Chapter 1

Derivatives

In calculus, the derivative of a function represents the rate at which the function is changing at a particular point. It is a fundamental concept that has vast applications in various fields, including physics.

1.1 Definition

The derivative of a function $f(x)$ at a point x is defined as the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1.1}$$

provided this limit exists. In words, the derivative of f at x is the limit of the rate of change of f at x as the change in x approaches zero. The derivative of a function is equal to the slope of the function. The derivative of a function, $f(x)$, is denoted as $f'(x)$ (read out loud as "f prime of x") or df/dx. The origin of this definition was shown in the previous chapter, Differentiation.

1.1.1 Estimating the Derivative

Consider the function $f(x) = x^2$. Suppose we want to write an equation for a line that is tangent to the curve at $x = 2$ (see figure 1.1). We already have a point that the line passes through: $(2, 4)$. To write an equation for the tangent line, we would need to know its slope, m.

We can estimate the slope by choosing points on either side of P, drawing a line through those points, and calculating the slope of that secant line (it is a secant line because it intersects the curve more than once). See figure 1.2 for a visualization.

As the points Q and R get closer to P, the better the estimate becomes.

Much scientific data is not described as continuous functions, but rather as discrete data points. Consider the following data of a falling object:

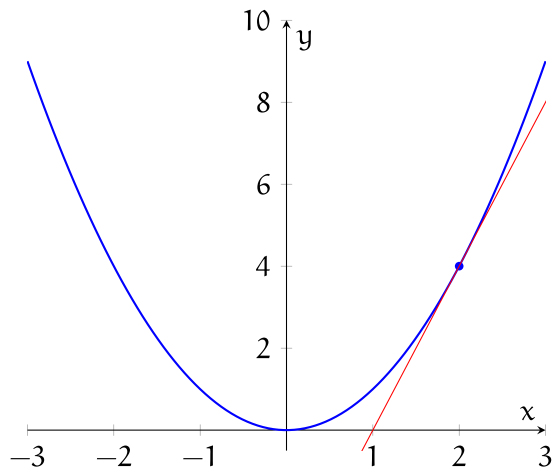

Figure 1.1: The red line is tangent to $f(x) = x^2$ at the point $(2, 4)$

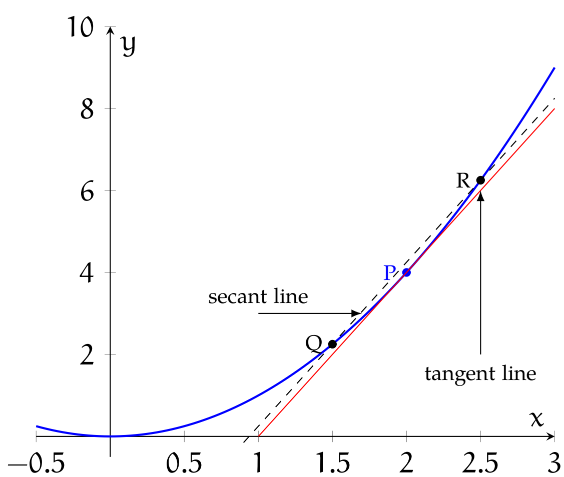

Figure 1.2: The slope of the secant line is approximately the slope of the tangent line

time (seconds)	height (m)
0	50
0.5	48.775
1	45.1
1.5	38.975
2	30.4
2.5	19.375
3	5.9

A graph of the data is shown in figure 1.3.

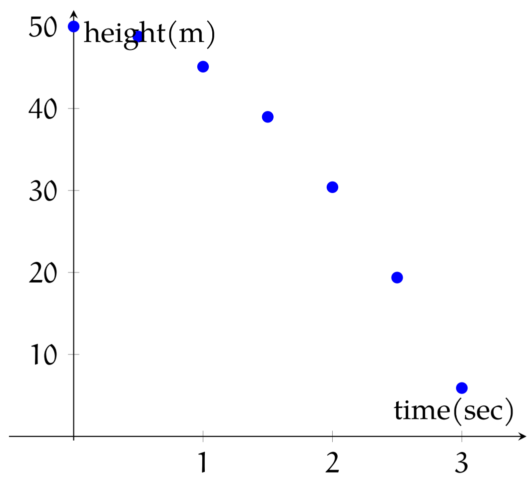

Figure 1.3: The height of a falling object over time

Suppose we wanted to estimate the velocity of the falling object at $t = 1.5$s. Recall that velocity is given by the change in position divided by the change in time. We can select data points on either side of $t = 1.5$s and use them to find the average velocity from $t = 1$s and $t = 2$s (see figure 1.4):

$$v = \frac{h_2 - h_2}{t_2 - t_1} = \frac{30.4\text{m} - 45.1\text{m}}{2\text{s} - 1\text{s}} = -14.7\frac{\text{m}}{\text{s}}$$

Example: A 1000-gallon tank drains from the bottom in 30 minutes. The volume left in the tank is recorded every 5 minutes, as shown in the data table below. Use the data to estimate $V'(15)$ and $V'(25)$, including appropriate units. At which time is the tank draining faster?

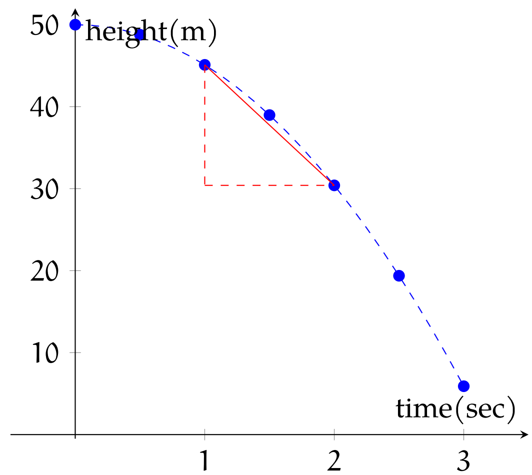

Figure 1.4: The slope of the line connecting the data points on either side of $t = 1.5$s is approximately the velocity of the falling object at $t =$ s

t (min)	V (gal)
5	694
10	444
15	250
20	111
25	28
30	0

Solution: To estimate $V'(15)$, we find the slope of the line connecting the data points on either side of $t = 15$:

$$V'(15) \approx \frac{111\text{gal} - 444textgal}{20\text{min} - 10\text{min}}$$

$$V'(15) \approx \frac{-333\text{gal}}{10\text{min}}$$

$$V'(15) \approx -33.3\frac{\text{gal}}{\text{min}}$$

And we can use the data at $t = 20$ and $t = 30$ to estimate $V'(25)$:

$$V'(25) \approx \frac{0\text{gal} - 111\text{gal}}{30\text{min} - 20\text{min}}$$

$$V'(25) \approx \frac{-111\text{gal}}{10\text{min}}$$

$$V'(25) \approx -11.1\frac{\text{gal}}{\text{min}}$$

Both answers are negative because the tank is emptying, and the tank is draining faster at $t = 15$ than at $t = 25$.

Exercise 1

[This question was originally presented as a free-response, calculator-allowed question on the 2012 AP Calculus BC Exam.] The temperature of water in a tub at time t is modeled by a function, W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. Values of $W(t)$ at selected times for the first 20 minutes are given in the table. Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of the problem.

t (minutes)	$W(t)$ (degrees Fahrenheit)
0	55.0
4	57.1
9	61.8
15	67.9
20	71.0

Working Space

Answer on Page 15

1.2 The Derivative as a Function

We have seen how to estimate the value of a derivative at a specific point on a graph. Suppose we wanted to describe the slope of a graph everywhere. That is: can we find a function, $g(x)$ that describes the slope of another function, $f(x)$, over the domain of f? Using the definition of a derivative, we can.

You have already seen an algorithm to find the derivatives of polynomial functions (see chapter Differentiating Polynomials). Recall that for a function, $f(x) = x^n$, the derivative is $f'(x) = nx^{n-1}$. Here is the proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

In order to expand the polynomial, $(x+h)^n$, we'll need to apply the Binomial Theorem, which tells us that:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n$$

Substituting this into our limit definition of a derivative, we see that:

$$f'(x) = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots nxh^{n-1} + h^n - x^n}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots nxh^{n-1} + h^n}{h}$$

$$f'(x) = \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots nxh^{n-2} + h^{n-1}$$

$$f'(x) = nx^{n-1}$$

Example: Use the limit definition of a derivative to find $f'(x)$ if $f(x) = 2x^3 - x^2$.

Solution: According to the limit definition, f' is:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{\left[2\,(x+h)^3 - (x+h)^2\right] - \left[2x^3 - x^2\right]}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\left[2\left(x^3 + 3hx^2 + 3h^2x + h^3\right) - \left(x^2 + 2xh + h^2\right)\right] - 2x^3 + x^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2x^3 - 2x^3 + 6hx^2 + 6h^2x + 6h^3 - x^2 + x^2 - 2xh - h^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{6hx^2 + 6h^2x + 6h^3 - 2xh - h^2}{h}$$

$$f'(x) = \lim_{h\to 0} 6x^2 + 6hx + 6h^2 - 2x - h = 6x^2 - 2x$$

Therefore, if $f(x) = 2x^3 - x^2$, then $f'(x) = 6x^2 - 2x$.

Exercise 2 Finding Functions for Derivatives

Use the limit definition of a derivative to find an equation for $f'(x)$.

1. $f(x) = mx + b$
2. $f(x) = \sqrt{16 - x}$
3. $f(x) = \frac{x^2-1}{2x-3}$

Working Space

Answer on Page 15

1.3 Applications in Mathematics

1.3.1 l'Hospital's Rule

Consider the function $h(x) = \frac{\ln x}{x-1}$ and suppose we are interested in the behavior of $h(x)$ around $x = 1$. If we apply the Quotient Rule, we get an indeterminate result:

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \frac{0}{0}$$

Looking at the graph of $h(x)$ (see figure 1.5), we can guess that $\lim_{x \to 1} \frac{\ln x}{x-1} = 1$.

Let's examine the numerator and denominator separately: we'll define $f(x) = \ln x$ and $g(x) = x - 1$ (see figure 1.6).

If we zoom in very far around $x = 1$, the graphs begin to look linear (see figure 1.7):

We can approximate these graphs as linear functions with slopes m_1 and m_2, so that the blue curve is approximated as $y = m_1(x-1)$ and the red curve is approximated as $y = m_2(x-1)$. The ratio of the functions would then be

$$\frac{m_1(x-1)}{m_2(x-1)} = \frac{m_1}{m_2}$$

which is the same as the ratio of the derivatives of our linear approximations. This suggests l'Hospital's rule, that the limit of a ratio is the same as the limit of the ratio of the

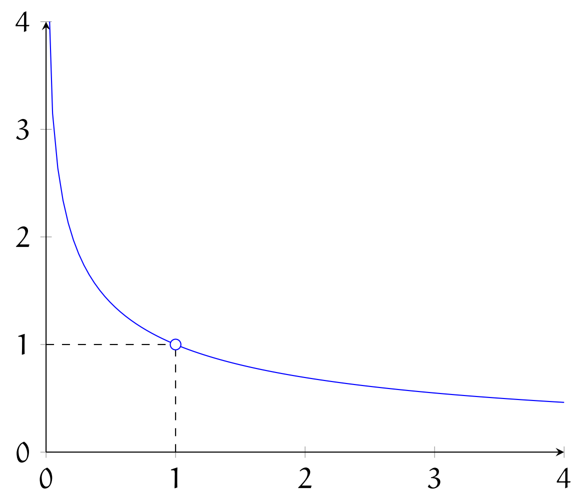

Figure 1.5: $h(x) = \frac{\ln x}{x-1}$

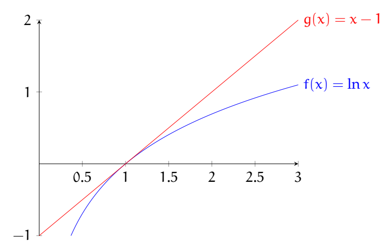

Figure 1.6: Examining each part of $\frac{\ln x}{x-1}$ separately

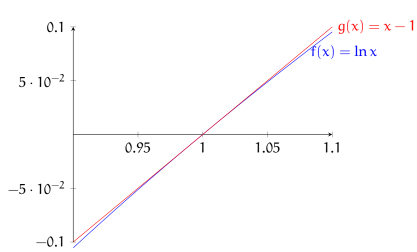

Figure 1.7: As we zoom in, the graph of $\ln x$ appears linear

derivatives for certain indeterminate forms:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

.

Let's apply l'Hospital's rule to our limit of $h(x)$:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

Notice our result with l'Hospital's rule matches our guess based on the graph of $h(x) = \frac{\ln x}{x-1}$.

L'Hospital's rule also applies to the indeterminate result $\frac{\pm\infty}{\pm\infty}$. For a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, l'Hospital's rule applies if:

1. the original limit is of the indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
2. f and g are differentiable on an interval containing a (but possibly not differentiable at a)
3. $g'(x) \neq 0$ on said interval

Example: Determine $\lim_{x \to \infty} \frac{e^x}{x^2}$.

Solution: We begin by evaluating the limit:

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \frac{e^\infty}{\infty^2} = \frac{\infty}{\infty}$$

This is an indeterminate form that we can apply l'Hospital's rule to:

$$= \lim_{x \to \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} x^2} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Evaluating this limit, we get another indeterminate form:

$$= \frac{e^\infty}{2 \cdot \infty} = \frac{\infty}{\infty}$$

Don't panic! We can apply l'Hospital's rule again (in fact, we can apply l'Hospital's rule as many times as needed to evaluate a limit, as long as we keep getting $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$):

$$= \lim_{x \to \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} 2x} = \lim_{x \to \infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

and therefore, $\lim_{x\to\infty} \frac{e^x}{x^2} = \infty$.

Exercise 3

What is $\lim_{x\to 0} \frac{\tan x - x}{x^3}$?

Working Space

Answer on Page 16

Exercise 4

Evaluate each of the following limits, using l'Hospital's rule where needed.

1. $\lim_{x\to 3} \frac{x-3}{x^2-9}$
2. $\lim_{x\to 1/2} \frac{6x^2+5x-4}{4x^2+16x-9}$
3. $\lim_{x\to 0^+} \frac{\ln x}{\sqrt{x}}$
4. $\lim_{x\to \pi} \frac{1+\cos x}{1-\cos x}$
5. $\lim_{x\to 1} \frac{x \sin x - 1}{2x^2-x-1}$

Working Space

Answer on Page 17

1.3.2 Mean Value Theorem

The Mean Value Theorem (MVT) states that on an interval $[a, b]$ where a continuous function f is differentiable on an open interval (a, b), there is at least one point where the tangent line to f has the same slope as a line connecting the points $(a, f(a))$ and $(b, f(b))$. Consider the graph of $f(x) = x^2$ (see figure 1.8). The line connecting the points $(-1, 1)$ and $(2, 4)$ has a slope of $\frac{1}{2}$. MVT tells us there must be *at least one* point, c, on the interval $x \in (-1, 2)$ where $f'(c) = \frac{1}{2}$. We can find this point by setting $f'(x)$ equal to $\frac{1}{2}$:

$$2x = \frac{1}{2} \to x = \frac{1}{4}$$

Examining the figure 1.8, you can see that the tangent at $f(\frac{1}{4})$ (the black line) is parallel to the red line connecting $(-1, f(-1))$ and $(2, f(2))$.

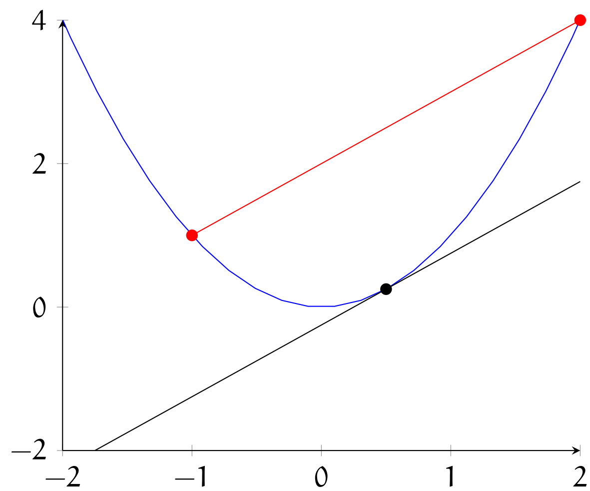

Figure 1.8: $f(x) = x^2$

Note that MVT doesn't tell us *where* $f'(x)$ is parallel to the line connecting $(a, f(a))$ and $(b, f(b))$, just that some value c exists that satisfies the condition.

Example: Consider a hammer thrown upwards at 5 $\frac{m}{s^2}$ on Earth (where the acceleration due to gravity is approximately $-9.8\frac{m}{s^2}$).

Solution: We can use the MVT to show that there must be some point in the hammer's path upwards where the velocity of the hammer is exactly equal to its average velocity as it flies through the air.

The hammer's rise can be described with the function $y(t) = 5t - 4.9t^2$. The hammer reaches its peak at approximately $t = 0.51$. So, we are looking for some value, c, such that

$$y'(c) = \frac{y(0.51) - y(0)}{0.51 - 0} = \frac{5(0.51) - 4.9(0.51^2)}{0.51} = \frac{1.2755}{0.51} = 2.5$$

Solving $y'(t) = 5 - 9.8t = 2.5$, we find that the c that satisfies the MVT is approximately 0.255. This result is illustrated in figure 1.9:

MVT Practice

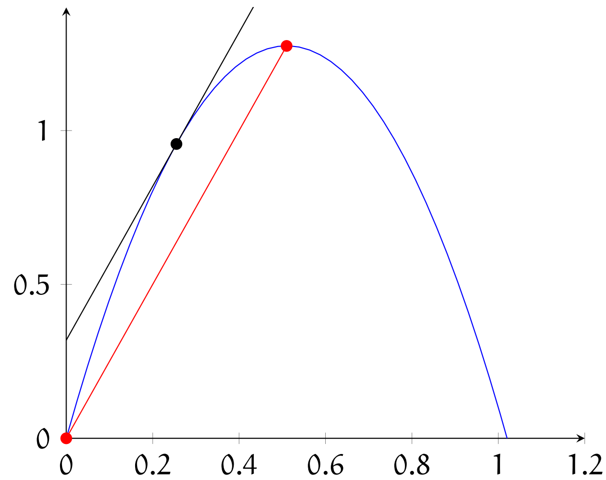

Figure 1.9: The height of a hammer tossed upwards at $5\frac{\text{m}}{\text{s}}$

Exercise 5

AT 3:30 PM, a car's speedometer reads $30\frac{\text{mi}}{\text{hr}}$. At 3:40 PM, it reads $50\frac{\text{mi}}{\text{hr}}$. Show that at some time between 3:30 and 3:340 PM, the car's acceleration is exactly $120\frac{\text{mi}}{\text{hr}^2}$.

Working Space

Answer on Page 17

Exercise 6

Find the number c that satisfies the MVT on the given interval.

(a) $f(x) = \sqrt{x}$, $[0, 4]$

(b)$f(x) = e^{-x}$, $[0, 2]$

(c)$f(x) = \ln x$, $[1, 4]$

Working Space

Answer on Page 17

1.4 Applications in Physics

In physics, derivatives play a vital role in describing how quantities change with respect to one another.

1.4.1 Velocity and Acceleration

In kinematics, the derivative of the position function with respect to time gives the velocity function, and further taking the derivative of the velocity function gives the acceleration function. For example, if $s(t)$ represents the position of an object at time t, then the velocity $v(t)$ and acceleration $a(t)$ are given by:

$$v(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} \tag{1.2}$$

Practice

A particle's motion is described by $s(t) = t^3 - 6t^2 + 6t$, where t is measured in seconds and s is measured in meters. Answer the following questions about the particle's motion:

Exercise 7

Find the velocity at time t.

Working Space

Answer on Page 19

Exercise 8

What is the velocity after 2s? After 4s?

Working Space

Answer on Page 19

Exercise 9

When is the particle at rest?

Working Space

Answer on Page 19

1.4.2 Force and Momentum

In mechanics, the derivative of the momentum of an object with respect to time gives the net force acting on the object, as stated by Newton's second law of motion:

$$F = \frac{dp}{dt} \tag{1.3}$$

where F is the force, p is the momentum, and t is the time.

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Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

To estimate the slope at $t = 12$, we can use the data at $t = 9$ and $t = 15$. The slope of the line connecting those points is approximate of the slope at $t = 12$.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{67.9 - 61.8}{15 - 9} = \frac{6.1}{6} = 1.017$$

The units for the numerator are degrees Fahrenheit and for the denominator are minutes. Therefore, the estimated slope has units of degrees Fahrenheit per minute. This represents the change in temperature of the water in the tub. When $t = 12$, the water in the tub is increasing in temperature at about 1 degree Fahrenheit per minute.

Answer to Exercise 2 (on page 7)

1.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[m\,(x+h) + b] - [mx + b]}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$$

2.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{16-x-h} - \sqrt{16-x}}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{\sqrt{16-x-h} - \sqrt{16-x}}{h} \cdot \frac{\sqrt{16-x-h} + \sqrt{16-x}}{\sqrt{16-x-h} + \sqrt{16-x}}$$
$$f'(x) = \lim_{h\to 0} \frac{(16-x-h) - (16-x)}{h\left(\sqrt{16-x-h} + \sqrt{16-x}\right)}$$
$$f'(x) = \lim_{h\to 0} \frac{-h}{h\left(\sqrt{16-x-h} + \sqrt{16-x}\right)} = \lim_{h\to 0} \frac{-1}{\sqrt{16-x-h} + \sqrt{16-x}}$$
$$f'(x) = \frac{-1}{\sqrt{16-x} + \sqrt{16-x}} = \frac{-1}{2\sqrt{16-x}}$$

3.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)^2-1}{2(x+h)-3} - \frac{x^2-1}{2x-3}}{h}$$

$$f'(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)\left[\frac{x^2+2xh+h^2-1}{2x+2h-3} - \frac{x^2-1}{2x-3}\right]$$

$$f'(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)\left[\frac{x^2+2xh+h^2-1}{2x+2h-3}\left(\frac{2x-3}{2x-3}\right) - \frac{x^2-1}{2x-3}\left(\frac{2x+2h-3}{2x+2h-3}\right)\right]$$

$$f'(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)\left[\frac{\left(x^2+2xh+h^2-1\right)(2x-3) - \left(x^2-1\right)(2x+2h-3)}{(2x-3)(2x+2h-3)}\right]$$

$$f'(x) = \left(\frac{1}{h}\right)\left[\frac{2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3\left(2x^3+2x^2h-3x^2-2x-2h+3\right)}{(2x-3)(2x+2h-3)}\right]$$

$$f'(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)\left[\frac{2x^2h+2xh^2-6xh-3h^2+2h}{(2x-3)(2x+2h-3)}\right]$$

$$f'(x) = \lim_{h\to 0}\frac{2x^2+2xh-6x-3h+2}{(2x-3)(2x+2h-3)} = \frac{2x^2-6x+2}{(2x-3)^2}$$

Answer to Exercise 3 (on page 10)

First, let's confirm that l'Hospital's rule applies here:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \frac{0-0}{0} = \frac{0}{0}$$

Therefore, we can apply l'Hospital's rule:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}x^3}$$

$$= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \frac{1-1}{0} = \frac{0}{0}$$

which is an indeterminate form. We apply l'Hospital's rule again:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\frac{d}{dx}(\sec^2 x - 1)}{\frac{d}{dx}3x^2}$$

$$= \lim_{x\to 0}\frac{2\tan x\sec^2 x}{6x} = \frac{2(0)(1^2)}{6\cdot 0} = \frac{0}{0}$$

which is also an indeterminate form. We apply l'Hospital's rule again:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\frac{d}{dx}(2\tan x\sec^2 x)}{\frac{d}{dx}6x}$$

$$= \lim_{x \to 0} \frac{2\sec^2 x[2\tan^2 x + \sec^2 x]}{6} = \frac{2 \cdot 1 \cdot [2 \cdot 0 + 1]}{6}$$
$$= \frac{2}{6} = \frac{1}{3}$$

Answer to Exercise 4 (on page 10)

1. $\lim_{x\to 3} \frac{x-3}{x^2-9} = \frac{0}{0}$, so we apply l'Hospital's rule. $\lim_{x\to 3} \frac{x-3}{x^2-9} = \lim_{x\to 3} \frac{1}{2x} = \frac{1}{6}$
2. $\lim_{x\to 1/2} \frac{6x^2+5x-4}{4x^2+16x-9} = \frac{0}{0}$, so we apply l'Hospital's rule. $\lim_{x\to 1/2} \frac{6x^2+5x-4}{4x^2+16x-9} = \lim_{x\to 1/2} \frac{12x+5}{8x+16} = \frac{11}{20}$
3. $\lim_{x\to 0^+} \frac{\ln x}{\sqrt{x}} = \frac{-\infty}{0} = -\infty$. This limit does not require l'Hospital's rule because it is evaluable
4. $\lim_{x\to 1} \frac{x\sin x-1}{2x^2-x-1} = \frac{1\cdot\sin 1-1}{2(1)^2-1-1} = \frac{0}{0}$, so we apply l'Hospital's rule: $\lim_{x\to 1} \frac{x\sin x-1}{2x^2-x-1} = \lim_{x\to 1} \frac{\frac{d}{dx}(x\sin x-1}{\frac{d}{dx}(2x^2-x-1} = \lim_{x\to 1} \frac{x\cdot\cos x-1+\sin x-1}{4x-1} = \frac{1\cdot\cos 0+\sin 0}{4-1} = \frac{1\cdot 1+0}{-3} = \frac{-1}{3}$.

Answer to Exercise 5 (on page 12)

The speed of a car must a continuous, differentiable function, since your car can't "jump" from one speed to another: it must smoothly accelerate from one speed to another. Therefore, the Mean Value Theorem applies. The average acceleration from 3:30 PM to 3:40 PM is given by:

$$\frac{\text{change in speed}}{\text{change in time}} = \frac{50\frac{mi}{hr} - 30\frac{mi}{hr}}{3:40PM - 3:30PM}$$

Simplifying and converting minutes to hours, we see the average acceleration is:

$$\frac{20\frac{mi}{hr}}{\frac{1}{6}hr} = 120\frac{mi}{hr^2}$$

Therefore, by MVT, there must be some time between 3:30 and 3:40 PM where the car's acceleration is exactly $120\frac{mi}{hr^2}$.

Answer to Exercise 6 (on page 12)

(a) For the domain given, $f(x)$ is defined and differentiable. Finding the slope of the secant line connecting the endpoints:

$$\frac{f(b)-f(a)}{b-a}=\frac{\sqrt{4}-\sqrt{0}}{4-0}=\frac{2}{4}=\frac{1}{2}$$

So we are looking for some number c such that $f'(c)=\frac{1}{2}$. Let's find $f'(x)$:

$$f'(x)=\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$$

Setting this equal to $\frac{1}{2}$ to find c:

$$f'(c)=\frac{1}{2\sqrt{c}}=\frac{1}{2}$$
$$\sqrt{c}=1$$
$$c=1$$

(b)For the domain given, $f(x)$ is defined and differentiable. Finding the slope of the secant line connecting the endpoints:

$$\frac{f(2)-f(0)}{2-0}=\frac{e^{-2}-e^{0}}{2}=\frac{1-e^{2}}{2e^{2}}\approx -0.432$$

And find $f'(x)$:

$$f'(x)=-e^{-x}$$

According to MVT, there must be some c such that $f'(c)\approx -0.432$:

$$-e^{-c}\approx -0.432$$
$$e^{-c}\approx 0.432$$
$$-c\approx \ln 0.432$$
$$c\approx -\ln 0.432\approx 0.839$$

(c) For the domain given, $f(x)$ is defined and differentiable. Finding the secant line connecting the endpoints:

$$\frac{f(b)-f(a)}{b-a}=\frac{\ln 4-\ln 1}{4-1}=\frac{\ln 4}{3}\approx 0.462$$

And find $f'(x)$:

$$f'(x)=\frac{1}{x}$$

According to MVT, there must be some c such that $f'(c) \approx 0.462$

$$f'(c) = \frac{1}{c} \approx 0.462$$

$$c \approx \frac{1}{0.462} = 2.164$$

Answer to Exercise 7 (on page 13)

Velocity is the derivative of position. Therefore, $v(t) = s'(t) = 3t^2 - 12t + 6$.

Answer to Exercise 8 (on page 13)

$$v(2) = 3(2)^2 - 12(2) + 6 = -6\frac{m}{s}$$

$$v(4) = 3(4)^2 - 12(4) + 6 = 6\frac{m}{s}$$

Answer to Exercise 9 (on page 14)

When the particle is at rest, $v(t) = 0$.

$$3t^2 - 12t + 6 = 0$$

$$3(t^2 - 4t + 2) = 0$$

$$t^2 - 4t + 2 = 0$$

This is not easily factorable, so we will use the quadratic formula:

$$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2} \approx 0.586, 3.414$$

Therefore, the particle is at rest at 0.586s and 3.414s.

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