Beginning Combinatorics

Discrete probability problems often include some counting. For example, we figured out that there were 36 different ways the two dice could land, but all of them summed to some number 2 through 12. How many different ways could three 8-sided dice come up? We would need to count them, right? As the numbers get bigger, we will need some tricks so that we don't need to write them all down and count them one-by-one.

The branch of mathematics that focuses on tricks for counting is called *combinatorics*.

How can we be sure that there were 36 different configurations for the two 6-sided dice? The first die could have come up as any one of six numbers. For each of those, the second could have come up with any one of six numbers. Thus, the number of possibilities is $6 \times 6 = 36$.

How many different configurations for three 8-sided dice? $8 \times 8 \times 8 = 8^3 = 512$.

What about seven dice, each with 20 sides? There would be $20^7 = 1,280,000,000$ configurations. See, aren't you glad we don't need to write them all down?

Now, let's say that six people (Anne, Brock, Carl, Dev, Edgar, and Fred) are going to run a race. You have to make a plaque that says who won first place, who won second place, and who won third. If you want to get all the possible plaques created beforehand, and just pull the right one out as soon as the race ends, how many plaques would you need to get engraved?

In this case, once someone has been given first place, they can't win second or third place. Thus, any of the 6 people can come in first, but once you have engraved that person's name on the plaque, there are only 5 people whose names can appear in second place. Once you have engraved that name, there are only 4 people whose names can appear in third place. Thus, you would get $6 \times 5 \times 4 = 120$ plaques engraved.

What if the plaque includes all 6 places? Then you would need $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ plaques engraved. We use this process often enough that we gave it a name. We say "I need 6 factorial plaques engraved." When we write a *factorial*, we use an exclamation point:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

We use the word "permutation" to mean a particular ordering. This rule says n items

can be ordered in $n!$ ways. Thus, mathematicians actually say "If you have a list of n items then we can generate $n!$ different permutations of those items". We will discuss this further in the next chapter.

In Python, there is a `factorial` function in the math library:

```
> python3
>>> import math
>>> math.factorial(6)
720
```

Handy, right? Now you don't need to write a loop to calculate factorials.

Remember when we only wanted the first three names on the plaque? We can solve that problem using factorials:

$$6 \times 5 \times 4 = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{6!}{3!}$$

This formulation makes it easy to figure out on any calculator with a "!" button.

The rule on this is to fill m positions from n items, it can be done this many ways:

Permutations

For permutations, where order is a factor, the formula for ^nP_m (n objects pick m)

$$\frac{n!}{(n-m)!}$$

1.0.1   Choose

Let's say that there are 12 kids in a classroom, and you need a team of 4 to wipe down the desks. How many different possible teams are there? You know that if you were giving out four different positions (Like the race gave out 1st, 2nd, and 3rd), the answer would be $12 \times 11 \times 10 \times 9$ or $12!/(12-4)!$.

However, once we pick the 4 people, we don't care what order they are in, right? In this problem, the team "Anne, Brad, Carl, and Don" is the same as the team "Carl, Don, Brad, and Anne".

Thus, the quantity $12!/(12-4)!$ is many times too large, because it counts each permutation separately. To get the right number, we just divide this by the number of possible permuations for a group of four people: $4!$

That gets us our answer: How many different teams of four can be chosen from 12 people?

$$\frac{12!}{(12-4)!4!} = 495$$

Generally, this formula, referred to as the Binomial coefficient. In combinatorics, we use this quantity a lot, so we have given it a name: *choose*

Combinations

For combinations, where order doesn't matter, the formula for ^nC_k (n objects choose k)

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We have also given it a notation. "12 choose 4" is written like this:

$$\binom{12}{4}$$

Python has the `math.comb` function:

```
> python3
>>> import math
>>> comb(12, 4)
495
```

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Answers to Exercises

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