Chapter 1

Change of Variables

Let's say that we are making ice spheres by filling water into balloons, and we tell you that the radius of the ice spheres is normally distributed with a mean of 0.7 cm and a standard deviation of 0.08 cm. You can then draw the probability distribution and cumulative distribution for that:

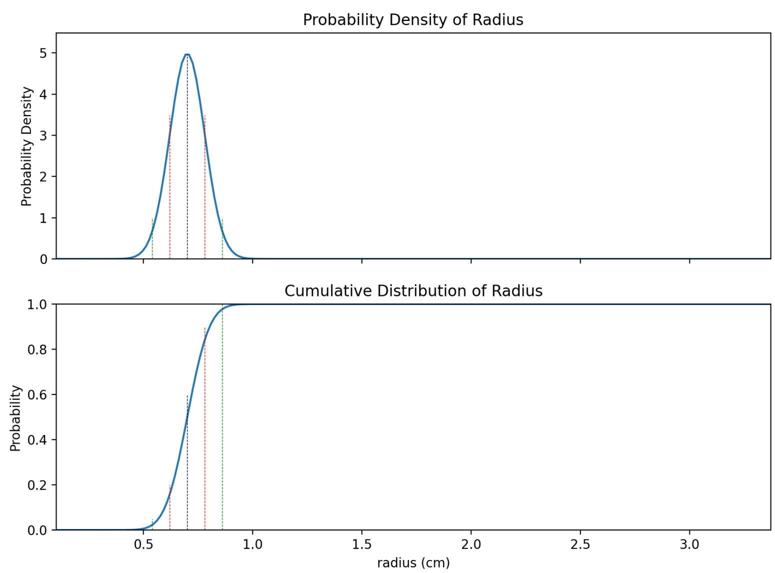

Figure 1.1: Ice sphere PDF and cumulative density function.

This includes lines indicating the mean and two standard deviations on each side.

Now, let's say we ask you what the cumulative distribution is for the *mass* of the balloons. A cubic centimeter of ice weighs about gram, so if you know the radius of a particular ice sphere, it is easy to compute the mass of it:

$$m = \frac{4}{3}\pi r^3$$

So, for example, if a sphere has a radius of 5cm, its mass in grams is $\frac{4\pi(0.7^3)}{3} \approx 1.44$ g.

Thinking about the graph of the cumulative distribution, if half the balloons have a radius less than 5 cm, then half of the balloons have a mass less than 523.6 g. For each point on the cumulative graph, we can use the radius of that point to compute the corresponding mass — the CDF gets stretched out:

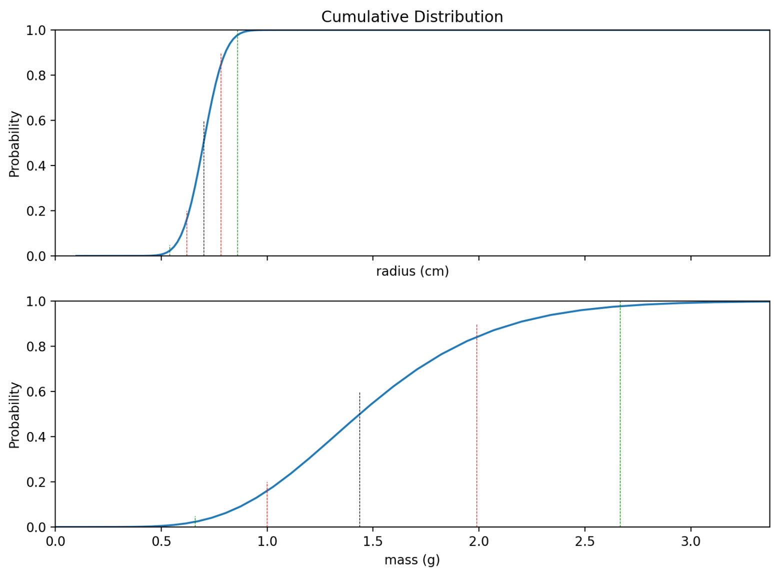

Figure 1.2: Cumulative distribution of both the radius and mass of the ice spheres.

If F is the original cumulative distribution function, and g is the function that maps the new variable (mass, in this case) to the old one (radius), then the new cumulative distribution function H is given by

$$H(m) = F(g(m))$$

In this case, F is the cumulative function for the normal distribution, with mean 0.7 and standard deviation of 0.08. g maps the mass to the radius:

$$g(m) = \left(\sqrt[3]{\frac{3}{4\pi}}\right) x^{\frac{1}{3}}$$

1.1 Making a Probability Density Function

Now, we know how to calculate a new cumulative distribution function using the new variable. However, we usually want a probability density.

Here is the CDF and the PDF of the mass of the ice spheres:

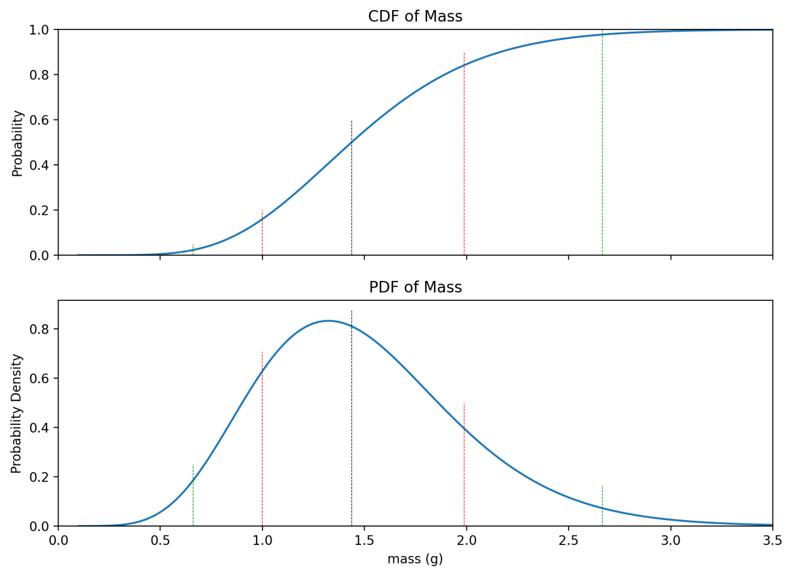

Figure 1.3: CDF and PDF of the masses.

Reminder: The probability density function is the derivative of the cumulative distribution function. We know the CDF is

$$H(m) = F(g(m))$$

By the chain rule:

$$H'(m) = F'(g(m))g'(m)$$

The function F is the cumulative distribution for the normal distribution with mean 0.7 and standard deviation of 0.08. So, we know its derivative:

$$F'(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $\mu = 0.7$ and $\sigma = 0.08$.

We've already said that

$$g(m) = \left(\sqrt[3]{\frac{3}{4\pi}}\right) m^{\frac{1}{3}}$$

Which is easy to differentiate:

$$g'(m) = \left(\frac{1}{3}\right)\left(\sqrt[3]{\frac{3}{4\pi}}\right) m^{-\frac{2}{3}}$$

Here, then, is the code to generate that last plot:

```
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt

# Constants
MEAN_RADIUS = 0.7
STD_RADIUS = 0.08

# Range to plot
MIN_MASS = 0.1
MAX_MASS = 3.5

# Number of points to plot
N = 200

# Needed for radius_for_mass and d_radius_for_mass
C = np.power(3 / (4 * np.pi), 1/3)

# In these three functions, x can
# be a number or a numpy array

def mass_for_radius(x):
  return 4 * np.pi * np.power(x, 3) / 3

def radius_for_mass(x):
  return C * np.power(x, 1/3)

# Derivative of radius_for_mass()
def d_radius_for_mass(x):
  return (C/3) * np.power(x,-2/3)

# Compute mean and 2 standard deviations in each direction
m_mean = mass_for_radius(MEAN_RADIUS)
m_minus_std = mass_for_radius(MEAN_RADIUS - STD_RADIUS)
m_plus_std = mass_for_radius(MEAN_RADIUS + STD_RADIUS)
m_minus_2std = mass_for_radius(MEAN_RADIUS - 2 * STD_RADIUS)
m_plus_2std = mass_for_radius(MEAN_RADIUS + 2 * STD_RADIUS)
```

```
# Make N possible values for mass
m_values = np.linspace(MIN_MASS, MAX_MASS, N)

# Compute g(m) for each of these masses
# That is: What is the radius for each of these masses?
r_values = radius_for_mass(m_values)

# Compute F(g(m)) for each of these masses
# That is: What is the cumulative distribution for each those radii?
cdf_values = norm.cdf(r_values, loc=MEAN_RADIUS, scale=STD_RADIUS)

# Compute g'(m) for each of these masses
dg_values = d_radius_for_mass(m_values)

# What is F'(g(m))g'(m)?
pdf_values = norm.pdf(r_values, loc=MEAN_RADIUS, scale=STD_RADIUS) * dg_values

# Sanity check: It should integrate to a little less then 1.0
dx = (MAX_MASS - MIN_MASS)/N
area_under_curve = pdf_values.sum() * dx
print(f"Integral from {MIN_MASS:.2f} to {MAX_MASS:.2f}: {area_under_curve:.3f}")

# Make a figure with two axes
fig, axs = plt.subplots(nrows=2, sharex=True, figsize=(10, 7), dpi=200)

# Draw the CDF on the second axix
axs[0].set_title("CDF of Mass")
axs[0].set_ylim(bottom=0.0, top=1.0)
axs[0].set_xlim(left=0.0, right=MAX_MASS)
axs[0].set_ylabel("Probability")
axs[0].plot(m_values, cdf_values)

# Add lines for mean, mean-std, and mean+std
axs[0].vlines(m_minus_2std, 0, 0.05, "g", linestyle="dashed",lw=0.5)
axs[0].vlines(m_minus_std, 0, 0.2, "r", linestyle="dashed",lw=0.5)
axs[0].vlines(m_mean, 0, 0.6, "k", linestyle="dashed",lw=0.5)
axs[0].vlines(m_plus_std, 0, 0.9, "r", linestyle="dashed",lw=0.5)
axs[0].vlines(m_plus_2std, 0, 1.0, "g", linestyle="dashed",lw=0.5)

# How high does the pdf go?
max_density = pdf_values.max()

# Draw the PDF on the second axix
axs[1].set_title("PDF of Mass")
axs[1].set_ylim(bottom=0.0, top=max_density * 1.1)
axs[1].set_xlim(left=0.0, right=MAX_MASS)
axs[1].set_xlabel("mass (g)")
axs[1].set_ylabel("Probability Density")
axs[1].plot(m_values, pdf_values)

# Add lines for mean, mean-std, and mean+std
```

```
axs[1].vlines(m_minus_2std, 0, max_density * .3, "g", linestyle="dashed",lw=0.5)
axs[1].vlines(m_minus_std, 0, max_density * .85 , "r", linestyle="dashed",lw=0.5)
axs[1].vlines(m_mean, 0, max_density * 1.05, "k", linestyle="dashed",lw=0.5)
axs[1].vlines(m_plus_std, 0, max_density * .6, "r", linestyle="dashed",lw=0.5)
axs[1].vlines(m_plus_2std, 0, max_density * .2, "g", linestyle="dashed",lw=0.5)
fig.savefig("pdf.png")
```

1.2 Decreasing Conversions

The last case (mass and radius) is relatively straightforward, because the function g is always increasing. What if we have a change of variables where g is decreasing? For example, $V = IR$ so $\frac{V}{R} = I$.

Let's say that you work at a lightbulb factory and you sample the lightbulbs to see what their resistance is. You find the resistances of the lightbulbs are normally distributed with a mean of 24 ohms and a standard deviation of 3 ohms. The voltage will be exactly 12 volts. What is the PDF of the currents that will pass through the lightbulbs?

$$I = \frac{12}{R}$$

so

$$g(x) = \frac{12}{x}$$

is the function that will convert the current to resistance. Taking the derivative, we get:

$$g'(i) = -\frac{12}{x^2}$$

FIXME finish this section with graphs and code.

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Appendix A

Answers to Exercises