Chapter 1

Introduction to Sequences

A sequence is a list of numbers in a particular order. {1, 3, 5, 7, 9} is a sequence. So is $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$. There are many types of sequences. We will present two of the most common types in this chapter: arithmetic and geometric sequences.

Sequences are generally represented like this:

$$a_1, a_2, a_3, a_4, ..., a_n, ...$$

The first number, a_1, is called the *first term*, a_2 is the *second term*, and a_n is the *nth term*. A sequence can be finite or infinite. If the sequence is infinite, we represent that with ellipses $(\cdots)$ at the end of the list, to indicate that there are more numbers.

We can also write formulas to represent a sequence. Take the first example, the finite sequence {1, 3, 5, 7, 9}. Notice that each term is two more than the previous term. We can define the sequence *recursively* by defining the n^{th} term as a function of the $(n-1)^{th}$ term. In our example, we see that $a_n = a_{n-1} + 2$ with $a_1 = 1$ for $1 \leq n \leq 5$. This is called a recursive formula, because you have to already know the $(n-1)^{th}$ term to find the n^{th} term.

Another way to write a formula for a sequence is to find a rule for the n^{th} term. In our example sequence, the first term is 1 plus 0 times 2, the second term is 1 plus 1 times 2, the third term is 1 plus 2 times 2, and so one. Did you notice the pattern? The n^{th} term is 1 plus (n-1) times 2. We can write this mathematically:

$$a_n = 1 + 2(n-1) \text{ for } 1 \leq n \leq 5$$

This is called the *explicit* formula because each term is explicitly defined. Notice, for the second way of writing a formula, we don't have to state what the first term is: the formula tells us.

1.1 Arithemtic sequences

Our first example sequence, {1, 3, 5, 7, 9} is a *finite*, *arithmetic* sequence. We know it is finite because there is a limited number of terms in the sequence (in this case, 5). How do we know it is arithmetic?

An arithmetic sequence is one where you add the same number every time to get the next term. Our example is an arithmetic sequence because you add 2 to get the next term every time. That number that you add is called the *common difference*, so we can say the sequence {1, 3, 5, 7, 9} has a common difference of 2. The common difference can be positive (in the case of an increasing arithmetic sequence) or negative (in the case of a decreasing arithmetic sequence). Formally, we can find the common difference of an arithmetic sequence by subtracting the $(n-1)^{th}$ term from the n^{th} term:

$$d = a_n - a_{n-1}$$

Exercise 1

Which of the following are arithmetic sequences? For the arithmetic sequences, find the common difference.

1. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, ...\}$
2. {5, 8, 11, 14, 17, ...}
3. {3, -1, -5, -9, ...}
4. {-1, 2, -3, 4, -5, 6, ...}

Working Space

Answer on Page 9

1.1.1 Formulas for arithmetic sequences

If you are given an arithmetic sequence, you can write an explicit or recursive formula. You can think of the formula as a function where the domain (input) is restricted to integers greater than or equal to one. Let's write explicit and recursive formulas for the sequence {3, -1, -5, -9, ...}.

For either type of formula, we need to identify the common difference. Since each term is 4 less than the previous term, the common difference is -4 (see figure 1.1). This means the n^{th} term is the $(n-1)^{th}$ term minus 4. The general form of a recursive formula is $a_n = a_{n-1} + d$, where d is the common difference. For our example, the common difference is -4, so we can write a recursive formula:

$$a_n = a_{n-1} - 4$$

However, this formula doesn't tell us what a_1 is! For recursive formulas, you have to specify the first term in the sequence. So the *complete* recursive formula for the sequence is:

$$a_n = a_{n-1} - 4$$
$$a_1 = 3$$

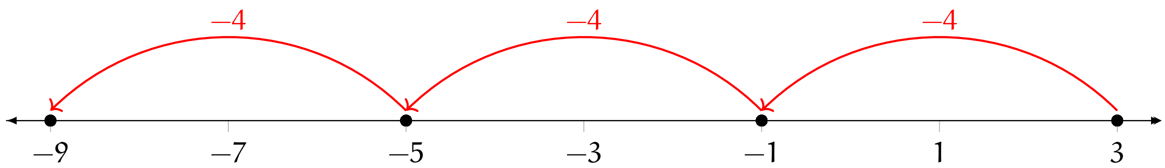

Figure 1.1: The common difference in the sequence {3, -1, -5, -9,...} is -4

Recursive formulas make it easy to see how each term is related to the next term. However, it is difficult to use recursive formulas to find a specific term. Say we wanted to know the 7^{th} term in the sequence. Well, from the formula, we know that

$$a_7 = a_6 - 4$$

What is a_6? Again, we see that

$$a_6 = a_5 - 4$$

Now we have to find a_5! If we keep going we see that:

$$a_5 = a_4 - 4$$
$$a_4 = a_3 - 4$$
$$a_3 = a_2 - 4$$
$$a_2 = a_1 - 4$$

Since we were told a_1, we can find a_2 and propagate our terms back up the chain to find a_7:

$$a_2 = 3 - 4 = -1$$
$$a_3 = a_2 - 4 = -1 - 4 = -5$$
$$a_4 = a_3 - 4 = -5 - 4 = -9$$
$$a_5 = a_4 - 4 = -9 - 4 = -13$$
$$a_6 = a_5 - 4 = -13 - 4 = -17$$
$$a_7 = a_6 - 4 = -17 - 4 = -21$$

So finally, we see that $a_7 = -21$. That was a lot of work! You can imagine that for higher n terms, such as the 100^{th} or 1000^{th} term, this method becomes cumbersome. This is where the explicit formula is more useful.

The general form of an explicit formula for an arithmetic sequence is

$$a_n = a_1 + d \times (n-1)$$

where d is the common difference. For our example sequence, {3, -1, -5, -9, ...}, the common difference is -4. So the explicit formula is

$$a_n = 3 + (-4)(n-1) = 3 - 4(n-1)$$

You may be tempted to distribute and simplify, which is fine and yields an equivalent formula:

$$a_n = 7 - 4n$$

Now, to find the 7^{th} term, all we have to do is substitute $n = 7$:

$$a_7 = 3 - 4(7-1) = 3 - 4(6) = 3 - 24 = -21$$

We get the same answer with much less effort!

Exercise 2

An arithmetic sequence is defined by the recursive formula $a_n = a_{n-1} + 5$ with $a_1 = -4$. Write the first 5 terms of the sequence and determine an explicit formula for the same sequence.

Working Space

Answer on Page 9

Exercise 3

The first four terms of an arithmetic sequence are $\{\pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}\}$. What is the common difference? Write explicit and recursive formulas for the infinite sequence.

Working Space

Answer on Page 9

1.2 Geometric sequences

Let's look at the other sequence given as an example at the beginning of the chapter: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$. How is each term related to the previous term? Well, $\frac{1}{4}$ is half of $\frac{1}{2}$, and $\frac{1}{8}$ is half of $\frac{1}{4}$, so each term is the previous term multiplied by $\frac{1}{2}$. When each term in a sequence is a multiple of the previous term, this is a *geometric* sequence. The number we multiply by each time (in our example, this is $\frac{1}{2}$) is called the *common ratio*. The common ratio can be positive or negative, but not zero.

An easy way to determine the common ratio (r) is to divide the n^{th} term by the $(n-1)^{th}$ term. In our example sequence, the first term is $\frac{1}{2}$ and the second is $\frac{1}{4}$.

$$r = \frac{a_2}{a_1} = \frac{1/4}{1/2} = \frac{1}{2}$$

which returns the common ratio we already identified, $r = \frac{1}{2}$.

If the common ratio is negative, then the sequence will "flip" back and forth from positive to negative. For example suppose there is a geometric sequence such that $a_1 = 1$ and $r = -2$. Then the first 5 terms are {1, -2, 4, -8, 16}. Whenever you see a sequence going back and forth from positive to negative, that means the common ratio is negative.

For positive common ratios, if $r > 1$, then the sequence is increasing. And if $r < 1$, the sequence is decreasing.

1.2.1 Formulas for geometric sequences

Like arithmetic sequences, we can write recursive and explicit formulas. For geometric sequences, the recursive formula has the general form:

$$a_n = r(a_{n-1})$$

where r is the common ratio and a_1 is specified. For our example sequence, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$, the recursive formula is:

$$a_n = \frac{1}{2}a_{n-1}$$

$$a_1 = \frac{1}{2}$$

In a geometric sequence, each term is the first term, a_1, multiplied by the common ratio, r, $n-1$ times. Therefore, the general form of an explicit formula for a geometric function is:

$$a_n = (a_1)r^{n-1}$$

Again, for our example sequence, $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$, so the explicit formula is:

$$a_n = (\frac{1}{2})(\frac{1}{2})^{(n-1)}$$

Exercise 4

Which of the following are geometric sequences? For each geometric sequence, determine the common ratio.

1. {2, -4, 6, -8,...}
2. {4, 2, 1, $\frac{1}{2}$, ...}
3. {-5, 25, -125, 525, ...}
4. {2, 0, -2, -4, ...}

Working Space

Answer on Page 9

Exercise 5

A geometric sequence is defined by the recursive formula $a_n = a_{n-1} \times \frac{3}{2}$ with $a_1 = 1$. Write the first 5 terms of the sequence and determine an explicit formula for the same sequence.

Working Space

Answer on Page 10

Exercise 6

The first four terms of a geometric sequence are {-4, 2, -1, $\frac{1}{2}$}. What is the common ratio? Write recursive and explicit formulas for the infinite sequence.

Working Space

Answer on Page 10

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

1. not arithmetic
2. arithmetic, common difference is 3
3. arithmetic, common difference is -4
4. not arithmetic

Answer to Exercise 2 (on page 4)

The first five terms are {-4, 1, 6, 11, 16} and an explicit formula is $a_n = -4 + 5(n - 1)$.

Answer to Exercise 3 (on page 4)

The common difference is $\frac{3\pi}{2} - \pi = \frac{\pi}{2}$. The recursive formula is $a_n = a_{n-1} + \frac{\pi}{2}$ with $a_1 = \pi$. The explicit formula is $a_n = \pi + \frac{\pi}{2}(n - 1)$.

Answer to Exercise 4 (on page 6)

1. not geometric
2. geometric sequence with common ratio $r = \frac{1}{2}$
3. geometric sequence with common ration $r = -5$
4. not geometric

Answer to Exercise 5 (on page 6)

The first 5 terms are $\{1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}\}$. An explicit formula for this sequence is $a_n = 1(\frac{3}{2})^{(n-1)}$.

Answer to Exercise 6 (on page 7)

The common ratio is $\frac{a_n}{a_{n-1}} = \frac{2}{-4} = -\frac{1}{2}$. A recursive formula would be $a_n = a_{n-1} \times -\frac{1}{2}$ with $a_1 = -4$. An explicit formula would be $a_n = (-4)(-\frac{1}{2})^{(n-1)}$.