

# Projections

The word “projection” has two main meanings in everyday life. One is a projection as a forecast or estimate of something in the future based on the current situation; another is the result of shining a light to cast a shadow or show a movie. Both these definitions apply to mathematical projection.

Projections are used in many fields, such as science, math, engineering, and finance. Here are a few examples:

- Investors evaluate risk and return of a portfolio by projecting an asset’s return onto a reference portfolio.
- Astronomers analyze the motion of stellar objects by projecting the object’s true motion onto the plane of the sky.
- Robotics engineers use projections to prevent robots from running into obstacles by projecting the robot’s position onto the optimal path.

Mathematically, a projection describes the relationship of one vector to another in terms of direction and orthogonality. Given two vectors,  $\mathbf{u}$  and  $\mathbf{v}$ , the projection of  $\mathbf{u}$  onto  $\mathbf{v}$  separates  $\mathbf{u}$  into two components. The first component signifies how much  $\mathbf{u}$  lies in the direction of  $\mathbf{v}$ . The second signifies the component of  $\mathbf{u}$  that is orthogonal (perpendicular) to  $\mathbf{v}$ .

The figure ?? depicts a projection. The perpendicular line dropped from the end of  $\mathbf{u}$  is the *orthogonal* component. The portion of  $\mathbf{u}$  that lies in the direction of  $\mathbf{v}$  is the blue segment.

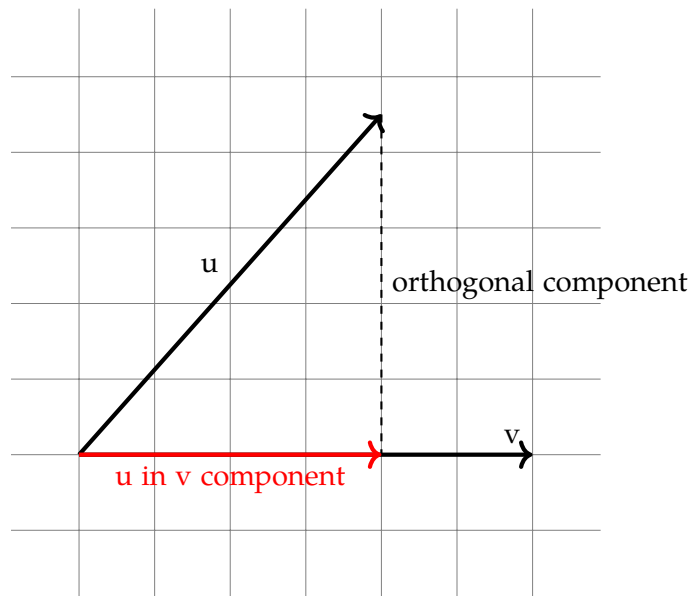


Figure 1.1: The projection of  $u$  onto  $v$

You can also think of a projection as the shadow cast by one vector onto the other by an overhead light. See Figure ??

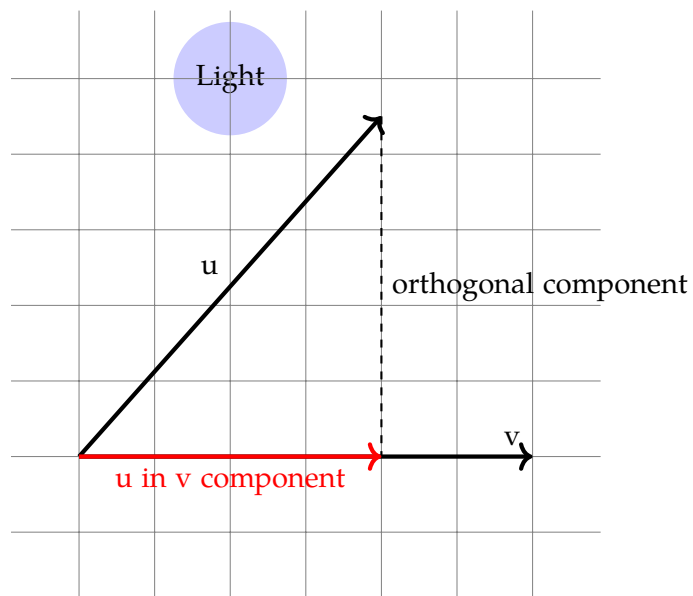


Figure 1.2: Projection of  $u$  onto  $v$  with a light included to simulate a shadow.

The projected vector can be in any direction and its length can extend beyond the vector onto which it is projecting. See Figure ??.

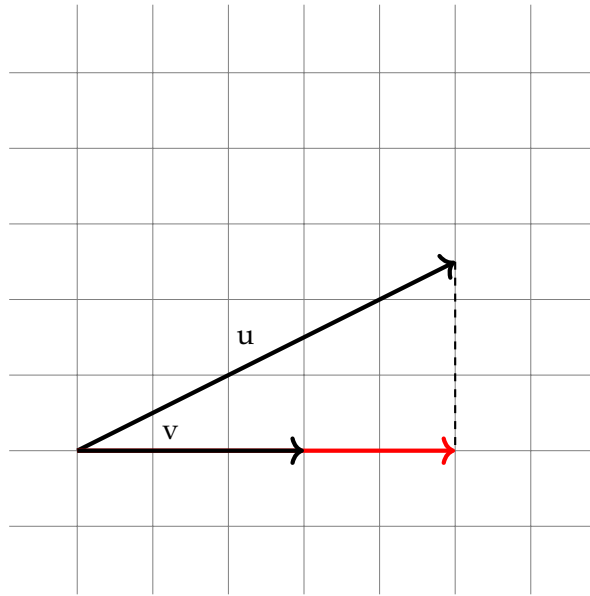


Figure 1.3: Projection vector extended beyond  $\mathbf{v}$ .

To calculate the projection of  $\mathbf{v}$  onto  $\mathbf{u}$ , use this formula:

$$\text{Projection of } \mathbf{u} \text{ onto } \mathbf{v}: \quad \text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$$

where  $\mathbf{u} \cdot \mathbf{v}$  is the dot product of  $\vec{\mathbf{u}}$  and  $\vec{\mathbf{v}}$ . Note that either form of the equation works, and the  $\mathbf{v}$  being multiplied by the dot product quotient cannot be cancelled because it is a vector, not a scalar.

Note that the denominator is the magnitude squared of vector  $\mathbf{v}$ .

$$\left( \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} \right)^2$$

You learned previously that this is the same as the dot product of a vector with itself.

$$\mathbf{v} \cdot \mathbf{v}$$

In the examples that follow, we will simplify to the dot product notation.

Let's look at a specific example:

$$\mathbf{u} = (1, 4, 6)$$

$$\mathbf{v} = (-2, 6, 2)$$

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}$$

$$\mathbf{proj}_v(\mathbf{u}) = \frac{(1, 4, 6) \cdot (-2, 6, 2)}{(-2, 6, 2) \cdot (-2, 6, 2)}(-2, 6, 2)$$

$$\mathbf{proj}_v(\mathbf{u}) = \left(\frac{34}{44}\right)(-2, 6, 2)$$

$$\mathbf{proj}_v(\mathbf{u}) = (-1.545, 4.64, 1.545)$$

As you work your way through this course, you will have a chance to apply the calculations you learn in this chapter to a variety of problems. Specifically, the next chapter shows how to transform a set of linearly independent vectors into a set of orthogonal ones. Projections are essential to that transformation.

## Exercise 1      Projections

Find the projection of **a** on **b** where:

$$\mathbf{a} = (1, 3)$$

$$\mathbf{b} = (-4, 6)$$

*Working Space*

*Answer on Page ??*

## 1.1 Projections in Python

Create a file called `projections.py` and enter this code:

```
import numpy as np

# define two vectors
a = np.array([1, 4, 6])
b = np.array([-2, 6, 2])

# use np.dot() to calculate the dot product
projection_a_on_b = (np.dot(a, b)/np.dot(b, b))*b

print("The projection of vector a on vector b is:", projection_a_on_b)
```

## 1.2 Where to Learn More

Watch this Introduction to Projections from Khan Academy from your digital resources:  
<https://rb.gy/yf0i3>

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*This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.*



# Answers to Exercises

## Answer to Exercise ?? (on page ??)

Compute dot product of **a** and **b**:

$$1 * -4 + 3 * 6 = -4 + 18 = 14$$

Compute the dot product of **b** and **b**

$$16 + 36 = 52$$

$$14/52 * (-4, 6) = (-1.076, 1.61)$$







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