

Volumes with Integrals

Suppose we wanted to know the volume of a theoretical irregular shape (we stipulate theoretical because, if you had this object and a large enough container, you could use displacement to determine the volume of the object). [fixme better intro]

1.1 Volume of a Sphere

Below, we will prove the volume of a sphere is given by $\frac{4}{3}\pi r^3$ using the integral method. Suppose we have a sphere of radius r centered at the origin (see figure ??).

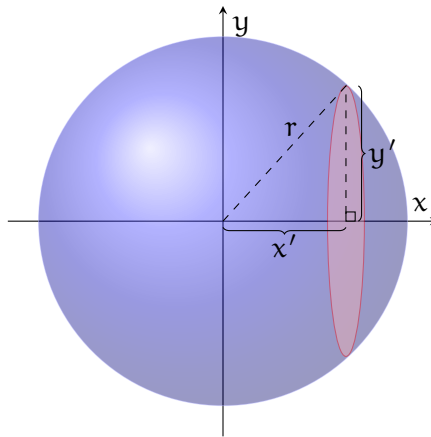


Figure 1.1: A vertical cross-section of a sphere

We begin by taking very thin vertical cross-sections. The radius of the cross-section is the height, y , of the sphere at the horizontal position, x . Since the edges of the cross-section lie on the sphere, we know the edge of the cross-section is distance r from the origin. Applying the Pythagorean theorem, we see that $r^2 = x^2 + y^2$, which implies that $y = \sqrt{r^2 - x^2}$. So, the area of the cross-section is given by $\pi y^2 = \pi(r^2 - x^2)$. If we imagine each cross section as having a width, dx , and taking the sum of all the cross sections from $x = -r$ to $x = r$, we can write an integral equal to the volume of the sphere:

$$V_{\text{sphere}} = \int_{-r}^r \pi(r^2 - x^2) dx$$

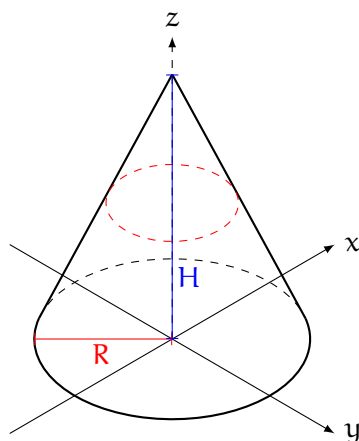
We can then evaluate that integral:

$$V_{\text{sphere}} = \pi \int_{-r}^r r^2 dx - \pi \int_{-r}^r x^2 dx$$

$$\begin{aligned}
 V_{\text{sphere}} &= \pi \left[r^2 x \right]_{x=-r}^{x=r} - \frac{\pi}{3} \left[x^3 \right]_{x=-r}^{x=r} \\
 V_{\text{sphere}} &= \pi \left[r^3 - (-r^3) \right] - \frac{\pi}{3} \left[r^3 - (-r^3) \right] \\
 V_{\text{sphere}} &= 2\pi r^3 - \frac{2\pi}{3} r^3 = \frac{4}{3} \pi r^3
 \end{aligned}$$

Exercise 1

Prove the volume of a regular cone is $\frac{\pi}{3} R^2 H$, where R is the radius of the base and H is the height of the cone. (Hint: A cone is a series of decreasing circles stacked on top of each other; see figure below.)



Working Space

Answer on Page ??

1.2 Volumes of Solids of Revolution

We can also find the volume of solids made by revolving a graph about the x or y -axis. Suppose the graph $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ were rotated vertically about the x -axis to form a solid. How could we find the volume of that solid? Well, we can imagine a rectangle of width dx and height y (see figure ??)

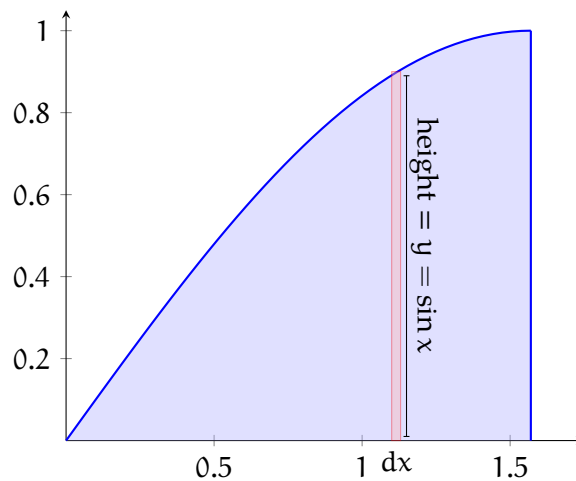


Figure 1.2: A cross section has width dx and height $y = \sin x$

If we rotate the plot vertically about the x -axis, the rectangle becomes a cylinder with radius $y = \sin x$ and height dx (see figure ??). Therefore, the volume of each cylindrical slice is $V_{\text{slice}} = \pi r^2 dx = \pi \cdot \sin^2 x dx$.

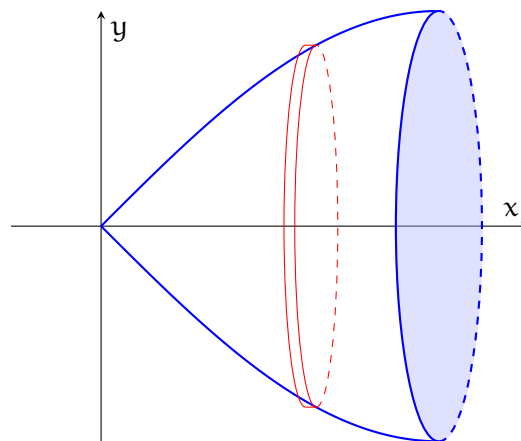


Figure 1.3: When rotated, the cross-section becomes a cylinder with radius $\sin x$ and width dx , which has a total volume of $\pi \sin^2 x dx$

We can find the total volume by integrating from 0 to $\pi/2$:

$$V = \pi \int_0^{\pi/2} \sin^2 x \, dx$$

Recall the half angle formula, $\sin^2 x = \frac{1}{2} (1 - \cos 2x)$. Substituting, we see that:

$$V = \frac{\pi}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$$

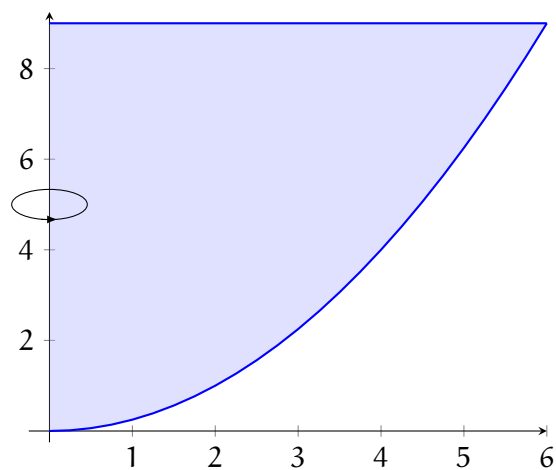
$$V = \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right) \Big|_{x=0}^{x=\pi/2}$$

$$V = \frac{\pi}{2} \left[\left(\pi/2 - \frac{1}{2} \sin \pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right]$$

$$V = \frac{\pi}{2} [\pi/2 - 0 - 0 + 0] = \frac{\pi^2}{4}$$

Exercise 2

Find the volume of a solid created by rotating the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$ about the y -axis. A graph is shown below.

*Working Space**Answer on Page ??*

Exercise 3

Let $f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$. Bird's eggs of various sizes can be modeled by rotating $f(x)$ about the x -axis, with different values of a , b , c , and d defining different sizes and shapes of eggs. For a domestic chicken, $a = -0.02$, $b = 0.03$, $c = 0.12$, and $d = 0.454$. For a mallard duck, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Use a calculator, such as a TI-89 or Wolfram Alpha, to determine which species lays a bigger egg.

Working Space

Answer on Page ??

1.2.1 Using donuts for solids of revolution

Sometimes there is space between the region we are rotating and the line we are rotating it about. Consider the region bounded between $y = 2x$ and $y = x^2$ (see figure ??):

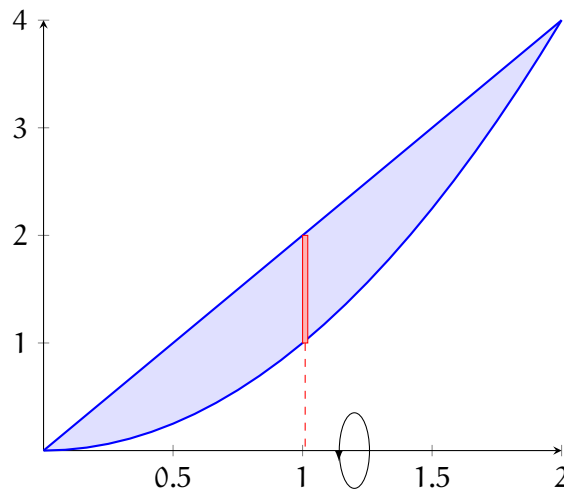


Figure 1.4: When rotated, the slices will become donuts with outer radius $2x$ and inner radius x^2

When rotated, the slices will take the form of donuts (or washers), the volume of which is $\pi (R^2 - r^2) dx$, where R is the outer radius and r is the inner radius. Therefore, in this case, the total volume of the rotated region is given by:

$$\begin{aligned}
 V &= \int_0^2 \pi \left[(2x)^2 - (x^2)^2 \right] dx \\
 V &= \pi \int_0^2 4x^2 - x^4 dx = \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{x=0}^{x=2} \\
 V &= \pi \left[\frac{4}{3}2^3 - \frac{1}{5}2^5 \right] = \pi \left[\frac{32}{3} - \frac{32}{5} \right] \\
 V &= \frac{64\pi}{15}
 \end{aligned}$$

Exercise 4

What is the volume of the region bounded by $y = x^2$ and $y = 2\sqrt{x}$ when rotated about the y -axis?

Working Space

Answer on Page ??

1.3 Volumes of Other Solids

You can also model a solid as a base defined by a function with cross-sections of specific shapes. Consider the function $y = x^2$ from $x = 0$ to $x = 2$ (see figure ??). Suppose the area between the curve, the y -axis, and the line $y = 4$ defines a base and each vertical cross-section is a square. So, the width of the each cross section is dx , the length is $4 - x^2$, and (because they are squares) the height in the z -plane is also $4 - x^2$. The volume of each cross-section is $V_{\text{slice}} = (4 - x^2)^2 dx$ and the total volume of the solid is:

$$V = \int_0^2 (4 - x^2)^2 dx$$

$$V = \int_0^2 (16 - 8x^2 + x^4) dx$$

$$V = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_{x=0}^{x=2}$$

$$V = 16(2) - \frac{8}{3}(2)^3 + \frac{1}{5}(2)^5 = \frac{256}{15} \approx 17.067$$

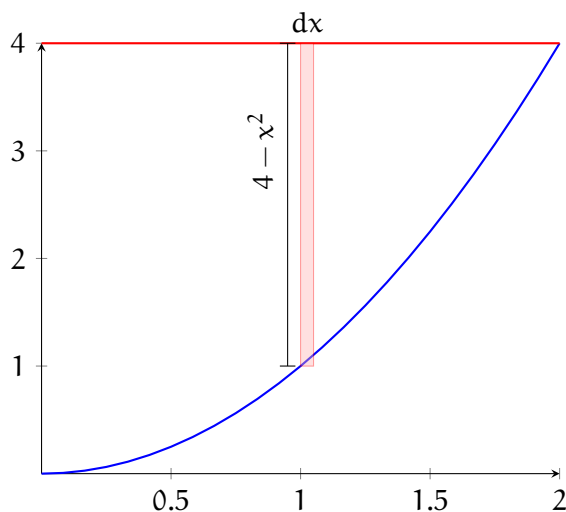


Figure 1.5: $y = x^2$ with a vertical cross-section

You can use a similar method for triangular, semi-circular, or any other shape cross-section. The trick is writing everything in terms of x (when your cross sections are vertical and have width dx) or y (when your cross section are horizontal and have length dy).

Exercise 5

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let R be the region in the first quadrant bounded above by the graph $y = \ln(3 - x)$, for $0 \leq x \leq 2$. R is the base of a solid for which each cross section perpendicular to the x -axis is square. What is the volume of the solid? Give your answer to 3 decimal places.

Working Space

Answer on Page ??

Exercise 6

Find the volume of a solid whose base is defined by the ellipse $9x^2 + 16y^2 = 25$ and is made up of isosceles-triangular cross-sections perpendicular to the x -axis (with the hypotenuse in the base of the solid).

Working Space

Answer on Page ??

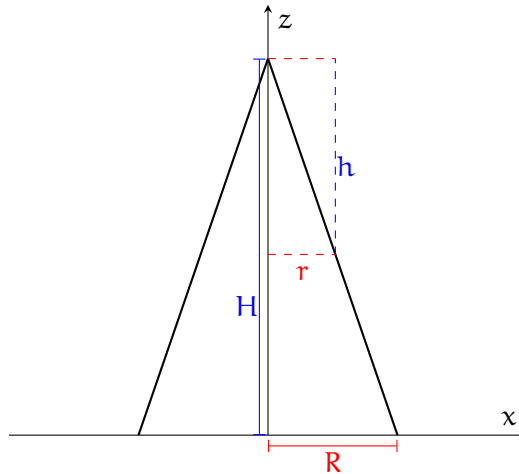
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Answers to Exercises

Answer to Exercise ?? (on page ??)

Imagine a side view of the cone (see figure below), an isosceles triangle with height H and base $2R$. If we take horizontal cross-sections, then each cross-section is a circle h from the top with a radius r . Because the triangles are similar (FIXME: better wording/explanation here), we also know that $\frac{H}{h} = \frac{R}{r}$. Therefore, we can define r in terms of h : $r = \frac{hR}{H}$ and the volume of each subsequent cross-section is $\pi r^2 dh = \pi \frac{h^2 R^2}{H^2} dh$. We start with $h = 0$ and end with $h = H$:

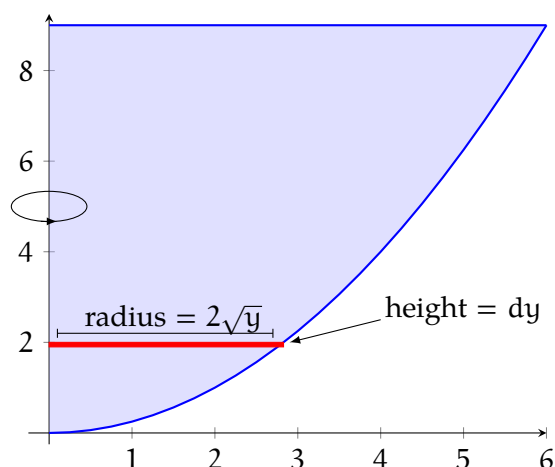
$$\begin{aligned} V_{\text{cone}} &= \int_0^H \pi \frac{h^2 R^2}{H^2} dh = \pi \frac{R^2}{H^2} \int_0^H h^2 dh \\ &= \pi \frac{R^2}{H^2} \left[\frac{1}{3} h^3 \right]_{h=0}^{h=H} = \pi \frac{R^2}{3H^2} [H^3 - 0^3] \\ &= \pi \frac{R^2}{3H^2} H^3 = \frac{\pi}{3} R^2 H \end{aligned}$$



Answer to Exercise ?? (on page ??)

If we are rotating about the y axis, we should make our slices horizontal, so their width is dy (see graph below). Then, the volume of each cylinder is given by $V = \pi r^2 dy$ and the total volume is given by:

$$\begin{aligned} V &= \int_0^9 \pi [2\sqrt{y}]^2 dy \\ V &= 4\pi \int_0^9 y dy = 2\pi y^2 \Big|_{y=0}^{y=9} \\ V &= 2\pi (9)^2 = 162\pi \end{aligned}$$



Answer to Exercise ?? (on page ??)

Since the graph is rotated around the x -axis, we will take vertical slices with width dx , and rotate them to make cylinders with radius $f(x)$ and height dx . The volume of each egg is given by:

$$\int_{-1}^1 \pi [f(x)]^2 dx$$

To determine our limits of integration, we note that $\sqrt{1-x^2} = 0$ (and therefore, $f(x) = 0$) when $x = \pm 1$.

For the chicken:

$$V_{\text{chickenegg}} = \pi \int_{-1}^1 \left[(-0.02x^3 + 0.03x^2 + 0.12x + 0.454) \sqrt{1-x^2} \right]^2 dx$$

For the mallard duck:

$$V_{\text{duckegg}} = \pi \int_{-1}^1 \left[\left(-0.06x^3 + 0.04x^2 + 0.1x + 0.54 \right) \sqrt{1-x^2} \right]^2 dx$$

Using a calculator, we find that $V_{\text{chickenegg}} \approx 0.897$ and $V_{\text{duckegg}} \approx 1.263$. Therefore, mallard ducks lay larger eggs than chickens do.

Answer to Exercise ?? (on page ??)

First, since we are revolving around the y -axis, we know our slices will have width dy . We will rewrite the functions as x in terms of y :

$$x = \sqrt{y}$$

$$x = \frac{y^2}{4}$$

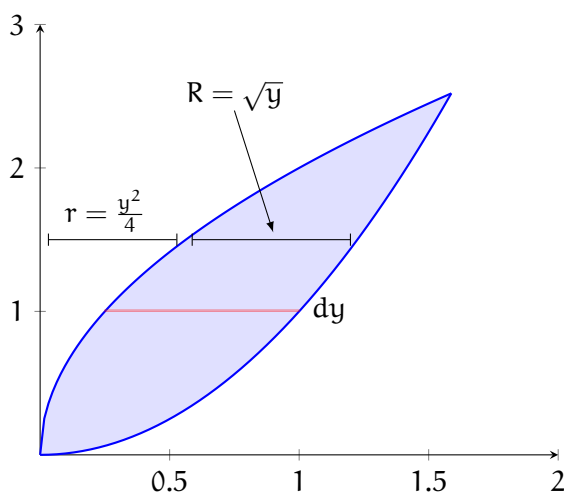
Setting them equal to each other to find the y -value at which they intercept:

$$\sqrt{y} = \frac{y^2}{4}$$

$$4 = \frac{y^2}{\sqrt{y}} = y^{3/2}$$

$$y = \sqrt[3]{4^2} = 2\sqrt[3]{2}$$

Examining a graph (shown below), we see that the outer radius is $x = \sqrt{y}$ and the inner radius is $x = \frac{y^2}{4}$.



So, the total volume of the solid of revolution is given by:

$$V = \pi \int_0^{2\sqrt[3]{2}} (\sqrt{y})^2 - \left(\frac{y^2}{4}\right)^2 dy$$

$$V = \pi \int_0^{2\sqrt[3]{2}} \left[y - \frac{y^4}{16} \right] dy$$

$$V = \pi \left[\frac{1}{2}y^2 - \frac{1}{80}y^5 \right]_{y=0}^{y=2\sqrt[3]{2}}$$

$$V = \pi \left[\frac{6}{5}2^{2/3} \right] \approx 5.9844$$

Answer to Exercise ?? (on page ??)

If each cross section is a square, then the volume of each cross section is given by $s^2 dx$, where s is the side length of the square. Since the side length is equal to the distance between the graph of y and the x -axis, we can see that $s = y = \ln(3 - x)$. And, therefore, the total volume of all the cross sections is given by $\int_0^2 [\ln(3 - x)]^2 dx$. Using a calculator, this integral evaluates to ≈ 1.029 .

Answer to Exercise ?? (on page ??)

Since the cross-sections are perpendicular to the x -axis, they will have width dx and we will integrate across the domain of the ellipse. Setting $y = 0$ to find the domain of the ellipse:

$$9x^2 = 25 \rightarrow x^2 = \frac{25}{9} \rightarrow x = \pm \frac{5}{3}$$

A right isosceles triangle with hypotenuse h has area $\frac{1}{4}h^2$. In this case, each triangle's hypotenuse is given by the distance between the top and bottom of the ellipse. The top of the ellipse is defined by $y = \frac{1}{4}\sqrt{25 - 9x^2}$ and the bottom by $y = -\frac{1}{4}\sqrt{25 - 9x^2}$. Therefore, the length of each hypotenuse is $\frac{1}{2}\sqrt{25 - 9x^2}$.

Then, each cross-section has a total volume of $\frac{1}{4}h^2 dx = \frac{1}{4} \left(\frac{1}{2}\sqrt{25 - 9x^2} \right)^2 dx$ and the volume of the solid is:

$$\begin{aligned} V_{\text{solid}} &= \int_{-5/3}^{5/3} \frac{1}{4} \left(\frac{1}{2}\sqrt{25 - 9x^2} \right)^2 dx \\ &= \frac{1}{4} \int_{-5/3}^{5/3} \frac{1}{4} (25 - 9x^2) dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{6} \int_{-5/3}^{5/3} (25 - 9x^2) \, dx = \frac{1}{16} \left[25x - 3x^3 \right]_{x=-5/3}^{x=5/3} \\ &= \frac{1}{16} \left[\left(25 \left(\frac{5}{3} \right) - 25 \left(\frac{-5}{3} \right) \right) - \left(3 \left(\frac{5}{3} \right)^3 - 3 \left(\frac{-5}{3} \right)^3 \right) \right] \\ &= \frac{1}{16} \left[\frac{250}{3} - \left(\frac{375}{27} + \frac{375}{27} \right) \right] = \frac{1}{16} \left[\frac{250}{3} - \frac{250}{9} \right] = \frac{1}{16} \left[\frac{750}{9} - \frac{250}{9} \right] = \frac{1}{16} \left[\frac{500}{9} \right] = \frac{125}{36} \end{aligned}$$



INDEX

solids of revolution, [3](#)

Volume Single Integral, [1](#)
 volume of a sphere, [1](#)

washer method, [7](#)