Chapter 1

Implicit Differentiation

Implicit differentiation is a technique in calculus for finding the derivative of a relation defined implicitly (that is, a relation between variables x and y that is not explicitly solved for one variable in terms of the other).

1.1 Implicit Differentiation Procedure

Consider an equation that defines a relationship between x and y:

$$F(x, y) = 0$$

To find the derivative of y with respect to x, we differentiate both sides of this equation with respect to x, treating y as an implicit function of x:[1]

$$\frac{d}{dx}F(x, y) = \frac{d}{dx}0$$

Applying the chain rule during the differentiation on the left side of the equation gives:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Finally, we solve for $\frac{dy}{dx}$ to find the derivative of y with respect to x:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

This result is obtained using the implicit differentiation method.

[1] This $\frac{d}{dx}$ form of the derivative is the same as y' said as taking the derivative of y with respect to x

1.2 Example

Consider the equation of a circle with radius r:

$$x^2 + y^2 = r^2$$

First, we will find $\frac{dy}{dx}$ without implicit differentiation. Newxt, we will apply implicit differentiation to get the same result.

1.2.1 Without Implicit Differentiation

First, we need to rearrange the equation to solve for y:

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

We take the derivative of y by applying the Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2 \pm \sqrt{r^2 - x^2}} \cdot (-2x) = \frac{-x}{\pm\sqrt{r^2 - x^2}}$$

Notice the denominator of this fraction is the same as the solution we found for y, $y = \pm\sqrt{r^2 - x^2}$. So, we can also represent this as:

$$\frac{dy}{dx} = \frac{-x}{y}$$

1.2.2 With Implicit Differentiation

With implicit differentiation, we assume y is a function of x and apply the Chain Rule.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[r^2]$$

For x^2 and r^2, we take the derivative as we normally would.[2] For y^2, we apply the Chain Rule, as outlined above.[3]

$$2x + 2y\frac{dy}{dx} = 0$$

[2] The $\frac{d}{dx}$ part disappears when taking the derivative of x, as the derivative of x with respect to x is just regular differentiation.

[3] Applying the chain rule is only allowed as y is not the variable we were taking the derivative with respect to.

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{-x}{y}$$

, which is the same result as we found without implicit differentiation.

1.3 Folium of Descartes

It was relatively easy to rearrange the equation for a circle to solve for y, but that is not always the case. To help you understand this better, we will consider a famous curve known as the *Folium of Descartes*, given by the equation,

$$x^3 + y^3 = 3xy.$$

The word *folium* comes from the Latin word for "leaf," describing the curve's distinctive looped shape. The curve is named after René Descartes, a French mathematician who originally presented the curve as a challenge to fellow mathematician Pierre de Fermat, asking him to find the tangent line to the curve. Fermat was able to solve the problem with ease!

The Folium of Descartes (seen in Figure **??**) is historically significant because unlike simpler curves such as circles or parabolas, it exhibits more complex behavior, including a self-intersection at the origin (the curve crosses itself!). This complexity makes it difficult to describe the curve using a single equation of the form $y = f(x)$.

To explicitly solve the equation $x^3 + y^3 = 3xy$ for y requires multiple expressions to fully capture the entire curve. As a result, the familiar techniques used for functions defined explicitly in terms of x are no longer sufficient. Instead, to find the slope of the tangent line at a point on the folium, we must use *implicit differentiation*.

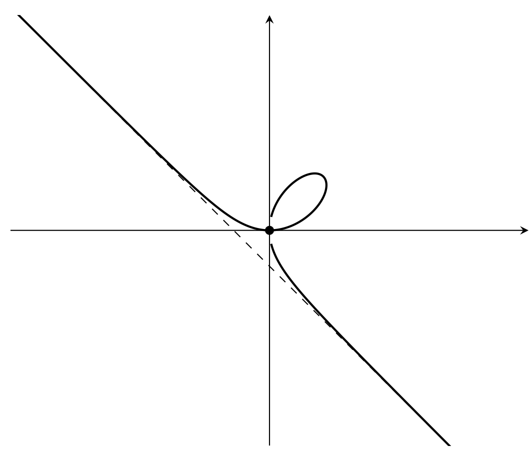

Figure 1.1: The Folium of Descartes $x^3 + y^3 = 3xy$ with asymptote $x + y + 1 = 0$.

1.3.1 Example: Tangent to Folium of Descartes

In this example, we will use implicit differentiation to easily find the tangent line at a point on the folium.

(a) Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

(b) Find the tangent to the folium $x^3 + y^3 = 3xy$ at the point $(2, 2)$

(c) Is there any place in the first quadrant where the tangent line is horizontal? If so, state the point(s).

Solution:

(a) $\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$
$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

Rearranging to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$
$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

(b) We already have the coordinate point, $(2, 2)$, so to write an equation for the tangent line, all we need is the slope. Substituting $x = 2$ and $y = 2$ into our result from part (a):

$$\frac{dy}{dx} = \frac{2 - 2^2}{2^2 - 2} = \frac{-2}{2} = -1$$

This is the slope, m. Using the point-slope form of a line, our tangent line is $y - 2 = -(x - 2)$.

(c) Recall that in the first quadrant, $x > 0$ and $y > 0$. We will set our solution for $\frac{dy}{dx}$ equal to 0:

$$\frac{y - x^2}{y + 2 - x} = 0$$

which implies that

$$y - x^2 = 0$$

Substituting $y = x^2$ into the original equation:

$$x^3 + (x^2)^3 = 3(x)(x^2)$$
$$x^3 + x^6 = 3x^3$$

Which simplifies to

$$x^6 = 2x^3$$

Since we have excluded $x = 0$ by restricting our search to the first quadrant, we can divide both sides by x^3:

$$x^3 = 2$$

$$x = \sqrt[3]{2} \approx 1.26$$

Substituting $x \approx 1.26$ into our equation for y:

$$y \approx 1.26^2 = 1.59$$

Therefore, the folium has a horizontal tangent line at the point $(1.26, 1.59)$.

1.4 Practice

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] If $\arcsin x = \ln y$, what is $\frac{dy}{dx}$?

Working Space

Answer on Page ??

Exercise 2

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] The points $(-1,-1)$ and $(1,-5)$ are on the graph of a function $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = x^2+y$. Use implicit differentiation to find $\frac{d^2y}{dx^2}$. Determine if each point is a local minimum, local maximum, or inflection point by substituting the x and y values of the coordinates into $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Working Space

Answer on Page ??

Appendix A

Answers to Exercises

Answer to Exercise ?? (on page ??)

Using implicit differentiation, we see that:

$$\frac{d}{dx}\arcsin x = \frac{d}{dx}\ln y$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{y}\frac{dy}{dx}$$

Multiplying both sides by y to isolate $\frac{dy}{dx}$, we find that:

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$$

Answer to Exercise ?? (on page ??)

First, we need to find $\frac{d^y}{dx^2}$:

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}y$$

$$= 2x + \frac{dy}{dx} = 2x + x^2 + y$$

At $(-1,-1)$, $\frac{dy}{dx} = (-1)^2 + (-1) = 0$ and $\frac{d^2y}{dx^2} = 2(-1) + (-1)^2 + (-1) = -2 < 0$. Since the slope of y is zero and the graph of y is concave down, $(-1,-1)$ is a local maximum. At $(1,-5)$, $\frac{dy}{dx} = 1^2 + -5 = -4 \neq 0$ and $\frac{d^2y}{dx^2} = 2(1) + 1^2 + (-5) = -2 \neq 0$. Since neither the first nor second derivative of y are zero, $(1,-5)$ is neither a local extrema nor an inflection point.

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