Euler's Method

How do computers approximate the solution to a differential equation that cannot be explicitly solved? Let's consider the differential equation

$$\frac{dy}{dx} = x + y \text{ with initial condition } y(0) = 1$$

.

This means the solution passes through the point $(0, 1)$. Additionally, the slope of the solution is $\frac{dy}{dx} = 0 + 1 = 1$ at that point. This means we can approximate the solution with the linear function $L(x) = x + 1$ (see figure 1.1). As you can see, near $(0, 1)$ the approximation is good, but as x increases, the divergence between the actual solution and the approximation grows.
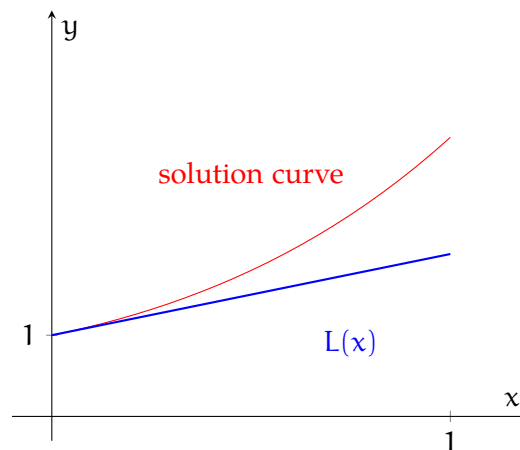

Figure 1.1: A first Euler approximation

How can we make a better approximation? Suppose we stop the first approximation at $x = 1.5$, re-evaluate $\frac{dy}{dx}$, and use that to make a second linear approximation. When $x = 0.5$, $L(x) = 0.5 + 1 = 1.5$. Taking the point $(0.5, 1.5)$, then $\frac{dy}{dx} = 0.5 + 1.5 = 2$. We can then write a second linear approximation, $L_2(x) = 2(x - 0.5) + 1.5 = 2x - 1 + 1.5 = 2x + 0.5$. As you can see (figure 1.2), this new approximation is closer than our first approximation. We call this an approximation with a step size of 0.5.
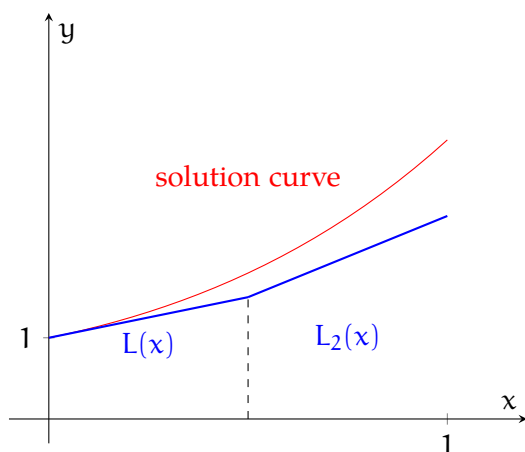
Figure 1.2: An Euler approximation with step size 0.5

We can improve this further by taking a step size of 0.25 (see figure 1.3). As the step size decreases and the step number increases, the approximation gets closer and closer to the true solution.
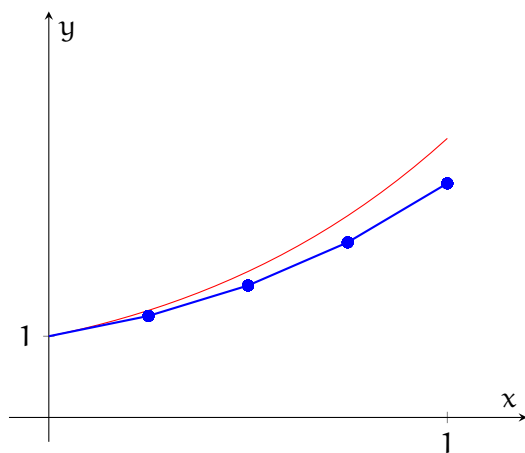
Figure 1.3: An Euler approximation with step size 0.25

In general, Euler's method is a numerical process similar to sketching a solution on a slope field. One begins at the given initial value, proceeds for a short step in the direction indicated by the slope field. You adjust the slope of your approximation based on the value of the slope field at the end of each step.

For a first-order differential equation, let $\frac{dy}{dx} = F(x, y)$ and $y(x_0) = y_0$. If we have step size h, then our successive x-values are $x_1 = x_0 + h$, $x_2 = x_1 + h$, etc. The differential equation tells us that the slope at x_0 is $F(x_0, y_0)$. So, $y_1 = y_0 + hF(x_0, y_0)$ (see figure 1.4).
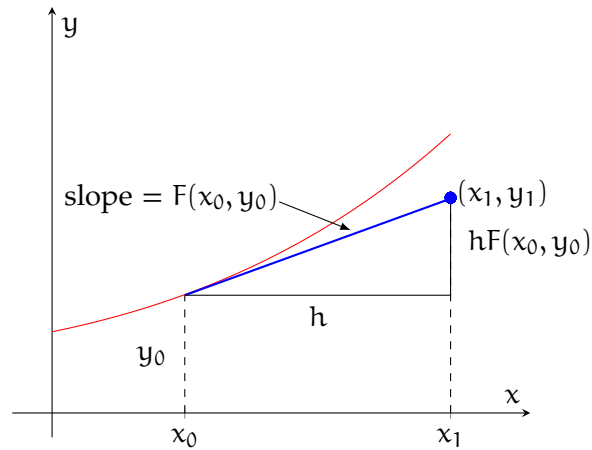
Figure 1.4: Visualization of Euler's method

Continuing, once we have found y_1, we can then define $x_2 = x_1 + h$ and $y_2 = y_1 + hF(x_1, y_1)$. And in general, for an initial-value problem when $\frac{dy}{dx} = F(x, y)$ and $y(x_0) = y_0$, we can make an approximation with step size h where:

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

where $n = 1, 2, 3, \cdots$.

Example: Use Euler's method with a step size of 0.2 to approximate the value of $y(1)$ if $\frac{dy}{dx} = 2x + y$ and $y(0) = 1$.

Solution: We are given $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = 2x + y$. This means we will need 5 steps to reach $x_5 = 1$. So, we know that:

$$y_1 = 1 + 0.2[2(0) + 1] = 1 + 0.2[1] = 1.2$$

$$y_2 = 1.2 + 0.2[2(0.2) + 1.2] = 1.2 + 0.2(1.6) = 1.52$$

$$y_3 = 1.52 + 0.2[2(0.4) + 1.52] = 1.984$$

We can continue in this manner. The results are shown in the table:

n	x_n	y_n	$F(x_n, y_n)$
0	0	1	1
1	0.2	1.2	1.6
2	0.4	1.52	2.32
3	0.6	1.984	3.184
4	0.8	2.6208	4.2208
5	1	3.46496	–

Therefore, $y(1) \approx 3.4696$.

Example: This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam. Let $y = f(x)$ be the solution to $\frac{dy}{dx} = x - y$ with initial condition $f(1) = 3$. What is the approximation of $f(2)$ obtained using Euler's method with two steps of equal length starting at $x = 1$?

Solution: The question asks that we use Euler's method two steps. The step size should be $h = \frac{x_2 - x_0}{2} = \frac{2-1}{2} = \frac{1}{2}$. Taking $x_0 = 1$ and $y_0 = 3$, we find that:

$$y_1 = y_0 + h\,[x_0 - y_0]$$

$$y_1 = 3 + \frac{1}{2}\,[1 - 3]$$

$$y_1 = 3 + \frac{1}{2}\,[-2] = 3 - 1 = 2$$

So our intermediate point is $(x_1, y_1) = (\frac{3}{2}, 2)$. Finding y_2:

$$y_2 = y_1 + h\,[x_1 - y_1]$$

$$y_2 = 2 + \frac{1}{2}\left[\frac{3}{2} - 2\right]$$

$$y_2 = 2 + \frac{1}{2}\left[\frac{-1}{2}\right] = 2 - \frac{1}{4} = \frac{7}{4}$$

So the approximate value of $f(2)$ is $\frac{7}{4}$.

Exercise 1

In the previous chapter on slope fields, we discussed the behavior of inductors in electronic circuits. As you may recall, capacitors also exhibit more complex behavior than regular resistors. Consider a circuit with a resistor and capacitor (see figure below). Let the resistor have resistance R ohms and the capacitor have capacitance C farads. By Kirchhoff's law, we know that:

$$RI + \frac{Q}{C} = V$$

where Q is the charge on each side of the capacitor and $\frac{Q}{C}$ is the voltage drop across the capacitor. Recall that current is the change in charge over time. Therefore, $I = \frac{dQ}{dt}$, and we can write the differential equation:

$$R\frac{dQ}{dt} + \frac{1}{C}Q = V$$

When the switch is first closed, there is no charge (that is, $Q(0) = 0$). If the resistor is 5Ω, the battery is 60V, and the capacitor is 0.05F, use Euler's method with a step size of 0.1 to estimate the charge after half a second.
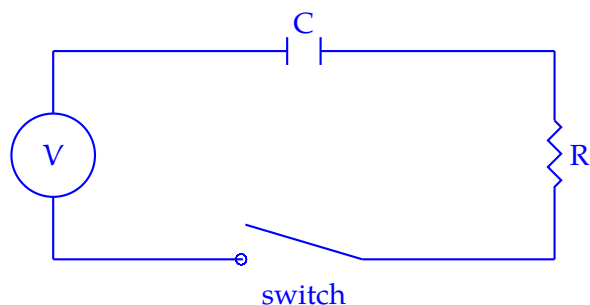
Exercise 2

[This problem was originally presented as a calculator-allowed, free-response question on the 2012 AP Calculus BC exam.] The function f is twice- differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table below. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

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Answers to Exercises

Answer to Exercise 1 (on page 5)

Substituting the given values, we find that $(5)\frac{dQ}{dT} + \frac{1}{0.05}Q = 60$. Solving for $\frac{dQ}{dt}$:

$$(5)\frac{dQ}{dt} + (20)Q = 60$$

$$\frac{dQ}{dt} + 4Q = 12$$

$$\frac{dQ}{dt} = 12 - 4Q$$

We also know that $Q(0) = 0$. Using Euler's method with step size $h = 0.1$, $Q(0.1) \approx Q(0) + h[12 - 4Q(0)] = 0 + 0.1[12 - 4(0)] = 1.2$. And $Q(0.2) \approx Q(0.1) + h[12 - 4Q(0.1)] = 1.2 + 0.1[12 - 4(1.2)] = 1.92$. And $Q(0.3) \approx Q(0.2) + h[12 - 4Q(0.2)] = 1.92 + 0.1[12 - 4(1.92)] = 2.352$. And $Q(0.4) \approx Q(0.3) + h[12 - 4Q(0.3)] = 2.352 + 0.1[12 - 4(2.352)] = 2.6112$. And finally, $Q(0.5) \approx Q(0.4) + h[12 - 4Q(0.4)] = 2.6112 + 0.1[12 - 4(2.6112)] = 2.76672$. Because we are finding a charge, the unit it Coulombs (C), so our final answer is $Q(0.5) \approx 2.77C$.

Answer to Exercise 2 (on page 7)

We are given $x_0 = 1$ and $x_2 = 1.4$. Therefore we will use step size $h = \frac{1.4-1}{2} = \frac{0.4}{2} = 0.2$. Taking $x_0 = 1$ and $y_0 = f(1) = 15$, we find y_1: $y_1 = y_0 + h \cdot f'(x_0) = 15 + 0.2 \cdot f'(1) = 15 + 0.2(8) = 15 + 1.6 = 16.6$. And then $y_2 = y_1 + h \cdot f'(x_1) = 16.6 + 0.2 \cdot f'(1.2) = 16.6 + 0.2(12) = 16.6 + 2.4 = 19$. Therefore, $f(1.4) \approx 19$.

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