Chapter 1

Partial Fractions

How can you add fractions with different denominators, like $\frac{1}{x}+\frac{2}{x+3}$? You would need to make the denominators the same; after that, you could just add the numerators. You achieve this by multiplying the numerator and denominator of each fraction by the denominator of the other fraction:

$$\frac{1}{x}+\frac{2}{x+3}=\frac{1}{x}\left(\frac{x+3}{x+3}\right)+\frac{2}{x+3}\left(\frac{x}{x}\right)$$

Recall that when the numerator and denominator of a fraction are the same, the fraction is equal to one. So, we are not changing the *value* of each fraction, since we are just multiplying by one. Continuing, we can perform the multiplication and see that:

$$\begin{aligned}\frac{1}{x}+\frac{2}{x+3}&=\frac{x+3}{x(x+3)}+\frac{2x}{x(x+3)}\\ &=\frac{(x+3)+2x}{x(x+3)}=\frac{3x+3}{x^2+3x}=\frac{3(x+1)}{x^2+3x}\end{aligned}$$

The inverse of this process is called **partial fraction decomposition** (or partial fraction expansion). This method has applications in many fields, but we will find it most useful as a tool to evaluate integrals in a later chapter.

Let $g(x)$ be a rational function such that

$$g(x)=\frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials. If $g(x)$ is proper (that is, the degree of P is less than the degree of $Q(x)$ then we can express $g(x)$ as the sum of simpler rational fractions. If $g(x)$ is improper (that is, the degree of P is greater than or equal to the degree of Q), then we must first perform long division to obtain a remainder, $R(x)$, where the degree of R is less than the degree of Q:

$$g(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$

1.0.1 Improper fractions

What is $\int \frac{x^3+x}{x-1}\,dx$. Using long division, we see that:

$$\frac{x^3+x}{x-1}=x^2+x+2+\frac{2}{x-1}$$

(see figure 1.1 for an explanation). Then we can also say that:

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

$$\begin{array}{r|l}
 & \quad x^2+x+2 \\ \cline{2-2}
x-1 & x^3+0x^2+x \\
 & -(x^3-x^2) \\ \cline{2-2}
 & \qquad\quad x^2+x \\
 & \qquad -(x^2-x) \\ \cline{2-2}
 & \qquad\qquad 2x \\
 & \qquad\quad -(2x-2) \\ \cline{2-2}
 & \qquad\qquad\quad 2
\end{array}$$

Figure 1.1: Evaluating $(x^3+x) \div (x-1)$ with the long division method

When you start with an improper fraction, use long division to reduce it to a term plus a proper fraction, then use the methods outlined below to further manipulate the proper fraction.

Exercise 1

Use long division to reduce the following improper rational functions to a term plus a proper rational fraction.

1. $\frac{x^4+x^3+2x^2+2x-3}{x^2-3x+2}$
2. $\frac{2x^3+5}{x^3-3x^2+2x-4}$
3. $\frac{3x^4-2x^3-x^2+1}{x^3-3x}$

Working Space

Answer on Page 7

1.0.2 Proper fractions

When the order of the numerator is less than or equal to the denominator, there are three further possibilities.

No repeated linear factors

In the first case, the denominator, $Q(x)$ is composed of distinct linear factors. In this case, we can say that $Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$, where no factor is repeated (including constant multiples). Then, there exists $A, B, C, \cdots$, such that:

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x + b_1} + \frac{B}{a_2x + b_2} + \cdots$$

Let's see an example of this by decomposing $\frac{4x^2-7x-12}{x(x+2)(x-3)}$. We start by defining A, B, and C, such that:

$$\frac{4x^2 - 7x - 12}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$

Multiplying both sides by $x(x+2)(x-3)$ we get:

$$4x^2 - 7x - 12 = A(x+2)(x-3) + B(x)(x-3) + C(x)(x+2)$$

We have 3 unknowns and only one equation! Don't worry — remember this equation is true for all x, so we can choose a convenient value of x to isolate each unknown in turn. Starting, let $x = 0$. Then:

$$4(0)^2 - 7(0) - 12 = A(0+2)(0-3) + B(0)(x-3) + C(0)(x+2)$$
$$-12 = A(2)(-3) + 0 + 0$$

Notice that the B and C disappear, and we can solve for A:

$$A = \frac{-12}{-6} = 2$$

We can solve for B by setting $x = -2$ and for C by setting $x = 3$ (notice, we've used all three zeroes of the denominator polynomial):

$$4(-2)^2 - 7(-2) - 12 = A(-2+2)(-2-3) + B(-2)(-2-3) + C(-2)(-2+2)$$
$$4(4) + 14 - 12 = 0 + B(-2)(-5) + 0$$
$$16 + 2 = 10B$$
$$B = \frac{9}{5}$$

and

$$4(3)^2 - 7(3) - 12 = A(3+2)(3-3) + B(3)(3-3) + C(3)(3+2)$$
$$4(9) - 21 - 12 = 0 + 0 + C(3)(5)$$
$$36 - 33 = 15C$$
$$C = \frac{1}{5}$$

We can then decompose our original fraction:

$$\frac{4x^2 - 7x - 12}{x(x+2)(x-3)} = \frac{2}{x} + \frac{9}{5(x+2)} + \frac{1}{5(x-3)}$$

You can check your answer by cross-multiplying and adding. You should get the same rational function back.

Repeated linear factors

The second case is if $Q(x)$ has repeated factors (such as $x^2+8x+16=(x+4)^2$). Suppose the first linear factor, (a_1x+b_1) is repeated r times (that is, $Q(x)$ contains the factor $(a_1x+b_1)^r$). Then, instead of $\frac{A}{a_1x+b_1}$, we should write:

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}$$

Let's look at a concrete example to see how this works:

Example: Decompose $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Solution: We start by defining:

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}$$

Multiplying both sides by $(x+1)^2(x+2)$:

$$x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2$$

Since there are only 2 roots to $(x+1)^2(x+2)$, we will use another method called "equating the coefficients" to find A, B, and C. We start by expanding the right side of the equation:

$$x^2+x+1=A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)$$

Distributing and combining, we find that:

$$x^2+x+1=Ax^2+3Ax+2A+Bx+2B+Cx^2+2Cx+C$$

$$x^2+x+1=(A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

For this equation to be true, we know that:

$$A+C=1$$

$$3A+B+2C=1$$

$$2A+2B+C=1$$

(That is, the coefficient for x^2 on the left, 1, must be equal to the coefficient for x^2 on the right, $(A+C)$, and so on.) We now have a system of 3 equations and 3 unknowns. When you solve for each, you should find that:

$$A=-2$$

$$B=1$$

$$C=3$$

Therefore,

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}$$

Irreducible quadratic factors

Sometimes, we cannot express a polynomial as the product of two linear statements (that is, terms in the form $ax + b$). Take $x^2 + 1$, which cannot be expressed as the product of real, linear terms. What do you do if something like $x^2 + 1$ is in the denominator? In this case, when we write an expression for $\frac{P(x)}{Q(x)}$, we include a term in the form:

$$\frac{Ax + B}{ax^2 + bx + c}$$

For example, we can write:

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}$$

Example: Decompose $\frac{2x^2-x+4}{x^3+4x}$

Solution: We begin by factoring the denominator:

$$x^3 + 4x = x(x^2 + 4)$$

Which cannot be factored further. Therefore, we define:

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

Which implies that:

$$2 = A + B$$

$$C = -1$$

$$4A = 4$$

Therefore, $A = 1$, $B = 1$, and $C = -1$ and we can say that:

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

Repeated irreducible quadratic factors

Lastly, the denominator might contain repeated irreducible quadratic factors. Similar to repeated linear factors, when setting up your partial fractions, instead of only writing

$$\frac{A}{ax^2 + bx + c}$$

For a quadratic factor that is repeated r times, your equation should include:

$$\frac{A_1}{ax^2 + bx + c} + \frac{A_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r}{(ax^2 + bx + c)^r}$$

Exercise 2

Decompose the following proper fractions

1. $\frac{x-4}{x^2+5x-6}$
2. $\frac{x^2+x+1}{(x^2+1)^2}$
3. $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Working Space

Answer on Page 7

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

1. $x^2 + 4x + 12\frac{30x-27}{x^2-3x+2}$
2. $2 + \frac{6x^2-4x+13}{x^3-3x^2+2x-4}$
3. $3x - 2 + \frac{8x^2-6x+1}{x^3-3x}$

Answer to Exercise 2 (on page 6)

1. $\frac{10}{7(x+6)} + \frac{-3}{7(x-1)}$
2. $\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}$
3. $\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$

Index