Chapter 1

# Rules for Finding Derivatives

Derivatives play a key role in calculus, providing us with a means of calculating rates of change and the slopes of curves. In this chapter, we present some common rules used to calculate derivatives.

## 1.1 Constant Rule

The derivative of a constant is zero. If $c$ is a constant and $x$ is a variable, then:

$$\frac{d}{dx}c = 0 \tag{1.1}$$

## 1.2 Power Rule

For any real number $n$, the derivative of $x^n$ is:

$$\frac{d}{dx}x^n = nx^{n-1} \tag{1.2}$$

## 1.3 Product Rule

The derivative of the product of two functions is:

$$\frac{d}{dx}(fg) = f'g + fg' \tag{1.3}$$

where $f'$ and $g'$ denote the derivatives of $f$ and $g$, respectively.

## 1.4 Quotient Rule

The derivative of the quotient of two functions is:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \tag{1.4}$$

## 1.5 Chain Rule

The derivative of a composition of functions is:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \tag{1.5}$$

## 1.6 Practice

### *Exercise 1*

If $f$ is the function given, find $f'$.

1. $f(x) = x \sin x$
2. $f(x) = (x^3 - \cos x)^5$
3. $f(x) = \sin^3 x$

*Working Space*

*Answer on Page 7*

### *Exercise 2*

Let $f(x) = 7x - 3 + \ln x$. Find $f'(x)$ and $f'(1)$

*Working Space*

*Answer on Page 7*

## Exercise 3

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] The position of a particle in the $xy$-plane is given by the parametric equations $x(t) = t^3 - 3t^2$ and $y(t) = 12t - 3t^2$. State a coordinate point $(x, y)$ at which the particle is at rest.

*Working Space*

*Answer on Page 7*

## Exercise 4

Let $f(x) = \sqrt{x^2 - 4}$ and $g(x) = 3x - 2$. Find the derivative of $f(g(x))$ at $x = 3$.

*Working Space*

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## Exercise 5

The particle's position on the $x$-axis is given by $x(t) = (t - a)(t - b)$, where $a$ and $b$ are constants and $a \neq b$. At what time(s) is the particle at rest?

*Working Space*

*Answer on Page 8*

### Exercise 6

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] Let $f(x) = \frac{x}{x+2}$. At what values of $x$ does $f$ have the property that the line tangent to $f$ has a slope of $\frac{1}{2}$?

*Working Space*

*Answer on Page 8*

### Exercise 7

For $t \geq 0$, the position of a particle moving along the x-axis is given by $x(t) = \sin t - \cos t$. (a) When does the velocity first equal $0$? (b) What is the acceleration at the time when the velocity first equals $0$?

*Working Space*

*Answer on Page 8*

### Exercise 8

The graph of $y = e^(\tan x) - 2$ crosses the x-axis at one point on the interval $[0, 1]$. What is the slope of the graph at this point?

*Working Space*

*Answer on Page 9*

### Exercise 9

The function $f$ is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.
(a) Find $f'(x)$.
(b) Write an equation for the line tangent to the graph at $x = -3$.

Working Space

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### Exercise 10

For $0 \leq t \leq 12$, a particle moves along the $x$-axis. The velocity of the particle at a time $t$ is given by $v(t) = \cos \frac{\pi}{6} t$. What is the acceleration of the particle at time $t = 4$?

Working Space

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### Exercise 11

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let $f$ and $g$ be the functions given by $f(x) = e^x$ and $g(x) = x^4$. On what intervals is the rate of change of $f(x)$ greater than the rate of change of $g(x)$?

Working Space

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## 1.7 Conclusion

These rules form the basis for calculating derivatives in calculus. Many more complex rules and techniques are built upon these fundamental rules.

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Appendix A

# Answers to Exercises

## Answer to Exercise 1 (on page 2)

1. $\frac{dy}{dx} = \frac{d}{dx}[x \sin x] = x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x = x(-\cos x) + \sin x(1) = \sin x - x\cos x$
2. By the chain rule, $f'(x) = 5(x^3 - \cos x)^4 \cdot \frac{d}{dx}(x^3 - \cos x) = 5(x^3 - \cos x)^4 \cdot (3x^2 + \sin x)$
3. By the chain rule, $f'(x) = \frac{d}{d(\sin x)}[\sin^3 x] \times \frac{d}{dx}\sin x = 3\sin^2 x \cdot \cos x$

## Answer to Exercise 2 (on page 2)

$f'(x) = \frac{d}{dx}(7x) - \frac{d}{dx}(3) + \frac{d}{dx}(\ln x) = 7 - 0 + \frac{1}{x} = 7 - \frac{1}{x}$ and $f'(1) = 7 - \frac{1}{1} = 6$

## Answer to Exercise 3 (on page 3)

The particle is at rest when $x'(t) = y'(t) = 0$. First, we find each of the derivatives:

$$x'(t) = 3t^2 - 6t$$
$$y'(t) = 12 - 6t$$

We can solve $y' = 0$ for $t$ and find that the $y$-velocity is $0$ when $t = 2$. Substituting $t = 2$ into our expression for $x'$, we find $x'(2) = 3(2)^2 - 6(2) = 0$. Therefore, the particle is at rest when $t = 0$. To find the $xy$-coordinate, we substitute $t = 2$ into $x(t)$ and $y(t)$:

$$x(2) = (2)^3 - 3(2)^2 = 8 - 12 = -4$$
$$y(2) = 12(2) - 6(2) = 24 - 12 = 12$$

Therefore, the particle is at rest when it is located at $(-4, 12)$.

## Answer to Exercise 4 (on page 3)

$f(g(x)) = \sqrt{(3x-2)^2 - 4} = \sqrt{9x^2 - 12x}$ and $\frac{d}{dx}f(g(x)) = \frac{18x-12}{2\sqrt{9x^2-12x}}$. Substituting $x = 3$, we find $f'(g(x)) = \frac{18(3)-12}{2\sqrt{9(3)^2-12(3)}} = \frac{42}{2\sqrt{45}} = \frac{21}{3\sqrt{5}} = \frac{7}{\sqrt{5}}$

## Answer to Exercise 5 (on page 3)

First, recall that the velocity of a particle is the derivative of its position function. Therefore, $v(t) = x'(t) = \frac{d}{dt}[(t-a)(t-b)]$. Applying the Product Rule for derivatives, we see that $v(t) = (t-a)(1) + (t-b)(1) = 2t - a - b$. To find the time(s) when the particle is at rest, we set $v(t) = 0$ and solve for $t$.

$$0 = 2t - a - b$$
$$2t = a + b$$
$$t = \frac{a+b}{2}$$

## Answer to Exercise 6 (on page 4)

The question is asking when the derivative of $f$ is $\frac{1}{2}$. We will take the derivative and set it equal to $\frac{1}{2}$.

$$f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{2}{(x+2)^2}$$
$$\frac{2}{(x+2)^2} = \frac{1}{2}$$
$$4 = (x+2)^2$$
$$\pm 2 = x + 2$$
$$x = 2 - 2 = 0 \text{ and } x = -2 - 2 = -4$$

## Answer to Exercise 7 (on page 4)

(a) Let $t_0$ be the time at which the particle is first at rest. The velocity of the particle is given by $v(t) = x'(t) = \cos t + \sin t$. Setting $v(t) = 0$, we find:

$$\cos t = -\sin t$$

which is true for $t = \frac{3\pi + 4n}{4}$, where $n$ is an integer. Therefore, the first time the velocity is $0$ is $t_0 = \frac{3\pi}{4}$.
(b) To find the acceleration at $t = \frac{3\pi}{4}$, we take the derivative of the velocity function to yield the acceleration function.

$$a(t) = v'(t) = -\sin t + \cos t$$

. Substituting $t = \frac{3\pi}{4}$, we find the acceleration is $-\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

## Answer to Exercise 8 (on page 4)

First, we find the $x$ such that $y = 0$

$$0 = e^{\tan x} - 2$$

$$2 = e^{\tan x}$$

$$\ln 2 = \tan x$$

$$x = \arctan(\ln 2) = \arctan 0.693 \approx 0.606$$

Then, we find the slope of the function at $x = 0.606$ by finding $y'(0.606)$

$$y' = e^{\tan x}(\sec x)^2 = \frac{e^{\tan x}}{(\cos x)^2}$$

$$y'(0.606) = \frac{e^{\tan 0.606}}{(\cos 0.606)^2} = 2.961$$

## Answer to Exercise 9 (on page 5)

(a) Apply the chain rule to find $f'(x)$

$$f'(x) = \frac{1}{2\sqrt{25 - x^2}} \cdot (-2x) = \frac{-x}{\sqrt{25 - x^2}}$$

.

(b) First, substitute $x = -3$ into $f'(x)$

$$f'(-3) = \frac{-(-3)}{\sqrt{25 - (-3)^2}} = \frac{3}{\sqrt{16}} = \frac{3}{4}$$

This is the slope of the line. To complete an equation for the tangent line, we need a point. We know the tangent line touches $f(x)$ at $x = -3$, so the tangent line must pass through the point $(-3, f(-3))$.

$$f(-3) = \sqrt{25 - (-3)^2} = 4$$

We use $m = \frac{3}{4}$ and the coordinate point $(x_1, y_1) = (-3, 16)$ to complete the equation $y - y_1 = m(x - x_1)$

$$y - 16 = \frac{3}{4}(x + 3)$$

## Answer to Exercise 10 (on page 5)

$$a(t) = v'(t) = -\frac{\pi}{6}\sin\frac{\pi}{6}t$$

$$a(4) = -\frac{\pi}{6}\sin\frac{2\pi}{3} = -\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{12}$$

## Answer to Exercise 11 (on page 5)

Recall that the rate of change of a function is given by the derivative of that function. Therefore, we are looking for the interval(s) where $f'(x) > g'(x)$. First, we find each derivative:

$$f'(x) = e^x$$

$$g'(x) = 4x^3$$

We are looking for $x$-values, such that $e^x > 4x^3$. This inequality can be restated as $e^x - 4x^3 > 0$. Using a calculator, you should find that $e^x - 4x^3 = 0$ when $x \approx 0.831$ and $x \approx 7.384$. We will check values on either side of and in the interval $x \in (0.831, 7.384)$ to determine the sign value of $e^x - 4x^3$. We know that when $x = 0$, $e^x - 4x^3 > 0$, when $x = 5$, $e^x - 4x^3 < 0$, and when $x = 10$, $e^x - 4x^3 > 0$. Therefore, $f'(x)$ is greater than $g'(x)$ on the open intervals $x \in (-\infty, 0.831) \cup (7.384, \infty)$.

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