CHAPTER 1

Applications of Double Integrals

1.1 Total Mass and Charge

Suppose there is a generic, thin layer (called a *lamina*) with a variable density that occupies an area B (see figure 1.1). Further, let the density of the lamina be described by a function, $\rho(x, y)$, which is continuous over B. For some small rectangle centered at (x, y), the density is given by:

$$\rho(x, y) = \frac{\Delta m}{\Delta A}$$

where Δm is the mass of the small rectangle and ΔA is the area. Then the mass of the rectangle is given by:

$$\Delta m = \rho(x, y)\Delta A$$

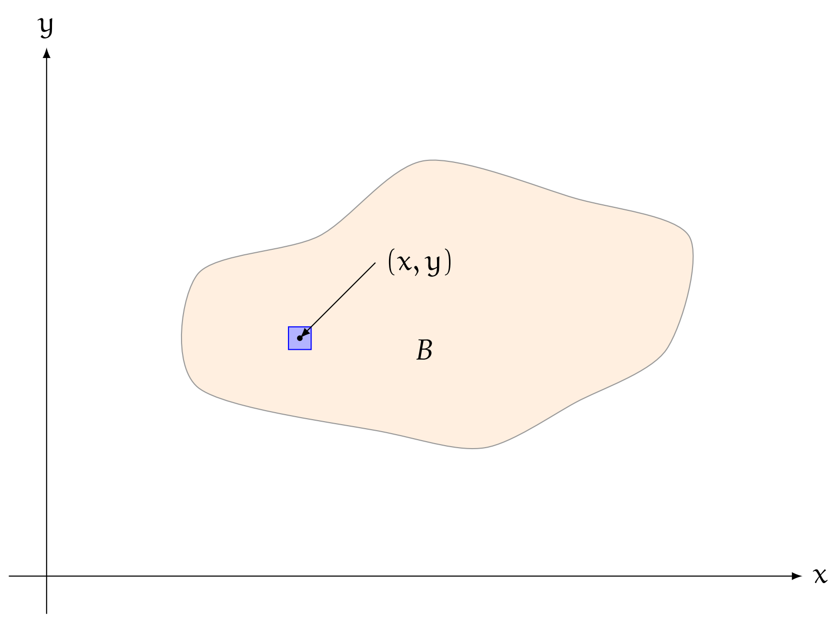

Figure 1.1: A generic lamina that occupies the region B

We can find the mass of the entire lamina by dividing it into many of these small rectangles and adding the masses of all the rectangles (see 1.2). Just like in previous examples, there is some point (x_{ij}^*, y_{ij}^*) in each rectangle, R_{ij}, such that the mass of the part of the lamina that occupies R_{ij} is $\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Adding all these masses yields:

$$m_{total} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

Taking the limit as $m, n \to \infty$ increases the number of rectangles to yield the true total mass:

$$m_{total} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_B \rho(x, y)\, dA$$

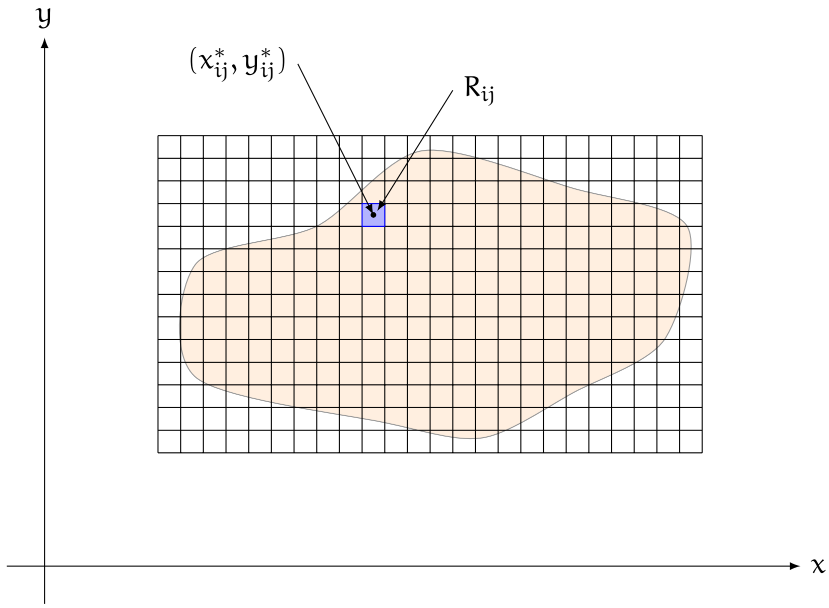

Figure 1.2: A generic lamina divided into many rectangles

Example: Find the total mass of a lamina that occupies the region $D = \{(x, y) \mid 1 \leq x \leq 3, 1 \leq y \leq 4\}$ with a density function $\rho(x, y) = 3y^2$.

Solution: We know that the total mass is given by:

$$\iint_D 3y^2\, dA$$

Applying Fubini's theorem, we see that:

$$\iint_D 3y^2\, dA = \int_1^3 \int_1^4 3y^2\, dy\, dx$$

$$= \int_1^3 \left[y^3\right]_{y=1}^{y=4} dx = \int_1^3 \left[4^3 - 1^3\right] dx$$

$$= \int_1^3 63\, dx = 63x|_{x=1}^{x=3} = 126$$

Exercise 1 Finding Total Mass

Find the mass of the lamina that occupies the region, D, and has the given density function, ρ.

1. $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}; \rho(x, y) = 1 + x^2 + y^2$
2. D is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

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This method applies not only to mass density, but any other type of density. Some examples could include animals per acre of forest, cells per square centimeter of petri dish, or people per city block. A density physicists are often interested in is charge density (that is, the amount of charge, Q, per unit area). Charge is measured in coulombs (C). Often, charge density is given by a function, $\sigma(x, y)$, in units of coulombs per area (such as cm^2 or m^2). If there is some region, D, with charge distributed across it such that the charge density can be described by a continuous function, $\sigma(x, y)$, then the total charge, Q, is given by:

$$Q = \iint_D \sigma(x, y)\, dA$$

Example: Charge is distributed over the region B shown in figure 1.3 such that the charge density is given by $\sigma(x, y) = xy$, measured in C/m^2. Find the total charge.

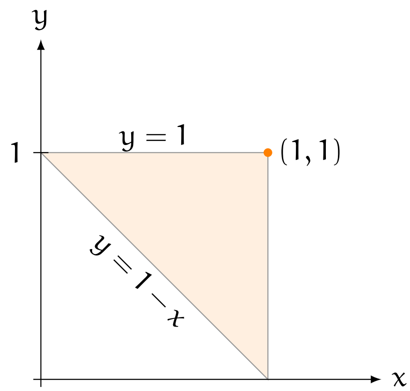


Figure 1.3: A triangular region over which charge is distributed such that $\sigma(x, y) = xy$

Solution: We know that total charge is given by:

$$Q = \iint_B xy \, dA$$

Examining figure 1.3, we see that:

$$\begin{aligned}
\iint_B xy \, dA &= \int_0^1 \int_{1-x}^1 xy \, dy \, dx \\
&= \int_0^1 \frac{x}{2} \left[y^2\right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2} \left[1^2 - (1-x)^2\right] dx \\
&= \frac{1}{2} \int_0^1 x \left(1 - 1 + 2x - x^2\right) dx = \frac{1}{2} \int_0^1 x \left(2x - x^2\right) dx \\
&= \frac{1}{2} \int_0^1 2x^2 - x^3 \, dx = \frac{1}{2} \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{x=0}^{x=1} = \frac{1}{2}\left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{24} C
\end{aligned}$$

1.2 Center of Mass

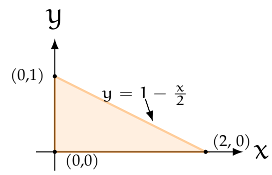
For a thin disk (lamina) of variable density in the xy-plane, the coordinates of the center of mass, $(\bar{x}, \bar{y})$, are given by:

$$\bar{x} = \frac{1}{m} \iint_D x\rho(x, y) \, dA$$
$$\bar{y} = \frac{1}{m} \iint_D y\rho(x, y) \, dA$$

where m is the total mass and ρ is the density of the lamina as a function of x and y.

Example: Find the center of mass of a triangular lamina with vertices at $(0, 0)$, $(2, 0)$, and $(0, 1)$ and a density function $\rho(x, y) = 2 + x + 3y$.

Solution: We begin by visualizing the region so we can determine if it is type I or type II:



Recall that the total mass is given by $m = \iint_D \rho(x, y)\, dA$. As shown above, we can define $D = \{(x, y)\,|0 \le x \le 2, 0 \le y \le 1 - \frac{x}{2}\}$:

$$m = \int_0^2 \int_0^{1-x/2} (2 + x + 3y)\ dy\, dx = \int_0^2 \left[2y + xy + \frac{3}{2}y^2\right]_{y=0}^{y=1-x/2} dx$$

$$= \int_0^2 \left[2(1 - \frac{x}{2}) + x(1 - \frac{x}{2}) + \frac{3}{2}(1 - \frac{x}{2})^2\right] dx = \int_0^2 \left[\frac{7}{2} - \frac{3x}{2} - \frac{x^2}{8}\right] dx$$

$$= \left[\frac{7x}{2} - \frac{3x^2}{4} - \frac{x^3}{24}\right]_{x=0}^{x=2} = \frac{7(2)}{2} - \frac{3(4)}{4} - \frac{8}{24} = 7 - 3 - \frac{1}{3} = \frac{11}{3}$$

Finding $\overline{x}$:

$$\overline{x} = \frac{1}{m} \iint_D x\,(2 + x + 3y)\ dA$$

$$\overline{x} = \frac{3}{11} \int_0^2 \int_0^{1-x/2} \left[2x + x^2 + 3xy\right] dy\, dx$$

$$\overline{x} = \frac{3}{11} \int_0^2 \left[2xy + x^2y + \frac{3}{2}xy^2\right]_{y=0}^{y=1-x/2} dx$$

$$\overline{x} = \frac{3}{11} \int_0^2 \left[\frac{7x}{2} - \frac{3x^2}{2} - \frac{x^3}{8}\right] dx$$

$$\overline{x} = \frac{3}{11} \left[\frac{7x^2}{4} - \frac{x^3}{2} - \frac{x^4}{32}\right]_{x=0}^{x=2}$$

$$\overline{x} = \frac{3}{11} \left[\frac{7(4)}{4} - \frac{8}{2} - \frac{16}{32}\right] = \frac{3}{11}\left(7 - 4 - \frac{1}{2}\right) = \frac{3}{11}\left(\frac{5}{2}\right) = \frac{15}{22}$$

We can similarly find $\overline{y}$:

$$\overline{y} = \frac{1}{m} \iint_D y\,(2 + x + 3y)\ dA$$

$$\overline{y} = \frac{3}{11} \int_0^2 \int_0^{1-x/2} \left[2y + xy + 3y^2\right] dy\, dx$$

$$\overline{y} = \frac{3}{11} \int_0^2 \left[y^2 + \frac{x}{2}y^2 + y^3\right]_{y=0}^{y=1-x/2} dx$$

$$\overline{y} = \frac{3}{11}\int_0^2 \left[\left(1-\frac{x}{2}\right)^2 + \frac{x}{2}\left(1-\frac{x}{2}\right)^2 + \left(1-\frac{x}{2}\right)^3\right] dx$$

$$\overline{y} = \frac{3}{11}\int_0^2 \left[2-2x+\frac{x^2}{2}\right] dx = \frac{3}{11}\left[2x - x^2 + \frac{x^3}{6}\right]_{x=0}^{x=2}$$

$$\overline{y} = \frac{3}{11}\left[2(2)-2(2)+\frac{8}{6}\right] = \frac{3}{11}\left(\frac{4}{3}\right) = \frac{4}{11}$$

Therefore, the center of mass $(\overline{x}, \overline{y})$ is $(\frac{15}{22}, \frac{4}{11})$.

Exercise 2 Center of Mass

Find the center of mass of:

1. A lamina that occupies the area enclosed by the curves $y = 0$ and $y = 2\sin x$ from $0 \leq x \leq \pi$ if its density is given by $\rho(x, y) = x$.
2. The region D if $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$; $\rho(x, y) = 1 + x^2 + y^2$
3. The triangular region D with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

Working Space

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1.3 Moment of Inertia

We can also use double integrals to find the **moment of inertia** of a lamina about a particular axis (we will extend this to three-dimensional objects in the next chapter on triple integrals). Recall that the moment of inertia for a particle with mass m a distance r from the axis of rotation is mr^2. Dividing a lamina into small pieces, we see that the moment of inertia of each piece about the x-axis is:

$$\left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

Where x_{ij}^* and y_{ij}^* are the x- and y-coordinates of the small piece. The moment of inertia of the entire lamina about the x-axis is then the sum of all the individual moments:

$$I_x = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_D y^2 \rho(x, y)\, dA$$

Similarly, the moment of inertia of a lamina about the y-axis is:

$$I_y = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_D x^2 \rho(x, y)\, dA$$

Example: Find the moment of inertia of a square centered at the origin with side length r and constant density ρ about the x-axis.

Solution: We can describe the square as the region bounded by $D = \{(x, y) | -r/2 \leq x \leq r/2, -r/2 \leq y \leq r/2\}$ with density function $\rho(x, y) = \rho$. Therefore, the moment of inertia about the x-axis is given by:

$$\begin{aligned}
I_x &= \int_{-r/2}^{r/2} \int_{-r/2}^{r/2} \rho y^2 \, dy \, dx \\
&= \rho \int_{-r/2}^{r/2} \left[\frac{1}{3} y^3\right]_{y=-r/2}^{y=r/2} dx \\
&= \frac{\rho}{3} \int_{-r/2}^{r/2} \left[\frac{r^3}{8} - \left(-\frac{r^3}{8}\right)\right] dx \\
&= \frac{\rho}{3} \int_{-r/2}^{r/2} \left[\frac{r^3}{4}\right] dx = \frac{r^3 \rho}{12} \int_{-r/2}^{r/2} 1 \, dx \\
&= \frac{r^3 \rho}{12} [x]_{x=-r/2}^{x=r/2} = \frac{r^3 \rho}{12} \cdot r = \frac{r^4 \rho}{12}
\end{aligned}$$

We can also find the moment of inertia about the origin, I_o. This is the moment of inertia for an object rotating in the xy-plane about the origin. The moment of inertia about the origin is the sum of the moments of inertia about the x- and y-axes:

$$I_o = I_x + I_y = \iint_D \left(x^2 + y^2\right) \rho(x, y)\, dA$$

Example: Find the moment of inertia about the origin of a disk with density $\rho(x, y) = b$, centered at the origin, with a radius of a. Show that this is equal to the expected moment of inertia, $\frac{1}{2}MR^2$, where M is the total mass of the disk and R is the radius of the disk.

Solution: Since we are examining a circle about the origin, the region can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$. Converting from Cartesian coordinates to polar coordinates:

$$\begin{aligned} I_o &= \iint_D \left(x^2 + y^2\right) b\, dA = \int_0^a \int_0^{2\pi} r \left(r^2\right) b\, d\theta\, dr \\ &= \int_0^a r^3\, dr \cdot \int_0^{2\pi} b\, d\theta = \frac{a^4}{4} \cdot (2\pi b) = \frac{\pi a^4 b}{2} \end{aligned}$$

The total mass of this disk is the density, b, multiplied by the area, πa^2. Therefore,

$$R = a$$

$$M = \pi a^2 b$$

Substituting into the result of our double integral, we see that:

$$\frac{\pi a^4 b}{2} = \left(\pi a^2 b\right) \cdot \left(\frac{a^2}{2}\right) = M \cdot \frac{R^2}{2} = \frac{1}{2} M R^2$$

1.3.1 Radius of Gyration

When modeling rotating objects, it can be helpful to have a simplified model. A spinning, continuous object can be modeled as a point mass by using the lamina's *radius of gyration*. The radius of gyration of a lamina about the origin is a radius, R, such that:

$$mR^2 = I_o$$

where m is the mass of the lamina and I is the moment of inertia of the lamina. Essentially, we are finding a radius such that if the lamina were shrunk down to a point mass and rotated about the axis at that radius, the moment of inertia would be the same.

We can also find radii of gyration about the x- and y-axes:

$$m\overline{\overline{y}}^2 = I_x$$

$$m\overline{\overline{x}}^2 = I_y$$

About the origin, $R = \sqrt{\overline{\overline{x}}^2 + \overline{\overline{y}}^2}$.

Example: Find the radius of gyration about the y-axis for a disk with density $\rho(x, y) = y$ if the disk has radius 2 and is centered at $(0, 2)$.

Solution: We are ultimately looking for a radius such that $m\overline{\overline{x}}^2 = I_y$, so we need to know the mass, m, and the moment of inertia about the y-axis, I_y. First, let's find the total mass, m, of the disk. We can describe the disk in polar coordinates as $D = \{(r, \theta) \mid 0 \le r \le 4\sin\theta, 0 \le \theta \le \pi\}$, and therefore the mass is given by:

$$m = \iint_D y\, dA = \int_0^{\pi}\int_0^{4\sin\theta} r\,(r\sin\theta)\, dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta \int_0^{4\sin\theta} r^2\, dr\, d\theta = \frac{1}{3}\int_0^{\pi} \sin\theta\, [4\sin\theta]^3\, d\theta$$

$$= \frac{64}{3}\int_0^{\pi} \sin^4\theta\, d\theta = \frac{64}{3}\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = \frac{64}{3}\left(\frac{1}{2}\right)^2 \int_0^{\pi} \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{16}{3}\left[(\theta - \sin 2\theta)_{\theta=0}^{\theta=\pi} + \int_0^{\pi} \frac{1+\cos 4\theta}{2}\, d\theta\right] = \frac{16}{3}\left[\pi + \frac{1}{2}\left(\theta + \frac{1}{4}\sin 4\theta\right)_{\theta=0}^{\theta=\pi}\right]$$

$$= \frac{16}{3}\left[\pi + \frac{\pi}{2}\right] = \frac{16}{3}\left(\frac{3\pi}{2}\right) = 8\pi$$

Now that we have found the mass, let's find the moment of inertia, I_y:

$$I_y = \iint_D x^2 y\, dA = \int_0^{\pi}\int_0^{4\sin\theta} r\,(r\cos\theta)^2\,(r\sin\theta)\, dr\, d\theta$$

$$= \int_0^{\pi}\left[\cos^2\theta\sin\theta\int_0^{4\sin\theta} r^4\, dr\right] d\theta = \int_0^{\pi}\cos^2\theta\sin\theta\left[\frac{1}{5}r^5\right]_{\theta=0}^{\theta=4\sin\theta} d\theta$$

$$= \frac{1024}{5}\int_0^{\pi}\cos^2\theta\sin^6\theta\, d\theta = \frac{1024}{5}\int_0^{\pi}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)^3 d\theta$$

$$= \frac{1024}{5}\left(\frac{1}{2}\right)^4\int_0^{\pi}(1+\cos 2\theta)(1-\cos 2\theta)(1-\cos 2\theta)^2\, d\theta$$

$$= \frac{64}{5}\int_0^{\pi}\left(1-\cos^2 2\theta\right)\left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{64}{5}\int_0^{\pi} 1 - 2\cos 2\theta + \cos^2 2\theta - \cos^2 2\theta + 2\cos^3 2\theta - \cos^4 2\theta\, d\theta$$

$$= \frac{64}{5}\int_0^{\pi} 1 - 2\cos 2\theta + 2\cos 2\theta\left(1-\sin^2 2\theta\right) - \left(\frac{1+\cos 4\theta}{2}\right)^2 d\theta$$

$$= \frac{64}{5}\int_0^{\pi} 1 + 2\cos 2\theta \sin^2 2\theta - \frac{1}{4}\left(1 + 2\cos 4\theta + \cos^2 4\theta\right)\, d\theta$$
$$= \frac{64}{5}\int_0^{\pi} 1 + 2\cos 2\theta \sin^2 2\theta - \frac{1}{4} - \frac{\cos 4\theta}{2} - \frac{1}{4}\left(\frac{1 + \cos 8\theta}{2}\right)\, d\theta$$
$$= \frac{64}{5}\left[\frac{5\theta}{8} + \frac{1}{3}\sin^3\theta - \frac{\sin 4\theta}{8} - \frac{\sin 8\theta}{64}\right]_{\theta=0}^{\theta=\pi} = \frac{64}{5}\cdot\frac{5\pi}{8} = 8\pi$$

We have found that $m = 8\pi$ and $I_y = 8\pi$. Substituting to find the radius of gyration:

$$m\bar{\bar{x}}^2 = I_y$$
$$(8\pi)\bar{\bar{x}}^2 = 8\pi$$
$$\bar{\bar{x}} = 1$$

Therefore, the radius of gyration about the y-axis is $\bar{\bar{x}} = 1$

Exercise 3 Moments of Inertia and Radii of Gyration

Find the requested moment of inertia and radius of gyration of the lamina with the given density function.

1. about the x-axis, $D = \{(x, y) \mid 1 \le x \le 4,\ 0 \le y \le 3\}$, $\rho(x, y) = xy$.
2. about the y-axis, D is enclosed by the curves $y = 0$ and $y = 2\cos x$ for $-\pi/2 \le x \le \pi/2$, $\rho(x, y) = x$.
3. about the origin, $D = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$, $\rho(r, \theta) = r$.

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1.4 Surface Area

We have already seen how to find the areas of surfaces of revolution using single-variable calculus. Now, we will use multivariable calculus to find the surface area of a generic, two-variable function, $z = f(x, y)$. Suppose a surface, S, is defined by the continuous, partially differentiable function, $f(x, y)$, over a rectangular region, R (see figure 1.4).

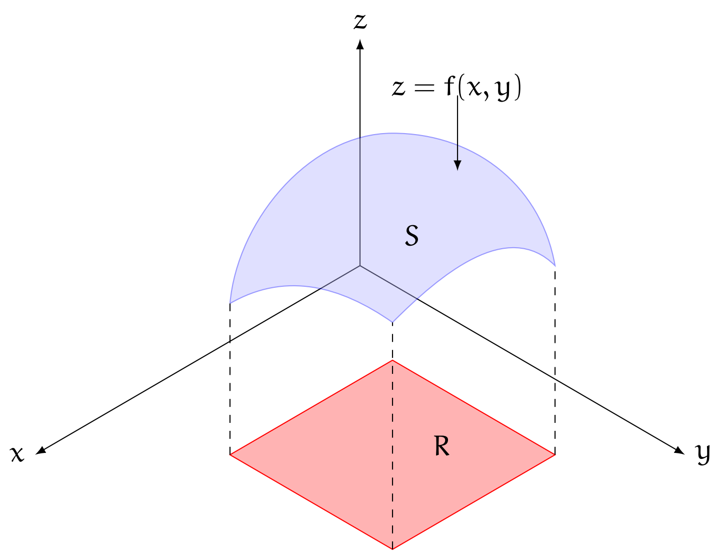


Figure 1.4: The graph of f over the region R creates a surface, S

We begin by dividing the region, R, into sub-rectangles, R_{ij}, each with area $\Delta A = \Delta x \Delta y$. Then, projecting upwards from the point closest to the origin, $(x_i, y_j, 0)$, we find a point on the surface, $P_{ij} = (x_i, y_j, f(x_i, y_j))$. Next, there is a small plane, ΔT_{ij}, tangent to the surface at P_{ij}, and the area of the tangent plane is approximately the same as the area of the surface over the sub-rectangle R_{ij} (see figure 1.5).

It follows that the total surface area of the surface, S, is the sum of all these little tangent surfaces as the number of tangent surfaces approaches infinity:

$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

How can we find an expression for ΔT_{ij}? We will define two vectors, $\mathbf{a}$ and $\mathbf{b}$, that are equal to the sides of ΔT_{ij} (see figure 1.6). Geometrically, the area of ΔT_{ij} is the absolute value of the cross product of the two vectors. Mathematically,

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$$

Recall the three unit vectors: $\mathbf{i}$ in the x-direction, $\mathbf{j}$ in the y-direction, and $\mathbf{k}$ in the z-direction. We can then describe $\mathbf{a}$ and $\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$:

$$\mathbf{a} = \Delta x \mathbf{i} + f_x(x_i, y_j) \Delta x \mathbf{k}$$

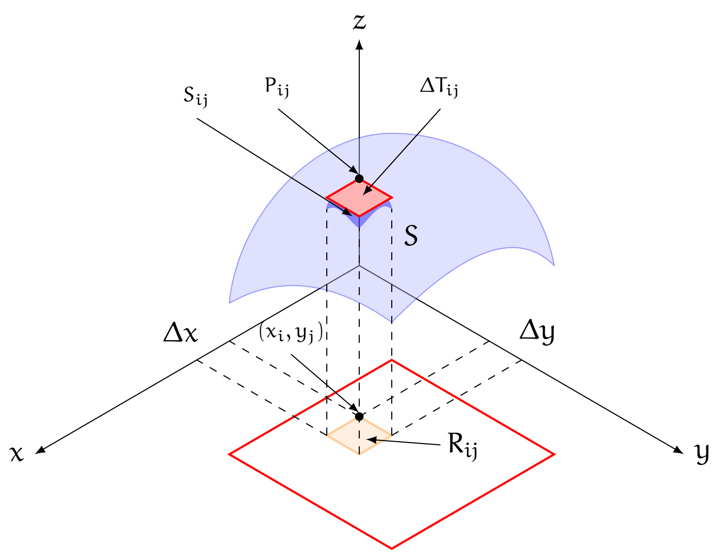


Figure 1.5: The tangent surface, ΔT_{ij}, is approximately the same surface area as the surface, S_{ij}, over the sub-rectangle, R_{ij}

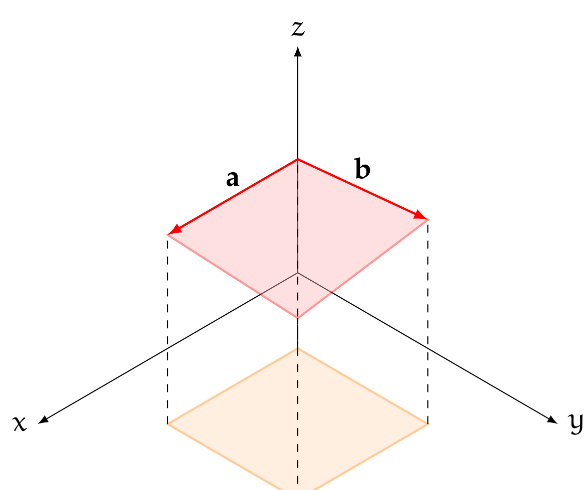


Figure 1.6: The vectors **a** and **b** define the sides of the tangent surface ΔT_{ij}

$$\mathbf{b} = \Delta y\mathbf{j} + f_y(x_i, y_j)\Delta y\mathbf{k}$$

(Recall that f_x is the partial derivative of $f(x, y)$ with respect to x, and f_y is the partial derivative with respect to y.) This is true because the partial derivative of f_x gives the slope of a tangent line parallel to the x-axis, and f_y parallel to the y-axis. We then find an expression for $|\mathbf{a} \times \mathbf{b}|$ (we've omitted some details here):

$$\mathbf{a} \times \mathbf{b} = -f_x(x_i, y_j)\Delta x\Delta y\mathbf{i} - f_y(x_i, y_j)\Delta x\Delta y\mathbf{j} + \Delta x\Delta y\mathbf{k}$$

Substituting $\Delta A = \Delta x\Delta y$:

$$\mathbf{a} \times \mathbf{b} = [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A$$

To find the area of ΔT_{ij}, we need to find the length of $\mathbf{a} \times \mathbf{b}$. Recall that we can use the Pythagorean theorem to find the length of a vector. For a 3-dimensional vector $\mathbf{v} = r\mathbf{i} + s\mathbf{j} + t\mathbf{k}$, it's length is given by:

$$|\mathbf{v}| = \sqrt{r^2 + s^2 + t^2}$$

Applying this, we find the length of $\mathbf{a} \times \mathbf{b}$ (which is the same as the area of ΔT_{ij}) is:

$$\Delta T_{ij} = |\,[-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A|$$

$$= \sqrt{(-f_x(x_i, y_j)\Delta A)^2 + (-f_y(x_i, y_j)\Delta A)^2 + (\Delta A)^2}$$

$$= \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\Delta A$$

So, the area of the entire surface over region R is:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\Delta A$$

This is the definition of a double integral; therefore, the surface area of a two-variable function, $f(x, y)$ over a region, R, where f_x and f_y are continuous, is:

$$A(S) = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Using the notation of partial derivatives, this is also expressed as:

$$A(S) = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example: Find the surface area of the part of the surface $z = 2 - y^2$ that lies over the triangle whose vertices are at $(0,0)$, $(0,4)$, and $(3,4)$.

Solution: We can define $R = \{(x,y) \mid 0 \le x \le 34y,\ 0 \le y \le 4\}$. Additionally,

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = -2y$$

Therefore, the area of the surface that lies above R is:

$$A(S) = \iint_R \sqrt{1 + 0^2 + (-2y)^2}\, dA = \int_0^4 \int_0^{\frac{3}{4}y} \sqrt{1+4y^2}\, dx\, dy$$

$$= \int_0^4 \sqrt{1+4y^2}\, [x]_{x=0}^{x=\frac{3}{4}y}\ dy = \frac{3}{4}\int_0^4 y\sqrt{1+4y^2}\, dy$$

Let $u = 1 + 4y^2$, then $du = (8y)dy$ and $(y)dy = \frac{du}{8}$. Substituting:

$$A(S) = \frac{3}{4}\int_{y=0}^{y=4} \frac{1}{8}\sqrt{u}\, du = \frac{3}{32}\left[\frac{2}{3}u^{3/2}\right]_{y=0}^{y=4}$$

$$= \frac{1}{16}\left[\left(1+4y^2\right)^{3/2}\right]_{y=0}^{y=4} = \frac{1}{16}\left[(65)^{3/2} - 1\right] \approx 32.69$$

Exercise 4 Surface Area of Two-Variable Functions

Find the area of the surface.

1. The part of the plane $9x+6y-3z+6=0$ that lies above the rectangle $[2,6]\times[1,4]$.
2. The part of the paraboloid in the circle $z = 2x^2 + 2y^2$ that lies under the plane $z = 32$.
3. The part of the surface $z = 3xy$ that lies in the cylinder $x^2 + y^2 = 4$.

Working Space

Answer on Page 23

1.5 Average Value

Recall that the average value of a one-variable function over the interval $x \in [a, b]$ is given by:

$$f_{ave} = \frac{1}{a-b} \int_a^b f(x)\, dx$$

For a two-variable function, the average value over a region, R, is given by:

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

Where $A(R)$ is the area of the two-dimensional region.

Example: Find the average value of $f(x, y) = xy^2$ over the rectangle with vertices at $(-2, 0)$, $(-2, 4)$, $(2, 4)$, and $(2, 0)$.

Solution: The rectangular region has an area of $(2-(-2)) \cdot (4-0) = 4 \cdot 4 = 16$. Therefore, the average value is given by:

$$f_{ave} = \frac{1}{16} \iint_R xy^2\, dA = \frac{1}{16} \int_{-2}^{2} \int_0^4 xy^2\, dy\, dx$$

$$= \frac{1}{16} \int_{-2}^{2} \frac{x}{3} y^3 \Big|_{y=0}^{y=4}\, dx = \frac{1}{16} \int_{-2}^{2} \frac{x}{3} \left(4^3\right)\, dx = \frac{4}{3} \int_{-2}^{2} x\, dx$$

$$= \frac{4}{3} \left(\frac{1}{2}\right) x^2 \Big|_{x=-2}^{x=2} = 0$$

Exercise 5 Average Value

Find the average value of the function over the region D:

1. $f(x, y) = x \sin y$, $D = [0, 2] \times [-\pi/2, \pi/2]$
2. $f(x, y) = x + y$, D is the circle with radius 1 centered at $(1, 0)$
3. $f(x, y) = xy$, D is the triangle with vertices at $(0, 0)$, $(2, 0)$, $(2, 2)$

Working Space

Answer on Page 24

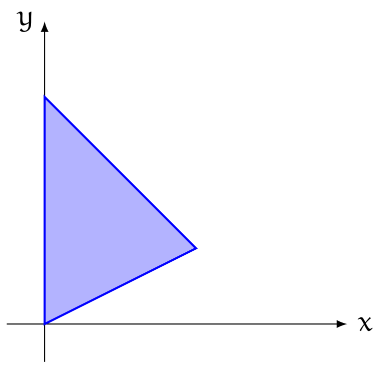
Appendix A

Answers to Exercises

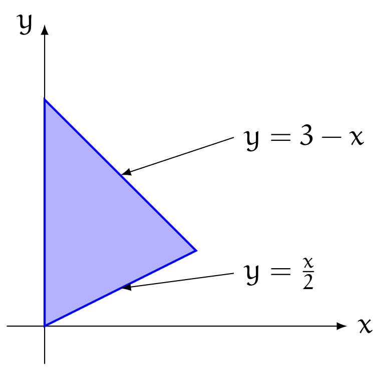
Answer to Exercise 1 (on page 3)

1. $\iint_D \left(1+x^2+y^2\right)\, dA = \int_0^4 \int_0^3 \left(1+x^2+y^2\right)\, dy\, dx = \int_0^4 \left[y + x^2 y + \frac{1}{3}y^3\right]_{y=0}^{y=3} dx = \int_0^4 \left[3 + 3x^2 + \frac{1}{3}(3)^3\right]$
$= \int_0^4 \left(12+3x^2\right)\, dx = \left[12x + x^3\right]_{x=0}^{x=4} = 12(4) + 4^3 = 112$

2. First, let's visualize this region, since it isn't a rectangle:



Let's divide the triangle horizontally and write equations for each of the sides that do not lie on the y-axis.



We see that we can describe region D as $D = \{(x, y) \mid 0 \leq x \leq 2, \frac{x}{2} \leq y \leq 3 - x\}$. Therefore $\iint_D (x+y)\, dA = \int_0^3 \int_{x/2}^{3-x} (x+y)\, dy\, dx = \int_0^3 \left[xy + \frac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx =$
$\int_0^3 \left[(x(3-x)) - (x(x/2)) + \frac{1}{2}\left((3-x)^2 - (x/2)^2\right)\right]\, dx$
$= \int_0^3 \left[\left(3x - x^2 - \frac{x^2}{2}\right) + \frac{1}{2}\left(9 - 6x + x^2 - \frac{x^2}{4}\right)\right]\, dx = \int_0^3 \left[-x^2 - \frac{x^2}{2} + \frac{x^2}{2} - \frac{x^2}{8} + 3x - 3x + \frac{9}{2}\right]\, dx$
$= \int_0^3 \left(-\frac{9x^2}{8} + \frac{9}{2}\right)\, dx = \left[\frac{9x}{2} - \frac{3x^3}{8}\right]_{x=0}^{x=2} = \frac{9(2)}{2} - \frac{3(8)}{8} = 9 - 3 = 6$

Answer to Exercise 2 (on page 7)

1. First, we find the total mass: $m = \int_0^\pi \int_0^{2\sin x} x \, dy \, dx = \int_0^\pi [xy]_{y=0}^{y=2\sin x} \, dx = \int_0^\pi 2x \sin x \, dx$.

   We apply integration by parts to evaluate the integral: $\int_0^\pi 2x \sin x \, dx = (-2x\cos x)\,|_{x=0}^{x=\pi} + \int_0^\pi 2\cos x \, dx = [-2\pi(-1)] - (0) + \sin x|_{x=0}^{x=\pi} = 2\pi + \sin\pi - \sin 0 = 2\pi$

   Now that we know $m = 2\pi$, we can find $\bar{x}$ and $\bar{y}$: $\bar{x} = \frac{1}{2\pi}\int_0^\pi \int_0^{2\sin x} x \cdot x \, dy \, dx = \frac{1}{2\pi}\int_0^\pi x^2 y|_{y=0}^{y=2\sin x} \, dx = \frac{1}{2\pi}\int_0^\pi x^2 \left(2\sin x\right) \, dx = \frac{1}{\pi}\int_0^\pi x^2 \sin x \, dx$.

   Applying integration by parts: $\frac{1}{\pi}\int_0^\pi x^2 \sin x \, dx = \frac{1}{\pi}\left[x^2\left(-\cos x\right)|_{x=0}^{x=\pi} - \int_0^\pi 2x\left(-\cos x\right) \, dx\right]$ $= \frac{1}{\pi}\left[\left(-\pi^2\cos\pi\right) + 2\int_0^\pi x\cos x \, dx\right] = \frac{1}{\pi}\left[\pi^2 + 2\int_0^\pi x\cos x \, dx\right] = \pi + \frac{2}{\pi}\int_0^\pi x\cos x \, dx$.

   Applying integration by parts again: $\pi + \frac{2}{\pi}\int_0^\pi x\cos x \, dx = \pi + \frac{2x\sin x}{\pi}|_{x=0}^{x=\pi} - \frac{2}{\pi}\int_0^\pi \sin x \, dx$ $= \pi - \frac{2}{\pi}\int_0^\pi \sin x \, dx = \pi + \frac{2}{\pi}[\cos x]_{x=0}^{x=\pi} = \pi + \frac{2}{\pi}[\cos\pi - \cos 0] = \pi + \frac{2}{\pi}(-1-1) = \pi - \frac{4}{\pi} = \bar{x}$

   And finding $\bar{y}$: $\bar{y} = \frac{1}{2\pi}\int_0^\pi \int_0^{2\sin x} y \cdot x \, dy \, dx = \frac{1}{2\pi}\int_0^\pi \left[\frac{1}{2}xy^2\right]_{y=0}^{y=2\sin x} \, dx = \frac{1}{4\pi}\int_0^\pi x\left[2\sin x\right]^2 \, dx$ $= \frac{1}{4\pi}\int_0^\pi 4x\sin^2 x \, dx = \frac{1}{\pi}\int_0^\pi x\sin^2 x \, dx = \frac{1}{\pi}\int_0^\pi x\frac{1-\cos(2x)}{2} \, dx = \frac{1}{\pi}\int_0^\pi \frac{x}{2} \, dx - \frac{1}{\pi}\int_0^\pi x\cos(2x) \, dx$ $= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{x=0}^{x=\pi} - \frac{1}{\pi}\left[\frac{1}{2}x\sin(2x)|_{x=0}^{x=\pi} - \frac{1}{2}\int_0^\pi \sin(2x) \, dx\right] = \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) - \frac{1}{2\pi}\left[\pi\sin(2\pi) - 0 + \frac{1}{2}\cos(2x)|_{x-0}^{x=\pi}\right]$ $= \frac{\pi}{4} - \frac{1}{2\pi}\left[\frac{1}{2}(\cos 2\pi - \cos 0)\right] = \frac{\pi}{4}$

   Therefore, the center of mass is found at $(\bar{x}, \bar{y}) = \left(\pi - \frac{4}{\pi}, \frac{\pi}{4}\right)$

2. We know from a previous question that the total mass of this lamina is 112 (see *Finding Total Mass*).

   Finding $\bar{x}$: $\bar{x} = \frac{1}{112}\int_0^4 \int_0^3 x\left(1 + x^2 + y^2\right) \, dy \, dx = \frac{1}{112}\int_0^4 \int_0^3 \left(x + x^3 + xy^2\right) \, dy \, dx$ $= \frac{1}{112}\int_0^4 \left[xy + x^3y + \frac{x}{3}y^3\right]_{y=0}^{y=3} \, dx = \frac{1}{112}\int_0^4 \left[3x + 3x^3 + 9x\right] \, dx = \frac{3}{112}\int_0^4 \left[4x + x^3\right] \, dx = \frac{3}{112}\left[2x^2 + \frac{x^4}{4}\right]_{x=0}^{x=4} = \frac{3}{112}\left[2(4)^2 - 2(0)^2 + \frac{4^4}{4} - \frac{0^4}{4}\right] = \frac{3}{112}[32 + 64] = \frac{3 \cdot 96}{112} = \frac{3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 7 \cdot 2 \cdot 2} = \frac{18}{7}$

   Finding $\bar{y}$: $\bar{y} = \frac{1}{112}\int_0^4 \int_0^3 y\left(1 + x^2 + y^2\right) \, dy \, dx = \frac{1}{112}\int_0^4 \int_0^3 \left[y + x^2y + y^3\right] \, y \, dx$ $= \frac{1}{112}\int_0^4 \left[\frac{y^2}{2} + \frac{x^2y^2}{2} + \frac{y^4}{4}\right]_{y=0}^{y=3} \, dx = \frac{1}{112}\int_0^4 \left[\frac{3^2}{2} + \frac{3^2x^2}{2} + \frac{3^4}{4}\right] \, dx = \frac{1}{112}\int_0^4 \left[\frac{99}{4} + \frac{9}{2}x^2\right] \, dx$ $= \frac{1}{112}\left[\frac{99}{4}x + \frac{3}{2}x^3\right]_{x=0}^{x=4} = \frac{3}{224}\left[\frac{33}{2}(4) + 4^3\right] = \frac{3}{224}(66 + 64) = \frac{3 \cdot 130}{224} = \frac{3 \cdot 65}{112} = \frac{195}{112}$

   Therefore, the center of mass of the rectangular region D is $\left(\frac{18}{7}, \frac{195}{112}\right)$

3. We know from a previous question (see *Finding Total Mass*) that the total mass of D is 6, and it can be described as $D = \{(x, y) \mid 0 \leq x \leq 2, \frac{x}{2} \leq y \leq 3 - x\}$

   Finding $\bar{x}$:

   $\bar{x} = \frac{1}{6}\int_0^2 \int_{x/2}^{3-x} x(x+y) \, dy \, dx = \frac{1}{6}\int_0^2 \int_{x/2}^{3-x} \left(x^2 + xy\right) \, dy \, dx = \frac{1}{6}\int_0^2 \left[x^2y + \frac{x}{2}y^2\right]_{y=x/2}^{y=3-x} \, dx$ $= \frac{1}{6}\int_0^2 \left[x^2\left(3 - x - \frac{x}{2}\right) + \frac{x}{2}\left((3-x)^2 - \left(\frac{x}{2}\right)^2\right)\right] \, dx = \frac{1}{6}\int_0^2 \left[3x^2 - x^3 - \frac{x^3}{2} + \frac{x}{2}\left(9 - 6x + x^2 - \frac{x^2}{4}\right)\right] \, dx$ $= \frac{1}{6}\int_0^2 \left[3x^2 - \frac{3}{2}x^3 + \frac{x}{2}\left(9 - 6x + \frac{3}{4}x^2\right)\right] \, dx = \frac{1}{6}\int_0^2 \left[3x^2 - \frac{3}{2}x^3 + \frac{9}{2}x - 3x^2 + \frac{3}{8}x^3\right] \, dx = \frac{1}{6}\int_0^2 \left[\frac{9}{2}x - \frac{9}{8}x^3\right] \, dx = \frac{1}{6}\left[\frac{9}{4}x^2 - \frac{9}{32}x^4\right]_{x=0}^{x=2} = \frac{1}{6}\left[\frac{9 \cdot 4}{4} - \frac{9 \cdot 16}{32}\right] = \frac{1}{6}\left[9 - \frac{9}{2}\right] = \frac{1}{6} \cdot \frac{9}{2} = \frac{9}{12} = \frac{3}{4}$

And finding $\overline{y}$:

$\overline{y} = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} y\,(x+y)\ dy\,dx = \frac{1}{6}\int_0^2\int_{x/2}^{3-x}\left(xy+y^2\right)\ dy\,dx = \frac{1}{6}\int_0^2\left[\frac{x}{2}y^2+\frac{1}{3}y^3\right]_{y=x/2}^{y=3-x}\ dx$
$= \frac{1}{6}\int_0^2\left[\frac{x}{2}\left((3-x)^2-\left(\frac{x}{2}\right)^2\right)+\frac{1}{3}\left((3-x)^3-\left(\frac{x}{2}\right)^3\right)\right]\ dx$
$= \frac{1}{6}\int_0^2\left[\frac{x}{2}\left(9-6x+x^2-\frac{x^2}{4}\right)+\frac{1}{3}\left(27-27x+9x^2-x^3-\frac{x^3}{8}\right)\right]\ dx$
$= \frac{1}{6}\int_0^2\left[\frac{x}{2}\left(9-6x+\frac{3x^2}{4}\right)+\frac{1}{3}\left(27-27x+9x^2-\frac{9x^3}{8}\right)\right]\ dx$
$= \frac{1}{6}\int_0^2\left[\frac{9}{2}x-3x^2+\frac{3}{8}x^3+9-9x+3x^2-\frac{3}{8}x^3\right]\ dx = \frac{1}{6}\int_0^2\left[9-\frac{9}{2}x\right]\ dx = \frac{1}{6}\left[9x-\frac{9}{4}x^2\right]_{x=0}^{x=2}$
$= \frac{3}{6}\left[3(2)-\frac{3}{4}(2)^2\right] = \frac{1}{2}\,(6-3) = \frac{1}{2}\cdot 3 = \frac{3}{2}$

Therefore, the center of mass is $(\overline{x},\overline{y}) = \left(\frac{3}{4},\frac{3}{2}\right)$

Answer to Exercise 3 (on page 12)

1.

$$I_x = \iint_D y^2\rho(x,y)\,dA = \int_1^4\int_0^3 y^2(xy)\,dy\,dx$$
$$= \int_1^4\int_0^3 xy^3\,dy\,dx = \int_1^4 x\left[\frac{1}{4}y^4\right]_{y=0}^{y=3} dx = \frac{1}{4}\int_1^4 81x\,dx$$
$$= \frac{81}{4}\left[\frac{1}{2}x^2\right]_{x=1}^{x=4} = \frac{81}{2}\left(4^2-1^2\right) = \frac{81}{2}\cdot 15 = \frac{1215}{2}$$

To find the radius of gyration, first we need to find the total mass:

$$m = \iint_D \rho(x,y)\,dA = \int_1^4\int_0^3 xy\,dy\,dx$$
$$= \int_1^4 \frac{x}{2}\left[y^2\right]_{y=0}^{y=3} dx = \frac{9}{2}\int_1^4 x\,dx = \frac{9}{2}\cdot\left(\frac{1}{2}\right)\cdot\left[x^2\right]_{x=1}^{x=4} = \frac{9}{4}\,[16-1] = \frac{135}{2}$$

Finding the radius of gyration about the x-axis:

$$I_x = m\overline{\overline{y}}^2$$
$$\frac{1215}{2} = \left(\frac{135}{2}\right)\overline{\overline{y}}^2$$
$$9 = \overline{\overline{y}}^2$$
$$\overline{\overline{y}} = 3$$

2.

$$I_y = \iint_D x^2\rho(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos x} x^2y\,dy\,dx$$

$$= \int_{-pi/2}^{\pi/2} x^2 \left[\frac{1}{2}y^2\right]_{y=0}^{y=2\cos x} dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} x^2 \left(2\cos x\right)^2 \, dx$$

$$= 2\int_{-\pi/2}^{\pi/2} x^2 \cos^2 x \, dx = 2\int_{-\pi/2}^{\pi/2} x^2 \left(\frac{1+\cos 2x}{2}\right) dx = \int_{-\pi/2}^{\pi/2} x^2 \, dx + \int_{-\pi/2}^{\pi/2} x^2 \cos 2x \, dx$$

$$= \frac{1}{3}\left[x^3\right]_{x=-\pi/2}^{x=\pi/2} + \frac{1}{2}x^2 \sin 2x|_{x=-\pi/2}^{x=\pi/2} - \int_{-\pi/2}^{\pi/2} \frac{1}{2}\sin 2x \left(2x\right) \, dx$$

$$= \frac{1}{3}\left[\left(\frac{\pi}{2}\right)^3 - \left(\frac{-\pi}{2}\right)^3\right] - \int_{-\pi/2}^{\pi/2} x \sin 2x \, dx$$

$$= \frac{1}{3}\left(\frac{2\pi^3}{8}\right) - \left(\left[-\frac{1}{2}x\cos 2x\right]_{x=-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} (-\frac{1}{2}\cos 2x) \, dx\right)$$

$$= \frac{\pi^3}{12} + \left(\frac{1}{2}\left(\frac{\pi}{2}\right)\cos\left(\pi\right) - \frac{1}{2}\left(\frac{-\pi}{2}\right)\cos\left(-\pi\right)\right) - \left[\frac{1}{4}\sin 2x\right]_{x=-\pi/2}^{\pi/2}$$

$$= \frac{\pi^3}{12} + \frac{\pi}{4}(-1) + \frac{\pi}{4}(-1) = \frac{\pi^3}{12} - \frac{\pi}{2}$$

In order to find the radius of gyration, we need to first know the total mass:

$$m = \iint_D \rho(x,y) \, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos x} y \, dy \, dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}y^2|_{y=0}^{y=2\cos x} dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} 4\cos^2 x \, dx = \int_{-\pi/2}^{\pi/2} 1 + \cos 2x \, dx$$

$$= \left[x + \frac{1}{2}\sin 2x\right]_{x=-\pi/2}^{\pi/2} = \pi$$

We can then find the radius of gyration about the y-axis:

$$m\bar{\bar{x}}^2 = I_y$$

$$\pi\bar{\bar{x}}^2 = \frac{\pi^3}{12} - \frac{\pi}{2}$$

$$\bar{\bar{x}} = \sqrt{\frac{\pi^2}{12} - \frac{1}{2}}$$

3.

$$I_o = \iint_D \left(x^2 + y^2\right)\rho(x,y) = \int_1^2 \int_0^{\pi} r(r^2)r \, d\theta \, dr$$

$$= \int_1^2 r^4 \theta|_{\theta=0}^{\theta=\pi} dr = \pi\int_1^2 r^4 \, dr = \frac{\pi}{5}r^5|_{r=1}^{r=2} = \frac{\pi}{5}\left(2^5 - 1\right) = \frac{\pi}{5}(31) = \frac{31\pi}{5}$$

We find the total mass:

$$m = \iint_D \rho(x,y) \, dA = \int_1^2 \int_0^{\pi} r^2 \, d\theta \, dr = \int_1^2 r^2 \theta|_{\theta=0}^{\theta=\pi} dr$$

$$= \pi \int_1^2 r^2\, dr = \frac{\pi}{3} r^3|_{r=1}^{r=2} = \frac{\pi}{3}\left(2^3 - 1\right) = \frac{\pi}{3}(7) = \frac{7\pi}{3}$$

To find the radius of gyration about the origin:

$$mR^2 = I_o$$

$$\left(\frac{7\pi}{3}\right) R^2 = \frac{31\pi}{5}$$

$$R^2 = \frac{31}{5} \cdot \frac{3}{7} = \frac{93}{35}$$

$$R = \sqrt{\overline{\overline{x}}^2 + \overline{\overline{y}}^2} = \sqrt{\frac{93}{35}}$$

Answer to Exercise 4 (on page 16)

1. Rearranging the formula for the plane, we find that $z = 3x + 2y + 2$. Therefore, $\partial z/\partial x = 3$ and $\partial z/\partial y = 2$. Then the surface area is given by:

$$A(S) = \int_2^6 \int_1^4 \sqrt{1 + 3^2 + 2^2}\, dy\, dx = \int_2^6 \sqrt{14} y|_{y=1}^{y=4}\, dx$$

$$= \int_2^6 3\sqrt{14}\, dx = 3\sqrt{14} x|_{x=2}^{x=6} = 12\sqrt{14}$$

2. The paraboloid intersects the plane when $2x^2 + 2y^2 = 32$, which is the circle of radius 4 centered at the origin. So, we are looking for the area of the surface $z = 2x^2 + 2y^2$ that lies above the region $R - \{(r, \theta) \mid 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$. The surface area is:

$$A(S) = \iint_R \sqrt{1 + (4x)^2 + (4y)^2}\, dA = \int_0^4 \int_0^{2\pi} r\sqrt{1 + (4r\cos\theta)^2 + (4r\sin\theta)^2}\, d\theta\, dr$$

$$= \int_0^4 \int_0^{2\pi} r\sqrt{1 + 16r^2}\, d\theta\, dr = \int_0^4 r\sqrt{1 + 16r^2}\, [\theta]_{\theta=0}^{\theta=2\pi}\, dr$$

$$= \int_0^4 2\pi r\sqrt{1 + 16r^2}\, dr$$

Let $u = 1 + 16r^2$, then $du = 32r(dr)$ and $r(dr) = du/32$. Substituting:

$$A(S) = \frac{2\pi}{32} \int_{r=0}^{r=4} \sqrt{u}\, du = \frac{\pi}{16}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_{r=0}^{r=4}$$

$$= \frac{\pi}{24}\left[\left(1 + 16r^2\right)^{3/2}\right]_{r=0}^{r=4} = \frac{\pi}{24}\left[(257)^{3/2} - 1\right] \approx 6470.15$$

3. The region, R, we are interested in is the circle of radius 2 centered at the origin of the xy-plane, described by $R = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$. Noting that $\partial z/\partial x = 3y$ and $\partial z/\partial y = 3x$, we see that the surface area is given by:

$$A(S) = \iint_R \sqrt{1+(3y)^2+(3x)^2}\,dA = \int_0^2 \int_0^{2\pi} r\sqrt{1+9r^2\sin^2\theta+9r^2\cos^2\theta}\,d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} r\sqrt{1+9r^2}\,d\theta\,dr = 2\pi \int_0^2 r\sqrt{1+9r^2}\,dr$$

Let $u = 1+9r^2$, then $du = 18r(dr)$, which means that $r(dr) = du/18$. Substituting:

$$A(S) = \frac{2\pi}{18}\int_{r=0}^{r=2} \sqrt{u}\,du = \frac{\pi}{9}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_{r=0}^{r=2}$$

$$= \frac{2\pi}{27}\left[(1+9(4))^{3/2}-1\right] = \frac{2\pi}{27}\left[(37)^{3/2}-1\right] \approx 52.14$$

Answer to Exercise 5 (on page 18)

1. The area of D is 2π. Therefore, the average value is:

$$\frac{1}{2\pi}\iint_D x\sin y\,dA = \frac{1}{2\pi}\int_0^2\int_0^{\pi} x\sin y\,dy\,dx$$

$$= \frac{1}{2\pi}\int_0^2 -x\cos y\big|_{y=0}^{y=\pi}\,dx = \frac{1}{2\pi}\int_0^2 -x\left(\cos\pi - \cos 0\right)\,dx$$

$$= \frac{1}{2\pi}\int_0^2 (-x)(-1-1)\,dx = \frac{1}{2\pi}\int_0^2 2x\,dx$$

$$= \frac{1}{2\pi}x^2\big|_{x=0}^{x=2} = \frac{2}{\pi}$$

2. Since D is a circle of radius $r = 1$, the area is $A = \pi r^2 = \pi$. D can be described with $D = \{(r,\theta) \mid 0 \le r \le 2\cos\theta,\ -\pi/2 \le \theta \le \pi/2\}$. Therefore, the average value of $f(x,y) = x+y$ over D is:

$$f_{ave} = \frac{1}{\pi}\iint_D (x+y)\,dA = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\cdot(r\cos\theta + r\sin\theta)\,dr\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta+\sin\theta)\left[\int_0^{2\cos\theta} r^2\,dr\right]\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta+\sin\theta)\cdot\left[\frac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta}\,d\theta$$

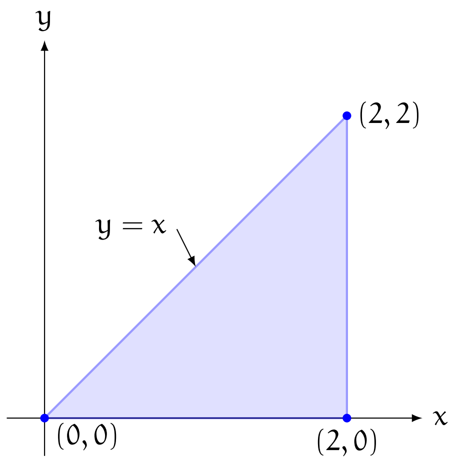
$$= \frac{8}{3\pi}\int_{-\pi/2}^{\pi/2} (\cos\theta + \sin\theta)\cos^3\theta \, d\theta = \frac{8}{3\pi}\int_{\pi/2}^{\pi/2} \left(\cos^4\theta + \sin\theta\cos^3\theta\right) \, d\theta$$

$$= \frac{8}{3\pi}\left[\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 \, d\theta - \left[\frac{1}{4}\cos^4\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2}\right]$$

$$= \frac{2}{3\pi}\int_{-\pi/2}^{\pi/2}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right) \, d\theta = \frac{2}{3\pi}\left[(\theta + \sin 2\theta)_{\theta=-\pi/2}^{\theta=\pi/2} + \int_{-\pi/2}^{\pi/2}\frac{1+\cos 4\theta}{2} \, d\theta\right]$$

$$= \frac{2}{3\pi}\left[\pi + \frac{1}{2}\left(\theta + \frac{1}{4}\sin 4\theta\right)_{\theta=-\pi/2}^{\theta=\pi/2}\right] = \frac{2}{3\pi}\left[\pi + \frac{1}{2}(\pi)\right] = \frac{2}{3\pi}\left(\frac{3\pi}{2}\right) = 1$$

3. Let's visualize D:



So, D can be described $D = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le x\}$. Additionally, D has area $A = \frac{1}{2}\left(2^2\right) = 2$. Therefore, the average value of $f(x, y) = xy$ over D is:

$$f_{ave} = \frac{1}{2}\iint_D (xy) \, dA = \frac{1}{2}\int_0^2\int_0^x (xy) \, dy \, dx$$

$$= \frac{1}{2}\int_0^2 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=x} dx = \frac{1}{4}\int_0^2 x^3 \, dx = \frac{1}{4}\left[\frac{1}{4}x^4\right]_{x=0}^{x=2}$$

$$= \frac{1}{16}\left(2^4\right) = 1$$

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