Chapter 1

Series

When writing a number with an infinite decimal, such as the Golden Ratio (also known as the Golden Number):

$$\phi = 1.618033988\cdots$$

The decimal system means we can rewrite the Golden Ratio (or any irrational number) as an infinite sum:

$$\phi = 1 + \frac{6}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{0}{10^4} + \frac{3}{10^5} + \cdots$$

You might recall from the chapter on Riemann Sums that we can represent the addition of many (or infinite) with big sigma notation:

$$\sum_{i=1}^{n} a_i$$

where i is the index as discussed in Sequences and n is the number of terms. For infinite sums, $n = \infty$.

1.1 Partial Sums

Let's quickly define a *partial sum*. A partial sum is where we only look at the first n terms of a series. For the general series, $\sum_{i=1}^{n} a_i$, the partial sums are:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\cdots$$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Example: A series is given by $\sum_{i=1}^{\infty}(\frac{-3}{4})^i$. What is the value of the partial sum s_4?

Solution: s_4 is the sum of the first 4 terms:

$$(\frac{-3}{4})^1 + (\frac{-3}{4})^2 + (\frac{-3}{4})^3 + (\frac{-3}{4})^4$$

$$= \frac{-3}{4} + \frac{9}{16} + \frac{-27}{64} + \frac{81}{256} = \frac{-75}{256}$$

1.2 Reindexing

Sometimes it is necessary to re-index series. This means changing what n the series starts at . In general,

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+1}^{\infty} a_{n-1} \text{ and } \sum_{n=i}^{\infty} a_n = \sum_{n=i-1}^{\infty} a_{n+1}$$

In other words, to increase the index by 1, you need to replace n with $(n-1)$ and do decrease the index by 1, you need to replace n with $(n+1)$. Let's visualize why this is true (see figure 1.1). Notice that for each series, the terms are the same. This is similar to shifting functions: to move the function to the left on the x-axis, you plot $f(x+1)$, and to move it to the right, $f(x-1)$.

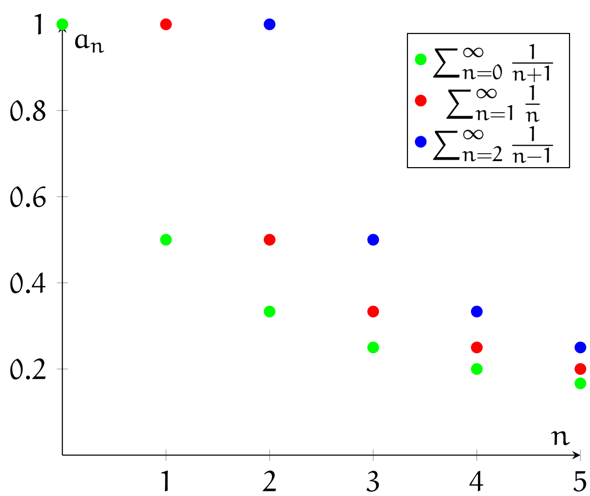

Figure 1.1: $\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

We can also prove each reindexing rule mathematically. Recall that

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

We also know that

$$\sum_{n=2}^{\infty} a_{n-1} = a_{2-1} + a_{3-1} + a_{4-1} + \cdots = a_1 + a_2 + a_3 + \cdots$$

Therefore, $\sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} a_{n-1}$.

Similarly,

$$\sum_{n=0}^{\infty} a_{n+1} = a_{0+1} + a_{1+1} + a_{2+1} + \cdots = a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

Example: Reindex the series $\sum_{n=3}^{\infty} \frac{n+1}{n^2-2}$ to begin with $n = 1$.

Solution: We are decreasing the index, so we will use $\sum_{n=i-1}^{\infty} a_{n+1} = \sum_{n=i}^{\infty} a_n$. We will apply this rule twice, to decrease the index from 3 to 1:

$$\sum_{n=2}^{\infty} \frac{(n+1)+1}{(n+1)^2-2} = \sum_{n=2}^{\infty} \frac{n+2}{(n+1)^2-2}$$

$$\sum_{n=1}^{\infty} \frac{(n+1)+2}{[(n+1)+1]^2-2} = \sum_{n=1}^{\infty} \frac{n+3}{(n+2)^2-2}$$

It is easier and faster to be able to reindex a series by more than one step at a time. Using the example above, we can write an even more general rule for reindexing:

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+j}^{\infty} a_{n-j}$$

where i and j are integers. (Then, to decrease the index, you would choose a j such that $j < 0$.)

1.3 Convergent and Divergent Series

Just like sequences, series can also be convergent or divergent. Consider the series $\sum_{i=1}^{\infty} i$. Given what you already know about the meaning of "convergent" and "divergent", guess whether $\sum_{i=1}^{\infty} i$ is convergent or divergent.

Let's determine the first few partial sums of the series (shown graphically in figure 1.2):

n	Terms	Partial Sum
1	1	1
2	1+2	3
3	1+2+3	6
4	1+2+3+4	10

As you can see, as n increases, the value of the partial sum increases without approaching a particular value. We can also see that the value of the first n terms summed together is $\frac{n(n+1)}{2}$. This means that as n approaches ∞, the sum also approaches ∞ and the series is divergent.

Obviously, for a series to not become overly large, the values of the terms should decrease as i increases (that is, each subsequent term is smaller than the one before it). Take the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. As i increases, $\frac{1}{2^i}$ decreases. Let's look at the first few partial sums of this series (shown graphically in figure 1.3):

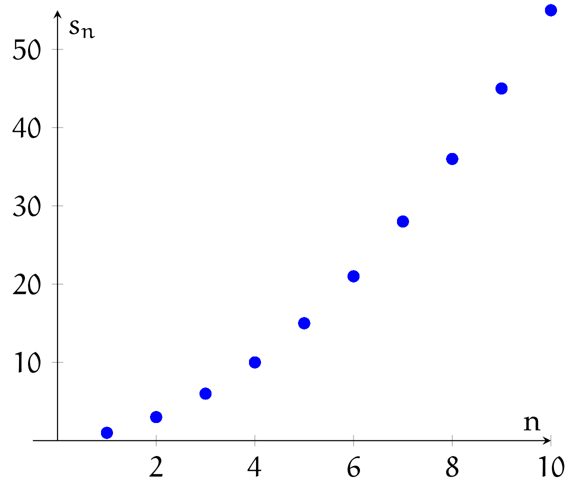

Figure 1.2: For the divergent series $\sum_{i=1}^{n} i$, the value of the partial sum increases to infinity as n increases

n	Terms	Partial Sum
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}+\frac{1}{4}$	$\frac{3}{4}$
3	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$	$\frac{7}{8}$
4	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$	$\frac{15}{16}$

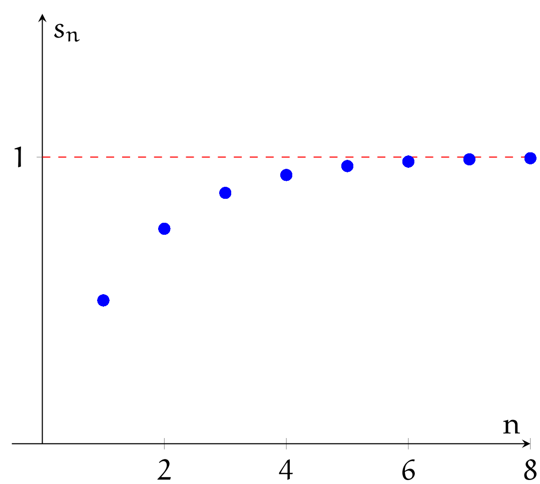

Figure 1.3: For the convergent series $\sum_{i=1}^{n} \frac{1}{2^i}$, the value of the partial sum approaches 1 as n increases

Do you see the pattern? The n^{th} partial sum is equal to $\frac{2^n-1}{2^n} = 1 - \frac{1}{2^n}$. And as n approaches ∞, the partial sum approaches 1. The series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is convergent.

Let's define the sequence $\{s_n\}$, where s_n is the n^{th} partial sum of a series:

$$s_n = \sum_{i=1}^{n} a_i$$

.

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n$ exists, then the series $\sum_{i=1}^{\infty} a_i$ is also convergent. And if the sequence $\{s_n\}$ is divergent, then the series $\sum_{i=1}^{\infty} a_i$ is also divergent.

Example: Is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent or divergent?

Solution: You may think that the series is convergent, since $\lim_{n\to\infty} \frac{1}{n} = 0$. Let's see if we can confirm this. We begin by looking at the partial sums s_2, s_4, s_8, and s_{16}:

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) >$$

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) = 1 + \frac{4}{2}$$

Notice that, in general, $s_{2^n} > 1 + \frac{n}{2}$ for $n > 1$. Taking the limit as $n \to \infty$, we see that $\lim_{n\to\infty} s_{2^n} > \lim_{n\to\infty} 1 + \frac{n}{2} = \infty$. Therefore, s_{2^n} also approaches ∞ as n gets larger and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

This example shows a very important point: A series whose terms decrease to zero as n gets large is not necessarily convergent. What we can say, though, is that if the limit as n approaches infinity of the terms of a series does not exist or is not zero, then the series is divergent (i.e., not convergent). This is called the **Test for Divergence**, and we will explore it further in the next chapter.

1.3.1 Properties of Convergent Series

We just saw that if $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges. The contrapositive statement gives a property of convergent series:

$$\text{If the series } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n\to\infty} = 0$$

If a series is made of other convergent series, it may be convergent. Recall, if a series is convergent, this means the $\lim_{n\to\infty}\sum_{i=1}^{n} a_i = L$. By the properties of limits, we can also say that the series multiplied by a constant is convergent:

$$\sum_{n=1}^{\infty} ca_n = c \cdot L = c\sum_{n=1}^{\infty} a_n$$

Suppose there is another convergent series such that $\lim_{n\to\infty}\sum_{i=1}^{n} b_i = M$. In this case, the sum of those series is also convergent. That is:

$$\sum_{n=1}^{\infty}(a_n + b_n) = L + M = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Similarly, the difference of the series is convergent:

$$\sum_{n=1}^{\infty}(a_n - b_n) = L - M = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

1.4 Geometric Series

A geometric series is the sum of a geometric sequence, and has the form:

$$\sum_{n=1}^{\infty} ar^n \text{ or } \sum_{n=1}^{\infty} ar^{n-1}$$

Where a is some constant and r is the common ratio. For $\sum_{n=1}^{\infty} ar^{n-1}$, a is also the first term.

Example: Write the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ in sigma notation.

Solution: We see that the first term is $a = 1$ and the common ratio is $\frac{1}{2}$, so we can write the series:

$$\sum_{n=1}^{\infty} 1(\frac{1}{2})^{n-1} = \sum_{n=1}^{\infty}(\frac{1}{2})^{n-1}$$

When are geometric series convergent? First, let's consider the case where $r = 1$. If this is true, then $s_n = a + a + a + \cdots + a = na$. As n approaches ∞, the sum will approach $\pm\infty$ (depending on whether a is positive or negative), and the series is divergent.

When $r \neq 1$, we can write s_n and rs_n:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

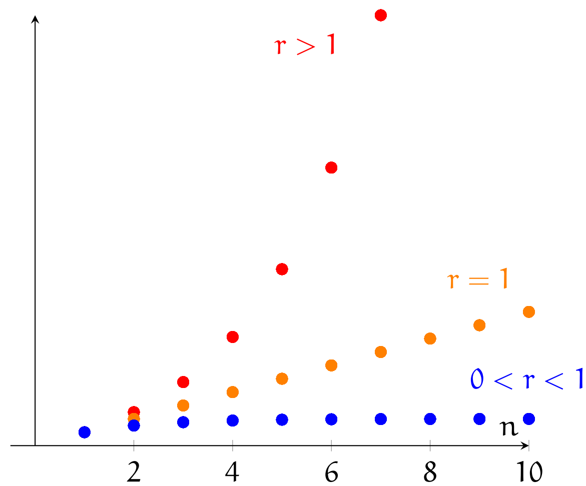

Figure 1.4: Geometric sequences are divergent if $r \geq 1$

Subtracting rs_n from s_n, we get:

$$s_n - rs_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n)$$
$$= a - ar^n$$

Solving for s_n, we find:

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

We take the limit as $n \to \infty$ to determine for what values of r the series converges:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r}$$
$$= \lim_{n\to\infty} \left[\frac{a}{1 - r} - \frac{ar^n}{1 - r}\right] = \frac{a}{1 - r} - \left(\frac{a}{1 - r}\right) \lim_{n\to\infty} r^n$$

This introduces the question: When is $\lim_{n\to\infty} r^n$ convergent? From the sequences chapter, we know this limit converges if $|r| < 1$ (that is, $-1 < r < 1$). If this is true, then $\lim_{n\to\infty} r^n = 0$ and

$$\lim_{n\to\infty} s_n = \frac{a}{1 - r}$$

(see figures 1.4 and 1.5 for a visual)

Example: Find the sum of the geometric series given by $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \cdots$.

Solution: The first term is $a = 2$, and each common ratio is $r = \frac{-1}{3}$. Since $|r| < 1$, we know that the series converges. We can calculate the value of the sum using the geometric series

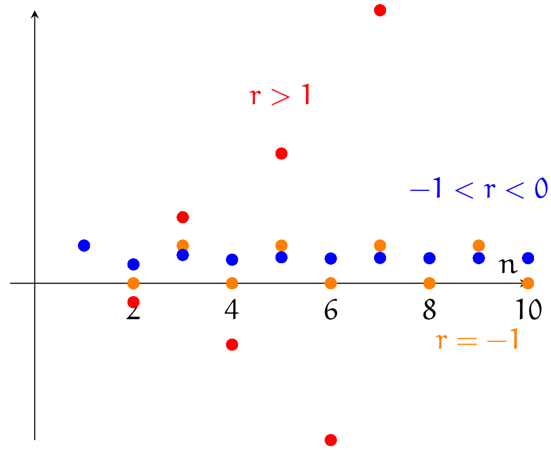

Figure 1.5: Geometric sequences are divergent if $r \leq 1$. Notice that for $r = -1$, the partial sums alternate between the initial term and zero.

formula:

$$\sum_{i=1}^{\infty} a(r)^{i-1} = \frac{a}{1-r}$$

$$\sum_{i=1}^{\infty} 2(\frac{-1}{3})^{i-1} = \frac{2}{1-\frac{-1}{3}} = \frac{2}{\frac{4}{3}} = \frac{6}{4} = 1.5$$

We can confirm this graphically (see figure 1.6). You can also write out the first several partial sequences. You should find the sums approach 1.5 as n increases.

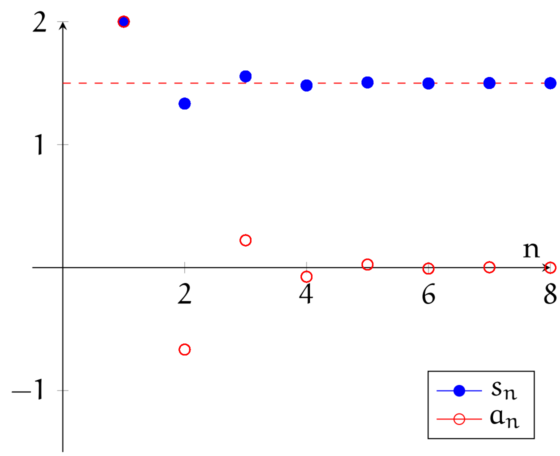

Figure 1.6: the n^{th} term and partial sums of $\sum_{i=1}^{n} 2(\frac{-1}{3})^{i-1}$

Example: What is the value of $\sum_{n=1}^{\infty} 2^{2n}5^{1-n}$

Solution: The key here is to re-write the series in the form $\sum_{n=1}^{\infty} ar^{n-1}$ so we can use the fact that convergent geometric series sum to $\frac{a}{1-r}$.

$$\sum_{n=1}^{\infty} 2^{2n} 5^{1-n} = \sum_{n=1}^{\infty} \left(2^2\right)^n \left(\frac{1}{5}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 4 \cdot (4)^{n-1} \left(\frac{1}{5}\right)^{n-1} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^{n-1}$$

Which is in the form $\sum_{n=1}^{\infty} ar^{n-1}$ with $a = 4$ and $r = \frac{4}{5}$. Since $|r| < 1$, the series converges to

$$\frac{a}{1-r} = \frac{4}{1-\frac{4}{5}} = \frac{4}{\frac{1}{5}} = 20$$

Exercise 1

Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

1. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$
2. $2 + 0.5 + 0.125 + 0.03125 + \cdots$
3. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$
4. $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

Working Space

Answer on Page 15

Exercise 2

Find a value of c such that $\sum_{n=0}^{\infty}(1+c)^{-n} = \frac{5}{3}$.

Working Space

Answer on Page 15

Exercise 3

For what values of p does the series $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converge?

Working Space

Answer on Page 15

1.5 p-series

A p-series takes the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and converges if $p > 1$ and diverges if $p \leq 1$. We won't prove this here, since it requires the application of a test you will learn about in the next chapter.

Example Write the series $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$. Is it convergent or divergent?

Solution: We see that $a_n = \frac{1}{\sqrt[3]{n}}$, so the infinite series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

. We see that this is a p-series with $p = \frac{1}{3}$. Since $p < 1$, the series is divergent.

Exercise 4

Euler found that the exact sum of the p-series where $p = 2$ is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

And that the exact sum of the p-series where $p = 4$ is:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use this and the properties of convergent series to find the sum of each of the following series:

1. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$
2. $\sum_{n=2}^{\infty} \frac{1}{n^2}$
3. $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$
4. $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$
5. $\sum_{n=1}^{\infty} \left(\frac{4}{n^2} + \frac{3}{n^4}\right)$

Working Space

Answer on Page 16

Exercise 5

For what values of k does the series $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ converge?

Working Space

Answer on Page 16

1.6 Alternating Series

An alternating series is one in which the terms alternate between positive and negative . Here is an example:

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$$

Alternating series are generally of the form

$$a_n=(-1)^n b_n \text{ or } a_n=(-1)^{n-1}b_n$$

Where b_n is positive (and therefore, $|a_n|=b_n$).

An alternating series is convergent if (i)$b_{n+1}\leq b_n$ and (ii)$\lim_{n\to\infty} b_n=0$. In other words, we say that if the absolute value of the terms of a series decrease towards zero, then the series converges. This is called the **Alternating Series Test**.

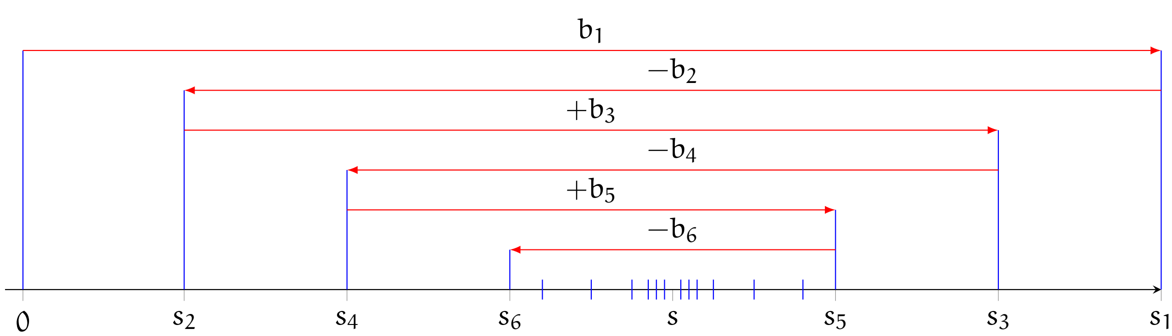

Figure 1.7: As n increases, s_n approaches s

Example: Is the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ convergent?

Solution: The Alternating series test states that an alternating series is convergent if $|a_{n+1}|<|a_n|$:

$$\left|\frac{(-1)^{n-1+1}}{n+1}\right|<\left|\frac{(-1)^{n-1}}{n}\right|$$

$$\frac{1}{n+1} < \frac{1}{n}$$

Since $|a_{n+1}| < |a_n|$ and the series is alternating, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Exercise 6

Test the following alternating series for convergence:

1. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$
2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

Working Space

Answer on Page 16

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 9)

1. We need to identify a and r. If we use the form $\sum_{n=1}^{\infty} ar^{n-1}$, then $a = 3$. To find the common ratio, we can evaluate $\frac{a_{n+1}}{a_n} = \frac{-4}{3}$. We can then write the series as $\sum_{n=1}^{\infty} 3\left(\frac{-4}{3}\right)^{n-1}$. In this case, $r = \frac{-4}{3}$ and $|r| \geq 1$, and therefore the series is divergent.
2. Following the process outlined above, we see that $a = 2$ and $r = \frac{1}{4}$. Therefore, the series is $\sum_{n=1}^{\infty} 2\left(\frac{1}{4}\right)^{n-1}$. Since $|r| < 1$, the series converges to $\frac{a}{1-r} = \frac{2}{1-1/4} = \frac{2 \cdot 4}{3} = \frac{8}{3}$
3. We need to rewrite the series into a standard from in order to identify a and r:
$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4(4)^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{-3}{4}\right)^{n-1}$$
So $r = \frac{-3}{4}$ and $|r| < 1$. Therefore, the series converges to $\frac{1/4}{1-(-3/4)} = \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}$
4. We need to rewrite the series into a standard from in order to identify a and r:
$$\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)^n}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)(e^2)^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} e^2\left(\frac{e^2}{6}\right)^{n-1}$$
Therefore, $r = \frac{e^2}{6} \approx 1.232$. Since $|r| > 1$, the series diverges.

Answer to Exercise 2 (on page 9)

We want to rewrite this as a geometric series of the form $\sum_{n=i}^{\infty} ar^{n-1}$, so we can use the fact that the sum of a convergent geometric series is $\frac{a}{1-r}$. $\sum_{n=0}^{\infty}(1+c)^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{1+c}\right)^n = \sum_{n=1}^{\infty}\left(\frac{1}{1+c}\right)^{n-1}$. This is a geometric series with $a = 1$ and $r = \frac{1}{1+c}$. So, the value of the series is $\frac{1}{1-\frac{1}{1+c}} = \frac{1}{\frac{c}{c+1}} = \frac{c+1}{c}$. Setting this equal to $\frac{5}{3}$ and solving for c, we find that $c = \frac{3}{2}$.

Answer to Exercise 3 (on page 10)

$-2 < p < 2$ Let's rewrite this geometric series into standard form: $\sum_{n=1}^{\infty}\left(\frac{p}{2}\right)^n = \sum_{n=1}^{\infty}\frac{p}{2}\left(\frac{p}{2}\right)^n$ which means $a = \frac{p}{2}$ and $r = \frac{p}{2}$. We know that geometric series converge if $|r| < 1$, so we set up an inequality and solve for p:

$$\left|\frac{p}{2}\right| < 1$$
$$-1 < \frac{p}{2} < 1$$
$$-2 < p < 2$$

Answer to Exercise 4 (on page 11)

1. Separating the terms, we see that $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4} = \sum_{n=1}^{\infty} \left(\frac{n^2}{n^4} + \frac{1}{n^4}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{6} + \frac{\pi^4}{90}$
2. Notice that this series starts at $n = 2$. By the properties of series, we know that $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$. Therefore, $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) - \frac{1}{1^2} = \frac{\pi^2}{6} - 1$
3. We can begin by reindexing this series: $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2}$. Similar to the previous problem, we also know that $\sum_{n=4}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) = \frac{\pi^2}{6} - \frac{49}{36}$
4. We can rewrite this series as $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4 = \sum_{n=1}^{\infty} (3^4)\frac{1}{n^4} = 81 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{81\pi^4}{90} = \frac{9\pi^4}{10}$
5. We can re-write the series as $\sum_{n=1}^{\infty} \left(\frac{4}{n^2} + \frac{3}{n^4}\right) = \sum_{n=1}^{\infty} \frac{4}{n^2} + \sum_{n=1}^{\infty} \frac{3}{n^4} = 4\sum_{n=1}^{\infty} \frac{1}{n^2} + 3\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{4\pi^2}{6} + \frac{3\pi^4}{90} = \frac{2\pi^2}{3} + \frac{\pi^4}{30}$

Answer to Exercise 5 (on page 12)

This is a p-series where $p = 2k$. We know that p-series converge for $p > 1$: $2k > 1 \to k > \frac{1}{2}$.

Answer to Exercise 6 (on page 13)

1. The series is convergent if $\left|\frac{(-1)^{n+1}3(n+1)}{4(n+1)-1}\right| < \left|\frac{(-1)^n 3n}{4n-1}\right|$ if $\frac{3n+3}{4n+4-1} < \frac{3n}{4n-1}$ and if $\frac{3n+3}{4n+3} < \frac{3n}{4n-1}$ if $(3n+3)(4n-1) < (3n)(4n+3)$ if $12n^2 + 12n - 3n - 3 < 12n^2 + 9n$ if $-3 < 0$ which is true. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is convergent.
2. The series is convergent if $\left|(-1)^{n+1+1}\frac{(n+1)^2}{(n+1)^3+1}\right| < \left|(-1)^{n+1}\frac{n^2}{n^3+1}\right|$, which is true if $\frac{(n+1)^2}{(n+1)^3+1} < \frac{n^2}{n^3+1}$ if $(n+1)^2(n^3+1) < (n^2)((n+1)^3+1)$ if $(n^2+2n+1)(n^3) <$

$(n^2)(n^3+3n^2+3n+1+1)$ if $n^5+2n^4+n^3 < n^5+3n^4+3n^3+2n^2$, which is true for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ is convergent.

3. The series is convergent if $\left|(-1)^{n-1+1}e^{2/(n+1)}\right| < \left|(-1)^{n-1}e^{2/n}\right|$, which is true if $e^{2/(n+1)} < e^{2/n}$, which is true if $\frac{2}{n+1} < \frac{2}{n}$ which is true for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1}e^{2/n}$ is convergent.

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