

## CHAPTER 1

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# Rules for Finding Derivatives

Derivatives play a key role in calculus, providing us with a means of calculating rates of change and the slopes of curves. In this chapter, we present some common rules used to calculate derivatives.

### 1.1 Constant Rule

The derivative of a constant is zero. If  $c$  is a constant and  $x$  is a variable, then:

$$\frac{d}{dx}c = 0 \quad (1.1)$$

### 1.2 Power Rule

For any real number  $n$ , the derivative of  $x^n$  is:

$$\frac{d}{dx}x^n = nx^{n-1} \quad (1.2)$$

### 1.3 Product Rule

The derivative of the product of two functions is:

$$\frac{d}{dx}(fg) = f'g + fg' \quad (1.3)$$

where  $f'$  and  $g'$  denote the derivatives of  $f$  and  $g$ , respectively.

### 1.4 Quotient Rule

The derivative of the quotient of two functions is:

$$\frac{d}{dx} \left( \frac{f}{g} \right) = \frac{f'g - fg'}{g^2} \quad (1.4)$$

## 1.5 Chain Rule

The derivative of a composition of functions is:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \quad (1.5)$$

## 1.6 Practice

### Exercise 1

If  $f$  is the function given, find  $f'$ .

Working Space

1.  $f(x) = x \sin x$
2.  $f(x) = (x^3 - \cos x)^5$
3.  $f(x) = \sin^3 x$

Answer on Page ??

### Exercise 2

Let  $f(x) = 7x - 3 + \ln x$ . Find  $f'(x)$  and  $f'(1)$ .

Working Space

Answer on Page ??

**Exercise 3**

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] The position of a particle in the  $xy$ -plane is given by the parametric equations  $x(t) = t^3 - 3t^2$  and  $y(t) = 12t - 3t^2$ . State a coordinate point  $(x, y)$  at which the particle is at rest.

**Working Space***Answer on Page ??***Exercise 4**

Let  $f(x) = \sqrt{x^2 - 4}$  and  $g(x) = 3x - 2$ . Find the derivative of  $f(g(x))$  at  $x = 3$ .

**Working Space***Answer on Page ??***Exercise 5**

The particle's position on the  $x$ -axis is given by  $x(t) = (t - a)(t - b)$ , where  $a$  and  $b$  are constants and  $a \neq b$ . At what time(s) is the particle at rest?

**Working Space***Answer on Page ??*

**Exercise 6**

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] Let  $f(x) = \frac{x}{x+2}$ . At what values of  $x$  does  $f$  have the property that the line tangent to  $f$  has a slope of  $\frac{1}{2}$ ?

**Working Space***Answer on Page ??***Exercise 7**

For  $t \geq 0$ , the position of a particle moving along the  $x$ -axis is given by  $x(t) = \sin t - \cos t$ . (a) When does the velocity first equal 0? (b) What is the acceleration at the time when the velocity first equals 0?

**Working Space***Answer on Page ??***Exercise 8**

The graph of  $y = e^{\tan x} - 2$  crosses the  $x$ -axis at one point on the interval  $[0, 1]$ . What is the slope of the graph at this point?

**Working Space***Answer on Page ??*

**Exercise 9**

The function  $f$  is defined by  $f(x) = \sqrt{25 - x^2}$  for  $-5 \leq x \leq 5$ .

- (a) Find  $f'(x)$ .  
(b) Write an equation for the line tangent to the graph at  $x = -3$ .

**Working Space***Answer on Page ??***Exercise 10**

For  $0 \leq t \leq 12$ , a particle moves along the  $x$ -axis. The velocity of the particle at a time  $t$  is given by  $v(t) = \cos \frac{\pi}{6}t$ . What is the acceleration of the particle at time  $t = 4$ ?

**Working Space***Answer on Page ??***Exercise 11**

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let  $f$  and  $g$  be the functions given by  $f(x) = e^x$  and  $g(x) = x^4$ . On what intervals is the rate of change of  $f(x)$  greater than the rate of change of  $g(x)$ ?

**Working Space***Answer on Page ??*

## 1.7 Conclusion

These rules form the basis for calculating derivatives in calculus. Many more complex rules and techniques are built upon these fundamental rules.

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*This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.*

## APPENDIX A

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# Answers to Exercises

### Answer to Exercise ?? (on page ??)

1.  $\frac{dy}{dx} = \frac{d}{dx}[x \sin x] = x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x = x(-\cos x) + \sin x(1) = \sin x - x \cos x$
2. By the chain rule,  $f'(x) = 5(x^3 - \cos x)^4 \cdot \frac{d}{dx}(x^3 - \cos x) = 5(x^3 - \cos x)^4 \cdot (3x^2 + \sin x)$
3. By the chain rule,  $f'(x) = \frac{d}{d(\sin x)}[\sin^3 x] \times \frac{d}{dx} \sin x = 3 \sin^2 x \cdot \cos x$

### Answer to Exercise ?? (on page ??)

$$f'(x) = \frac{d}{dx}(7x) - \frac{d}{dx}(3) + \frac{d}{dx}(\ln x) = 7 - 0 + \frac{1}{x} = 7 - \frac{1}{x} \text{ and } f'(1) = 7 - \frac{1}{1} = 6$$

### Answer to Exercise ?? (on page ??)

The particle is at rest when  $x'(t) = y'(t) = 0$ . First, we find each of the derivatives:

$$x'(t) = 3t^2 - 6t$$

$$y'(t) = 12 - 6t$$

We can solve  $y' = 0$  for  $t$  and find that the  $y$ -velocity is 0 when  $t = 2$ . Substituting  $t = 2$  into our expression for  $x'$ , we find  $x'(2) = 3(2)^2 - 6(2) = 0$ . Therefore, the particle is at rest when  $t = 0$ . To find the  $xy$ -coordinate, we substitute  $t = 2$  into  $x(t)$  and  $y(t)$ :

$$x(2) = (2)^3 - 3(2)^2 = 8 - 12 = -4$$

$$y(2) = 12(2) - 6(2) = 24 - 12 = 12$$

Therefore, the particle is at rest when it is located at  $(-4, 12)$ .

### Answer to Exercise ?? (on page ??)

$f(g(x)) = \sqrt{(3x - 2)^2 - 4} = \sqrt{9x^2 - 12x}$  and  $\frac{d}{dx} f(g(x)) = \frac{18x - 12}{2\sqrt{9x^2 - 12x}}$ . Substituting  $x = 3$ , we find  $f'(g(x)) = \frac{18(3) - 12}{2\sqrt{9(3)^2 - 12(3)}} = \frac{42}{2\sqrt{45}} = \frac{21}{3\sqrt{5}} = \frac{7}{\sqrt{5}}$

## Answer to Exercise ?? (on page ??)

First, recall that the velocity of a particle is the derivative of its position function. Therefore,  $v(t) = x'(t) = \frac{d}{dt}[(t-a)(t-b)]$ . Applying the Product Rule for derivatives, we see that  $v(t) = (t-a)(1) + (t-b)(1) = 2t - a - b$ . To find the time(s) when the particle is at rest, we set  $v(t) = 0$  and solve for  $t$ .

$$0 = 2t - a - b$$

$$2t = a + b$$

$$t = \frac{a+b}{2}$$

## Answer to Exercise ?? (on page ??)

The question is asking when the derivative of  $f$  is  $\frac{1}{2}$ . We will take the derivative and set it equal to  $\frac{1}{2}$ .

$$\begin{aligned}f'(x) &= \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{2}{(x+2)^2} \\ \frac{2}{(x+2)^2} &= \frac{1}{2} \\ 4 &= (x+2)^2 \\ \pm 2 &= x+2 \\ x = 2 - 2 &= 0 \text{ and } x = -2 - 2 = -4\end{aligned}$$

## Answer to Exercise ?? (on page ??)

(a) Let  $t_0$  be the time at which the particle is first at rest. The velocity of the particle is given by  $v(t) = x'(t) = \cos t + \sin t$ . Setting  $v(t) = 0$ , we find:

$$\cos t = -\sin t$$

which is true for  $t = \frac{3\pi+4n}{4}$ , where  $n$  is an integer. Therefore, the first time the velocity is 0 is  $t_0 = \frac{3\pi}{4}$ .

(b) To find the acceleration at  $t = \frac{3\pi}{4}$ , we take the derivative of the velocity function to yield the acceleration function.

$$a(t) = v'(t) = -\sin t + \cos t$$

. Substituting  $t = \frac{3\pi}{4}$ , we find the acceleration is  $-\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4} = \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

## Answer to Exercise ?? (on page ??)

First, we find the  $x$  such that  $y = 0$

$$0 = e^{\tan x} - 2$$

$$2 = e^{\tan x}$$

$$\ln 2 = \tan x$$

$$x = \arctan(\ln 2) = \arctan 0.693 \approx 0.606$$

Then, we find the slope of the function at  $x = 0.606$  by finding  $y'(0.606)$

$$y' = e^{\tan x} (\sec x)^2 = \frac{e^{\tan x}}{(\cos x)^2}$$

$$y'(0.606) = \frac{e^{\tan 0.606}}{(\cos 0.606)^2} = 2.961$$

## Answer to Exercise ?? (on page ??)

(a) Apply the chain rule to find  $f'(x)$

$$f'(x) = \frac{1}{2\sqrt{25-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{25-x^2}}$$

(b) First, substitute  $x = -3$  into  $f'(x)$

$$f'(-3) = \frac{-(-3)}{\sqrt{25 - (-3)^2}} = \frac{3}{\sqrt{16}} = \frac{3}{4}$$

This is the slope of the line. To complete an equation for the tangent line, we need a point. We know the tangent line touches  $f(x)$  at  $x = -3$ , so the tangent line must pass through the point  $(-3, f(-3))$ .

$$f(-3) = \sqrt{25 - (-3)^2} = 4$$

We use  $m = \frac{3}{4}$  and the coordinate point  $(x_1, y_1) = (-3, 16)$  to complete the equation  $y - y_1 = m(x - x_1)$

$$y - 16 = \frac{3}{4}(x + 3)$$

**Answer to Exercise ?? (on page ??)**

$$a(t) = v'(t) = -\frac{\pi}{6} \sin \frac{\pi}{6} t$$
$$a(4) = -\frac{\pi}{6} \sin \frac{2\pi}{3} = -\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{12}$$

**Answer to Exercise ?? (on page ??)**

Recall that the rate of change of a function is given by the derivative of that function. Therefore, we are looking for the interval(s) where  $f'(x) > g'(x)$ . First, we find each derivative:

$$f'(x) = e^x$$

$$g'(x) = 4x^3$$

We are looking for  $x$ -values, such that  $e^x > 4x^3$ . This inequality can be restated as  $e^x - 4x^3 > 0$ . Using a calculator, you should find that  $e^x - 4x^3 = 0$  when  $x \approx 0.831$  and  $x \approx 7.384$ . We will check values on either side of and in the interval  $x \in (0.831, 7.384)$  to determine the sign value of  $e^x - 4x^3$ . We know that when  $x = 0$ ,  $e^x - 4x^3 > 0$ , when  $x = 5$ ,  $e^x - 4x^3 < 0$ , and when  $x = 10$ ,  $e^x - 4x^3 > 0$ . Therefore,  $f'(x)$  is greater than  $g'(x)$  on the open intervals  $x \in (-\infty, 0.831) \cup (7.384, \infty)$ .



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