Chapter 1

Transforming Functions

Recall how we could translate, mirror, or rotate shapes and they would still be congruent shapes? We can do the same with functions, but the functions are not always equal. Let's say we gave you the graph of a function f, like this:

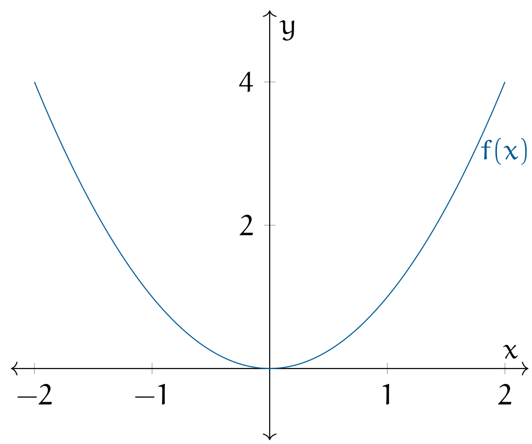


Figure 1.1: A graph of x^2.

We then tell you that the function is $g(x) = f(x) + 1.5$. Can you guess what the graph of g would look like? It is the same graph, just translated up 1.5:

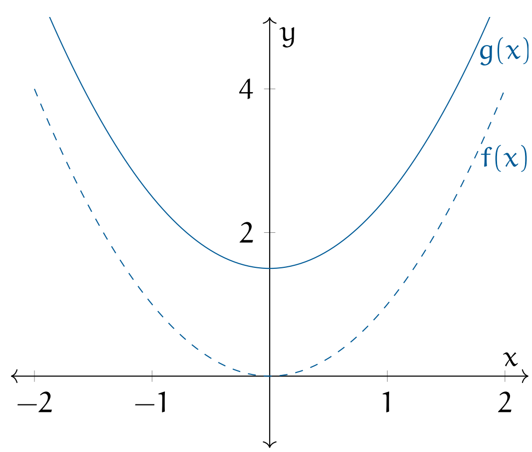


Figure 1.2: Shifted graph of $f(x) + 1.2$.

There are four kinds of transformations that we do all the time:

- Translation up and down in the direction of y axis (the one you just saw)
- Translation left and right in the direction of the x axis
- Scaling up and down along the y axis
- Scaling up and down along the x axis

Next, we will demonstrate each of the four using the graph of $\sin(x)$.

1.1 Translation up and down

When you add a positive constant to a function, you translate the whole graph up that much. A negative constant translates it down.

Here is the graph of $\sin(x) - 0.5$:

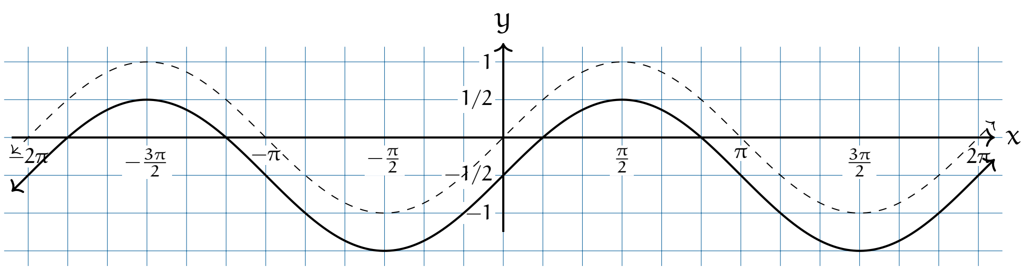


Figure 1.3: A graph of $\sin(x)$ dashed and $\sin(x) - 0.5$ in solid

1.2 Translation left and right

When you add a positive number to x before running it through f, you translate the graph to the left by that amount. Adding a negative number translates the graph to the right.

Here is the graph of $\sin(x - \pi/6)$:

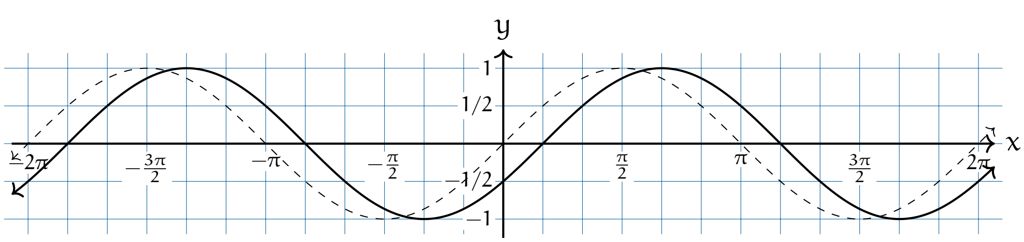


Figure 1.4: A graph of $\sin(x)$ in dashed lines and $\sin(x - \pi/6)$ in solid lines.

Notice the sign:

- Adding to x before processing with the function translates the graph to the *left*.
- Subtracting from x before processing with the function translates the graph to the *right*

1.3 Scaling up and down in the y direction

To scale the function up and down, you multiply the result of the function by a constant. If the constant is larger than 1, it stretches the function up and down.

Here is $y = 2\sin(x)$:

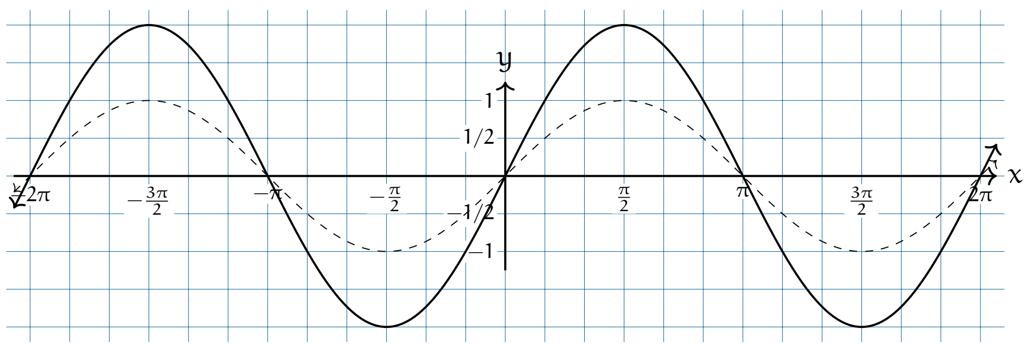


Figure 1.5: Scaling up and down in the y-direction requires a scalar outside of the function.

With a wave like this, we speak of its *amplitude*, which you can think of as its height. The baseline that this wave oscillates around is zero. The maximum distance that it gets from that baseline is its amplitude. Thus, the amplitude here has been increased from 1 to 2.

If you multiply by a negative number, the function gets flipped. Here is $y = -0.5\sin(x)$:

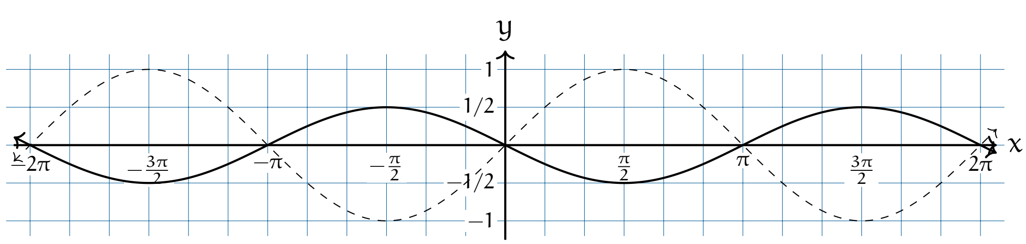


Figure 1.6: Scaling down and flipping or mirroring the function using $y = -0.5\sin(x)$.

Amplitude is never negative. Thus, the amplitude of this wave is 0.5.

1.4 Scaling up and down in the x direction

If you multiply x by a number larger than 1 before running it through the function, the graph gets compressed toward zero.

Here is $y = \sin(3x)$:

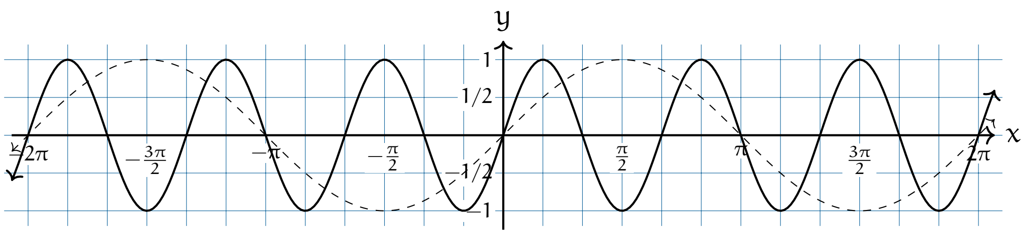


Figure 1.7: A solid $\sin(3x)$ compared to a solid $\sin(x)$ graph.

The distance between two peaks of a wave is known as its *wavelength*. The original wave had a wavelength of 2π. The compressed wave has a wavelength of $2\pi/3$.

If you multiply x by a number smaller than 1, it will stretch the function out, away from the y axis.

If you multiply x by a negative number, it will flip the function around the y axis.

Here is $y = 2^{(-0.5x)}$. Notice that it has flipped around the y axis and is stretched out along the x axis.

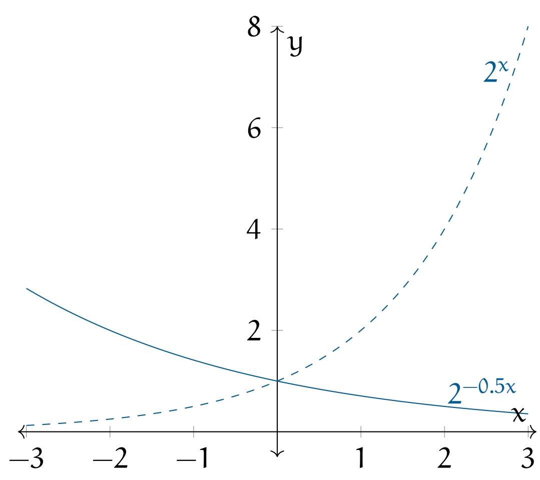


Figure 1.8: A base function 2^x in dashes compared to $2^{-0.5x}$ in solid.

Reflection over *x*-axis	$(x, y) \to (x, -y)$
Reflection over *y*-axis	$(x, y) \to (-x, y)$
Translation	$(x, y) \to (x + a, y + b)$
Dilation	$(x, y) \to (kx, ky)$
Rotation 90° counterclockwise	$(x, y) \to (-y, x)$
Rotation 180°	$(x, y) \to (-x, -y)$

Figure 1.9: A table of different transforms on a function

1.5 Order is important!

We can combine these transformations. This allows us, for example, to translate a function up 2, then scale along the y axis by 3.

Here is $y = 2.0(\sin(x) + 1)$:

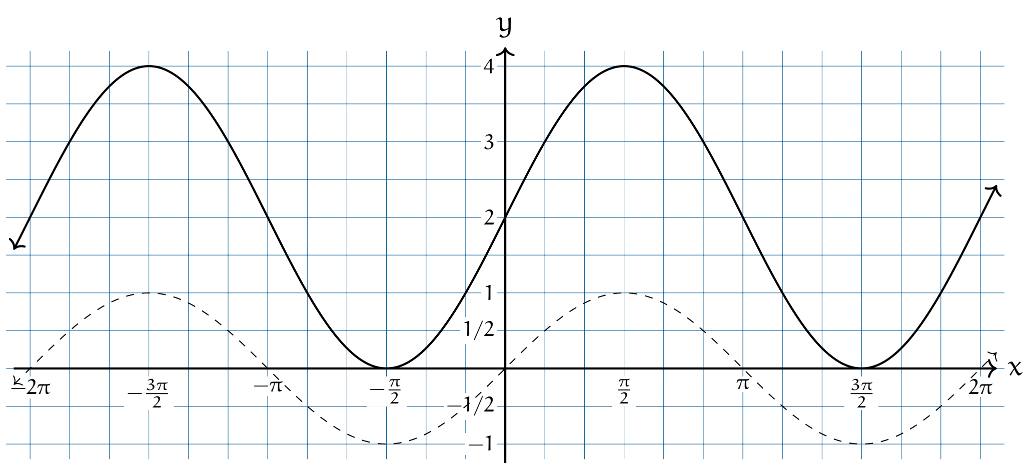


Figure 1.10: A graph of the base function $\sin(x)$ and the transformed function $2(\sin(x)+1)$

A function is often a series of steps. Here are the steps in $f(x) = 2(\sin(x) + 1)$:

1. Take the sine of x
2. Add 1 to that
3. Multiply that by 2

Note that this function can be distributed into $f(x) = 2\sin(x) + 2$ by bringing the 2 in both of the steps.

What if we change the order? Here are the steps in $g(x) = 2\sin(x) + 1$:

1. Take the sine of x
2. Multiply that by 2
3. Add 1 to that

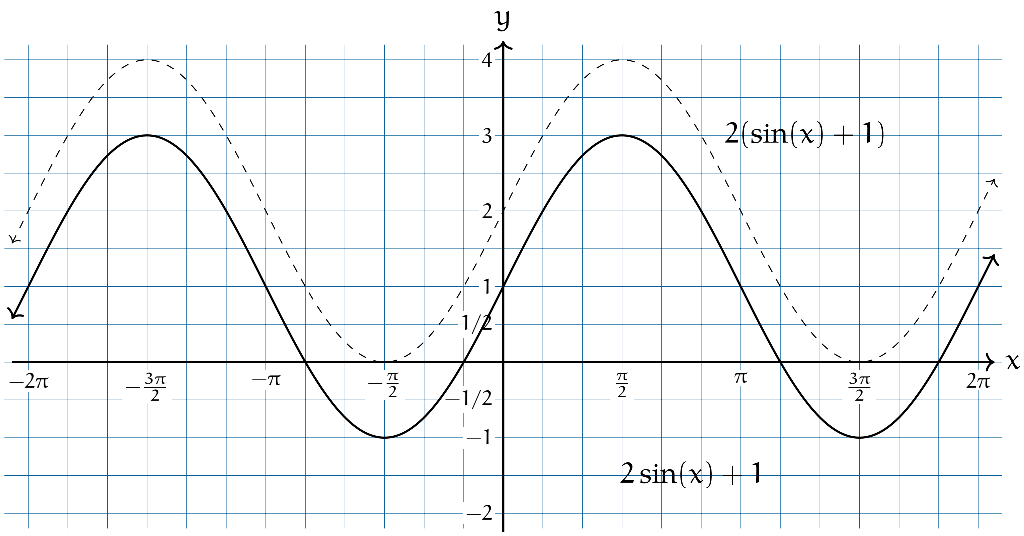
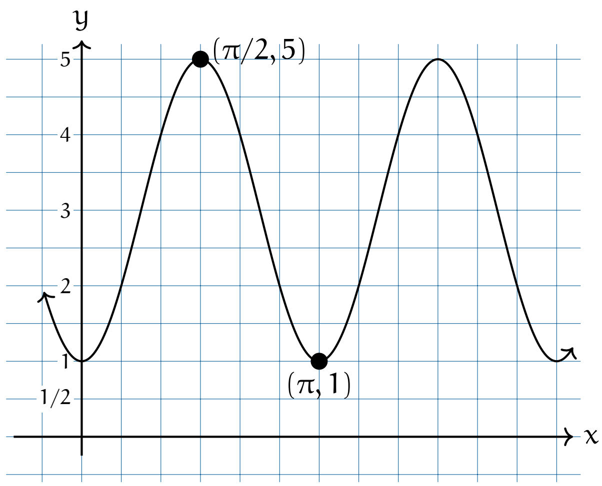


Figure 1.11: The functions $2\sin(x) + 1$ in solid compared to $2(\sin(x) + 1)$ in dashes.

The moral: You can do multiple transformations of your function, but the order in which you do them is important.

Exercise 1 Transforms

Find a function that creates a sine wave such that the top of the first crest is at the point $(\frac{\pi}{2}, 5)$ and the bottom of the trough that follows is at $(\pi, 1)$.



Working Space

Answer on Page 9

1.6 Effects on even and odd Functions

Now that you've worked with transformations like shifts and scalings, you can use symmetry to quickly assess how those changes affect a function.

1. Vertical shifts (like $f(x)+c$) do not change whether a function is even or odd — but they break symmetry about the origin or y-axis. Example: $x^2 + 1$ is still even, but it's not symmetric about the origin.
2. Horizontal shifts (like $f(x-c)$) also typically destroy even/odd symmetry. Example: $f(x) = \sin(x)$ is odd. But $f(x) = \sin(x - \pi/2)$ is neither even nor odd.
3. Vertical scalings (like $a \cdot f(x)$) preserve symmetry type. Example: x^2 versus $3x^2$.
4. Horizontal scalings (like $f(bx)$) preserve symmetry as long as the center of the transformation stays aligned. Example: $f(x) = x^3$ is odd. $f(x) = (2x)^3$ is still odd.

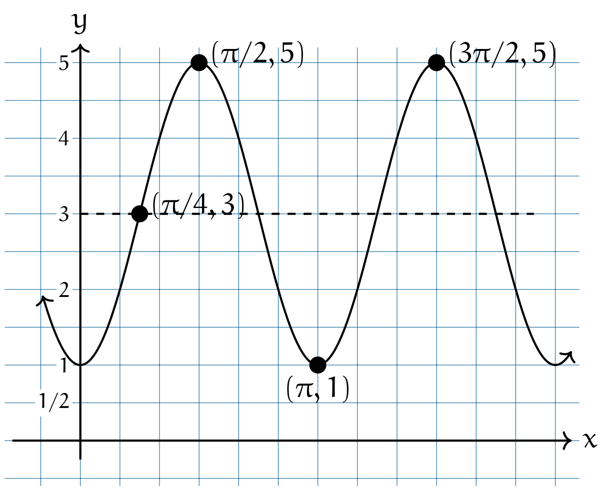
Remember these key ideas:

1. Reflecting an even function over the y-axis? It remains unchanged.
2. Reflecting an odd function over the origin? Still unchanged.
3. But any shift left or right often breaks the symmetry, so check $f(-x)$ again if you've applied one.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 7)



This wave has an amplitude of 2; its baseline has been translated up to 3.

This wave has wavelength of π. A sine wave usually has a wavelength of 2π, so we need to compress the x axis by a factor of 2.

The wave first crosses its baseline at $pi/4$. The sine wave starts by crossing its baseline, so we need to translate the curve right by $\pi/4$.

$$f(x) = 2\sin(2x - \frac{\pi}{4}) + 3$$