

# Optimization

Optimization is a branch of mathematics that involves finding the best solution from all feasible solutions. In the field of operations research, optimization plays a crucial role. Whether it is minimizing costs, maximizing profits, or reducing the time taken to perform a task, optimization techniques are employed to make decisions effectively and efficiently.

In this chapter, the primary goal is to find the maximum or minimum value of a function, often referred to as the objective function, by finding the point on the graph where the derivative is equal to zero. This is usually referred to as the critical point!

At a critical point where the derivative is zero, the graph of the function has a horizontal tangent line. At such a point, the function may be reaching a peak or a valley.

Not every critical point corresponds to a maximum or minimum. Some critical points may instead be points of inflection. This is when the second derivative of the point is also equal to zero! You will see this in a few examples!

## 1.1 Extrema

There are two types of extrema: relative extrema and absolute extrema.

A relative maximum or minimum occurs when the function reaches a peak or valley compared to nearby values. This does not guarantee that the value is the highest or lowest value of the function overall.

An absolute maximum or minimum is the greatest or least value that the function takes on a given interval.

An absolute maximum or minimum can occur either at a critical point or at an endpoint of the interval.

For this reason, when solving optimization problems on a restricted domain, it is important to test both critical points and endpoints.

## 1.2 The Second Derivative Test

Once a critical point has been found, we often want to determine whether it corresponds to a maximum or a minimum. One method for doing this is the Second Derivative Test.

If a function has a critical point at  $x = c$ , then:

- if the second derivative is positive at  $x = c$ , the function is concave up and the point is a relative minimum;
- if the second derivative is negative at  $x = c$ , the function is concave down and the point is a relative maximum;
- if the second derivative is zero at  $x = c$ , the test is inconclusive.

If the second derivative is always positive or always negative on an interval, then any critical point in that interval must be an absolute minimum or absolute maximum, respectively.

## 1.3 Solving Optimization Problems

Optimization problems apply the ideas of derivatives, critical points, and extrema to real situations. Although the context of each problem may be different, the method used to solve them follows the same general pattern.

## 1.4 General Strategy for Solving Optimization Problems

Most calculus optimization problems follow these steps:

1. Identify the quantity to be optimized.
2. Write an equation for the objective function. Sometimes, you might be given an equation already. In that case, you will skip this step. For word problems, you will typically be required to write the equations yourself.
3. Use the given constraints to rewrite the function using one variable.
4. Determine the appropriate domain.
5. Find the derivative of the function.
6. Find critical points by setting the derivative equal to zero.
7. Interpret the result in the context of the problem.

When solving optimization problems, some common mistakes include forgetting to define variables, not using the constraints of the problem appropriately, misinterpreting or failing to interpret the final answer, and giving answers without appropriate units. For word problems, be very vigilant about how your answer relates back to the question given! It serves as a small check for accuracy.

### 1.4.1 Example

Consider the function

$$f(x) = x^2 - 4x + 1.$$

We want to find the maximum or minimum value of this function.

First, take the derivative of the function:

$$f'(x) = 2x - 4.$$

Next, set the derivative equal to zero to find the critical point:

$$2x - 4 = 0.$$

Solving for  $x$  gives

$$x = 2.$$

Now determine whether this critical point corresponds to a maximum or a minimum by using the second derivative.

The second derivative is

$$f''(x) = 2.$$

Because the second derivative is positive, the function is concave up, and the critical point corresponds to a minimum.

To find the minimum value of the function, substitute  $x = 2$  back into the original function:

$$f(2) = 2^2 - 4(2) + 1 = -3.$$

Therefore, the function has a minimum value of  $-3$  at  $x = 2$ .

**Exercise 1**

Consider the function

$$f(x) = x^2 - 6x + 5.$$

*Working Space*

Find the maximum or minimum value of the function using calculus.

*Answer on Page 9*

Now, let's look at a word example:

**Example**

The cost, in dollars, of producing  $x$  units of a product is given by the function

$$C(x) = 2x^2 - 24x + 100.$$

Find the number of units that should be produced in order to minimize the cost, and determine the minimum cost.

To minimize the cost, take the derivative of the cost function:

$$C'(x) = 4x - 24.$$

Set the derivative equal to zero to find the critical point:

$$4x - 24 = 0.$$

Solving for  $x$  gives

$$x = 6.$$

Next, take the second derivative:

$$C''(x) = 4.$$

Since the second derivative is positive, the cost function is concave up, and the critical point corresponds to a minimum.

Substitute  $x = 6$  back into the original cost function:

$$C(6) = 2(6)^2 - 24(6) + 100 = 28.$$

Therefore, the cost is minimized when 6 units are produced, and the minimum cost is \$28.

### Exercise 2

A rectangular enclosure is to be built using 60 units of fencing. Three sides of the enclosure require fencing, while the fourth side is along a wall and does not require fencing.

Let  $x$  represent the length of the side parallel to the wall and  $y$  represent the width of the enclosure.

Find the dimensions of the enclosure that minimize the amount of fencing used.

*Working Space*

*Answer on Page 9*

## 1.5 Helpful Table for Optimization Problems

Condition at a point $x = c$	What it tells you	What kind of point it could be
$f'(c) \neq 0$	The graph has a non-horizontal tangent	Not a maximum or minimum
$f'(c) = 0$	The graph has a horizontal tangent	Possible maximum, minimum, or neither
$f'(c)$ does not exist	The graph may have a corner, cusp, or vertical tangent	Possible maximum or minimum so run second derivative test to check!
$f'(c) = 0$ and $f''(c) > 0$	Graph is concave up at $c$	Local minimum
$f'(c) = 0$ and $f''(c) < 0$	Graph is concave down at $c$	Local maximum
$f'(c) = 0$ and $f''(c) = 0$	Second derivative test fails	Could be max, min, or neither

## 1.6 Using Python to Visualize Your Optimization Problem

You may be familiar with the use of Python to find the derivative of functions. We are going to use Python to supplement your knowledge of Optimization problems! Below is a script that can be used to visualize what your world problems may be asking you to do! This script will help you with finding the first derivative, performing the second derivative test, and graphs of your equations showing these points.

```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt

# 1) Define the variable
x = sp.symbols('x')

# 2) Define the objective function (STUDENTS EDIT THIS)
f = 20*x - x**2

# 3) Derivatives
f_prime = sp.diff(f, x)
f_double_prime = sp.diff(f_prime, x)

# 4) Solve  $f'(x) = 0$ 
critical_point = sp.solve(f_prime, x)[0]

# 5) Classify using second derivative
second_derivative_value = f_double_prime.subs(x, critical_point)

if second_derivative_value > 0:
    classification = "minimum"
elif second_derivative_value < 0:
    classification = "maximum"
else:
    classification = "inconclusive"

print("f(x) =", f)
print("f'(x) =", f_prime)
print("f''(x) =", f_double_prime)
print("Critical point:", critical_point)
print("Classification:", classification)

# 6) Plot
f_num = sp.lambdify(x, f, "numpy")

X = np.linspace(0, 20, 400)
Y = f_num(X)

xc = float(critical_point)
yc = float(f_num(xc))

plt.plot(X, Y)
```

```
plt.scatter([xc], [yc])
plt.title("Optimization")
plt.xlabel("x")
plt.ylabel("f(x)")
plt.grid(True)

plt.annotate(
    classification,
    (xc, yc),
    xytext=(xc + 1, yc),
    arrowprops=dict(arrowstyle="->")
)

plt.show()
```

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*This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.*



# Answers to Exercises

## Answer to Exercise 1 (on page 4)

First, take the derivative of the function:

$$f'(x) = 2x - 6.$$

Set the derivative equal to zero to find the critical point:

$$2x - 6 = 0.$$

Solving this equation gives

$$x = 3.$$

Next, take the second derivative:

$$f''(x) = 2.$$

Since the second derivative is positive, the function is concave up, and the critical point corresponds to a minimum.

Finally, substitute  $x = 3$  back into the original function:

$$f(3) = 3^2 - 6(3) + 5 = -4.$$

Therefore, the function has a minimum value of  $-4$  at  $x = 3$ .

## Answer to Exercise 2 (on page 5)

Because only three sides require fencing, the total amount of fencing is given by

$$x + 2y = 60.$$

Solving this equation for  $x$  gives

$$x = 60 - 2y.$$

The area of the enclosure is

$$A = xy.$$

Substituting for  $x$  yields

$$A(y) = y(60 - 2y) = 60y - 2y^2.$$

Take the derivative:

$$A'(y) = 60 - 4y.$$

Set the derivative equal to zero:

$$60 - 4y = 0.$$

Solving for  $y$  gives

$$y = 15.$$

The second derivative is

$$A''(y) = -4.$$

Since the second derivative is negative, this critical point corresponds to a maximum area.

Substituting  $y = 15$  into the constraint gives

$$x = 60 - 2(15) = 30.$$

Therefore, the enclosure has dimensions 30 units by 15 units.



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