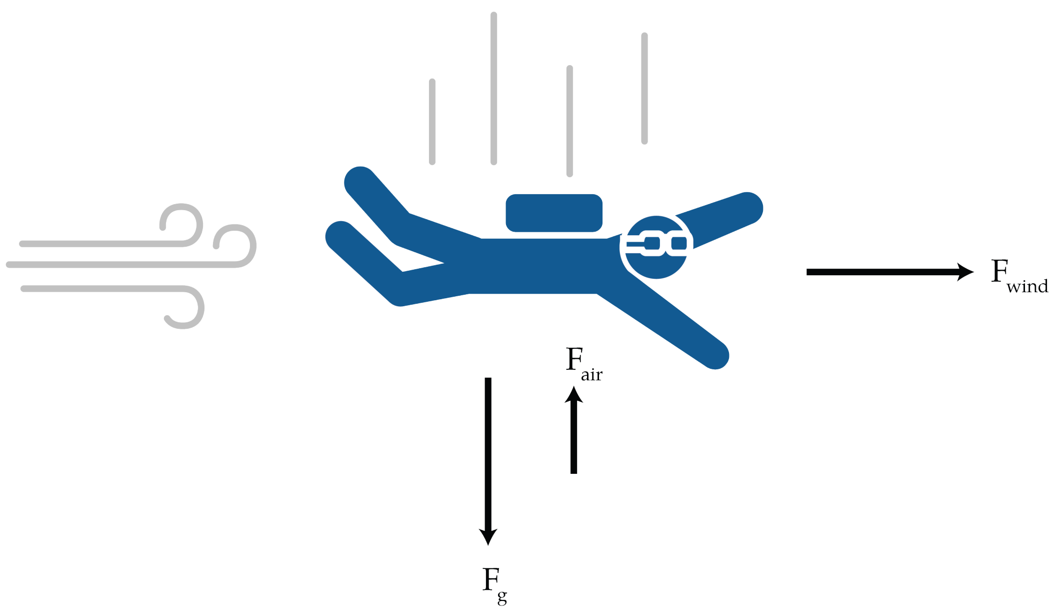
Chapter 1

Vectors

We have talked a some about forces, but in the calculations that we have done, we have only talked about the magnitude of a force. It is equally important to talk about its direction. To do the math on things with a magnitude and a direction (like forces), we need vectors.

For example, if you jump out of a plane (hopefully with a parachute), several forces with different magnitudes and directions will be acting upon you. Gravity will push you straight down. That force will be proportional to your weight. If there were a wind from the west, it would push you toward the east. That force will be proportional to the square of the speed of the wind and approximately proportional to your size. Once you are falling, there will be resistance from the air that you are pushing through — that force will point in the opposite direction from the direction you are moving and will be proportional to the square of your speed.



To figure out the net force (which will tell us how we will accelerate), we will need to add these forces together. To do this, we need to learn to do math with vectors.

1.1 Adding Vectors

A vector is typically represented as a list of numbers, with each number representing a particular dimension. For example, if you are creating a 3-dimensional vector representing a force, it will have three numbers representing the amount of force in each of the three axes. For example, if a force of one newton is in the direction of the x-axis, you might represent the vector as $v = [1, 0, 0]$. Another vector might be $u = [0.5, 0.9, 0.7]$. You can see examples of 2-dimensional and 3-dimensional vectors in figures 1.1 and 1.2.

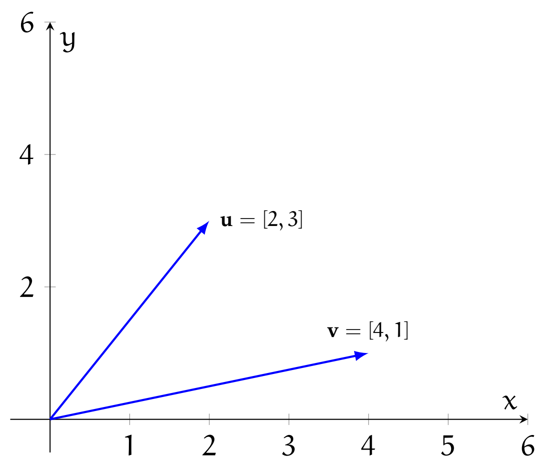


Figure 1.1: 2-dimensional vectors, **u** and **v**

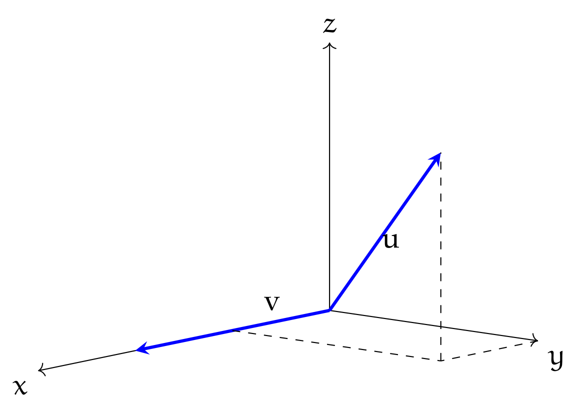


Figure 1.2: 3-dimensional vectors, **u** and **v**

Thinking visually, when we add to vectors, we put the starting point second vector at the ending point of the first vector. This is illustrated for 2-dimensional vectors in figure 1.3 and for 3-dimensional vectors in figure 1.4.

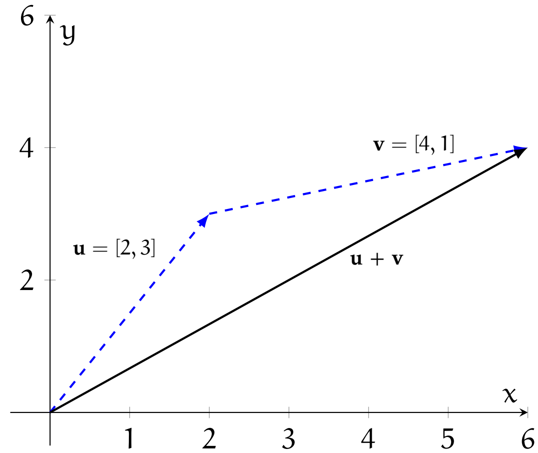


Figure 1.3: A visual representation of adding 2-dimensional vectors, **u** and **v**

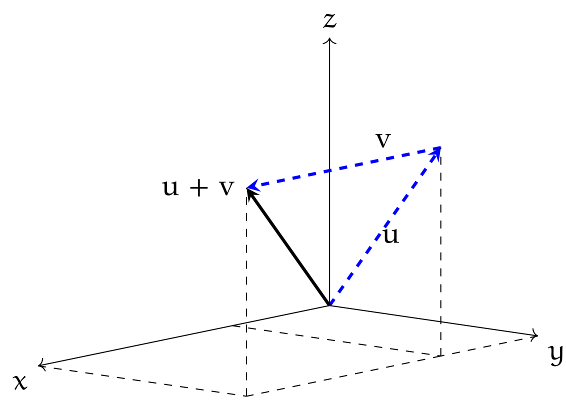


Figure 1.4: A visual representation of adding 3-dimensional vectors, **u** and **v**

If you know the vectors, you will just add them element-wise:

$$u + v = [0.5, 0.9, 0.7] + [1.0, 0.0, 0.0] = [1.5, 0.9.0.7]$$

These vectors have 3 components, so we say they are *3-dimensional*. Vectors can have any number of components. For example, the vector $[-12.2, 3, \pi, 10000]$ is 4-dimensional.

You can only add two vectors if they have the same dimension.

$$[12, -4] + [-1, 5] = [11, 1]$$

Addition is commutative; if you have two vectors a and b, then $a + b$ is the same as $b + a$.

Addition is also associative: If you have three vectors a, b, and c, it doesn't matter which order you add them in. That is, $a + (b + c) = (a + b) + c$.

A 1-dimensional vector is just a number. We say it is a *scalar*, not a vector.

Exercise 1 Adding vectors

Add the following vectors:

- $[1, 2, 3] + [4, 5, 6]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi]$

Working Space

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Exercise 2 Adding Forces

You are adrift in space, near two different stars. The gravity of one star is pulling you towards it with a force of $[4.2, 5.6, 9.0]$ newtons. The gravity of the other star is pulling you towards it with a force of $[-100.2, 30.2, -9.0]$ newtons. What is the net force?

Working Space

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1.2 Multiplying a vector with a scalar

It is not uncommon to multiply a vector by a scalar. For example, a rocket engine might have a force vector v. If you fire 9 engines in the exact same direction, the resulting force vector would be $9v$.

Visually, when we multiply a vector u by a scalar a, we get a new vector that goes in the

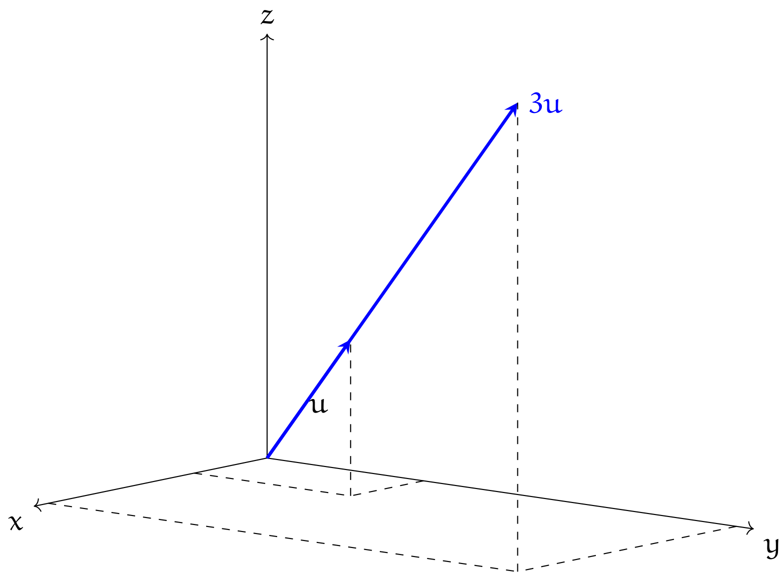

Figure 1.5: To multiply vectors, the vector gets stretched in the same direction a amount

same direction as u but has a magnitude a times as long as u. A visual is presented in figure 1.5.

When you multiply a vector by a scalar, you simply multiply each of the components by the scalar:

$$3 \times [0.5, 0.9, 0.7] = [1.5, 2.7, 3.6]$$

Exercise 3 Multiplying a vector and a scalar

Simplify the following expressions:

- $2 \times [1, 2, 3]$
- $[-1, -2, -3, -4] \times -2$
- $\pi[\pi, 2\pi, 3\pi]$

Working Space

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Note that when you multiply a vector times a negative number, the new vector points in the opposite direction (see figure 1.6).

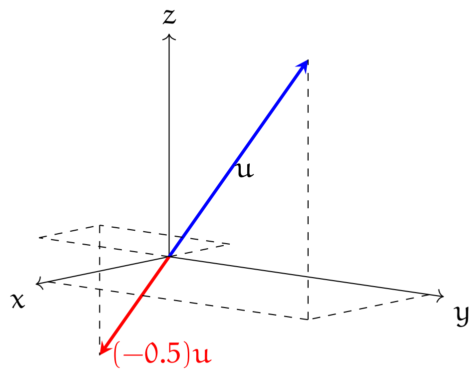


Figure 1.6: Multiplying a vector by a negative number reverses the direction of the vector.

1.3 Vector Subtraction

As you might guess, when you subtract one vector from another, you just do element-wise subtraction:

$$[4, 2, 0] - [3, -2, 9] = [1, 4, -9]$$

So, $u - v = u + (-1v)$.

Visually, you reverse the one that is being subtracted (see figure 1.7):

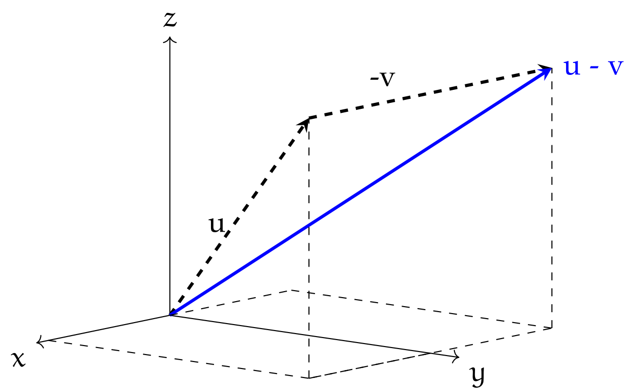


Figure 1.7: To subtract a vector, you reverse it, then add the reversed vector.

1.4 Magnitude of a Vector

The *magnitude* of a vector is just its length. We write the magnitude of a vector v as $|v|$.

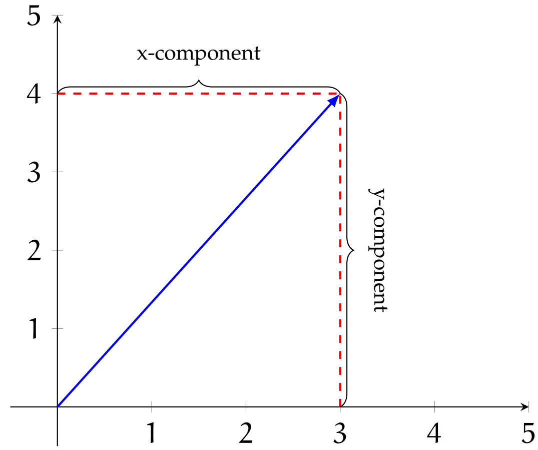


Figure 1.8: The magnitude of a vector can be thought of as the length of a hypotenuse of a right triangle.

We compute the magnitude using the pythagorean theorem. If $v = [3, 4, 5]$, then

$$|v| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$$

(You might notice that the notation for the magnitude is exactly like the notation for absolute value. If you think of a scalar as a 1-dimensional vector, the absolute value and the magnitude are the same. For example, the absolute value of -5 is 5. If you take the magnitude of the one-dimensional vector $[-5]$,you get $\sqrt{25} = 5$.)

Where does this equation come from? Consider a 2-dimensional vector, $\mathbf{v} = [3, 4]$. This means the the vector represents 3 units in the x-direction, and 4 units in the y-direction. We can then visualize a right triangle, with the vector being the hypotenuse and the legs being the x- and y-components of the vector (see figure 1.8). As you recall, the length of the hypotenuse of a right triangle is the square root of the sum of the squares of the legs. That is:

$$c = \sqrt{a^2 + b^2}$$

Where c is the length of the hypotenuse and a and b are the lengths of the legs. See Figure **??**.

We won't prove it here, but this method holds for higher-dimension vectors as well.

Magnitude of Vectors

For an n-dimensional vector, $\mathbf{v} = [x_1, x_2, x_3, \cdots, x_n]$, the magnitude of the vector is given by:

$$|\mathbf{v}| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

Notice that if you scale up a vector, its magnitude scales by the same amount. For example:

$$|7[3, 4, 5]| = 7\sqrt{50} \approx 7 \times 7.07$$

Here is why that is true. Suppose we have a vector, $\mathbf{u} = [a, b, c]$. Then the magnitude of $\mathbf{u}$ is given by:

$$|\mathbf{u}| = \sqrt{a^2 + b^2 + c^2}$$

If we scale $\mathbf{u}$ to create $\mathbf{v}$ such that $\mathbf{v} = k\mathbf{u} = [ka, kb, kc]$, where k is some constant. Then the magnitude of $\mathbf{v}$ is given by:

$$|\mathbf{v}| = \sqrt{(ka)^2 + (kb)^2 + (kc)^2}$$

We can expand and simplify this equation:

$$|\mathbf{v}| = \sqrt{k^2a^2 + k^2b^2 + k^2c^2}$$

$$|\mathbf{v}| = \sqrt{k^2\left(a^2 + b^2 + c^2\right)}$$

$$|\mathbf{v}| = \left(\sqrt{k^2}\right)\sqrt{a^2 + b^2 + c^2}$$

$$|\mathbf{v}| = |k|\sqrt{a^2 + b^2 + c^2} = |k|\,|\mathbf{u}|$$

So, if you scale a vector, the magnitude of the resulting vector is the absolute value of the scale factor times the magnitude of the original vector.

The rule then is: If you have any vector v and any scalar k:

$$|kv| = |k||v|$$

1.4.1 Unit Vectors

A *unit vector* is a vector whose magnitude is 1. For any non-zero vector $\vec{\mathbf{v}}$, the unit vector $\vec{\mathbf{u}}$ pointing in the same direction is

$$\vec{\mathbf{u}} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

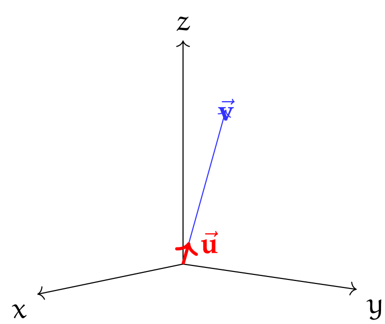
For example, if $\mathbf{v} = [3,4,5]$ then

$$|\mathbf{v}| \;=\; \sqrt{3^2+4^2+5^2} \;=\; \sqrt{50},$$

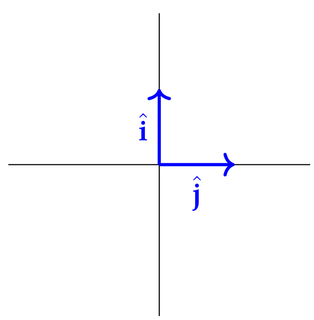
so

$$\vec{\mathbf{u}} \;=\; \frac{1}{\sqrt{50}}[3,4,5] \;=\; \left[\tfrac{3}{\sqrt{50}}, \tfrac{4}{\sqrt{50}}, \tfrac{5}{\sqrt{50}}\right].$$

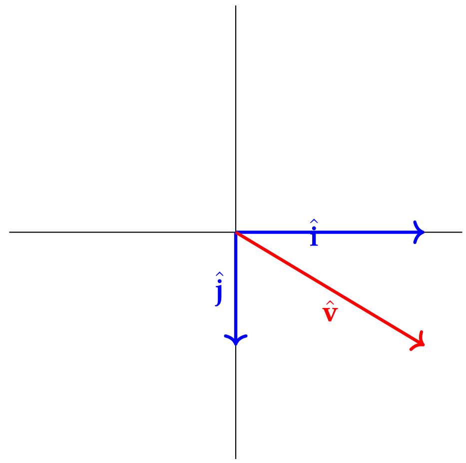
$$\boxed{\widehat{\mathbf{u}} = \frac{\mathbf{v}}{|\mathbf{v}|}}, \qquad |\vec{\mathbf{u}}| = 1.$$



Unit vectors are often represented by placing a hat over the vector. In standard vector calculus, the variables $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{i}}$ for the x, y, and z variables respectively.



You may see vectors provided to you in the form $\mathbf{v} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$. You may say this as '5 times the unit vector in the x-direction plus the -3 times the unit vector in the y-direction'. The same may apply for 3D vectors as well.



Exercise 4 Magnitude of a Vector

Find the magnitude of the following vectors:

- $[1, 1, 1]$
- $[-5, -5, -5]$ (that is the same as $-5 \times [1, 1, 1]$)
- $[3, 4, -4] + [-2, -3, 5]$

Working Space

Answer on Page 13

1.5 Vectors in Python

NumPy is a library that allows you to work with vectors in Python. You might need to install it on your computer. This is done with `pip`. `pip3` installs things specifically for Python 3.

```
pip3 install NumPy
```

We can think of a vector as a list of numbers. There are also grids of numbers known as *matrices*. NumPy deals with both in the same way, so it refers to both of them as arrays.

The study of vectors and matrices is known as *Linear Algebra*. Some of the functions we need are in a sublibrary of NumPy called `linalg`.

As a convention, everyone who uses NumPy, imports it as *np*.

Create a file called `first_vectors.py`:

```
import NumPy as np

# Create two vectors
v = np.array([2,3,4])
u = np.array([-1,-2,3])
print(f"u = {u}, v = {v}")

# Add them
w = v + u
print(f"u + v = {w}")

# Multiply by a scalar
w = v * 3
print(f"v * 3 = {w}")

# Get the magnitude
# Get the magnitude
mv = np.linalg.norm(v)
mu = np.linalg.norm(u)
print(f"|v| = {mv}, |u| = {mu}")
```

When you run it, you should see:

```
> python3 first_vectors.py
u = [-1 -2  3], v = [2 3 4]
u + v = [1 1 7]
v * 3 = [ 6  9 12]
|v| = 5.385164807134504, |u| = 3.7416573867739413
```

1.5.1 Formatting Floats

The numbers 5.385164807134504 and 3.7416573867739413 are pretty long. You probably want them rounded off after a couple of decimal places.

Numbers with decimal places are called *floats*. In the placeholder for your float, you can specify how you want it formatted, including the number of decimal places.

Change the last line to look like this:

```
print(f"|v| = {mv:.2f}, |u| = {mu:.2f}")
```

When you run the code, it will be neatly rounded off to two decimal places:

```
|v| = 5.39, |u| = 3.74
```

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

- $[1, 2, 3] + [4, 5, 6] = [5, 7, 9]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7] = [3, 3, 3, 3]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi] = [\pi, \pi, \pi]$

Answer to Exercise 2 (on page 4)

To get the net force, you add the two forces:

$$F = [4.2, 5.6, 9.0] + [-100.2, 30.2, -9.0] = [-96, 35.8, 0.0] \text{ newtons}$$

Answer to Exercise 3 (on page 5)

- $2 \times [1, 2, 3] = [2, 4, 6]$
- $[-1, -2, -3, -4] \times -3 = [3, 6, 9, 12]$
- $\pi[\pi, 2\pi, 3\pi] = \pi^2, 2\pi^2, 3\pi^2]$

Answer to Exercise 4 (on page 10)

- $|[1, 1, 1]| = \sqrt{3} \approx 1.73$
- $|[-5, -5, -5]| = |-5 \times [1, 1, 1]| = 5\sqrt{3} \approx 8.66$
- $|[3, 4, 5] + [-2, -3, -4]| = |[1, 1, 1]| = \sqrt{3} \approx 1.73$

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