Chapter 1

Definite Integrals

Integrals are a fundamental concept in calculus. They are used to calculate areas, volumes, and many other things. A definite integral calculates the net area between the function and the x-axis over a given interval.

Recall that you can use a Riemann sum to estimate the area under a function, and that as we increase the number of subintervals, the estimated area approaches the actual area. In sigma notation we can express a Riemann sum as

$$\sum_{i=1}^{n} f(x_i)\Delta x \tag{1.1}$$

1.1 Definition

The definite integral of a function $f(x)$ over an interval $[a, b]$ is defined as the limit of a Riemann sum as n approaches ∞:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{1.2}$$

where x_i^* is a sample point in the i^{th} subinterval of a partition of $[a, b]$, $\Delta x = \frac{b-a}{n}$ is the width of each subinterval, and the limit is taken as the number of subintervals n approaches infinity.

Exercise 1

Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$$

as an integral on the interval $[0, \pi]$. Do not solve.

Working Space

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1.2 Positive and Negative Areas

What if the function dips below the x-axis? We consider that area negative. In other words, it represents a *decrease* as opposed to an increase. Consider an oscillating object where $v(t) = \sin \pi x$ (figure 1.1). From $t = 0$ to $t = 1$, the velocity is positive, which means the object is moving *away from* the starting position. This is a positive displacement. From $t = 1$ to $t = 2$, the velocity is negative. What does this tell you about the direction the object is moving and its displacement during this time period? A negative velocity means the object is moving *back towards* the starting position.

In general, areas above the x-axis are positive, while areas below the x-axis are negative.

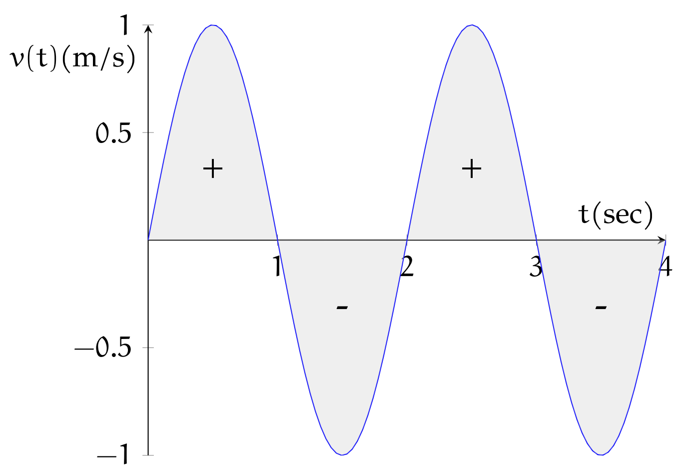


Figure 1.1: velocity of an oscillating object

1.3 Properties of Integrals

There are several important properties of integrals that will help us evaluate more complex integrals in the future. The following examples apply when $f(x)$ is continuous or has a finite number of jump discontinuities on the interval $a \leq x \leq b$.

1.3.1 What happens when $a = b$?

What if the endpoints of the integral are the same? Let's consider $\int_a^b x^2\,dx$, and take the limit as $b \to a$ (shown in figure 1.2). As you can see, as b approaches a, the calculated area decreases. Intuitively, we can guess that when $b = a$, then the width of the area (Δx) is zero, and therefore the area is also zero. Let's prove this formally.

Recall that $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. To evaluate the integral when $b = a$, we will take the limit of the limit:

$$\lim_{b\to a} \lim_{n\to\infty} f(x_i)\frac{b-a}{n}$$

This can be rewritten as

$$\lim_{b\to a}(b-a) \lim_{n\to\infty} \frac{f(x_i)}{n}$$

We know that $\lim_{b\to a}(b-a) = (a-a) = 0$, and therefore

$$\int_a^a f(x)\,dx = 0 \cdot \lim_{n\to\infty} \frac{f(x_i)}{n} = 0$$

This is true for any function.

Note that when $a = b$, the integral just becomes a line. From geometry we also know that a line has no area, as the width or dx is 0.

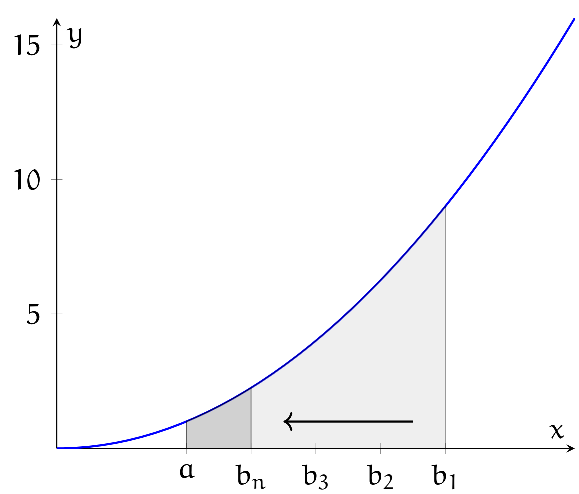


Figure 1.2: As b gets closer to a, the area represented by the integral decreases

1.3.2 The integral of a constant

When the function we are integrating is a constant (that is, it takes the form $f(x) = C$), the area is simply $(b - a) \cdot C$. This is shown graphically in figure 1.3.

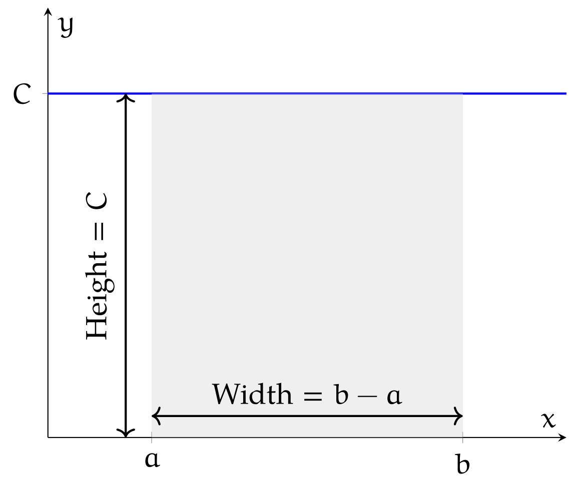


Figure 1.3: $\int_a^b f(x)\,dx = (b - a) \cdot C$

Since $f(x) = C$ is a horizontal line, the area under $f(x)$ is simply a rectangle. As you can see in figure 1.3, the width of the rectangle is $b - a$ and the height is C. To find the area of a rectangle, we multiply the width by the height, and therefore $\int_a^b C\,dx = (b - a) \cdot C$.

1.3.3 The integral of a function multiplied by a constant

How is $\int_a^b f(x)\,dx$ related to $\int_a^b C \cdot f(x)\,dx$? Intuitively, we know that multiplying a function by a constant, C, vertically stretches the graph by a factor of C. In turn, the area under the curve increases by a factor of C. Imagine a simple shape, like a triangle. If we keep the base of the triangle the same (analogous to the integral being over the same interval) and make the triangle three times taller (analogous to multiplying the function we're integrating by a factor of $C = 3$), then we would expect the total area of the triangle to be 3 times greater. Therefore, $\int_a^b C \cdot f(x)\,dx = C \int_a^b f(x)\,dx$.

1.3.4 Integrals of sums and differences of functions

If a function can be described as a sum of two other functions, then the integral of the original function is the same as the sum of the integrals of the two other functions. Concretely, we say $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. Figure 1.4 shows $f(x) = x + 2$, $g(x) = 4x^3 - 12x^2 + 10x$, and $f(x) + g(x)$. As you can see, the area under $f(x) + g(x)$ is equal to the area under $f(x)$ (the red area) plus the area under $g(x)$ (the diagonal lined area).

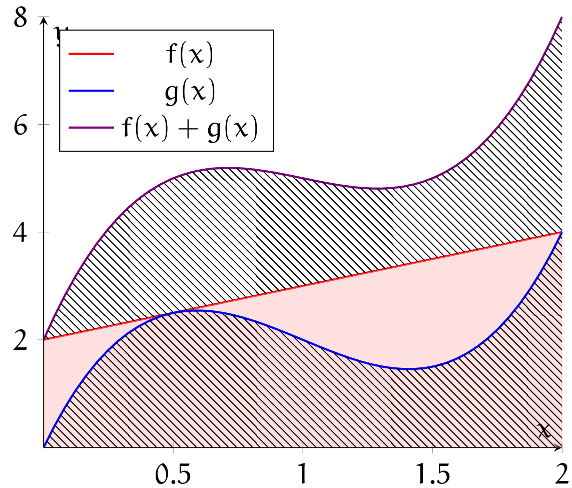


Figure 1.4: The intergral of $f(x) + g(x)$ is equal to the integral of $f(x)$ plus the integral of $g(x)$

Mathematically, we can prove this by recalling that the limit of a sum is the sum of the limits:

$$\int_a^b f(x) + g(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\Delta x$$

$$= \lim_{n\to\infty} [\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} g(x_i)\Delta x]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Similar to the addition property, the integral of the difference between two function is equal to the difference of the integrals of two functions.

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

. This is more difficult to visualize than addition, but we can easily prove it by applying the constant multiple and addition properties. Let's define $f(x) - g(x) = f(x) + (-g(x))$:

$$\int_a^b f(x) - g(x)\,dx = \int_a^b f(x) + (-g(x))\,dx$$

By the addition property,

$$= \int_a^b f(x)\,dx + \int_a^b -g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b (-1)\cdot g(x)\,dx$$

By the constant multiple property:

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

1.3.5 Integrals of adjacent areas

If c is some x-value between a and b, then $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$. This is shown graphically in figure 1.5. The total area from $x = a$ to $x = b$ is equal to the red area (the integral from a to c) plus the blue area (the integral from c to b).

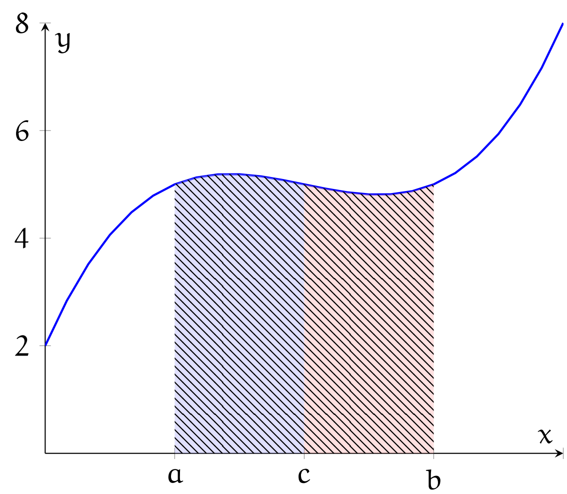


Figure 1.5: $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

1.3.6 Estimating the value of an integral

Suppose we need to know the area under a complex function. We can estimate a range for the value of the integral if we can bookend the function over the interval we are interested. Suppose there is some value m such that $f(x) \geq m$, and some other value M such that $f(x) \leq M$ on the interval we are interested in (see figure 1.6). The total area under $f(x)$ is the light blue plus the darker blue. The total area under $y = M$ is the darker blue, plus the light blue, plus the white area. The darker blue area under the curve has total area $m \cdot (b - a)$ and the rectangle under $y = M$ has total area $M \cdot (b - a)$ (since these are both integrals of a constant, which we learned about above). The actual area under our function is more than just the dark blue area, but less than the total area under $y = M$. Therefore, $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ if $m \leq f(x)$ and $M \geq f(x)$ on the interval $x \in [a, b]$.

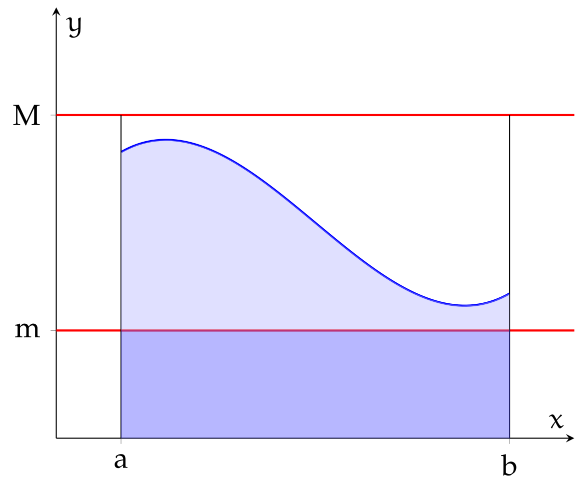


Figure 1.6: $m \leq f(x) \leq M$

1.3.7 Other Properties of Integrals

If $f(x) \geq 0$ over the for $a \leq x \leq b$, then $\int_a^b f(x)\,dx > 0$. We can make an intuitive, geometric argument to support this claim. Recall that areas above the x-axis are considered positive. If $f(x) \geq 0$, then all the area of the integral lies above the x-axisj therefore, the total area must be positive.

Similarly, if $f(x) \geq g(x)$ on the interval $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ (see figure 1.7). The entire area under $g(x)$ is contained in the area under $f(x)$. Therefore, $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

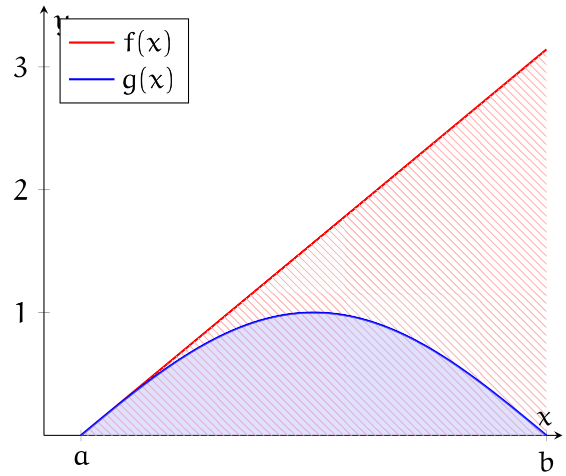


Figure 1.7: $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Lastly, we see what happens when we switch a and b. While it is unusual to integrate from right to left (that is, in a case where $a > b$), this property will be useful. Recall that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n}$$

What is $\int_b^a f(x)\,dx$? Substituting, we see that

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(a-b)}{n}$$

Noting that $(a-b) = -(b-a)$ we see:

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{-(b-a)}{n}$$

$$= (-1)\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n}$$

$$= (-1)\int_a^b f(x)\,dx$$

Therefore, it is true that $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

1.4 Applications in Physics

We have already seen that the area under a velocity function is displacement, and the area under an acceleration function is change in velocity (Riemann Sums). We can use integrals to determine the change in position of an object over a given time frame. If we *also* know the object's starting position, then we can state the object's ending position. Consider the graph of an object's velocity in figure 1.8:

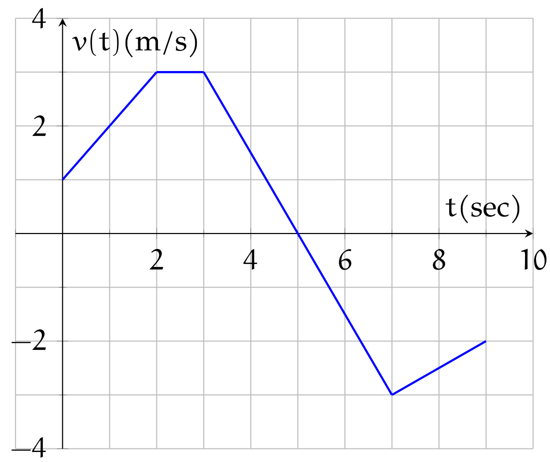


Figure 1.8: Velocity of an object from $t = 0$ to $t = 9$

We can determine the net displacement of the object from $t = 0$ to $t = 9$ by evaluating $\int_0^9 v(t)\,dt$. Since the definite integral is equal to the area under the curve, we need to find the total area. As the function consists of straight lines, we will leave the explicit calculation of the area as an exercise for the student. You should find that the total positive area (above the x-axis) is 10 meters, and the total negative area (below the x-axis) is 8 meters. Therefore, the object's displacement over the specified time interval is $10 - 8 = 2$ meters.

When you push on something to move it, you are applying a force over a distance (assuming you are strong enough to move it!). The integral of force as a function of distance is the *work* done on that object. Work is the change in kinetic energy (KE) of an object. Mathematically, this is

$$\int_a^b F(x)\,dx = \Delta KE = \frac{1}{2}m(v_f^2 - v_i^2)$$

.

If you integrate the force as a function of time, that is *impulse*. Impulse is the change in momentum (p) of the object. Mathematically, this is

$$\int_a^b F(t)\,dt = \Delta p = m(v_f - v_i)$$

.

Example problem: You push a 3 kg box with force $F(x) = 0.5x$, where x is measured in meters and F is measured in Newtons. If the box was initially at rest, what is its speed when it reaches the 2 meter mark? (Hint: $KE = \frac{1}{2}mv^2$.)

Solution: Change in kinetic energy is the area under a force-distance curve. We can plot the force applied to the box from d=0 to d=2 (see figure 1.9):

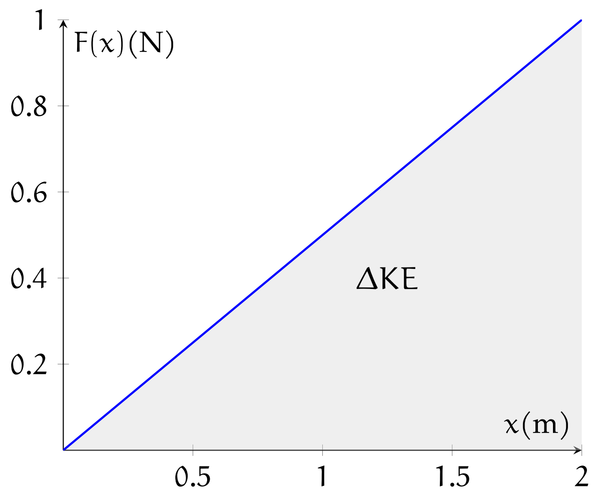


Figure 1.9: Force applied to a box over a distance; the shaded area represents the change in kinetic energy.

Given that the box's initial velocity is $0\frac{m}{s}$, we know that the initial kinetic energy (KE) is 0J. This implies that $KE_f = \Delta KE$. We can find ΔKE from the shaded area:

$$\Delta KE = \frac{1}{2}(2m)(1N) = 1J = KE_f$$

Solving for the final velocity:

$$KE_f = 1J = \frac{1}{2}(3kg)(v^2)$$

$$2J = (3kg)(v^2)$$

$$\frac{2}{3}\frac{m^2}{s^2} = v^2$$

$$v = \sqrt{\frac{2}{3}\frac{m^2}{s^2}} \approx 0.816\frac{m}{s}$$

1.5 Practice Exercises

Exercise 2

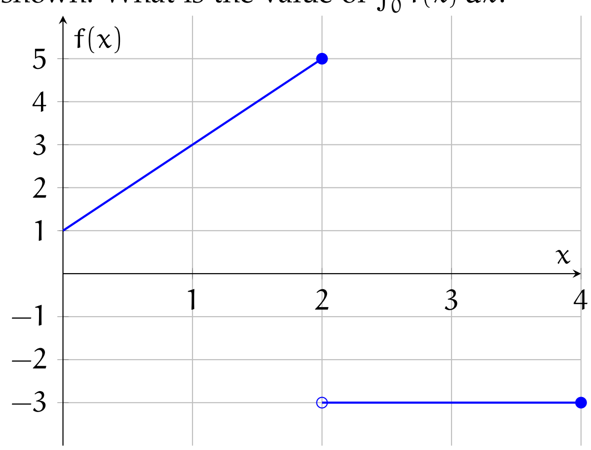
Given that $\int_0^1 x^2\,dx = \frac{1}{3}$, use the properties of integrals to evaluate $\int_0^1 (5-6x^2)\,dx$.

Working Space

Answer on Page 15

Exercise 3

[This question was originally presented as a multiple-choice problem on the 2012 AP Calculus BC exam.] The graph of f is shown. What is the value of $\int_0^4 f(x)\,dx$?

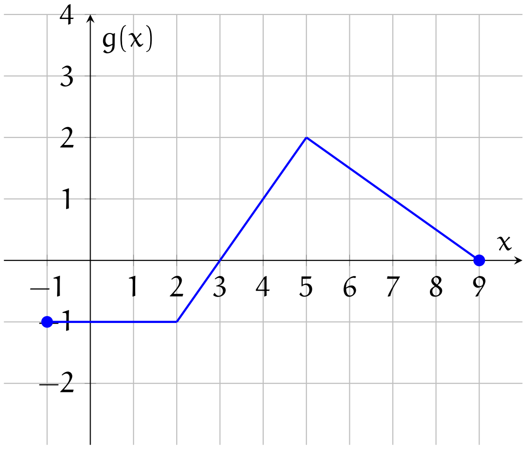


Working Space

Answer on Page 15

Exercise 4

[This question was originally presented as a multiple-choice problem on the 2012 AP Calculus BC exam.] The graph of the piecewise function $g(x)$ is shown. What is the value of $\int_{-1}^{9} 3g(x) + 2\, dx$?



Working Space

Answer on Page 16

Exercise 5

[This question was originally presented as a calculator-allowed, multiple- choice question on the 2012 AP Calculus BC exam.] If $f'(x) > 0$ for all real numbers and $\int_4^7 f(x)\,dx = 0$, which of the following could be a table of values for the function f?

(A)

x	$f(x)$
4	-4
5	-3
7	0

(B)

x	$f(x)$
4	-4
5	-2
7	5

(C)

x	$f(x)$
4	-4
5	6
7	3

(D)

x	$f(x)$
4	0
5	0
7	0

(E)

x	$f(x)$
4	0
5	4
7	6

Working Space

Answer on Page 16

APPENDIX A

Answers to Exercises

Answer to Exercise 1 (on page 2)

Following the structure shown in the formal definition of a definite integral, we can set $f(x) = x^3 + x \sin x$ and rewrite the limit of the sum as $\lim_{n\to\infty} \sum_{i=1}^{n} f(x)\Delta x = \int_0^\pi f(x)\,dx$. Therefore, the full definite integral would be written as $\int_0^\pi (x^3 + x \sin x)\,dx$.

Answer to Exercise 2 (on page 11)

By property 6, we know that

$$\int_0^1 (5 - 6x^2)\,dx = \int_0^1 5\,dx - \int_0^1 6x^2\,dx$$

By property 5, we know that

$$\int_0^1 5\,dx - \int_0^1 6x^2\,dx = \int_0^1 5\,dx - 6\int_0^1 x^2\,dx$$

By property 3, we know that

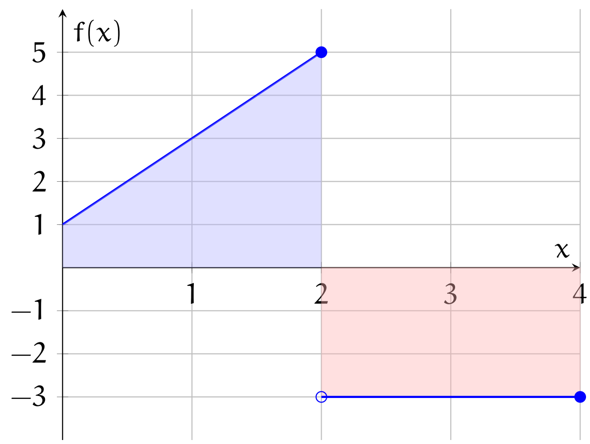
$$\int_0^1 5\,dx = 5(1 - 0) = 5$$

Putting it all together, we see that

$$\int_0^1 (5 - 6x^2)\,dx = 5 - 6(\frac{1}{3}) = 5 - 2 = 3$$

Answer to Exercise 3 (on page 11)

We can break the integral into two parts: from $x = 0$ to $x = 2$ (shaded in blue), and from $x = 2$ to $x = 4$ (shaded in red). The blue portion is a trapezoid, so it has a total area of $\frac{1}{2}(b_1 + b_2)(h) = \frac{1}{2}(1 + 5)(2) = 6$. Because it is above the x-axis, the area is positive. The red portion is a rectangle and has a total area of $2 \times 3 = 6$ and is *negative*, because it lies below the x-axis. Therefore, the total area is $6 + -6 = 0$.



Answer to Exercise 4 (on page 12)

Using the properties of integrals, we can rewrite $\int_{-1}^{9} 3g(x) + 2\,dx$ as $3\int_{-1}^{9} g(x)\,dx + 2(9 - (-1))$. From the graph, we can determine $\int_{-1}^{9} g(x)\,dx = 2.5$. Therefore, $\int_{-1}^{9} 3g(x) + 2\,dx = 3(2.5) + 2(10) = 27.5$.

Answer to Exercise 5 (on page 13)

(B). If $f'(x) > 0$ for all x, then f must be increasing for $4 < x < 7$. Since (C) decreases from $x = 5$ to $x = 7$, we can eliminate it. We can also eliminate (D), since $f(4) = f(5) = f(7)$, which implies either the slope of f is zero or changes from positive to negative. If $\int_4^7 f(x)\,dx =$, then some portion of $f(x)$ lies above the $x-axis$, while some other portion lies below (we must have positive and negative areas for the sum to be zero). This eliminates (A) and (E), since the integral of (A) would have a negative value and the integral of (E) would have a positive value. This leaves (B).

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