Chapter 1

Convergence Tests for Series

1.1 Test for Divergence

Recall from the previous chapter that if the terms of a series do not approach zero as n approaches infinity, then the series is divergent. This is the Test for Divergence, and there are two possible outcomes. For a series $\sum_{n=1}^{\infty} a_n$:

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then the series diverges}$$

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then the test is inconclusive}$$

It is important to remember that the Test for Divergence cannot tell us conclusively that a series converges. Rather, it only identifies series that are divergent.

Example: Apply the Test for Divergence to the series $\sum_{n=1}^{\infty} \sqrt{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution: $\lim_{n\to\infty} \sqrt{n} = \infty \neq 0$. Therefore, the series $\sum_{n=1}^{\infty} \sqrt{n}$ is divergent.

$\lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ may be divergent or convergent. This is the harmonic series, which we proved to be divergent in the previous chapter. This is a good example that demonstrates that just because $\lim_{n\to\infty} a_n = 0$ does not mean the series is convergent.

1.2 The Integral Test

We were able to determine the exact value of some infinite series because it was possible to write the n^{th} partial sum, s_n, in terms of n. For example, we determined that the n^{th} partial sum of $\sum_{i=1}^{n} \frac{1}{2^i}$ is $s_n = 1 - \frac{1}{2^n}$. However, it is not always possible to do this. How can we estimate the value of an infinite series in cases where we can't explicitly write s_n in terms of n?

Consider the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$. The first few terms are:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

The series is decreasing, but is it convergent? Let's plot this series on an xy-plane (see figure 1.1).

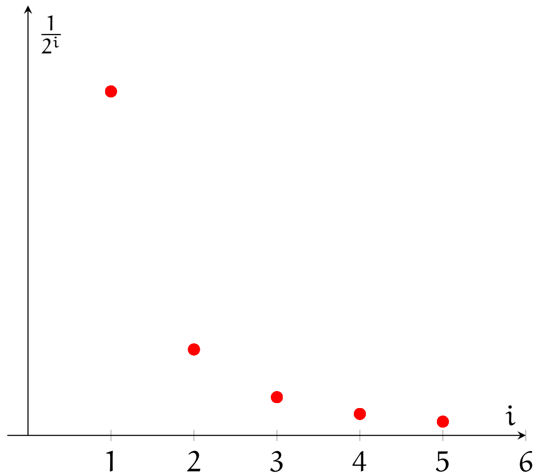

Figure 1.1: The first 5 terms of $\sum_{i=1}^{\infty} \frac{1}{2^i}$

We can overlay the function $y = \frac{1}{2^x}$ (figure 1.2). We can draw rectangles of width 1 and height $\frac{1}{x^2}$ (see figure 1.3). The area of the first n rectangles is equal to the n^{th} partial sum.

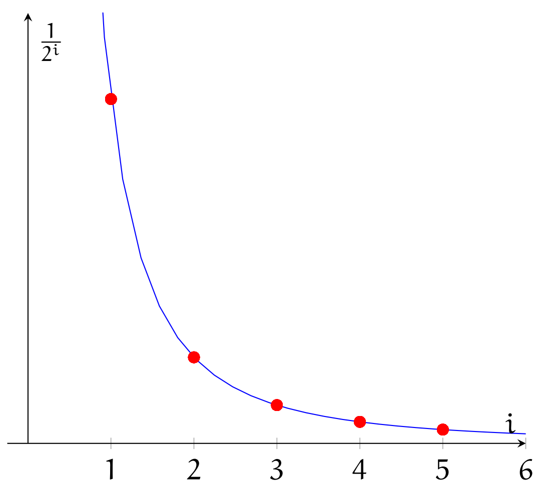

Figure 1.2: The first 5 terms of $\sum_{i=1}^{\infty} \frac{1}{2^i}$ lie on the curve $y = \frac{1}{x^2}$

This should remind you of a Riemann sum. Since the total area of the rectangles is less than the area under the curve, we can state:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < \int_0^{\infty} \frac{1}{x^2}\, dx$$

We can exclude the first rectangle and also state that:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < 1 + \int_1^{\infty} \frac{1}{x^2}\, dx$$

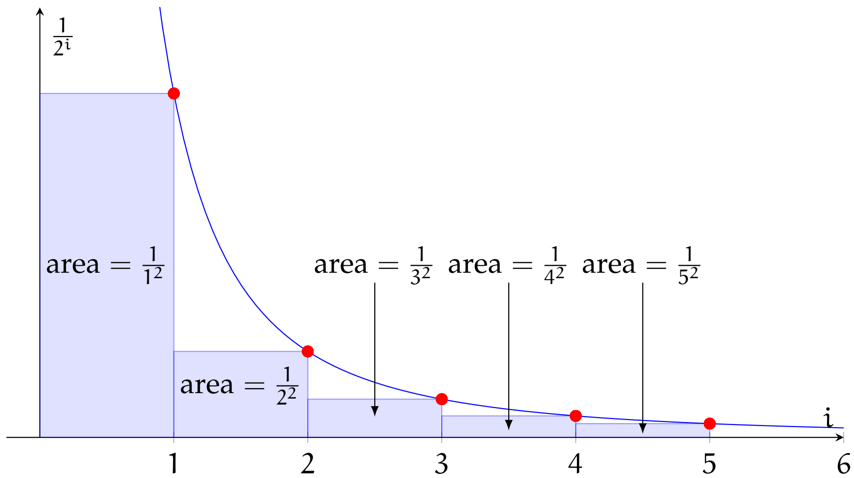

Figure 1.3: The partial sum $\sum_{i=1}^{n=5} \frac{1}{2^i}$ is equal to the area of the rectangles

We can evaluate this integral:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[\int_1^t \frac{1}{x^2}\,dx\right]$$

$$= \lim_{t\to\infty} \frac{-1}{x}\Big|_{x=1}^{t} = \lim_{t\to\infty}\left(\frac{-1}{t}\right) - \frac{-1}{1} = 0 - (-1) = 1$$

Therefore:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < 1 + 1 = 2$$

This means the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is bounded above. Since the series is also monotonic (each term is positive, so the value of the sum increases as n increases), we can state that the sum is convergent!

Let's look at a divergent example: $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}}$. Again, we will make a visual, but this time we will draw rectangles that lie above the curve $y = \frac{1}{\sqrt{x}}$ (see figure 1.4). In this case, $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}} > \int_1^\infty \frac{1}{\sqrt{x}}\,dx$. Let's evaluate the integral:

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\left[\int_1^t \frac{1}{\sqrt{x}}\,dx\right]$$

$$= \lim_{t\to\infty}\left[2\sqrt{x}\right]_{x=1}^{t} = \lim_{t\to\infty}\left(2\sqrt{t}\right) - 2\sqrt{1} = \infty - 2 \to \text{divergent}$$

Since the integral diverges to infinity and the series is greater than the integral, the series must also diverge to infinity. This is another case where a monotonic decreasing series is not convergent!

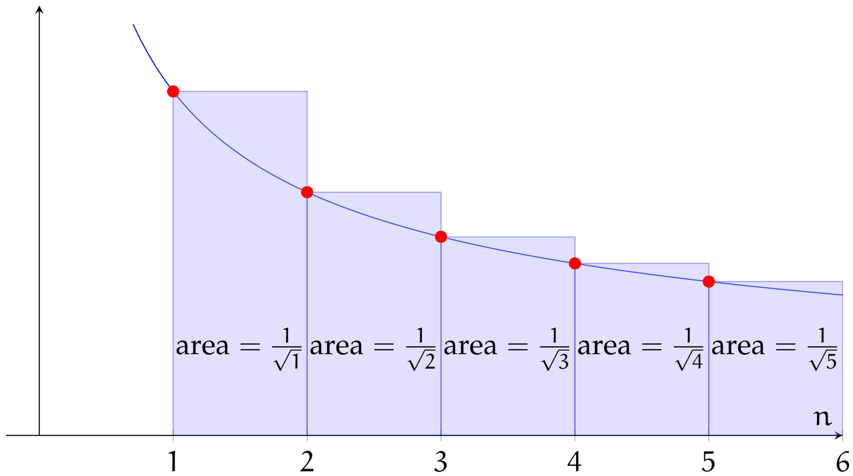

Figure 1.4: $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$

This leads us to the **Integral Test**. If f is a continuous, positive, decreasing function on the interval $x \in [1, \infty)$ and $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\, dx$ is convergent. Subsequently, if $\int_1^{\infty} f(x)\, dx$ is divergent, then the series is also divergent.

Example: Is the series $\sum_{i=1}^{\infty} \frac{1}{n^2+1}$ convergent or divergent?

Solution: To apply the integral test, we define $f(x) = \frac{1}{x^2+1}$, which is a positive, decreasing function on the interval $x \in [1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\, dx$$

$$= \lim_{t\to\infty} [\arctan x]_{x=1}^{t} = \lim_{t\to\infty} (\arctan t) - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Because the integral $\int_1^{\infty} \frac{1}{x^2+1}\, dx$ converges, so does the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Exercise 1

Use the integral test to determine if the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} 2n^{-3}$
2. $\sum_{n=1}^{\infty} \frac{5}{3n-1}$
3. $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$

Working Space

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Exercise 2

Apply the Integral Test to show that p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ are convergent only when $p > 1$ (hint: consider the cases $p \leq 0$, $0 < p < 1$, $p = 1$ and $p > 1$).

Working Space

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1.2.1 Using Integrals to Estimate the Value of a Series

Recall that $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots = s$ and that the n^{th} partial sum, often represented as s_n, is $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$. We can then define the n^{th} remainder $R_n = s - s_n$. Expanding s and s_n, we see that:

$$R_n = [a_1 + a_2 + \cdots + a_{n-1} + a_n + a_{n+1} + \cdots] - [a_1 + a_2 + \cdots + a_{n-1} + a_n]$$

$$R_n = [a_1 - a_1] + [a_2 - a_2] + \cdots + [a_{n-1} - a_{n-1}] + [a_n - a_n] + a_{n-1} + a_{n-2} + \cdots$$

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Just like the integral test, suppose there is some continuous, positive, decreasing function, such that $a_n = f(n)$. We can then represent R_n as the right Riemann sum with width $\Delta x = 1$ from $x = n$ to ∞. Since the rectangles are below the curve (see figure 1.5), we can state that $R_n \leq \int_n^{\infty} f(x)\,dx$.

Similarly, we can represent R_n as the left Riemann sum with width $\Delta x = 1$ from $x = n+1$ to ∞. This time, the rectangles are above the curve (see figure 1.6), and we can state that $R_n \geq \int_{n+1}^{\infty} f(x)\,dx$. Putting this all together, we have an estimate for the remainder, R_n, from the integral test:

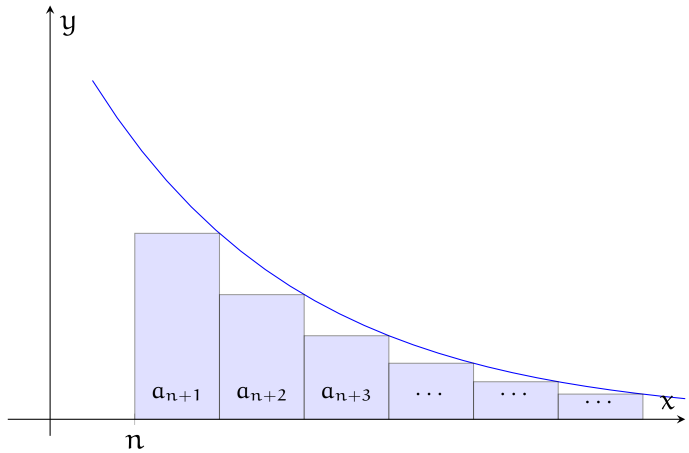

Figure 1.5: $R_n \leq \int_n^\infty f(x)\,dx$

Suppose there is a function such that $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. Then, $\int_{n+1}^\infty f(x)\,dx \leq R_n \leq \int_n^\infty f(x)\,dx$, where R_n is $s - s_n$.

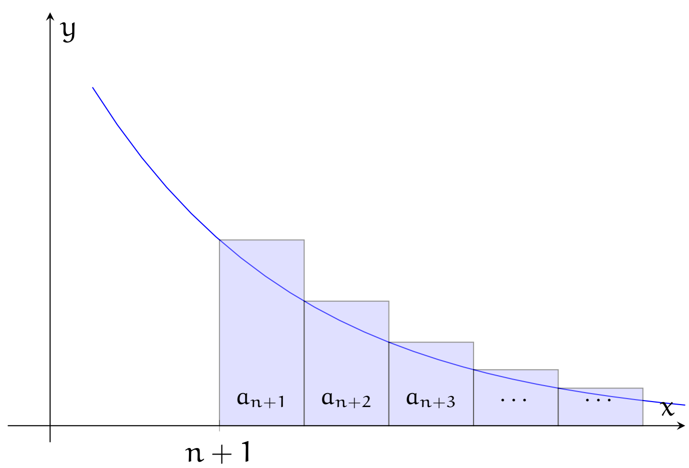

Figure 1.6: $R_n \geq \int_{n+1}^\infty f(x)\,dx$

Example: Approximate the sum of the series $\sum_{n=1}^\infty \frac{3}{n^3}$ by finding the 10^{th} partial sum. Estimate the error of this approximation.

Solution: Using a calculator, you can find the 10^{th} partial sum:

$$\sum_{n=1}^{10} \frac{3}{n^3} = \frac{3}{1^3} + \frac{3}{2^3} + \frac{3}{3^3} + \cdots + \frac{3}{10^3} \approx 3.593 = s_{10}$$

Recall that the remainder, R_{10}, is the difference between the actual sum, s, and the partial sum, s_{10}. Using the integral test to estimate the remainder, we can state that:

$$R_{10} \leq \int_{10}^{\infty} \frac{3}{x^3}\, dx = \frac{3}{2(10)^2} = \frac{3}{200} = 0.015$$

Therefore, the size of the error is at most 0.015.

Example: How many terms are required for the error to be less than 0.0001 for the sum presented above?

Solution: We are looking for an n such that $R_n \leq 0.0001$. Recalling that $R_n \leq \int_n^{\infty} \frac{3}{x^3}\, dx$, we need to find an n such that $\int_n^{\infty} \frac{3}{x^3}\, dx \leq 0.0001$.

$$\int_n^{\infty} \frac{3}{x^3}\, dx \leq 0.0001$$

$$\frac{-1}{6x^2}\Big|_{x=n}^{\infty} \leq 0.0001$$

$$\lim_{x\to\infty} \frac{-1}{6x^2} - \frac{-1}{6n^2} \leq 0.0001$$

$$0 + \frac{1}{6n^2} = \frac{1}{6n^2} \leq 0.0001$$

$$1 \leq 0.0006n^2$$

$$1667 \leq n^2$$

$$40.8 \leq n \to n = 41$$

Therefore, $s - s_{41} \leq 0.0001$ and the partial sum $\Sigma_{n=1}^{41} \frac{3}{n^3}$ is less than 0.0001 from the value of the infinite sum $\sum_{n=1}^{\infty} \frac{3}{n^3}$.

Exercise 3

Working Space

1. Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
2. Estimate the error from using s_{10} as an approximation of the series.
3. Use $s_n + \int_{n+1}^{\infty} \frac{1}{x^4}\,dx \le s \le s_n + \int_{n}^{\infty} \frac{1}{x^4}\,dx$ to give an improved estimate of the sum.
4. The actual value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is $\frac{\pi^4}{90}$. Compare your estimate with the actual value.
5. Find a value of n such that s_n is within 0.00001 of the sum.

Answer on Page 22

1.3 Comparison Tests

In comparison tests, we compare a series to a known convergent or divergent series. Take the series $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$. This is similar to $\sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a geometric series that converges to $\frac{1}{2}$. Notice that:

$$\frac{1}{3^n + 3} < \frac{1}{3^n}$$

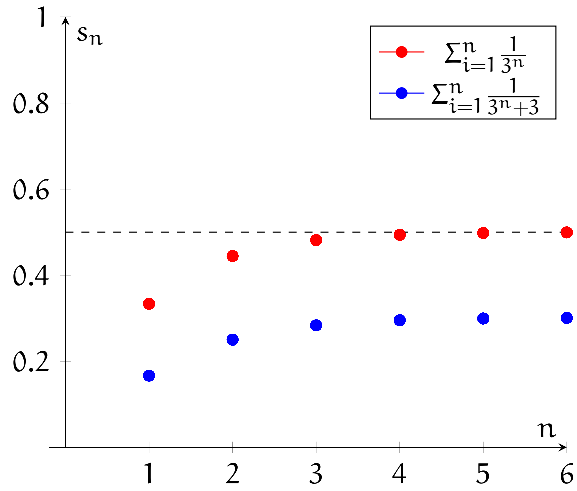

Figure 1.7: $\sum_{i=1}^{n} \frac{1}{3^n+3} < \Sigma_{i=1}^{n} \frac{1}{3^n}$ for all n

Which implies that

$$\sum_{n=1}^{\infty} \frac{1}{3^n+3} < \Sigma_{n=1}^{\infty} \frac{1}{3^n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent, it follows that $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$ is also convergent (see figure 1.7). As you can see, since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ approaches $\frac{1}{2}$, $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$ must be $\leq \frac{1}{2}$ and therefore convergent.

1.3.1 The Direct Comparison Test

F For the **Direct Comparison Test**, we compare the terms a_n to b_n directly. Take $\sum a_n$ and $\sum b_n$ to be series with positive terms. Then,

1. If $a_n \leq b_n$ and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
2. If $a_n \geq b_n$ and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

We already discussed above why the first part is true. The second part follows a similar argument: If a_n is greater than b_n, then you can imagine that as $\sum b_n$ grows and diverges, it is pushing upwards on $\sum a_n$, meaning that $\sum a_n$ must also diverge. Consider the series $\sum_{n=1}^{\infty} \frac{2 \ln n}{n}$. For $n \geq 2$, $2 \ln n > 1$, and therefore if $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\sum_{n=1}^{\infty} \frac{2 \ln n}{n}$ must also diverge. We recognize the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore, $\sum_{n=1}^{\infty} \frac{2 \ln n}{n}$ is also divergent (see figure 1.8).

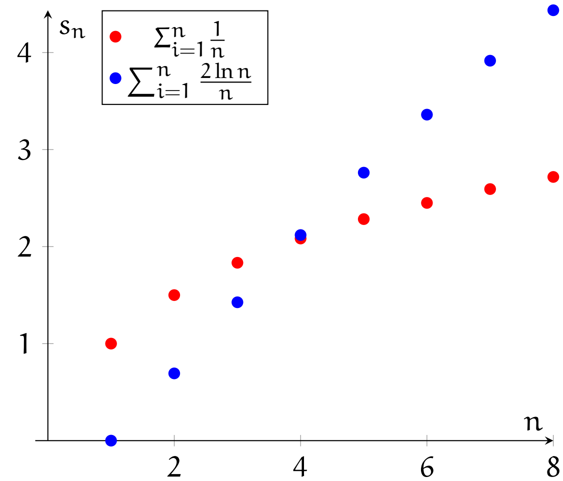

Figure 1.8: $\sum_{i=1}^{n} \frac{2\ln n}{n} > \sum_{i=1}^{n} \frac{1}{n}$ for $n \geq 4$

1.3.2 The Limit Comparison Test

Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. We may want to compare this to the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. The direct comparison test isn't helpful here, since $\frac{1}{2^n-1} > \frac{1}{2^n}$, so $\sum_{n=1}^{\infty} \frac{1}{2^n}$ doesn't put a cap on $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ like our earlier example (see figure 1.7). In a case such as this, we can use the **Limit Comparison Test**, which states that:
If $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both series diverge.

Let's apply this to the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. We know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, since it is a geometric series with $r < 1$.

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{2^n-1} \cdot \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = \frac{1}{1-0} = 1 > 0$$

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

In general, comparison tests are most useful for series resembling geometric or p-series. When choosing a p-series to compare the unknown series to, choose p such that the order of your p series is the same as the order of the unknown series.

Example: What p-series should one compare the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ to?

Solution: We can determine the order of $\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ by looking at the highest-order terms in the numerator and denominator:

$$\frac{\sqrt{n^3}}{n^3} = \frac{n^{3/2}}{n^3} = \frac{1}{n^{3/2}}$$

So, we should compare $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

Example: Is $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ convergent or divergent?

Solution: We have already determine that we should compare this series to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. To apply the limit test, we need to evaluate

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n^3+1}}{3n^3+4n^2+2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^6+n^3}}{3n^3+4n^2+2} = \frac{1}{3} > 0$$

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ is convergent because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

Exercise 4

Use the Comparison Test or the Limit Comparison Test to determine if the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$
2. $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$
3. $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$

Working Space

Answer on Page 22

1.4 Ratio and Root Tests for Convergence

1.4.1 Absolute Convergence

Suppose there is a series $\sum_{n=1}^{\infty} a_n$, then there is a corresponding series $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent**.

Example: Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Is this series absolutely convergent?

Solution: We examine the corresponding series where we take the absolute value of each term:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We can identify $\sum_{n=1}^{\infty} \frac{1}{n^2}$ as a convergent p-series. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can state that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

Example Is the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ absolutely convergent?

Solution We consider the sum of the absolute values of the terms:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

You should recognize this as the harmonic series, which is divergent. When a series is convergent but the corresponding series of absolute values is not, we call it **conditionally convergent**.

We won't prove the theorem here, but it is useful to know that if a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent. You can prove this yourself using the Comparison Test.

Exercise 5

Is the series given by

$$\frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^3} + \cdots$$

convergent or divergent?

Working Space

Answer on Page 23

Exercise 6

Determine whether each of the following series is absolutely or conditionally convergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$
2. $\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{n^2+4}$

Working Space

Answer on Page 23

1.4.2 The Ratio Test

The ratio test compares the $(n+1)^{\text{th}}$ term of a series to the n^{th} term and takes the limit as $n \to \infty$ of the absolute value of this ratio:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

There are three possible outcomes of the ratio test:

1. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
2. If $L = 1$, then the ratio test is inconclusive and we cannot draw any conclusions about whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.
3. If $L > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Apply the ratio test to determine if $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ is convergent or divergent.

Solution:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{3^{n+1}}\cdot\frac{3^n}{n^3}$$

$$= \lim_{n\to\infty}\frac{(n+1)^3}{n^3}\cdot\frac{3^n}{3\cdot 3^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\cdot\frac{1}{3} = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

Since $L < 1$, the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ is absolutely convergent.

The ratio test is most useful for series that contain factorials, constants raised to the n^{th} power, or other products.

Exercise 7

[This question was originally presented as a multiple-choice, no-calculator problem on the 2012 AP Calculus BC exam.] Which of the following series are convergent?

1. $\sum_{n=1}^{\infty}\frac{8^n}{n!}$
2. $\sum_{n=1}^{\infty}\frac{n!}{n^{100}}$
3. $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)}$

Working Space

Answer on Page 24

Exercise 8

[This question was originally presented as a multiple-choice, calculator-allowed problem on the 2012 AP Calculus BC exam.]If the series $\Sigma_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n, which of the following statements must be true? Explain why.

1. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$
2. $|a_n| < 1$ for all n
3. $\sum_{n=1}^{\infty} a_n = 0$
4. $\sum_{n=1}^{\infty} n a_n$ diverges
5. $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges

Working Space

Answer on Page 24

1.4.3 Root Test

The root test examines the behavior of the n^{th} root of a_n as $n \to \infty$. Similar to the ratio test, there are three possible outcomes:

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the Root Test is inconclusive.

The root test is best when there is a term or terms raised to the n^{th} power. Consider the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$:

Example: Is the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ convergent or divergent?

Solution: Since a_n consists of terms raised to the n^{th} power, we will apply the root test for convergence:

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

Therefore, by the root test, the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent.

Exercise 9

Use the Root Test to determine whether the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{n^2-4}\right)^n$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\ln n)^n}$
3. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

Working Space

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1.5 Strategies for Testing Series

When testing series for convergence, we want to choose a test based on the form of the series. While you may by tempted to try each test one-by-one until you find an answer, this quickly becomes cumbersome and time-consuming. Additionally, if you plan to take an AP Calculus exam, you need to be able to quickly choose an appropriate test as to conserve the time you have available for the exam. Here are some tips:

1. Check if the series is a p-series $\left(\sum_{n=1}^{\infty} \frac{1}{n^p}\right)$. If so, then if $p > 1$, the series converges.

Otherwise, the series diverges.

2. If the series is not a p-series, check to see if you can write it as a geometric series ($\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^{n}$). Recall that geometric series are convergent if $|r| < 1$ and divergent otherwise.

3. If the series can't be written as a p-series or geometric series, but has a similar form, consider the comparison tests (the Direct Comparison Test and the Limit Comparison Test). When choosing a p-series to compare your series to, follow the guidelines outlined in the Comparison Tests section above.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then apply the Test for Divergence to show the series is divergent. REMEMBER: $\lim_{n\to\infty} a_n \neq 0$ implies the series $\sum_{n=1}^{\infty} a_n$ is divergent, but $\lim_{n\to\infty} a_n = 0$ **does not** necessarily imply the series $\sum_{n=1}^{\infty} a_n$ is convergent.

5. If the series is alternating (has $(-1)^n$ or $(-1)^{n-1}$ in the term), the Alternating Series test may provide an answer.

6. The Ratio Test is excellent for series with factorials, other products, or constants to the n^{th} power. Remember that the Ratio Test will be inconclusive for p-series, rational functions of n, and algebraic functions of n.

7. If a_n is of the form $(b_n)^n$, use the Root Test.

8. If $a_n = f(n)$ where $f(n)$ is continuous, positive, and decreasing and you can evaluate $\int_1^{\infty} f(x)\,dx$, use the Integral Test.

You don't need to treat this as a checklist, where you check for every condition. Rather, you should use this as a guide to quickly determine the convergence test most likely to be useful.

Exercise 10

Choose an appropriate test to determine if the series is convergent of divergent. Apply the test and classify the series as convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$
2. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
3. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
4. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
5. $\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)^n$

Working Space

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APPENDIX A

Answers to Exercises

Answer to Exercise 1 (on page 5)

1. The function $2x^{-3}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^\infty 2x^{-3}\, dx = \lim_{t\to\infty} \int_1^t 2x^{-3}\, dx = \lim_{t\to\infty} \left[-x^{-2}\right]_{x=1}^{t} = \lim_{t\to\infty}(-t^{-2}) - -(1)^{-2} = 0+1 = 1$. Since the integral $\int_1^\infty 2x^{-3}\, dx$ converges, the series $\sum_{n=1}^\infty 2n^{-3}$ is also convergent.

2. The function $\frac{5}{3x+1}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^\infty \frac{5}{3x-1}\, dx = \lim_{t\to\infty} \int_1^t \frac{5}{3x-1}\, dx$ Using u-substitution to evaluate the integral, we set $u = 3x - 1$ and find that $du = 3dx \to dx = \frac{du}{3}$. Substituting, $\int_1^t \frac{5}{3x-1}\, dx = \int_{x=1}^{x=t} \frac{5}{3}\frac{1}{u}\, du$. Evaluating the integral, $\int_{x=1}^{x=t} \frac{5}{3}\frac{1}{u}\, du = \frac{5}{3}\ln u|_{x=1}^{x=t} = \frac{5}{3}\ln 3x + 1|_1^t$. Substituting this back into the limit, $\int_1^\infty \frac{5}{3x-1}\, dx = \lim_{t\to\infty} \frac{5}{3}\ln 3x + 1|_1^t = \lim_{t\to\infty}[\frac{5}{3}\ln 3t + 1] - \frac{5}{3}\ln 4 = \infty - \frac{5}{3}\ln 4 = \infty$. Therefore, the integral $\int_1^\infty \frac{5}{3x-1}\, dx$ is divergent and so is the series $\sum_{n=1}^\infty \frac{5}{3n-1}$.

3. The function $\frac{x}{3x^2+1}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^\infty \frac{x}{3x^2+1}\, dx = \lim_{t\to\infty} \int_1^t \frac{x}{3x^2+1}\, dx$. Applying the substitution $u = 3x^2+1$ and $\frac{du}{6} = x\, dx$, we see that $\lim_{t\to\infty} \int_1^t \frac{x}{3x^2+1}\, dx = \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{6u}\, du = \lim_{t\to\infty} \frac{1}{6}\ln u|_{x=1}^{x=t} = \lim_{t\to\infty} \frac{1}{6}\ln 3x^2 + 1|_1^t = \lim_{t\to\infty} \left[\frac{1}{6}\ln 3t^2 + 1\right] - \frac{1}{6}\ln 4 = \infty$. Therefore, the integral $\int_1^\infty \frac{x}{3x^2+1}\, dx$ is divergent, and so is the series $\sum_{n=1}^\infty \frac{n}{3n^2+1}$.

Answer to Exercise 2 (on page 6)

1. If $p \leq 0$, then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, and the series fails the Test for Divergence. Therefore, a p-series is divergent if $p \leq 0$.

2. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on the interval $x \in [1, \infty)$, and we can apply the integral test. So, we want to know, when is $\int_1^\infty \frac{1}{x^p}\, dx$ convergent? When $p = 1$, $\int_1^\infty \frac{1}{x^p}\, dx = \ln x|_{x=1}^{x=\infty} = \lim_{t\to\infty} \ln t - \ln 1 = \infty$ and the integral and p-series are both divergent.

3. What about when $0 < p < 1$? In this case, the integral $\int_1^\infty \frac{1}{x^p}\, dx = \lim_{t\to\infty} \int_1^t x^{-p}\, dx = \lim_{t\to\infty} \frac{1}{1-p}x^{1-p}|_{x=1}^{x=t} = \lim_{t\to\infty} \frac{1}{1-p}\frac{1}{x^{p-1}} = \left(\frac{1}{1-p}\right)\left[\lim_{t\to\infty}\left(\frac{1}{t^{p-1}}\right) - 1\right]$. When $0 < p < 1$, then $1 - p > 0$ is positive and $\lim_{t\to\infty} \frac{1}{t^{p-1}} = \lim_{t\to\infty} t^{1-p} = \infty$ and the integral diverges. Therefore, p-series are divergent for $0 < p < 1$.

4. When $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx = \left(\frac{1}{1-p}\right)\left[\lim_{t\to\infty}\left(\frac{1}{t^{p-1}}\right) - 1\right]$. When $p > 1$, $p-1 > 0$ and $\lim_{t\to\infty}\frac{1}{t^{p-1}} = 0$. Therefore, $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$ when $p > 1$, and therefore the p-series is convergent when $p > 1$.

Answer to Exercise 3 (on page 9)

1. $s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \cdots + \frac{1}{10^4} \approx 1.082037$.
2. $R_{10} \leq \int_{10}^\infty \frac{1}{x^4}\,dx = \frac{-1}{3x^3}\Big|_{x=10}^\infty = \lim_{x\to\infty}\frac{-1}{3x^3} - \frac{-1}{3\cdot 10^3} = \frac{1}{3000} = 0.000333$. Therefore, the error is less than 0.000333.
3. Given $s_{10} \approx 1.082037$, we can say that $1.082037 + \int_{n+1}^\infty \frac{1}{x^4}\,dx \leq s \leq 1.082037 + \int_n^\infty \frac{1}{x^4}\,dx$. Using a calculator to evaluate each integral, we see that: $1.082037 + 0.000250 \leq s \leq 1.082037 + 0.000333$ and therefore the sum is between 1.082287 and 1.082370.
4. Writing the actual value as a decimal, $\frac{\pi^4}{90} \approx 1.082323$, which is in the estimate window from the previous part.
5. We are looking for an n such that $\int_n^\infty \frac{1}{x^4}\,dx \leq 0.00001$. $\lim_{x\to\infty}\frac{-1}{3x^3} - \frac{-1}{3n^3} = \frac{1}{3n^3} \leq 0.00001$. $100,000 \leq 3n^3$. $33,333.33 \leq n^3$. $32.183 \leq n$. Since n must be an integer, $n = 33$ gives $R_n \leq 0.00001$.

Answer to Exercise 4 (on page 13)

1. This is similar to $\sum_{n=1}^\infty \frac{1}{n}$, which is divergent. Unfortunately, $\frac{1}{n} > \frac{1}{\sqrt{n^2+1}}$, so we can't use the direct comparison test. We will try the limit comparison test:

$$\lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}\cdot\frac{n}{1}\right) = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n^2}} = \frac{1}{1+0} = 1 > 0$$

Therefore, since $\sum_{n=1}^\infty \frac{1}{n}$ diverges, so does $\sum_{n=1}^\infty \frac{1}{\sqrt{n^2+1}}$.

2. This series is similar to the convergent geometric series $\sum_{n=1}^\infty \left(\frac{9}{10}\right)^n$. Given that:

$$\left(\frac{9}{10}\right) = \frac{9^n}{10^n} < \frac{9^n}{3+10^n}$$

Since $\frac{9^n}{3+10^n} < \left(\frac{9}{10}\right)^n$ and $\sum_{n=1}^\infty \left(\frac{9}{10}\right)^n$ is convergent, by the direct comparison test, $\sum_{n=1}^\infty \frac{9^n}{3+10^n}$ is also convergent.

3. We can compare this to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Noting that $\sin^2 n \leq 1$:

$$\frac{n \sin^2 n}{1 + n^3} < \frac{n \sin^2 n}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

Because $\frac{n \sin^2 n}{1+n^3} \leq \frac{1}{n^2}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can state by the direct comparison test that $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$ is also convergent.

Answer to Exercise 5 (on page 14)

We can write the series as $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$. Since n is real, we know that $n^2 > 0$ and we can say that $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$. Additionally, $|\cos n| \leq 1$ for all n, and therefore $\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$. We know the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. And since we have shown that $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$, by the comparison test $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent. Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and therefore convergent.

Answer to Exercise 6 (on page 15)

1. Conditionally Convergent. $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{3n+2}\right| = \sum_{n=1}^{\infty} \frac{1}{3n+2}$ Applying the integral test to this sum: $\int_1^{\infty} \frac{1}{3x+2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{3x+2}\,dx = \left[\frac{1}{3}\ln 3x + 2\right]_{x=1}^{t} = \lim_{t\to\infty} [\ln 3x + 2] - \ln 3(1) - 2 = \infty - 0 = \infty$. Since $\int_1^{\infty} \frac{1}{3x+2}\,dx$ is divergent, $\sum_{n=1}^{\infty} \frac{1}{3n+2}$ is divergent, and $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$ is conditionally convergent.

2. Absolutely Convergent. $\sum_{n=1}^{\infty} \left|\frac{\sin n}{4^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{4^n}$. Applying the integral test to $\sum_{n=1}^{\infty} \frac{1}{4^n}$: $\int_1^{\infty} \frac{1}{4^x}\,dx = \lim_{t\to\infty} \frac{-1}{4^x \ln 4}\Big|_{x=1}^{t} = \lim_{t\to\infty} \left[\frac{-1}{4^t \ln 4}\right] - \frac{-1}{4^1 \ln 4} = 0 + \frac{1}{4 \ln 4} = \frac{1}{4 \ln 4}$. Since $\int_1^{\infty} \frac{1}{4^x}\,dx$ is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is also convergent. And since $\sum_{n=1}^{\infty} \left|\frac{\sin n}{4^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{4^n}$, $\sum_{n=1}^{\infty} \left|\frac{\sin n}{4^n}\right|$ is also convergent, which shows that $\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$ is absolutely convergent.

3. Conditionally Convergent. We are asking if the series $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{2n}{n^2+4}\right|$ is convergent. $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{2n}{n^2+4}\right| = \sum_{n=1}^{\infty} \frac{2n}{n^2+4}$ We will apply the Limit Comparison test and compare this series to the known, divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n\to\infty} \frac{\frac{2n}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2}{n^2+4} = 2 > 0$. Therefore, by the Limit Comparison test, $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{2n}{n^2+4}\right|$ is divergent AND $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{n^2+4}$ is conditionally convergent.

Answer to Exercise 7 (on page 16)

Series 1 and 3 converge

1. We apply the ratio test: $\lim_{n\to\infty}\left|\frac{\frac{8^{n+1}}{(n+1)!}}{\frac{8^n}{n!}}\right| = \lim_{n\to\infty}\frac{8\cdot 8^n}{(n+1)(n!)}\cdot\frac{n!}{8^n} = \lim_{n\to\infty}\frac{8}{n+1} = 0.$
   Therefore, the series converges.

2. We apply the ratio test: $\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{100}}}{\frac{n!}{n^{100}}}\right| = \lim_{n\to\infty}\frac{(n+1)n!}{(n+1)^{100}}\cdot\frac{n^{100}}{n!} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{100}.$
   $(n+1) = \lim_{n\to\infty}\frac{n^{100}}{(n+1)^{99}} = \infty$. Therefore, the series diverges.

3. We apply the comparison test: $\frac{n+1}{(n)(n+2)(n+3)} = \frac{n}{(n)(n+2)(n+3)} + \frac{1}{(n)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} + \frac{1}{(n)(n+2)(n+3)} = \frac{1}{n^2+5n+6} + \frac{1}{n^3+5n^2+6n} \leq \frac{1}{n^2}+\frac{1}{n^3}$. The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^3}$ are both convergent, because they are p-series with $p > 1$. Having established that $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{n^3}$ and that $\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{n^3}$ converges, by the comparison test we can state that $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)}$ converges.

Answer to Exercise 8 (on page 17)

1. This is not necessarily true. For a convergent series, the result of the ratio test is $L < 1$, so the limit could be $\neq 0$.

2. This is not necessarily true. Consider the geometric series $\sum_{n=1}^{\infty} 2(\frac{1}{2})^{n-1}$. This series is convergent because the common ratio is less than one, but the first term is $2(\frac{1}{2})^0 = 2 > 1$.

3. This is not necessarily true. Again, consider the geometric series $\sum_{n=1}^{\infty} 2(\frac{1}{2})^{n-1}$, which converges to $4 \neq 0$.

4. This is not necessarily true. Consider the p-series $\sum_{n=1}^{\infty}\frac{1}{n^4}$. Then the series $\sum_{n=1}^{\infty} n\frac{1}{n^4} = \sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent.

5. This must be true. By the comparison test, $\sum_{n=1}^{\infty}\frac{a_n}{n} \leq \sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} a_n$ converges, so much $\sum_{n=1}^{\infty} na_n$.

Answer to Exercise 9 (on page 18)

1. $\lim_{n=1}^{\infty}\sqrt[n]{\left|\left(\frac{3n^2+1}{n^2-4}\right)^n\right|} = \lim_{n=1}^{\infty}\frac{3n^2+1}{n^2-4} = 3 > 1$. Therefore, the series $\sum_{n=1}^{\infty}\left(\frac{3n^2+1}{n^2-4}\right)^n$ is

divergent.

2. $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = \frac{1}{\infty} = 0 < 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ is convergent.

3. $\lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^{n^2}\right|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$. Therefore, $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ is divergent.

Answer to Exercise 10 (on page 20)

1. Divergent. Since there is a constant to the n^{th} power and an algebraic function of n, we will try the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)^2} \cdot \frac{n^2}{e^n} = \lim_{n\to\infty} \frac{e^n \cdot e}{e^n} \cdot \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} e \cdot \left(\frac{n}{n+1}\right)^2 = e \cdot 1^2 = e > 1$. Therefore, $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ is divergent.

2. Convergent. Since there is a factorial, we will try the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n\to\infty} \frac{3 \cdot 3^n}{3^n} \cdot \frac{n!}{(n+1)n!} \cdot \left(\frac{n+1}{n}\right)^2 = \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)n^2} = \lim_{n\to\infty} \frac{3(n+1)}{n^2} = 0 < 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ is convergent.

3. Divergent. Since $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\, dx$ can be integrated, we will apply the integral test. $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{\ln x}}\, dx$. Setting $u = \ln x$, then $du = \frac{dx}{x}$ and $\frac{1}{x\sqrt{\ln x}} dx = \frac{1}{\sqrt{u}} du$. Then we can say that $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\, dx = \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{1}{\sqrt{u}}\, du = \lim_{t\to\infty} \left(\frac{-1}{2}\right) \sqrt{u}\Big|_{x=2}^{x=t} = \lim_{t\to\infty} \left(\frac{-1}{2}\right) \sqrt{\ln x}\Big|_2^t = \left(\frac{-1}{2}\right) \lim_{t\to\infty} \sqrt{\ln t} - \left(\frac{-1}{2}\right) \sqrt{\ln 2} = \infty$. Since the integral diverges, so does the series.

4. Convergent. Since this series has terms to the n^{th} power, we will try the Root Test. $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{n+1}\right)^{n^2}\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n+2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$ Therefore, by the root test, the series is convergent.

5. Convergent. This series also has terms raised to the n^{th} power, we will try the Root Test again. $\lim_{n\to\infty} \sqrt[n]{\left|\left(\sqrt[n]{2}-1\right)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\sqrt[n]{2}-1\right)^n} = \lim_{n\to\infty} \left(\sqrt[n]{2}-1\right)^{n/n} = \lim_{n\to\infty} \left(\sqrt[n]{2}-1\right) = \lim_{n\to\infty} 2^{1/n} - 1 = 1 - 1 = 0 < 1$. Therefore, the series converges.

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