Chapter 1

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) is a theorem that connects the concept of differentiating a function with the concept of integrating a function. This theorem is divided into two parts.

1.1 First Part

The first part of the Fundamental Theorem of Calculus states that if f is a continuous real-valued function defined on a closed interval $[a, b]$ and F is the function defined, for all x in $[a, b]$, by:

$$F(x) = \int_a^x f(t)\,dt \tag{1.1}$$

Then, F is uniformly continuous and differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all x in (a, b). (That is $F(x)$ is the antiderivative of $f(x)$.)

1.2 Second Part

The second part of the Fundamental Theorem of Calculus states that if f is a real-valued function defined on a closed interval $[a, b]$ that admits an antiderivative F on $[a, b]$, and f is integrable on $[a, b]$ (it need not be continuous), then

$$\int_a^b f(t)\,dt = F(b) - F(a). \tag{1.2}$$

We will also use shorthand as follows:

$$\int_a^b f(t)\,dt = F(t)|_a^b \tag{1.3}$$

Which means "$F(t)$ evaluated from $t = a$ to $t = b$".

1.3 FTC and Definite Integrals

Let f be a function that is continuous on the interval $x \in [a, b]$ and $g(x)$ is given by:

$$g(x) = \int_a^x f(t)\,dt$$

So, g is continuous on $[a, b]$ and differentiable on (a, b). Additionally,

$$g'(x) = f(x)$$

Proof: Let x and $x + h$ be in (a, b). Then,

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

Recall from the chapter on definite integrals that we can split the first integral, rewriting it as:

$$g(x+h) - g(x) = \left[\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right] - \int_a^x f(t)\,dt$$

$$g(x+h) - g(x) = \int_x^{x+h} f(t)\,dt$$

And for $h \neq 0$:

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Since f is continuous, there is some u in (a, b), such that $f(u) = m$, where m is the minimum value of f on the interval (a, b). Similarly, there is also some v, such that $f(v) = M$, where M is the maximum value (see figure ??). We can then state the true inequality that:

$$mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$$

Therefore, (assuming $h > 0$):

$$f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v)$$

Substituting the equation above for the integral, we see that:

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v)$$

If we let h approach zero, then the window that u and v are in collapses and u and v both approach x. Therefore,

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x)$$

Recall also that

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

Taking the limit as $h \to 0$ of the whole, inequality becomes the Squeeze Theorem:

$$\lim_{h \to 0} f(u) \leq \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \leq \lim_{h \to 0} f(v)$$

$$f(x) \leq g'(x) \leq f(x)$$

Therefore, if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$. Notice it doesn't matter what a is!

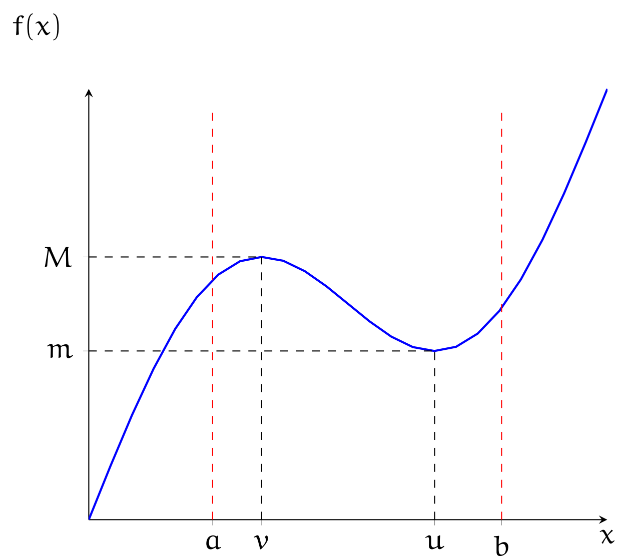

Figure 1.1: $f(v) = M$, the maximum value, and $f(u) = m$, the minimum value on the interval $x \in [a, b]$

Exercise 1

[This question was originally presented as a no-calculator, multiple-choice problem on the 2012 AP Calculus BC Exam.] Let g be a continuously differentiable function with $g(1) = 6$ and $g'(1) = 3$. What is the value of $\lim_{x \to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6}$?

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) 2
(E) The limit does not exist

Working Space

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1.4 The Meaning of the FTC

What the Fundamental Theorem of Calculus is really saying is that differentiation and integration are opposite processes. Mathematically, we can say

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This may seem clunky, but many useful functions are defined this way. Consider the Fresnel function: $S(x) = \int_0^x \sin\frac{\pi t^2}{2}\,dt$. Originally used in optics, this equation is also used by civil engineers to design road and railway curves. According to FTC, then, $S'(x) = \sin\frac{\pi t^2}{2}$.

We can also apply the Chain Rule when taking derivatives of integrals. Let $f(x) =$

$\int_1^{x^4} \sec t \, dt$. What is $f'(x)$? First, let us define $u = x^4$. By the Chain Rule,

$$\begin{aligned} \frac{d}{dx} \int_0^{x^4} \sec t \, dt &= \frac{d}{dx} \int_0^{u} \sec t \, dt \\ &= \frac{d}{du} \left[\int_0^{u} \sec t \, dt \right] \frac{du}{dx} \\ &= \sec u \frac{du}{dx} \end{aligned}$$

Noting that $\frac{du}{dx} = \frac{d}{dx} x^4 = 4x^3$,

$$f'(x) = \sec x^4 (4x^3)$$

1.4.1 FTC Practice

Exercise 2

Use the Fundamental Theorem of Calculus to find the derivative of the function.

1. $g(x) = \int_0^x \sqrt{t + t^3} \, dt$
2. $F(x) = \int_x^0 \sqrt{1 + \sec t} \, dt$
3. $h(x) = \int_1^{e^x} \ln t \, dt$
4. $y = \int_{\sqrt{x}}^{\frac{\pi}{4}} \theta \tan \theta \, d\theta$

Working Space

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1.5 Using Antiderivatives to Evaluate Definite Integrals

In everyday English, the FTC states that the integral from a to b of a function is the antiderivative of that function evaluated from a to b. In the previous chapter, the integrals presented were of linear functions where the area under the curve could be equally calculated by hand. The FTC connects integrals to antiderivatives, allowing us to evaluate more complex integrals. Consider the following example:

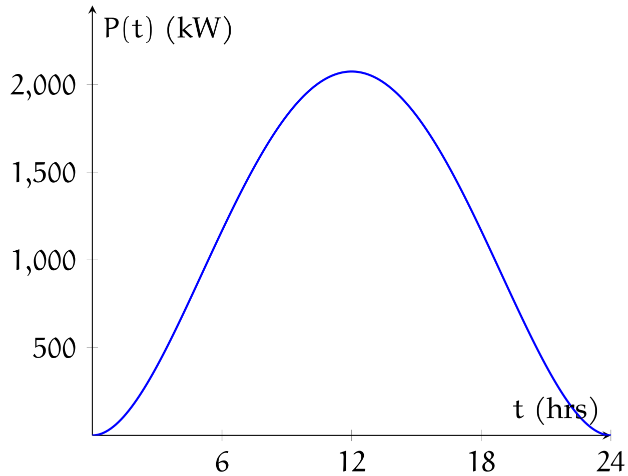

Figure 1.2: Power consumption of a household in a day

The power consumption of a household can be modeled as $P(t) = \frac{1}{10}t^2(t-24)^2$ from $t = 0$ to $t = 24$, where P is measured in watts and t is measured in hours ($t = 0$ is midnight). The total energy the household uses is given by $\int_0^{24} P(t)\,dt$. As you can see from the graph (see figure 1.2), we cannot simply use our geometry skills to determine the area under the curve.

To determine the total energy use, we need to evaluate $\int_0^{24} \frac{1}{10}t^2(t-24)^2\,dt$. First, we expand the polynomial:

$$\begin{aligned} E_{tot} &= \frac{1}{10}\int_0^{24} t^2(t^2 - 48t + 576)\,dt = \frac{1}{10}\int_0^{24} t^4 - 48t^3 + 576t^2\,dt \\ &= \frac{1}{10}\int_0^{24} t^4\,dt - \frac{24}{5}\int_0^{24} t^3\,dt + \frac{288}{5}\int_0^{24} t^2\,dt \end{aligned}$$

Using the power rule to determine the antiderivatives of t^4, t^3, and t^2, we see:

$$= \frac{1}{10}\left[\frac{1}{5}t^5\right]_0^{24} - \frac{24}{5}\left[\frac{1}{4}t^4\right]_0^{24} + \frac{288}{5}\left[\frac{1}{3}t^3\right]_0^{24} = 26542.1\text{Whr} = 26.5421\text{kWhr}$$

1.5.1 Definite Integrals Practice

Exercise 3

Evaluate the following integral:

1. $\int_1^4 t^{-3/2}\,dt$

Working Space

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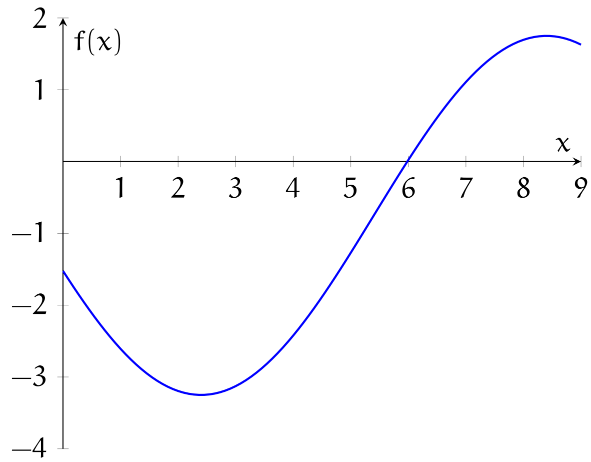
Exercise 4

[This question was originally presented as a multiple-choice, no-calculator problem on the 2012 Calculus BC exam.] The graph of a differentiable function f is shown in the graph. $h(x) = \int_0^x f(t)\,dt$. Rank the relative values of $h(6)$, $h'(6)$, and $h''(6)$ from lowest to highest.



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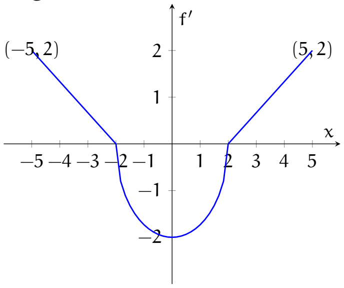
Exercise 5

The graph of f' is shown in the graph and consists of a semi-circle and two line segments. If $f(2) = 1$, then what is $f(-5)$?



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Exercise 6

[This question was originally presented as a calculator-allowed, multiple- choice question on the 2012 AP Calculus BC exam.] A particle moves along a line so that its acceleration for $t \geq 0$ is given by $a(t) = \frac{t+3}{\sqrt{t^3+1}}$. If the particle's velocity at $t = 0$ is 5, what is the velocity of the particle at $t = 3$?

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1.6 Average Value of a Function

The average value of a function, f, on an interval $[a, b]$ is given by:

Average Value of a Function

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example: Find the average value of $f(x) = 3 + x^2$ on the interval $[-2, 1]$.

Solution: Taking $a = -2$ and $b = 1$, we have:

$$f_{avg} = \frac{1}{1-(-2)}\int_{-2}^{1}\left[3 + x^2\right]\,dx$$

$$f_{avg} = \frac{1}{3}\int_{-2}^{1}\left[3 + x^2\right]\,dx$$

$$f_{avg} = \frac{1}{3}\left[3x + \frac{1}{3}x^3\right]_{x=-2}^{x=1}$$

$$f_{avg} = \frac{1}{3}\left[\left(3\cdot 1 + \frac{1}{3}\cdot 1^3\right) - \left(3\cdot(-2) + \frac{1}{3}\cdot(-2)^3\right)\right]$$

$$f_{avg} = \frac{1}{3}\left[\left(3 + \frac{1}{3}\right) - \left(-6 - \frac{8}{3}\right)\right]$$

$$f_{avg} = \frac{1}{3}\left[\frac{10}{3} + 6 + \frac{8}{3}\right] = \frac{1}{3}(12) = 4$$

Therefore, the average value of $f(x) = 3 + x^2$ on the interval $[-2, 1]$ is 4.

Exercise 7

[This question was originally presented as a multiple-choice, calculator- allowed problem on the 2012 AP Calculus BC Exam.] What is the average value of $y = \sqrt{\cos x}$ on the interval $0 \leq x \leq \frac{\pi}{2}$?

Working Space

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Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

(D) 2. First, we try to compute the limit directly: $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6} = \frac{\int_1^1 g(t)\,dt}{6-6} = \frac{0}{0}$, which is undefined. Because g is continuous and differentiable, we can apply L'Hospital's rule. $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6} = \lim_{x\to 1} \frac{d}{dx}\left[\frac{\int_1^x g(t)\,dt}{g(x)-6}\right] = \lim_{x\to 1} \frac{g(x)}{g'(x)} = \frac{g(6)}{g'(6)} = \frac{6}{3} = 2$.

Answer to Exercise 2 (on page 5)

1. $g'(x) = \sqrt{x + x^3}$
2. $F(x) = -\int_0^x \sqrt{1 + \sec t}\,dt$ and therefore $F'(x) = -\sqrt{1 + \sec x}$
3. setting $u = e^x$ and noting $\frac{du}{dx} = e^x$, then $h'(x) = \frac{d}{du}\int_1^u \ln t\,dt(\frac{du}{dx})$ Taking the derivative and substituting for $\frac{du}{dx}$, we find $h'(x) = \ln u \cdot e^x = \ln(e^x) \cdot e^x = x \cdot e^x$
4. $y = -\int_{\frac{\pi}{4}}^{\sqrt{x}} \theta \tan\theta\,d\theta$. Setting $u = \sqrt{x}$ and noting that $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, we see that $y' = -\frac{d}{du}[\int_{\frac{\pi}{4}}^{u} \theta \tan\theta\,d\theta]\frac{du}{dx} = u \tan u \cdot \frac{1}{2\sqrt{x}} = -\sqrt{x}\tan\sqrt{x} \cdot \frac{1}{2\sqrt{x}} = \frac{-\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}} = -\frac{1}{2}\tan\sqrt{x}$

Answer to Exercise 3 (on page 7)

1. The antiderivative of $t^{-3/2}$ is $\frac{-2}{\sqrt{t}}$. Therefore, the integral is equal to $[\frac{-2}{\sqrt{t}}]_1^4 = \frac{-2}{\sqrt{4}} - \frac{-2}{\sqrt{1}} = -1 + 2 = 1$.

Answer to Exercise 4 (on page 7)

According to FTC, $h'(x) = f(x)$ and $h''(x) = f'(x)$. Examining the graph, we see that the curve lies below the x-axis for $0 < x < 6$, which means that $h(6) = \int_0^6 f(t)\,dt < 0$. $h'(6) = f(6) = 0$ and $h''(6) = f'(6) > 0$. Therefore, $h(6) < h'(6) < h''(6)$.

Answer to Exercise 5 (on page 8)

We know that $f(2) = \int_{-5}^{2} f'(x)\,dx + f(-5)$. Examining the graph, we know that $\int_{-5}^{2} f'(x)\,dx = frac12(3)(2) - \frac{1}{2}\pi(2^2)$ (the area of the triangle above the x-axis less the area of the semi-circle below the axis). Therefore, $f(-5) = f(2) - \int_{-5}^{2} f'(x)\,dx = 1 - (3 - 2\pi) = 2\pi - 2$

Answer to Exercise 6 (on page 8)

11.71. The particle's velocity will be given by its initial velocity plus the integral of its acceleration over the time period. Therefore, $v(3) = v(0) + \int_0^3 a(t)\,dt = 5 + \int_0^3 \frac{t+3}{\sqrt{t^3+1}}\,dt \approx 5 + 6.71 = 11.71$.

Answer to Exercise 7 (on page 9)

We begin by setting up the integral for average value of a function with $a = 0$, $b = \frac{\pi}{2}$, and $f(x) = \sqrt{\cos x}$:

$$\frac{1}{\pi/2 - 0} \int_0^{\pi/2} \sqrt{\cos x}\,dx$$
$$= \frac{2}{\pi} \int_0^{\pi/2} \sqrt{\cos x}\,dx$$

There's not an obvious way to evaluate this integral by hand. Luckily, this question allows for the use of a calculator. Entering this integral into a calculator (such as a TI-89 or Wolfram Alpha), we find that:

$$\frac{2}{\pi} \int_0^{\pi/2} \sqrt{\cos x}\,dx \approx 0.763$$

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