Chapter 1

Implicit Differentiation

Implicit differentiation is a technique in calculus for finding the derivative of a relation defined implicitly, that is, a relation between variables x and y that is not explicitly solved for one variable in terms of the other.

1.1 Implicit Differentiation Procedure

Consider an equation that defines a relationship between x and y:

$$F(x, y) = 0$$

To find the derivative of y with respect to x, we differentiate both sides of this equation with respect to x, treating y as an implicit function of x:

$$\frac{d}{dx}F(x, y) = \frac{d}{dx}0$$

Applying the chain rule during the differentiation on the left side of the equation gives:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Finally, we solve for $\frac{dy}{dx}$ to find the derivative of y with respect to x:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

This result is obtained using the implicit differentiation method.

1.2 Example

Consider the equation of a circle with radius r:

$$x^2 + y^2 = r^2$$

First, we'll find $\frac{dy}{dx}$ the without implicit differentiation. Then, we'll apply implicit differentiation to get the same result.

1.2.1 Without Implicit Differentiation

First, we need to re-arrange the equation to solve for y:

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

We take the derivative of y by applying the Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2 \pm \sqrt{r^2 - x^2}} \cdot (-2x) = \frac{-x}{\pm\sqrt{r^2 - x^2}}$$

Notice the denominator of this fraction is the same as the solution we found for y, $y = \pm\sqrt{r^2 - x^2}$. So we can also represent this as:

$$\frac{dy}{dx} = \frac{-x}{y}$$

1.2.2 With Implicit Differentiation

With implicit differentiation, we assume y is a function of x and apply the Chain Rule.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[r^2]$$

For x^2 and r^2, we take the derivative as we normally would. For y^2, we apply the Chain Rule as outlined above.

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{-x}{y}$$

which is the same result as we found without implicit differentiation.

1.3 Folium of Descartes

It was relatively easy to rearrange the equation for a circle to solve for y, but that is not always the case. Consider the equation for the folium of Descartes (yes, that Descartes!):

$$x^3 + y^3 = 3xy$$

It is much more difficult to isolate y in this equation. In fact, were we to do so, we would need 3 separate equations to completely describe the original equation.

1.3.1 Example: Tangent to Folium of Descartes

In this example, we will use implicit differentiation to easily find the tangent line at a point on the folium.

(a) Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

(b) Find the tangent to the folium $x^3 + y^3 = 3xy$ at the point $(2, 2)$

(c) Is there any place in the first quadrant where the tangent line is horizontal? If so, state the point(s).

Solution:

(a) $\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$
$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

Rearranging to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$
$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

(b) We already have the coordinate point, $(2, 2)$, so to write an equation for the tangent line all we need is the slope. Substituting $x = 2$ and $y = 2$ into our result from part (a):

$$\frac{dy}{dx} = \frac{2 - 2^2}{2^2 - 2} = \frac{-2}{2} = -1$$

This is the slope, m. Using the point-slope form of a line, our tangent line is $y - 2 = -(x - 2)$.

(c) Recall that in the first quadrant, $x > 0$ and $y > 0$. We will set our solution for $\frac{dy}{dx}$ equal to 0:

$$\frac{y - x^2}{y + 2 - x} = 0$$

which implies that

$$y - x^2 = 0$$

Substituting $y = x^2$ into the original equation:

$$x^3 + (x^2)^3 = 3(x)(x^2)$$

$$x^3 + x^6 = 3x^3$$

Which simplifies to

$$x^6 = 2x^3$$

Since we have excluded $x = 0$ by restricting our search to the first quadrant, we can divide both sides by x^3:

$$x^3 = 2$$

$$x = \sqrt[3]{2} \approx 1.26$$

Substituting $x \approx 1.26$ into our equation for y:

$$y \approx 1.26^2 = 1.59$$

Therefore, the folium has a horizontal tangent line at the point $(1.26, 1.59)$.

1.4 Practice

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] If $\arcsin x = \ln y$, what is $\frac{dy}{dx}$?

Working Space

Answer on Page 7

Exercise 2

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] The points $(-1,-1)$ and $(1,-5)$ are on the graph of a function $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = x^2 + y$. Use implicit differentiation to find $\frac{d^2y}{dx^2}$. Determine if each point is a local minimum, local maximum, or inflection point by substituting the x and y values of the coordinates into $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Working Space

Answer on Page 7

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

Using implicit differentiation, we see that:

$$\frac{d}{dx}\arcsin x = \frac{d}{dx}\ln y$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{y}\frac{dy}{dx}$$

Multiplying both sides by y to isolate $\frac{dy}{dx}$, we find that:

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$$

Answer to Exercise 2 (on page 5)

First, we need to find $\frac{d^y}{dx^2}$:

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}y$$

$$= 2x + \frac{dy}{dx} = 2x + x^2 + y$$

At $(-1,-1)$, $\frac{dy}{dx} = (-1)^2 + (-1) = 0$ and $\frac{d^2y}{dx^2} = 2(-1) + (-1)^2 + (-1) = -2 < 0$. Since the slope of y is zero and the graph of y is concave down, $(-1,-1)$ is a local maximum. At $(1,-5)$, $\frac{dy}{dx} = 1^2 + -5 = -4 \neq 0$ and $\frac{d^2y}{dx^2} = 2(1) + 1^2 + (-5) = -2 \neq 0$. Since neither the first nor second derivative of y are zero, $(1,-5)$ is neither a local extrema nor an inflection point.

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