1])
```

You can also use np.diag() to create a diagonal matrix from a 1D array. In this case, do not pass a second paramenter.

```
Q = np.array([1, 2, 3])
DiagArray = np.diag(Q)
print(DiagArray)
```

When you run it you should see;

```
[[1 0 0]
 [0 2 0]
 [0 0 3]]
```

Python has functions for extracting upper and lower triangular matrices. Try these:

```
print(np.triu(W))
print(np.tril(W))
```

You should see:

```
[[ 1  2  3  4]
 [ 0  6  7  8]
 [ 0  0 -6 -5]
 [ 0  0  0 -1]]

[[ 1  0  0  0]
 [ 5  6  0  0]
 [-8 -7 -6  0]
 [-4 -3 -2 -1]]
```

1.7 Conclusion

In this module, we have explored the fundamental concepts of matrices, including their types, properties, and how they can be used to represent and solve systems of linear equations. We have also learned how to perform row operations to convert matrices into

Row-Echelon Form (REF) and Reduced Row-Echelon Form (RREF), which are essential techniques in linear algebra. In the next chapter **??**, we will talk deeper about the concepts of vector spaces and subspaces and how to find RREF, building on our understanding of matrices and their applications.

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua. org/) for more details.

Answers to Exercises

Answer to Exercise 1 (on page 2)

$$A = \begin{bmatrix} -4 & 9 & -8 \\ -1 & 0 & 6 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \implies \begin{bmatrix} -4 & 9 & -8 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Answer to Exercise 2 (on page 13)

Notice immediately that $R_2 = 2R_1$, which automatically means redundancy.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 1 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow -R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \boxed{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

The RREF is:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us the system:

$$x_1 + x_3 = 1,$$
$$x_2 - x_3 = 1.$$

So,

$$x_1 = 1 - x_3, \qquad x_2 = 1 + x_3$$

making x_3 free. We can write this as: $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $x_3 \in \mathbb{R}$ Equivalently, as a

set:

$$\left\{ \begin{pmatrix} 1 - x_3 \\ 1 + x_3 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$$

Since x_3 can be anything, there are infinitely many solutions. Equivalently, noticing that the row of all zeros implies infinitely many solution is a valid step to come to an answer.

Answer to Exercise 3 (on page 13)

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{3} R_3} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_3} \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

This gives us a unique solution:

$$x_1 = -\frac{5}{3}, \quad x_2 = 0, \quad x_3 = \frac{1}{3}$$

or as a vector $\vec{x} = \begin{bmatrix} -\frac{5}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$

Answer to Exercise 4 (on page 13)

Notice that one row of the matrix states $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, which implies $0x_1 + 0x_2 + 0x_3 = 1$, which is not possible. Therefore the matrix represents an inconsistent system and there is no solution.

Answer to Exercise 5 (on page 14)

The RREF of this matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that there are two rows of zeroes, and the first column has a 1. This implies two dependent rows ($R_2 = 2R_1 \quad R_3 = 3R_1$), and with only 1 pivot column, $\text{rank}(A) = 1$.

Answer to Exercise 6 (on page 14)

By definition in this chapter:

$$\text{rank}(A) = \text{number of pivots} = 3$$

A 4×6 matrix has $n = 6$ columns, so

$$\text{\# free variables} = n - \text{rank}(A) = 6 - 3 = 3.$$

So, free variables $= 3$

Answer to Exercise 7 (on page 15)

There are $n = 4$ variables (x_1, x_2, x_3, x_4).

1. Pivot columns are columns $1, 2, 4$, so the pivot columns are x_1, x_2, x_4.

2. Free variables: the only non-pivot column is column 3, so the free variable is x_3

3. The dimension of the solution set is the number of free variables: 1. Equivalently, dimension $= n - \text{rank} = 4 - 3 = 1$

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