Applications of Matrices and Systems of Linear Equations

In the chapter on linear combinations, we saw that we can linearly combine vectors to create other vectors. Consider 3 vectors:

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

We can write a linear combination of these vectors:

$$c\mathbf{x} + d\mathbf{y} + e\mathbf{z}$$

Which we can expand to show the vectors:

$$c \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + e \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c - d - 2e \\ 2c + 2d + e \\ d \end{bmatrix}$$

We can also represent this combination with a matrix where each column is one of the vectors:

$$\begin{bmatrix} -1 & -1 & -2 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -c - d - 2e \\ 2c + 2d + e \\ d \end{bmatrix}$$

1.1   Trail Mix for Mars

Let's look at an applied problem. Three astronauts, Pat, Kai, and River, are getting ready for a trip to Mars. NASA food service is preparing trail mix for the voyage, tailored to each astronaut's taste. The chef needs to submit a budget based on the cost of the trail mix for each astronaut. The mix is made up of raisins, almonds, and chocolate.

Pat prefers a raisins:almonds:chocolate ratio of 6:10:4, Kai likes 2:3:15, and River wants 14:1:5. The chef can buy a kg of raisins for $7.50, a kg of almonds for $14.75, and a kg of chocolate for $22.25. Assuming each astronaut will get 20 kg of trail mix, which astronaut will cost more to feed?

First, set up a matrix to represent the raisins:almonds:chocolate ratios. (Conveniently, these ratios already add to 20.)

$$\text{MixRatios} = \begin{bmatrix} 6 & 10 & 4 \\ 2 & 3 & 15 \\ 14 & 1 & 5 \end{bmatrix}$$

Use a vector to represent the cost of each item:

$$\text{IngredientCost} = \begin{bmatrix} 7.50 \\ 14.75 \\ 22.25 \end{bmatrix}$$

To find the cost of trail mix for each astronaut, we simply find the dot product between the mix ratios and the ingredient costs to get:

Pat = \$281.50
Kai = \$615.50
River = \$231.00

Exercise 1   Vector Matrix Multiplication

Multiply the array A with the vector v. Compute this by hand, and make sure to show your computations.

$$A = \begin{bmatrix} 1 & -2 & 3 & 5 \\ -4 & 2 & 7 & 1 \\ 3 & 3 & -9 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 2 \\ 2 \\ 6 \\ -1 \end{bmatrix}$$

Working Space

Answer on Page ??

Exercise 2   **Using Vector Matrix Multiplication**

A college professor offers three different methods of determining a student's final grade. In method A, the student's grade is 20% based on attendance, 50% homework, 15% midterm, and 15% final. This professor knows many students can learn the material without attending every class, so with method B the student's grade is 50% homework, 20% midterm, and 30% final. Last, the professor knows some students don't do the homework but still show they understand the material by doing well on the tests. With method C, a student's grade is 40% midterm and 60% final. The professor uses whatever method results in the highest grade to determine each student's final grade.

Suppose Suzy has attended 65% of classes, has an average homework grade of 30%, earned a 80% on the midterm, and earned a 75% on the final. What final grade will her professor post?

Working Space

Answer on Page **??**

1.1.1   Vector-Matrix Multiplication in Python

Most real-world problems use very large matrices, where it becomes impractical to do calculations by hand. As long as you understand how matrix-vector multiplication is performed, you will be equipped to use a computing language, like Python, to do the calculations for you.

Create a file called `vectors_matrices.py` and enter this code:

```
# import the python module that supports matrices
import numpy as np
```

```
# create an array
a = np.array([[5, 1, 3, -2],
              [1, -1, 8, 4],
              [6, 2, 1, 3]])

# create a vector
b = np.array([1, 2, 3, -8])

# calculate the dot product
print(a.dot(b))
```

When you run it, you should see:

```
[16, 6, 8]
```

1.2   Where to Learn More

Watch this video from Khan Academy about matrix-vector products: https://rb.gy/frga5

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Answers to Exercises

Answer to Exercise ?? (on page ??)

$$Av = (11 \ 37 \ -43)$$

Answer to Exercise ?? (on page ??)

The different methods can be represented in a matrix:

$$\begin{bmatrix} 0.20 & 0.50 & 0.15 & 0.15 \\ 0 & 0.50 & 0.20 & 0.30 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}$$

And Suzy's individual grades can be represented by a vector:

$$\begin{bmatrix} 65 \\ 30 \\ 80 \\ 75 \end{bmatrix}$$

To see the results of the three different methods, we can multiply the matrix and the vector:

$$\begin{bmatrix} 0.20 & 0.50 & 0.15 & 0.15 \\ 0 & 0.50 & 0.20 & 0.30 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix} \cdot \begin{bmatrix} 65 \\ 30 \\ 80 \\ 75 \end{bmatrix} = \begin{bmatrix} 0.2(65) + 0.5(30) + 0.15(80) + 0.15(75) \\ 0(65) + 0.5(30) + 0.2(80) + 0.3(75) \\ 0(65) + 0(30) + 0.4(80) + 0.6(75) \end{bmatrix}$$

Which yields:

$$\begin{bmatrix} 51.25 \\ 53.5 \\ 77 \end{bmatrix}$$

Since method C yields the highest grade, the professor will post a final grade of 77.