Chapter 1

Trigonometric Identities

1.1 The Unit Circle

There are some values of $\sin\theta$ and $\cos\theta$ that will be useful to know off the top of your head. The Unit Circle will help you in this memorization process (see figure 1.1). When a circle of radius 1 is centered at the origin, the Cartesian coordinates of any point on the circle correspond to the values of cosine and sine of the angle above the horizontal (how far you've rotated from the positive x-axis).

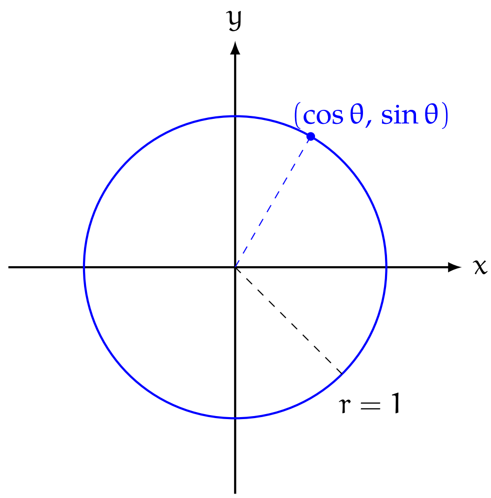

Figure 1.1: The Unit Circle is a circle with radius 1 centered at the origin

Let's take a closer look at a triangle in the first quadrant to see why this is true. Imagine some point on the circle, (x_o, y_o). Drawing a line from that point back to the origin creates an angle θ between the imaginary line and the positive x-axis (see figure 1.2). Extending an imaginary vertical down to $(x_o, 0)$, then an imaginary horizontal from $(x_o, 0)$ to the origin, creates a right triangle. What can we say about the legs of the triangle?

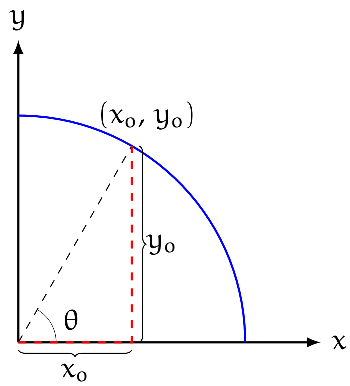

Figure 1.2: Drawing a line from any point on the circle to the origin creates an angle with the horizontal

Recall SOH-CAH-TOA from a previous chapter. This acronym tells us that, for a right triangle, the sine of an angle is given by the ratio of the length of the leg opposite the angle to the hypotenuse. In our case, then, $\sin\theta = \frac{y_o}{1} = y_o$. [Remember: We are dealing with the Unit Circle, which has a radius of one. Examining figure 1.2 shows you that the hypotenuse of the imaginary triangle is the same as the circle's radius.] This means that the y-coordinate of any point on the Unit Circle is the sine of the angle of rotation from the horizontal.

Exercise 1

In a similar manner as we did with $\sin\theta$ above, prove the x-coordinate of any point on the unit circle is equal to $\cos\theta$, where θ is the angle of rotation from the horizontal.

Working Space

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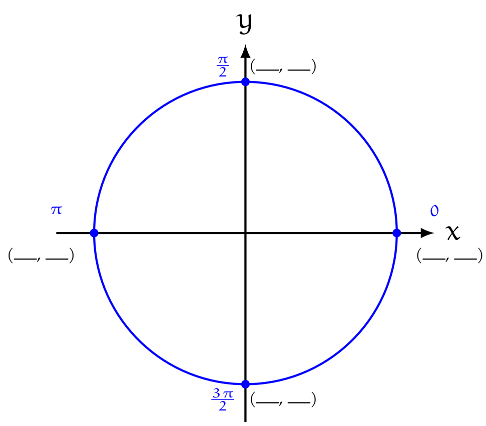
From these exercises, we can see that each (x, y) coordinate on the circle is equal to $(\cos(\theta), \sin(\theta))$.

Exercise 2

Fill in the unit circle with the coordinates for $\theta = 0$, $\pi/2$, π, and $3\pi/2$. Use this to determine:

1. $\sin \frac{\pi}{2}$
2. $\cos \frac{3\pi}{2}$
3. $\sin \pi$
4. $\cos -\pi$



Working Space

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1.1.1 Exact Values of Key Angles

We will examine two triangles. First, a 30-60-90 triangle, then a 45-45-90 triangle. As shown in figure 1.3, you can get a 30-60-90 triangle with hypotenuse 1 by dividing an equilateral triangle in half. We will label the horizontal leg of the 30-60-90 triangle A and the vertical leg B.

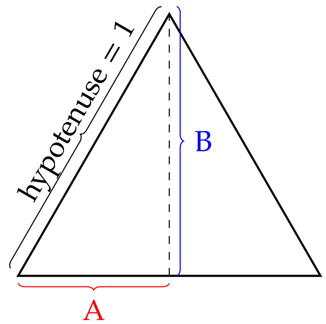


Figure 1.3: A 30-60-90 triangle is made by vertically bisecting an equilateral triangle

From the figure, we see that the length of A is half that of the hypotenuse, which in this case is $\frac{1}{2}$. This means the $\cos 60° = \cos \frac{\pi}{3} = \frac{1}{2}$. To find the length of side B, we can use the Pythagorean theorem:

$$B^2 = C^2 - A^2, \text{ where C is the hypotenuse}$$

$$B^2 = 1^2 - \left(\frac{1}{2}\right)^2$$

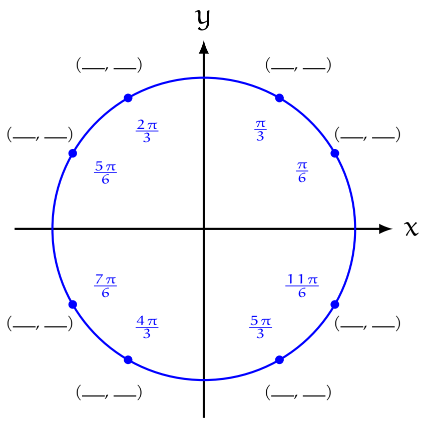
$$B^2 = \frac{3}{4}$$

$$B = \frac{\sqrt{3}}{2}$$

Therefore, $\sin 60° = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Exercise 3

Use symmetry to complete the blank unit circle below. (Hint: We just showed that the (x, y) coordinate for $\frac{\pi}{3}$ is $(1/2, \sqrt{3}/2)$).



Working Space

Answer on Page 14

Now we will look at a 45-45-90 triangle (see figure 1.4), which will allow us to complete our Unit Circle. Recall that a 45-45-90 triangle is an isosceles triangle in addition to being a right triangle. This means both the legs are the same length. Using the Pythagorean theorem, we would say $A = B$. We also know that $C = 1$, since our triangle is inscribed in the unit circle. Let's find A:

$$A^2 + B^2 = C^2$$
$$A^2 + A^2 = 1^2$$
$$2A^2 = 1$$
$$A^2 = \frac{1}{2}$$
$$A = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, each leg has a length of $\sqrt{2}/2$, and the (x, y) coordinates for $\theta = 45° = \pi/4$ are $(\sqrt{2}/2, \sqrt{2}/2)$.

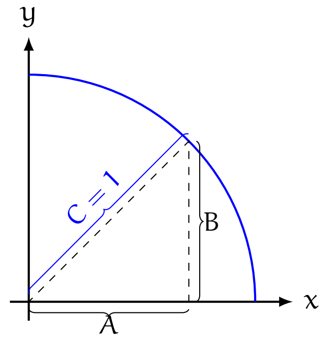
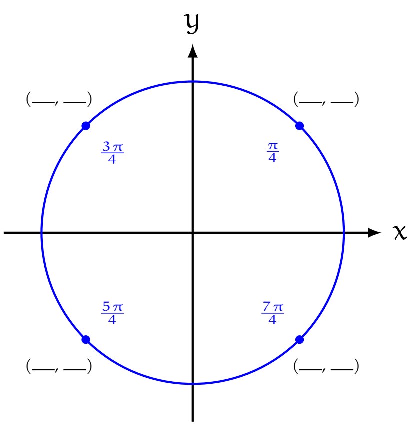


Figure 1.4: The two legs of a 45-45-90 triangle are the same length

Exercise 4

Use symmetry to complete the blank unit circle below.



Working Space

Answer on Page 14

Exercise 5

Without a calculator and using only your completed unit circles, determine the value requested (angles are given in radians unless otherwise indicated).

1. $\cos \frac{3\pi}{2}$
2. $\sin \frac{\pi}{4}$
3. $\sin -\frac{\pi}{6}$
4. $\cos \frac{4\pi}{3}$
5. $\sin \frac{3\pi}{4}$
6. $\cos -\frac{\pi}{3}$
7. $\sin 45°$
8. $\sin 270°$
9. $\sin -60°$
10. $\sin 150°$

Working Space

Answer on Page 14

1.2 Sum and Difference Formulas

Consider 4 points on the unit circle: B at $(1, 0)$, Q at some angle β, P at some angle α, and A at angle $\alpha - \beta$ (see figure 1.5).

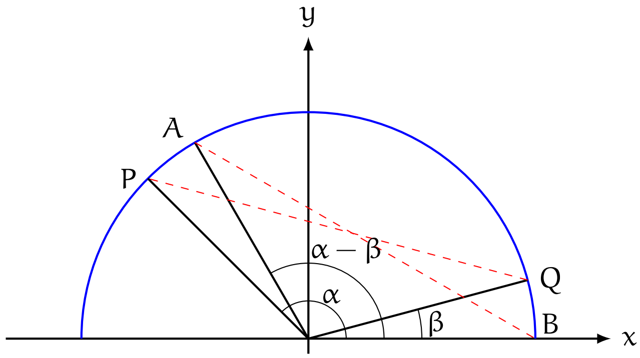


Figure 1.5: $\overline{AB} = \overline{PQ}$

The distance from P to Q is the same as the distance from A to B, since ΔPOQ is a rotation of ΔAOB. Because this is a Unit Circle, $P = (\cos\alpha, \sin\alpha)$, $Q = (\cos\beta, \sin\beta)$, and $A = (\cos\alpha - \beta, \sin\alpha - \beta)$. Let's use the distance formula to find the length of $\overline{PQ}$:

$$\overline{PQ} = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} =$$

$$\sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} =$$

$$\sqrt{\left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

Recall that for any angle, θ, $\sin^2\theta + \cos^2\theta = 1$. Substituting this identity, we see that:

$$\overline{PQ} = \sqrt{1 + 1 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta} = \sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta}$$

Let's leave this simplified equation for $\overline{PQ}$ alone for the moment and similarly find $\overline{AB}$:

$$\overline{AB} = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2} =$$

$$\sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)} =$$

$$\sqrt{\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) + 1 - 2\cos(\alpha - \beta)}$$

$$= \sqrt{2 - 2\cos(\alpha - \beta)} = \overline{AB}$$

Recall that we've established $\overline{AB} = \overline{PQ}$. We can set the statements equal to each other:

$$\sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta} = \sqrt{2 - 2\cos(\alpha - \beta)}$$

Squaring both sides and subtracting 2, we find:

$$-2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta = -2\cos(\alpha - \beta)$$

Finally, we can divide both sides by negative 2 to get the difference of angles formula for cosine:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

There are similar formulas for the sine and cosine of the sum of two angles, and for the sine of the difference of two angles, which we won't derive here.

Sum and Difference Formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Exercise 6

Without a calculator, find the exact value requested:

1. $\sin\frac{\pi}{12}$
2. $\cos\frac{7\pi}{12}$
3. $\tan\frac{13\pi}{12}$ (hint: $\tan\theta = \sin\theta / \cos\theta$)

Working Space

Answer on Page 15

1.3 Double and Half Angle Formulas

We can easily derive a formula for twice an angle by letting $\alpha = \beta$ for a sum formula.

Example: Derive a formula for $\cos 2\theta$ in terms of trigonometric functions of θ.

Solution: Using the sum formula for cosine, we see that:

$$\cos 2\theta = \cos(\theta + \theta)$$

$$= \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

Noting that $\sin^2\theta = 1 - \cos^2\theta$: $\boxed{\cos 2\theta = 2\cos^2\theta - 1}$ Alternatively, we could note that $\cos^2\theta = 1 - \sin^2\theta$: $\boxed{\cos 2\theta = 1 - 2\sin^2\theta}$

Or additionally, $\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$

Exercise 7

Derive a formula for $\sin 2\theta$ in terms of trigonometric functions of θ.

Working Space

Answer on Page 15

We can use these double-angle formulas for find half-angle formulas. Consider the double-angle formula for cosine:

$$\cos 2\theta = 2\cos^2\theta - 1$$

Let $\theta = \alpha/2$, then:

$$\cos\alpha = 2\cos^2(\alpha/2) - 1$$

Rearranging to solve for $\cos(\alpha/2)$:

$$2\cos^2(\alpha/2) = \cos\alpha + 1$$

$$\cos^2(\alpha/2) = \frac{\cos\alpha + 1}{2}$$

$$\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Exercise 8

Derive a formula for $\sin(\alpha/2)$.

Working Space

Answer on Page 15

There are two identities that will be very useful for integrals in a future chapter:

Squared Trigonometric Identities

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

These are just specific re-writings of the half-angle identities.

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

APPENDIX A

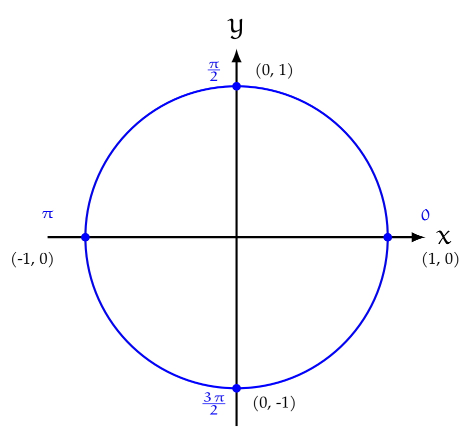
Answers to Exercises

Answer to Exercise 1 (on page 2)

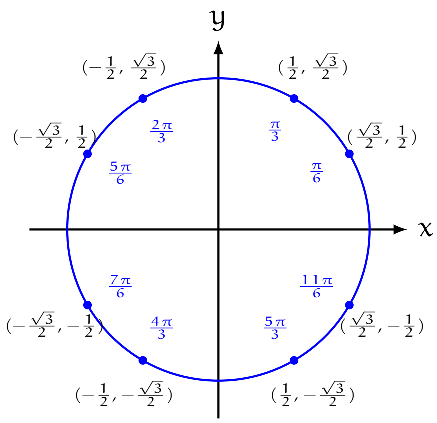
We know that for a right triangle, $\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$. For a right triangle inscribed in the Unit Circle, the adjacent leg is parallel to the x-axis and has the same length as the x-value of the coordinate point on the circle. Additionally, the length of the hypotenuse is 1. Therefore, $\cos\theta = \frac{x_o}{1} = x_o$.

Answer to Exercise 2 (on page 3)

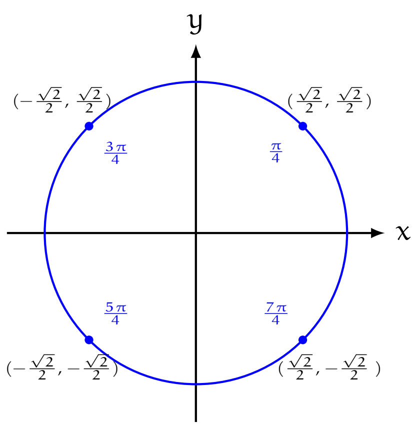
1. $\sin\frac{\pi}{2} = 1$
2. $\cos\frac{3\pi}{2} = 0$
3. $\sin\pi = 0$
4. $\cos -\pi = -1$ (Negative angles are measured clockwise from the x-axis, so $\theta = -\pi$ is at the same angle as $\theta = \pi$.)



Answer to Exercise 3 (on page 5)



Answer to Exercise 4 (on page 6)



Answer to Exercise 5 (on page 7)

1. 0
2. $\sqrt{2}/2$
3. $-1/2$
4. $-1/2$
5. $\sqrt{2}/2$
6. $1/2$

7. $\sqrt{2}/2$
8. -1
9. $-\sqrt{3}/2$
10. $1/2$

Answer to Exercise 6 (on page 9)

1. $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$
2. $\cos(7\pi/12) = \cos(4\pi/12 + 3\pi/12) = \cos(\pi/3 + \pi/4) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \left(\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$
3. $\tan(13\pi/12) = \frac{\sin(13\pi/12)}{\cos(13\pi/12)}$ First, we will find $\sin(13\pi/12)$: $\sin(13\pi/12) = \sin(3\pi/12 + 10\pi/12) = \sin(\pi/4 + 5\pi/6) = \sin\frac{\pi}{4}\cos\frac{5\pi}{6} + \cos\frac{\pi}{4}\sin\frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} = \frac{\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$. Next we find $\cos(13\pi/12) = \cos(\pi/4 + 5\pi/6) = \cos\frac{\pi}{4}\cos\frac{5\pi}{6} - \sin\frac{\pi}{4}\sin\frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{-\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-\sqrt{6}-\sqrt{2}}{4}$. And therefore $\tan(13\pi/12) = \frac{\sin 13\pi/12}{\cos 13\pi/12} = \frac{\sqrt{2}-\sqrt{6}}{4}\cdot\frac{4}{-\sqrt{6}-\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}\cdot\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{6-2\sqrt{12}+2}{6-2} = \frac{8-4\sqrt{3}}{4} = 2-\sqrt{3}$

Answer to Exercise 7 (on page 10)

$\sin 2\theta = \sin\theta + \theta = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

Answer to Exercise 8 (on page 11)

Similar to $\cos(\alpha/2)$, we begin with the double angle formula for cosine, but another version:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

Substituting $\theta = \alpha/2$:

$$\cos\alpha = 1 - 2\sin^2(\alpha/2)$$

And rearranging to solve for $\sin(\alpha/2)$:

$$\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$