

CHAPTER 1

Rules for Finding Derivatives

Derivatives play a key role in calculus, providing us with a means of calculating rates of change and the slopes of curves. In this chapter, we present some common rules used to calculate derivatives.

1.1 Constant Rule

The derivative of a constant is zero. If c is a constant and x is a variable, then:

$$\frac{d}{dx}c = 0 \quad (1.1)$$

1.2 Power Rule

For any real number n , the derivative of x^n is:

$$\frac{d}{dx}x^n = nx^{n-1} \quad (1.2)$$

1.3 Product Rule

The derivative of the product of two functions is:

$$\frac{d}{dx}(fg) = f'g + fg' \quad (1.3)$$

where f' and g' denote the derivatives of f and g , respectively.

1.4 Quotient Rule

The derivative of the quotient of two functions is:

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{f'g - fg'}{g^2} \quad (1.4)$$

1.5 Chain Rule

The derivative of a composition of functions is:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \quad (1.5)$$

1.6 Practice

Exercise 1

If f is the function given, find f' .

Working Space

1. $f(x) = x \sin x$
2. $f(x) = (x^3 - \cos x)^5$
3. $f(x) = \sin^3 x$

Answer on Page 7

Exercise 2

Let $f(x) = 7x - 3 + \ln x$. Find $f'(x)$ and $f'(1)$.

Working Space

Answer on Page 7

Exercise 3

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] The position of a particle in the xy -plane is given by the parametric equations $x(t) = t^3 - 3t^2$ and $y(t) = 12t - 3t^2$. State a coordinate point (x, y) at which the particle is at rest.

Working Space*Answer on Page 7***Exercise 4**

Let $f(x) = \sqrt{x^2 - 4}$ and $g(x) = 3x - 2$. Find the derivative of $f(g(x))$ at $x = 3$.

Working Space*Answer on Page 7***Exercise 5**

The particle's position on the x -axis is given by $x(t) = (t - a)(t - b)$, where a and b are constants and $a \neq b$. At what time(s) is the particle at rest?

Working Space*Answer on Page 8*

Exercise 6

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] Let $f(x) = \frac{x}{x+2}$. At what values of x does f have the property that the line tangent to f has a slope of $\frac{1}{2}$?

Working Space**Answer on Page 8****Exercise 7**

For $t \geq 0$, the position of a particle moving along the x -axis is given by $x(t) = \sin t - \cos t$. (a) When does the velocity first equal 0? (b) What is the acceleration at the time when the velocity first equals 0?

Working Space**Answer on Page 8****Exercise 8**

The graph of $y = e^{\tan x} - 2$ crosses the x -axis at one point on the interval $[0, 1]$. What is the slope of the graph at this point?

Working Space**Answer on Page 9**

Exercise 9

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.

- (a) Find $f'(x)$.
(b) Write an equation for the line tangent to the graph at $x = -3$.

Working Space*Answer on Page 9***Exercise 10**

For $0 \leq t \leq 12$, a particle moves along the x -axis. The velocity of the particle at a time t is given by $v(t) = \cos \frac{\pi}{6}t$. What is the acceleration of the particle at time $t = 4$?

Working Space*Answer on Page 10***Exercise 11**

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let f and g be the functions given by $f(x) = e^x$ and $g(x) = x^4$. On what intervals is the rate of change of $f(x)$ greater than the rate of change of $g(x)$?

Working Space*Answer on Page 10*

1.7 Conclusion

These rules form the basis for calculating derivatives in calculus. Many more complex rules and techniques are built upon these fundamental rules.

This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.

APPENDIX A

Answers to Exercises

Answer to Exercise 1 (on page 2)

1. $\frac{dy}{dx} = \frac{d}{dx}[x \sin x] = x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x = x(-\cos x) + \sin x(1) = \sin x - x \cos x$
2. By the chain rule, $f'(x) = 5(x^3 - \cos x)^4 \cdot \frac{d}{dx}(x^3 - \cos x) = 5(x^3 - \cos x)^4 \cdot (3x^2 + \sin x)$
3. By the chain rule, $f'(x) = \frac{d}{d(\sin x)}[\sin^3 x] \times \frac{d}{dx} \sin x = 3 \sin^2 x \cdot \cos x$

Answer to Exercise 2 (on page 2)

$$f'(x) = \frac{d}{dx}(7x) - \frac{d}{dx}(3) + \frac{d}{dx}(\ln x) = 7 - 0 + \frac{1}{x} = 7 - \frac{1}{x} \text{ and } f'(1) = 7 - \frac{1}{1} = 6$$

Answer to Exercise 3 (on page 3)

The particle is at rest when $x'(t) = y'(t) = 0$. First, we find each of the derivatives:

$$\begin{aligned}x'(t) &= 3t^2 - 6t \\y'(t) &= 12 - 6t\end{aligned}$$

We can solve $y' = 0$ for t and find that the y -velocity is 0 when $t = 2$. Substituting $t = 2$ into our expression for x' , we find $x'(2) = 3(2)^2 - 6(2) = 0$. Therefore, the particle is at rest when $t = 0$. To find the xy -coordinate, we substitute $t = 2$ into $x(t)$ and $y(t)$:

$$\begin{aligned}x(2) &= (2)^3 - 3(2)^2 = 8 - 12 = -4 \\y(2) &= 12(2) - 6(2) = 24 - 12 = 12\end{aligned}$$

Therefore, the particle is at rest when it is located at $(-4, 12)$.

Answer to Exercise 4 (on page 3)

$f(g(x)) = \sqrt{(3x - 2)^2 - 4} = \sqrt{9x^2 - 12x}$ and $\frac{d}{dx} f(g(x)) = \frac{18x - 12}{2\sqrt{9x^2 - 12x}}$. Substituting $x = 3$, we find $f'(g(x)) = \frac{18(3) - 12}{2\sqrt{9(3)^2 - 12(3)}} = \frac{42}{2\sqrt{45}} = \frac{21}{3\sqrt{5}} = \frac{7}{\sqrt{5}}$

Answer to Exercise 5 (on page 3)

First, recall that the velocity of a particle is the derivative of its position function. Therefore, $v(t) = x'(t) = \frac{d}{dt}[(t-a)(t-b)]$. Applying the Product Rule for derivatives, we see that $v(t) = (t-a)(1) + (t-b)(1) = 2t - a - b$. To find the time(s) when the particle is at rest, we set $v(t) = 0$ and solve for t .

$$0 = 2t - a - b$$

$$2t = a + b$$

$$t = \frac{a+b}{2}$$

Answer to Exercise 6 (on page 4)

The question is asking when the derivative of f is $\frac{1}{2}$. We will take the derivative and set it equal to $\frac{1}{2}$.

$$\begin{aligned} f'(x) &= \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{2}{(x+2)^2} \\ \frac{2}{(x+2)^2} &= \frac{1}{2} \\ 4 &= (x+2)^2 \\ \pm 2 &= x+2 \\ x = 2 - 2 &= 0 \text{ and } x = -2 - 2 = -4 \end{aligned}$$

Answer to Exercise 7 (on page 4)

(a) Let t_0 be the time at which the particle is first at rest. The velocity of the particle is given by $v(t) = x'(t) = \cos t + \sin t$. Setting $v(t) = 0$, we find:

$$\cos t = -\sin t$$

which is true for $t = \frac{3\pi+4n}{4}$, where n is an integer. Therefore, the first time the velocity is 0 is $t_0 = \frac{3\pi}{4}$.

(b) To find the acceleration at $t = \frac{3\pi}{4}$, we take the derivative of the velocity function to yield the acceleration function.

$$a(t) = v'(t) = -\sin t + \cos t$$

. Substituting $t = \frac{3\pi}{4}$, we find the acceleration is $-\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4} = \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

Answer to Exercise 8 (on page 4)

First, we find the x such that $y = 0$

$$0 = e^{\tan x} - 2$$

$$2 = e^{\tan x}$$

$$\ln 2 = \tan x$$

$$x = \arctan(\ln 2) = \arctan 0.693 \approx 0.606$$

Then, we find the slope of the function at $x = 0.606$ by finding $y'(0.606)$

$$y' = e^{\tan x} (\sec x)^2 = \frac{e^{\tan x}}{(\cos x)^2}$$

$$y'(0.606) = \frac{e^{\tan 0.606}}{(\cos 0.606)^2} = 2.961$$

Answer to Exercise 9 (on page 5)

(a) Apply the chain rule to find $f'(x)$

$$f'(x) = \frac{1}{2\sqrt{25-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{25-x^2}}$$

(b) First, substitute $x = -3$ into $f'(x)$

$$f'(-3) = \frac{-(-3)}{\sqrt{25 - (-3)^2}} = \frac{3}{\sqrt{16}} = \frac{3}{4}$$

This is the slope of the line. To complete an equation for the tangent line, we need a point. We know the tangent line touches $f(x)$ at $x = -3$, so the tangent line must pass through the point $(-3, f(-3))$.

$$f(-3) = \sqrt{25 - (-3)^2} = 4$$

We use $m = \frac{3}{4}$ and the coordinate point $(x_1, y_1) = (-3, 16)$ to complete the equation $y - y_1 = m(x - x_1)$

$$y - 16 = \frac{3}{4}(x + 3)$$

Answer to Exercise 10 (on page 5)

$$\begin{aligned}a(t) &= v'(t) = -\frac{\pi}{6} \sin \frac{\pi}{6} t \\a(4) &= -\frac{\pi}{6} \sin \frac{2\pi}{3} = -\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{12}\end{aligned}$$

Answer to Exercise 11 (on page 5)

Recall that the rate of change of a function is given by the derivative of that function. Therefore, we are looking for the interval(s) where $f'(x) > g'(x)$. First, we find each derivative:

$$f'(x) = e^x$$

$$g'(x) = 4x^3$$

We are looking for x -values, such that $e^x > 4x^3$. This inequality can be restated as $e^x - 4x^3 > 0$. Using a calculator, you should find that $e^x - 4x^3 = 0$ when $x \approx 0.831$ and $x \approx 7.384$. We will check values on either side of and in the interval $x \in (0.831, 7.384)$ to determine the sign value of $e^x - 4x^3$. We know that when $x = 0$, $e^x - 4x^3 > 0$, when $x = 5$, $e^x - 4x^3 < 0$, and when $x = 10$, $e^x - 4x^3 > 0$. Therefore, $f'(x)$ is greater than $g'(x)$ on the open intervals $x \in (-\infty, 0.831) \cup (7.384, \infty)$.



INDEX

chain rule, [2](#)

constant rule, [1](#)

power rule, [1](#)

product rule, [1](#)

quotient rule, [1](#)