Contents

Chapter 1

Volumes with Integrals

Suppose we wanted to know the volume of a theoretical irregular shape (we stipulate theoretical because, if you had this object and a large enough container, you could use displacement to determine the volume of the object). [fixme better intro]

1.1 Volume of a Sphere

Below we will prove the volume of a sphere is given by $\frac{4}{3}\pi r^3$ using the integral method. Suppose we have a sphere of radius r centered at the origin (see figure 1.1)

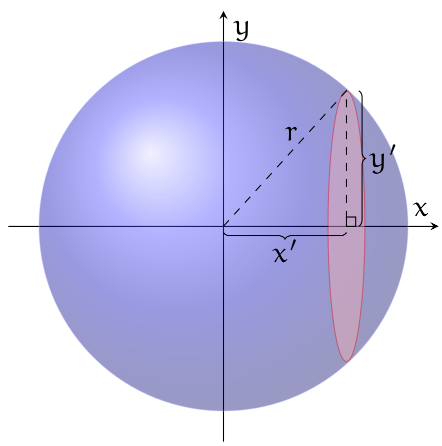


Figure 1.1: A vertical cross-section of a sphere

We begin by taking very thin vertical cross-sections. The radius of the cross-section is the height, y, of the sphere at the horizontal position, x. Since the edges of the cross-section lie on the sphere, we know the edge of the cross-section is distance r from the origin. Applying the Pythagorean theorem, we see that $r^2 = x^2 + y^2$, which implies that $y = \sqrt{r^2 - x^2}$. And then the area of the cross-section is given by $\pi y^2 = \pi(r^2 - x^2)$. If we imagine each cross section as having a width, dx, and taking the sum of all the cross sections from $x = -r$ to $x = r$, we can write an integral equal to the volume of the sphere:

$$V_{\text{sphere}} = \int_{-r}^{r} \pi(r^2 - x^2)\, dx$$

And we can evaluate that integral:

$$V_{\text{sphere}} = \pi \int_{-r}^{r} r^2\, dx - \pi \int_{-r}^{r} x^2\, dx$$

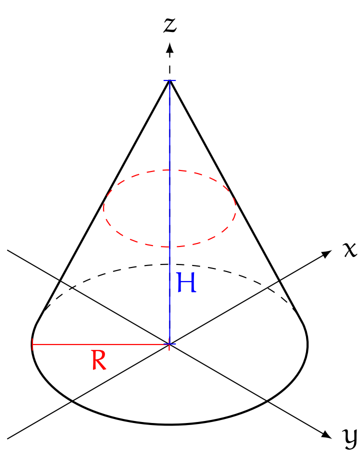
$$V_{\text{sphere}} = \pi \left[r^2 x\right]_{x=-r}^{x=r} - \frac{\pi}{3}\left[x^3\right]_{x=-r}^{x=r}$$

$$V_{\text{sphere}} = \pi \left[r^3 - (-r^3)\right] - \frac{\pi}{3}\left[r^3 - (-r^3)\right]$$

$$V_{\text{sphere}} = 2\pi r^3 - \frac{2\pi}{3} r^3 = \frac{4}{3}\pi r^3$$

Exercise 1

Prove the volume of a regular cone is $\frac{\pi}{3}R^2H$, where R is the radius of the base and H is the height of the cone. (Hint: a cone is a series of decreasing circles stacked on top of each other, see figure below.)



Working Space

Answer on *Page 69*

1.2 Volumes of Solids of Revolution

We can also find the volume of solids made by revolving a graph about the x or y-axis. Suppose the graph $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ were rotated vertically about the x-axis to form a solid. How could we find the volume of that solid? Well, we can imagine a rectangle of width dx and height y (see figure 1.2)

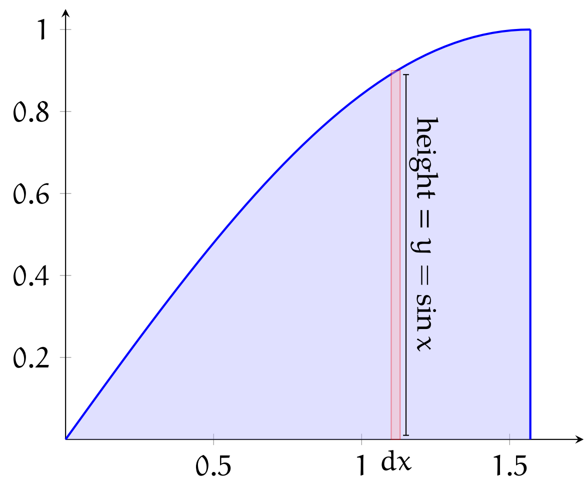


Figure 1.2: A cross section has width dx and height $y = \sin x$

If we rotate the plot vertically about the x-axis, the rectangle becomes a cylinder with radius $y = \sin x$ and height dx (see figure **??**). And, therefore, the volume of each cylindrical slice is $V_{slice} = \pi r^2 dx = \pi \cdot \sin^2 x dx$.

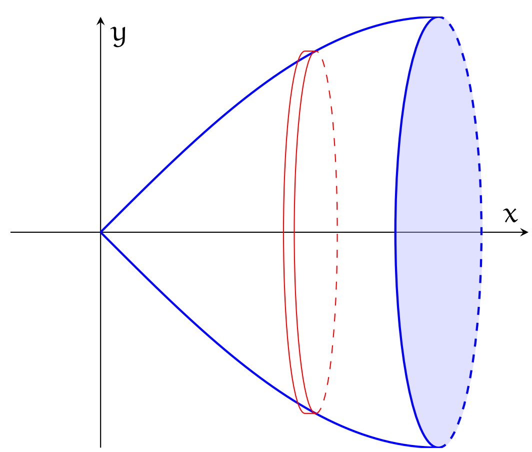


Figure 1.3: When rotated, the cross-section becomes a cylinder with radius $\sin x$ and width dx, which has a total volume of $\pi \sin^2 x dx$

We can find the total volume by integrating from 0 to $\pi/2$:

$$V = \pi \int_0^{\pi/2} \sin^2 x \, dx$$

Recall the half angle formula, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Substituting, we see that:

$$V = \frac{\pi}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$$

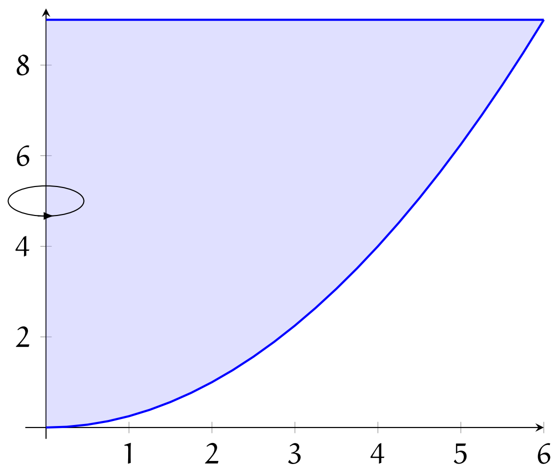
$$V = \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right) \Big|_{x=0}^{x=\pi/2}$$

$$V = \frac{\pi}{2} \left[\left(\pi/2 - \frac{1}{2} \sin \pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right]$$

$$V = \frac{\pi}{2} [\pi/2 - 0 - 0 + 0] = \frac{\pi^2}{4}$$

Exercise 2

Find the volume of a solid created by rotating the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$ about the y-axis. A graph is shown below.



Working Space

Answer on Page 70

Exercise 3

Let $f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$. Bird's eggs of various sizes can be modeled by rotating $f(x)$ about the x-axis, with different values of a, b, c, and d defining different sizes and shapes of eggs. For a domestic chicken, $a = -0.02$, $b = 0.03$, $c = 0.12$, and $d = 0.454$. For a mallard duck, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Use a calculator, such as a TI-89 or Wolfram Alpha, to determine which species lays a bigger egg.

Working Space

Answer on Page 70

1.2.1 Using donuts for solids of revolution

Sometimes there is space between the region we are rotating and the line we are rotating it about. Consider the region bounded between $y = 2x$ and $y = x^2$ (see figure 1.4):

When rotated, the slices will take the form of donuts (or washers), the volume of which is $\pi\left(R^2 - r^2\right) dx$, where R is the outer radius and r is the inner radius. Therefore, in this case, the total volume of the rotated region is given by:

$$V = \int_0^2 \pi \left[(2x)^2 - \left(x^2\right)^2\right] dx$$

$$V = \pi \int_0^2 4x^2 - x^4 \, dx = \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{x=0}^{x=2}$$

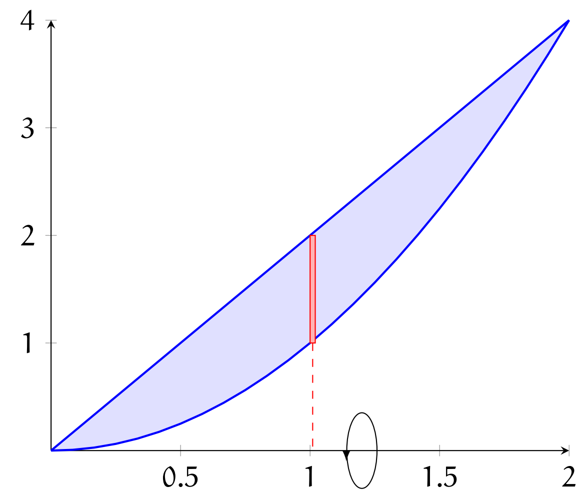


Figure 1.4: When rotated, the slices will become donuts with outer radius $2x$ and inner radius x^2

$$V = \pi\left[\frac{4}{3}2^3 - \frac{1}{5}2^5\right] = \pi\left[\frac{32}{3} - \frac{32}{5}\right]$$

$$V = \frac{64\pi}{15}$$

Exercise 4

What is the volume of the region bounded by $y = x^2$ and $y = 2\sqrt{x}$ when rotated about the y-axis?

Working Space

Answer on *Page 71*

1.3 Volumes of Other Solids

You can also model a solid as a base defined by a function with cross-sections of specific shapes. Consider the function $y = x^2$ from $x = 0$ to $x = 2$ (see figure 1.5). Suppose the area between the curve, the y-axis, and the line $y = 4$ defines a base and each vertical cross-section is a square. Then the width of the each cross section is dx, the length is $4 - x^2$, and (because they are squares) the height in the z-plane is also $4 - x^2$. Then the volume of each cross-section is $V_{slice} = \left(4 - x^2\right)^2 dx$ and the total volume of the solid is:

$$V = \int_0^2 \left(4 - x^2\right)^2 \, dx$$

$$V = \int_0^2 \left(16 - 8x^2 + x^4\right) \, dx$$

$$V = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{x=0}^{x=2}$$

$$V = 16(2) - \frac{8}{3}(2)^3 + \frac{1}{5}(2)^5 = \frac{256}{15} \approx 17.067$$

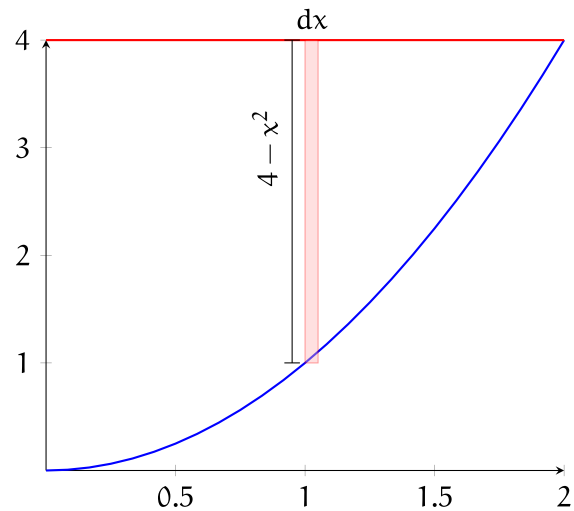


Figure 1.5: $y = x^2$ with a vertical cross-section

You can use a similar method for triangular, semi-circular, or any other shape cross-section. The trick is writing everything in terms of x (when you cross sections are vertical and have width dx) or y (when your cross section are horizontal and have length dy).

Exercise 5

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let R be the region in the first quadrant bounded above by the graph $y = \ln(3-x)$, for $0 \leq x \leq 2$. R is the base of a solid for which each cross section perpendicular to the x-axis is square. What is the volume of the solid? Give your answer to 3 decimal places.

Working Space

Answer on Page 72

Exercise 6

Find the volume of a solid whose base is defined by the ellipse $9x^2 + 16y^2 = 25$ and is made up of isosceles-triangular cross-sections perpendicular to the x-axis (with the hypotenuse in the base of the solid).

Working Space

Answer on Page 72

Chapter 2

Double Integrals over Rectangular Regions

fixme change intro to reflect shorter chapter In this chapter, we extend this powerful idea into higher dimensions using the tools of multiple integration. While single integration enables us to calculate the area under a curve or the volume under a surface, multiple integration allows us to calculate volumes in three dimensions, and even hypervolumes in higher dimensions.

We start by discussing double integration, which allows us to find the volume under a surface in three dimensions. This method involves slicing the solid into infinitesimally small columns, and summing the volumes of these columns.

Next, we'll cover triple integration, a tool that lets us find the volume of more complicated solids in three-dimensional space. The idea is similar to double integration.

To properly implement these techniques, we'll also discuss the different coordinate systems that can be used in multiple integration, such as rectangular, cylindrical, and spherical coordinates, and when it's advantageous to use one system over another.

By the end of this chapter, you will have a deeper understanding of the techniques of multiple integration and how to apply them to find the volumes of various types of solids. The methods we study here will serve as a foundation for many topics in higher mathematics and physics, including electromagnetism, fluid dynamics, and quantum mechanics.

2.1 Double Integrals

Double integrals extend single-variable integration to functions of two variables, allowing us to calculate quantities like area, volume, and mass over a two-dimensional region. By integrating a function across a specified domain in the xy-plane, they help analyze how a quantity changes in both dimensions. Common in physics, engineering, and economics, double integrals involve setting up limits for the region and performing two successive integrations, often tailored to the region's geometry. We begin by discussing double integrals over rectangular regions, then extending that discussion to regions of any general shape. Finally, we discuss applications of double integrals.

2.1.1 Over Rectangular Regions

Suppose there is some function, $z = f(x, y)$, that is defined over the rectangular region, R, defined by $R = [a, b] \times [c, d] = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, and f is such that $f \geq 0$ for all $(x, y) \in \mathbb{R}$. Then the graph of f is a surface that lies above the rectangular region, R (see figure 2.1).

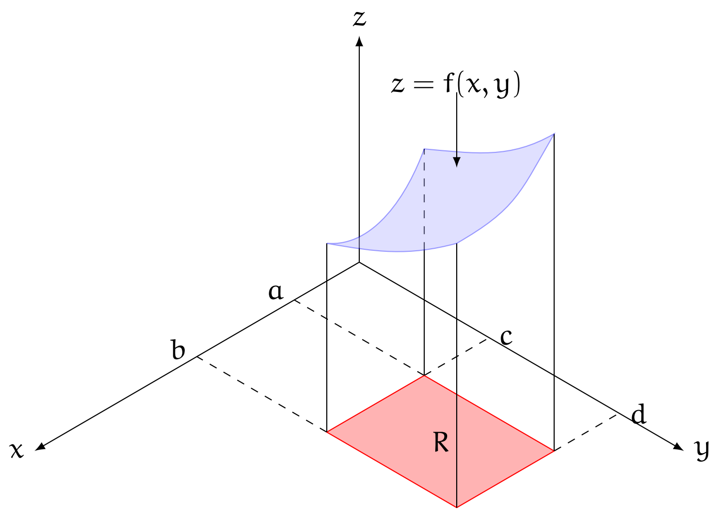


Figure 2.1: The graph of f over the region R

Let us call the solid that fills the space between the xy-plane and the surface $z = f(x, y)$ S. Formally, this is written as

$$S = \{(x, y, z) \in \mathbb{R}^3 | 0 \leq z \leq f(x, y), (x, y) \in \mathbb{R}\}$$

How can we find the volume of the solid, S? We will apply what we learned about Riemann sums and definite integrals in two dimensions to this three dimensional problem.

First, we divide R into rectangular subregions. We do this by dividing the interval $[a, b]$ into m subintervals with width $\Delta x = (b - a)/m$ and the interval $[c, d]$ into n subintervals with width $\Delta y = (d - c)/n$. Drawing lines through these divisions parallel to the x- and y-axes, we create a field of subrectangles, each with area $\Delta A = \Delta x \Delta y$ (see figure 2.2). Each subrectangle is defined by:

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] - \{(x, y)|x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

Since $f(x, y)$ in continuous over the R, there is some point, (x_{ij}^*, y_{ij}^*), equal to the average value of $f(x, y)$ over the subrectangle. Then we can approximate the volume between the xy-plane and $z = f(x, y)$ over the subrectangle as a column with base area ΔA and height

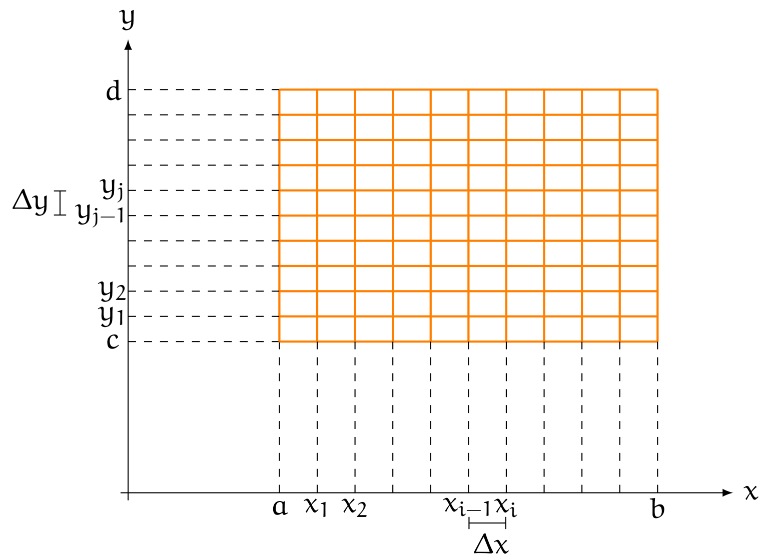


Figure 2.2: The region, R, on the xy-plane divided into subrectangles

$f(x^*_{ij}, y^*_{ij})$ (seefigure 2.3) and the volume of the column is given by:

$$V_{ij} = f(x^*_{ij}, y^*_{ij})\Delta A$$

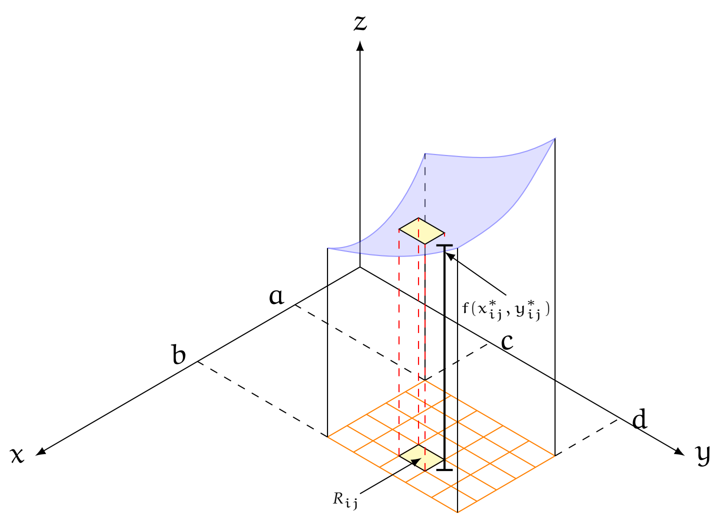


Figure 2.3: A single column with base ΔA and height $f(x^*_{ij}, y^*_{ij})$

And therefore the approximate volume of the solid, S, that lies between the region, R, and $z = f(x, y)$ is the sum of all the columns over i and j:

$$V_S \approx \sum_{i=1}^{n} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A$$

And just like with the area under a curve, we get the true volume by taking the limit as $n \to \infty$, which becomes a **double integral**:

Volume of a Solid over a Region

$$V_S = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R f(x,y)\, dA$$

2.2 Iterated Integrals

To be able to evaluate the double integral as outlined above, we must first discuss iterated integrals. Iterated integrals happen when you evaluate two single integrals, one inside the other. Consider some function, $g(x,y)$. We could integrate that function from $x = q$ to $x = r$ thusly:

$$\int_q^r g(x,y)\, dx$$

Notice that we are integrating with respect to x, so y terms will be treated as constants (recall partial differentiation: this is the opposite process). Let's call the result of this first integral $A(y)$:

$$A(y) = \int_q^r g(x,y)\, dx$$

We can then integrate the resulting function, $A(y)$, from $y = s$ to $y = t$:

$$\int_s^t A(y)\, dy = \int_s^t \left[\int_q^r g(x,y)\, dx\right]\, dy$$

This is called an **iterated integral**. When evaluating iterated integrals, we work from the inside out. You can also write it without the brackets:

$$\int_s^t \int_q^r g(x,y)\, dx\, dy$$

Example: evaluate the iterated integral $\int_0^3 \int_1^2 xy^2\, dy\, dx$.

Solution: We can re-write this to more explicitly show the inner and outer integrals:

$$\int_0^3 \left[\int_1^2 xy^2\, dy\right]\, dx$$

As you can see, the inner integral is with respect to y. Let's isolate and evaluate the inner integral:

$$\int_1^2 xy^2\, dy = x\int_1^2 y^2\, dy = x\left[\frac{1}{3}y^2\right]_{y=1}^{y=2}$$
$$= \frac{x}{3}\left[2^3 - 1^3\right] = \frac{x}{3}[8-1] = \frac{7x}{3}$$

We were able to move x outside the integral because when we are integrating with respect to a specific variable (in this case, y), other variables are treated as constants. Now we can substitute $\int_1^2 xy^2\, dy = \frac{7x}{3}$ into the iterated integral:

$$\int_0^3 \left[\int_1^2 xy^2\, dy\right] dx = \int_0^3 \left[\frac{7x}{3}\right] dx$$
$$= \frac{7}{3}\left[\frac{1}{2}x^2\right]_{x=0}^{x=3} = \frac{7}{6}\left[3^2 - 0^2\right] = \frac{7\cdot 9}{6} = \frac{21}{2}$$

Exercise 7 Order of Evaluating Iterated Integrals

Show that $\int_0^3\int_1^2 xy^2\, dy\, dx = \int_1^2\int_0^3 xy^2\, dx\, dy$.

Working Space

Answer on Page 73

Exercise 8 Evaluating Iterated Integrals

Evaluate the following iterated integrals.

1. $\int_0^1 \int_1^2 (x + e^{-y})\ dx\,dy$
2. $\int_{-3}^3 \int_0^{\pi/2} \left(2y + y^2 \cos x\right)\ dx\,dy$
3. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3 \theta\, d\theta\, dt$

Working Space

Answer on Page 73

2.3 Fubini's Theorem for Double Integrals

Fubini's theorem states that for a function, f, that is continuous over the rectangular region, R, the double integral of f over the region $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$ is equal to the iterated integral of f with respect to x and y. This is expressed mathematically below:

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x, y)\,|a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Exercise 9 Applying Fubini's Theorem

Rewrite the following double integrals as iterated integrals.

1. $\iint_R \frac{xy^2}{x^2+1}\ dA,\ R = \{(x,y)|0 \leq x \leq 1, -3 \leq y \leq 3\}$
2. $\iint_R \frac{\sec\theta}{\sqrt{1+t^2}}\ dA,\ R = \{(\theta,t)|0 \leq \theta \leq \frac{\pi}{4}, 0 \leq t \leq 1\}$

Working Space

Answer on Page 74

Exercise 10

Evaluate the double integral.

1. $\iint_R \frac{xy^2}{x^2+1}\, dA$, $R = \{(x, y) \mid 0 \le x \le 2, \ -3 \le y \le 3\}$
2. $\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\, dA$, $R = \{(\theta, t) \mid 0 \le \theta \le \pi/3, \ 0 \le t \le \frac{1}{2}\}$
3. $\iint_R x \sin(x+y)\, dA$, $R = [0, \pi/6] \times [0, \pi/3]$

Working Space

Answer on Page 74

Chapter 3

Double Integrals Over Non-Rectangular Regions

Now that we've seen how to evaluate double integrals over rectangular regions, let's consider non-rectangular regions. Suppose we are interested in the integral of a function, $f(x, y)$, over a region, D, exists such that it can be bounded by inside a rectangular region, R (see figure 3.1). We can then define a new function:

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

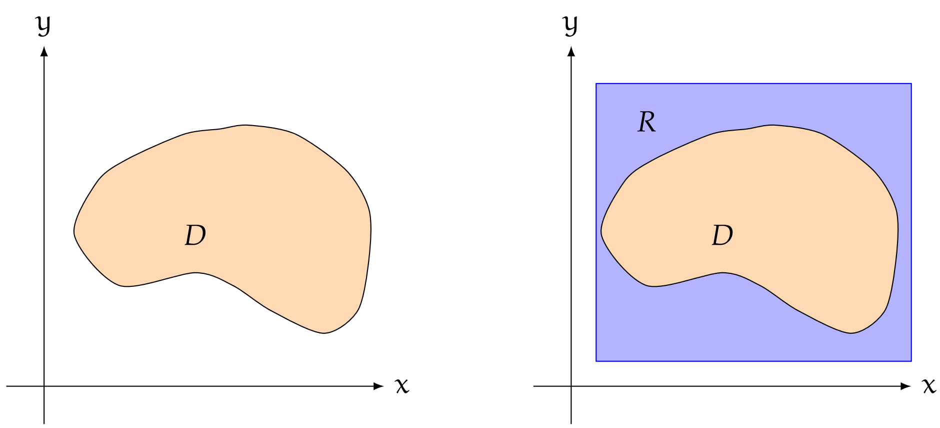


Figure 3.1: We can find a rectangular region, R, that completely encloses D

Then, we can see that:

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA$$

Which makes sense intuitively, since integrating over F outside of D doesn't contribute anything to the integral, and the integral of F inside D is equal to the integral of f inside D. In general, there are two types of regions for D. A region is **type I** if it lies between two continuous functions of x and can be defined thusly:

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

Some type I regions are shown in figure 3.2. To evaluate $\iint_D f(x, y)\, dA$, we begin by choosing a rectangle $R = [a, b] \times [c, d]$ such that D is completely contained in R. We again

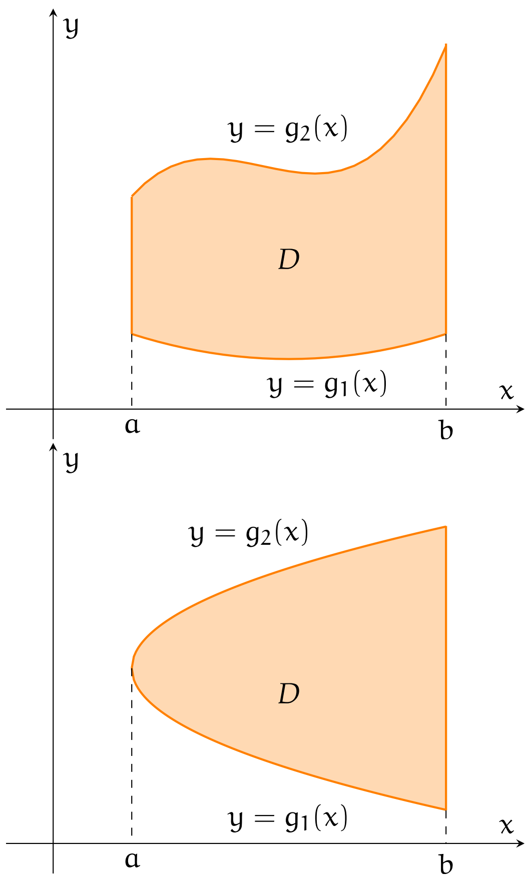


Figure 3.2: Two examples of type I domains

define $F(x, y)$ such that $F(x, y) = f(x, y)$ on D and $F = 0$ outside of D. Then, by Fubini's theorem:

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Since $F(x, y) = 0$ when $y \leq g_1(x)$ or $y \geq g_2(x)$, we know that:

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

Substituting this into the iterated integral above, we see that for a type I region $D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$,

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Another way to visualize the double integral over a type I region is shown in figure 3.3. For any value of $x \in [a, b]$, we know that $g_1(x) \leq y \leq g_2(x)$. The inner integral represents moving along one blue line from $y = g_1(x)$ to $y = g_2(x)$ and integrating with respect to y. Then, for the outer integral, we integrate with respect to x, which is represented by moving the line from $x = a$ to $x = b$.

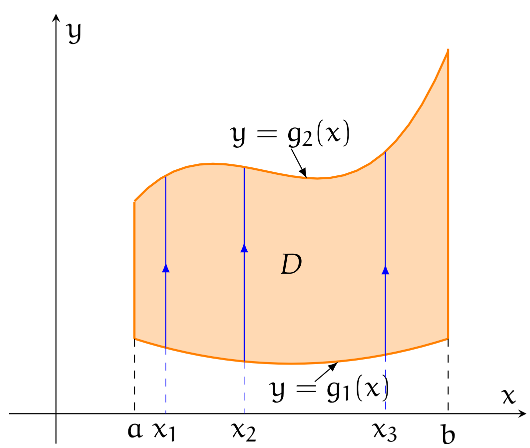


Figure 3.3: On type I domains, for a given value of x, $g_1(x) \leq y \leq g_2(x)$

A **type II** region is a region such that we can define the limits of x in terms of y (see figure 3.4). That is, a type II region can be defined as:

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y0 \leq x \leq h_2(y)\}$$

And in a similar manner to above, we can show that:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

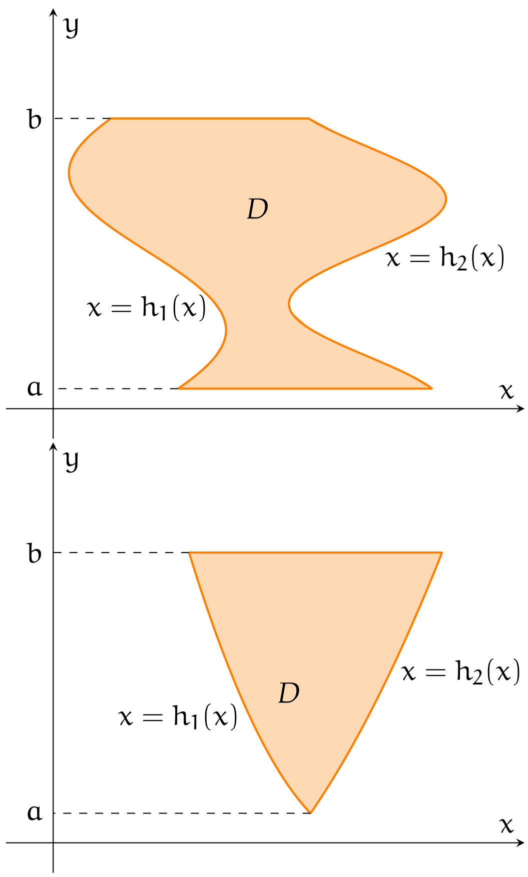


Figure 3.4: Two examples of type II domains

You can annotate type II regions with horizontal lines to show that, for a given y values, all x values in the region are contained in $h_1(y) \leq x \leq h_2(y)$ (see figure 3.5).

3.1 Determining Region Type

Many regions can be described as either type I or type II. Consider the region between the curves $y = \frac{3}{2}(x-1)$ and $y = \frac{1}{2}(x-1)^2$ (see figure 3.6).[fix me classifying domains examples and explanations]

Example: Evaluate $\iint_D (2x+y)\,dA$, where D is the region bounded by the parabolas $y = 3x^2$ and $y = 2 + x^2$. Region D is shown in figure 3.7.

Solution:This is a type I region, since for a given x, $y \in \left[3x^2, 2+x^2\right]$. We can define region

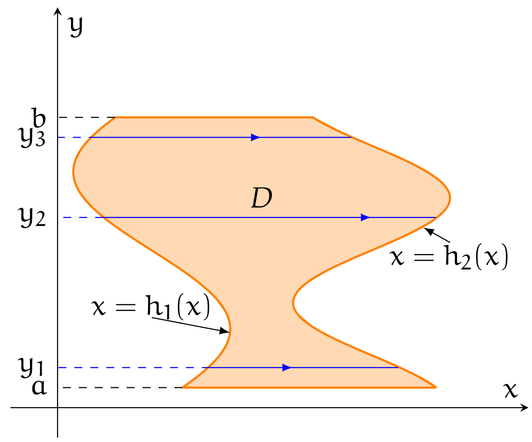


Figure 3.5: On type II domains, for a given value of y, $h_1(y) \leq x \leq h_2(y)$

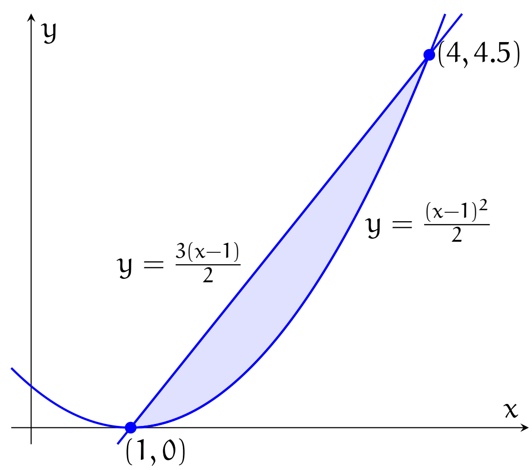


Figure 3.6: The region that lies between $y = \frac{(x-1)^2}{2}$ and $y = \frac{3(x-1)}{2}$ can be classified as type I or type II

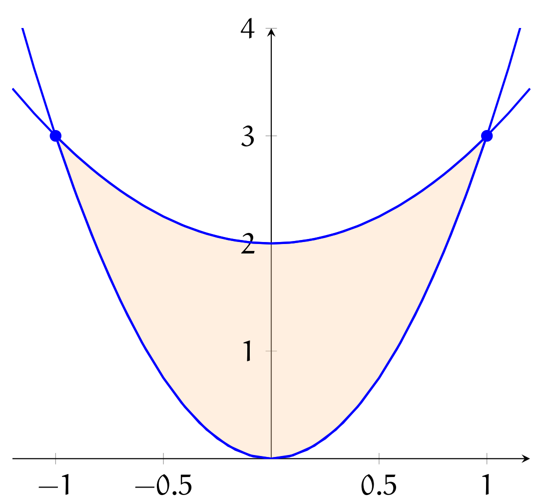


Figure 3.7: Region D is bounded above by $y = 2 + x^2$ and below by $y = 3x^2$

D as $D = \{(x, y) \mid -1 \leq x \leq 1, 3x^2 \leq y \leq 2 + x^2\}$. Therefore,

$$\iint_D (2x + y)\, dA = \int_{-1}^{1} \int_{3x^2}^{2+x^2} (2x + y)\ dy\, dx$$

$$= \int_{-1}^{1} \left[\int_{3x^2}^{2+x^2} 2x\, dy + \int_{3x^2}^{2+x^2} y\, dy \right] dx$$

$$= \int_{-1}^{1} \left[2xy|_{y=3x^2}^{y=2+x^2} + \frac{1}{2} y^2|_{y=3x^2}^{y=2+x^2} \right] dx$$

$$= \int_{-1}^{1} \left[2x \left(2 + x^2 - 3x^2 \right) + \frac{1}{2} \left((2 + x^2)^2 - (3x^2)^2 \right) \right] dx$$

$$= \int_{-1}^{1} \left[2 + 4x + 2x^2 - 4x^3 - 4x^4 \right] dx$$

$$= \left[2x + 2x^2 + \frac{2}{3}x^3 - x^4 - \frac{4}{5}x^5 \right]_{x=-1}^{x=1}$$

$$= \left(2 + 2 + \frac{2}{3} - 1 - \frac{4}{5} \right) - \left(-2 + 2 - \frac{2}{3} - 1 + \frac{4}{5} \right)$$

$$= 4 + \frac{4}{3} - \frac{8}{5} = \frac{56}{15}$$

Exercise 11 Double Integrals over Non-Rectangular Regions

Working Space

Evaluate the double integral.

1. $\iint_D e^{-y^2}\, dA$, $D = \{(x, y) \mid 0 \leq y \leq 3, 0 \leq x \leq 2y\}$.
2. $\iint_D x \sin y\, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 2$.
3. $\iint_D (2y - x)\, dA$, D is bounded by the circle with center at the origin and radius 3.

Answer on Page 76

3.2 Double Integrals in Other Coordinate Systems

Consider a region composed of a semi-circular ring (see figure **??**). Describing the region in Cartesian coordinates is complicated: you would have to split it into 3 regions (see figure ...). However, in polar coordinates, we can describe the whole region in one statement:

$$D = \{(r, \theta) \mid 1 \leq r \leq 4,\ 0 \leq \theta \leq \pi\}$$

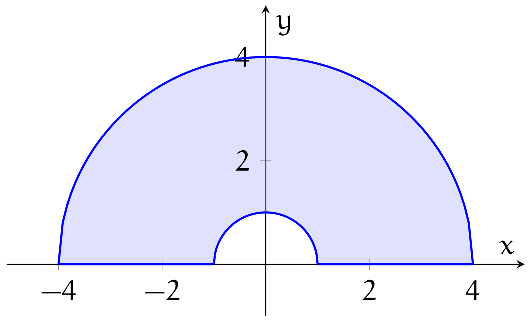


Figure 3.8: A semi-circular ring

There are many instances where a region is simpler to describe in polar coordinates, so how do we take double integrals in polar coordinates? Suppose we want to integrate some function, $f(x,y)$, over a polar rectangle described by $D = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$ (see figure 3.9). Similar to Cartesian coordinates, we can divide this region in to many smaller polar rectangles, with each subrectangle defined by $D_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{i-1} \le \theta \le \theta_i\}$. And the center of each subrectangle has polar coordinates (r_i^*, θ_j^*), where:

$$r_i^* - \frac{1}{2}\left(r_{i-1} + r_i\right)$$

$$\theta_j^* = \frac{1}{2}\left(\theta_{j-1} + \theta_j\right)$$

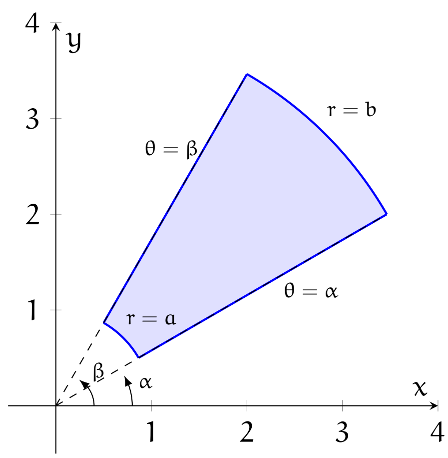


Figure 3.9: A polar rectangle described by $D = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$

Each subrectangle is a larger radius sector minus a smaller radius sector, each with the same central angle, $\Delta\theta = \theta_j - \theta_{j-1}$. Then the total area of each subrectangle is given by:

$$\Delta A_i = \frac{1}{2}\left(r_i\right)^2 \delta\theta - \frac{1}{2}\left(r_{i-1}\right)^2 \Delta\theta = \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta$$

Substituting $\left(r_i^2 - r_{i-1}^2\right) = (r_i + r_{i-1})(r_i - r_{i-1})$, we see that:

$$\Delta A_i = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta$$

Recall that we have defined $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$. Additionally, $\Delta r = r_i - r_{i-1}$. Substituting this, we find a simplified expression for the area of each subrectangle:

$$\Delta A_i = r_i^* \Delta r \Delta\theta$$

And therefore the Riemann sum of $f(x, y)$ over the region is:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_i$$

(Recall that to convert from Cartesian to polar coordinates, we use $x = r\cos\theta$ and $y = r\sin\theta$). Substituting for δA_i:

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta\theta$$

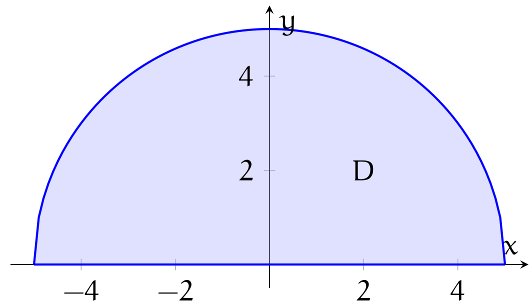
Taking the limit as $n \to \infty$, the Riemann sum becomes the double integral:

$$\int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

And therefore, if f is continuous on the polar rectangle $a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, then:

$$\iint_D f(x, y)\, dA = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

Example: Evaluate $\iint_D x^2 y \, dA$, where D is the semi-circle shown below.



Solution: Since the region is a semi-circle with radius 5, we can describe D as $D = \{(r, \theta) \mid 0 \leq r \leq 5,\ 0 \leq \theta \leq \pi\}$. Therefore,

$$\iint_D x^2 y \, dA = \int_0^{\pi} \int_0^5 (r\cos\theta)^2 (r\sin\theta)\, r \, d\theta \, dr$$

$$= \int_0^{\pi} \int_0^5 r^4 \cos^2\theta \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \cos^2\theta \sin\theta \left[\frac{1}{5} r^5\right]_{r=0}^{r=5} d\theta$$

$$= \int_0^{\pi} \cos^2\theta \sin\theta \frac{5^5}{5} \, d\theta = 625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

Using u-substitution, let $u = \cos\theta$. Then $-du = \sin\theta d\theta$ and therefore:

$$625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = 625 \int_{\theta=0}^{\theta=\pi} -u^2 \, du$$

$$= -625 \frac{1}{3} u^3 |_{\theta=0}^{\theta=\pi} = -625 \frac{1}{3} \left(\cos^3\theta\right) |_{\theta=0}^{\theta=\pi}$$

$$= -\frac{625}{3} \left[(-1)^3 - (1)^3\right] = -\frac{625}{3}(-2) = \frac{1250}{3}$$

Exercise 12 Changing to Polar Coordinates

Evaluate the following iterated integrals by converting to polar coordinates:

1. $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$
2. $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy$
3. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

Working Space

Answer on Page 76

Exercise 13 Using Polar Coordinates in Multiple Integration

Find the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy-plane.

Working Space

Answer on Page 78

Exercise 14 The volume of a pool

A circular swimming pool has a 40-ft diameter. The depth of the pool is constant along the north-south axis and increases from 3 feet at the west end to 10 feet at the east end. What is the total volume of water in the pool?

Working Space

Answer on Page 79

CHAPTER 4

Applications of Double Integrals

4.1 Total Mass and Charge

Suppose there is a generic, thin layer (called a *lamina*) with a variable density that occupies an area B (see figure 4.1). Further, let the density of the lamina be described by a function, $\rho(x, y)$, which is continuous over B. For some small rectangle centered at (x, y), the density is given by:

$$\rho(x, y) = \frac{\Delta m}{\Delta A}$$

Where Δm is the mass of the small rectangle and ΔA is the area. Then the mass of the rectangle is given by:

$$\Delta m = \rho(x, y)\Delta A$$

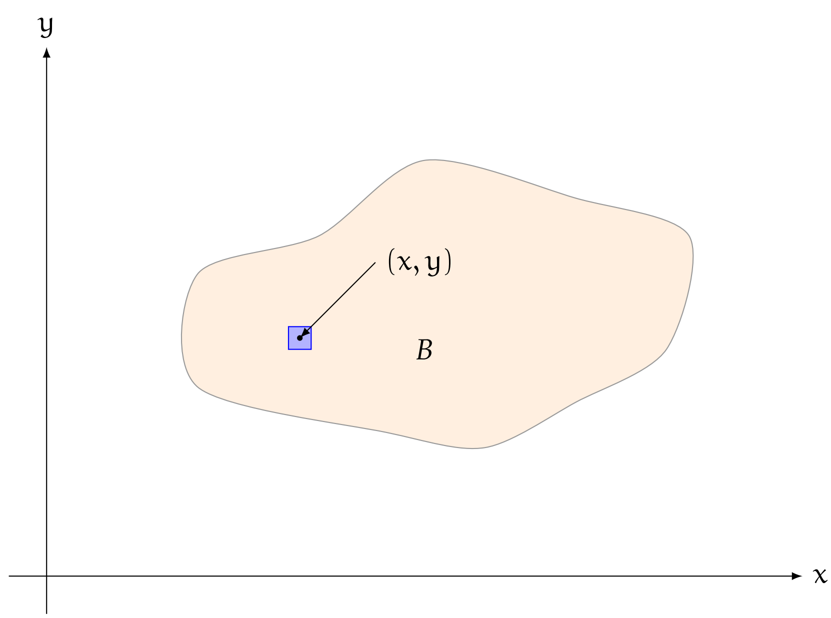


Figure 4.1: A generic lamina that occupies the region B

We can find the mass of the entire lamina by dividing it into many of these small rectangles and adding the masses of all the rectangles (see 4.2). Just like in previous examples, there is some point (x_{ij}^*, y_{ij}^*) in each rectangle, R_{ij}, such that the mass of the part of the lamina that occupies R_{ij} is $\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Adding all these masses yields:

$$m_{total} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

Taking the limit as $m, n \to \infty$ increases the number of rectangles to yield the true total mass:

$$m_{total} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_B \rho(x, y)\, dA$$

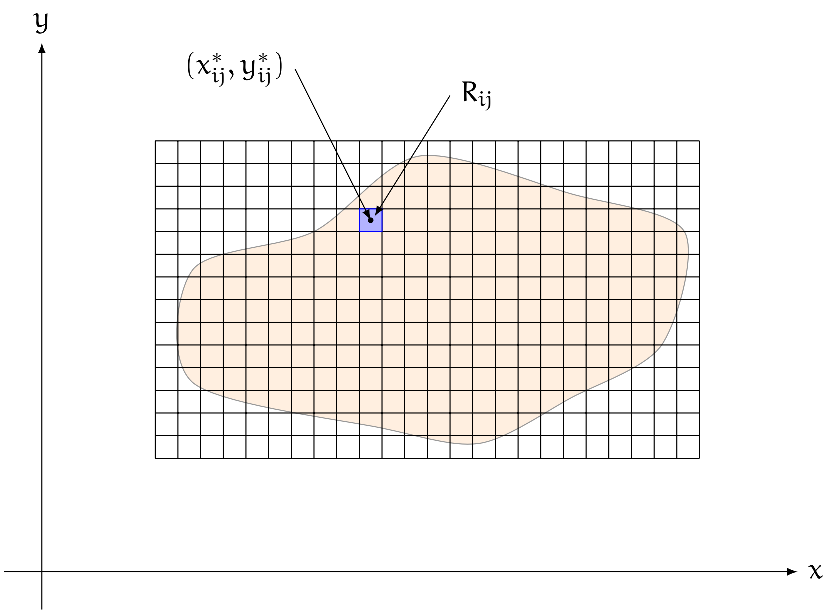


Figure 4.2: A generic lamina divided into many rectangles

Example: Find the total mass of a lamina that occupies the region $D = \{(x, y) \mid 1 \leq x \leq 3, 1 \leq y \leq 4\}$ with a density function $\rho(x, y) = 3y^2$.

Solution: We know that the total mass is given by:

$$\iint_D 3y^2\, dA$$

Applying Fubini's theorem, we see that:

$$\iint_D 3y^2\, dA = \int_1^3 \int_1^4 3y^2\, dy\, dx$$

$$= \int_1^3 \left[y^3\right]_{y=1}^{y=4} dx = \int_1^3 \left[4^3 - 1^3\right]\, dx$$

$$= \int_1^3 63\,dx = 63x\Big|_{x=1}^{x=3} = 126$$

Exercise 15 Finding Total Mass

Find the mass of the lamina that occupies the region, D, and has the given density function, ρ.

1. $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$; $\rho(x, y) = 1 + x^2 + y^2$
2. D is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

Working Space

Answer on Page 80

This method applies not only to mass density, but any other type of density. Some examples could include animals per acre of forest, cells per square centimeter of petri dish, or people per city block. A density physicists are often interested in is charge density (that is, the amount of charge, Q, per unit area). Charge is measured in coulombs (C). Often, charge density is given by a function, $\sigma(x, y)$, in units of coulombs per area (such as cm^2 or m^2). If there is some region, D, with charge distributed across it such that the charge density can be described by a continuous function, $\sigma(x, y)$, then the total charge, Q, is given by:

$$Q = \iint_D \sigma(x, y)\,dA$$

Example: Charge is distributed over the region B shown in figure 4.3 such that the charge density is given by $\sigma(x, y) = xy$, measured in C/m^2. Find the total charge.

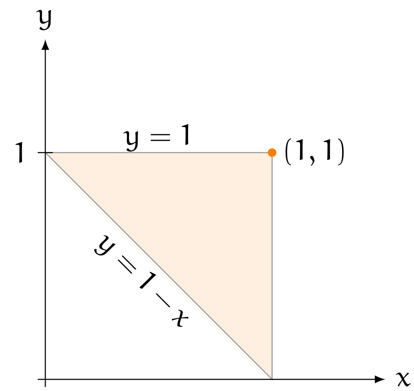


Figure 4.3: A triangular region over which charge is distributed such that $\sigma(x, y) = xy$

Solution: We know that total charge is given by:

$$Q = \iint_B xy \, dA$$

Examining figure 4.3, we see that:

$$\iint_B xy \, dA = \int_0^1 \int_{1-x}^1 xy \, dy \, dx$$

$$= \int_0^1 \frac{x}{2} \left[y^2\right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2} \left[1^2 - (1-x)^2\right] dx$$

$$= \frac{1}{2} \int_0^1 x \left(1 - 1 + 2x - x^2\right) dx = \frac{1}{2} \int_0^1 x \left(2x - x^2\right) dx$$

$$= \frac{1}{2} \int_0^1 2x^2 - x^3 \, dx = \frac{1}{2} \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{x=0}^{x=1} = \frac{1}{2}\left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{24} C$$

4.2 Center of Mass

For a thin disk (lamina) of variable density in the xy-plane, the coordinates of the center of mass, $(\overline{x}, \overline{y})$, are given by:

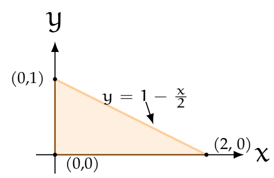
$$\overline{x} = \frac{1}{m} \iint_D x\rho(x, y) \, dA$$

$$\overline{y} = \frac{1}{m} \iint_D y\rho(x, y) \, dA$$

Where m is the total mass and ρ is the density of the lamina as a function of x and y.

Example: Find the center of mass of a triangular lamina with vertices at $(0,0)$, $(2,0)$, and $(0,1)$ and a density function $\rho(x,y) = 2 + x + 3y$.

Solution: We begin by visualizing the region so we can determine if it is type I or type II:



Recall that the total mass is given by $m = \iint_D \rho(x,y)\,dA$. As shown above, we can define $D = \{(x,y)\,|0 \le x \le 2, 0 \le y \le 1 - \frac{x}{2}\}$:

$$m = \int_0^2 \int_0^{1-x/2} (2 + x + 3y)\ dy\,dx = \int_0^2 \left[2y + xy + \frac{3}{2}y^2\right]_{y=0}^{y=1-x/2} dx$$

$$= \int_0^2 \left[2(1 - \frac{x}{2}) + x(1 - \frac{x}{2}) + \frac{3}{2}(1 - \frac{x}{2})^2\right] dx = \int_0^2 \left[\frac{7}{2} - \frac{3x}{2} - \frac{x^2}{8}\right] dx$$

$$= \left[\frac{7x}{2} - \frac{3x^2}{4} - \frac{x^3}{24}\right]_{x=0}^{x=2} = \frac{7(2)}{2} - \frac{3(4)}{4} - \frac{8}{24} = 7 - 3 - \frac{1}{3} = \frac{11}{3}$$

Finding $\overline{x}$:

$$\overline{x} = \frac{1}{m}\iint_D x\,(2 + x + 3y)\ dA$$

$$\overline{x} = \frac{3}{11}\int_0^2 \int_0^{1-x/2} \left[2x + x^2 + 3xy\right] dy\,dx$$

$$\overline{x} = \frac{3}{11}\int_0^2 \left[2xy + x^2y + \frac{3}{2}xy^2\right]_{y=0}^{y=1-x/2} dx$$

$$\overline{x} = \frac{3}{11}\int_0^2 \left[\frac{7x}{2} - \frac{3x^2}{2} - \frac{x^3}{8}\right] dx$$

$$\overline{x} = \frac{3}{11}\left[\frac{7x^2}{4} - \frac{x^3}{2} - \frac{x^4}{32}\right]_{x=0}^{x=2}$$

$$\overline{x} = \frac{3}{11}\left[\frac{7(4)}{4} - \frac{8}{2} - \frac{16}{32}\right] = \frac{3}{11}\left(7 - 4 - \frac{1}{2}\right) = \frac{3}{11}\left(\frac{5}{2}\right) = \frac{15}{22}$$

And we can similarly find $\overline{y}$:

$$\overline{y} = \frac{1}{m}\iint_D y\,(2 + x + 3y)\ dA$$

$$\overline{y} = \frac{3}{11}\int_0^2\int_0^{1-x/2}\left[2y + xy + 3y^2\right]\,dy\,dx$$

$$\overline{y} = \frac{3}{11}\int_0^2\left[y^2 + \frac{x}{2}y^2 + y^3\right]_{y=0}^{y=1-x/2}\,dx$$

$$\overline{y} = \frac{3}{11}\int_0^2\left[\left(1-\frac{x}{2}\right)^2 + \frac{x}{2}\left(1-\frac{x}{2}\right)^2 + \left(1-\frac{x}{2}\right)^3\right]\,dx$$

$$\overline{y} = \frac{3}{11}\int_0^2\left[2 - 2x + \frac{x^2}{2}\right]\,dx = \frac{3}{11}\left[2x - x^2 + \frac{x^3}{6}\right]_{x=0}^{x=2}$$

$$\overline{y} = \frac{3}{11}\left[2(2) - 2(2) + \frac{8}{6}\right] = \frac{3}{11}\left(\frac{4}{3}\right) = \frac{4}{11}$$

Therefore, the center of mass $(\overline{x}, \overline{y})$ is $(\frac{15}{22}, \frac{4}{11})$.

Exercise 16 Center of Mass

Find the center of mass of

1. a lamina that occupies the area enclosed by the curves $y = 0$ and $y = 2\sin x$ from $0 \leq x \leq \pi$ if its density is given by $\rho(x, y) = x$.
2. the region D if $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$; $\rho(x, y) = 1 + x^2 + y^2$
3. The triangular region D with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

Working Space

Answer on Page 81

4.3 Moment of Inertia

We can also use double integrals to find the **moment of inertia** of a lamina about a particular axis (we will extend this to 3-dimensional objects in the next chapter on triple integrals). Recall that the moment of inertia for a particle with mass m a distance r from the axis of rotation is mr^2. Dividing a lamina into small pieces, we see that the moment of inertia of each piece about the x-axis is:

$$\left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

Where x_{ij}^* and y_{ij}^* are the x- and y-coordinates of the small piece. The moment of inertia of the entire lamina about the x-axis is then the sum of all the individual moments:

$$I_x = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_D y^2 \rho(x, y)\ dA$$

Similarly, the moment of inertia of a lamina about the y-axis is:

$$I_y = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_D x^2 \rho(x, y)\ dA$$

Example: Find the moment of inertia of a square centered at the origin with side length r and constant density ρ about the x-axis.

Solution: We can describe the square as the region bounded by $D = \{(x, y) | -r/2 \leq x \leq r/2, -r/2 \leq y \leq r/2\}$ with density function $\rho(x, y) = \rho$. Therefore, the moment of inertia about the x-axis is given by:

$$\begin{aligned}
I_x &= \int_{-r/2}^{r/2} \int_{-r/2}^{r/2} \rho y^2\, dy\, dx \\
&= \rho \int_{-r/2}^{r/2} \left[\frac{1}{3} y^3\right]_{y=-r/2}^{y=r/2} dx \\
&= \frac{\rho}{3} \int_{-r/2}^{r/2} \left[\frac{r^3}{8} - \left(-\frac{r^3}{8}\right)\right] dx \\
&= \frac{\rho}{3} \int_{-r/2}^{r/2} \left[\frac{r^3}{4}\right] dx = \frac{r^3 \rho}{12} \int_{-r/2}^{r/2} 1\, dx \\
&= \frac{r^3 \rho}{12} [x]_{x=-r/2}^{x=r/2} = \frac{r^3 \rho}{12} \cdot r = \frac{r^4 \rho}{12}
\end{aligned}$$

We can also find the moment of inertia about the origin, I_o. This is the moment of inertia for an object rotating in the xy-plane about the origin. The moment of inertia about the origin is the sum of the moments of inertia about the x- and y-axes:

$$I_o = I_x + I_y = \iint_D \left(x^2 + y^2\right) \rho(x, y)\, dA$$

Example: Find the moment of inertia about the origin of a disk with density $\rho(x, y) = b$, centered at the origin, with a radius of a. Show this is equal to the expected moment of inertia, $\frac{1}{2}MR^2$, where M is the total mass of the disk and R is the radius of the disk.

Solution: Since we are examining a circle about the origin, the region can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$. Converting from Cartesian coordinates to polar coordinates:

$$I_o = \iint_D \left(x^2 + y^2\right) b\, dA = \int_0^a \int_0^{2\pi} r \left(r^2\right) b\, d\theta\, dr$$

$$= \int_0^a r^3\, dr \cdot \int_0^{2\pi} b\, d\theta = \frac{a^4}{4} \cdot (2\pi b) = \frac{\pi a^4 b}{2}$$

The total mass of this disk is the density, b, multiplied by the area, πa^2. Therefore,

$$R = a$$

$$M = \pi a^2 b$$

Substituting into the result of our double integral, we see that:

$$\frac{\pi a^4 b}{2} = \left(\pi a^2 b\right) \cdot \left(\frac{a^2}{2}\right) = M \cdot \frac{R^2}{2} = \frac{1}{2} M R^2$$

4.3.1 Radius of Gyration

When modeling rotating objects, it can be helpful to have a simplified model. A spinning, continuous object can be modeled as a point mass by using the lamina's *radius of gyration*. The radius of gyration of a lamina about the origin is a radius, R, such that:

$$mR^2 = I_o$$

where m is the mass of the lamina and I is the moment of inertia of the lamina. Essentially, we are finding a radius such that if the lamina were shrunk down to a point mass and rotated about the axis at that radius, the moment of inertia would be the same.

We can also find radii of gyration about the x- and y-axes:

$$m\bar{\bar{y}}^2 = I_x$$
$$m\bar{\bar{x}}^2 = I_y$$

About the origin, $R = \sqrt{\bar{\bar{x}}^2 + \bar{\bar{y}}^2}$.

Example: Find the radius of gyration about the y-axis for a disk with density $\rho(x, y) = y$ if the disk has radius 2 and is centered at $(0, 2)$.

Solution: We are ultimately looking for a radius such that $m\bar{\bar{x}}^2 = I_y$, so we need to know the mass, m, and the moment of inertia about the y-axis, I_y. First, let's find the total mass, m, of the disk. We can describe the disk in polar coordinates as $D = \{(r, \theta) \mid 0 \le r \le 4\sin\theta, 0 \le \theta \le \pi\}$, and therefore the mass is given by:

$$m = \iint_D y\, dA = \int_0^{\pi} \int_0^{4\sin\theta} r\,(r\sin\theta)\; dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta \int_0^{4\sin\theta} r^2\, dr\, d\theta = \frac{1}{3}\int_0^{\pi} \sin\theta\,[4\sin\theta]^3\; d\theta$$

$$= \frac{64}{3}\int_0^{\pi} \sin^4\theta\, d\theta = \frac{64}{3}\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = \frac{64}{3}\left(\frac{1}{2}\right)^2 \int_0^{\pi} \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{16}{3}\left[(\theta - \sin 2\theta)_{\theta=0}^{\theta=\pi} + \int_0^{\pi} \frac{1+\cos 4\theta}{2}\, d\theta\right] = \frac{16}{3}\left[\pi + \frac{1}{2}\left(\theta + \frac{1}{4}\sin 4\theta\right)_{\theta=0}^{\theta=\pi}\right]$$

$$= \frac{16}{3}\left[\pi + \frac{\pi}{2}\right] = \frac{16}{3}\left(\frac{3\pi}{2}\right) = 8\pi$$

Now that we have found the mass, let's find the moment of inertia, I_y:

$$I_y = \iint_D x^2 y\, dA = \int_0^{\pi} \int_0^{4\sin\theta} r\,(r\cos\theta)^2\,(r\sin\theta)\; dr\, d\theta$$

$$= \int_0^{\pi} \left[\cos^2\theta \sin\theta \int_0^{4\sin\theta} r^4\, dr\right] d\theta = \int_0^{\pi} \cos^2\theta\sin\theta \left[\frac{1}{5}r^5\right]_{\theta=0}^{\theta=4\sin\theta} d\theta$$

$$= \frac{1024}{5}\int_0^{\pi} \cos^2\theta \sin^6\theta\, d\theta = \frac{1024}{5}\int_0^{\pi} \left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)^3 d\theta$$

$$= \frac{1024}{5}\left(\frac{1}{2}\right)^4 \int_0^{\pi} (1+\cos 2\theta)\,(1-\cos 2\theta)\,(1-\cos 2\theta)^2\; d\theta$$

$$= \frac{64}{5}\int_0^{\pi} \left(1 - \cos^2 2\theta\right)\left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{64}{5}\int_0^{\pi} 1 - 2\cos 2\theta + \cos^2 2\theta - \cos^2 2\theta + 2\cos^3 2\theta - \cos^4 2\theta \, d\theta$$

$$= \frac{64}{5}\int_0^{\pi} 1 - 2\cos 2\theta + 2\cos 2\theta\left(1 - \sin^2 2\theta\right) - \left(\frac{1+\cos 4\theta}{2}\right)^2 \, d\theta$$

$$= \frac{64}{5}\int_0^{\pi} 1 + 2\cos 2\theta \sin^2 2\theta - \frac{1}{4}\left(1 + 2\cos 4\theta + \cos^2 4\theta\right) \, d\theta$$

$$= \frac{64}{5}\int_0^{\pi} 1 + 2\cos 2\theta \sin^2 2\theta - \frac{1}{4} - \frac{\cos 4\theta}{2} - \frac{1}{4}\left(\frac{1+\cos 8\theta}{2}\right) \, d\theta$$

$$= \frac{64}{5}\left[\frac{5\theta}{8} + \frac{1}{3}\sin^3\theta - \frac{\sin 4\theta}{8} - \frac{\sin 8\theta}{64}\right]_{\theta=0}^{\theta=\pi} = \frac{64}{5}\cdot\frac{5\pi}{8} = 8\pi$$

We have found that $m = 8\pi$ and $I_y = 8\pi$. Substituting to find the radius of gyration:

$$m\bar{\bar{x}}^2 = I_y$$

$$(8\pi)\bar{\bar{x}}^2 = 8\pi$$

$$\bar{\bar{x}} = 1$$

Therefore, the radius of gyration about the y-axis is $\bar{\bar{x}} = 1$

Exercise 17 Moments of Inertia and Radii of Gyration

Find the requested moment of inertia and radius of gyration of the lamina with the given density function.

1. about the x-axis, $D = \{(x, y) \mid 1 \leq x \leq 4,\ 0 \leq y \leq 3\}$, $\rho(x, y) = xy$.
2. about the y-axis, D is enclosed by the curves $y = 0$ and $y = 2\cos x$ for $-\pi/2 \leq x \leq \pi/2$, $\rho(x, y) = x$.
3. about the origin, $D = \{(r, \theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$, $\rho(r, \theta) = r$.

Working Space

Answer on Page 82

4.4 Surface Area

We have already seen how to find the areas of surfaces of revolution using single-variable calculus. Now, we will use multivariable calculus to find the surface area of a generic, two-variable function, $z = f(x, y)$. Suppose a surface, S, is defined by the continuous, partially differentiable function, $f(x, y)$, over a rectangular region, R (see figure 7.3).

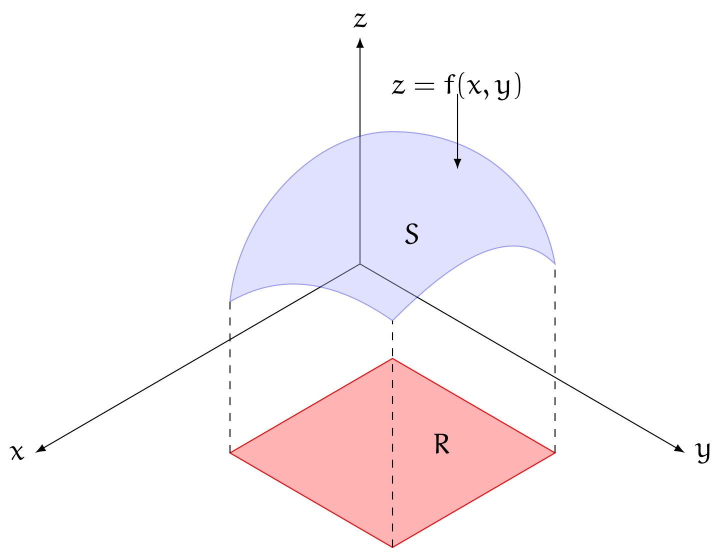


Figure 4.4: The graph of f over the region R creates a surface, S

We begin by dividing the region, R, into sub-rectangles, R_{ij}, each with area $\Delta A = \Delta x \Delta y$. Then projecting upwards from the point closest to the origin, $(x_i, y_j, 0)$, we find a point on the surface, $P_{ij} = (x_i, y_j, f(x_i, y_j))$. Then there is a small plane, ΔT_{ij}, tangent to the surface at P_{ij}, and the area of the tangent plane is approximately the same as the area of the surface over the sub-rectangle R_{ij} (see figure 4.5).

It follows that the total surface area of the surface, S, is the sum of all these little tangent surfaces as the number of tangent surfaces approaches infinity:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

How can we find an expression for ΔT_{ij}? We will define two vectors, $\mathbf{a}$ and $\mathbf{b}$, that are equal to the sides of ΔT_{ij} (see figure 4.6). Geometrically, the area of ΔT_{ij} is the absolute value of the cross product of the two vectors. Mathematically,

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$$

Recall the three unit vectors: $\mathbf{i}$ in the x-direction, $\mathbf{j}$ in the y-direction, and $\mathbf{k}$ in the z-

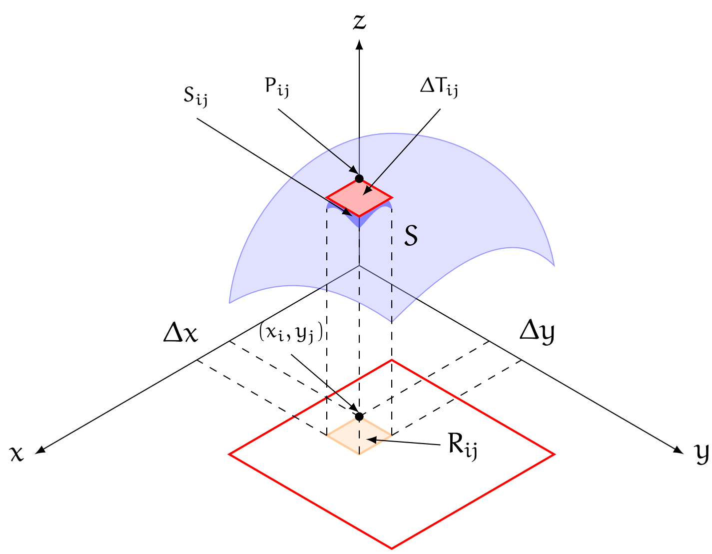


Figure 4.5: The tangent surface, ΔT_{ij}, is approximately the same surface area as the surface, S_{ij}, over the sub-rectangle, R_{ij}

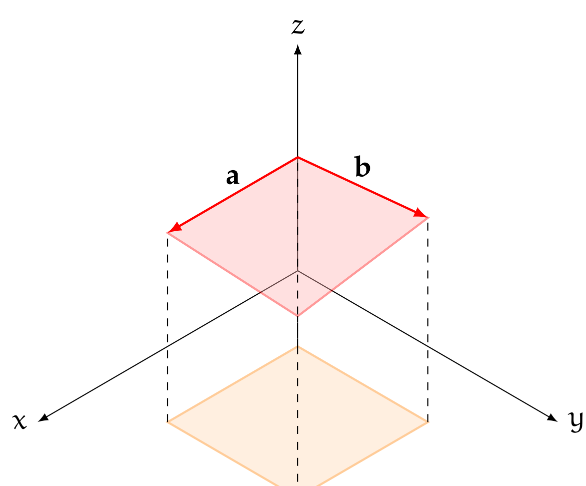


Figure 4.6: The vectors **a** and **b** define the sides of the tangent surface ΔT_{ij}

direction. Then we can describe **a** and **b** in terms of **i**, **j**, and **k**:

$$\mathbf{a} = \Delta x\mathbf{i} + f_x(x_i, y_j)\Delta x\mathbf{k}$$

$$\mathbf{b} = \Delta y\mathbf{j} + f_y(x_i, y_j)\Delta y\mathbf{k}$$

(Recall that f_x is the partial derivative of $f(x, y)$ with respect to x, and f_y is the partial derivative with respect to y.) This is true because the partial derivative of f_x gives the slope of a tangent line parallel to the x-axis, and f_y parallel to the y-axis. Then we find an expression for $|\mathbf{a} \times \mathbf{b}|$ (we've omitted some details here):

$$\mathbf{a} \times \mathbf{b} = -f_x(x_i, y_j)\Delta x\Delta y\mathbf{i} - f_y(x_i, y_j)\Delta x\Delta y\mathbf{j} + \Delta x\Delta y\mathbf{k}$$

Substituting $\Delta A = \Delta x\Delta y$:

$$\mathbf{a} \times \mathbf{b} = [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A$$

To find the area of ΔT_{ij}, we need to find the length of $\mathbf{a} \times \mathbf{b}$. Recall that we can use the Pythagorean theorem to find the length of a vector. For a 3-dimensional vector $\mathbf{v} = r\mathbf{i} + s\mathbf{j} + t\mathbf{k}$, it's length is given by:

$$|\mathbf{v}| = \sqrt{r^2 + s^2 + t^2}$$

Applying this, we find the length of $\mathbf{a} \times \mathbf{b}$ (which is the same as the area of ΔT_{ij} is:

$$\Delta T_{ij} = |\,[-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A|$$

$$= \sqrt{(-f_x(x_i, y_j)\Delta A)^2 + (-f_y(x_i, y_j)\Delta A)^2 + (\Delta A)^2}$$

$$= \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\Delta A$$

And then the area of the entire surface over region R is:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\Delta A$$

This is the definition of a double integral, and therefore the surface area of a two-variable function, $f(x, y)$ over a region, R, where f_x and f_y are continuous, is:

$$A(S) = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Using the notation of partial derivatives, this is also expressed as:

$$A(S) = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example: Find the surface area of the part of the surface $z = 2 - y^2$ that lies over the triangle whose vertices are at $(0, 0)$, $(0, 4)$, and $(3, 4)$.

Solution: We can define $R = \{(x, y) \mid 0 \leq x \leq 34y,\ 0 \leq y \leq 4\}$. Additionally,

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = -2y$$

And therefore the area of the surface that lies above R is:

$$A(S) = \iint_R \sqrt{1 + 0^2 + (-2y)^2}\, dA = \int_0^4 \int_0^{\frac{3}{4}y} \sqrt{1 + 4y^2}\, dx\, dy$$

$$= \int_0^4 \sqrt{1 + 4y^2}\, [x]_{x=0}^{x=\frac{3}{4}y}\ dy = \frac{3}{4} \int_0^4 y\sqrt{1 + 4y^2}\, dy$$

Let $u = 1 + 4y^2$, then $du = (8y)dy$ and $(y)dy = \frac{du}{8}$. Substituting:

$$A(S) = \frac{3}{4} \int_{y=0}^{y=4} \frac{1}{8}\sqrt{u}\, du = \frac{3}{32}\left[\frac{2}{3}u^{3/2}\right]_{y=0}^{y=4}$$

$$= \frac{1}{16}\left[\left(1 + 4y^2\right)^{3/2}\right]_{y=0}^{y=4} = \frac{1}{16}\left[(65)^{3/2} - 1\right] \approx 32.69$$

Exercise 18 Surface Area of Two-Variable Functions

Find the area of the surface.

1. The part of the plane $9x+6y-3z+6=0$ that lies above the rectangle $[2,6]\times[1,4]$.
2. The part of the paraboloid in the circle $z=2x^2+2y^2$ that lies under the plane $z=32$.
3. The part of the surface $z=3xy$ that lies in the cylinder $x^2+y^2=4$.

Working Space

Answer on Page 84

4.5 Average Value

Recall that the average value of a one-variable function over the interval $x\in[a,b]$ is given by:

$$f_{ave}=\frac{1}{a-b}\int_a^b f(x)\,dx$$

For a two-variable function, the average value over a region, R, is given by:

$$f_{ave}=\frac{1}{A(R)}\iint_R f(x,y)\,dA$$

Where $A(R)$ is the area of the two-dimensional region.

Example: Find the average value of $f(x,y)=xy^2$ over the rectangle with vertices at $(-2,0)$, $(-2,4)$, $(2,4)$, and $(2,0)$.

Solution: The rectangular region has an area of $(2-(-2))\cdot(4-0)=4\cdot 4=16$. Therefore, the average value is given by:

$$f_{ave}=\frac{1}{16}\iint_R xy^2\,dA=\frac{1}{16}\int_{-2}^{2}\int_0^4 xy^2\,dy\,dx$$

$$= \frac{1}{16}\int_{-2}^{2}\frac{x}{3}y^3|_{y=0}^{y=4}\,dx = \frac{1}{16}\int_{-2}^{2}\frac{x}{3}\left(4^3\right)\,dx = \frac{4}{3}\int_{-2}^{2}x\,dx$$
$$= \frac{4}{3}\left(\frac{1}{2}\right)x^2|_{x=-2}^{x=2} = 0$$

Exercise 19 Average Value

Find the average value of the function over the region D:

1. $f(x, y) = x \sin y$, $D = [0, 2] \times [-\pi/2, \pi/2]$
2. $f(x, y) = x + y$, D is the circle with radius 1 centered at $(1, 0)$
3. $f(x, y) = xy$, D is the triangle with vertices at $(0, 0)$, $(2, 0)$, $(2, 2)$

Working Space

Answer on Page 86

Chapter 5

Multivariate Distributions

The world of probability and statistics doesn't limit itself to the study of single variables. Often, we are interested in the interconnections, relationships, and associations among several variables. In such a scenario, the univariate distributions that we have studied so far become inadequate. To comprehend the joint behavior of these variables and to uncover the underlying patterns of dependency, we must turn to the realm of multivariate distributions.

This chapter aims to introduce the reader to the concept of multivariate probability distributions. These are probability distributions that take into account and describe the behavior of more than one random variable. We will start our exploration with a discussion on the joint probability mass and density functions. These functions extend the concepts of probability mass and density functions for one variable to the situation where we have multiple variables.

Next, we will explore important properties of joint distributions, including the concept of marginal distribution and conditional distribution, which allow us to explore the probability of a subset of variables while conditioning on, or ignoring, other variables. We will also introduce the idea of independence of random variables in the multivariate context.

Subsequently, we will discuss some commonly used multivariate distributions such as the multivariate normal distribution, and the multivariate Bernoulli and binomial distributions. These specific distributions will provide us with practical tools for modelling multivariate data.

Finally, we will delve into covariance and correlation, two key measures that give us a sense of how two variables change together. Understanding these measures is critical for capturing the relationships in multivariate data.

Chapter 6

The Multivariate Normal Distribution

Let's say that you are gathering statistics on a species of snail. You have found 89 snails. Every snail has been weighted and had its shell measured. You've created a table like this:

Weight	Diameter
4.4769 grams	2.2692 cm
5.4755 grams	2.1973 cm
4.1183 grams	2.52928 cm
...	...
3.0522 grams	1.7822 cm

You have been told that for any particular species of snail, the weight and shell diameter are typically normally distributed. So you compute the mean of each:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

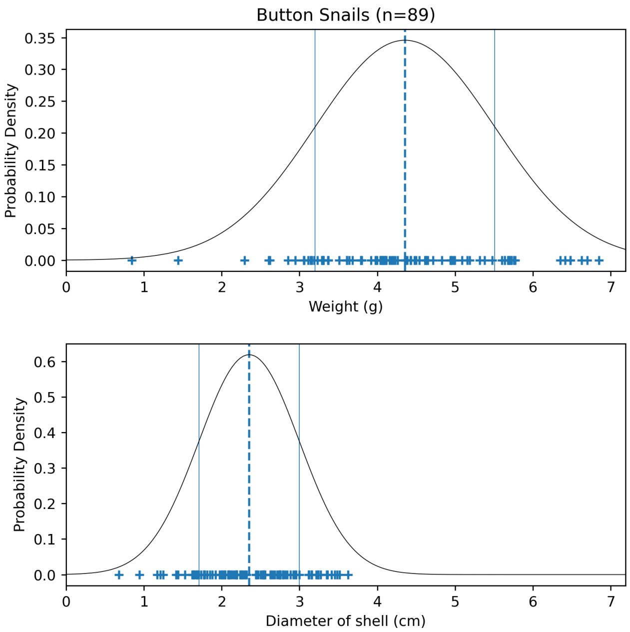
Your snails have a mean mass off 4.25 grams and a mean shell diameter of 2.35 centimeters. Now you know where the center of each normal distribution will be.

To know how wide each normal distribution is, you need to compute the variance:

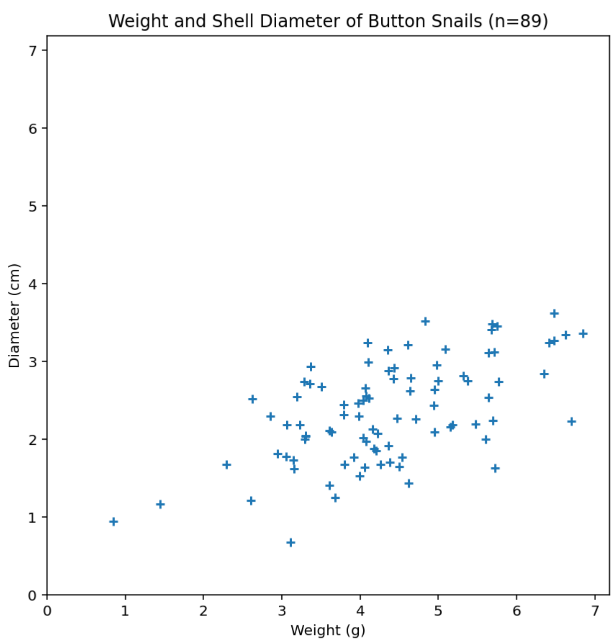
$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

(Reminder: The standard deviation is the square root of σ^2)

Plotting what you have looks like this:



Nice! But then you think: "Hmm. Maybe the mass and the diameter of the shell are related. It seems like a heavy snail might tend to have a larger shell." So you make a scatter plot:



Sure enough, there is some correlation between the mass of the snail and the diameter of its shell.

There is a form of the normal distribution that can deal with multiple variables and their correlations. This is known as the *multivariate normal* or the *Gaussian distribution*.

In a multivariate normal distribution, the mean (often denoted μ) is a vector containing the mean of every variable. For your snails, $\mu = [4.25, 2.35]$.

Just as in the single-variable normal distribution, measurements near the mean are what you are most likely to observe. In a single-variable normal distribution, the standard deviation tells us how fast that likelihood falls off as we move away from the mean. In multivariate normal distributions, we need something a little more expressive: a matrix.

6.1 The Covariance Matrix

We can think of the data as a list of vectors. For example, if we have n snails and d properties that we have measured, we would get a list like this:

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,d} \\ x_{2,1} & x_{2,2} & \dots & x_{2,d} \\ \dots & \dots & & \dots \\ x_{n,1} & x_{n,2} & \dots & x_{b,d} \end{bmatrix}$$

Each row represents one snail. Each column represents one property.

Note that to make μ (the mean vector), you just take the mean of each column of this data matrix. (For convenience, we will use μ_i to refer to the mean of column i.)

We usually use Σ (the uppercase Greek sigma) to represent a covariance matrix. Σ is a $d \times d$ matrix. The entries on the diagonal of Σ are just the variance of each property. So we can caluculate the entry at row j, column j like this:

$$\Sigma_{j,j} = \frac{\sum_{i=1}^{n}(x_{i,j} - \mu_j)^2}{n}$$

(Yes, we are using Σ to represent both summation and the covariance matrix here. It can be confusing, but you will be able to tell from context how it is being used.)

The other entries, $\Sigma_{j,k}$ where $j \neq k$, are the *covariance* between property j and property k. If the covariance is a positive number, property j and property k are correlated. For example, in your snails, weight and diameter have a positive covariance: If the snail is heavier than average, it tends to have a diameter that is larger than average.

If the covariance is a negative number, when property j is greater than average, property k tends to be less than average. For example, the number of times a restaurant mops the floor in a week has a negative covariance with how much bacteria lives on the floor.

How do we compute the covariance?

$$\Sigma_{j,k} = \frac{\sum_{i=1}^{n}(x_{i,j} - \mu_j)(x_{i,k} - \mu_k)}{n}$$

Note that $\Sigma_{j,k} = \Sigma_{k,j}$, so the matrix is symmetric.

6.1.1 Computing Mean and Covariance in Python

If you have a numpy array where each row represents one sample and each column represents one property, it is really easy to compute μ and Σ:

```
# Read in the data matrix, each row represents one snail
snails = ...

# Compute mu
mean_vector = snails.mean(axis=0)
print(f"Mean = {mean_vector}")

# Compute Sigma
covariance_matrix = np.cov(snails, rowvar=False)
print(f"Covariance = {covariance_matrix}")
```

6.2 Multivariate Normal Probability Density

Now that we have good estimates of μ and Σ, how can we compute the probability density?

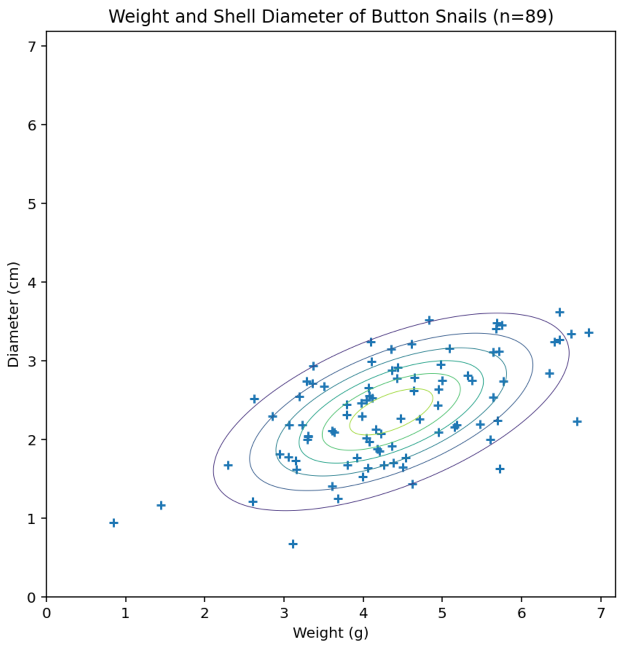
When you were working with just one variable x, you computed the probability density using the mean μ and the variance σ^2 like this:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

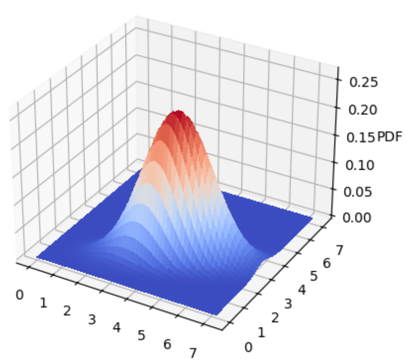
The probability density function of a d-dimensional multivariate normal distribution with a mean of μ and a covariance matrix of Σ is given by:

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d|\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$$

If we draw lines to show where μ is and contours to show lines of equal probability density, we get a plot like this:



Or, we can do a 3D plot:



A probability density always integrates to 1.0, so the volume under the surface must be 1.0.

6.2.1 Multivariate Normal Probability Density in Python

The scipy library has a class that represents the multivariate normal. Here is how you could compute the probability density for a particular snail:

```
import numpy as np
from scipy.stats import multivariate_normal

...Compute mean_vector and covariance_matrix...

# Get the probability density at [3 grams, 2 cm]
x = np.array([3.0, 2.0])
pd = multivariate_normal.pdf(x, mean=mean_vector, cov=covariance_matrix)
print(f"The probability density at {x} is {pd}")
```

Or maybe you would like to generate weights and diameters for a fictional population of 700 snails:

```
new_snails = multivariate_normal.rvs(mean=mean_vector, cov=covariance_matrix, size=700)
print(f"My fictional snails:\n{new_snails}")
```

Chapter 7

Numerical Double Integration

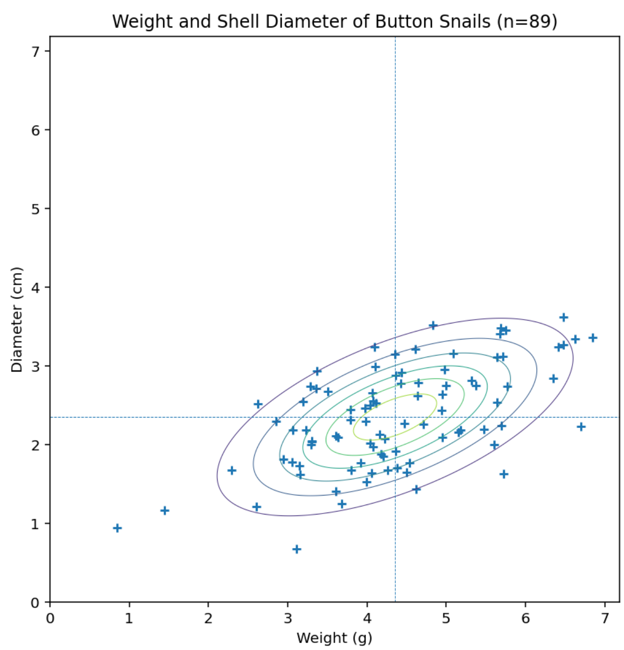
In an earlier chapter, we gave an example of the multivariate normal distribution: the mass and shell diameter of a population of snails.

Starting with a table of measurements, we estimated that the mean vector μ was $\begin{bmatrix} 4.25\text{g} & 2.35\text{cm} \end{bmatrix}$

Then we computed the covariance matrix:

$$\Sigma = \begin{bmatrix} 1.33 & 0.443 \\ 0.443 & 0.416 \end{bmatrix}$$

Plotting the data points, the mean, and the equal-density contours looked like this:



We have a great formula for computing the probability density for any mass/diameter combination:

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$$

Can you answer the following question? "If I pick a random snail off the floor of the ocean, what is the probability that its mass is between 3 and 4 grams and its diameter is between 1.5 and 2.0 centimeters?"

If we think of the probability density as a surface, this question is really "What is the volume under the surface in the rectangular patch $3 <= x_1 <= 4$ and $1.5 <= x_2 <= 2.0$?" Which you should recognize as a double integration problem:

$$P = \int_{1.5}^{2.0} \int_{3}^{4} \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})\right) dx_1 dx_2$$

Sadly, however, no one has ever been able to find the antiderivative of the multivariable probability density function, so no one can solve this problem.

Instead, we use Reimmann sums to find an approximate solution. This is known as *numerical integration*.

(After spending so much time learning the techniques for integration, it may be disappointing to hear this: For a lot of real-world problems, there is no way to find an antiderivative, so we end up doing numerical integration much more often than most people realize.)

7.1 Reimann Sums on 2-Dimensional Domains

When doing Reimann sums on function that takes a single real number, you summed the area of the rectangles under the function to approximate the integral:

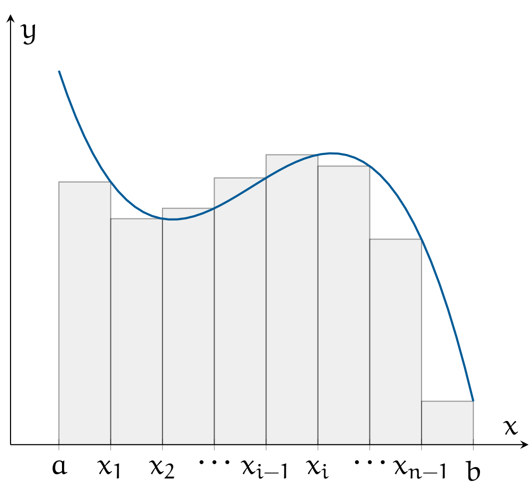


Figure 7.1: A representative function divided into n rectangles of equal width, with rectangle height determined by the right endpoint of the subinterval

Here you are finding the volume under a function that takes two variables (the probability density is based on the mass and diameter of the shell).

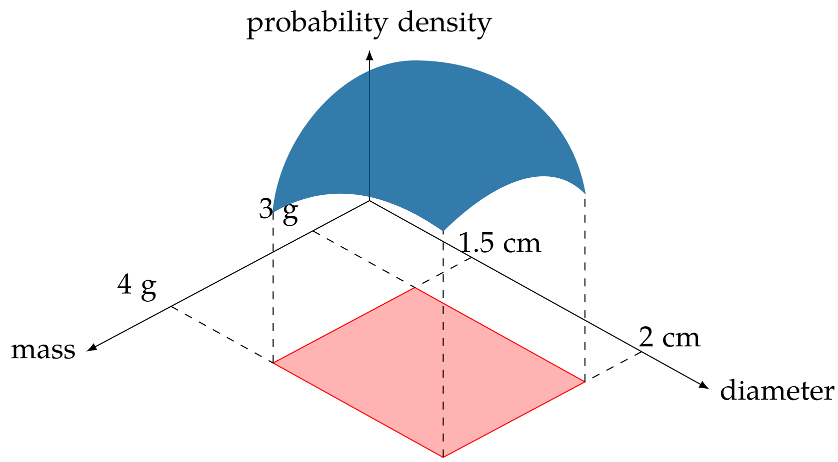


Figure 7.2: The probability is the volume under a surface for a region

To do the Reimann sum, we can break the range of the mass into n_1 equally sized intervals and the range of the diameter into n_2 equally sized intervals. For the diagram, we will just break each range into three, but you will get more accurate estimate as the intervals get smaller.

Now you will be calculating the volume of rectangular solids and summing up those volumes. Here are a few of the rectangular volumes:

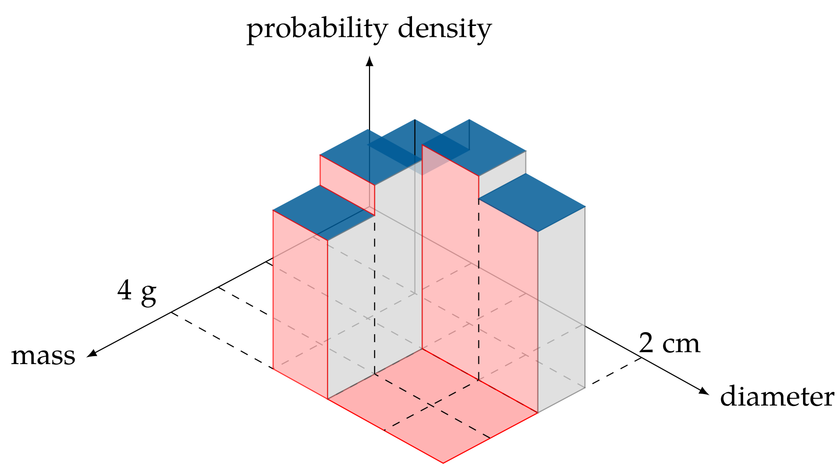


Figure 7.3: Reimann Sum

7.2 Numerical Integration in Python

Here's the code:

```
import numpy as np
from scipy.stats import multivariate_normal

# pip3 install scipy
weight_lower_limit = 3.0
weight_upper_limit = 4.0
weight_slices = 100

diameter_lower_limit = 1.5
diameter_upper_limit = 2.0
diameter_slices = 100

# What's the average weight and diameter
mean_vector = np.array([4.35559489,2.3526593 ])
print(f"Mean [weight, diameter] = {mean_vector}")

# Do heavier snails tend to have bigger shells?
covariance_matrix = np.array([[1.33099714,0.44309754],
 [0.44309754,0.41603925]])
print(f"Covariance = \n{covariance_matrix}")

# Create a multivariate normal distribution
rv = multivariate_normal(mean_vector, covariance_matrix)

delta_weight = (weight_upper_limit - weight_lower_limit) / weight_slices
delta_diameter = (diameter_upper_limit - diameter_lower_limit) / diameter_slices

sum = 0.0
# Step though each different weight
for i in range(weight_slices):
    # What is the weight in the middle of this slice?
    current_weight = weight_lower_limit + (i + 0.5) * delta_weight

    for j in range(diameter_slices):
        # What is the diameter in the middle of this slice?
        current_diameter = diameter_lower_limit + (j + 0.5) * delta_diameter

        # What is the probability density there?
        prob_density = rv.pdf((current_weight, current_diameter))

        # What is the volume under that for this tiny square
        sum += prob_density * delta_weight * delta_diameter

print(
    f"\nThe probability that the weight is between {weight_lower_limit} and {weight_upper_limit}"
)
print(
    f"and that the diameter is between {diameter_lower_limit} and {diameter_upper_limit}"
)
print(f"is about {sum * 100.0:.4f}%")
```

This should get you the following output:

```
> python3 num_integration.py
Mean [weight, diameter] = [4.35559489 2.3526593 ]
Covariance =
[[1.33099714 0.44309754]
 [0.44309754 0.41603925]]

The probability that the weight is between 3.0 and 4.0
and that the diameter is between 1.5 and 2.0
is about 7.8316%
```

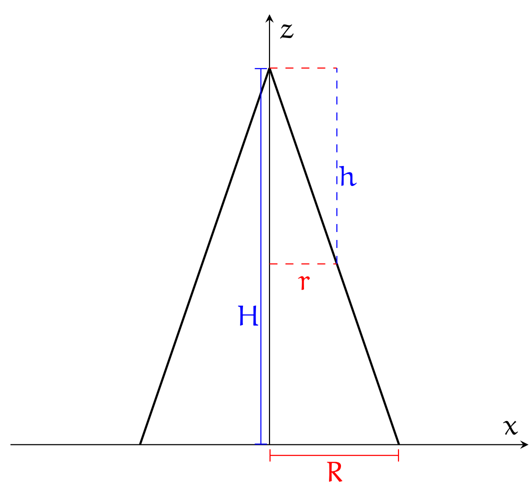
Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

Imagine a side view of the cone (see figure below), an isosceles triangle with height H and base $2R$. If we take horizontal cross-sections, then each cross-section is a circle h from the top with a radius r. Because the triangles are similar (fix me better wording/explanation here), we also know that $\frac{H}{h} = \frac{R}{r}$. Therefore, we can define r in terms of h: $r = \frac{hR}{H}$ and the volume of each subsequent cross-section is $\pi r^2 dh = \pi \frac{h^2 R^2}{H^2} dh$. We start with $h = 0$ and end with $h = H$:

$$\begin{aligned} V_{cone} &= \int_0^H \pi \frac{h^2 R^2}{H^2}\, dh = \pi \frac{R^2}{H^2} \int_0^H h^2\, dh \\ &= \pi \frac{R^2}{H^2} \left[\frac{1}{3} h^3\right]_{h=0}^{h=H} = \pi \frac{R^2}{3H^2} \left[H^3 - 0^3\right] \\ &= \pi \frac{R^2}{3H^2} H^3 = \frac{\pi}{3} R^2 H \end{aligned}$$



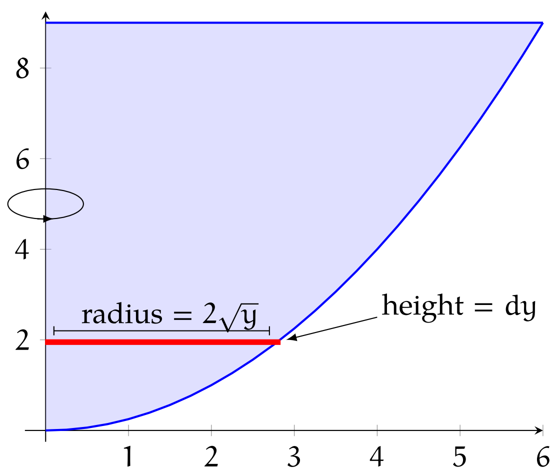
Answer to Exercise 2 (on page 7)

If we are rotating about the y axis, we should make our slices horizontal so their width is dy (see graph below). Then the volume of each cylinder is given by $V = \pi r^2 dy$ and the total volume is given by:

$$V = \int_0^9 \pi \left[2\sqrt{y}\right]^2 \, dy$$

$$V = 4\pi \int_0^9 y \, dy = 2\pi y^2 |_{y=0}^{y=9}$$

$$V = 2\pi (9)^2 = 162\pi$$



Answer to Exercise 3 (on page 8)

Since the graph is rotated around the x-axis, we will take vertical slices with width dx and rotate them to make cylinders with radius $f(x)$ and height dx. Then the volume of each egg is given by:

$$\int_{-1}^{1} \pi \left[f(x)\right]^2 \, dx$$

To determine our limits of integration, we note that $\sqrt{1-x^2} = 0$ (and therefore, $f(x) = 0$) when $x = \pm 1$.

For the chicken,

$$V_{chickenegg} = \pi \int_{-1}^{1} \left[\left(-0.02x^3 + 0.03x^2 + 0.12x + 0.454\right) \sqrt{1-x^2}\right]^2 \, dx$$

And for the mallard duck,

$$V_{duckegg} = \pi \int_{-1}^{1} \left[\left(-0.06x^3 + 0.04x^2 + 0.1x + 0.54 \right) \sqrt{1 - x^2} \right]^2 dx$$

Using a calculator, we find that $V_{chickenegg} \approx 0.897$ and $V_{duckegg} \approx 1.263$. Therefore, mallard ducks lay larger eggs than chickens do.

Answer to Exercise ?? (on page 10)

First, since we are revolving around the y-axis, we know our slices will have width dy. We will re-write the functions as x in terms of y:

$$x = \sqrt{y}$$

$$x = \frac{y^2}{4}$$

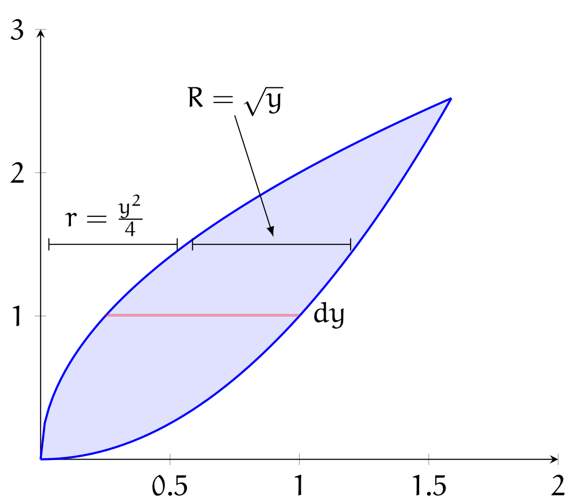
Setting them equal to each other to find the y-value at which they intercept:

$$\sqrt{y} = \frac{y^2}{4}$$

$$4 = \frac{y^2}{\sqrt{y}} = y^{3/2}$$

$$y = \sqrt[3]{4^2} = 2\sqrt[3]{2}$$

Examining a graph (shown below), we see that the outer radius is $x = \sqrt{y}$ and the inner radius is $x = \frac{y^2}{4}$.



Then the total volume of the solid of revolution is given by:

$$V = \pi \int_0^{2\sqrt[3]{2}} (\sqrt{y})^2 - \left(\frac{y^2}{4}\right)^2 \, dy$$

$$V = \pi \int_0^{2\sqrt[3]{2}} \left[y - \frac{y^4}{16}\right] dy$$

$$V = \pi \left[\frac{1}{2}y^2 - \frac{1}{80}y^5\right]_{y=0}^{y=2\sqrt[3]{2}}$$

$$V = \pi \left[\frac{6}{5}2^{2/3}\right] \approx 5.9844$$

Answer to Exercise 5 (on page 12)

If each cross section is a square, then the volume of each cross section is given by $s^2 dx$, where s is the side length of the square. Since the side length is equal to the distance between the graph of y and the x-axis, we can see that $s = y = \ln(3 - x)$. And, therefore, the total volume of all the cross sections is given by $\int_0^2 [\ln(3 - x)]^2 \, dx$. Using a calculator, this integral evaluates to ≈ 1.029.

Answer to Exercise 6 (on page 13)

Since the cross-sections are perpendicular to the x-axis, they will have width dx and we will integrate across the domain of the ellipse. Setting $y = 0$ to find the domain of the ellipse:

$$9x^2 = 25 \to x^2 = \frac{25}{9} \to x = \pm\frac{5}{3}$$

A right isosceles triangle with hypotenuse h has area $\frac{1}{4}h^2$. In this case, each triangle's hypotenuse is given by the distance between the top and bottom of the ellipse. The top of the ellipse if defined by $y = \frac{1}{4}\sqrt{25 - 9x^2}$ and the bottom by $y = -\frac{1}{4}\sqrt{25 - 9x^2}$. Therefore, the length of each hypotenuse is $\frac{1}{2}\sqrt{25 - 9x^2}$.

Then, each cross-section has a total volume of $\frac{1}{4}h^2 dx = \frac{1}{4}\left(\frac{1}{2}\sqrt{25 - 9x^2}\right)^2 dx$ and the volume of the solid is:

$$V_{solid} = \int_{-5/3}^{5/3} \frac{1}{4}\left(\frac{1}{2}\sqrt{25 - 9x^2}\right)^2 dx$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{-5/3}^{5/3} \frac{1}{4}\left(25-9x^2\right)\, dx \\
&= \frac{1}{6}\int_{-5/3}^{5/3} \left(25-9x^2\right)\, dx = \frac{1}{16}\left[25x-3x^3\right]_{x=-5/3}^{x=5/3} \\
&= \frac{1}{16}\left[\left(25\left(\frac{5}{3}\right)-25\left(\frac{-5}{3}\right)\right)-\left(3\left(\frac{5}{3}\right)^3-3\left(\frac{-5}{3}\right)^3\right)\right] \\
&= \frac{1}{16}\left[\frac{250}{3}-\left(\frac{375}{27}+\frac{375}{27}\right)\right] = \frac{1}{16}\left[\frac{250}{3}-\frac{250}{9}\right] = \frac{1}{16}\left[\frac{750}{9}-\frac{250}{9}\right] = \frac{1}{16}\left[\frac{500}{9}\right] = \frac{125}{36}
\end{aligned}
$$

Answer to Exercise ?? (on page 19)

We have already shown that $\int_0^3\int_1^2 xy^2\, dy\, dx = \frac{21}{2}$. We will evaluate $\int_1^2\int_0^3 xy^2\, dx\, dy$ and see if we get the same result.

$$
\begin{aligned}
\int_0^3 xy^2\, dx &= y^2\int_0^3 x\, dx = y^2\left[\frac{1}{2}x^2\right]_{x=0}^{x=3} \\
&= \frac{y^2}{2}\left[3^2-0^2\right] = \frac{9y^2}{2}
\end{aligned}
$$

Substituting this back into the iterated integral:

$$
\begin{aligned}
\int_1^2\int_0^3 xy^2\, dx\, dy &= \int_1^2 \frac{9y^2}{2}\, dy = \frac{9}{2}\int_1^2 y^2\, dy \\
&= \frac{9}{2}\left[\frac{1}{3}y^3\right]_{y=1}^{y=2} = \frac{9}{2}\cdot\frac{1}{3}\left[2^3-1^3\right] \\
&= \frac{3}{2}(8-1) = \frac{21}{2}
\end{aligned}
$$

Answer to Exercise 8 (on page 20)

1. Answer: $\frac{5}{2}-\frac{1}{e}$. Solution: $\int_0^1\int_1^2 \left(x+e^{-y}\right)\, dx\, dy = \int_0^1 \left(\frac{1}{2}x^2+xe^{-y}\right)\Big|_{x=1}^{x=2} dy = \int_0^1 \left(2-\frac{1}{2}+2e^{-y}-e^{-y}\right)$ d $\int_0^1 \left(\frac{3}{2}+e^{-y}\right)\, dy = \left[\frac{3}{2}y-e^{-y}\right]_{y=0}^{y=1} = \left(\frac{3}{2}(1)-e^{-1}\right)-\left(\frac{3}{2}(0)-e^0\right) = \frac{5}{2}-\frac{1}{e}$
2. Answer: 18. Solution: $\int_{-3}^3\int_0^{\pi/2} \left(2y+y^2\cos x\right)\, dx\, dy = \int_{-3}^3 \left[2xy+y^2\sin x\right]_{x=0}^{x=\pi/2}\, dy = \int_{-3}^3 \left[\left(\pi y+y^2\right)-(0+0)\right]\, dy = \int_{-3}^3 \left(\pi y+y^2\right)\, dy = \left[\frac{\pi}{2}y^2+\frac{1}{3}y^3\right]_{y=-3}^{y=3} = \left(\frac{\pi}{2}(9)+\frac{1}{3}(27)\right)-\left(\frac{\pi}{2}(9)+\frac{1}{3}(-27)\right) = 9-(-9) = 18$

3. Answer: 6. Solution: $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\theta \, d\theta \, dt = \left(\int_0^3 t^2 \, dt\right) \times \left(\int_0^{\pi/2} \sin^3\theta \, d\theta\right) = \left[\frac{1}{3}t^3\right]_{t=0}^{t=3} \times \left(\int_0^{\pi/2} \sin\theta \sin^2\theta \, d\theta\right) = 9\int_0^{\pi/2} \sin\theta \left(1-\cos^2\theta\right) \, d\theta = 9\left[\int_0^{\pi/2} \sin\theta \, d\theta - \int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta\right] = 9\left[(-\cos\theta)\,|_{\theta=0}^{\theta=\pi/2} + \left(\frac{1}{3}\cos^3\theta\right)|_{\theta=0}^{\theta=\pi/2}\right] = 9\left[-(-\cos 0) + (-\frac{1}{3}\cos^3 0)\right] = 9\left(1-\frac{1}{3}\right) = 9\left(\frac{2}{3}\right) = 6$

Answer to Exercise 9 (on page 21)

1. $\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1} \, dy \, dx$ OR $\int_{-3}^3 \int_0^1 \frac{xy^2}{x^2+1} \, dx \, dy$
2. $\int_0^{\pi/4} \int_0^1 \frac{\sec\theta}{\sqrt{1+t^2}} \, dt \, d\theta$ OR $\int_0^1 \int_0^{\pi/4} \frac{\sec\theta}{\sqrt{1+t^2}} \, d\theta \, dt$

Answer to Exercise 10 (on page 22)

1. $\iint_R \frac{xy^2}{x^2+1} \, dA$, $R = \{(x,y)\,|0 \le x \le 2, -3 \le y \le 3\} = \int_0^2 \int_{-3}^3 \frac{xy^2}{x^2+1} \, dy \, dx = \int_0^2 \frac{x}{x^2+1} \, dx \cdot \int_{-3}^3 y^2 \, dy$. To evaluate the integral with respect to x, we use the u-substitution $u = x^2+1$, $(x)dx = \frac{1}{2}du$: $\int_0^2 \frac{x}{x^2+1} \, dx \cdot \int_{-3}^3 y^2 \, dy = \int_{x=0}^{x=2} \frac{1}{2}\frac{1}{u} \, du \cdot \int_{-3}^3 y^2 \, dy = \frac{1}{2}\ln|u||_{x=0}^{x=2} \cdot \frac{1}{3}\left[y^3\right]_{y=-3}^{y=3} = \frac{1}{2}\left[\ln\left(2^2+1\right) - \ln\left(0^2+1\right)\right] \cdot \frac{1}{3}\left[3^3 - (-3)^3\right] = \frac{1}{2}\ln 5 \cdot \frac{1}{3}\left(27-(-27)\right) = \frac{\ln 5}{2}\frac{54}{3} = 9\ln 5$
2. $\iint_R \frac{\tan\theta}{\sqrt{1-t^2}} \, dA$, $R = \{(\theta,t)\,|\,0 \le \theta \le \pi/3,\ 0 \le t \le \frac{1}{2}\} = \int_0^{\pi/3} \int_0^{1/2} \frac{\tan\theta}{\sqrt{1-t^2}} \, dt \, d\theta = \left[\int_0^{\pi/3} \tan\theta \, d\theta\right] \cdot \left[\int_0^{1/2} \frac{1}{\sqrt{1-t^2}} \, dt\right]$. Recall that $\frac{d}{dt}\arcsin t = \frac{1}{\sqrt{1-t^2}}$. Applying FTC, then $\left[\int_0^{\pi/3} \tan\theta \, d\theta\right] \cdot \left[\int_0^{1/2} \frac{1}{\sqrt{1-t^2}} \, dt\right] = \left[\int_0^{\pi/3} \tan\theta \, d\theta\right] \cdot \left[\arcsin t\right]_{t=0}^{t=1/2} = \left[\int_0^{\pi/3} \tan\theta \, d\theta\right] \cdot \left[\arcsin\frac{1}{2} - \arcsin 0\right] = \left[\int_0^{\pi/3} \tan\theta \, d\theta\right] \cdot \left[\frac{\pi}{6}\right] = \frac{\pi}{6}\int_0^{\pi/3} \frac{\sin\theta}{\cos\theta} \, d\theta$. To evaluate this final integral, we use the u-substitution $u = \cos\theta$ and $-du = \sin\theta d\theta$: $\frac{\pi}{6}\int_0^{\pi/3} \frac{\sin\theta}{\cos\theta} \, d\theta = -\frac{\pi}{6}\int_{\theta=0}^{\theta=\pi/3} \frac{1}{u} \, du = -\frac{\pi}{6}\ln u|_{\theta=0}^{\theta=\pi/3} = -\frac{\pi}{6}\left[\ln\left(\cos\theta\right)\right]_{\theta=o}^{\theta=\pi/3} = \frac{\pi}{6}\left[\ln\left(\cos 0\right) - \ln\left(\cos\frac{\pi}{3}\right)\right] = \frac{\pi}{6}\left[\ln 1 - \ln\frac{1}{2}\right] = \frac{\pi}{6}\ln\frac{1}{1/2} = \frac{\pi}{6}\ln 2$
3. $\iint_R x\sin\left(x+y\right) dA$, $R = [0,\pi/6] \times [0,\pi/3] = \int_0^{\pi/6} \int_0^{\pi/3} x\sin\left(x+y\right) dy \, dx$. Recall the sum formula for sine:

   $$\sin\left(x+y\right) = \sin x \cos y + \cos x \sin y$$

   We can substitute this into our iterated integral:

   $$\int_0^{\pi/6} \int_0^{\pi/3} x\sin\left(x+y\right) dy \, dx = \int_0^{\pi/6} \int_0^{\pi/3} x\left[\sin x \cos y + \cos x \sin y\right] \, dy \, dx$$

$$= \int_0^{\pi/6} \left[\int_0^{\pi/3} x \sin x \cos y \, dy + \int_0^{\pi/3} x \cos x \sin y \, dy \right] \, dx$$

Let us designate $\int_0^{\pi/3} x \sin x \cos y \, dy$ as integral **A** and $\int_0^{\pi/3} x \cos x \sin y \, dy$ as integral **B**. First, we will evaluate integral **A**:

$$\int_0^{\pi/3} x \sin x \cos y \, dy = x \sin x \int_0^{\pi/3} \cos y \, dy$$

$$= x \sin x \, [\sin y]_{y=0}^{y=\pi/3} = x \sin x \left[\sin \frac{\pi}{3} - \sin 0 \right]$$

$$= x \sin x \left(\frac{\sqrt{3}}{2} \right) = \frac{x\sqrt{3}}{2} \sin x$$

Next we evaluate integral **B**:

$$\int_0^{\pi/3} x \cos x \sin y \, dy = x \cos x \int_0^{\pi/3} \sin y \, dy$$

$$= x \cos x \, [-\cos y]_{y=0}^{y=\pi/3} = x \cos x \left[-\cos \frac{\pi}{3} - (-\cos 0) \right]$$

$$= x \cos x \left[-\frac{1}{2} - (-1) \right] = \frac{x}{2} \cos x$$

Substituting back in for integrals **A** and **B**:

$$\int_0^{\pi/6} \left[\int_0^{\pi/3} x \sin x \cos y \, dy + \int_0^{\pi/3} x \cos x \sin y \, dy \right] \, dx = \int_0^{\pi/6} \left[\frac{x\sqrt{3}}{2} \sin x + \frac{x}{2} \cos x \right] \, dx$$

$$= \frac{\sqrt{3}}{2} \int_0^{\pi/6} x \sin x \, dx + \frac{1}{2} \int_0^{\pi/6} x \cos x \, dx$$

Again, we will designate $\int_0^{\pi/6} x \sin x \, dx$ as integral **C** and $\int_0^{\pi/6} x \cos x \, dx$ as integral **D**. We start by using integration by parts to evaluate integral **C**:

Let $u = x$ and $dv = \sin x dx$. Then $v = -cos x$ and $du = dx$ and therefore:

$$\int_0^{\pi/6} x \sin x \, dx = [x \, (-\cos x)]_{x=0}^{x=\pi/6} - \int_0^{\pi/6} (-\cos x) \, dx$$

$$= \left[\frac{\pi}{6} \left(-\cos \frac{\pi}{6} \right) \right] - [0 \, (-\cos 0)] + \sin x |_{x=0}^{x=\pi/6}$$

$$= -\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} - 0 + \sin \frac{\pi}{6} - \sin 0 = \frac{1}{2} - \frac{\pi\sqrt{3}}{12} = \frac{6 - \pi\sqrt{3}}{12}$$

Next, we will use integration by parts to evaluate integral **D**. Let $u = x$ and $dv =$

$\cos x dx$. Then $du = dx$ and $v = \sin x$ and therefore:

$$\int_0^{\pi/6} x \cos x\, dx = [x \sin x]_{x=0}^{x=\pi/6} - \int_0^{\pi/6} \sin x\, dx$$

$$= \left[\frac{\pi}{6} \sin \frac{\pi}{6} - 0 \sin 0\right] - (-\cos x)\,|_{x=0}^{x=\pi/6} = \frac{\pi}{6} \cdot \frac{1}{2} + \cos \frac{\pi}{6} - \cos 0$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

Substituting back in for integrals **C** and **D**:

$$\frac{\sqrt{3}}{2} \int_0^{\pi/6} x \sin x\, dx + \frac{1}{2} \int_0^{\pi/6} x \cos x\, dx = \frac{\sqrt{3}}{2} \left(\frac{6 - \pi\sqrt{3}}{12}\right) + \frac{1}{2} \left(\frac{\pi + 6\sqrt{3} - 12}{12}\right)$$

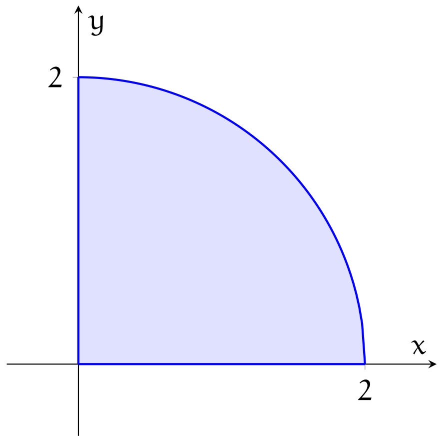
$$= \frac{6\sqrt{3} - 3\pi + \pi + 6\sqrt{3} - 12}{24} = \frac{6\sqrt{3} - 6 - \pi}{12}$$

Answer to Exercise 11 (on page 29)

1. $\iint_D e^{-y^2}\, dA = \int_0^3 \int_0^{2y} e^{-y^2}\, dx\, dy = \int_0^3 \left[e^{-y^2} x|_{x=0}^{x=2y}\right] dy = \int_0^3 2y e^{-y^2}\, dy = -e^{-y^2}|_{y=0}^{y=3} = 1 - e^{-9} \approx 0.9999$
2. $\iint_D x \sin y\, dA = \int_0^2 \int_0^{x^2} x \sin y\, dy\, dx = \int_0^2 x \int_0^{x^2} \sin y\, dy\, dx = \int_0^2 x\, [-\cos y]_{y=0}^{y=x^2}$
$= \int_0^2 x \left(\cos 0 - \cos x^2\right) dx = \int_0^2 \left(x - x \cos x^2\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{2} \sin x^2\right]_{x=0}^{x=2} = \frac{1}{2}(2)^2 - \frac{1}{2} \left(\sin 2^2 - \sin 0\right)$
$= 2 - \frac{1}{2} (\sin 4 - 0) = 2 - \frac{\sin 4}{2} \approx 2.378$
3. We can describe the region as $D = \{(x, y) \mid -3 \le x \le -3, -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2}\}$.
Therefore, $\iint_D (2y - x)\, dA = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (2x - y)\, dy\, dx = \int_{-3}^3 \left[2xy - \frac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$
$= \int_{-3}^3 \left[2x \left(\sqrt{9 - x^2} + \sqrt{9 - x^2}\right) - \frac{1}{2} \left(9 - x^2 - \left(9 - x^2\right)\right)\right] dx = \int_{-3}^3 4x\sqrt{9 - x^2}\, dx$. Let $u = 9 - x^2$, then $du = -2x$ and $4x = -2du$. Substituting, $\int_{-3}^3 4x\sqrt{9 - x^2}\, dx = \int_{x=-3}^{x=3} -2\sqrt{u}\, du = -2 \cdot \frac{2}{3} u^{3/2}|_{x=-3}^{x=3} = -\frac{4}{3} \left[\left(9 - x^2\right)\right]_{x=-3}^{x=3} = 0$

Answer to Exercise 12 (on page 33)

1. Let's visualize the region in the xy-plane:



The region is a quarter-circle that can be described with $D = \{(r, \theta) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq \pi/2\}$. Then we can re-write the integral in polar coordinates:

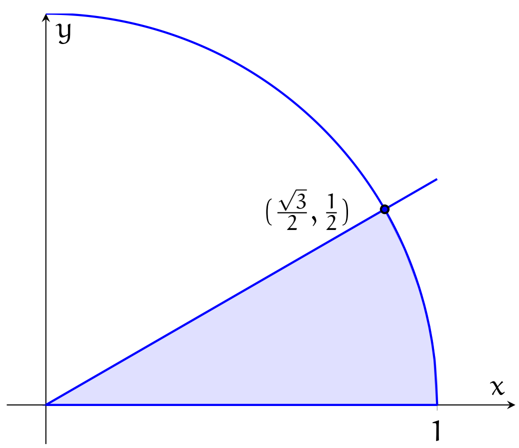
$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^2 re^{-r^2}\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=2} d\theta = \int_0^{\pi/2} \left(-\frac{1}{2}\right) \left[e^{-4} - 1\right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 1 - e^{-4}\, d\theta = \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \int_0^{\pi/2} 1\, d\theta$$

$$= \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \theta\Big|_{\theta=0}^{\theta=\pi/2} = \frac{\pi}{4}\left(1 - \frac{1}{e^4}\right)$$

2. The region is bounded by the x-axis, the line $y = x/\sqrt{3}$, and the circle $x^2 + y^2 = 1$:



We see that the region defined in polar coordinates is $D = \{(r, \theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/6\}$. And therefore:

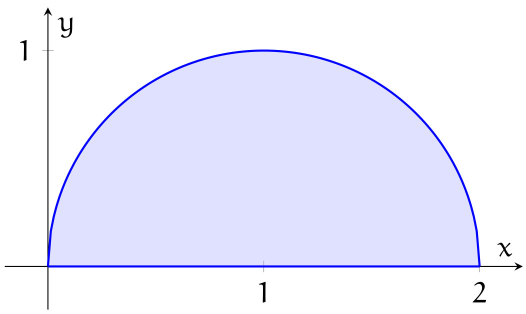
$$\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy = \int_0^{\pi/6} \int_0^1 r\,(r\cos\theta)\,(r\sin\theta)^2\, dr\, d\theta$$

$$= \int_0^{\pi/6} \left[\cos\theta \sin^2\theta\right] \, d\theta \cdot \int_0^1 r^4 \, dr$$

$$= \left(\frac{1}{3} \sin^3\theta \Big|_{\theta=0}^{\theta=\pi/6} \right) \cdot \left(\frac{1}{5} r^5 \Big|_{r=0}^{r=1} \right)$$

$$= \frac{1}{15} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{120}$$

3. Visualizing the region:



We see that the region is the top half of a circle of radius 1 centered at (1, 0). In polar coordinates, this region is $D = \{(r, \theta) \mid 0 \le r \le 2\cos\theta, 0 \le \theta \le \pi/2\}$. And therefore:

$$\int_0^2 \int_0^{\sqrt{2x - x^2}} \sqrt{x^2 + y^2} \, dy \, dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r\sqrt{r^2} \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \, dr \, d\theta = \int_0^{\pi/2} \frac{1}{3} \left[r^3\right]_{r=0}^{r=2\cos\theta} \, d\theta$$

$$= \frac{8}{3} \int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{8}{3} \int_0^{\pi/2} \cos\theta \left(1 - \sin^2\theta\right) \, d\theta$$

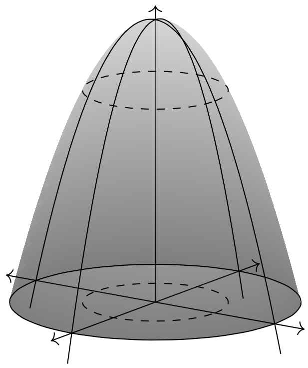
$$= \frac{8}{3} \left[\int_0^{\pi/2} \cos\theta \, d\theta - \int_0^{\pi/2} \cos\theta \sin^2\theta \, d\theta \right]$$

$$= \frac{8}{3} \left[(\sin\theta)_{\theta=0}^{\theta=\pi/2} - \left(\frac{1}{3} \sin^3\theta \right)_{\theta=0}^{\theta=\pi/2} \right]$$

$$= \frac{8}{3} \left[(1 - 0) - \frac{1}{3} \left(1^3 - 0^3\right) \right] = \frac{8}{3} \cdot \frac{2}{3} = \frac{16}{9}$$

Answer to Exercise 13 (on page 34)

We are finding the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy-plane.



We can use polar coordinates to simplify the double integral. In polar coordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so $x^2 + y^2 = r^2$. The volume under the surface and above the xy-plane is given by

$$V = \int\int (4 - r^2) r \, dr \, d\theta, \tag{3.1}$$

where r ranges from 0 to 2 (since $4 - r^2 \geq 0$ if $0 \leq r \leq 2$) and θ ranges from 0 to 2π.

Hence,

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[2r^2 - \frac{1}{4} r^4 \right]_0^2 d\theta \\ &= \int_0^{2\pi} (8 - 4) \, d\theta \\ &= \int_0^{2\pi} 4 \, d\theta \\ &= [4\theta]_0^{2\pi} \\ &= 8\pi. \end{aligned}$$

So the volume of the solid is 8π cubic units.

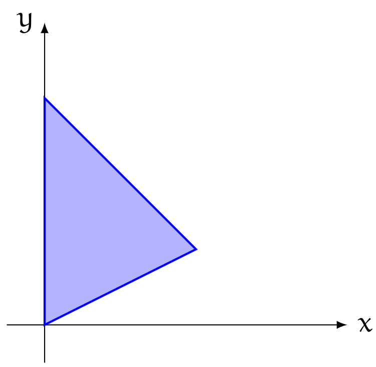
Answer to Exercise 14 (on page 35)

Let's describe the footprint of the pool as a 20-foot radius circle centered at the origin (that is, a region $D = \{(r, \theta) \mid 0 \leq r \leq 20, \ 0 \leq \theta \leq 2\pi\}$). Further, let's take north-south as parallel to the y-axis and east-west as parallel to the x-axis. Then the depth of water is then given by $z = f(x, y) = \frac{7}{40}x + \frac{13}{2}$ over the footprint of the pool. And the total volume of water is given by:

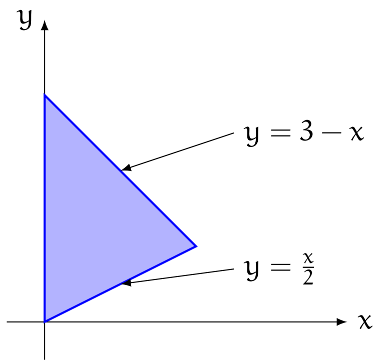
$$\begin{aligned}
&\int_0^{2\pi}\int_0^{20} r\left(\frac{7}{40}r\cos\theta + \frac{13}{2}\right)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{20}\left[\frac{7}{40}r^2\cos\theta + \frac{13}{2}r\right]\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\frac{7\cos\theta}{40}\int_0^{20} r^2\,dr + \frac{13}{2}\int_0^{20} r\,dr\right]\,d\theta \\
&= \int_0^{2\pi}\left[\frac{7\cos\theta}{40}\left(\frac{1}{3}r^3\right)_{r=0}^{r=20} + \frac{13}{2}\left(\frac{1}{2}r^2\right)_{r=0}^{r=20}\right]\,d\theta \\
&= \int_0^{2\pi}\left[\frac{1400}{3}\cos\theta + 1300\right]\,d\theta = \left[\frac{1400}{3}\sin\theta + 1300\theta\right]_{\theta=0}^{\theta=2\pi} \\
&= 2600\pi \text{ cubic feet}
\end{aligned}$$

Answer to Exercise 15 (on page 39)

1. $\iint_D \left(1 + x^2 + y^2\right)\,dA = \int_0^4\int_0^3 \left(1 + x^2 + y^2\right)\,dy\,dx = \int_0^4 \left[y + x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=3}\,dx = \int_0^4 \left[3 + 3x^2 + \frac{1}{3}(3)^3\right]\,dx$ $= \int_0^4 \left(12 + 3x^2\right)\,dx = \left[12x + x^3\right]_{x=0}^{x=4} = 12(4) + 4^3 = 112$
2. First, let's visualize this region, since it isn't a rectangle:



Let's divide the triangle horizontally and write equations for each of the sides that do not lie on the y-axis.



We see that we can describe region D as $D = \{(x, y) \mid 0 \le x \le 2, \frac{x}{2} \le y \le 3 - x\}$. Therefore $\iint_D (x + y)\ dA = \int_0^3 \int_{x/2}^{3-x} (x + y)\ dy\, dx = \int_0^3 \left[xy + \frac{1}{2}y^2\right]_{y=x/2}^{y=3-x}\ dx = \int_0^3 \left[(x(3-x)) - (x(x/2)) + \frac{1}{2}\left((3-x)^2 - (x/2)^2\right)\right]\ dx$
$= \int_0^3 \left[\left(3x - x^2 - \frac{x^2}{2}\right) + \frac{1}{2}\left(9 - 6x + x^2 - \frac{x^2}{4}\right)\right]\ dx = \int_0^3 \left[-x^2 - \frac{x^2}{2} + \frac{x^2}{2} - \frac{x^2}{8} + 3x - 3x + \frac{9}{2}\right]\ dx$
$= \int_0^3 \left(-\frac{9x^2}{8} + \frac{9}{2}\right)\ dx = \left[\frac{9x}{2} - \frac{3x^3}{8}\right]_{x=0}^{x=2} = \frac{9(2)}{2} - \frac{3(8)}{8} = 9 - 3 = 6$

Answer to Exercise 16 (on page 43)

1. First, we find the total mass: $m = \int_0^\pi \int_0^{2\sin x} x\, dy\, dx = \int_0^\pi [xy]_{y=0}^{y=2\sin x}\ dx = \int_0^\pi 2x \sin x\, dx$.

   We apply integration by parts to evaluate the integral: $\int_0^\pi 2x \sin x\, dx = (-2x\cos x)\,|_{x=0}^{x=\pi} + \int_0^\pi 2\cos x\, dx = [-2\pi(-1)] - (0) + \sin x|_{x=0}^{x=\pi} = 2\pi + \sin\pi - \sin 0 = 2\pi$

   Now that we know $m = 2\pi$, we can find $\bar{x}$ and $\bar{y}$: $\bar{x} = \frac{1}{2\pi}\int_0^\pi \int_0^{2\sin x} x \cdot x\, dy\, dx = \frac{1}{2\pi}\int_0^\pi x^2 y|_{y=0}^{y=2\sin x}\ dx = \frac{1}{2\pi}\int_0^\pi x^2\,(2\sin x)\ dx = \frac{1}{\pi}\int_0^\pi x^2 \sin x\, dx$.

   Applying integration by parts: $\frac{1}{\pi}\int_0^\pi x^2 \sin x\, dx = \frac{1}{\pi}\left[x^2\,(-\cos x)\,|_{x=0}^{x=\pi} - \int_0^\pi 2x\,(-\cos x)\ dx\right]$
   $= \frac{1}{\pi}\left[\left(-\pi^2\cos\pi\right) + 2\int_0^\pi x\cos x\, dx\right] = \frac{1}{\pi}\left[\pi^2 + 2\int_0^\pi x\cos x\, dx\right] = \pi + \frac{2}{\pi}\int_0^\pi x\cos x\, dx$.

   Applying integration by parts again: $\pi + \frac{2}{\pi}\int_0^\pi x\cos x\, dx = \pi + \frac{2x\sin x}{\pi}|_{x=0}^{x=\pi} - \frac{2}{\pi}\int_0^\pi \sin x\, dx$
   $= \pi - \frac{2}{\pi}\int_0^\pi \sin x\, dx = \pi + \frac{2}{\pi}[\cos x]_{x=0}^{x=\pi} = \pi + \frac{2}{\pi}[\cos\pi - \cos 0] = \pi + \frac{2}{\pi}(-1-1) = \pi - \frac{4}{\pi} = \bar{x}$

   And finding $\bar{y}$: $\bar{y} = \frac{1}{2\pi}\int_0^\pi \int_0^{2\sin x} y \cdot x\, dy\, dx = \frac{1}{2\pi}\int_0^\pi \left[\frac{1}{2}xy^2\right]_{y=0}^{y=2\sin x}\ dx = \frac{1}{4\pi}\int_0^\pi x\,[2\sin x]^2\ dx$
   $= \frac{1}{4\pi}\int_0^\pi 4x\sin^2 x\, dx = \frac{1}{\pi}\int_0^\pi x\sin^2 x\, dx = \frac{1}{\pi}\int_0^\pi x\frac{1-\cos(2x)}{2}\ dx = \frac{1}{\pi}\int_0^\pi \frac{x}{2}\ dx - \frac{1}{\pi}\int_0^\pi x\cos(2x)\ dx$
   $= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{x=0}^{x=\pi} - \frac{1}{\pi}\left[\frac{1}{2}x\sin(2x)|_{x=0}^{x=\pi} - \frac{1}{2}\int_0^\pi \sin(2x)\, dx\right] = \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) - \frac{1}{2\pi}\left[\pi\sin(2\pi) - 0 + \frac{1}{2}\cos(2x)|_{x-0}^{x=\pi}\right]$
   $= \frac{\pi}{4} - \frac{1}{2\pi}\left[\frac{1}{2}(\cos 2\pi - \cos 0)\right] = \frac{\pi}{4}$

   And therefore the center of mass is found at $(\bar{x}, \bar{y}) = \left(\pi - \frac{4}{\pi}, \frac{\pi}{4}\right)$

2. We know from a previous question that the total mass of this lamina is 112 (see *Finding Total Mass*).

   Finding $\bar{x}$: $\bar{x} = \frac{1}{112}\int_0^4 \int_0^3 x\left(1 + x^2 + y^2\right)\ dy\, dx = \frac{1}{112}\int_0^4 \int_0^3 \left(x + x^3 + xy^2\right)\ dy\, dx$

$= \frac{1}{112}\int_0^4 \left[xy + x^3y + \frac{x}{3}y^3\right]_{y=0}^{y=3} dx = \frac{1}{112}\int_0^4 \left[3x + 3x^3 + 9x\right] dx = \frac{3}{112}\int_0^4 \left[4x + x^3\right] dx = \frac{3}{112}\left[2x^2 + \frac{x^4}{4}\right]_{x=0}^{x=4} = \frac{3}{112}\left[2(4)^2 - 2(0)^2 + \frac{4^4}{4} - \frac{0^4}{4}\right] = \frac{3}{112}\left[32 + 64\right] = \frac{3\cdot 96}{112} = \frac{3\cdot2\cdot2\cdot2\cdot2\cdot2\cdot3}{2\cdot2\cdot7\cdot2\cdot2} = \frac{18}{7}$

Finding $\overline{y}$: $\overline{y} = \frac{1}{112}\int_0^4\int_0^3 y\left(1 + x^2 + y^2\right) dy\, dx = \frac{1}{112}\int_0^4\int_0^3 \left[y + x^2y + y^3\right] y\, dx$

$= \frac{1}{112}\int_0^4 \left[\frac{y^2}{2} + \frac{x^2y^2}{2} + \frac{y^4}{4}\right]_{y=0}^{y=3} dx = \frac{1}{112}\int_0^4 \left[\frac{3^2}{2} + \frac{3^2x^2}{2} + \frac{3^4}{4}\right] dx = \frac{1}{112}\int_0^4 \left[\frac{99}{4} + \frac{9}{2}x^2\right] dx$

$= \frac{1}{112}\left[\frac{99}{4}x + \frac{3}{2}x^3\right]_{x=0}^{x=4} = \frac{3}{224}\left[\frac{33}{2}(4) + 4^3\right] = \frac{3}{224}(66 + 64) = \frac{3\cdot 130}{224} = \frac{3\cdot 65}{112} = \frac{195}{112}$

Therefore the center of mass of the rectangular region D is $\left(\frac{18}{7}, \frac{195}{112}\right)$

3. We know from a previous question (see *Finding Total Mass*) that the total mass of D is 6 and it can be described as $D = \{(x, y) \mid 0 \le x \le 2, \frac{x}{2} \le y \le 3 - x\}$

Finding $\overline{x}$:

$\overline{x} = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} x(x+y)\, dy\, dx = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} \left(x^2 + xy\right) dy\, dx = \frac{1}{6}\int_0^2 \left[x^2y + \frac{x}{2}y^2\right]_{y=x/2}^{y=3-x} dx$

$= \frac{1}{6}\int_0^2 \left[x^2\left(3 - x - \frac{x}{2}\right) + \frac{x}{2}\left((3-x)^2 - \left(\frac{x}{2}\right)^2\right)\right] dx = \frac{1}{6}\int_0^2 \left[3x^2 - x^3 - \frac{x^3}{2} + \frac{x}{2}\left(9 - 6x + x^2 - \frac{x^2}{4}\right)\right] dx$

$= \frac{1}{6}\int_0^2 \left[3x^2 - \frac{3}{2}x^3 + \frac{x}{2}\left(9 - 6x + \frac{3}{4}x^2\right)\right] dx = \frac{1}{6}\int_0^2 \left[3x^2 - \frac{3}{2}x^3 + \frac{9}{2}x - 3x^2 + \frac{3}{8}x^3\right] dx = \frac{1}{6}\int_0^2 \left[\frac{9}{2}x - \frac{9}{8}x^3\right] dx = \frac{1}{6}\left[\frac{9}{4}x^2 - \frac{9}{32}x^4\right]_{x=0}^{x=2} = \frac{1}{6}\left[\frac{9\cdot 4}{4} - \frac{9\cdot 16}{32}\right] = \frac{1}{6}\left[9 - \frac{9}{2}\right] = \frac{1}{6}\cdot\frac{9}{2} = \frac{9}{12} = \frac{3}{4}$

And finding $\overline{y}$:

$\overline{y} = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} y(x+y)\, dy\, dx = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} \left(xy + y^2\right) dy\, dx = \frac{1}{6}\int_0^2 \left[\frac{x}{2}y^2 + \frac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx$

$= \frac{1}{6}\int_0^2 \left[\frac{x}{2}\left((3-x)^2 - \left(\frac{x}{2}\right)^2\right) + \frac{1}{3}\left((3-x)^3 - \left(\frac{x}{2}\right)^3\right)\right] dx$

$= \frac{1}{6}\int_0^2 \left[\frac{x}{2}\left(9 - 6x + x^2 - \frac{x^2}{4}\right) + \frac{1}{3}\left(27 - 27x + 9x^2 - x^3 - \frac{x^3}{8}\right)\right] dx$

$= \frac{1}{6}\int_0^2 \left[\frac{x}{2}\left(9 - 6x + \frac{3x^2}{4}\right) + \frac{1}{3}\left(27 - 27x + 9x^2 - \frac{9x^3}{8}\right)\right] dx$

$= \frac{1}{6}\int_0^2 \left[\frac{9}{2}x - 3x^2 + \frac{3}{8}x^3 + 9 - 9x + 3x^2 - \frac{3}{8}x^3\right] dx = \frac{1}{6}\int_0^2 \left[9 - \frac{9}{2}x\right] dx = \frac{1}{6}\left[9x - \frac{9}{4}x^2\right]_{x=0}^{x=2}$

$= \frac{3}{6}\left[3(2) - \frac{3}{4}(2)^2\right] = \frac{1}{2}(6 - 3) = \frac{1}{2}\cdot 3 = \frac{3}{2}$

Therefore, the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{3}{4}, \frac{3}{2}\right)$

Answer to Exercise 17 (on page 48)

1.

$$I_x = \iint_D y^2\rho(x, y)\, dA = \int_1^4\int_0^3 y^2(xy)\, dy\, dx$$

$$= \int_1^4\int_0^3 xy^3\, dy\, dx = \int_1^4 x\left[\frac{1}{4}y^4\right]_{y=0}^{y=3} dx = \frac{1}{4}\int_1^4 81x\, dx$$

$$= \frac{81}{4}\left[\frac{1}{2}x^2\right]_{x=1}^{x=4} = \frac{81}{2}\left(4^2 - 1^2\right) = \frac{81}{2}\cdot 15 = \frac{1215}{2}$$

To find the radius of gyration, first we need to find the total mass:

$$m = \iint_D \rho(x, y)\, dA = \int_1^4 \int_0^3 xy\, dy\, dx$$

$$= \int_1^4 \frac{x}{2} \left[y^2\right]_{y=0}^{y=3} dx = \frac{9}{2} \int_1^4 x\, dx = \frac{9}{2} \cdot \left(\frac{1}{2}\right) \cdot \left[x^2\right]_{x=1}^{x=4} = \frac{9}{4} [16 - 1] = \frac{135}{2}$$

Finding the radius of gyration about the x-axis:

$$I_x = m\bar{\bar{y}}^2$$

$$\frac{1215}{2} = \left(\frac{135}{2}\right) \bar{\bar{y}}^2$$

$$9 = \bar{\bar{y}}^2$$

$$\bar{\bar{y}} = 3$$

2.

$$I_y = \iint_D x^2 \rho(x, y)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos x} x^2 y\, dy\, dx$$

$$= \int_{-pi/2}^{\pi/2} x^2 \left[\frac{1}{2} y^2\right]_{y=0}^{y=2\cos x} dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} x^2 \left(2\cos x\right)^2\, dx$$

$$= 2\int_{-\pi/2}^{\pi/2} x^2 \cos^2 x\, dx = 2\int_{-\pi/2}^{\pi/2} x^2 \left(\frac{1 + \cos 2x}{2}\right)\, dx = \int_{-\pi/2}^{\pi/2} x^2\, dx + \int_{-\pi/2}^{\pi/2} x^2 \cos 2x\, dx$$

$$= \frac{1}{3} \left[x^3\right]_{x=-\pi/2}^{x=\pi/2} + \frac{1}{2} x^2 \sin 2x\Big|_{x=-\pi/2}^{x=\pi/2} - \int_{-\pi/2}^{\pi/2} \frac{1}{2} \sin 2x \left(2x\right)\, dx$$

$$= \frac{1}{3} \left[\left(\frac{\pi}{2}\right)^3 - \left(\frac{-\pi}{2}\right)^3\right] - \int_{-\pi/2}^{\pi/2} x \sin 2x\, dx$$

$$= \frac{1}{3} \left(\frac{2\pi^3}{8}\right) - \left(\left[-\frac{1}{2} x \cos 2x\right]_{x=-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} (-\frac{1}{2} \cos 2x)\, dx\right)$$

$$= \frac{\pi^3}{12} + \left(\frac{1}{2}\left(\frac{\pi}{2}\right) \cos\left(\pi\right) - \frac{1}{2}\left(\frac{-\pi}{2}\right) \cos\left(-\pi\right)\right) - \left[\frac{1}{4} \sin 2x\right]_{x=-\pi/2}^{\pi/2}$$

$$= \frac{\pi^3}{12} + \frac{\pi}{4}(-1) + \frac{\pi}{4}(-1) = \frac{\pi^3}{12} - \frac{\pi}{2}$$

In order to find the radius of gyration, we need to first know the total mass:

$$m = \iint_D \rho(x, y)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos x} y\, dy\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2} y^2 \Big|_{y=0}^{y=2\cos x} dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} 4\cos^2 x \, dx = \int_{-\pi/2}^{\pi/2} 1 + \cos 2x \, dx$$

$$= \left[x + \frac{1}{2} \sin 2x \right]_{x=-\pi/2}^{\pi/2} = \pi$$

Then we can find the radius of gyration about the y-axis:

$$m\bar{\bar{x}}^2 = I_y$$

$$\pi\bar{\bar{x}}^2 = \frac{\pi^3}{12} - \frac{\pi}{2}$$

$$\bar{\bar{x}} = \sqrt{\frac{\pi^2}{12} - \frac{1}{2}}$$

3.

$$I_o = \iint_D \left(x^2 + y^2\right) \rho(x,y) = \int_1^2 \int_0^\pi r(r^2) r \, d\theta \, dr$$

$$= \int_1^2 r^4 \theta \Big|_{\theta=0}^{\theta=\pi} dr = \pi \int_1^2 r^4 \, dr = \frac{\pi}{5} r^5 \Big|_{r=1}^{r=2} = \frac{\pi}{5}\left(2^5 - 1\right) = \frac{\pi}{5}(31) = \frac{31\pi}{5}$$

We find the total mass:

$$m = \iint_D \rho(x,y) \, dA = \int_1^2 \int_0^\pi r^2 \, d\theta \, dr = \int_1^2 r^2 \theta \Big|_{\theta=0}^{\theta=\pi} dr$$

$$= \pi \int_1^2 r^2 \, dr = \frac{\pi}{3} r^3 \Big|_{r=1}^{r=2} = \frac{\pi}{3}\left(2^3 - 1\right) = \frac{\pi}{3}(7) = \frac{7\pi}{3}$$

To find the radius of gyration about the origin:

$$mR^2 = I_o$$

$$\left(\frac{7\pi}{3}\right) R^2 = \frac{31\pi}{5}$$

$$R^2 = \frac{31}{5} \cdot \frac{3}{7} = \frac{93}{35}$$

$$R = \sqrt{\bar{\bar{x}}^2 + \bar{\bar{y}}^2} = \sqrt{\frac{93}{35}}$$

Answer to Exercise 18 (on page 53)

1. Rearranging the formula for the plane, we find that $z = 3x + 2y + 2$ and therefore

$\partial z/\partial x = 3$ and $\partial z/\partial y = 2$. Then the surface area is given by:

$$A(S) = \int_2^6 \int_1^4 \sqrt{1+3^2+2^2}\, dy\, dx = \int_2^6 \sqrt{14} y|_{y=1}^{y=4}\, dx$$

$$= \int_2^6 3\sqrt{14}\, dx = 3\sqrt{14}x|_{x=2}^{x=6} = 12\sqrt{14}$$

2. The paraboloid intersects the plane when $2x^2+2y^2 = 32$, which is the circle of radius 4 centered at the origin. So, we are looking for the area of the surface $z = 2x^2 + 2y^2$ that lies above the region $R - \{(r, \theta) \mid 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$. The surface area is:

$$A(S) = \iint_R \sqrt{1+(4x)^2+(4y)^2}\, dA = \int_0^4 \int_0^{2\pi} r\sqrt{1+(4r\cos\theta)^2+(4r\sin\theta)^2}\, d\theta\, dr$$

$$= \int_0^4 \int_0^{2\pi} r\sqrt{1+16r^2}\, d\theta\, dr = \int_0^4 r\sqrt{1+16r^2}\, [\theta]_{\theta=0}^{\theta=2\pi}\, dr$$

$$= \int_0^4 2\pi r\sqrt{1+16r^2}\, dr$$

Let $u = 1 + 16r^2$, then $du = 32r(dr)$ and $r(dr) = du/32$. Substituting:

$$A(S) = \frac{2\pi}{32} \int_{r=0}^{r=4} \sqrt{u}\, du = \frac{\pi}{16}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_{r=0}^{r=4}$$

$$= \frac{\pi}{24}\left[\left(1+16r^2\right)^{3/2}\right]_{r=0}^{r=4} = \frac{\pi}{24}\left[(257)^{3/2} - 1\right] \approx 6470.15$$

3. The region, R, we are interested in is the circle of radius 2 centered at the origin of the xy-plane, described by $R = \{(r, \theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$. Noting that $\partial z/\partial x = 3y$ and $\partial z/\partial y = 3x$, we see that the surface area is given by:

$$A(S) = \iint_R \sqrt{1+(3y)^2+(3x)^2}\, dA = \int_0^2 \int_0^{2\pi} r\sqrt{1+9r^2\sin^2\theta+9r^2\cos^2\theta}\, d\theta\, dr$$

$$= \int_0^2 \int_0^{2\pi} r\sqrt{1+9r^2}\, d\theta\, dr = 2\pi \int_0^2 r\sqrt{1+9r^2}\, dr$$

Let $u = 1 + 9r^2$, then $du = 18r(dr)$, which means that $r(dr) = du/18$. Substituting:

$$A(S) = \frac{2\pi}{18} \int_{r=0}^{r=2} \sqrt{u}\, du = \frac{\pi}{9}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_{r=0}^{r=2}$$

$$= \frac{2\pi}{27}\left[(1+9(4))^{3/2} - 1\right] = \frac{2\pi}{27}\left[(37)^{3/2} - 1\right] \approx 52.14$$

Answer to Exercise 19 (on page 54)

1. The area of D is 2π. Therefore the average value is:

$$\frac{1}{2\pi}\iint_D x\sin y\, dA = \frac{1}{2\pi}\int_0^2\int_0^\pi x\sin y\, dy\, dx$$

$$= \frac{1}{2\pi}\int_0^2 -x\cos y|_{y=0}^{y=\pi}\, dx = \frac{1}{2\pi}\int_0^2 -x\left(\cos\pi-\cos 0\right)\, dx$$

$$= \frac{1}{2\pi}\int_0^2(-x)(-1-1)\, dx = \frac{1}{2\pi}\int_0^2 2x\, dx$$

$$= \frac{1}{2\pi}x^2|_{x=0}^{x=2} = \frac{2}{\pi}$$

2. Since D is a circle of radius $r=1$, the area is $A=\pi r^2=\pi$. D can be described with $D=\{(r,\theta)\mid 0\le r\le 2\cos\theta,\ -\pi/2\le\theta\le\pi/2\}$. Therefore, the average value of $f(x,y)=x+y$ over D is:

$$f_{ave} = \frac{1}{\pi}\iint_D\left(x+y\right)\, dA = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\cdot\left(r\cos\theta+r\sin\theta\right)\, dr\, d\theta$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta+\sin\theta)\left[\int_0^{2\cos\theta} r^2\, dr\right]\, d\theta$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta+\sin\theta)\cdot\left[\frac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta}\, d\theta$$

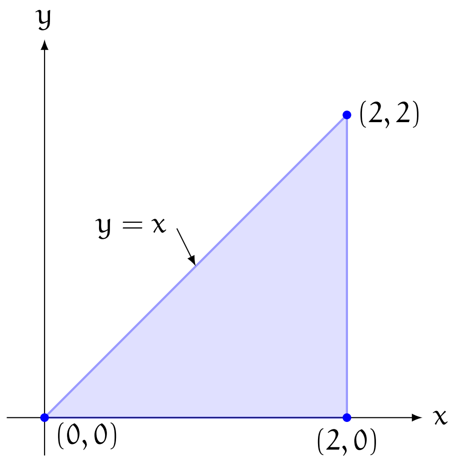
$$= \frac{8}{3\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta+\sin\theta)\cos^3\theta\, d\theta = \frac{8}{3\pi}\int_{\pi/2}^{\pi/2}\left(\cos^4\theta+\sin\theta\cos^3\theta\right)\, d\theta$$

$$= \frac{8}{3\pi}\left[\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2\, d\theta - \left[\frac{1}{4}\cos^4\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2}\right]$$

$$= \frac{2}{3\pi}\int_{-\pi/2}^{\pi/2}\left(1+2\cos 2\theta+\cos^2 2\theta\right)\, d\theta = \frac{2}{3\pi}\left[(\theta+\sin 2\theta)_{\theta=-\pi/2}^{\theta=\pi/2} + \int_{-\pi/2}^{\pi/2}\frac{1+\cos 4\theta}{2}\, d\theta\right]$$

$$= \frac{2}{3\pi}\left[\pi+\frac{1}{2}\left(\theta+\frac{1}{4}\sin 4\theta\right)_{\theta=-\pi/2}^{\theta=\pi/2}\right] = \frac{2}{3\pi}\left[\pi+\frac{1}{2}\left(\pi\right)\right] = \frac{2}{3\pi}\left(\frac{3\pi}{2}\right) = 1$$

3. Let's visualize D:



So, D can be described $D = \{(x, y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq x\}$. Additionally, D has area $A = \frac{1}{2}\left(2^2\right) = 2$. Therefore, the average value of $f(x, y) = xy$ over D is:

$$f_{ave} = \frac{1}{2}\iint_D (xy)\ dA = \frac{1}{2}\int_0^2\int_0^x (xy)\ dy\,dx$$

$$= \frac{1}{2}\int_0^2 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=x} dx = \frac{1}{4}\int_0^2 x^3\,dx = \frac{1}{4}\left[\frac{1}{4}x^4\right]_{x=0}^{x=2}$$

$$= \frac{1}{16}\left(2^4\right) = 1$$

Index