Chapter 1

u-Substitution

U-Substitution, also known as the method of substitution, is a technique used to simplify the process of finding antiderivatives and integrals of complicated functions. The method is similar to the chain rule for differentiation in reverse.

Suppose we have an integral of the form:

$$\int f(g(x)) \cdot g'(x)\, dx \tag{1.1}$$

The u-substitution method suggests letting a new variable u be equal to the inside function $g(x)$, i.e.,

$$u = g(x) \tag{1.2}$$

Next, the differential of u, du, is given by:

$$du = g'(x)\, dx \tag{1.3}$$

Substituting u and du back into the integral gives us a simpler integral:

$$\int f(u)\, du \tag{1.4}$$

This new integral can often be simpler to evaluate. Once the antiderivative of $f(u)$ is found, we can substitute $u = g(x)$ back into the antiderivative to get the antiderivative of the original function in terms of x.

The method of u-substitution is a powerful tool for evaluating integrals, especially when combined with other techniques like integration by parts, partial fractions, and trigonometric substitutions.

Example: Find $\int 2x^2 \cos\left(x^3 - 3\right) dx$.

Solution: Integrating $\cos\left(x^3 - 3\right)$ isn't so straightforward, so let's try the substitution

$u = x^3 - 3$. Then:

$$du = 3x^2 \ dx$$

We don't have $3x^2$ in the integral, but we do have $2x^2$:

$$\frac{2}{3}du = 2x^2 \ dx$$

Substituting:

$$\int 2x^2 \cos\left(x^3 - 3\right) dx = \int \frac{2}{3}\cos(u)\, du$$
$$= \frac{2}{3}\sin(u) + C$$

Now that we have the antiderivative of $f(u)$, we can back-substitute in for u:

$$\frac{2}{3}\sin(u) + C = \frac{2}{3}\sin\left(x^3 - 3\right) + C$$

We can check our answer by taking its derivative: we should get the original integrand back:

$$\frac{d}{dx}\left[\frac{2}{3}\sin\left(x^3 - 3\right) + C\right] = \frac{2}{3}\cos\left(x^3 - 3\right) \cdot \left[\frac{d}{dx}\left(x^3 - 3\right)\right]$$
$$= \frac{2}{3}\cos\left(x^3 - 3\right) \cdot \left(3x^2\right) = 2x^2\cos\left(x^3 - 3\right)$$

Sometimes, the right substitution takes a little thinking. Consider the following example:

Example: Find $\int \sqrt{x^2 - 1}\ x^5\, dx$.

Solution: We can guess that $u = x^2 - 1$ could be an appropriate substitution, as that is what is under the square root. What to do with x^5? First, let's look at how the u-substitution for $x^2 - 1$ works out:

$$u = x^2 - 1$$
$$du = 2x \ dx$$
$$\frac{du}{2} = x \ dx$$

Then we will need to use one of the x's in x^5 for the square root u-substitution. What can we do with the remaining x^4? Well, we see that if $u = x^2 - 1$, then $u + 1 = x^2$ and therefore $(u+1)^2 = x^4$. Substituting this all in:

$$\int \sqrt{x^2 - 1}\ x^5\, dx = \int x^4 \sqrt{x^2 - 1}\ x\, dx$$

$$= \frac{1}{2} \int (u+1)^2 \sqrt{u} \, du$$

We can expand this to find the antiderivative:

$$= \frac{1}{2} \int \left(u^2 + 2u + 1\right) u^{1/2} \, du = \frac{1}{2} \int u^{5/2} + 2u^{3/2} + u^{1/2} \, du$$

$$= \frac{1}{2} \left[\frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C$$

$$= \frac{1}{7}u^{7/2} + \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{7}\left(x^2 - 1\right)^{7/2} + \frac{2}{5}\left(x^2 - 1\right)^{5/2} + \frac{1}{3}\left(x^2 - 1\right)^{3/2} + C$$

Exercise 1 Indefinite Integrals and u-substitution

Use u-substitution to evaluate the following indefinite integrals. Confirm your answer by taking the derivative of the result.

1. $\int \sin x \sqrt{1 + \cos x}\, dx$
2. $\int \frac{\cos(\pi/x)}{x^2}\, dx$
3. $\int 2x^2 \left(9 - x^3\right)^{2/3}\, dx$
4. $\int 3x^2 \sqrt{1 + x}\, dx$
5. $\int \frac{3x^2}{x^3 - 1}\, dx$

Working Space

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1.1 The Substitution Rule for Definite Integrals

How to we use u-substitution for definite integrals? We will apply the fundamental theorem of calculus to answer this question. We define f and F such that F is the antiderivative of f. Then:

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = F(g(x))|_a^b = F(g(b)) - F(g(a))$$

This represents the method of finding the indefinite antiderivative and evaluating from the original limits of integration.

We can also see that:

$$F(g(b)) - F(g(a)) = F(u)|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\, du$$

Therefore, if g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then:

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

This represents *changing the limits of integration* into the new variable, u, then evaluating the integral. While the second method is preferable, both methods yield the same answer.

Example: Evaluate $\int_0^5 \sqrt{3x+1}\, dx$ using both methods outlined above.

Solution: We start with the first method. We will use the substitution $u = 3x + 1$, and therefore $du/3 = dx$:

$$\int_0^5 \sqrt{3x+3}\, dx = \frac{1}{3}\int_{x=0}^{x=5} \sqrt{u}\, du$$

(We write the limits as $x = \cdots$ to remind us the limits are for x, not u.)

$$\frac{1}{3}\int_{x=0}^{x=5} \sqrt{u}\, du = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_{x=0}^{x=5}$$

Now we substitute back in for u and evaluate:

$$\frac{2}{9}(3x+1)^{3/2}\,|_0^5 = \frac{2}{9}(16)^{3/2} - \frac{2}{9}(1)^{3/2}$$

$$= \frac{2}{9}(64-1) = \frac{2 \cdot 63}{9} = 14$$

Let's compare this to the second, preferred method. We already know the u-substitution we'll make, so next we need to find $g(0)$ and $g(5)$ (recall that we choose u such that $u = g(x)$):

$$g(x) = 3x + 1$$

$$g(0) = 1$$

$$g(5) = 16$$

Now we can make our substitution *and* change the limits of integration:

$$\int_0^5 \sqrt{3x+1}\,dx = \frac{1}{3}\int_1^{16} \sqrt{u}\,du$$

$$= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^{16} = \frac{2}{9}\left[16^{3/2} - 1^{3/2}\right]$$

$$= \frac{2}{9}(64-1) = 14$$

With the second method, we get the same answer in fewer steps.

Exercise 2 Definite Integrals and u-substitution

Working Space

Use u-substitution to evaluate the following definite integrals.

1. $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$
2. $\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}}\,dx$
3. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$
4. $\int_0^{\pi/6} \frac{\sin x}{\cos^2 x}\,dx$
5. $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

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Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

1. Let $u = 1 + \cos x$. Then $du = -\sin x\ dx$ and $-du = \sin x\ dx$. Substituting:

$$\int \sin x\sqrt{1+\cos x}\,dx = \int -\sqrt{u}\,du = -\frac{2}{3}u^{3/2} + C$$

$$= -\frac{2}{3}\left(1+\cos x\right)^{3/2} + C$$

Taking the derivative:

$$\frac{d}{dx}\left[-\frac{2}{3}\left(1+\cos x\right)^{3/2} + C\right] = -\frac{2}{3}\left[\frac{d}{dx}\left(1+\cos x\right)^{3/2} + \frac{d}{dx}C\right]$$

$$= -\frac{2}{3}\left[\frac{3}{2}\left(1+\cos x\right)^{1/2}\cdot\frac{d}{dx}\left(1+\cos x\right)\right] = -1\sqrt{1+\cos x}\cdot\left(-\sin x\right) = \sin x\sqrt{1+\cos x}$$

2. Let $u = \pi/x$. Then $du = (-\pi/x^2)dx$ and $-du/\pi = (1/x^2)dx$. Substituting:

$$\int \frac{\cos\left(\pi/x\right)}{x^2}\,dx = -\frac{1}{\pi}\int \cos u\,du = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\left(\pi/x\right) + C$$

Taking the derivative:

$$\frac{d}{dx}\left[-\frac{1}{\pi}\sin\left(\pi/x\right) + C\right] = -\frac{1}{\pi}\left[\frac{d}{dx}\sin\left(\pi/x\right) + \frac{d}{dx}C\right]$$

$$= -\frac{1}{\pi}\left[\cos\left(\pi/x\right)\cdot\frac{d}{dx}\left(\frac{\pi}{x}\right)\right] = -\frac{1}{\pi}\left[\cos\left(\pi/x\right)\cdot\left(\frac{-\pi}{x^2}\right)\right]$$

$$= \frac{\cos\left(\pi/x\right)}{x^2}$$

3. Let $u = 9 - x^3$. Then $du = -3x^2\ dx$ and $-\frac{2}{3}du = 2x^2\ dx$. Substituting:

$$\int 2x^2\left(9-x^3\right)^{2/3}\ dx = -\frac{2}{3}\int (u)^{2/3}\ du$$

$$= -\frac{2}{3}\left[\frac{3}{5}u^{5/3} + C\right] = -\frac{2}{5}\left(9-x^3\right)^{5/3} + C$$

Taking the derivative:

$$\frac{d}{dx}\left[-\frac{2}{5}\left(9-x^3\right)^{5/3}+C\right]=-\frac{2}{5}\left[\frac{d}{dx}\left(9-x^3\right)^{5/3}\right]+\frac{d}{dx}C$$

$$=-\frac{2}{5}\left[\frac{5}{3}\left(9-x^3\right)^{2/3}\cdot\frac{d}{dx}\left(9-x^3\right)\right]=-\frac{2}{3}\left(9-x^3\right)^{2/3}\left(-3x^2\right)=2x^2\left(9-x^3\right)^{2/3}$$

4. Let $u=1+x$. Then $du=dx$. Additionally, $u-1=x$ and $x^2=(u-1)^2$. Substituting:

$$\int 3x^2\sqrt{1+x}\,dx=\int 3\left(u-1\right)^2\sqrt{u}\,du=3\int\left(u^2-2u+1\right)\sqrt{u}\,du$$

$$=3\int u^{5/2}-2u^{3/2}+u^{1/2}\,du=3\left[\frac{2}{7}u^{7/2}-\frac{2\cdot 2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C\right]$$

$$=\frac{6}{7}u^{7/2}-\frac{12}{5}u^{5/2}+2u^{3/2}+C$$

$$=\frac{6}{7}\left(1+x\right)^{7/2}-\frac{12}{5}\left(1+x\right)^{5/2}+2\left(1+x\right)^{3/2}+C$$

Taking the derivative:

$$\frac{d}{dx}\left[\frac{6}{7}\left(1+x\right)^{7/2}-\frac{12}{5}\left(1+x\right)^{5/2}+2\left(1+x\right)^{3/2}+C\right]$$

$$=\frac{6}{7}\left(\frac{7}{2}\left(1+x\right)^{5/2}\right)-\frac{12}{5}\left(\frac{5}{2}\left(1+x\right)^{3/2}\right)+2\left(\frac{3}{2}\left(1+x\right)^{1/2}\right)$$

$$=3\left(1+x\right)^{5/2}-6\left(1+x\right)^{3/2}+3\left(1+x\right)^{1/2}$$

$$=\left[3\left(1+x\right)^2-6\left(1+x\right)+3\right]\left(1+x\right)^{1/2}$$

$$=\left[3\left(1+2x+x^2\right)-6-6x+3\right]\sqrt{1+x}$$

$$=\left[3+6x+3x^2-6-6x+3\right]\sqrt{1+x}=3x^2\sqrt{1+x}$$

5. Let $u=x^3-1$. Then $du=3x^2\,dx$. Substituting:

$$\int\frac{3x^2}{x^3-1}\,dx=\int\frac{1}{u}\,du=\ln u+C$$

$$=\ln\left(x^3-1\right)+C$$

Taking the derivative:

$$\frac{d}{dx}\left[\ln\left(x^3-1\right)+C\right]=\frac{d}{dx}\ln\left(x^3-1\right)+\frac{d}{dx}C$$

$$=\frac{1}{x^3-1}\cdot\left[\frac{d}{dx}\left(x^3-1\right)\right]=\frac{3x^2}{x^3-1}$$

Answer to Exercise 2 (on page 6)

1. Let $g(x) = u = \sin x$. Then $du = \cos x\, dx$. Additionally, $g(0) = \sin 0 = 0$ and $g(\pi/2) = \sin(\pi/2) = 1$. Substituting and changing the limits of integration:

$$\int_0^{\pi/2} \cos x \sin(\sin x)\, dx = \int_0^1 \sin u\, du = -\cos u|_0^1 = \cos 0 - \cos 1 = 1 - \cos 1$$

2. Let $g(x) = u = 1 + 2x$. Then $du = 2\, dx$ and $\frac{du}{2} = dx$. Additionally, $g(0) = 1$ and $g(13) = 1 + 2(13) = 27$. Substituting and changing the limits of integration:

$$\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}}\, dx = \frac{1}{2}\int_1^{27} \frac{1}{\sqrt[3]{u^2}}\, du = \frac{1}{2}\int_1^{27} u^{-2/3}\, du$$

$$= \frac{1}{2}\left[3u^{1/3}\right]_1^{27} = \frac{3}{2}\left[\sqrt[3]{27} - \sqrt[3]{1}\right] = \frac{3}{2}(3-1) = 3$$

3. Let $g(x) = u = 1/x$. Then $du = (-1/x^2)dx$ and $-du = dx/x^2$. Additionally, $g(1) = 1$ and $g(2) = 1/2$. Substituting and changing the limits of integration:

$$\int_1^2 \frac{e^{1/x}}{x^2}\, dx = -\int_1^{1/2} e^u\, du = \int_{1/2}^1 e^u\, du$$

$$= e^u|_{1/2}^1 = e - \sqrt{e}$$

4. Let $g(x) = u = \cos x$. Then $du = -\sin x\, dx$ and $-du = \sin x\, dx$. Additionally, $g(0) = \cos 0 = 1$ and $g(\pi/6) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. Substituting and changing the limits of integration:

$$\int_0^{\pi/6} \frac{\sin x}{\cos^2 x}\, dx = -\int_1^{\sqrt{3}/2} \frac{1}{u^2}\, du = \int_{\sqrt{3}/2}^1 \frac{1}{u^2}\, du$$

$$= -\frac{1}{u}\Big|_{\sqrt{3}/2}^1 = \frac{2}{\sqrt{3}} - 1 = \frac{2\sqrt{3}-3}{3}$$

5. Let $g(x) = u = 1 + 2x$. Then $du = 2\, dx$ and $\frac{du}{2} = dx$. And if $u = 1 + 2x$, then $x = \frac{u-1}{2}$. Additionally, $g(0) = 1$ and $g(4) = 9$. Substituting and changing the limits of integration:

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\, dx = \frac{1}{2}\int_1^9 \frac{\frac{u-1}{2}}{\sqrt{u}}\, du = \frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}\, du$$

$$= \frac{1}{4}\int_1^9 \sqrt{u} - \frac{1}{\sqrt{u}}\, du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9$$

$$= \frac{1}{4}\left[\frac{2}{3}\left(9^{3/2} - 1^{3/2}\right) - 2\left(\sqrt{9} - \sqrt{1}\right)\right] = \frac{1}{4}\left[\frac{2}{3}(26) - 2(2)\right]$$

$$= \frac{1}{4}\left[\frac{52}{3} - 4\right] = \frac{1}{4}\left[\frac{52-12}{3}\right] = \frac{1}{4}\left[\frac{40}{3}\right] = \frac{10}{3}$$

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