Chapter 1

Reflection

What happens when light hits a mass?

In a previous chapter, we talked about light as a wave, and we mentioned that each color in the rainbow is a different wavelength. You can also think of light as particles of energy called *photons*. Every photon comes with an amount of energy that determines what color it is. When we are talking about light interacting with objects, your intuition will be right more often if you think of light as a beam of photons.

When a photon comes from the sun and hits an object, one of several things can happen:

- The energy of the photon is absorbed by the object. It makes the object a little warmer. If a large proportion of photons hitting the mass are absorbed like this, we say the object is "black".
- The photon bounces off the object. If the surface is very smooth, the photons bounce in a predictable manner, and we call this *reflection* and we say the object is "shiny".
- If the surface is rough and the photons are not absorbed, the photons are scattered in random directions. We call this *diffusion*. If most of the photons hitting an object are bounced in random directions, we say that the object is "white".
- The photon passes through the mass. If the mass has smooth surfaces and a consistent composition, the photons will pass throught the mass in a predictable manner. We say that the mass is "transparent".
- If the photons pass through, but in an unpredictable, scattering manner, we say the mass is "translucent".

No object absorbs every photon, but chemists are always coming up with "blacker" materials. Vantablack, for example, is a super-black paint that absorbs 99.965% of all photons in the visible spectrum.

No object reflects every photon, but a mirror is pretty close. Let's talk about reflections in a mirror.

1.1 Reflection

When a beam of light hits a mirror, it bounces off the mirror at the same angle it approached from. That is, if it approaches nearly perpendicular to the mirror, it departs nearly perpendicular to the mirror. If it hits the mirror at a glancing angle, it departs at an angle close to the mirror's surface. A good video to see this in action can be found here: https://www.youtube.com/shorts/qA_VQZfTiUA

1.1.1 Law of Reflection

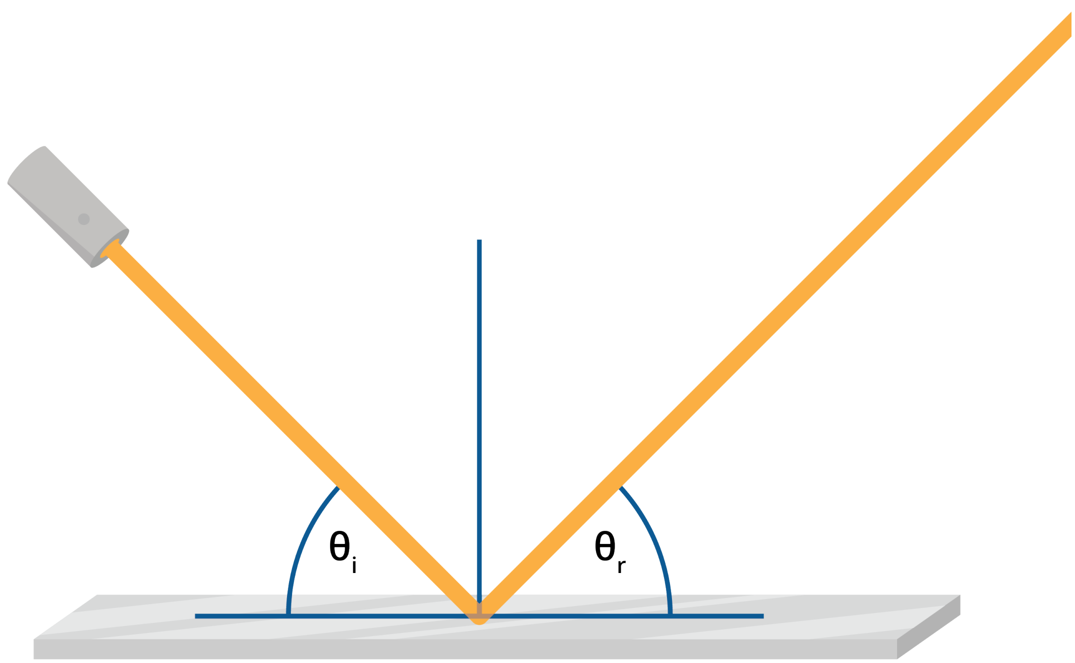


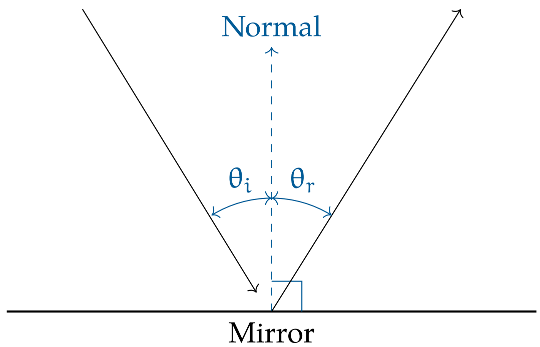
Figure 1.1: The angle of *incidence* is equal to the angle of *reflection*.

Law of Reflection

The angle of incidence, denoted as θ_i, is equal to the angle of reflection, denoted as θ_r. This law can be mathematically expressed as:

$$\theta_i = \theta_r$$

where θ_i is the angle between the incident light ray and the normal to the surface, and θ_r is the angle between the reflected light ray and the normal.



Exercise 1 Law of Reflection

You are standing 4 meters from a mirror hung on a wall. The bottom of the mirror is the same height as your chin, so you can't see your whole body. You stick a piece of masking tape to your body.

You walk forward until you are only 3 meters from the mirror, then put a piece of masking tape on your body at the new cut-off point. Is the new masking tape higher or lower on your body?

Working Space

Answer on Page 17

Exercise 2 Photons and Color

Working Space

There are red photons.

Are there black photons?

Are there white photons?

Are there yellow photons?

Answer on Page 17

1.2 Ellipses and Curved Mirrors

Flat mirrors are common and useful, but things get more interesting once you bend the mirrors. In this section, we are going to talk about a few different kinds of curved mirrors.

1.2.1 The Reflective Properties of Circles and Spheres

For example, if you were inside a circular room (a cylinder, actually), you could imagine standing in the center and pointing a flash light in any horizontal direction. The beam of light would bounce right back to you. See Figure 1.2

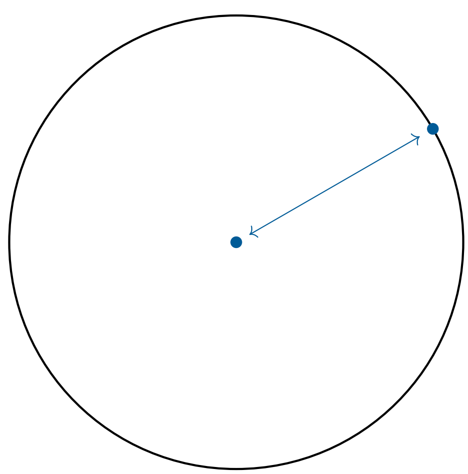


Figure 1.2: A circle would reflect back onto itself.

How do you know this? Because the tangent line is always perpendicular to the radius to the point of tangency:

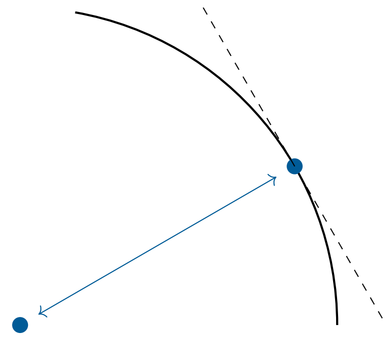


Figure 1.3: The reflection on a circle uses a radius which hits a tangent line as a perpendicular bisector.

You could create a spherical room with mirror walls. You would create a platform in the center where you could stand, and if you pointed your flashlight in any direction, its beam of light would shine back at you.

1.2.2 Ellipses and Ellipsoids

Intuitively, you know what an ellipse is: an oval. However, the ellipse is actually an oval with some special properties. This is a good time to talk about those properties.

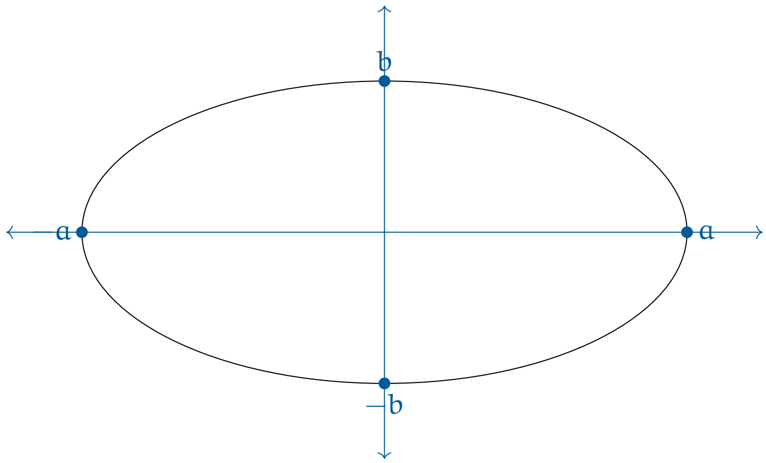
Mathematicians talk about a *standard* ellipse. A standard ellipse is centered on the origin $(0,0)$ and its long axis is parallel with the x-axis or the y-axis.

Equation for a Standard Ellipse

To be precise, a standard ellipse is the set of points (x,y) that are solutions to the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Note that $(a,0), (-a,0), (0,b), (0,-b)$ are all part of the set. The complete set looks like this:



The area contained inside this ellipse is given by

$$A = \pi ab$$

We can now talk about two special points: the *foci*. Each focal point is on the long axis of the ellipse. Let's assume for a moment that $a > b$. (Everything works the same if $b > a$, but it gets confusing if we try to deal with both cases simultaneously.)

If p is a point on the ellipse, the distance from p to focal point 1 plus the distance from p to focal point 2 is always $2a$.

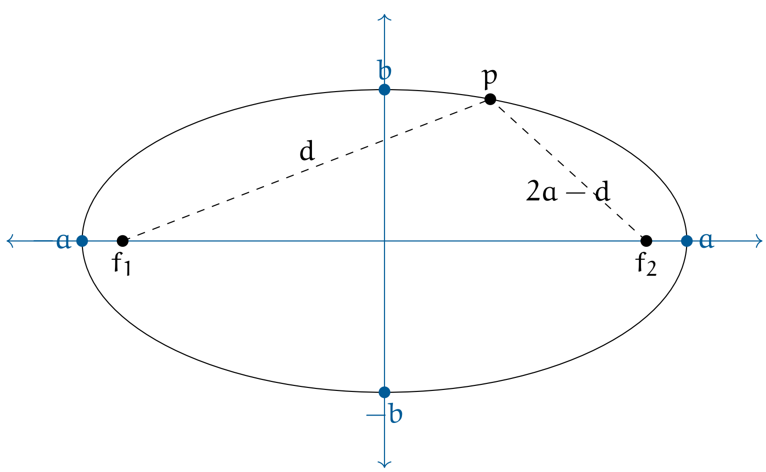


Figure 1.4: The both foci and point p forms a triangle with legs $2a - d$ or d.

How do we find the foci? We know they are on the long axis and that they are symmetrical across the short axis. All we need to know is how far are they from the short axis.

Distance from Center to the Foci

If you have an ellipse with a long axis that extends a from the center, and a short axis that extends b from the center, the foci lie on the long axis and are c from the center. Where

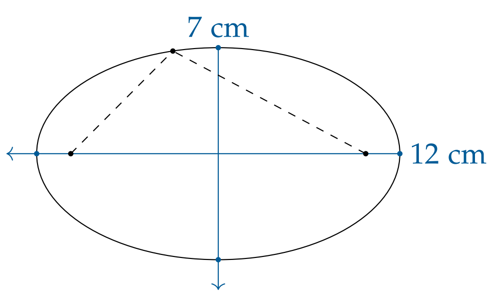
$$c = \sqrt{a^2 - b^2}$$

Exercise 3 Foci of an ellipse

Working Space

You need to draw an ellipse that is 12 cm long and 7 cm wide. You have a string, two pushpins, a ruler, and a pencil. Using the ruler, you draw two perpendicular axes.

You will stick one pin at each focal point. Each end of the string will be tied to a push pin. Using the pencil to keep the string taut, you will draw an ellipse.



How far from the short axis are the pushpins placed?

How long is the string between them?

Answer on Page 17

The Reflective Property of Ellipses

Here is something else that is wonderful about an ellipse: Pick any point p on the ellipse. Draw a line from p to each focal point. Draw the line tangent to the ellipse at p. You will see that the angle between the tangent and the line to focal point 1 is equal to the angle between the tangent and the line to focal point 2. See Figure 1.5

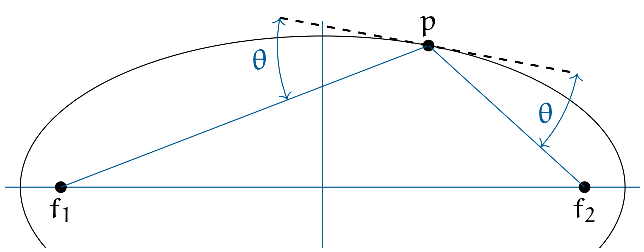


Figure 1.5: Reflective Property of Ellipses.

This is known as "The Reflective Property of Ellipses". Imagine you and your friend Fred are at an ellipse-shaped skating rink, and the edge of the rink is mirrored. You sit at one focal point and your friend sits at the other. If you point a flashlight at the mirror (in any direction!), the beam will bounce off the wall and head directly for Fred.

If Fred ducks out of the way, the beam will bounce again and head back to you.

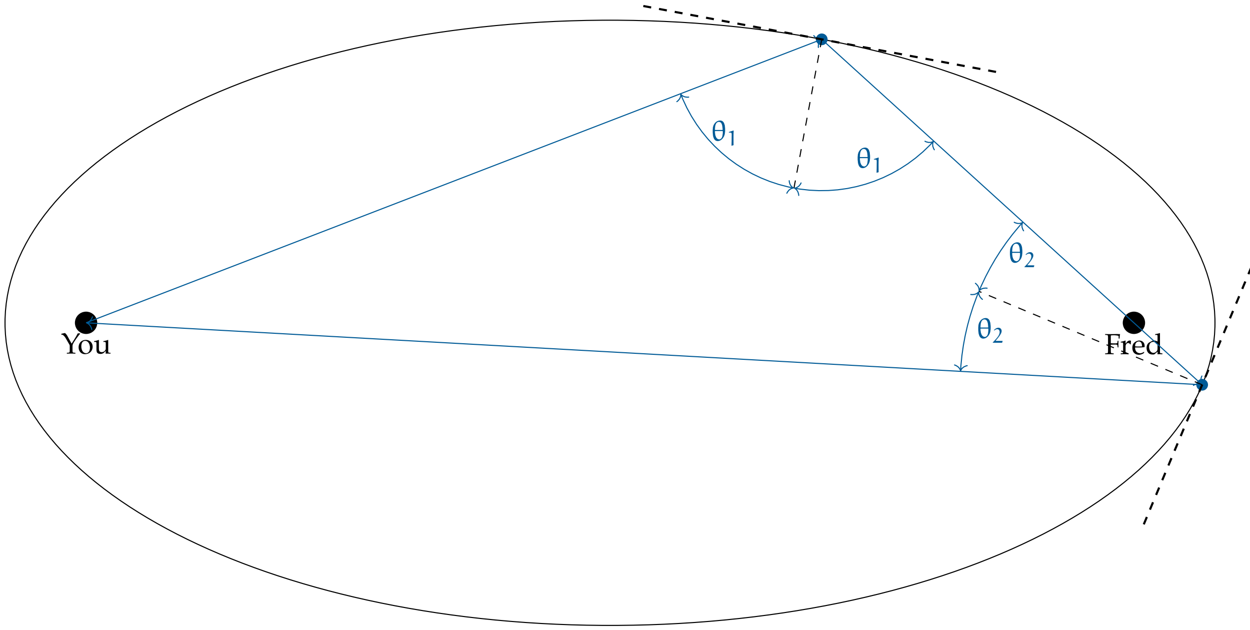


Figure 1.6: Your laser intersects Fred.

This will work for sound as well. If you whisper while on the focal point, Fred (at the other focal point) will hear you surprisingly well, because all the soundwaves that hit the wall will bounce (just like the light) straight at Fred.

1.2.3 Elliptical Orbits

One more fun fact about ellipses: We often imagine the planets traveling in circular orbits with the sun at the center — they actually travel in elliptical orbits, with the sun as one of the focal points.

The Earth is closest to the sun around January 3rd: 147 million km.

The Earth is farthest from the sun around July 3rd: 152 million km.

(Note that these dates are not the same as the solstices: The southern hemisphere is tilted the most toward the sun around December 21 and tilted most away around June 21.)

1.2.4 Ellipsoids

Just as we can pull the ideas of a circle into three dimensions to make a sphere, we can extend the ideas of the ellipse into three dimensions to talk about ellipsoids. Ellipsoids are like blimps.

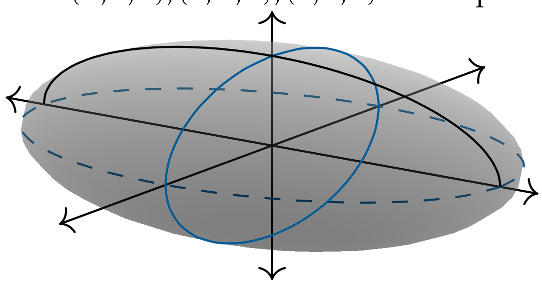
The standard ellipsoids are centered at the origin and aligned with the three axes.

Equation for a Standard Ellipsoid

To be precise, a standard ellipse is the set of points (x, y, z) that satify the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Note that $(a, 0, 0), (0, b, 0), (0, 0, c)$ are all part of the set. The complete set looks like



this:

The volume bounded by this ellipsoid is

$$V = \frac{4}{3}\pi abc$$

Of course, a, b, and c can be any positive number, but in the real world, we find ourselves working regularly with ellipsoids where two of the numbers are the same.

Oblate Spheroid

If two axes have the same length and one is shorter, you get something that looks like a sphere compressed in one direction — like a pumpkin. These are called *oblate spheroids*.

The Earth is actually an oblate spheroid; the axis that goes through the north and south pole is shorter than the axes that pass through the equator. How much shorter? Just a little, relatively speaking. The equator is 6,378 km from the center of the Earth; the north pole is 21 km closer.

Prolate Spheroid

If two axes have the same length and one is longer, you get something that looks like a sphere stretched in one direction — like a rugby ball. It is called a *prolate spheroid*.

Like an ellipse, prolate spheroids have two focal points.

Focal Points of a Prolate Spheroid

If the long axis has a radial length of a and the two shorter axes have radial length b, then the focal points are on the long axis. The distance from the center to the focal point is given by

$$c = \sqrt{a^2 - b^2}$$

For any point p on the prolate spheroid, the sum of the distances from p to the focus points will always be $2a$.

It has the reflective property: A photon shot in any any direction from one focal point will bounce off the wall and head directly at the other.

Exercise 4 Volume of Ruby Ball

Working Space

Some jokesters once thought it would be fun to make something that looked like a rugby ball, but made out of lead.

A rugby ball is about 30 cm long and has a circumference of 60 cm at its midpoint. A cubic centimeter of lead has a mass of 11.34 grams.

How much would a solid (not hollow) lead ruby ball weigh?

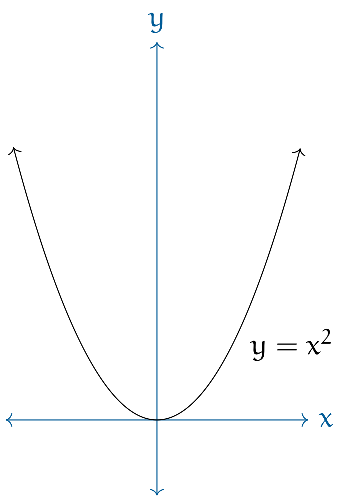
Answer on Page 18

1.2.5 Parabolas and Parabolic Reflectors

You are familiar with quadratic functions:

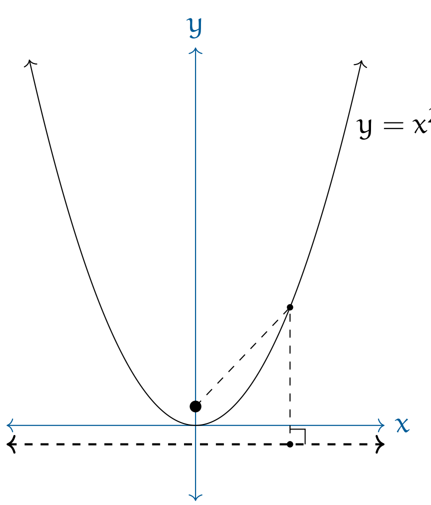
$$y = ax^2 + bx + c$$

If a is not zero, the graph of a quadratic is a curved line called a *parabola*. The first parabola that most mathematicians think of is the graph of $y = x^2$:



Every parabola has a *focus* and a *directix*. The focus is a point on the parabola's axis of symmetry. The directrix is a line perpendicular to the axis of symmetry. Every point on the parabola is equal distance from the focus and the directrix.

For the graph of $y = x^2$, the focus is the point $(0, \frac{1}{4})$. The directrix is the line $y = -\frac{1}{4}$:



For example, the point $(1, 1)$ is on this parabola. It is $5/4$ from the directorix. How far is it from the focus? 1 horizontally and $3/4$ vertically.

$$\sqrt{1^2 + \left(\frac{3}{4}\right)^2} = \frac{5}{4}$$

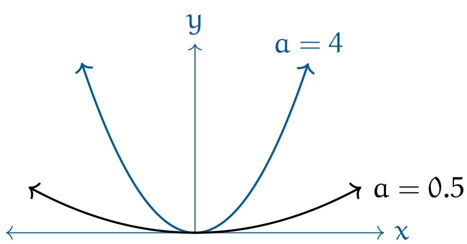
Thus, we have confirmed that $(1, 1)$ is equal distances from the focus and the directrix.

When we think about a parabola and its properties, we usually rotate and translate it to be symmetric around the y-axis, flip it so that it is low in the middle and rising on both sides, and push it up or down until the low point is is on the x-axis.

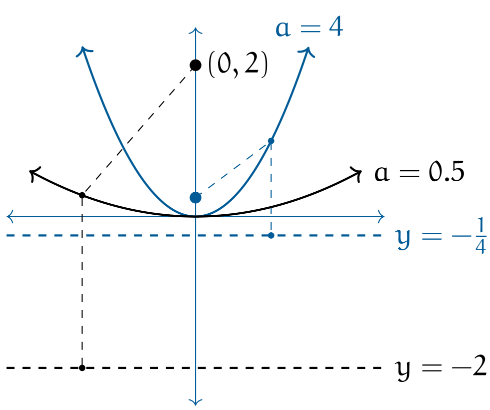
Then, they can all be written:

$$y = \frac{a}{4}x^2$$

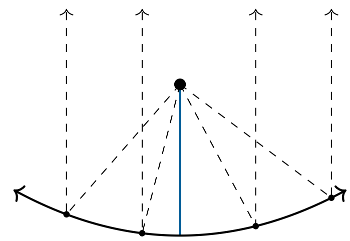
where $a > 0$. If a is small, the parabola opens wider.



The focus is at $(0, \frac{1}{a})$ and the directorix is the line $y = -\frac{1}{a}$.



Reflective Property of a Parabola

Assume you have a parabola-shaped mirror. A beam of light shot from the focus will bounce off the mirror in the direction of the axis of symmetry: 

This is why your flashlight has a parabolic mirror. The lightbulb is at the focus, so any photons that hit the mirror are redirected straight forward.

(Note that in the real world, we use parabolic dishes: a parabola rotated around its axis of symmetry.)

The reflection works exactly the same in reverse. There are solar cookers that are big parabolic mirrors. They let you put a pot on the focus point. You move the dish until its axis of symmetry is pointed at the sun.

You will also see a lot of antennas have parabolic dishes. Note that photons that come in parallel to the axis of symmetry are redirected to a single point: where the receiver is.

Figure 1.7: A real life parabolic dish as a satellite.

Sometimes in a science museum, you will see two parabolic dishes far apart and pointed at each other. One person speaks with their mouth at the focus of one. The other person listens with their ear at the focus of the other. Even though you are very far apart, it sounds like they are really, really close.

Figure 1.8: Two parabolic dishes send waves between eachother.

This is because the speaker's parabolic wall focuses the sound energy in a nice beam the size of the wall pointed straight at the listener's parabolic wall. The listener's wall focuses the energy of that beam at the listener's ear.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

Assuming the mirror is truly vertical and the floor is truly horizontal, the new cut off should be exactly the same as the old one: It should be below your chin the same amount that your eyes are above your chin.

Illustration Here

Answer to Exercise 2 (on page 4)

Are there white photons? No. What we call "white" is a blend of photons that are several different colors.

Some people like to say white light is the combination of all visible colors of photons in equal amounts. That seems oddly specific and unusual.

Maybe it is better to imagine it from the human experience of white light. In our eyes, we have three different types of color-sensing cones, which generally correspond to the the red, green, and blue regions of the spectrum. When all three are excited about equal amounts, humans experience that as white. On your computer screen, for example, what you see as white is just a blend of three colors of photons: a red, a green, and a blue.

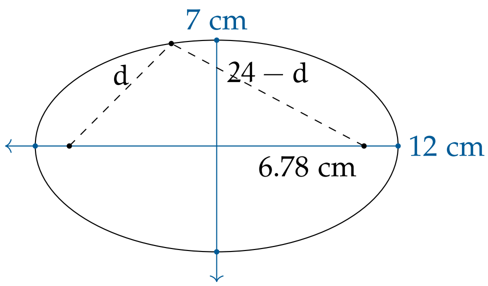
Are there black photons? No. What we call "black" is an absence of photons in the visible range.

Are there yellow photons? Yes! There is a region of the color spectrum that is yellow. It has a wavelength of about 527 nm. Photons at this energy level excite both our green-sensitive and red-sensitive cones (we well expand on this in the eye chapter!). Your computer monitor does not actually create light with a 527 nm wavelength. Instead, it creates red light and green light, which our eyes interpret as yellow.

Answer to Exercise 3 (on page 7)

The length of the string is easy: $2 \times 12 = 24$ cm.

The distance from the center to the focal point is $\sqrt{12^2 - 7^2} \approx 6.78$ cm.



Answer to Exercise 4 (on page 11)

We need the distance from the center out to each of the three axes. We know that $a = \left(\frac{1}{2}\right) 30 = 15$ cm.

We can calculate the b and c (which are equal) using the circumference given: $2b\pi = 60$, so $c = b \approx 9.55$ cm.

The volume, then is

$$V = \frac{4}{3}\pi(15)(9.55)(9.55) \approx 5,730 \text{ cubic centimeters}$$

The mass would be $5,730 \times 11.34 = 64,973$ grams or about 65 kg.

Index