Chapter 1

Complex Numbers

Complex numbers are an extension of real numbers, which in turn are an extension of rational numbers. In mathematics, the set of complex numbers is a number system that extends the real number line to a full two dimensions, using the imaginary unit, which is denoted by i, with the property that $i^2 = -1$.

1.1 Definition

A complex number is a number of the form $a + bi$, where a and b are real numbers, and i is the imaginary unit, with the property that $i^2 = -1$. The real part of the complex number is a, and the imaginary part is b.

1.2 Why Are Complex Numbers Necessary?

Complex numbers are essential to many fields of science and engineering. Here are a few reasons why:

1.2.1 Roots of Negative Numbers

In the real number system, the square root of a negative number does not exist, because there is no real number that you can square to get a negative number. The introduction of the imaginary unit i, which has the property that $i^2 = -1$, allows us to take square roots of negative numbers and gives rise to complex numbers.

1.2.2 Polynomial Equations

The fundamental theorem of algebra states that every non-constant polynomial equation with complex coefficients has a complex root. This theorem guarantees that polynomial equations of degree n always have n roots in the complex plane.

1.2.3 Physics and Engineering

In physics and engineering, complex numbers are used to represent waveforms in control systems, in quantum mechanics, and many other areas. Their properties make many mathematical manipulations more convenient.

1.3 Adding Complex Numbers

The addition of complex numbers is straightforward. If we have two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, their sum is defined as:

$$z_1 + z_2 = (a + c) + (b + d)i \tag{1.1}$$

In other words, you add the real parts to get the real part of the sum, and add the imaginary parts to get the imaginary part of the sum.

1.4 Multiplying Complex Numbers

The multiplication of complex numbers is a bit more involved. If we have two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, their product is defined as:

$$z_1 \cdot z_2 = (a + bi) \cdot (c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i \tag{1.2}$$

Note the last term comes from $i^2 = -1$. You multiply the real parts and the imaginary parts just as you would in a binomial multiplication, and remember to replace i^2 with -1.

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`) for more details.

Appendix A

Answers to Exercises

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