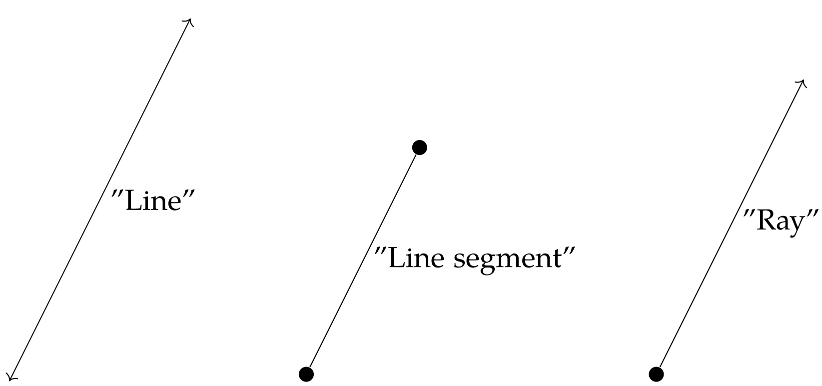
Contents

Chapter 1

Angles

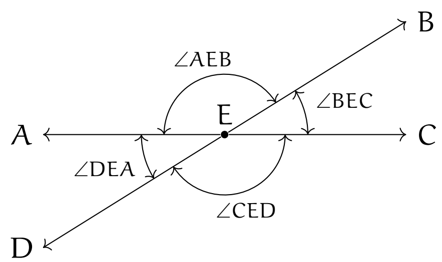
In the following recommend videos, the narrator talks about lines, line segments, and rays. When mathematicians talk about *lines*, they mean a straight line that goes forever in two directions. And if you pick any two points on that line; the space between them is a *line segment*. If you take any line, pick a point on that line and discard all the points on one side of the point, that is a *ray*. All three have no width.



Watch the following videos from Khan Academy:

- Introduction to angles: https://youtu.be/H-de6Tkxej8
- Measuring angles in degrees: https://youtu.be/92aLiyeQj0w

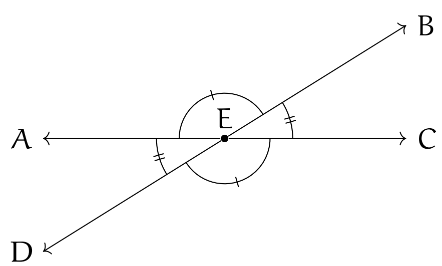
When two lines cross, they form four angles:



What do we know about those angles?

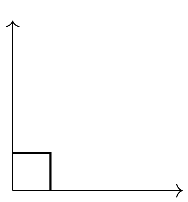
- The sum of any two adjacent angles add to be 180°. So, for example, $m\angle AEB + m\angle BEC = 180°$. We use the phrase "add to be 180°" so often that we have a special word for it: *supplementary*.
- The sum of all four angles is 360°.
- Angles opposite each other are equal. So, for example, $m\angle AEB = m\angle CED$.

In a diagram, to indicate that two angles are equal we often put hash marks in the angle:

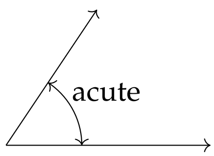


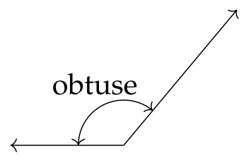
Here the two angles with a single hash mark are equal and the two angles with double hash marks are equal.

When two lines are perpendicular, the angle between them is 90° and we say they meet at a *right angle*. When drawing diagrams, we indicate right angles with an elbow:

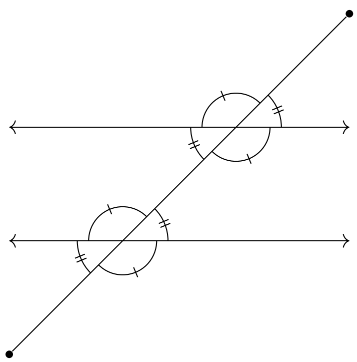


When an angle is less than 90°, it is said to be *acute*. When an angle is more than 90°, it is said to be *obtuse*.





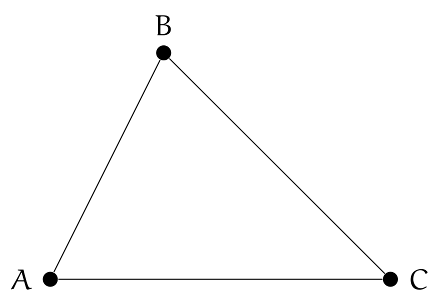
If two lines are parallel, line segments that intersect both lines, form the same angles with each line:



Chapter 2

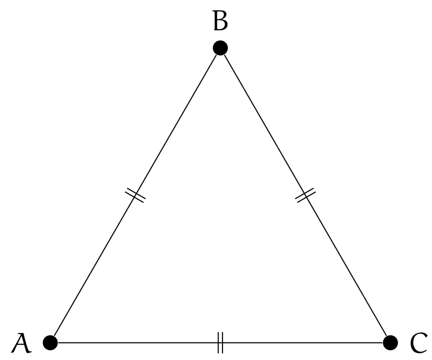
Introduction to Triangles

Connecting any three points with three line segments will get you a triangle. Here is the triangle ABC which was created by connecting three points A, B, and C:

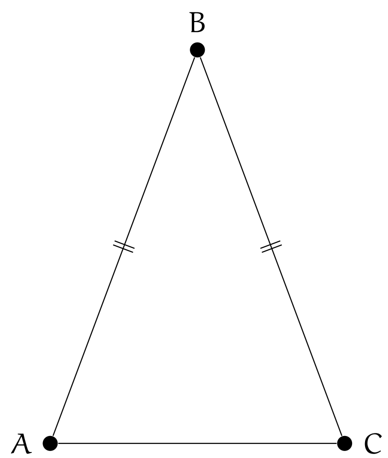


2.1 Equilateral and Isosceles Triangles

We talk a lot about the length of the sides of triangles. If all three sides of the triangle are the same length, we say it is an *equilateral triangle*:

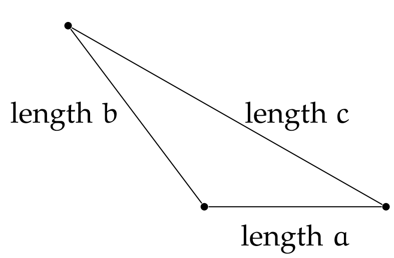


If only two sides of the triangle are the same length, we say it is an *isosceles triangle*:



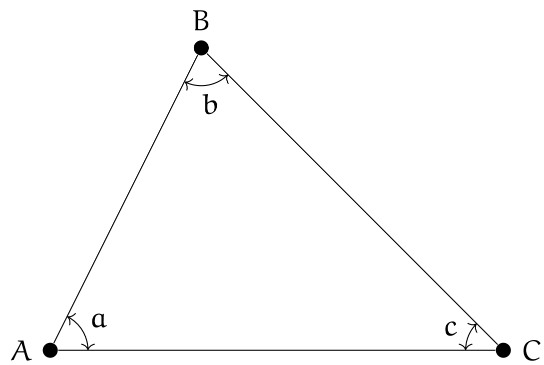
The shortest distance between two points is always the straight line between them. Thus, you can be certain that the length of one side will *always* be less than the sum of the lengths of the remaining two sides. This is known as the *triangle inequality*.

For example, in this diagram c must be less than $a + b$.

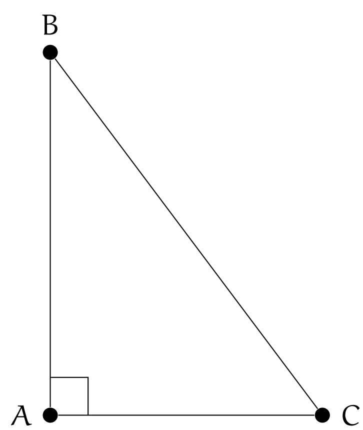


2.2 Interior Angles of a Triangle

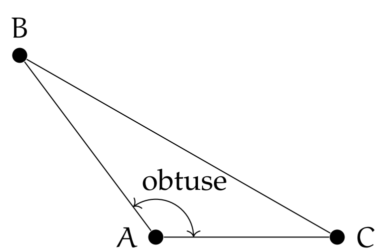
We also talk a lot about the interior angles of a triangle:



A triangle where one of the interior angles is a right angle is said to be a *right triangle*:



If a triangle has an obtuse interior angle, it is said to be an *obtuse triangle*:



If all three interior angles of a triangle are less than 90°, it is said to be an *acute triangle*.

The measures of the interior angles of a triangle always add up to 180°. For example, if we know that a triangle has interior angles of 37° and 56°, we know that the third interior angle is 87°.

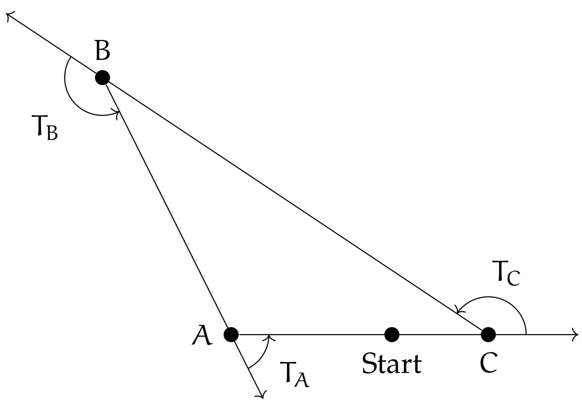
Exercise 1 Missing Angle

One interior angle of a triangle is 92°. The second angle is 42°. What is the measure of the third interior angle?

Working Space

Answer on Page 25

How can you know that the sum of the interior angles is 180°? Imagine that you started on the edge of a triangle and walked all the way around to where you started. (going counter-clockwise.) You would turn three times to the left:



After these three turns, you would be facing the same direction that you started in. Thus $T_A + T_B + T_C = 360°$. The measures of the interior angles are a, b, and c. Notice that a and T_A are supplementary. So we know that:

- $T_A = 180 - a$
- $T_B = 180 - b$
- $T_C = 180 - c$

So we can rewrite the equation above as

$$(180 - a) + (180 - b) + (180 - c) = 360°$$

Which is equivalent to

$$a + b + c = 360°$$

Exercise 2 Interior Angles of a Quadrilateral

Any four-sided polygon is a *quadrilateral*. Using the same "walk around the edge" logic, what is the sum of the interior angles of any quadrilateral?

Working Space

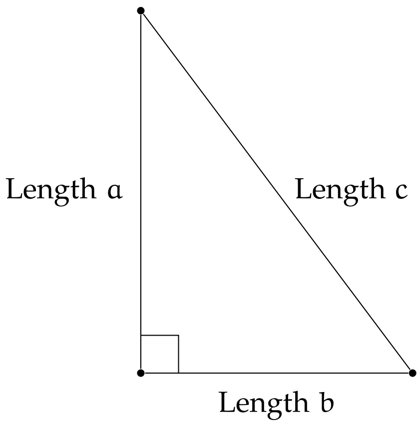
Answer on Page 25

Chapter 3

Pythagorean Theorem

Watch's Khan Academy's Intro to the Pythagorean Theorem video at https://youtu.be/AA6RfgP-AHU.

If you have a right triangle, the edges that touch the right angle are called *the legs*. The third edge, which is always the longest, is known as *the hypotenuse*. The Pythagorean Theorem gives us the relationship between the length of the legs and the length of the hypotenuse.



The Pythagorean Theorem tells us that $a^2 + b^2 = c^2$.

For example, if one leg has a length of 3 and the other has a length of 4, then $a^2 + b^2 = 3^2 + 4^2 = 25$. Thus c^2 must equal 25. So you know the hypotenuse must be of length 5.

(In reality, it rarely works out to be such a tidy number. For example, what is the length of the hypotenuse if the two legs are 3 and 6? $a^2 + b^2 = 3^2 + 6^2 = 45$. The length of the hypotenuse is the square root of that: $\sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$, which is approximately 6.708203932499369.)

Exercise 3 Find the Missing Length

What is the missing measure?

Leg 1 = 6, Leg 2 = 8, Hypotenuse = ? (It should be a whole number.)

Leg 1 = 5, Leg 2 = ?, Hypotenuse = 13 (It should be a whole number.)

Leg 1 = ?, Leg 2 = 15, Hypotenuse = 17 (It should be a whole number.)

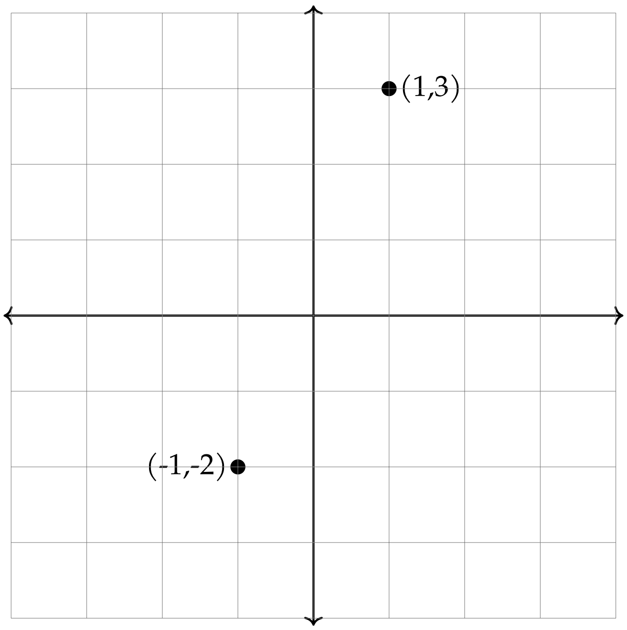
Leg 1 = 3, Leg 2 = 3, Hypotenuse = ? (It is an irrational number. Give the exact answer and then use a calculator to get an approximation.)

Working Space

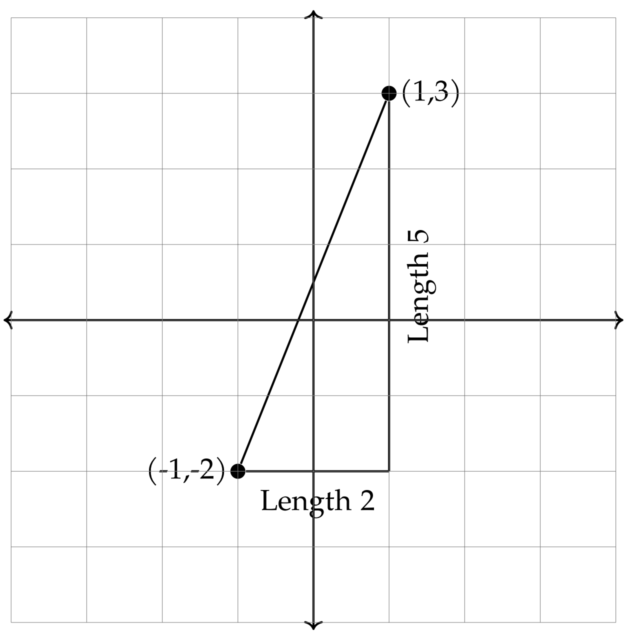
Answer on Page 25

3.1 Distance between Points

What is the distance between these two points?



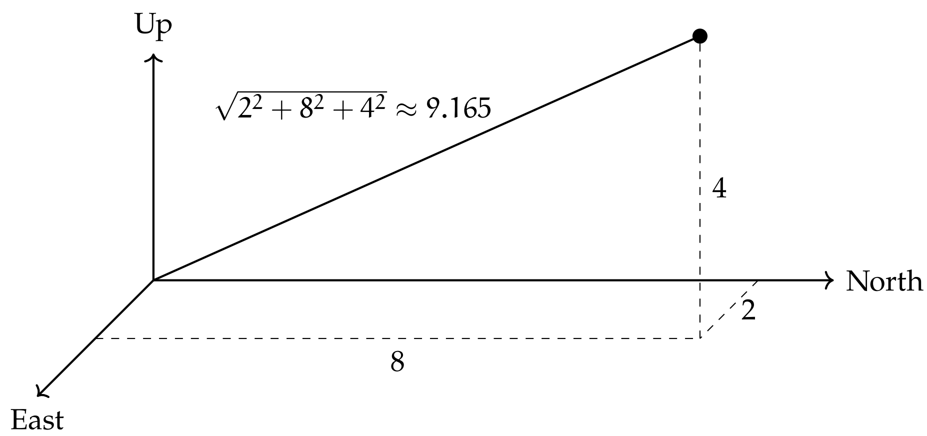
We can draw a right triangle and use the Pythagorean Theorem:



The distance between the two points is $\sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.385165$. That is, you square the change in x and add it to the square of the change in y. The distance is the square root of that sum.

3.2 Distance in 3 Dimensions

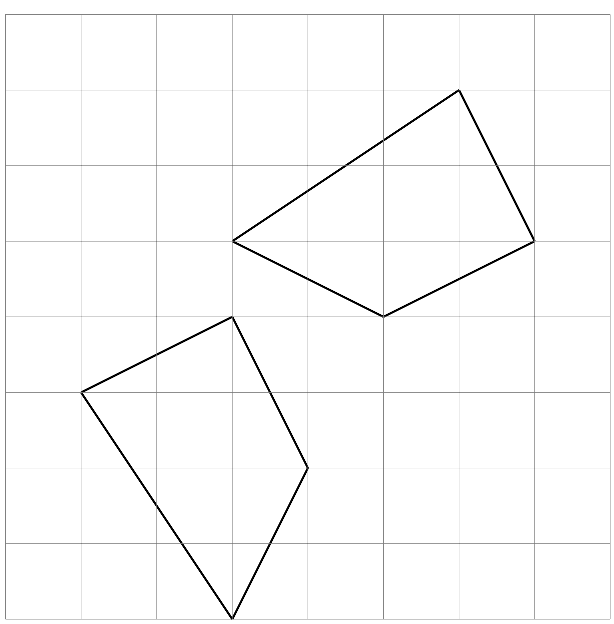
What if the point is in three-dimensional space? That is, you move 2 meters East, 8 meters North, and 4 meters up in the air. How far are you from where you started? You just square each, sum them, and take the square root: $\sqrt{2^2+8^2+4^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$ meters.



CHAPTER 4

Congruence

Look at this picture of two geometric figures.



They are the same shape, right? If you cut one out with scissors, it would lay perfectly on top of the other. In geometry, we say they are *congruent*.

What is the official definition of "congruent"? Two geometric figures are congruent if you can transform one into the other using only rigid transformations.

You might be wondering now, what are rigid transformations? A transformation is *Rigid* if it doesn't change the distances between the points or the measure of the angles between the lines, they form. These are all rigid transformations:

- Translations
- Rotations
- Reflections

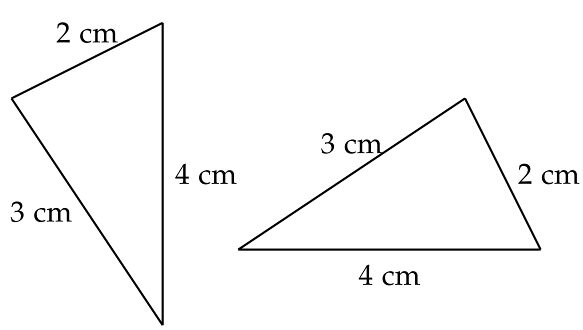
Once again imagine cutting out one figure with scissors and trying to match it with the second figure, your actions are rigid transformations:

- Translations - sliding the cutout left and right and up and down
- Rotations - rotating the cutout clockwise and counterclockwise
- Reflection - flipping the piece of paper over

A transformation is rigid if it is some combination of translations, rotations, and reflections.

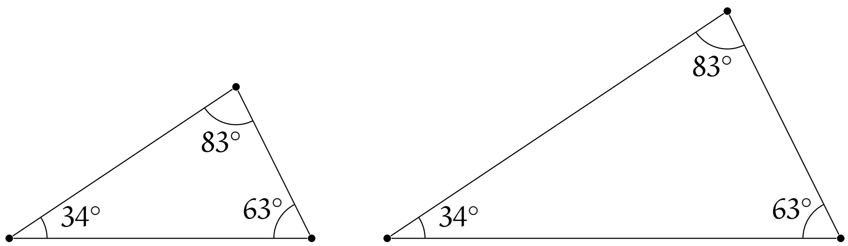
4.1 Triangle Congruency

If the sides of two triangles have the same length, the triangles must be congruent:



To be precise, the Side-Side-Side Congruency Test says that two triangles are congruent if three sides in one triangle are the same length as the corresponding sides in the other. We usually refer to this as the SSS test.

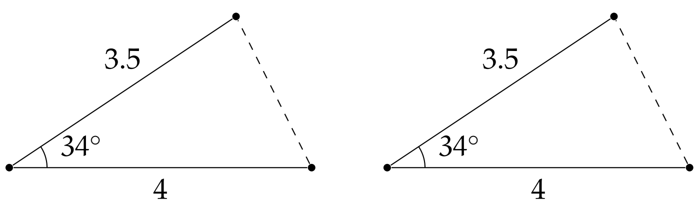
Note that two triangles with all three angles equal are not necessarily congruent. For example, here are two triangles with the same interior angles, but they are different sizes:



These triangles are not congruent, but they are *similar*. Meaning they have the same shape, but are not necessarily the same size.

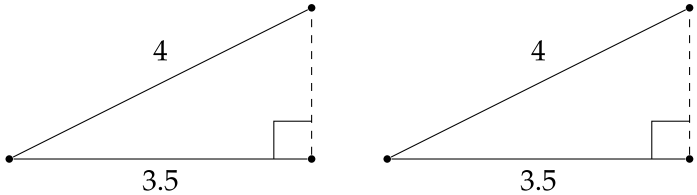
Therefore, if you know two angles of a triangle, you can calculate the third. So it makes sense to say "If two triangles have two angles that are equal, they are similar triangles." And if two similar triangles have one side that is equal in length, they must be the same size – so they are congruent. Thus, the Side-Angle-Angle Congruency Test says that two triangles are congruent if two angles and one side match.

What if you know that two triangles have two sides that are the same length and that the angle between them is also equal?



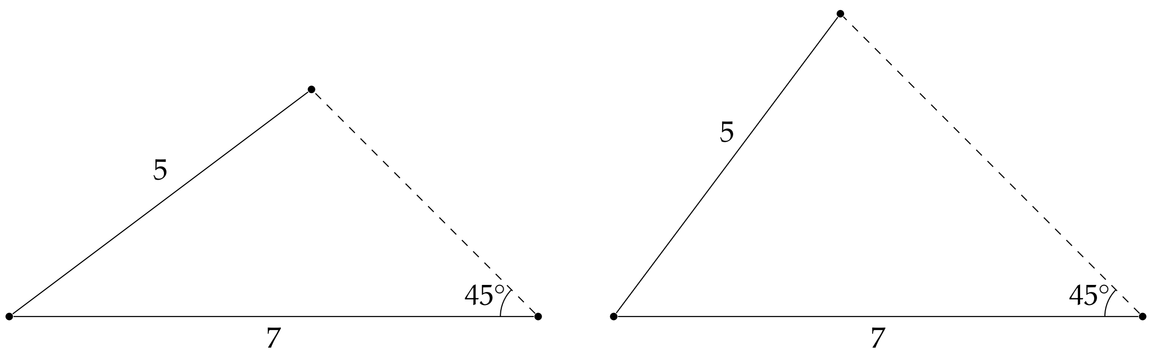
Yes, they must be congruent. This is the Side-Angle-Size Congruency Test.

What if the angle isn't the one between the two known sides? If it is a right angle, you can be certain the two triangles are congruent. (How do I know? Because the Pythagorean Theorem tells us that we can calculate the length of the third side. There is only one possibility, thus all three sides must be the same length.)

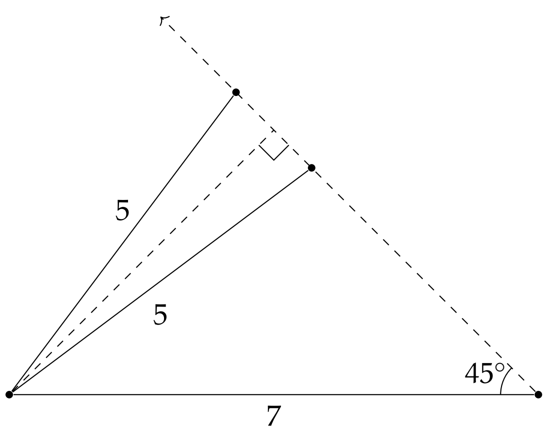


In this case, the third side of each triangle must be $\sqrt{4^2 - 3.5^2} \approx 1.9$.

What if the know angle is less than 90°? *The triangles are not necessarily congruent.* For example, let's say that there are two triangles with sides of length 5 and 7 and that the corresponding angle (at the end of the side of length 7) on each is 45°. Two different triangles satisfy this:



Let's see this another way by laying one triangle on top of the other:



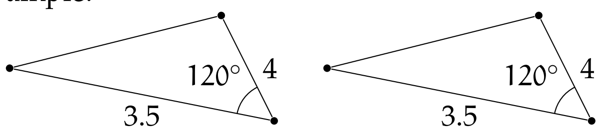
So there is *not* a general Side-Side-Angle Congruency Test.

Here, then, is the list of common congruency tests:

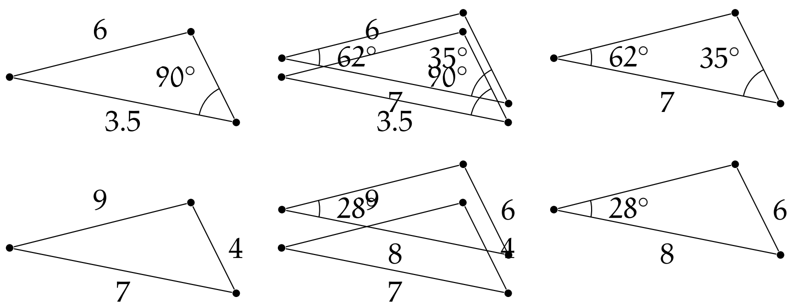
- Side-Side-Side: All three sides have the same measure
- Side-Angle-Angle: Two angles and one side have the same measure
- Side-Angle-Side: Two sides and the angle between them have the same measure
- Side-Side-Right: They are right triangles and have two sides have the same measure

Exercise 4 Congruent Triangles

Ted is terrible at drawing triangles: he always draws them exactly the same. Fortunately, he has marked these diagrams with the sides and angles that he measured. For each pair of triangles, write if you know them to be congruent and which congruency test proves it. For example:



(These drawings are clearly not accurate, but you are told the measurements are.) The answer is "Congruent by the Side-Angle-Side test."



Working Space

Answer on *Page 26*

Chapter 5

Parallel and Perpendicular

Two vectors are said to be parallel if they have the same or opposite direction. In simpler terms, if two vectors are pointing in the same direction (even if their magnitudes differ), they are considered parallel. For example, imagine you have a vector representing the direction and speed of a car moving north. If you have another vector representing the direction and speed of a different car also moving north, these vectors are parallel.

On the other hand, if two vectors point in completely opposite directions, they are still considered parallel. For instance, if one vector represents a car moving north and the other represents a car moving south, these vectors are parallel but in opposite directions.

Perpendicular vectors, as the name suggests, are vectors that intersect each other at a right angle, forming a 90-degree angle. If we imagine a sheet of paper, drawing a horizontal vector and a vertical vector on that paper would create perpendicular vectors. In this case, the horizontal vector represents left-right direction, while the vertical vector represents up-down direction. Perpendicular vectors are often seen in geometric shapes, such as squares and rectangles, where their sides intersect at right angles.

A fundamental property of perpendicular vectors is that their dot product is zero. The dot product is a mathematical operation that measures the extent to which two vectors align with each other. When two vectors are perpendicular, their dot product is always zero. This property provides a useful tool for determining whether two given vectors are perpendicular.

Understanding parallel and perpendicular vectors is essential in various areas of mathematics and physics. For example, in geometry, knowledge of perpendicular vectors helps us determine whether lines are perpendicular or parallel. In physics, vectors can represent forces, velocities, or displacements, and identifying parallel or perpendicular vectors aids in analyzing motion and forces acting on objects.

In summary, parallel vectors have the same or opposite direction, while perpendicular vectors intersect at a right angle. Recognizing these relationships between vectors enables us to solve problems involving geometry, physics, and many other fields. As you delve deeper into the exciting world of vectors, keep an eye out for parallel and perpendicular relationships, as they often hold valuable insights and solutions.

APPENDIX A

Answers to Exercises

Answer to Exercise 1 (on page 10)

$180° - (92° + 42°) = 46°$

Answer to Exercise 2 (on page 11)

$360°$

Answer to Exercise 3 (on page 14)

10 because $6^2 + 8^2 = 10^2$

12 because $5^2 + 12^2 = 13^2$

8 because $8^2 + 15^2 = 17^2$

$3\sqrt{2} \approx 4.24$ because $3^2 + 3^2 = \left(3\sqrt{2}\right)^2$

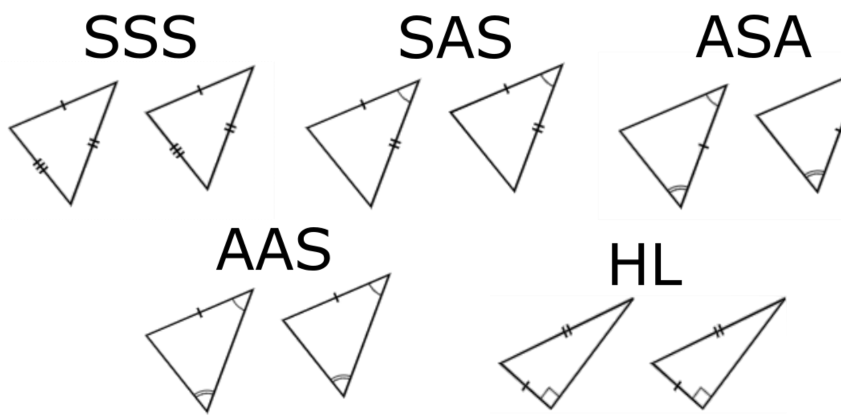
Answer to Exercise 4 (on page 21)

Congruent by the Side-Side-Right Congruency Test.

Congruent by the Side-Side-Side Congruency Test.

Congruent by the Side-Angle-Angle Congruency Test.

We don't know if they are congruent. The measured angle is not between the measured sides.



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