Chapter 1

Introduction to Discrete Probability

Before we begin talking about discrete probably, let's address the word *discrete* vs *discreet*. They sound exactly the same, but "discrete" means "individually separate and distinct" and "discreet" means "careful about what other people know". So, you might say, "You can think of light as a continuous wave or as a blast of discrete particles." And you might say, "Please go get the box of doughnuts from the kitchen. Oh, and there are a lot of hungry people in the house, so be discreet."

When we are talking about probabilities, some problems deal with discrete quantities like "What is the probability that I will throw these three dice and the numbers that roll face up sum to 9?" There are also problems that deal with continuous properties like "What is the probability that the next bird to fly over my house will weigh between 97.2 and 98.1 grams?" In this module, we are going to focus on the probability problems that deal with discrete quantities.

Watch Khan Academy's Introduction to Probability at `https://youtu.be/uzkc-qNVoOk`.

Let's say that you have a cloth sack filled with 100 marbles; 99 are red and 1 is white. If you reach in without looking and pull out one marble, you will probably pull out a red one. We say that "There is a 1 in 100 chance that you would pull out a white marble." Or we can use percentages and say "There is a 1% chance that you will pull out a white marble." Or we can use decimals and say "There is a 0.01 probability that you will pull out a white marble." In probability, we often talk about the probability of certain events. "Pulling out a white marble" is an event, and we can give it a symbol like W. In equations, we use p to mean "the probability of". Thus, we can say "There is a 0.01 probability that you will pull out a white marble," which becomes the equation

$$p(W) = 0.01$$

1.1 The Probability of All Possibilities is 1.0

We know that you are either going to pull out a red marble or a white marble, so the probability of a white marble being pulled and the probability of a red marble being pulled must add up to 100%. Therefore, the odds of pulling out a red marble must be 99%, or 0.99. If we let the event "Pull out a red marble" be given by the symbol R, we can say:

$$p(R) = 1.0 - P(W) = 1.0 - 0.01 = 0.99$$

Now, let's say that you take a marble from the bag, then toss a coin. What is the probability that you will pull a white marble, then get heads on the coin? It is the product of the two probabilities: $0.01 \times 0.5 = 0.005$, so one-half of a one percent chance. Do the probabilities still sum to 1?

- White and Heads $= 0.01 \times 0.5 = 0.005$
- White and Tails $= 0.01 \times 0.5 = 0.005$
- Red and Heads $= 0.99 \times 0.5 = 0.495$
- Red and Tails $= 0.99 \times 0.5 = 0.495$

Yes, the probabilitites of all the possibilities still add to 1.

1.2 Independence

In the last section, you learned that the probability of two events ("Pulling a red marble from the bag" and "Getting tails in a coin toss") is the product of the probability of each event: $0.99 \times 0.5 = 0.495$.

This is true if the two events are *independent*; that is, the outcome of one doesn't change the probability of the other. The example we gave is independent: It doesn't matter what ball you pull from the bag, the outcome of the coin toss will always be 50-50.

What are two events that are not independent? The probability that a person is a professional basketball player and the probability that someone wears a shoe that is size 13 or larger is *not* independent. After all, height is an advantage in basketball, and most tall people also have large feet. So, if you know someone is a basketball player, they likely wear large shoes.

Exercise 1 Rolling Dice

If you have three 6-sided dice to roll, what is the probability that you will roll a 5 on all three dice?

Working Space

Answer on Page 11

Exercise 2 Flipping Coins

If you have five coins to flip, what is the probability that at least one coin will come up heads?

Working Space

Answer on Page 11

1.3 Why 7 is the most likely sum of two dice

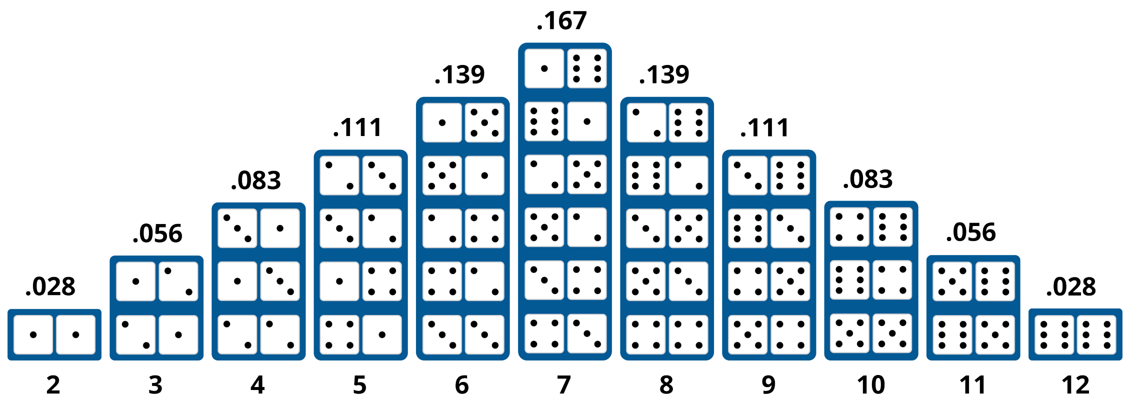
If you roll two dice, the sum will be 2, 12, or any number in between. It is very tempting to assume that the likelihood of any of those numbers is the same. In fact, the probability of a 2 is $\frac{1}{36} \approx 3\%$ and the probability of a 7 is $\frac{1}{6} \approx 17\%$. A 7 is six times more likely than a 12! Why?

When you roll the first die, there are six possibilities with equal probability. When you roll the second die, there are six possibilities with equal probability. So, there are a total of 36 possible events with equal probabilities: 1 then 1, 1 then 2, 2 then 1, 1 then 3, 3 then 1, and so on. Only one of these (1 then 1) adds to 2. However, six of these sum to 7: 1 then 6, 6 then 1, 2 then 5, 5 then 2, 3 then 4, and 4 then 3. So, a 7 is six times more likely than a 2.

Here is the complete table:

Sum							Count	Probability
2	1,1						1	1/36
3	1,2	2,1					2	1/18
4	1,3	2,2	3,1				3	1/12
5	1,4	2,3	3,2	4,1			4	1/9
6	1,5	2,4	3,3	4,2	5,1		5	5/36
7	1,6	2,5	3,4	4,3	5,2	6,1	6	1/6
8		2,6	3,5	4,4	5,3	6,2	5	5/36
9			3,6	4,5	5,4	6,3	4	1/9
10				4,6	5,5	6,4	3	1/22
11					5,6	6,5	2	1/18
12						6,6	1	1/36

If you don't believe these numbers, you could roll a pair of dice hundreds of times and make a histogram. However, it would be a tedious and time-consuming task — just the sort of thing that we make computers do for us.



1.4 Random Numbers and Python

You are going to write a simulation of rolling dice in Python. To do this, you will need to generate a random sequence of numbers. The numbers will need to be in the range 1 to 6, and they will need to appear in the sequence with the same frequency. We say the sequence will follow *the uniform distribution*. In other words, the probability is uniformly distributed among the 6 possibilities.

Start python and try a few of the different ways to generate random numbers:

```
> python3
>>> import random
>>> random.random()  # Generates a random floating point number between 0 and 1
0.6840892758539989
>>> randrange(5)      # Generates an integer in the range 0 - 4
2
>>> x = ['Rock', 'Paper', 'Scissors']
>>> random.choice(x)   # Pick a random entry from the sequence
'Paper'
>>> x
['Rock', 'Paper', 'Scissors']
>>> random.shuffle(x)   # Shuffle the order of the sequence
>>> x
['Scissors', 'Paper', 'Rock']
>>> a = list(range(30))
>>> a
```

```
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,
  16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 29]
>>> random.sample(a, 10) # Return 10 randomly chosen items from the sequence
[8, 7, 20, 9, 25, 13, 23, 11, 14, 16]
```

Clearly, Python has many ways to do things that look random. However, here's the hidden truth: they aren't really random. The computer that you are using can't generate random data. Instead, it uses tricks to create data that looks random; we call this *pseudorandom* data. Good pseudorandom algorithms are very important for cryptography and data security.

What if you want real random data? Some companies that are using the decay of radioactive materials to generate real random data. You can pay to download it. For our purposes, Python's pseudorandom numbers are quite sufficient.

If we generate two random numbers in the range 1 through 6 and add them together, we will have simulated rolling a pair of dice. Like this:

```
>>> a = random.randrange(6) + 1
>>> b = random.randrange(6) + 1
>>> a + b
8
```

First, let's write a program that just rolls the dice 100 times and shows the result. Make a file `dice.py`:

```
import random

roll_count = 100

for i in range(roll_count):
    a = random.randrange(6) + 1
    b = random.randrange(6) + 1
    roll = a + b
    print(f"Toss {i}: {a} + {b} = {roll}")
```

When you run it, you should see something like:

```
> python3 dice.py
Toss 0: 6 + 6 = 12
Toss 1: 4 + 4 = 8
Toss 2: 4 + 2 = 6
Toss 3: 4 + 6 = 10
Toss 4: 4 + 4 = 8
```

```
...
Toss 98: 5 + 2 = 7
Toss 99: 5 + 2 = 7
```

Now we want to count occurrences of each possible outcome. Let's use an array of integers. We will start with an array of zeros. When we roll a 3, we will add 1 to item 3 in the array. (We can never roll a zero or a one, so those two entries will always be zero.)

```
import random

roll_count = 100

# Make an array containing 13 zeros
counts = [0] * 13

for i in range(roll_count):
    a = random.randrange(6) + 1
    b = random.randrange(6) + 1
    roll = a + b
    print(f"Toss i: a + b = roll")

    # Increment the count for roll
    counts[roll] += 1

print(f"Counts: counts")
```

When you run this, at the end you will see a count for each possible outcome:

```
...
Toss 98: 3 + 2 = 5
Toss 99: 6 + 1 = 7
Counts: [0, 0, 2, 6, 16, 11, 13, 14, 11, 11, 6, 9, 1]
```

What was the count that we expected? For example, we expected to see a 2 about once every 36 rolls, right? It might be nice to compare our count to what we expected. Add a few more lines, and we are going to increase the number of rolls. You will probably want to delete the line that prints each roll separately:

```
import random

# Can't ever be 0 or 1
p = [0.0, 0.0, 1/36, 1/18, 1/12, 1/9, 5/36, 1/6, 5/36, 1/9, 1/12, 1/18, 1/36]
roll_count = 1000
```

```
# Make an array containing 13 zeros
counts = [0] * 13

for i in range(roll_count):
    a = random.randrange(6) + 1
    b = random.randrange(6) + 1
    roll = a + b

    # Increment the count for roll
    counts[roll] += 1

for i in range(2,13):
    print(f"i appeared counts[i] times, expected p[i] * roll_count:.1f")
```

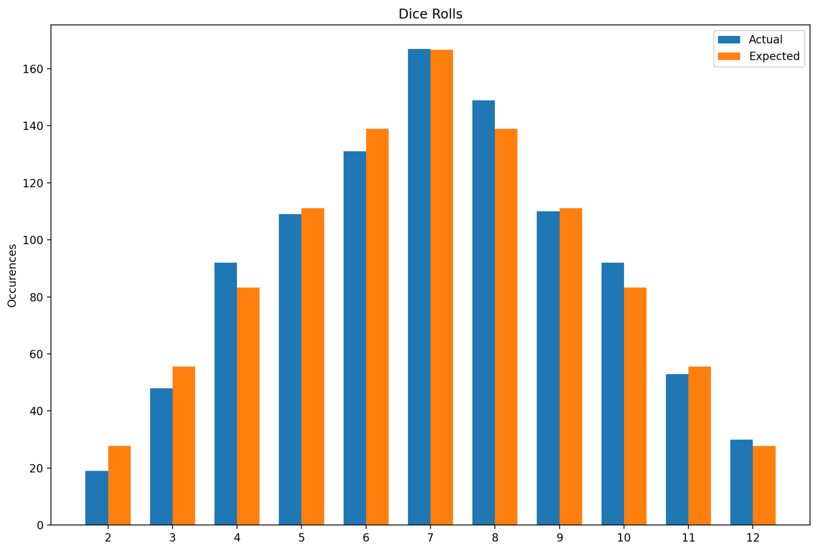
Now you should see something like:

```
2 appeared 39 times, expected 27.8
3 appeared 55 times, expected 55.6
4 appeared 84 times, expected 83.3
5 appeared 110 times, expected 111.1
6 appeared 160 times, expected 138.9
7 appeared 176 times, expected 166.7
8 appeared 124 times, expected 138.9
9 appeared 93 times, expected 111.1
10 appeared 87 times, expected 83.3
11 appeared 49 times, expected 55.6
12 appeared 23 times, expected 27.8
```

Whenever you are dealing with random numbers, the outcome will seldom be *exactly* what you expected. In this case, however, you should see that your predictions are pretty close.

1.4.1 Making a bar graph

A bar graph is a nice way to look at quantities like this. Let's make a bar graph that shows the actual count and the expected count:



We need to describe the set of rectangles, to do this we will loop through each possible roll (2 - 12) and put data in four lists for each:

```
import random
import matplotlib.pyplot as plt

# Can't ever be 0 or 1
p = [0.0, 0.0, 1/36, 1/18, 1/12, 1/9, 5/36, 1/6, 5/36, 1/9, 1/12, 1/18, 1/36]
roll_count = 1000

# Make an array containing 13 zeros
counts = [0] * 13

for i in range(roll_count):
    a = random.randrange(6) + 1
    b = random.randrange(6) + 1
    roll = a + b

    # Increment the count for roll
    counts[roll] += 1

# Gather data for bar chart
bar_width = 0.35
expected = []
actual_starts = []
expected_starts = []
labels = []
```

```
actual = []
for i in range(2,13):
    expected.append(p[i] * roll_count)
    actual.append(counts[i])
    actual_starts.append(i - bar_width/2)
    expected_starts.append(i + bar_width/2)
    labels.append(i)

fig, ax = plt.subplots()

# Create the bars
ax.bar(actual_starts, actual, bar_width, label='Actual')
ax.bar(expected_starts, expected, bar_width, label='Expected')
ax.set_xticks(labels)

# Provide labels
ax.set_ylabel('Occurences')
ax.set_title('Dice Rolls')
ax.legend()
plt.show()
```

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

probability of all 5's $= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \left(\frac{1}{6}\right)^3 = \frac{1}{216} \approx 0.0046$

Answer to Exercise 1 (on page 3)

probability of at least one heads = 1.0 - probability of all tails $= 1.0 - \left(\frac{1}{2}\right)^5 = 1.0 - \frac{1}{32} = \frac{31}{32} = \approx 0.97$

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