Chapter 1

Functions and Their Graphs

Functions are a major part of science, engineering, and math. You can think of a function as a machine: you put something into the machine, it processes it, and out comes something else: a product. Just as we often use the variable x to stand in for a number, we often use the variable f to stand in for a function.

For example, we might ask, "Let the function f be defined like this:

$$f(x) = -5x^2 + 12x + 2$$

What is the value of $f(3)$?"

You would run the number 3 through "the machine": $-5(3^2)+12(3)+2 = -7$. The answer would be "$f(3)$ is 7".

However, some functions are not defined for every possible input. For example:

$$f(x) = \frac{1}{x}$$

This is defined for any x except 0, because you can't divide 1 by 0. The set of values that a function can process is called its *domain*.

Exercise 1 Domain of a function

Let the function f be given by $f(x) = \sqrt{x-3}$. What is its domain?

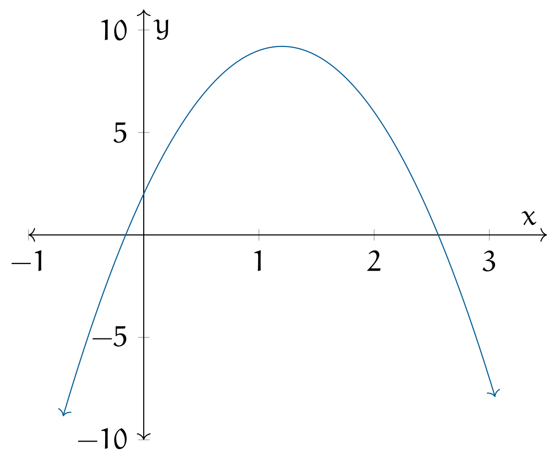
Working Space

Answer on Page 9

1.1 Graphs of Functions

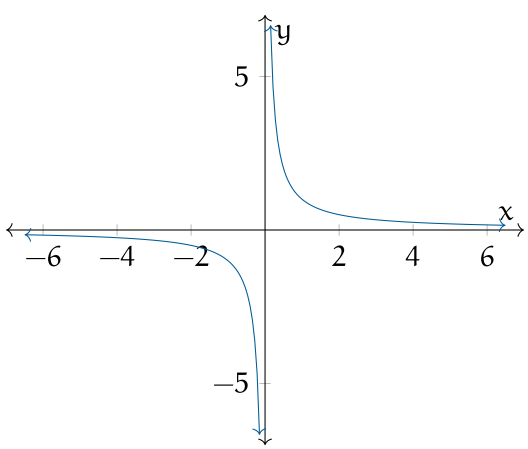
If you have a function, f, its graph is the set of pairs (x, y) such that $y = f(x)$. We usually draw a picture of this set, called a *graph*. The graph not only includes the picture, but also the values of x and y used to create it.

Here is the graph of the function $f(x) = -5x^2 + 12x + 2$:



(Note that this is just part of the graph; it goes infinitely in both directions. Remember your vectors!)

Here is the graph of the function $f(x) = \frac{1}{x}$:



Exercise 2 Draw a graph

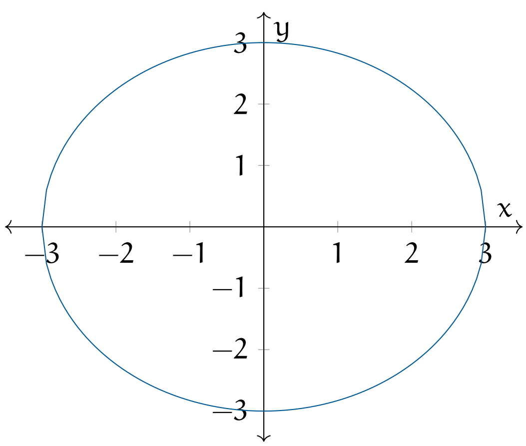
Working Space

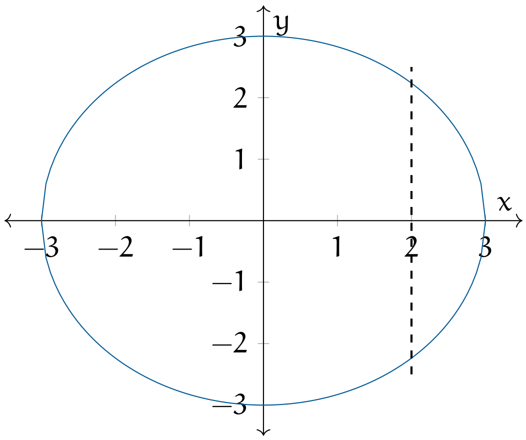
Let the function f be given by $f(x) = -3x + 3$. Sketch its graph.

Answer on Page 9

1.2 Can this be expressed as a function?

Note that not all sets can be expressed as graphs of functions. For example, here is the set of points (x, y) such that $x^2 + y^2 = 9$:



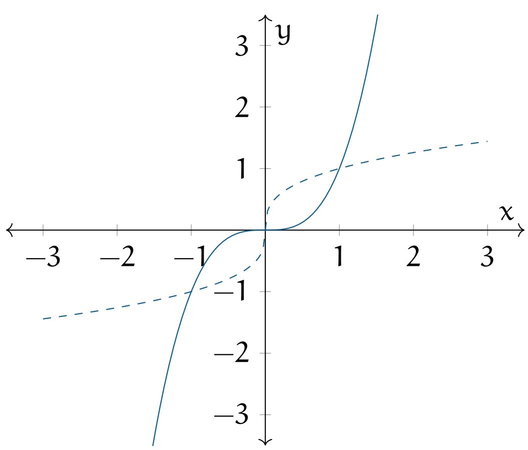
This cannot be the graph of a function, because what would $f(0)$ be? 3 or -3? This set fails what we call "the vertical line test": If any vertical line contains more than one point from the set, it isn't the graph of a function. For example, the vertical line $x = 2$ would cross the graph twice: 

1.3 Inverses

Some functions have inverse functions. If a function f is a machine that turns number x into y, the inverse (usually denoted f^{-1}) is the machine that turns y back into x.

For example, let $f(x) = 5x + 1$. Its inverse is $f^{-1}(x) = (x - 1)/5$. (Spot check it: $f(3) = 16$ and $f^{-1}(16) = 3$)

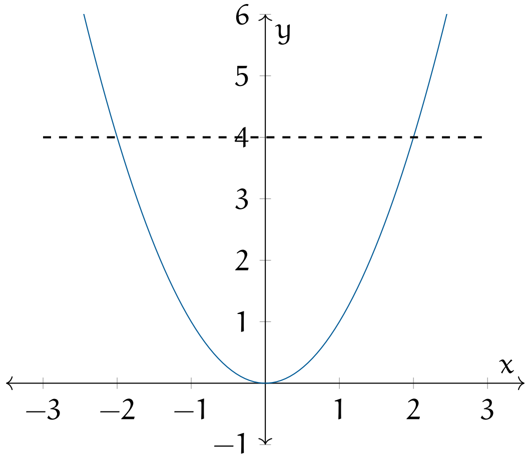
Does the function $f(x) = x^3$ have an inverse? Yes, $f^{-1}(x) = \sqrt[3]{x}$. Let's plot the function (solid line) and its inverse (dashed):



The inverse is the same as the function, just with its axes swapped. This tells us how to solve for an inverse: We swap x and y and solve for y.

For example, if you are given the function $f(x) = 5x + 1$, its graph is all (x, y) such that $y = 5x + 1$. The graph of its inverse is all (x, y), such that $x = 5y + 1$. This menas you solve for y: $y = (x - 1)/5$.

Not every function has an inverse. For example, $f(x) = x^2$. Note that $f(2) = f(-2) = 4$. What would $f^{-1}(4)$ be? 2 or -2? This implies the "horizontal line test": If any horizontal line contains more than one point of a function's graph, that function has no inverse. If a function passes the horizontal line test, it is called "one-to-one", meaning there is exactly one x that gives each y.



In some problems, you need an inverse, but you don't need the whole domain, so you trim the domain to a set you can define an inverse on. This allow you to make claims such as "If we restrict the domain to the nonnegative numbers, the function $f(x) = x^2 - 5$ has an inverse: $f^{-1}(x) = \sqrt{x+5}$.

This raises the question: What is the domain of the inverse function f^{-1}?

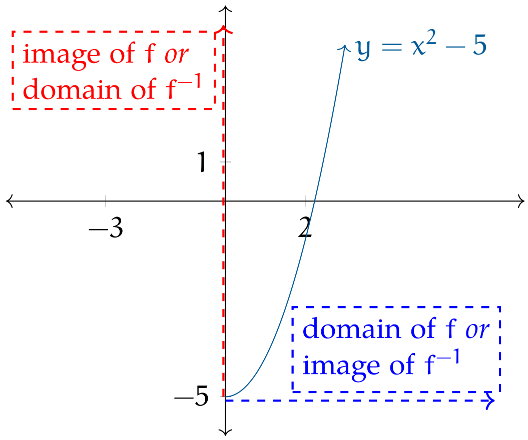
If we let X be the domain of f, we can run every member of X through "the machine" and gather them in a set on the other side. This set would be the *image* of the f "machine". (This is the *range* of f.)

What is the image of $f(x) = x^2 - 5$? It is the set of all real numbers greater than or equal to -5. We write this:

$$\{x \in \mathbb{R} | x \geq -5\}$$

Now we can say: **The image of the function is the domain of the inverse function.**

In our example, we can use any number greater than or equal to -5 as input into the inverse function.



Exercise 3 Find the inverse

Let $f(x) = (x-3)^2 + 2$. Sketch the graph.

Using all the real numbers as a domain, does this function have an inverse?

How would you restrict the domain to make the function invertible?

What is the inverse of that restricted function?

What is the domain of the inverse?

Working Space

Answer on Page 9

Exercise 4

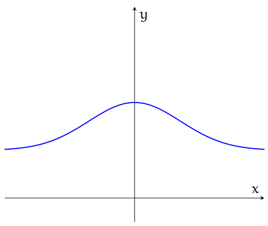
A function is given by a table of values, a graph, or a written description. Determine whether it is one-to-one.

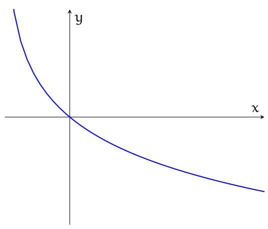
1.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

2.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

3. 

4. 

5. $f(t)$ is the height of a football t seconds after kickoff

6. $v(t)$ is the velocity of a dropped object

Working Space

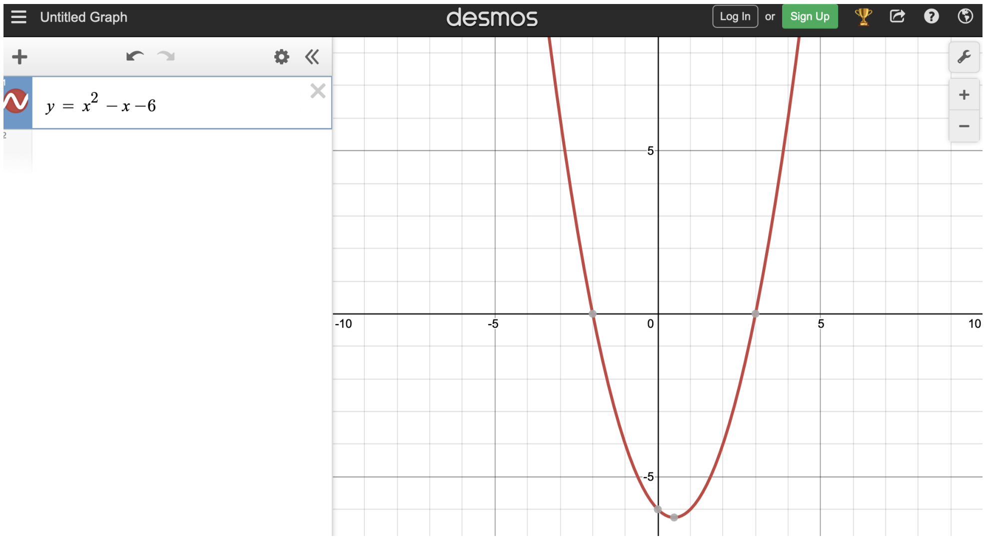
Answer on Page 10

1.4 Graphing Calculators

One really easy way to understand your function better is to use a graphing calculator. Desmos is a great, free online graphing calculator.

In a web browser, go to Desmos: `https://www.desmos.com/calculator`

In the field on the left, enter the function $y = x^2 - x - 6$. (For the exponent, just prefix it with a caret symbol: “x2”.)



Appendix A

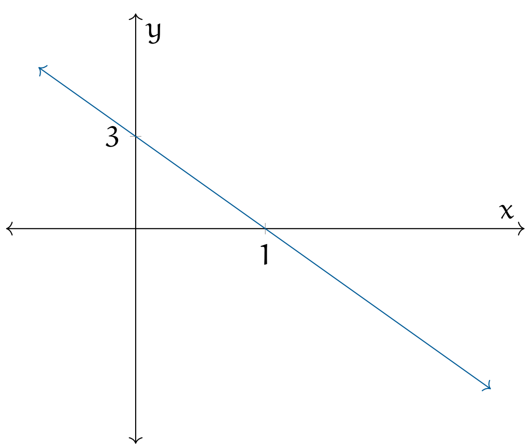
Answers to Exercises

Answer to Exercise 1 (on page 1)

You can only take the square root of nonnegative numbers, so the function is only defined when $x - 3 \geq 0$. Thus, the domain is all real numbers greater than or equal to 3.

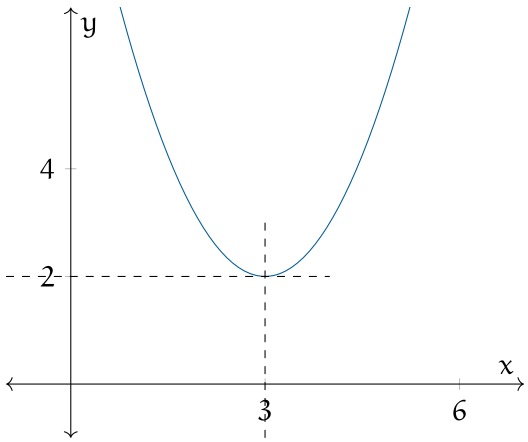
Answer to Exercise 2 (on page 3)

The graph of this function is a line, its slope is -3, and it intersects the y axis at $(0, 3)$.



Answer to Exercise 3 (on page 6)

This graph is the graph of $y = x^2$ that has been moved to the right by three units and up two units:



To prevent any horizontal line from containing more than one point of the graph, you would need to use the left or the right side — either $\{x \in \mathbb{R} | x \leq 3\}$ or $\{x\mathbb{R} | x \geq 3\}$. Most people will choose the right side; the rest of the solution will assume that you did too.

To find the inverse we swap x and y: $x = (y-3)^2 + 2$

Next, we solve for y to get the inverse: $y = \sqrt{x-2} + 3$

You can take the square root of nonnegative numbers. So the function $f^{-1}(x) = \sqrt{x-2}+3$ is defined whenever x is greater than or equal to 2.

Answer to Exercise 4 (on page 7)

1. This function is not one-to-one. From $x = 3$ to $x = 4$, the function increases from 3.6 to 5.3, which means it must pass through $f(x_1) = 4.0$. From $x = 4$ to $x = 5$, the function decreases from 5.3 to 2.8, which means it must pass through $f(x_2) = 4.0$ again.
2. This function is not one-to-one by a similar argument in the above solution
3. This function is not one-to-one, because it fails the horizontal line test
4. This function is one-to-one, because it passes the horizontal line test
5. $f(t)$ would not be one-to-one because the football must pass through each height (except the peak height) both on the way up and on the way back down
6. $v(t)$ would be one-to-one because a falling object only speeds up. Therefore, every time has a unique speed.