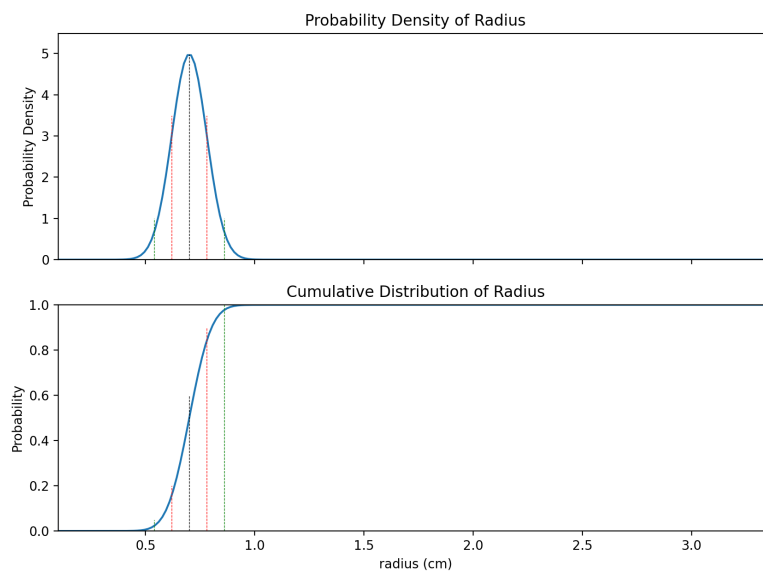


CHAPTER 1

Change of Variables

Let's say that we are making ice spheres, and we tell you that the radius of the ice spheres is normally distributed with a mean of 0.7 cm and a standard deviation of 0.08 cm. You can then draw the probability distribution and cumulative distribution for that:



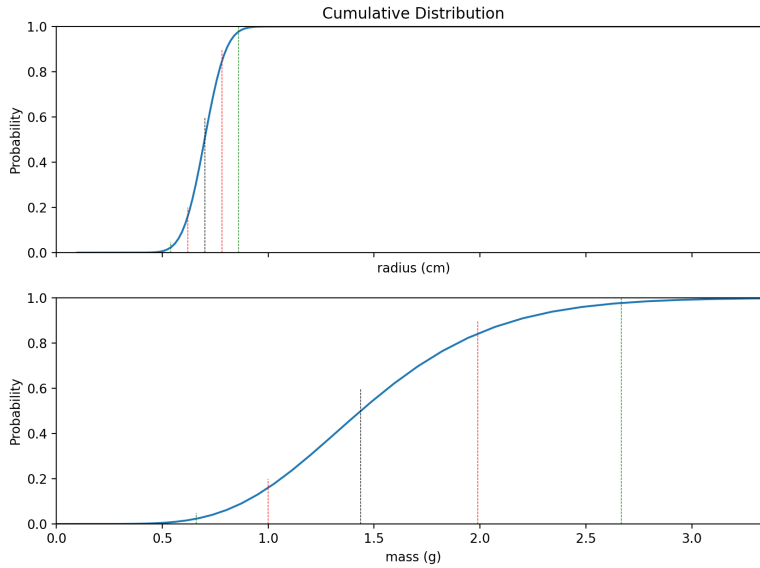
This includes lines indicating the mean and two standard deviations on each side.

Now, let's say we ask you what the cumulative distribution is for the *mass* of the balloons. A cubic centimeter of ice weighs about gram, so if you know the radius of a particular ice sphere, it is easy to compute the mass of it:

$$m = \frac{4}{3}\pi r^3$$

So, for example, if a sphere has a radius of 5cm, its mass in grams is $\frac{4\pi(0.7^3)}{3} \approx 1.44$ g.

Thinking about the graph of the cumulative distribution, if half the balloons have a radius less than 5 cm, then half of the balloons have a mass less than 523.6 g. For each point on the cumulative graph, we can use the radius of that point to compute the corresponding mass — the CDF gets stretched out:



If F is the original cumulative distribution function, and g is the function that maps the new variable (mass, in this case) to the old one (radius), then the new cumulative distribution function H is given by

$$H(m) = F(g(m))$$

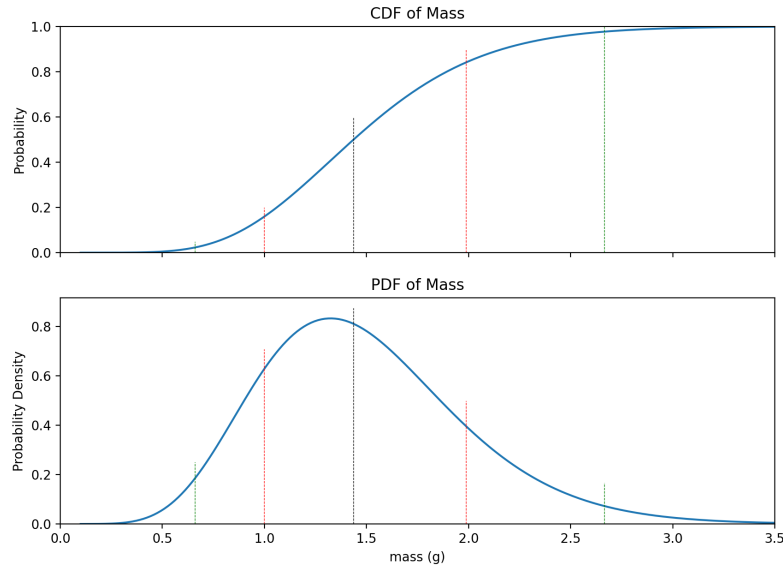
In this case, F is the cumulative function for the normal distribution, with mean 0.7 and standard deviation of 0.08. g maps the mass to the radius:

$$g(m) = \left(\sqrt[3]{\frac{3}{4\pi}} \right) m^{\frac{1}{3}}$$

1.1 Making a Probability Density Function

Now, we know how to calculate a new cumulative distribution function using the new variable. However, we usually want a probability density.

Here is the CDF and the PDF of the mass of the ice spheres:



Reminder: The probability density function is the derivative of the cumulative distribution function. We know the CDF is

$$H(m) = F(g(m))$$

By the chain rule:

$$H'(m) = F'(g(m))g'(m)$$

The function F is the cumulative distribution for the normal distribution with mean 0.7 and standard deviation of 0.08. So, we know its derivative:

$$F'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $\mu = 0.7$ and $\sigma = 0.08$.

We've already said that

$$g(m) = \left(\sqrt[3]{\frac{3}{4\pi}} \right) m^{\frac{1}{3}}$$

Which is easy to differentiate:

$$g'(m) = \left(\frac{1}{3}\right) \left(\sqrt[3]{\frac{3}{4\pi}}\right) m^{-\frac{2}{3}}$$

Here, then, is the code to generate that last plot:

```
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt

# Constants
MEAN_RADIUS = 0.7
STD_RADIUS = 0.08

# Range to plot
MIN_MASS = 0.1
MAX_MASS = 3.5

# Number of points to plot
N = 200

# Needed for radius_for_mass and d_radius_for_mass
C = np.power(3 / (4 * np.pi), 1/3)

# In these three functions, x can
# be a number or a numpy array

def mass_for_radius(x):
    return 4 * np.pi * np.power(x, 3) / 3

def radius_for_mass(x):
    return C * np.power(x, 1/3)

# Derivative of radius_for_mass()
def d_radius_for_mass(x):
    return (C/3) * np.power(x, -2/3)

# Compute mean and 2 standard deviations in each direction
m_mean = mass_for_radius(MEAN_RADIUS)
m_minus_std = mass_for_radius(MEAN_RADIUS - STD_RADIUS)
m_plus_std = mass_for_radius(MEAN_RADIUS + STD_RADIUS)
m_minus_2std = mass_for_radius(MEAN_RADIUS - 2 * STD_RADIUS)
m_plus_2std = mass_for_radius(MEAN_RADIUS + 2 * STD_RADIUS)

# Make N possible values for mass
m_values = np.linspace(MIN_MASS, MAX_MASS, N)

# Compute g(m) for each of these masses
```

```
# That is: What is the radius for each of these masses?
r_values = radius_for_mass(m_values)

# Compute  $F(g(m))$  for each of these masses
# That is: What is the cumulative distribution for each those radii?
cdf_values = norm.cdf(r_values, loc=MEAN_RADIUS, scale=STD_RADIUS)

# Compute  $g'(m)$  for each of these masses
dg_values = d_radius_for_mass(m_values)

# What is  $F'(g(m))g'(m)$ ?
pdf_values = norm.pdf(r_values, loc=MEAN_RADIUS, scale=STD_RADIUS) * dg_values

# Sanity check: It should integrate to a little less than 1.0
dx = (MAX_MASS - MIN_MASS)/N
area_under_curve = pdf_values.sum() * dx
print(f"Integral from {MIN_MASS:.2f} to {MAX_MASS:.2f}: {area_under_curve:.3f}")

# Make a figure with two axes
fig, axs = plt.subplots(nrows=2, sharex=True, figsize=(10, 7), dpi=200)

# Draw the CDF on the second axis
axs[0].set_title("CDF of Mass")
axs[0].set_ylim(bottom=0.0, top=1.0)
axs[0].set_xlim(left=0.0, right=MAX_MASS)
axs[0].set_ylabel("Probability")
axs[0].plot(m_values, cdf_values)

# Add lines for mean, mean-std, and mean+std
axs[0].vlines(m_minus_2std, 0, 0.05, "g", linestyle="dashed",lw=0.5)
axs[0].vlines(m_minus_std, 0, 0.2, "r", linestyle="dashed",lw=0.5)
axs[0].vlines(m_mean, 0, 0.6, "k", linestyle="dashed",lw=0.5)
axs[0].vlines(m_plus_std, 0, 0.9, "r", linestyle="dashed",lw=0.5)
axs[0].vlines(m_plus_2std, 0, 1.0, "g", linestyle="dashed",lw=0.5)

# How high does the pdf go?
max_density = pdf_values.max()

# Draw the PDF on the second axis
axs[1].set_title("PDF of Mass")
axs[1].set_ylim(bottom=0.0, top=max_density * 1.1)
axs[1].set_xlim(left=0.0, right=MAX_MASS)
axs[1].set_xlabel("mass (g)")
axs[1].set_ylabel("Probability Density")
axs[1].plot(m_values, pdf_values)

# Add lines for mean, mean-std, and mean+std
axs[1].vlines(m_minus_2std, 0, max_density * .3, "g", linestyle="dashed",lw=0.5)
axs[1].vlines(m_minus_std, 0, max_density * .85, "r", linestyle="dashed",lw=0.5)
axs[1].vlines(m_mean, 0, max_density * 1.05, "k", linestyle="dashed",lw=0.5)
axs[1].vlines(m_plus_std, 0, max_density * .6, "r", linestyle="dashed",lw=0.5)
axs[1].vlines(m_plus_2std, 0, max_density * .2, "g", linestyle="dashed",lw=0.5)
```

```
fig.savefig("pdf.png")
```

1.2 Decreasing Conversions

The last case (mass and radius) is relatively straightforward, because the function g is always increasing. What if we have a change of variables where g is decreasing? For example, $V = IR$ so $\frac{V}{R} = I$.

Let's say that you work at a lightbulb factory and you sample the lightbulbs to see what their resistance is. You find the resistances of the lightbulbs are normally distributed with a mean of 24 ohms and a standard deviation of 3 ohms. The voltage will be exactly 12 volts. What is the PDF of the currents that will pass through the lightbulbs?

$$I = \frac{12}{R}$$

so

$$g(x) = \frac{12}{x}$$

is the function that will convert the current to resistance. Taking the derivative, we get:

$$g'(i) = -\frac{12}{x^2}$$

This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.

Answers to Exercises

