Chapter 1

Sequences in Calculus

We introduced sequences in a previous chapter. Now, we will examine them in more detail in the context of calculus. You already know about arithmetic and geometric sequences, but not all sequences can be classified as arithmetic or geometric. Take the famous Fibonacci sequence, {1, 1, 2, 3, 5, 8, ...}, which can be explicitly defined as $a_n = a_{n-1} + a_{n-2}$, with $a_1 = a_2 = 1$. There is no common difference or common ratio, so the Fibonacci sequence is not arithmetic or geometric. Another example is $a_n = \sin \frac{n\pi}{6}$, which will cycle through a set of values.

Sequences have many real-world applications, including compound interest and modeling population growth. In later chapters, you will learn that the sum of all the values in a sequence is a series and how to use series to describe functions. In order to be able to do all that, we first need to talk in depth about sequences.

Some sequences are defined explicitly, like $a_n = \sin \frac{n\pi}{6}$, while others are defined recursively, like $a_n = a_{n-1} + a_{n-2}$.

Example: Write the first 5 terms for the explicitly defined sequence $a_n = \frac{n}{n+1}$.

Solution: We can construct a table to keep track of our work:

n	work	a_n
1	$\frac{1}{1+1}$	$\frac{1}{2}$
2	$\frac{2}{2+1}$	$\frac{2}{3}$
3	$\frac{3}{3+1}$	$\frac{3}{4}$
4	$\frac{4}{4+1}$	$\frac{4}{5}$
5	$\frac{5}{5+1}$	$\frac{5}{6}$

So, the first five terms are $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, and $\frac{5}{6}\}$.

Exercise 1

Write the first five terms for each sequence.

1. $a_n = \frac{2^n}{2n+1}$
2. $a_n = \cos \frac{n\pi}{2}$
3. $a_1 = 1$, $a_{n+1} = 5a_n - 3$
4. $a_1 = 6$, $a_{n+1} = \frac{a_n}{n+1}$

Working Space

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1.1 Convergence and Divergence

You can visualize a sequence on an xy-plane or a number line. Figures 1.1 and 1.2 show visualizations of the sequence $a_n = \frac{n}{n+1}$. To visualize this on the xy-plane, we take points such that $x = n$ and $y = a_n$, where n is a positive integer. What do you notice about this sequence? As n increases, a_n gets closer and closer to 1.

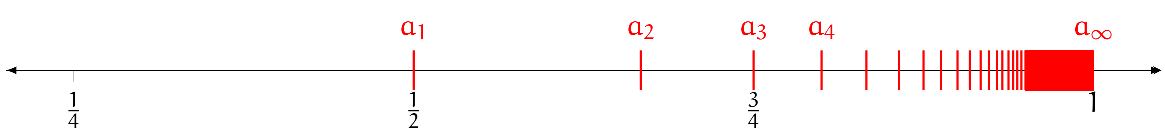

Figure 1.1: $a_n = \frac{n}{n+1}$ on a number line

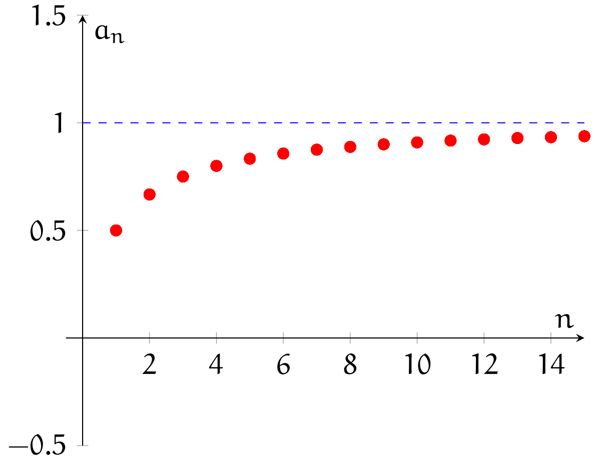

Figure 1.2: $a_n = \frac{n}{n+1}$ on an xy-plane

Because a_n approaches a specific number as $n \to \infty$, we call the series $a_n = \frac{n}{n+1}$ *convergent*. We prove a sequence is convergent by taking the limit as n approaches ∞. If the limit exists and approaches a specific number, the sequence is convergent. If the limit does not exist or approaches $\pm\infty$, the sequence is divergent.

We can see graphically that $\lim_{n\to\infty} \frac{n}{n+1} = 1$, so that sequence is convergent. What about $b_n = \frac{n}{\sqrt{10+n}}$? Is b_n convergent or divergent?

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{n/n}{\sqrt{\frac{10}{n^2} + \frac{n}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

Therefore, the sequence $b_n = \frac{n}{\sqrt{10+n}}$ is divergent.

Here is another example of a divergent sequence: $c_n = \sin \frac{n\pi}{2}$. The graph is shown in figure 1.3. As you can see, the value of c_n oscillates between 1, 0, and -1 without approaching a specific number. This means that c_n does not approach a particular number as $n \to \infty$ and the sequence is divergent.

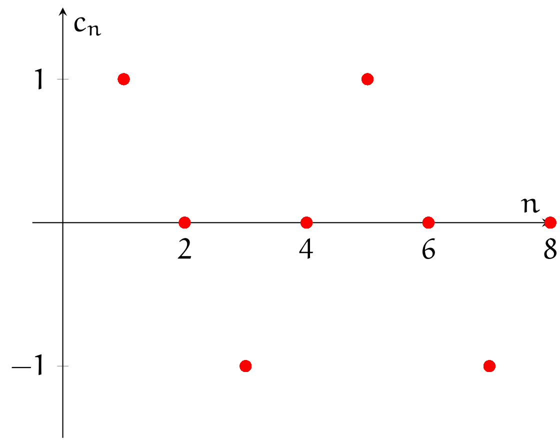

Figure 1.3: $c_n = \sin \frac{n\pi}{2}$ on an xy-plane

Exercise 2

Classify each sequence as convergent or divergent. If the sequence is convergent, find the limit as $n \to \infty$.

1. $a_n = \frac{3+5n^2}{n+n^2}$
2. $a_n = \frac{n^4}{n^3-2n}$
3. $a_n = 2 + (0.86)^n$
4. $a_n = \cos \frac{n\pi}{n+1}$
5. $a_n = \sin n$

Working Space

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1.2 Evaluating limits of sequences

Recall that a sequence can be considered a function where the domain is restricted to positive integers. If there is some $f(x)$ such that $a_n = f(n)$ when n is an integer, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$ (see figure 1.4). This means that all the rules that apply to the limits of functions also apply to the limits of sequences, including the Squeeze Theorem and l'Hôpital's rule.

Example: What is $\lim_{n\to\infty} \frac{\ln n}{n}$?

Solution: First, we will try to compute the limit directly:

$$\lim_{n\to\infty} \frac{\ln n}{n} =$$

$$\frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n} = \frac{\infty}{\infty}$$

This is undefined, but fits the criteria for L'Hospital's rule:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln n}{\frac{d}{dn}n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

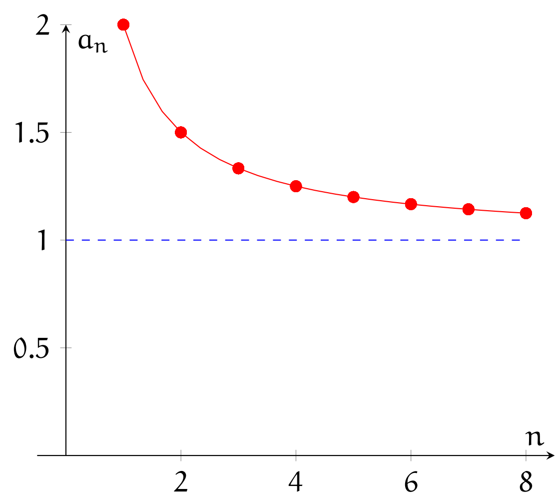

Figure 1.4: The limit of the function is the same as the limit of the sequence

Example: Is the sequence $a_n = \frac{n!}{n^n}$ convergent or divergent?

Solution: First trying to take the limit directly, we see that:

$$\lim_{n\to\infty} \frac{n!}{n^n} = \frac{\infty}{\infty}$$

which is undefined. Because the factorial cannot be described as a continuous function, we can't use l'Hôpital's rule. We can examine this sequence graphically (see figure 1.5) and mathematically. We examine it mathematically by writing out a few terms to get an idea of what happens to a_n as n gets large:

$$a_1 = \frac{1!}{1^1} = 1$$

$$a_2 = \frac{2!}{2^2} = \frac{1 \cdot 2}{2 \cdot 2}$$
$$a_3 = \frac{3!}{3^3} = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$
$$\cdots$$
$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdots n}$$

From examining the graph in figure 1.5, we can guess that $\lim_{n\to\infty} a_n = 0$. Let's prove that mathematically. We can rewrite our expression for a_n as n gets larger:

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n})$$

The expression inside the parentheses is less than 1; therefore, $0 < a_n < \frac{1}{n}$. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, by Squeeze Theorem, we know that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. Therefore, the sequence $a_n = \frac{n!}{n^n}$ is convergent.

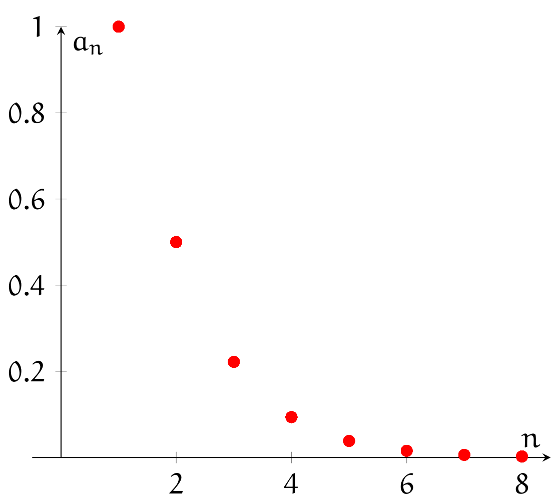

Figure 1.5: $a_n = \frac{n!}{n^n}$

[[FIX ME intro]] If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. For example, what is $\lim_{n\to\infty} \sin \frac{\pi}{n}$? Well, we know that $\lim_{n\to\infty} \frac{\pi}{n} = 0$ and that the sine function is continuous at 0. Therefore, $\lim_{n\to\infty} \sin \frac{\pi}{n} = \sin \lim_{n\to\infty} \frac{\pi}{n} = \sin 0 = 0$.

1.3 Monotonic and Bounded sequences

Just like functions, sequences can be increasing or decreasing. A sequence is increasing if $a_n < a_{n+1}$ for $n \geq 1$. Similarly, a sequence is decreasing if $a_n > a_{n+1}$ for $n \geq 1$. If a sequence is strictly increasing or decreasing, it is called *monotonic*.

The sequence $a_n = \frac{1}{n+6}$ is decreasing. We prove this formally by comparing a_n to a_{n+1}:

$$\frac{1}{n+6} > \frac{1}{(n+1)+6} = \frac{1}{n+7}$$

Example: Is the sequence $a_n = \frac{n}{n^2+1}$ increasing or decreasing?

Solution: First, we find an expression for a_{n+1}:

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$$

Since the degree of n is greater in the denominator, we have a guess that the sequence is decreasing. To prove this, we check if $a_n > a_{n+1}$ is true:

$$\frac{n}{n^2+1} > \frac{n+1}{n^2+2n+2}$$

We can cross-multiply, because $n > 0$ and the denominators are positive:

$$(n)(n^2+2n+2) > (n+1)(n^2+1)$$

$$n^3+2n^2+2n > n^3+n^2+n+1$$

Subtracting (n^3+n^2+n) from both sides we see that:

$$n^2+n > 1$$

Which is true for all $n \geq 1$. Therefore, $a_n > a_{n+1}$ for all $n \geq 1$ and the sequence is decreasing.

A sequence is *bounded above* if there is some number M such that $a_n \leq M$ for all $n \geq 1$. A sequence is *bounded below* if there is some other number m such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above and below, then it is a *bounded sequence*.

Not all bounded sequences are convergent. Take our earlier example of $a_n = \sin\frac{n\pi}{6}$. This sequence is bounded, since we can say that $-1 \leq a_n \leq 1$ for all n. However, $a_n = \sin\frac{n\pi}{6}$ is divergent because $\lim_{n\to\infty}\sin\frac{n\pi}{6}$ does not exist (see figure 1.6). Additionally, not all monotonic sequences are convergent. Consider $b_n = 2^n$ (shown in figure 1.7). This is monotonically increasing (that is, $b_n > b_{n-1}$ for all n), but $\lim_{n\to\infty} 2^n = \infty$ and the sequence is divergent.

A sequence must be convergent if it is **both** monotonic and bounded. Why is this? Recall that to be bounded, a sequence must be bounded above and below, which means there is some m and some M such that $m \leq a_n \leq M$ for all n. If the sequence is increasing,

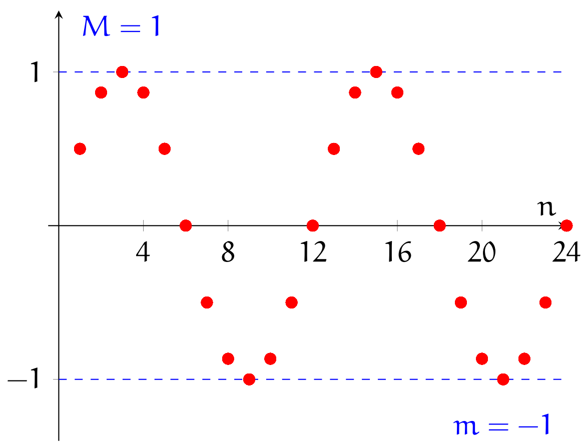

Figure 1.6: The sequence $a_n = \sin \frac{n\pi}{6}$ is bounded and divergent

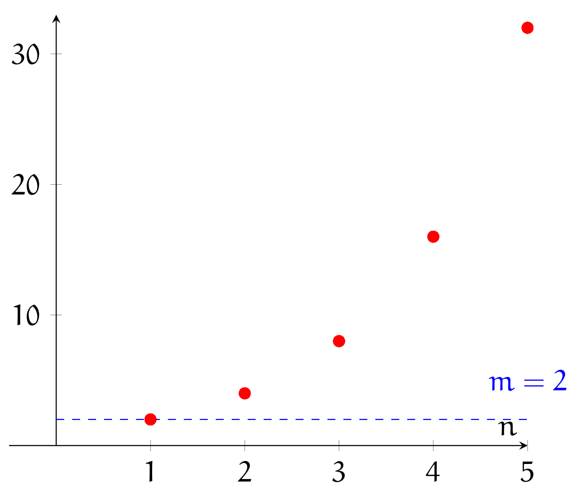

Figure 1.7: The sequence $b_n = 2^n$ is bounded below, monotonically increasing, and divergent

the terms must get close to but not exceed M. Likewise, if the sequence is decreasing, the terms must get close to, but not be less than m.

Example: Is the sequence given by $a_1 = 4$ and $a_{n+1} = \frac{1}{2}(a_n + 7)$ bounded above, below, both, or neither?

Solution: We start by calculating the first several terms:

Term	Work	Value
a_1	$a_1 = 4$	4
a_2	$= \frac{1}{2}(4+7)$	5.5
a_3	$= \frac{1}{2}(5.5+7)$	6.25
a_4	$= \frac{1}{2}(6.25+7)$	6.625
a_5	$= \frac{1}{2}(6.625+7)$	6.8125
a_6	$= \frac{1}{2}(6.8125+7)$	6.90625
a_7	$= \frac{1}{2}(6.90625+7)$	6.953125
a_8	$= \frac{1}{2}(6.953125+7)$	6.9765625

The sequence is increasing, so it is bounded below by the initial term, $a_1 = 4$, and we can state that $a_n \geq 4$. Examining the computed terms, we see that $a_n \to 7$ as n grows larger. We can guess that this sequence is bounded above, with $a_n \leq 7$. We can prove this by induction. Suppose that there is some k such that $a_k < 7$ (which is true for a_1, etc.). Then,

$$\begin{aligned} a_k &< 7 \\ a_k + 7 &< 14 \\ \frac{1}{2}(a_k + 7) &< \frac{1}{2}(14) \\ a_{k+1} &< 7 \end{aligned}$$

Therefore, $a_n < 7$ for all n and the sequence is bounded above. Because the sequence is monotonic and bounded, we know the sequence is convergent and, therefore, that the limit of a_n as $n \to \infty$ exists.

1.4 Applications of Sequences

1.4.1 Compound Interest

You previously learned about compound interest and modeled the accumulation of compound interest by $P_n = P_0(1 + r)^n$, where P_0 is the principal investment, r is the yearly interest rate, and n is the number of elapsed years. This sequence describes the value of an investment accumulating interest, but most people add to their savings on a regular schedule. We can write a sequence to model the value of a savings account that the owner makes regular deposits into.

Example: Suppose you open a savings account with an initial deposit of $3,000 and you plan to deposit an additional $1,200 at the end of every year. If your savings account has an annual interest rate of 3.25%, how long will it take you to save $10,000?

Solution: We can write a recursive definition for the sequence. At the end of each year, the account will gain the interest on the entirety of the previous year's balance plus $1200:

$$P_n = P_{n-1}(1 + 0.0325) + \$1200$$

With an initial investment $P_0 = \$3000$. We can write out the first few terms to find how many years it will take to save $10,000:

Year	Savings
0	$3,000
1	$4,297.50
2	$5,637.17
3	$7,020.38
4	$8,448.54
5	$9.923.12
6	$11,445.62

The accumulation of interest with deposits is better described by a sequence than a function. That iss because the deposits are happening at discrete times, not continuously.

Exercise 3

You invest $1500 at 5%, compounded annually. Write an explicit formula that describes the value of your investment every year. What will your investment be worth after 10 years? Is the sequence convergent or divergent? Explain.

Working Space

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1.4.2 Population Growth

Sequences can be used to model a reproducing population that is being occasionally culled from or added to. Similar to compound interest, a population of living things (plants, animals, fungi, etc.) reproducing at a rate *r* can be modeled with an exponential function:

$$P_n = P_0(1 + r)^n$$

where P_0 is the initial population, r is the yearly reproductive rate, and n is the number of years elapsed.

Example: Suppose the population of deer in a national park is estimated to be 6,500. If the deer reproduce at a rate of 8% per year and wolves hunt and kill 500 deer per year, how many deer will be in the park in 5 years?

Solution: We can write a recursive sequence:

$$P_n = P_{n-1}(1 + 0.08) - 500$$

$$P_0 = 6500$$

And calculate P_5 (we round to the nearest whole number because half of a deer is not a living deer):

Year		Population
1	$6500(1.08) - 500$	6520
2	$6520(1.08) - 500$	6542
3	$6542(1.08) - 500$	6565
4	$6565(1.08) - 500$	6590
5	$6590(1.08) - 500$	6617

There will be 6617 deer in the park after 5 years.

Exercise 4

A farmer keeps his pond stocked with fish. If the fish are eaten by predators at a rate of 5% per month and the farmer can afford to restock the pond with 10 fish every 6 months. If the farmer starts with 100 fish, how many total fish will he have lost to predation after 4 years?

Working Space

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This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

1. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$
2. 0, -1, 0, 1, 0
3. 1, 2, 7, 32, 157
4. 6, 3, 1, $\frac{1}{4}, \frac{1}{20}$

Answer to Exercise 2 (on page 4)

1. convergent, 5
2. divergent
3. convergent, 2
4. convergent, -1
5. divergent

Answer to Exercise 3 (on page 10)

Out principal is $P = 1500$ and the interest rate is $r = 0.06$. After n years, your investment will be worth $a_n = 1500(1.06)^n$. For $n = 10$, your investment will be valued at $a_{10} = \$1500(1.06)^{10} = \$2686.27$ (that's over \$1000 in interest!). To determine if the sequence is convergent or divergent, we examine the limit as $n \to \infty$:

$$\lim_{n\to\infty} 1500(1.06)^n = 1500 \cdot \lim_{n\to\infty}(1.06)^n = 1500 \cdot \infty = \infty$$

The sequence is divergent.

Answer to Exercise 4 (on page 11)

The number of fish in the pond is:

$$P_n = P_{n-1}(0.95)^6 + 50$$

$$P_0 = 100$$

where n is the number of 6-month periods that have passed. The four-year period is given by $1 \leq n \leq 8$. The amount lost to predation every 6 months is given by $P_{n-1}(1 - 0.95^6)$.

n	Fish Population	Lost to Predators
0	100	
1	84	26
2	71	22
3	62	19
4	56	17
5	51	15
6	48	14
7	45	13
8	43	12

Adding up all the fish lost to predators, we find that over 4 years, the farmer loses 138 fish.

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