Chapter 1

First and Second Derivatives and the Shape of a Function

1.1 Using first derivatives to describe a function

1.1.1 Critical Values

Let's re-examine our graph showing the height of a hammer tossed in the air:

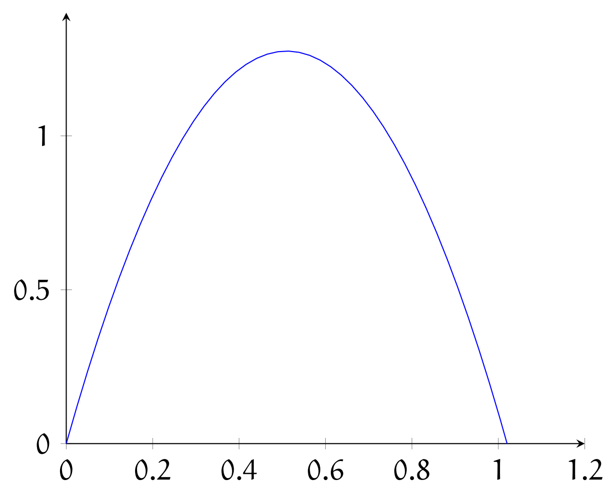

Figure 1.1: Height of a hammer over time

As you can see, the hammer reaches its peak around $t \approx 0.5s$ (see figure 1.1). Let's add tangent lines just before and after the peak of the hammer's path, so we can more easily examine how the slope of the graph changes:

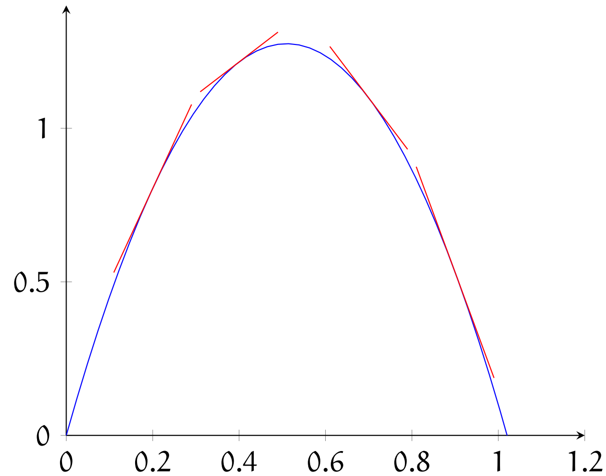

Figure 1.2: height of a hammer over time

In figure 1.2, we see that the slope changes from positive to negative as t increases. That implies that $f'(t)$ also changes from positive to negative. In fact, at the highest point of the hammer's flight, the slope (and therefore $f'(t)$) is exactly zero! In general,

1. If $f'(x) > 0$ (positive slope) on an interval, then $f(x)$ is increasing on that interval.
2. If $f'(x) < 0$ (negative slope) on an interval, then $f(x)$ is decreasing on that interval.

Example 1: Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing.

Solution: We want to find the intervals where $f'(x) > 0$. First, we take the derivative to find $f'(x)$:

$$f'(x) = 12x^3 - 12x^2 - 24x$$

It will be easier to analyze the value of $f'(x)$ if we factor it so:

$$f'(x) = 12x(x-2)(x+1)$$

To determine where $f'(x) > 0$, we start by finding where $f'(x) = 0$ (in this case, this is true when $x = -1, 0, 2$). These values of x are called *critical values*, and we will use them to divide $f'(x)$ into intervals. (Critical values are also called critical numbers, and we will use both in this text.) On each of these intervals, $f'(x)$ must be always positive or always negative. This is shown in the graph below:

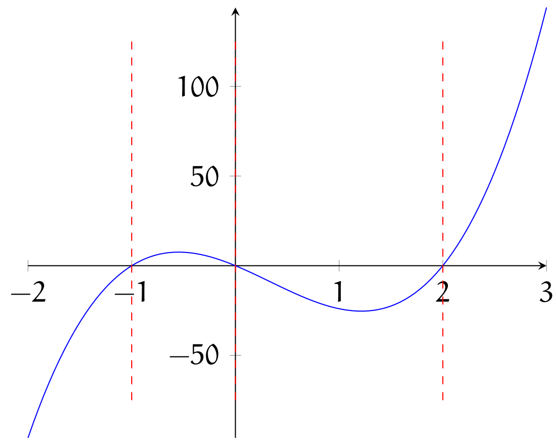

Figure 1.3: $f'(x)$ with critical values

As you can see in figure 1.3, $f'(x) > 0$ on two intervals: $x \in (-1, 0)$ and $x \in (2, \infty)$. These are open intervals because $f'(x) = 0$ at $x = -1$, $x = 0$, and $x = 2$. But what if we had a more complex function, or didn't have the resources to graph it?

We can use a table to help us analyze the value of $f'(x)$ (and therefore the behavior of $f(x)$). For each interval around the critical values, we can determine if $f'(x)$ is positive or negative by noting the value of the factors of $f'(x)$, which are $12x$, $x - 2$, and $x + 1$ in this case. For example, for $x < -1$, $12x < 0$, $(x - 2) < 0$, and $(x + 1) < 0$. Three negatives multiplied together is also negative. Therefore, for $x < -1$, $f'(x)$ is negative and $f(x)$ is decreasing. We can analyze all of the intervals similarly and log the results in a table:

x	$12x$	$x - 2$	$x + 1$	$f'(x)$	$f(x)$
$x < -1$	negative	negative	negative	negative	decreasing
$-1 < x < 0$	negative	negative	positive	positive	increasing
$0 < x < 2$	positive	negative	positive	negative	decreasing
$2 < x$	positive	positive	positive	positive	increasing

Notice the table method yields the same result as examining the graph: $f(x)$ is increasing for $x \in (0, -1)$ and $x \in (2, \infty)$, which can also be written as $x \in (-1, 0) \cup (2, \infty)$.

Exercise 1

Let g be the function given by $g(x) = x^2e^{kx}$, where k is a constant. For what value(s) of k does g have a critical value at $x = \frac{2}{3}$?

Working Space

Answer on Page 21

1.1.2 Local Extrema

Examine the graphs of x^2, $\sin x$, and $y = \sqrt{4 - x^2}$ below. Each has a dot at a local extreme (either a local minimum or local maximum). Sketch what you think the tangent line to the graph would be at each local extreme. Use this to estimate the value of the derivative at that point.

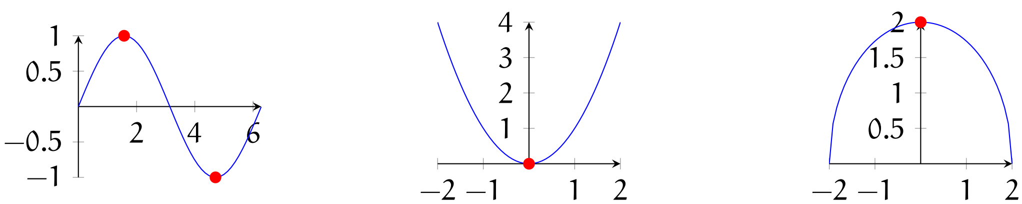

Figure 1.4: Three functions with highlighted points.

You should notice that all of the tangent lines are horizontal. Since the tangent lines at these local extrema have a slope of 0, that tells us $f'(x) = 0$ at these points as well. In fact, for *all* local minima and maxima, the value of the derivative is zero at that point. However, the converse statement is not necessarily true; just because the derivative is zero at some $x = c$, it does not mean there is a local extrema at $f(c)$. Consider $f(x) = x^3 + 3$, shown in figure 1.5:

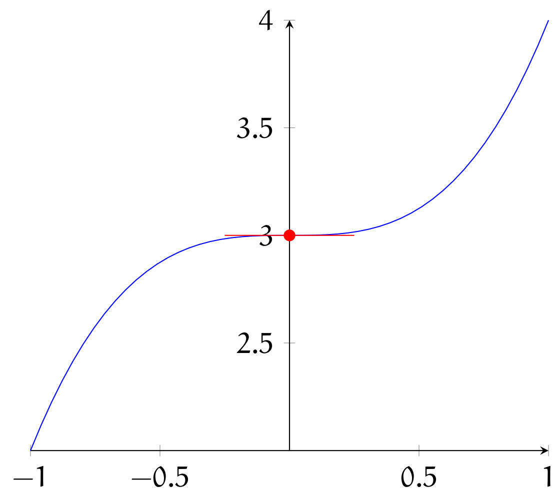

Figure 1.5: $f(x) = x^3 + 3$

At $x = 0$, $f'(x) = 0$, but there is not a local extreme. For a local extreme to exist, the graph of $f(x)$ must change from increasing to decreasing, or vice versa. Look closely at figure 1.5: the function is increasing for $x < 0$ and $x > 0$. Another way of saying this is to note that the graph of $f'(x)$ touches but does not cross the x-axis in this case:

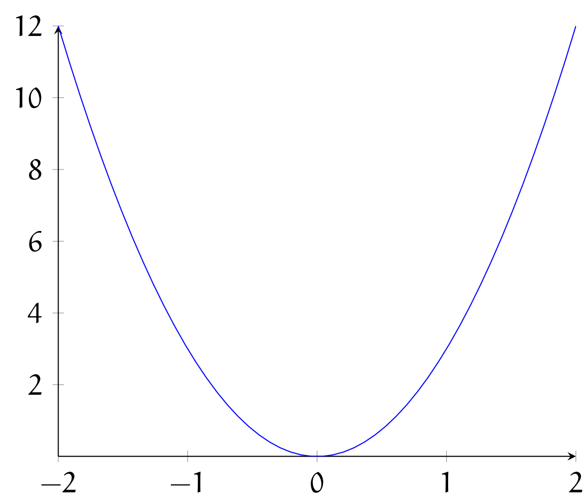

Figure 1.6: $f'(x) = 3x^2$

If $f(x)$ changes from increasing to decreasing, then $f'(x)$ is changing from positive to negative (i.e. crossing the x-axis). Look at the derivative of $f(x) = \sin x$, $f'(x) = \cos x$, presented in figure 1.7. The x-values where local extrema exist on $f(x)$ are marked in red (recall $\sin x = \pm 1$ when $x = \frac{n\pi}{2}$):

As you can see, local extrema are indicated when $f'(x)$ crosses the x-axis. If $f'(x)$ is negative to the left of $x = c$ and positive to the right, then $f(x)$ has a local minimum at $x = c$. On the other hand, if $f'(x)$ is positive to the left of $x = c$ and negative to the right, then $f(x)$ has a local maximum at $x = c$. Any value of $x = c$ where $f'(c) = 0$ is called a **critical number** or a **critical value**. Values where $f(c)$ does not exist are also a critical numbers.

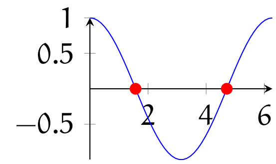

Figure 1.7: $f'(x) = \cos x$

1.1.3 Practice: Interval of Increasing and Decreasing, Local Extrema

Exercise 2

Let f be the function given by $f(x) = 300x - x^3$. On which of the following intervals is f increasing?

Working Space

Answer on Page 21

Exercise 3

Find the intervals on which $f(x) = x^3 - 3x^2 - 9x + 4$ is increasing or decreasing. Then, find all local minimum and/or maximum values of $f(x)$.

Working Space

Answer on Page 22

1.1.4 Global Extrema

Now that we've learned how to identify local minima and maxima, let's expand the discussion to include global extrema. A global extreme is an absolute minimum or maximum value of a function over a particular interval or the entire domain of the function. Let's examine the graph of $f(x) = x^4 - 5x^3 + 6x^2$ over the domain $x \in [-1, 4]$.

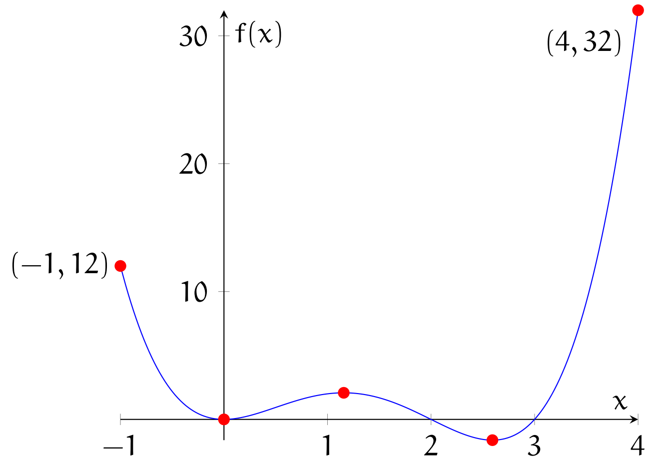

Figure 1.8: Graph of $f(x) = x^4 - 5x^3 + 6x^2$

As you can see in figure 1.8, $f(x)$ has two local minima and one local maximum. Additionally, the endpoints are labeled. To determine the *global* extrema, we need to examine the any local extrema (identified here graphically, but you can also identify them mathematically using that you learned in the "Local Extrema" subsection) **and** the endpoints of the domain (or the function's behavior at $\pm\infty$, if you are Notet restricted to a specific domain).

In the case of $f(x) = x^4 - 5x^3 + 6x^2, \ \mathrm{for} x \in [-1, 4]$, the global maximum value is 32 at $x = 4$ and the global minimum is -1.623 at $x = 2.593$.

If a function is continuous on an interval, then there must exist a global maximum and global minimum on that interval. These global extrema may also be local extrema (as is the case for $f(2.593)$ in the example above) or not (as is the case for $f(4)$). Applying the Closed Interval Method is a straightforward way to identify global (absolute) extrema.

To find the global extrema of a continuous function, f, on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from steps 1 and 2 is the absolute maximum; the smallest of the values is the absolute minimum.

Let's use the Closed Interval Method to determine the global extrema for the function $g(x) = x - 3\sin x$ on the interval $x \in [0, 2\pi]$.

To find the value of g at any critical numbers, we must first identify the critical numbers. Recall that critical numbers are values where the first derivative of the function is 0 or does not exist. To find critical numbers, we set g' equal to 0:

$$g'(x) = 1 - 3\cos x = 0$$

$$3\cos x = 1$$
$$\cos x = \frac{1}{3}$$
$$x = 1.23, 5.052$$

Now, we substitute these critical numbers back into $g(x)$:

$$g(1.23) \approx -1.60$$

$$g(5.052) = 7.881$$

Now we need to check the endpoints:

$$g(0) = 0 - 3 * 0 = 0$$

$$g(2\pi) = 2\pi - 3 * 0 = 2\pi \approx 6.28$$

The results are presented in the table below:

x	$g(x)$
0	0
1.23	-1.60
5.052	7.881
6.28	6.28

Therefore, for $g(x) = x - 3\sin x$ on the interval $x \in [0, 2\pi]$, the global maximum is $g(5.052) = 7.881$ and the global minimum is $g(1.23) = -1.60$.

1.1.5 Practice: Global Extrema

Exercise 4

Let f be the function defined by $f(x) = \frac{\ln x}{x}$. What is the absolute maximum value of f?

Working Space

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Exercise 5

Find the global minimum and maximum values on the stated interval.

1. $f(x) = 12 + 4x - x^2$, $[0, 5]$
2. $f(t) = \frac{\sqrt{t}}{1+t^2}$, $[0, 2]$
3. $f(t) = 2\cos t + \sin 2t$, $[0, \frac{\pi}{2}]$
4. $f(x) = \ln x^2 + x + 1$, $[-1, 1]$

Working Space

Answer on Page 23

1.2 Sketching f from f'

Now that we know how the shape of f is related to the value of f', we can predict the shape of f if we are given f'. Take the example $f'(x) = -(x-1)(x-5)$, shown in figure 1.9:

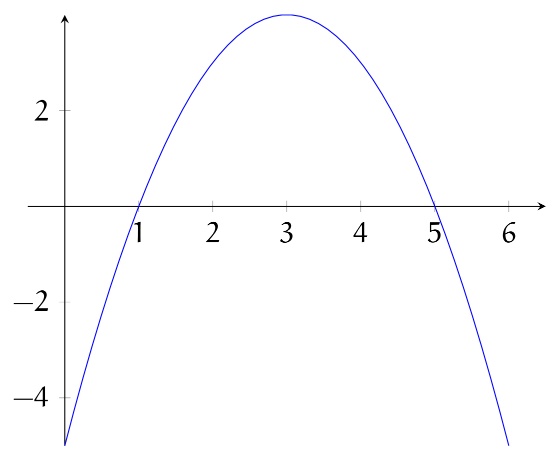

Figure 1.9: Graph of $f' = -(x-1)(x-5)$

Using the graph of f', we can construct an approximate sketch of f. First, let's identify the critical numbers. Where does $f' = 0$? Take a second to examine the graph of f' above and jot down what you think the critical numbers are.

You should recall that critical numbers are x-values where $f' = 0$. Examining the graph of f', we see that $f' = 0$ at $x = 1$ and $x = 5$. We can now use a table to describe the behavior of f:

x	$x-1$	$x-5$	f'	behavior of f
$x < 1$	negative	negative	negative	decreasing
$x = 1$	zero	negative	zero	local minimum
$1 < x < 5$	positive	negative	positive	increasing
$x = 5$	positive	zero	zero	local maximum
$x > 5$	positive	positive	negative	decreasing

We can use this information to sketch a possible graph of f. We start by noting the location of local extrema:

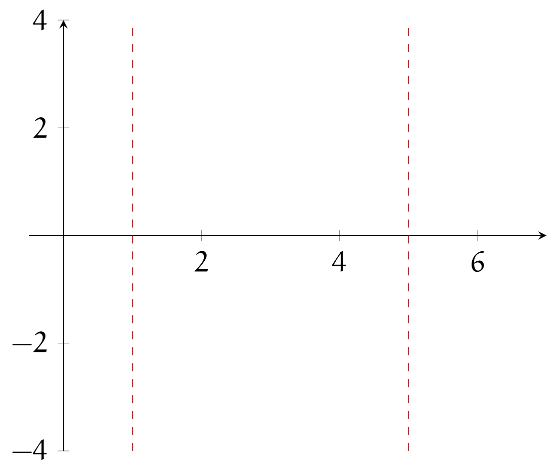

Figure 1.10: Possible graph of f

We know there is a local minimum at $x = 1$ and a local maximum at $x = 5$. We can add sketches around these values to indicate what we know about f:

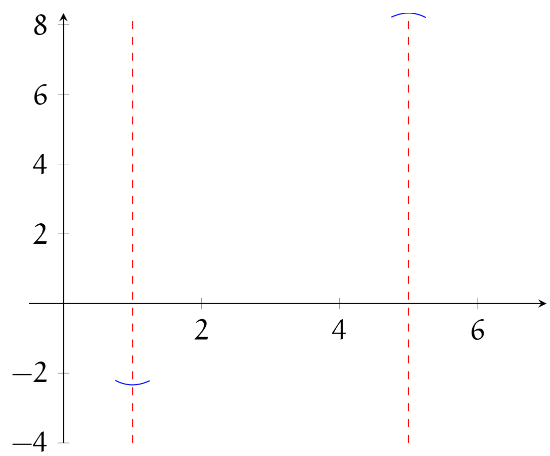

Figure 1.11: Possible graph of f

Lastly, we know f is increasing on $1 < x < 5$ and decreasing everywhere else, so we fill in the space between our local extrema:

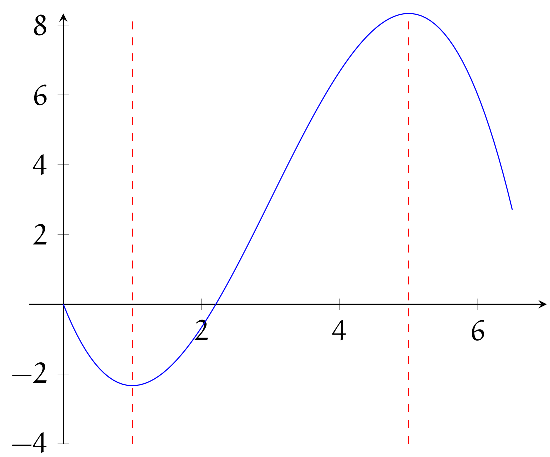

Figure 1.12: Possible graph of f

However, figure 1.12 is only a *possible* graph of f. Analyzing f' reveals the shape of f, but not how high or low it is on the y-axis. Recall that the derivative of a constant is zero. Therefore, any $+c$ (where c is a constant) is lost when taking the derivative. So, there are many sketches of f that fulfill the behavior of f indicated by f'. You can see several of the possible sketches for f in figure 1.13.

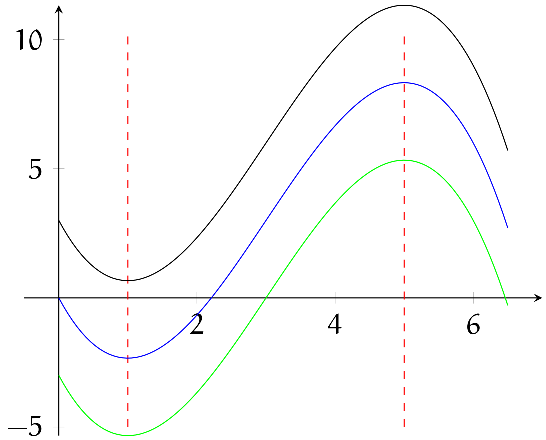

Figure 1.13: Possible graphs of f

1.2.1 Practice Sketching f from f′

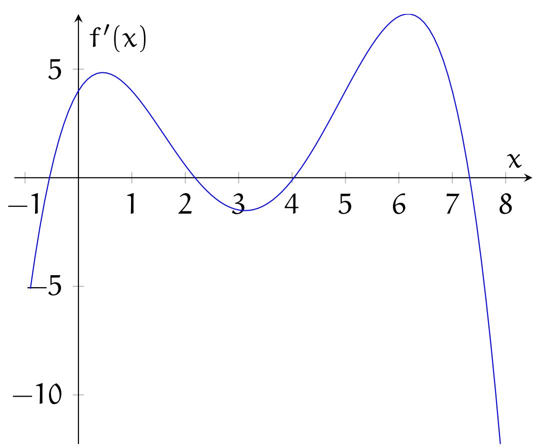

Figure 1.14: Graph of $f'(x)$

Exercise 6

Use figure 1.14 to answer the following questions:

1. On what approximate intervals is f increasing or decreasing?
2. At what approximate values of x does f have a local maximum or minimum?
3. Sketch a possible graph of f in the space below:

Working Space

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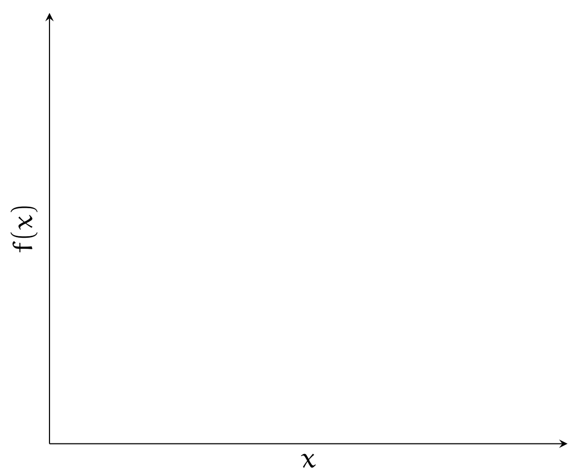
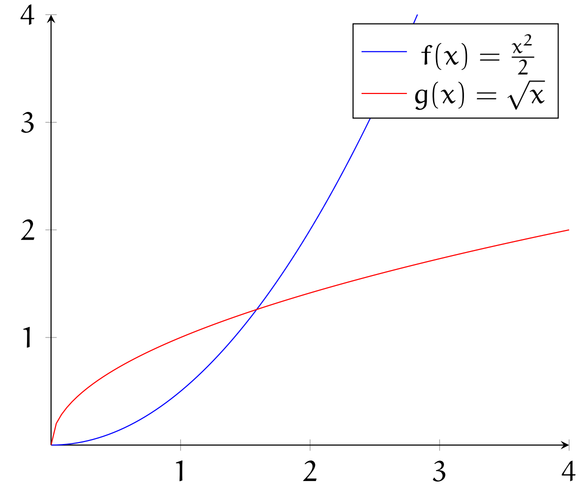


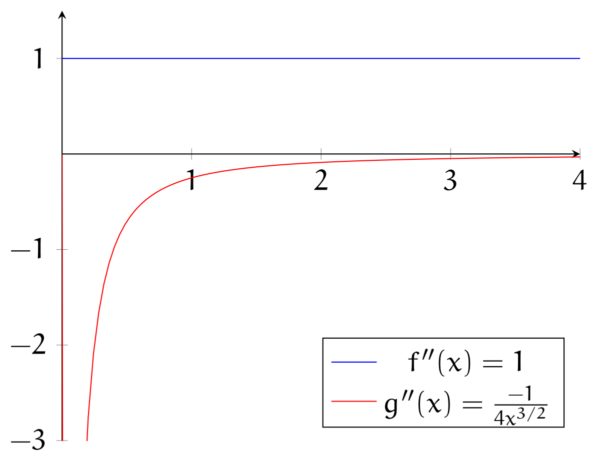
1.3 Using second derivatives to describe a function

1.3.1 Concavity

Let's examine two increasing functions, $f(x) = \frac{x^2}{2}$ and $g(x) = \sqrt{x}$:

Even though both of these functions are increasing, they have different shapes. $f(x)$ looks like a bowl. On the other hand, $g(x)$ looks like an upside-down bowl. These shapes are called *concave up* (in the case of $f(x)$) and *concave down* (in the case of $g(x)$). Both functions





are increasing on the interval $x \in [0, 4]$, and therefore both $f'(x)$ and $g'(x)$ are positive on the stated interval. Let's look at their second derivatives, $f''(x)$ and $g''(x)$:

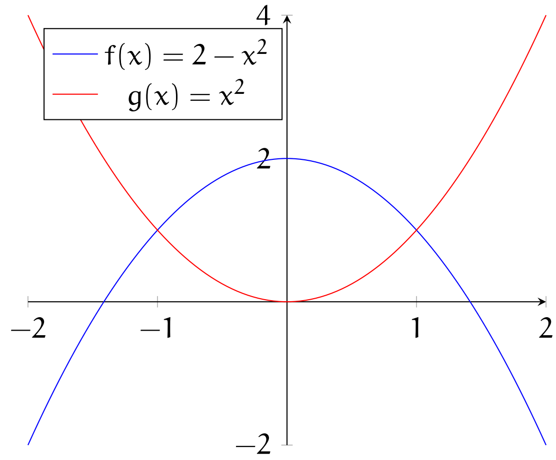
As you can see, $f''(x) > 0$ and $g''(x) < 0$. The second derivative tells us if a function is concave up or concave down. In general:

1. If $f''(x) > 0$ for all x in a given interval, then the graph of f is concave up on the interval.

2. If $f''(x) < 0$ for all x in a given interval, then the graph of f is concave down on the interval.

Additionally, the second derivative can help us determine if there is a local minimum or maximum at critical numbers. Look at the graphs of $f(x) = 2 - x^2$ and $g(x) = x^2$, which both have first derivatives equal to 0 at $x = 0$:

When the graph is concave up, there is a local minimum where the first derivative equals



0. When the graph is concave down, there is a local maximum where the first derivative equals 0. This is summarized with the Second Derivative Test:

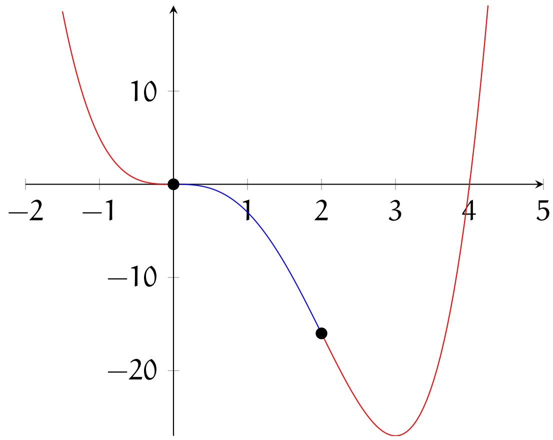
Suppose f'' is continuous near c. Then,

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

1.3.2 Inflection Points

If f is concave up when $f'' > 0$ and concave down when $f'' < 0$, what about when $f'' = 0$? This is the value at which f changes from concave up to concave down (or vice versa), which is called an *inflection point*. Similar to local extrema with f', if there is an inflection point at $x = c$, then $f''(c) = 0$, but the converse is not necessarily true. To check if $x = c$ is an inflection point, then f'' should change signs on either side of $x = c$ (either from positive to negative to from negative to positive).

Look at the graph of $f(x) = x^4 - 4x^3$. The concave up areas are shown in red, and the concave down in blue:



Let's examine f'' to confirm the inflection points are at $(0,0)$ and $(2,-16)$. First, we note that $f''(x) = 12x^2 - 24x$. Factoring, we see that $f''(x) = 12x(x-2)$, which has zeroes at $x = 0$ and $x = 2$. For $x < 0$, $f'' > 0$, and for $0 < x < 2$, $f'' < 0$; therefore, there is an inflection point in f at $(0,0)$.

Exercise 7

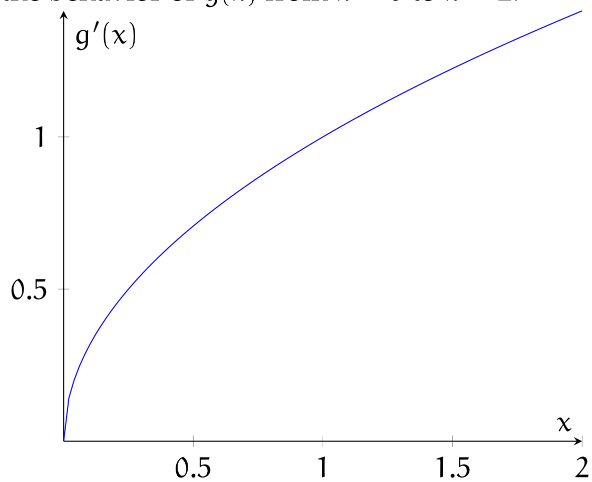
Prove that the other inflection point for $f(x) = x^4 - 4x^3$ is $(2,-16)$.

Working Space

Answer on Page 25

Exercise 8

The graph below shows $g'(x)$. Describe the behavior of $g(x)$ from $x = 0$ to $x = 2$.



Working Space

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Exercise 9

[This question was originally presented as a calculator-allowed, multiple-choice problem on the 2012 AP Calculus BC exam.] For $-1.5 < x < 1.5$, let f be a function with first derivative given by $f'(x) = e^{(x^4-2x^2+1)} - 2$. State the interval(s) (to three decimal places) for which f is concave down.

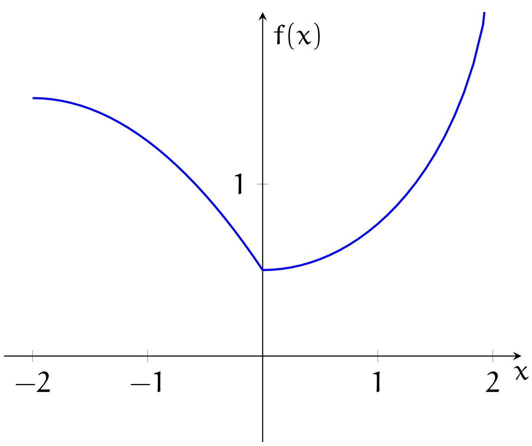
Working Space

Answer on Page 26

Exercise 10

[The following problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Consider the function, f, whose graph is shown below. Classify each of the following statements as true or false and explain.

1. $f' > 0$ for $x \in (-2, 0)$.
2. f is differentiable at $x = 0$.
3. $f'' > 0$ for $x \in (0, 2)$
4. f has a critical value at $x = 0$



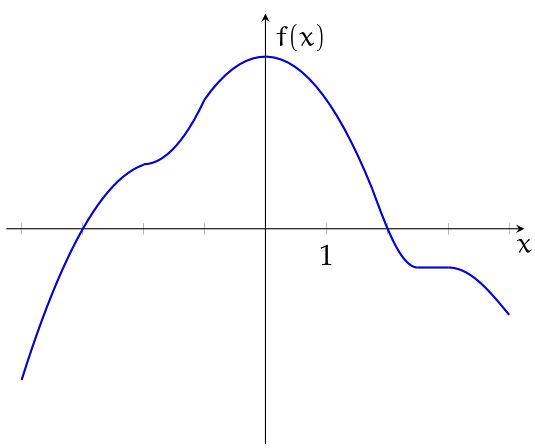
Working Space

Answer on Page 27

Exercise 11

[The following problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] The graph of f', the derivative of f, is shown below. Classify each of the following statements as true or false and explain your answer.

1. f has a relative minimum at $x = -3$.
2. The graph of f has a point of inflection at $x = -2$.
3. The graph of f is concave down for $0 < x < 4$.



Working Space

Answer on Page 27

Exercise 12

[The following problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Let f be a function that is twice differentiable on $-2 < x < 2$ and satisfies the conditions in the table below. If $f(x) = f(-x)$, what are the x-coordinates of the points of inflection of the graph of f on $-2 < x < 2$?

	$0 < x < 1$	$1 < x < 2$
$f(x)$	Positive	Negative
$f'(x)$	Negative	Negative
$f''(x)$	Negative	Positive

Working Space

Answer on Page 27

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

Recall that critical values are values of x where $g'(x) = 0$ or is undefined. We need to find an expression for $g'(x)$, set it equal to zero when $x = \frac{2}{3}$, and solve for k.

$$g'(x) = x^(2)[k * \exp kx] + \exp kx[2x]$$

$$g'(\frac{2}{3}) = (\frac{2}{3})^2[k * \exp \frac{2k}{3}] + \frac{4}{3} \exp \frac{2k}{3} = 0$$

$$\frac{4k}{9} e^{\frac{2k}{3}} + \frac{4}{3} e^{\frac{2k}{3}} = 0$$

$$(\frac{4k}{9} + \frac{4}{3}) e^{\frac{2k}{3}} = 0$$

There are no real values of k such that $e^{\frac{2k}{3}} = 0$, therefore, we will examine the other factor:

$$\frac{4k}{9} + \frac{4}{3} = 0$$

$$\frac{4k}{9} = \frac{-4}{3}$$

$$\frac{k}{3} = -1$$

$$k = -3$$

Therefore, $g(x)$ has a critical value at $x = \frac{2}{3}$ when $k = -3$.

Answer to Exercise 2 (on page 6)

First, we will find f' and set it equal to zero:

$$f'(x) = 300 - 3x^2 = 0$$

$$300 = 3x^2 \rightarrow x = \pm\sqrt{100} = \pm 10$$

(Note: $f'(x) = 3(10 - x)(10 + x)$, which implies roots at $x = \pm 10$. Now, we will evaluate the value of $f'(x)$ for $x < -10$, $-10 < x < 10$, and $x > 10$.

Value of x	(10-x)	(10+x)	$f'(x)$	$f(x)$ behavior
$x < -10$	positive	negative	negative	decreasing
$-10 < x < 10$	positive	positive	positive	increasing
$x > 10$	negative	positive	negative	decreasing

Therefore, the function is increasing on the interval $x \in [-10, 10]$ because $f'(x) > 0$ for $x \in [-10, 10]$.

Answer to Exercise 3 (on page 6)

Given $f(x) = x^3 - 3x^2 - 9x + 4$, it follows that $f'(x) = 3x^2 - 6x - 9$. Factoring, we find that $f'(x) = 9(x - 3)(x + 1)$ and $f'(x) = 0$ when $x = 3$ and $x = -1$. We construct our table to help us analyze the value of $f'(x)$ and behavior of $f(x)$ on the whole domain of the function:

Value of x	$(x-3)$	$(x+1)$	$f'(x)$	$f(x)$ behavior
$x < -1$	negative	negative	positive	increasing
$-1 < x < 3$	negative	positive	negative	decreasing
$x > 3$	positive	positive	positive	increasing

So, $f(x)$ is increasing for $x \in (-\infty, -1) \cup (3, \infty)$ and decreasing for $x \in (-1, 3)$. Since $f'(-1) = 0$ and changes from positive to negative, $f(x)$ has a local maximum at $x = -1$. And since $f'(3) = 0$ and changes from negative to positive, $f(x)$ has a local minimum at $x = 3$.

Answer to Exercise 4 (on page 8)

First, we identify any critical numbers:

$$f'(x) = \frac{x * (\frac{1}{x}) - \ln x * 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Recall that critical numbers are values where $f'(x) = 0$ or does not exist. We might identify $x = 0$ as a critical number, but the presence of $\ln x$ limits the domain of the function to $x \in (0, \infty)$, excluding $x = 0$. For all $x \in (0, \infty)$, $f'(x)$ exists. So, we look for values where $f'(x) = 0$.

$$\frac{1 - \ln x}{x^2} = 0$$

$$1 - \ln x = 0$$

$$1 = \ln x$$

$$x = e$$

Finding the value of $f(x)$ at $x = e$:

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

Because the domain of $f(x)$ is on an *open interval*, instead of checking the endpoints directly, we'll take the limits as x approaches 0 and ∞.

$$\lim_{x \to 0} \frac{\ln x}{x} = -\infty < \frac{1}{e}$$

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0 < \frac{1}{e}$$

Therefore, the absolute maximum values of $f(x) = \frac{\ln x}{x}$ is $\frac{1}{e}$ at $x = e$.

Answer to Exercise 5 (on page 9)

1. $f'(x) = 4 - 2x$ and to find the critical numbers, we set $f'(x) = 0$:

   $$4 - 2x = 0$$

   $$x = 2$$

   We evaluate $f(x)$ at $x = 0, 2, 5$:

   $$f(0) = 12 + 4(0) - 0^2 = 12$$

   $$f(2) = 12 + 4(2) - 2^2 = 12 + 8 - 4 = 16$$

   $$f(5) = 12 + 4(5) - 5^2 = 12 + 20 - 25 = 7$$

   Therefore, the global maximum is $f(2) = 16$ and the global minimum is $f(5) = 7$.

2. $f(t) = \dfrac{\sqrt{t}}{1 + t^2}$, $[0, 2]$.

   We write $f(t) = t^{1/2}(1 + t^2)^{-1}$ and differentiate:

   $$f'(t) = \frac{1 - 3t^2}{2\sqrt{t}(1 + t^2)^2}.$$

   Critical points occur when the numerator is zero (and t is in the interval):

   $$1 - 3t^2 = 0 \quad \Rightarrow \quad t = \frac{1}{\sqrt{3}}.$$

   We evaluate $f(t)$ at $t = 0, \frac{1}{\sqrt{3}}, 2$:

   $$f(0) = 0, \qquad f\left(\frac{1}{\sqrt{3}}\right) = \frac{3^{3/4}}{4}, \qquad f(2) = \frac{\sqrt{2}}{5}.$$

Since $\frac{3^{3/4}}{4}$ is the largest of these and 0 is the smallest, the global maximum is

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{3^{3/4}}{4},$$

and the global minimum is

$$f(0) = 0.$$

3. $f(t) = 2\cos t + \sin(2t)$, $\left[0, \frac{\pi}{2}\right]$.
   We differentiate:
   $$f'(t) = -2\sin t + 2\cos(2t).$$
   Set $f'(t) = 0$:
   $$-2\sin t + 2\cos(2t) = 0 \quad \Rightarrow \quad \cos(2t) = \sin t.$$
   Use $\cos(2t) = 1 - 2\sin^2 t$:
   $$1 - 2\sin^2 t = \sin t \quad \Rightarrow \quad 2\sin^2 t + \sin t - 1 = 0.$$
   Let $u = \sin t$:
   $$2u^2 + u - 1 = 0 \quad \Rightarrow \quad u = \frac{1}{2} \text{ or } u = -1.$$
   On $\left[0, \frac{\pi}{2}\right]$, $\sin t = \frac{1}{2}$ gives $t = \frac{\pi}{6}$ (and $\sin t = -1$ is not in this interval).
   Evaluate $f(t)$ at $t = 0, \frac{\pi}{6}, \frac{\pi}{2}$:
   $$f(0) = 2\cos 0 + \sin 0 = 2,$$
   $$f\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2},$$
   $$f\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{\pi}{2}\right) + \sin(\pi) = 0 + 0 = 0.$$
   Hence the global maximum is
   $$f\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2},$$
   and the global minimum is
   $$f\left(\frac{\pi}{2}\right) = 0.$$

4. $f(x) = \ln(x^2 + x + 1)$, $[-1, 1]$.
   Differentiate:
   $$f'(x) = \frac{2x + 1}{x^2 + x + 1}.$$
   To find critical points, set the numerator equal to 0:
   $$2x + 1 = 0 \quad \Rightarrow \quad x = -\frac{1}{2}.$$
   Evaluate $f(x)$ at $x = -1, -\frac{1}{2}, 1$:
   $$f(-1) = \ln(1 - 1 + 1) = \ln 1 = 0,$$

$$f\left(-\frac{1}{2}\right) = \ln\left(\left(-\frac{1}{2}\right)^2 - \frac{1}{2} + 1\right) = \ln\left(\frac{1}{4} - \frac{1}{2} + 1\right) = \ln\left(\frac{3}{4}\right),$$
$$f(1) = \ln(1 + 1 + 1) = \ln 3.$$

Since $\ln\left(\frac{3}{4}\right) < 0 < \ln 3$, the global minimum is

$$f\left(-\frac{1}{2}\right) = \ln\left(\frac{3}{4}\right),$$

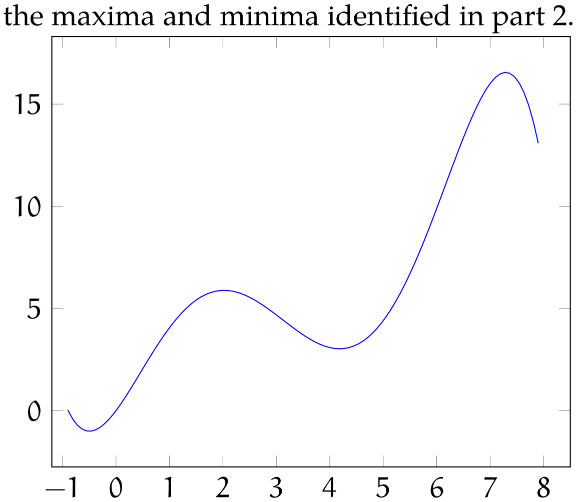
and the global maximum is

$$f(1) = \ln 3.$$

Answer to Exercise 6 (on page 12)

[Your answers are meant to be estimates; anything within ± 0.1 of the given answers are reasonable estimates.]

1. $f(x)$ is increasing on the intervals $x \in (-0.5, 2.2) \cup (4, 7.3)$. $f(x)$ is decreasing on the intervals $x \in (-\infty, -0.5) \cup (2.2, 4) \cup (7.3, \infty)$.
2. $f(x)$ has local maxima at $x = 2.2, 7.3$ and local minima at $x = -0.5, 4$.
3. Your sketch should show the maxima and minima identified in part 2. One possible solution is shown below.



Answer to Exercise 7 (on page 15)

Noting that $f''(2) = 0$, we examine the value of f'' around $x = 2$. For $0 < x < 2$, $f'' < 0$, which indicates f is concave down in the domain $x \in (0, 2)$. For $x > 2$, $f'' > 0$, which indicates f is concave up. Therefore, there is an inflection point at $x = 2$ for f. Recalling that $f(x) = x^4 - 4x^3$, we find the coordinate of the inflection point by substituting $x = 2$:

$$f(2) = 2^4 - 4 * 2^3 = 16 - 4 * 8 = 16 - 32 = -16$$

Therefore, $f(x)$ has an inflection point at $(2, -16)$.

Answer to Exercise 8 (on page 16)

According to the graph, g' is positive and increasing. Therefore, g is increasing (because g' is positive) and concave up (because g' is increasing, and therefore g'' is positive).

Answer to Exercise 9 (on page 16)

Since the question asks about concavity, we need to examine the second derivative:

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left[e^{(x^4-2x^2+1)} - 2\right]$$

$$f''(x) = \left(x^4 - 2x^2 + 1\right) e^{(x^4-2x^2+1)} \left(4x^3 - 4x\right)$$

The second derivative equals zero when $x^4 - 2x^2 + 1 = (x^2 - 1)^2 = 0$ or $4x^3 - 4x = 4(x)(x^2 - 1) = 0$, which gives roots $x = 0$, $x = 1$, and $x = -1$. So the intervals we need to test are $(-1.5, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, 1.5)$. To test $x \in (-1.5, -1)$, we will substitute $x = -1.25$ into $f''(x)$:

$$f''(-1.25) = -3.85928 < 0$$

Therefore, $f(x)$ is concave down on the interval $x \in (-1.5, -1)$. Next, we test $x \in (-1, 0)$:

$$f''(-0.5) = 2.63258 > 0$$

So, we eliminate $x \in (-1, 0)$. Next, we test $x \in (0, 1)$:

$$f''(0.5) = -2.63258 < 0$$

And $f(x)$ is concave down on the interval $x \in (0, 1)$. Finally, we test the interval $x \in (1, 1.5)$:

$$f''(1.25) = 3.85928 > 0$$

Which eliminates that interval. Therefore, $f(x)$ is concave down on the intervals $x \in (-1.5, -1)$ and $x \in (0, 1)$.

Answer to Exercise 10 (on page 17)

1. False. For $x \in (-2, 0)$, the slope of $f(x)$ is negative, which implies that $f'(x) < 0$ for $x \in (-2, 0)$.
2. False. The graph comes to a point at $x = 0$, therefore $\lim_{x \to 0^+} f'(x) \neq \lim_{x \to 0^-} f'(x)$, which means the limit does not exist and f is not differentiable at $x = 0$.
3. True. The graph of $f(x)$ is concave up for $x \in (0, 2)$, which means the second derivative is positive.
4. True. Recall that critical values are where derivatives equal 0 or do not exist. Since we have established that $f(x)$ does not exist at $x = 0$, then there is a critical value at $x = 0$.

Answer to Exercise 11 (on page 18)

1. True. $f'(3) = 0$ and f' has a positive slope, which means there is a local extreme and f is concave up at $x = 3$. Therefore, there is a local minimum at $x = 3$.
2. False. Though it appears that $f'' = 0$ at $x = -2$, the slope of f' is positive before and after. Therefore, f'' does not cross the x-axis and there is not an inflection point at $x = -2$.
3. True. For $0 < x < 4$, the slope of f' is negative, which means f'' is negative, which means f is concave down.

Answer to Exercise 12 (on page 19)

The graph of f has inflection points at $x = -1$ and $x = 1$. Since $f(x) = f(-x)$, we can expand the table to include the entire window we are investigating:

	$-2 < x < -1$	$-1 < x < 0$	$0 < x <$
$f(x)$	Negative	Positive	Positiv
$f'(x)$	Negative	Negative	Negativ
$f''(x)$	Positive	Negative	Negativ

Recall that inflection points occur when f'' changes from positive to negative or from negative to positive. Examining the table, we see that the sign of f'' changes at $x = -1$ and $x = 1$.

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