Chapter 1

Differential Equations

Differential equations are equations involving an unknown function and its derivatives. They play a crucial role in mathematics, physics, engineering, economics, and other disciplines due to their ability to describe change over time or in response to changing conditions.

1.1 Ordinary Differential Equations

An ordinary differential equation (ODE) involves a function of a single independent variable and its derivatives. The order of an ODE is determined by the order of the highest derivative present in the equation. An example of a first-order ODE is:

$$\frac{dy}{dx} + y = x \tag{1.1}$$

Here, y is the function of the independent variable x, and $\frac{dy}{dx}$ represents its first derivative.

A real-world example of the application of differential equations is an oscillating spring (or any harmonic motion). When a spring is stretched, the restoring force (the force pulling or pushing it back to its neutral position) is proportional to the distance by which the spring has been stretched (see Figure 1.1). Mathematically, we say that

$$\text{restoring force} = -kx$$

where k is the positive spring constant (the stiffer a spring, the greater k).

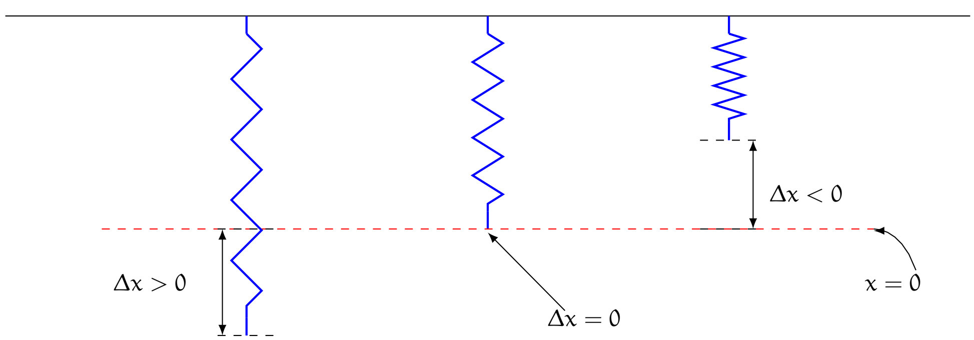


Figure 1.1: A spring can have a positive or negative displacement

Recall that Newton's Second Law tells us that force is equal to mass times acceleration, and that acceleration is the second derivative of position. We can then write the differential equation:

$$m\frac{d^2x}{dt^2} = -kx$$

This is called a **second-order differential equation**, because it involves second-order derivatives. The order of a differential equation is the same as the highest order of derivative in the equation. We can further rewrite the equation to isolate the second derivative:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

In everyday language, this is saying that the second derivative is proportional to the original function, just negative. There are two trigonometric functions that have this property, take a second to see if you remember and write down your guess.

The sine and cosine functions both have the property $\frac{d^2x}{dt^2} \propto -x(t)$ (recall that $\propto$ means "proportional to").

Example: Assuming $x(t)$ is a sine function, solve the second-order differential equation $\frac{d^2x}{dt^2} = \frac{-k}{m}x$.

Solution: Let $x(t) = \sin Ct$. Then $\frac{dx}{dt} = C\cos Ct$ and $\frac{d^2x}{dt^2} = -C^2 \sin t$. This implies that $C^2 = \frac{k}{m}$ and $C = \pm\sqrt{\frac{k}{m}}$. So, a solution to the differential equation $\frac{d^2x}{dt^2} = \frac{-k}{m}x$ is $x(t) = \sin\sqrt{\frac{k}{m}}t$.

1.1.1 Population Growth

Another real-world application of differential equations is modeling population growth. Under ideal conditions (unlimited food, no predators, disease-free, etc.), the population of a species grows at a rate proportional to the current population size. We can identify two variables:

$$t = \text{ time (the independent variable)}$$

$$P = \text{ the number pf individuals in the population (the dependent variable)}$$

So, what is the rate of growth? Recall that a rate is change over time. In that case, the rate of growth is given by $\frac{dP}{dt}$. If the rate of growth is proportional to the population, then we can write a first-order differential equation:

$$\frac{dP}{dt} = kP$$

where k is a proportionality constant. This is called **natural growth** or **logarithmic growth**. To find a solution, we must answer the question: What function's derivative is a constant multiple of itself? Recall that we have seen that the derivative of the exponential function e^{kt} is ke^{kt}. Setting $P(t) = Ce^{kt}$ (where C is some constant), we see that the derivative is $\frac{dP}{dt} = kCe^{kt} = kP(t)$ (see figure 1.2). You can determine C from initial conditions.

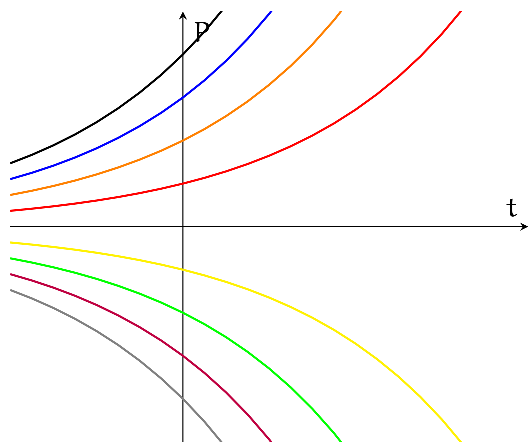


Figure 1.2: Several solutions to $\frac{dP}{dt} = kP$

Example: Suppose a population of bacteria has an initial population of 100 bacteria. If the bacteria's growth rate is given by $\frac{dP}{dt} = 2P$ (where t is in hours), how many bacteria are present after 4 hours?

Solution: We have seen that the solution to $\frac{dP}{dt} = 2P$ is $P(t) = Ce^{2t}$. We can then use the given initial condition to find C:

$$P(0) = 100 = Ce^{2\cdot 0} = C \cdot 1 = C$$

Which means that the complete solution is:

$$P(t) = 100e^{2t}$$

To answer the question, we need to find $P(4)$:

$$P(4) = 100e^{2\cdot 4} = 100e^{8} \approx 298096$$

As stated above, this model works well for populations under specific, ideal conditions. However, there are very few environments in which these conditions are met. Real animals suffer from disease, are hunted by predators, and have limited food supplies. Most environments have a maximum number of animals they can support, which ecologists call a **carrying capacity**. Let us call the carrying capacity of an environment M. So, the population growth can be modeled by the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

This is called a **logistic differential growth model**. Notice that if P is small, then $\frac{dP}{dt} \approx kP$. This makes sense: If the population is very small compared to the carrying capacity, the conditions are nearly ideal, and so growth should be nearly ideal too. On the other hand, if the population ever goes *above* the carrying capacity, the $\frac{dP}{dt} < 0$ and the population will decrease back below the carrying capacity (see figure 1.3). Notice that if the initial population is $P_0 = M$, then $\frac{dP}{dt} = kP(1-1) = 0$ and the population is stable at $P(t) = M$. We call this an **equilibrium solution**. Can you logically find the other equilibrium solution?

If there are no animals to begin with, then there are none to reproduce, and $P(t) = 0$. This is the other equilibrium solution. Notice that when the population is in equilibrium, then the rate of change is zero. Mathematically, to find equilibrium solutions, we can set $\frac{dP}{dt} = 0$ and solve for P.

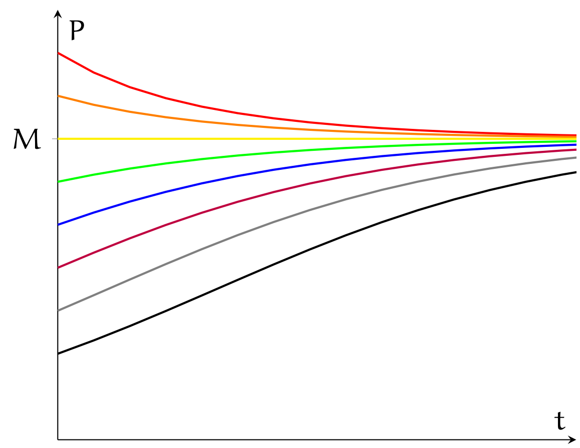


Figure 1.3: Several solutions to $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

Exercise 1

A population is modeled by the differential equation $\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$.

1. What is the carrying capacity of the environment?
2. For what values of P is the population increasing?
3. For what values of P is the population decreasing?
4. What are the equilibrium solutions?

Working Space

Answer on Page 11

Exercise 2

[This problem was originally presented as a calculator-allowed, free response question on the 2012 AP Calculus BC exam.] Let k be a positive constant. Which of the following is a logistic differential equation?
(a) $\frac{dy}{dt} = kt$
(b) $\frac{dy}{dt} = ky$
(c) $\frac{dy}{dt} = kt(1-t)$
(d) $\frac{dy}{dt} = ky(1-t)$
(e) $\frac{dy}{dt} = ky(1-y)$

Working Space

Answer on Page 11

1.1.2 Separable Differential Equations

Sometimes, differential equations can be explicitly solved. A first-order differential equation is separable if $\frac{dy}{dx}$ can be written as a function of x times a function of y. Symbolically, a differential equation is separable if it takes the form

$$\frac{dy}{dx} = g(x)f(y)$$

The equations may be solvable by separating the x from the y and integrating each side. For our generic form, we can separate the variables thusly if $f(y) \neq 0$:

$$\frac{dy}{dx}\frac{1}{f(y)} = g(x)$$

$$\frac{1}{f(y)}dy = g(x)dx$$

Integrating both sides:

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

.

Let's look at the example $\frac{dy}{dx} = \frac{x^2}{y}$. We can separate the variables by multiplying both sides by $y\,dx$:

$$y\,dy = x^2 dx$$

Integrating both sides:

$$\int y\,dy = \int x^2\,dx$$

$$\frac{1}{2}y^2 + C_1 = \frac{1}{3}x^3 + C_2$$

We can combine the constants by defining $C = C_2 - C_1$. Making this substitution and solving for y, we find:

$$y^2 = \frac{2}{3}x^3 + 2C$$

$$y = \sqrt{\frac{2}{3}x^3 + 2C}$$

Noting that $2C$ is also a constant (which we will call K for convenience), we find the general solution is

$$y = \sqrt{\frac{2}{3}x^3 + K}$$

A graph showing the solution for several values of K is in figure 1.4.

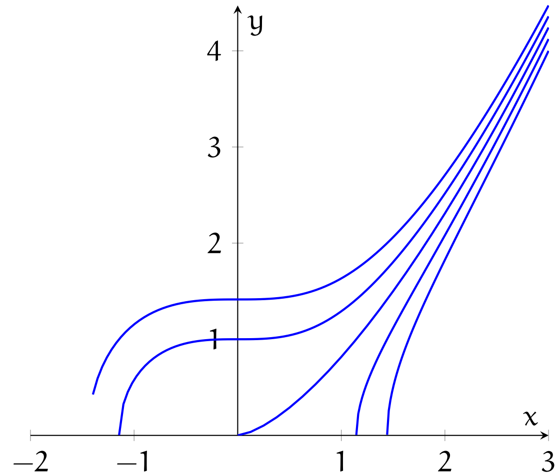


Figure 1.4: Several possible solutions to $\frac{dy}{dx} = \frac{x^2}{y}$

It is not always possible to solve for y explicitly in terms of x. The practice problem below is an example of this.

Exercise 3

Solve the differential equation $\frac{dy}{dx} = \frac{3x^2}{2y+\sin y}$.

Working Space

Answer on Page 11

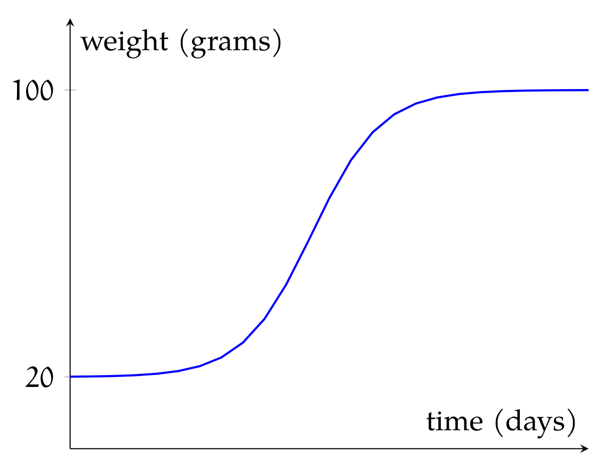
Exercise 4

[This problem was originally presented as a calculator-allowed, free response question on the 2012 AP Calculus BC exam.] The rate at which a baby bird gains mass is proportional to the difference between its adult mass and its current mass. At time $t = 0$, when the bird is first weighed, its mass is 20 grams. If $B(t)$ is the mass of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

Let $y = B(t)$ be the solution to the differential equation with initial condition $B(0) = 20$.

1. Is the bird gaining mass faster when it masses 40 grams or when it masses 70 grams? Explain your reasoning.
2. Find $\frac{d^2B}{dt^2}$ in terms of B. Use it to explain why the graph of B cannot resemble the graph shown below.
3. Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



Working Space

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Exercise 5

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] If $P(t)$ is the size of a population at time t, which of the following differential equations describes *linear* growth in the size of the population?
(a) $\frac{dP}{dt} = 200$
(b) $\frac{dP}{dt} = 200t$
(c) $\frac{dP}{dt} = 100t^2$
(d) $\frac{dP}{dt} = 200P$
(e) $\frac{dP}{dt} = 100P^2$

Working Space

Answer on Page 12

1.2 Partial Differential Equations

Partial differential equations (PDEs), on the other hand, involve a function of multiple independent variables and their partial derivatives. An example of a PDE is the heat equation, a second-order PDE:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1.2}$$

In this equation, $u = u(x, t)$ is a function of the two independent variables x and t, $\frac{\partial u}{\partial t}$ is the first partial derivative of u with respect to t, and $\frac{\partial^2 u}{\partial x^2}$ is the second partial derivative of u with respect to x.

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Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

1. 4200
2. Logically, we can say that the population will increase if it is below the carrying capacity (that is, $P < 4200$), but we can also prove it mathematically: $\frac{dP}{dt} < 0 \to 1.2P\left(1 - \frac{P}{4200}\right) < 0 \to P\left(1 - \frac{P}{4200}\right) < 0$. Since we are talking about population, we can assume that $P > 0$ and continue: $1 - \frac{P}{4200} < 0 \to 1 < \frac{P}{4200} \to 4200 < P$, which is the result we expected.
3. Similarly, we know the population should be decreasing when P is greater than the carrying capacity of 4200.
4. The equilibrium solutions can be found by setting $\frac{dP}{dt} = 0$ and solving. The solutions are $P(t) = 0$ and $P(t) = 4200$.

Answer to Exercise 2 (on page 5)

Recall that logistic differential equations are of the form $\frac{dy}{dt} = ky(1 - \frac{y}{m})$ where y is a function and t is the independent variable. (e) is the only logistic differential equation, with $m = 1$.

Answer to Exercise 3 (on page 7)

$$\frac{dy}{dx}dx = \frac{3x^2}{2y + \sin y}dx$$
$$(2y + \sin y)(dy) = \frac{3x^2}{2y + \sin y}(2y + \sin y)(dx)$$
$$(2y + \sin y)dy = (3x^2)dx$$
$$\int 2y\, dy + \int \sin y\, dy = \int 3x^2\, dx$$
$$y^2 - \cos y = x^3 + C$$

Answer to Exercise 4 (on page 8)

1. Since $\frac{dB}{dt}$ depends only on B, we can use the given masses to find the rate of growth for each mass. $\frac{dB}{dt}(40) = \frac{1}{5}(100 - 40) = \frac{1}{5}(60) = 12$ and $\frac{dB}{dt}(70) = \frac{1}{5}(100 - 70) = \frac{1}{5}(30) = 6$. Since $\frac{dB}{dt}$ is greater when $B = 40$, the baby bird is gaining mass faster when it has a mass of 40 grams.

2. $\frac{d^2B}{dt^2} = \frac{d}{dt}\left(\frac{dB}{dt}\right) = \frac{d}{dt}\left[\frac{1}{5}(100 - B)\right] = \frac{1}{5}\left(-\frac{dB}{dt}\right) = \frac{-1}{5}\left[\frac{1}{5}(100 - B)\right] = -\frac{1}{25}(100 - B)$. For $20 < B < 100$, $\frac{d^2B}{dt^2} < 0$ and the graph of B should be concave down. The graph shown has a concave up portion, so it cannot represent $B(t)$.

3. $\frac{dB}{dt} = \frac{1}{5}(100 - B) \to \frac{dB}{100-B} = \frac{1}{5}dt \to \int (100 - B)\, dB = \int \frac{1}{5}\, dt \to -\ln 100 - B = \frac{t}{5} + C \to e^{\frac{-t}{5} + C} = 100 - B \to ke^{\frac{-t}{5}} = 100 - B \to B(t) = 100 - ke^{\frac{-t}{5}}$. Setting $B(0) = 20$ to find k: $20 = 100 - ke^0 \to 20 = 100 - k \to k = 80$. So, the particular solution is $B(t) = 100 - 80e^{\frac{-t}{5}}$

Answer to Exercise 5 (on page 10)

(A). (a), (b), and (c) are all separable equations. But only the solution to A is linear ($P(t) = 200t + C$). (d) is logarithmic, or natural growth and (e) is also not linear.

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