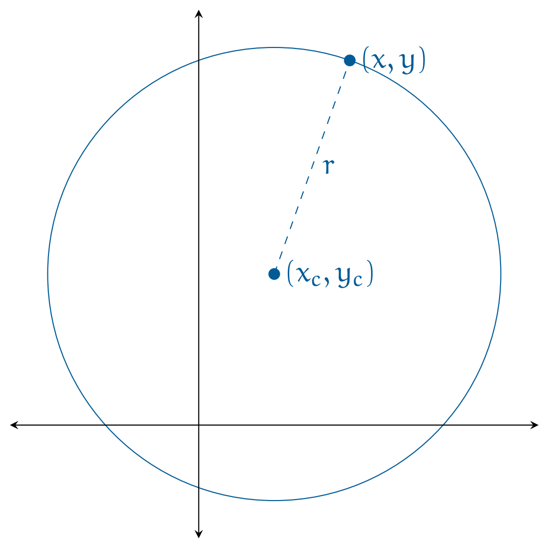
Chapter 1

Circles

A circle is the set of points (x, y) that are a particular distance r from a particular point (x_c, y_c). We say that r is the *radius* and (x_c, y_c) is the *center*.



Area and Radius

If the radius of a circle is r, the area of its interior (a) is given by

$$a = \pi r^2$$

Exercise 1 Area of a Circle

The paint you have says "One liter covers 6 square meters."

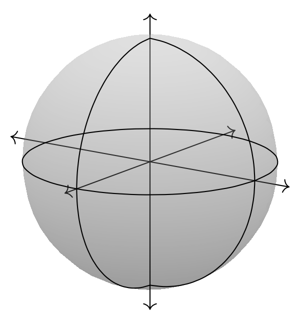
You are painting the top of a circular table with a radius of 3 meters.

How much paint will you need?

Working Space

Answer on Page 13

Note that a circle lives in a particular plane. In 3D, the points (x, y, z) that are a particular distance r from a particular point (x_c, y_c, z_c) are a sphere:



The distance all the way across the middle of a circle (or a sphere) is its *diameter*. The diameter is always twice the radius.

For the rest of the chapter, we will be talking about circles, points, and lines *in a plane*.

Circumference and Diameter

The circumference (c) of a circle is the distance around the circle. If the diameter is d,

$$c = \pi d$$

Exercise 2 Circumference

Using a tape measure, you figure out that the circumference of a tree in your yard is 64 cm.

Assuming the trunk is basically circular, what is its diameter?

Working Space

Answer on Page 13

Exercise 3 Splitting a Pie

A pie has a radius of 13 cm. 7 friends all want equal sized sectors. You have a tape measure to assist you.

How many centimeters will each outer crust be?

Working Space

Answer on Page 13

1.1 Arc Length

Previously, you learned that angles can be measured in degrees and radians. A circle is 360^o (see figure 1.1).

This means a circle is also 2π radians:

$$360^\circ \cdot \frac{\pi}{180^\circ} = 2\pi$$

You may be wondering: why is it that there are π radians in a 180° angle? A radian is

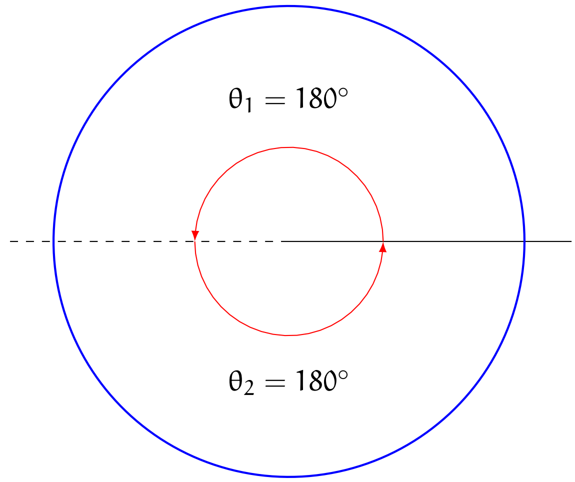


Figure 1.1: The total internal angle of a circle is $\theta_1 + \theta_2 = 360°$

defined such that one radian is the angle at the center of a circle which defines an arc of the circumference equal to the radius of the circle (see figure 1.2).

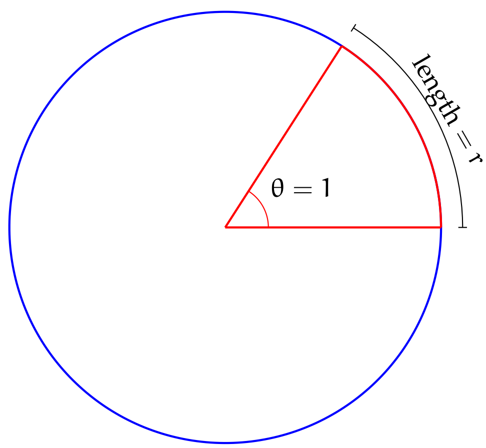
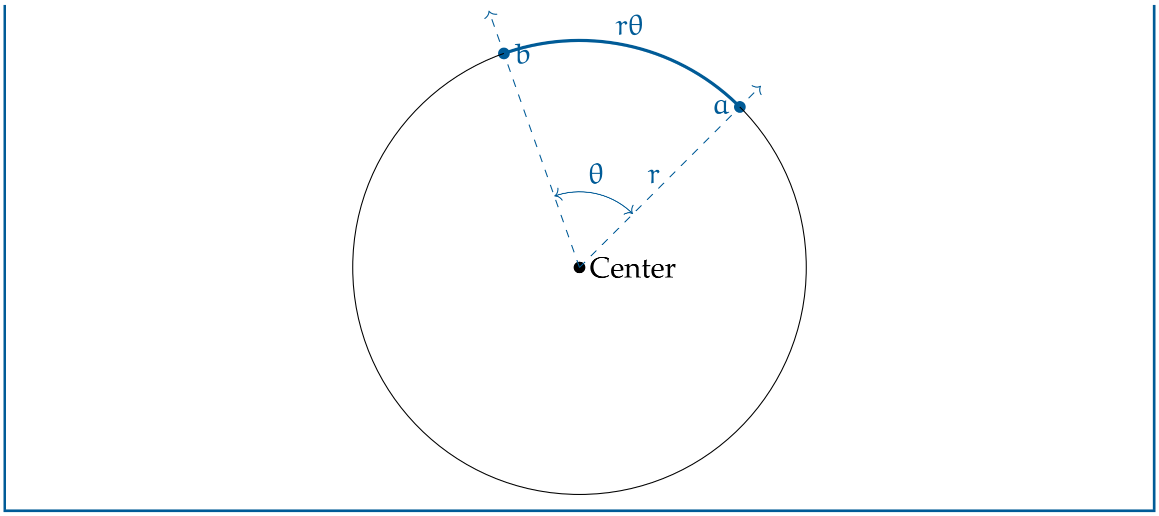


Figure 1.2: When the center angle is 1 radian, the length of the arc is equal to the radius of the circle

This makes it very straightforward to find the lengths of arcs if we know the center angle in radians. The arc length is just $r\theta$, where θ is the center angle in radians.

Length of an Arc

If you have two points a and b on a circle, the ray from the center through a and the ray from the center through b form an angle. If θ is the angle in radians and r is the radius of the circle, the distance from a to b on the circle is $r\theta$.



This shows us why π radians $= 180°$. Recall the formula for circumference: $c = \pi d$, where d is the diameter of the circle. Since the diameter is twice the radius, we can also say that $c = 2\pi r$, where r is the radius of the circle. The circumference of the circle is just an arc where the central angle is the entirety of the circle. Since we know that the length of an arc is $r\theta$, we can find the total internal angle of a circle in radians:

$$2\pi r = r\theta$$

$$\theta = 2\pi$$

This is how we know $360° = 2\pi$ radians.

Exercise 4 Angle of Rotation

A car tire has a radius of approximately 25 centimeters. If you roll your car forward 10 cm, by how many radians has your tire rotated?

Working Space

Answer on Page 14

Exercise 5 Arc Length Ranking

Rank the following arc lengths from longest to shortest (the central angle that defines the arc and the radius of the circle are provided:

1. central angle of $\frac{\pi}{4}$ and a radius of 2 cm
2. central angle of π and a radius of 1 cm
3. central angle of $\frac{\pi}{10}$ and a radius of 5 cm
4. central angle of $\frac{3\pi}{4}$ and a radius of 3 cm

Working Space

Answer on Page 14

Exercise 6 Arc Length

Working Space

You have been asked to find the radius of a very large cylindrical tank. You have a tape measure, but it is only 15 meters long and doesn't reach all the way around the tank.

However, you have a compass. So you stick one end of the tape measure to the side of the tank and measure the orientation of the wall at that point. You then walk the 15 meters and measure the orientation of the wall there.

You find that 15 meters represents 72 degrees of arc.

What is the radius of the tank in meters?

Answer on Page 14

1.2 Sector Area

We already know the area of a circle is given by $A = \pi r^2$. What about a piece of a circle? Let's start with a straightforward example:

Example: A pizza with a radius of 15 cm is divided into 6 equal pieces. What is the area of each piece?

Solution: First, we find the area of the entire pizza:

$$A = \pi r^2$$

$$A = \pi(18 \text{ cm})^2$$

$$A = 324\pi \text{ cm}^2 \approx 1018 \text{ cm}^2$$

Then, we divide by 6, since the pieces of equal sizes:

$$A_{\text{piece}} = \frac{A}{6} = \frac{324\pi\ \text{cm}^2}{6} = 54\pi\ \text{cm}^2$$

Let's use this to write a general formula for the area of a sector defined by a central angle θ (see figure 1.3). We know that when a circle is divided into 6 equal sectors, the central angle of each sector is $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$. Additionally, we know the area of each sector is the total area divided by 6: $A_{\text{sector}} = \frac{\pi r^2}{6} = \frac{\pi}{6} r^2 = \frac{\theta}{2} r^2$.

Area of a sector

For a sector whose corner is at the center of a circle, the area is given by $A_{\text{sector}} = \frac{\theta}{2} r^2$, where θ is the central angle and r is the radius.

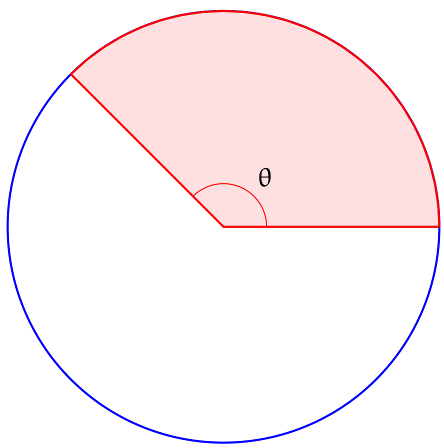


Figure 1.3: The area of a sector with central angle θ is $\frac{\theta}{2} r^2$

Exercise 7 Area of a sector

You are tasked with painting a large, circular logo on the side of a building. If a liter of paint covers covers 6 square meters and the logo is 5 meters wide, how many liters of red paint will you need to paint a sector whose central angle is $\frac{3\pi}{4}$ radians?

Working Space

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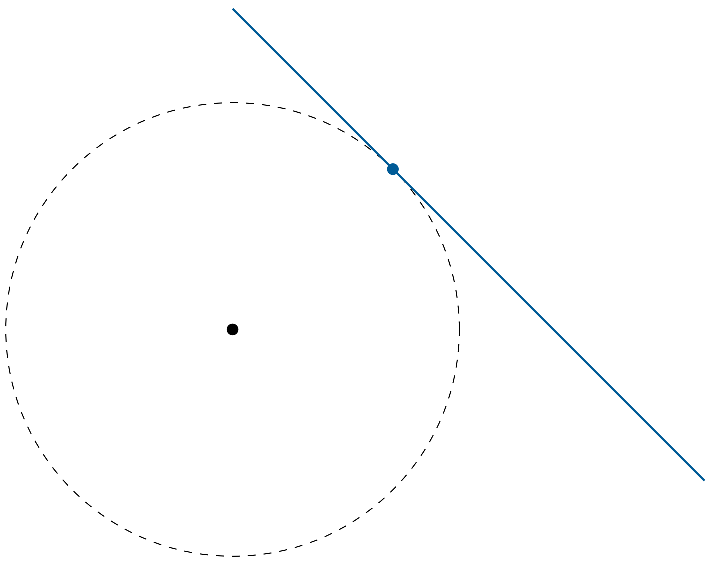
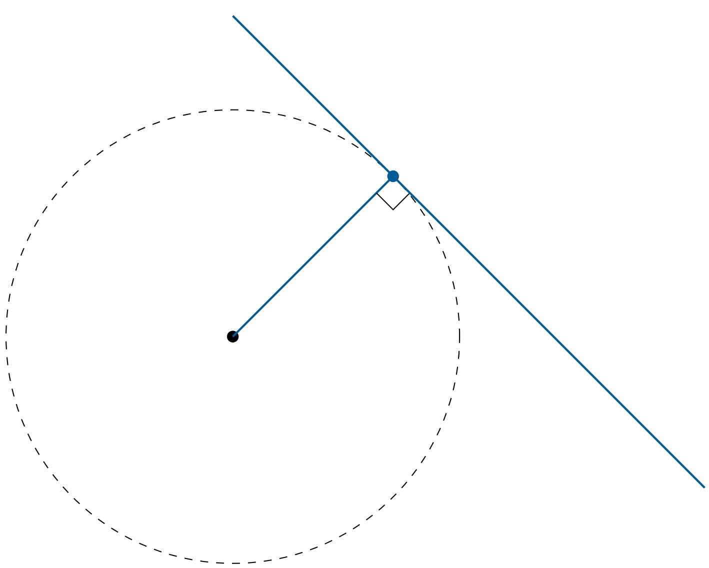
1.3 Tangents

A line that is *tangent* to a circle touches it at exactly one point:



The tangent line is always perpendicular to the radius to the point of tangency:



Exercise 8 Painting a Comet

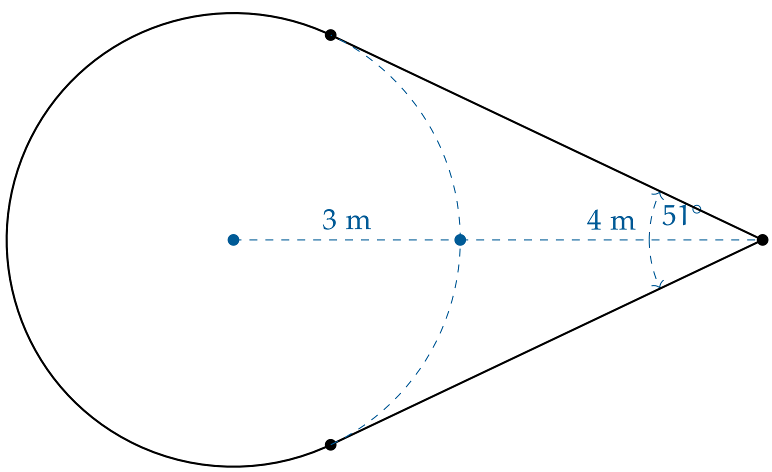
Working Space

You have been asked to paint a comet and its tail in yellow on the floor of a gymnasium.

A liter of yellow paint covers 6 square meters.

First you draw a circle with a radius of 3 meters. You then mark a point D on the floor 7 meters from the center of the circle. Then you draw two tangent lines that pass through D.

You use a protractor to measure the angle at which the tangent lines meet: about 51°



Before you paint the area contained by the circle and the two tangent lines, how much paint will you need?

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This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

The table has a radius of 3 meters.

So the area of its top is $3^2\pi \approx 28.27$.

$$28.27 \text{ square meters } \left(\frac{1 \text{ liter}}{6 \text{ square meters}}\right) = 4.72 \text{ liters}$$

Answer to Exercise 2 (on page 3)

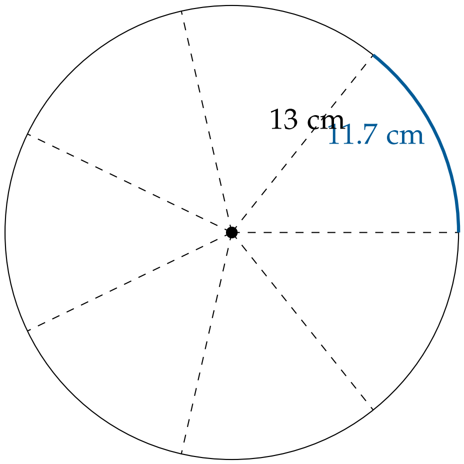
The diameter is

$$\frac{c}{\pi} = \frac{64}{\pi} \approx 20.37 \text{ centimeters}$$

Answer to Exercise 3 (on page 3)

The circumference of the pie is $26\pi \approx 81.7$ centimenters.

The length of the crust for each piece would be about $\frac{81.7}{7} = 11.7$ cm.



Answer to Exercise 4 (on page 5)

If you roll forward by 10 cm, that means you move along the edge of your tire such that the arc length is 10 cm. So, we are looking for a central angle such that $r\theta = 10$ cm. Substituting $r = 25$ cm and solving for θ: $\theta = \frac{10 \text{ cm}}{25 \text{ cm}} = 0.4$ radians.

Answer to Exercise 5 (on page 6)

1. $\frac{\pi}{4} \cdot 2 \text{ cm} = \frac{\pi}{2} \text{ cm}$
2. $\pi \cdot 1 \text{ cm} = \pi \text{ cm}$
3. $\frac{\pi}{10} \cdot 5 \text{ cm} = \frac{\pi}{2} \text{ cm}$
4. $\frac{3\pi}{4} \cdot 3 \text{ cm} = \frac{9\pi}{4} \text{ cm}$

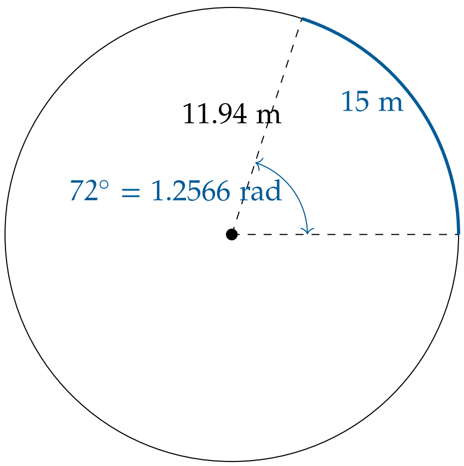
Therefore, from longest to shortest are 4, (1,3), 2 (1 and 3 are the same length).

Answer to Exercise 6 (on page 7)

$$72 \text{ degrees} \left(\frac{2\pi \text{ radians}}{360 \text{ degrees}}\right) \approx 1.2566 \text{ radians}$$

$$15 = 1.2566r$$

$$r = 11.94 \text{ meters}$$

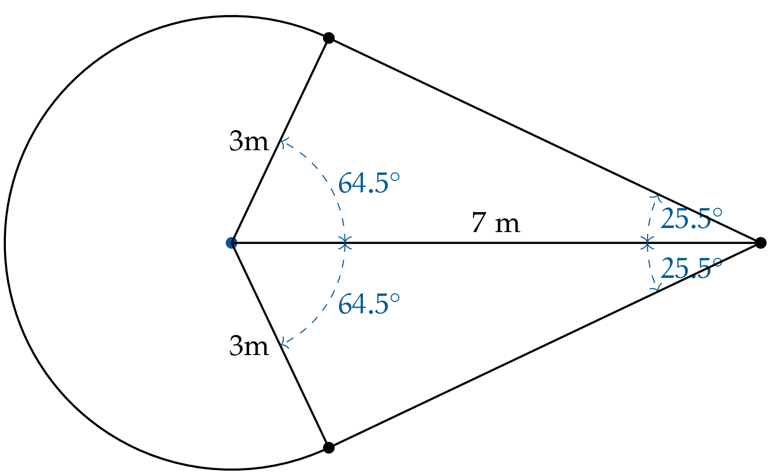


Answer to Exercise 7 (on page 8)

If the logo is 5 meters wide, the diameter is 5 meters and the radius is 2.5 meters. Using the formula for the area of a sector: $A_{sector} = \frac{1}{2}\frac{3\pi}{4}(21.5 \text{ m})^2 \approx 7.363 \text{ m}^2$. Since a liter covers 6 m^2, you will need $\frac{7.363}{6} \approx 1.227$ L of paint.

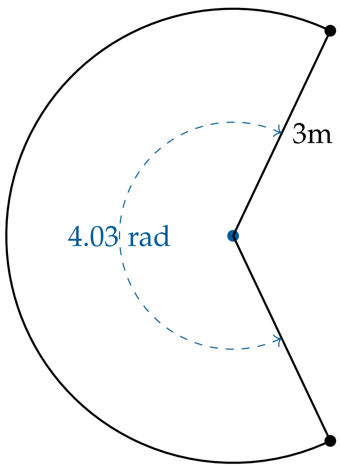
Answer to Exercise 8 (on page 10)

The trick here is to take advantage of the fact that the tangent is perpendicular to the radius to make right triangles:



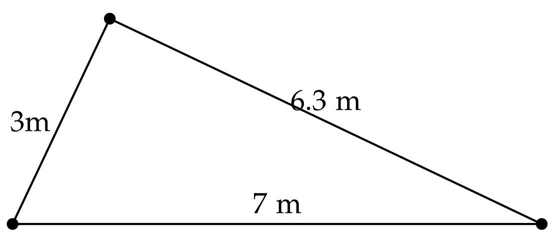
The sector has radius 3 and represents $360 - 2(64.5) = 231° \approx 4.03$ radians.

We are finding the area of this piece:



The area of this piece is $(4.03)(3^2) = 36.27$ square meters.

If a right triangle has a hypotenuse of 7m and one leg is 3m, the other leg is $\sqrt{7^2 - 3^2} = 2\sqrt{10} \approx 6.3$ m.



A right triangle with legs of 3m and 6.3m has an area of 9.45 square meters.

There are two of them, so the total area is $36.27 + 2(18.9) = 74.07$ square meters.

Six square meters per liter, so you need $\frac{74.07}{6} = 12.35$ liters of paint.

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