Chapter 1

Subspaces

Recall that, in Chapter **??**, we established that all linear systems can be represented in matrix form as $A\vec{x} = \vec{b}$. In this chapter, we will explore the concept of subspaces, which are fundamental to understanding the structure of solutions to linear systems.

Quickly, let's review some vocabulary. The zero vector, denoted as $\vec{0}$, is the vector where all components are zero. It will be a column vector of appropriate size $n \times 1$.

If $A\vec{x} = \vec{0}$, where $\vec{b} = \vec{0}$, then the system is called **homogeneous**. If $A\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$, then the system is called **non-homogeneous**.

1.1 What is a Subspace?

A subspace V of $\mathbb{R}^n$ is a subset V of $\mathbb{R}^n$ satisfying the following properties:

- The zero vector $\vec{0}$ is in the subspace.
- If $\vec{u}$ and $\vec{v}$ are in the subspace, then their sum $\vec{u} + \vec{v}$ is also in the subspace.
- If $\vec{u}$ is in the subspace and c is a scalar, then the scalar multiple $c\vec{u}$ is also in the subspace.

In short, a subspace is exactly the set of all linear combinations of some collection of vectors. Additionally, every subpace is a span.

Subspace Span

If V is a subspace of $\mathbb{R}^n$, then there exists a set of vectors

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$$

such that

$$V = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$$

That is, every vector in V can be written as a linear combination of vectors in V.

Because V is closed under:

- vector addition, and
- scalar multiplication,

any linear combination of vectors in V must also lie in V.

So if $\vec{u}, \vec{v} \in V$ and $a, b \in \mathbb{R}$, then
$$a\vec{u} + b\vec{v} \in V$$

The converse is also true: any span is a subspace.

$$\text{span}(V) \text{ is a subspace of } \mathbb{R}^n$$

For example, vectors $\vec{v_1} = [1, 0]$ and $\vec{v_2} = [0, 1]$ span all of $\mathbb{R}^2$, since they can be scaled to fit all of $\mathbb{R}^2$.

- the span v_1 is a line through $\vec{0}$
- the span v_1, v_2 is a line or plane through $\vec{0}$
- the span of v_1, v_2, v_3 is a line or plane through $\vec{0}$, or all of $\mathbb{R}^3$

We need to review and reinforce some vocabulary that will come up often during this section. Let's look at Basis and Dimension.

1.2 Basis and Dimension

A basis of a subspace V is a set of linearly independent vectors that span V.

Basis

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \in \mathbb{R}^n$ is called a **basis** for $\mathbb{R}^n$ if

- the vectors span $\mathbb{R}^n$, and
- the vectors are linearly independent.

For example, a basis for $\mathbb{R}^3$ is given by the standard unit vectors:

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note that the span of V is all of $\mathbb{R}^3$, and the vectors in V are linearly independent.

For a subspace V, there are often many possible bases. For example, another basis for $\mathbb{R}^3$ is given by the vectors:

$$W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Put simply, a basis is a smallest possible set of vectors that can be used to build every vector in the space, with no redundancy. No vector in the basis can be written as a linear combination of the others.

A standard basis is a special type of basis that is often used for $\mathbb{R}^n$.

Standard Basis

The **standard basis** for $\mathbb{R}^n$ is the set of vectors

$$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\},$$

where $\vec{e}_i$ is the vector in $\mathbb{R}^n$ with a 1 in the i-th position and 0 in all other positions.

An example of this is the standard basis for $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Recall we may often simplify this to i hat and j hat notation:

$$\left\{\hat{i}, \hat{j}\right\} \qquad c_1\hat{i} + c_2\hat{j}$$

The standard basis consists of the vectors that point along the coordinate axes. Each standard basis vector measures exactly one coordinate.

The number of vectors in the basis is called the dimension of the subspace.

Dimension

The **dimension** of a vector space V is the number of vectors in any basis for V. If V has a basis consisting of k vectors, then we say that V has dimension k, and write

$$\dim(V) = k.$$

In particular, the dimension of $\mathbb{R}^n$ is n, since the standard basis contains exactly n vectors.

Properties of Dimension:

- A basis of a subspace $\mathbb{R}^n$ contains exactly n vectors.
- If W is a linear subspace of V, then $\dim(W) \leq \dim(V)$.
- If V is a finite-dimensional vector space and W is a linear subspace of V with $\dim(W) = \dim(V)$, then $W = V$.
- The space $\mathbb{R}^n$ has the standard basis $\{e_1, \ldots, e_n\}$, where e_i is the i-th column of the corresponding identity matrix. Therefore, $\mathbb{R}^n$ has dimension n.

1.3 Nullspace

We now examine a fundamental example of a subspace: the **nullspace** of a matrix. Recall that a subspace is a subset of $\mathbb{R}^n$ that is closed under vector addition and scalar multiplication and contains the zero vector.

The nullspace of a matrix A, denoted $\text{Null}(A)$, is the set of all vectors $\vec{x}$ such that

$$A\vec{x} = \vec{0}.$$

That is, the nullspace is precisely the solution set of the homogeneous system $A\vec{x} = \vec{0}$.

Nullspace

The nullspace of a matrix A is defined as

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}.$$

The $\vec{x}$, then represents all vectors that get flattened to origin $(\vec{0})$.

Because the equation $A\vec{x} = \vec{0}$ is homogeneous, its solution set always contains the zero vector. Moreover, if $\vec{x}_1$ and $\vec{x}_2$ are solutions, then any linear combination of the form

$$a\vec{x}_1 + b\vec{x}_2$$

is also a solution. For this reason, the nullspace of a matrix is always a subspace of $\mathbb{R}^n$.

1.3.1 Linear Combinations and Span

Recall that a **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is any vector of the form

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n,$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ are scalars.

The set of all possible linear combinations of a collection of vectors is called their **span**. If a subspace can be written as the span of one or more vectors, those vectors describe all possible directions within the subspace.

1.3.2 Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

To find the nullspace of A, we solve the homogeneous system $A\vec{x} = \vec{0}$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This corresponds to the system of equations

$$x_1 + 2x_2 = 0, \qquad 2x_1 + 4x_2 = 0.$$

Solving for x_1 in terms of x_2 gives

$$x_1 = -2x_2.$$

Notice that x_1 can be written in terms of x_2. This implies *linear dependence*, the fact that the vectors can be written as combinations of each other. Thus, every vector in the nullspace can be written in the form

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Therefore, the nullspace of A is

$$\text{Null}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

Geometrically, this nullspace is a line through the origin in $\mathbb{R}^2$, which is a one-dimensional subspace of $\mathbb{R}^2$. The nullspace contains all scalar multiples of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. This tells us that the solutions to the homogeneous system $A\vec{x} = \vec{0}$ form a line in $\mathbb{R}^2$. All vectors along this line produce the zero vector when substituted into the equation $A\vec{x} = \vec{0}$.

1.3.3 Example

Find a basis for the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 6 \end{bmatrix}.$$

First, find the RREF form of A. Notice immediately that $R_2 = 2R_1$ and $R_4 = 2R_3$ in A, so we will definitely have free rows. We get that

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our $\vec{x}$ vector has to be the same size as our column, so it must live in $\mathbb{R}^4$. This gives us $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

So finding $A\vec{x} = \vec{0}$ provides the systems of equations:

$$\begin{cases} x_1 - 3x_3 - 6x_4 = 0 \\ x_2 + x_3 + 3x_4 = 0 \end{cases}$$

We can rewrite this as:

$$\begin{cases} x_1 = 3x_3 + 6x_4 \\ x_2 = -x_3 - 3x_4 \end{cases} \implies x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ -3 \\ 0 \\ -1 \end{bmatrix}$$

So a basis for the nullspace is formed by

$$\text{Null}(A) = \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Exercise 1 Finding a Basis for a Nullspace

Working Space

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 3 & -1 & 1 & 7 & 0 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$$

When put in reduced row echelon form,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a Find a basis for the nullspace of A.

b Find a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ in the nullspace of A such that $x_1 = -3$, $x_3 = 1$, and $x_4 = 2$.

Answer on Page 13

Key Idea: Nullspace and Linear Independence

Solving for the *nullspace* of a matrix tells you whether a set of vectors is **linearly independent or linearly dependent**.

- If the nullspace contains *only* the zero vector, then the vectors are **linearly independent**.
- If the nullspace contains a *nonzero* vector, then the vectors are **linearly dependent**.

Practical test (via RREF):

- If every column has a *pivot*, the columns are linearly independent.
- If one or more columns lack a pivot, the columns are linearly dependent.

Equivalently, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the columns of A are linearly dependent.

1.4 Row Space

The row space of a matrix A, denoted $\text{Row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A. Each row vector can be viewed as a vector in $\mathbb{R}^n$, and the row space consists of all linear combinations of these row vectors.

Row Space

The row space of a matrix A is defined as

$$\text{Row}(A) = \text{span}\{\text{row}_1, \text{row}_2, \ldots, \text{row}_m\},$$

where row_i represents the i-th row of the matrix A. The row space consists of all directions in $\mathbb{R}^n$ that can be built from the rows of the matrix. Note that we are observing the matrix before it is transformed into RREF or altered in any way.

Another way to think about it is that the row space represents all possible linear combinations of the equations represented by the rows of the matrix A. Each row is "tested" on by $\vec{x}$ to produce a component of the output vector $A\vec{x}$. Equivalently, the row space is the set of all vectors that can be formed by adding and scaling the rows of A.

When we compute:

$$A\vec{x}$$

each row of A gets dotted with $\vec{x}$ to produce a component of the output vector.

If each of the rows of A are denoted as $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$, then $A\vec{x}$,

$$\begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

The row space, then, is a subspace of $\mathbb{R}^n$ that captures all possible linear combinations of the rows of A. The row space consists of all possible tests on $\vec{x}$ that can be performed by the rows of A.

$$A = \begin{bmatrix} —\vec{r}_1— \\ —\vec{r}_2— \\ \vdots \\ —\vec{r}_m— \end{bmatrix} \quad \text{with } \vec{r}_i \in \mathbb{R}^n$$

Why does the row space live in $\mathbb{R}^n$?

- $\vec{x}$ has n components and lives in $\mathbb{R}^n$, and
- each row $\vec{r}_i$ has n components and lives in $\mathbb{R}^n$

1.4.1 Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

The row vectors of A are

$$\vec{r}_1 = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}, \quad \vec{r}_2 = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$$

dotting them with an $\vec{x}$ of appropriate size gives

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \end{bmatrix}$$

The row space of A is the span of its row vectors:

$$\text{Row}(A) = \text{span}\left\{ \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \right\}$$

1.5 Column Space

The column space of a matrix A, denoted $\text{Col}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A. Each column of A can be viewed as a vector in $\mathbb{R}^m$, and the column space consists of all linear combinations of these column vectors.

Column Space

The **column space** of a matrix A is defined as

$$\text{Col}(A) = \text{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\},$$

where $\vec{c}_i$ represents the i-th column of the matrix A. The column space consists of all vectors in $\mathbb{R}^m$ that can be formed as linear combinations of the columns of A.

Equivalently, the column space can be described as the set of all possible outputs of the matrix:

$$\text{Col}(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}.$$

The columns of the *original* matrix A that correspond to pivot columns in $\text{RREF}(A)$ span the column space of A. We will discuss this idea further in a future chapter, but it corresponds to the fact that the pivot columns are linearly independent and form a basis for the column space.

1.5.1 Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{bmatrix}.$$

After applying elementary row operations, we find that the RREF of A is

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there is a free row, there is a *dependent* row in the original matrix. The pivot columns are columns 1, 2, and 4. Thus, a basis for the column space of A is given by the original columns 1, 2, and 4:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 7 \\ 5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 9 \\ 1 \\ 8 \end{bmatrix}.$$

Thus the subspace spans $\mathbb{R}^4$ and consists of all linear combinations of these three vectors.

Exercise 2 Nullspace, Rowspace, and Columnspace

Working Space

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

a Compute the reduced row echelon form of A.

b Identify the pivot columns and free columns of A. What does this tell you about the solutions?

c Find a basis for the **nullspace** $N(A)$. What is the dimension of $N(A)$?

d Find a basis for the **row space** $R(A)$. What is the dimension of $R(A)$?

e Find a basis for the **column space** $C(A)$. State clearly where the vectors for the basis come from.

Answer on Page 14

1.6 Summary

In this chapter, we have established a few different common subspaces: the Nullspace, Rowspace, and Columnspace. Each of these subspaces captures different structural information about a matrix and the system of equations it represents. The nullspace describes all input vectors that are sent to the zero vector, the column space contains all possible outputs of the matrix, and the row space encodes the independent constraints imposed by the system.

We have now set the stage for a new way of viewing matrices. Rather than thinking of a matrix solely as a collection of numbers or equations, we can begin to interpret a matrix as a transformation – a rule that takes input vectors and maps them to output vectors.

In the next few chapters, we will build on our understanding of subspaces to study how matrices act on vectors and reshape space, providing a unifying viewpoint for many of the ideas introduced so far. We will look at the geometric implications of matrices and how they *transform* space. This is the basis of many computer graphic programs, like calculating lengths of shadows in VR games or simulations!

Before ending this chapter, take a look at this graphic. It shows the relationship between the subspaces we talked about, plus a new one: Null(A^T), the transpose of the nullspace. If matrix A has size $m \times n$, then A^T has size $n \times m$. Note which subspaces live in $\mathbb{R}^n$ and which live in $\mathbb{R}^m$.

$$
\begin{tikzcd}
\begin{matrix} N(A) \\ R(A) \end{matrix} \; \mathbb{R}^n \arrow[r, bend left, "\underset{m\times n}{A}"] & \mathbb{R}^m \; \begin{matrix} N\left(A^\top\right) \\ C(A) \end{matrix} \arrow[l, bend left, "\underset{n\times m}{A^\top}"]
\end{tikzcd}
$$

$$N(A) \perp R(A) \qquad\qquad N\left(A^\top\right) \perp C(A)$$
$$N(A) \cdot R(A) = 0 \qquad\qquad N\left(A^\top\right) \cdot C(A) = 0$$

Instead of asking what vectors solve $A\vec{x} = \vec{0}$, we will begin asking:

What does the matrix A do to an arbitrary vector?

The big take away: A matrix of size $m \times n$ can be viewed as a rule that takes vectors in $\mathbb{R}^n$ as inputs and produces vectors in $\mathbb{R}^m$as outputs.

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 7)

a Solving $A\vec{x} = \vec{0}$ gives us two equations:

$$\begin{cases} x_1 + 2x_4 - x_5 = 0 \\ x_2 - x_3 - x_4 - 3x_5 = 0 \end{cases} \implies \begin{cases} x_1 = -2x_4 + x_5 \\ x_2 = x_3 + x_4 + 3x_5 \end{cases}$$

So we establish x_3, x_4, and x_5 as bound or fixed variables. So, every vector in the nullspace has the form

$$\vec{x} = \begin{bmatrix} -2x_4 + x_4 \\ x_3 + x_4 + 3x_4 \\ x_3 \\ x_4 \\ x_4 \end{bmatrix}$$

or, equivalently

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So a basis can be formed by the set

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b We are given a vector $\vec{x} = \begin{bmatrix} -3 \\ x_2 \\ 1 \\ 2 \\ x_5 \end{bmatrix}$.

Recall that we have $x_1 = -2x_4 + x_5$, so inputting our knowns we can say $-3 = -2(2) + x_5 \implies = x_5 = 1$.

We can then solve $x_2 = x_3 + x_4 + 3x_5$ for x_2:

$$x_2 = 1 + 2 + 3(1) = 6$$

These two variables give us the $\vec{x}$

$$\vec{x} = \begin{bmatrix} -3 \\ 6 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Answer to Exercise 2 (on page 11)

a
$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b Columns 1, 2, and 3 are pivot columns (containing a pivot). Column 4 is a non-pivot or free column. This tells you that there are 3 fixed variables, and 1 free one. The nullspace will be 1 dimensional.

c We have the system of equations:

$$\begin{aligned} x_1 + x_4 &= 0, \\ x_2 + x_4 &= 0, \\ x_3 &= 0. \end{aligned}$$

Letting x_4 be free, $\vec{x} = x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. The $N(A) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ with a dimension of 1.

d The row space is the subspace spanned by the rows of A. A convenient basis for the row space is given by the nonzero rows of rref(A). Thus a basis is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \right\}.$$

These three rows are linearly independent, and they span the row space because row reduction did not create any new row space, just simplified the set. The rowspace has rowspace$(A) = 3$.

e For the column space, we must take columns from the *original* matrix A. The usual rule is: pivot columns of the original A form a basis for $C(A)$. Since the pivot columns are $1,\ 2,\ 3$, a basis is the set of the first three columns of A

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

and the column space is the span of each original pivot column:

$$C(A) = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$$

Index