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Chapter 1

Implicit Differentiation

Implicit differentiation is a technique in calculus for finding the derivative of a relation defined implicitly, that is, a relation between variables x and y that is not explicitly solved for one variable in terms of the other.

1.1 Implicit Differentiation Procedure

Consider an equation that defines a relationship between x and y:

$$F(x, y) = 0$$

To find the derivative of y with respect to x, we differentiate both sides of this equation with respect to x, treating y as an implicit function of x:

$$\frac{d}{dx}F(x, y) = \frac{d}{dx}0$$

Applying the chain rule during the differentiation on the left side of the equation gives:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Finally, we solve for $\frac{dy}{dx}$ to find the derivative of y with respect to x:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

This result is obtained using the implicit differentiation method.

1.2 Example

Consider the equation of a circle with radius r:

$$x^2 + y^2 = r^2$$

First, we'll find $\frac{dy}{dx}$ the without implicit differentiation. Then, we'll apply implicit differentiation to get the same result.

1.2.1 Without Implicit Differentiation

First, we need to re-arrange the equation to solve for y:

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

We take the derivative of y by applying the Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2 \pm \sqrt{r^2 - x^2}} \cdot (-2x) = \frac{-x}{\pm\sqrt{r^2 - x^2}}$$

Notice the denominator of this fraction is the same as the solution we found for y, $y = \pm\sqrt{r^2 - x^2}$. So we can also represent this as:

$$\frac{dy}{dx} = \frac{-x}{y}$$

1.2.2 With Implicit Differentiation

With implicit differentiation, we assume y is a function of x and apply the Chain Rule.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[r^2]$$

For x^2 and r^2, we take the derivative as we normally would. For y^2, we apply the Chain Rule as outlined above.

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{-x}{y}$$

which is the same result as we found without implicit differentiation.

1.3 Folium of Descartes

It was relatively easy to rearrange the equation for a circle to solve for y, but that is not always the case. Consider the equation for the folium of Descartes (yes, that Descartes!):

$$x^3 + y^3 = 3xy$$

It is much more difficult to isolate y in this equation. In fact, were we to do so, we would need 3 separate equations to completely describe the original equation.

1.3.1 Example: Tangent to Folium of Descartes

In this example, we will use implicit differentiation to easily find the tangent line at a point on the folium.

(a) Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

(b) Find the tangent to the folium $x^3 + y^3 = 3xy$ at the point $(2, 2)$

(c) Is there any place in the first quadrant where the tangent line is horizontal? If so, state the point(s).

Solution:

(a) $\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

Rearranging to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

(b) We already have the coordinate point, $(2, 2)$, so to write an equation for the tangent line all we need is the slope. Substituting $x = 2$ and $y = 2$ into our result from part (a):

$$\frac{dy}{dx} = \frac{2 - 2^2}{2^2 - 2} = \frac{-2}{2} = -1$$

This is the slope, m. Using the point-slope form of a line, our tangent line is $y-2 = -(x-2)$.

(c) Recall that in the first quadrant, $x > 0$ and $y > 0$. We will set our solution for $\frac{dy}{dx}$ equal to 0:

$$\frac{y - x^2}{y + 2 - x} = 0$$

which implies that

$$y - x^2 = 0$$

Substituting $y = x^2$ into the original equation:

$$x^3 + (x^2)^3 = 3(x)(x^2)$$
$$x^3 + x^6 = 3x^3$$

Which simplifies to

$$x^6 = 2x^3$$

Since we have excluded $x = 0$ by restricting our search to the first quadrant, we can divide both sides by x^3:

$$x^3 = 2$$
$$x = \sqrt[3]{2} \approx 1.26$$

Substituting $x \approx 1.26$ into our equation for y:

$$y \approx 1.26^2 = 1.59$$

Therefore, the folium has a horizontal tangent line at the point $(1.26, 1.59)$.

1.4 Practice

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] If $\arcsin x = \ln y$, what is $\frac{dy}{dx}$?

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Exercise 2

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] The points $(-1,-1)$ and $(1,-5)$ are on the graph of a function $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = x^2+y$. Use implicit differentiation to find $\frac{d^2y}{dx^2}$. Determine if each point is a local minimum, local maximum, or inflection point by substituting the x and y values of the coordinates into $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Working Space

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Chapter 2

Related Rates

In calculus, related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. The technique used to solve these problems is known as "related rates" because one rate is related to another rate.

2.1 Steps to solve related rates problems

2.1.1 Step 1: Understand the problem

First, read the problem carefully. Understand what rates are given and what rate you need to find.

2.1.2 Step 2: Draw a diagram

For most problems, especially geometry problems, drawing a diagram can be very helpful.

2.1.3 Step 3: Write down what you know

Write down the rates that you know and the rate that you need to find.

2.1.4 Step 4: Write an equation

Write an equation that relates the quantities in the problem. This equation will be your main tool to solve the problem.

2.1.5 Step 5: Differentiate both sides of the equation

Now you can use calculus. Differentiate both sides of the equation with respect to time.

2.1.6 Step 6: Substitute the known rates and solve for the unknown

Now that you have an equation that relates the rates, substitute the known rates into the equation and solve for the unknown rate.

2.2 Example

Here is an example of a related rates problem:

A balloon is rising at a constant rate of 5 m/s. A boy is cycling towards the balloon along a straight path at 15 m/s. If the balloon is 100 m above the ground, find the rate at which the distance from the boy to the balloon is changing when the boy is 40 m from the point on the ground directly beneath the balloon.

The problem can be modeled with a right triangle where the vertical side is the height of the balloon, the horizontal side is the distance of the boy from the point on the ground directly beneath the balloon, and the hypotenuse is the distance from the boy to the balloon.

Let x be the distance of the boy from the point on the ground directly beneath the balloon, y the height of the balloon above the ground, and z the distance from the boy to the balloon. From the Pythagorean theorem, we have

$$z^2 = x^2 + y^2 \tag{2.1}$$

Differentiating both sides with respect to time t gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \tag{2.2}$$

Given that $\frac{dx}{dt} = -15$ m/s (the boy is moving towards the point beneath the balloon), $\frac{dy}{dt} = 5$ m/s (the balloon is rising), $x = 40$ m, $y = 100$ m, we can substitute these into the equation and solve for $\frac{dz}{dt}$.

Chapter 3

Multivariate Functions

A real-valued multivariate function is a function that takes multiple real variables as input and produces a single real output.

We generally denote such a function as $f : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is the domain and $\mathbb{R}$ is the co-domain.

For example, consider a function f that takes two variables x and y:

$$f(x, y) = x^2 + y^2$$

Here, $f : \mathbb{R}^2 \to \mathbb{R}$ takes an ordered pair (x, y) from the 2-dimensional real coordinate space, squares each, and adds them to produce a real number.

In a similar way, a function $g : \mathbb{R}^3 \to \mathbb{R}$ could take three variables x, y, and z, and might be defined as:

$$g(x, y, z) = x^2 + y^2 + z^2$$

Here, the function squares each of the input variables and then adds them to produce a real number.

These functions are ”real-valued” because their outputs are real numbers, and ”multivariate” because they take multiple variables as inputs.

The concepts of limits, continuity, differentiability, and integrability can all be extended to multivariate functions, although they become more complex because we now have to consider different directions in which we approach a point, not just from the left or right as in the univariate case. For example, the partial derivative is the derivative of the function with respect to one variable, holding the others constant. It is one of the basic concepts in the calculus of multivariate functions.

For example, given the function $f(x, y) = x^2 + y^2$, the partial derivatives of f are computed as:

$$\frac{\partial f}{\partial x}(x, y) = 2x$$

$$\frac{\partial f}{\partial y}(x, y) = 2y$$

Chapter 4

Partial Derivatives and Gradients

This chapter will introduce you to partial derivatives and gradients, equipping you with the tools to study functions of multiple variables. We will explore how these concepts provide valuable insights into optimization, vector calculus, and various fields of science and engineering.

Partial derivatives come into play when dealing with functions that depend on multiple variables. Unlike ordinary derivatives that consider changes along a single variable, partial derivatives focus on how a function changes concerning each individual variable while holding the others constant. In essence, partial derivatives measure the rate of change of a function with respect to one variable while keeping the other variables fixed.

The notation for a partial derivative of a function $f(x, y, \dots)$ with respect to a specific variable, say x, is denoted as $\frac{\partial f}{\partial x}$. Similarly, $\frac{\partial f}{\partial y}$ represents the partial derivative with respect to y, and so on. It is essential to remember that when taking partial derivatives, we treat the other variables as constants during the differentiation process.

The gradient is a vector that combines the partial derivatives of a function. It provides a concise representation of the direction and magnitude of the steepest ascent or descent of the function. The gradient vector points in the direction of the greatest rate of increase of the function. By understanding the gradient, we gain insights into optimizing functions and finding critical points where the function reaches maximum or minimum values.

Throughout this chapter, we will explore the following key topics related to partial derivatives and gradients:

- Calculating partial derivatives: We will delve into the techniques and rules for computing partial derivatives of various functions, including polynomials, exponential functions, and trigonometric functions. We will also explore higher-order partial derivatives and mixed partial derivatives.
- Interpreting partial derivatives: Understanding the geometric and physical interpretations of partial derivatives is essential. We will discuss the notion of tangent planes, directional derivatives, and the relationship between partial derivatives and local linearity.
- Gradient vectors and their properties: We will introduce the gradient vector and its properties, such as its connection to the direction of steepest ascent, its relationship

with partial derivatives, and how it relates to level curves and level surfaces.

- Applications of partial derivatives and gradients: We will explore various applications of these concepts, including optimization problems, constrained optimization, tangent planes, linear approximations, and their relevance in fields like physics, economics, and engineering.

By grasping the concepts of partial derivatives and gradients, you will unlock a powerful mathematical framework for analyzing and optimizing functions of multiple variables. These tools will equip you to tackle advanced calculus problems and gain deeper insights into the behavior of functions in diverse fields.

4.1 Calculating Partial Derivatives

For a function of two variables, $f(x, y)$, we can take the derivative with respect to x or with respect to y. These are called the *partial derivatives* of f. Formally, the partial derivatives are defined as:

Limit Definition of Partial Derivatives

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Let's consider a polynomial function of two variables: $f(x, y) = 3x^2 + y^3 + 4xy$. We will use the limit definition to find the partial derivative with respect to x, then compare this to what we already know about derivatives of single-variable functions. Recall that if we can describe a function as a sum of two other functions, the derivative of the original function is the same as the sum of the derivatives of the other functions. That is,

$$\text{if } f(x) = g(x) + h(x)$$
$$\text{then } f'(x) = g'(x) + h'(x)$$

Let's then define $r(x, y) = 3x^2$, $s(x, y) = y^3$, and $t(x, y) = 4xy$. And so $f(x, y) = r(x, y) + s(x, y) + t(x, y)$, which means $f_x(x, y) = r_x(x, y) + s_x(x, y) + t_x(x, y)$. Then,

$$\begin{aligned} f_x(x, y) &= \lim_{h \to 0} \frac{r(x+h, y) - r(x, y)}{h} + \lim_{h \to 0} \frac{s(x+h, y) - s(x, y)}{h} + \lim_{h \to 0} \frac{t(x+h, y) - t(x, y)}{h} \\ &= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} + \lim_{h \to 0} \frac{y^3 - y^3}{h} + \lim_{h \to 0} \frac{4(x+h)y - 4xy}{h} \end{aligned}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + h^2 - 3x^2}{h} + 0 + \lim_{h \to 0} \frac{4xy + 4hy - 4xy}{h}$$

Notice that $s_x(x, y) = 0$. This term only had y, and its derivative with respect to x is zero. Continuing,

$$f_x(x, y) = \lim_{h \to 0} \frac{6xh + h^2}{h} + \lim_{h \to 0} \frac{4hy}{h} = \lim_{h \to 0} 6x + h + \lim_{h \to 0} 4y$$
$$= 6x + 4y$$

As you can see, $r_x(x, y) = 6x$ and $t_x(x, y) = 4y$. Recall the polynomial rule for single derivatives. The derivative of $3x^2$ is $6x$, which is also what we see with the partial derivative in this case. What about the other term, $4xy$? Well, we know the derivative of bx, where b is a constant, is b. The partial derivative of $4xy$ with respect to x being $4y$ suggests the rule for determining partial derivatives:

Rule for Finding Partial Derivatives of $f(x, y)$

1. To find the partial derivative with respect to x, f_x, treat y as a constant and differentiate with respect to x.
2. To find the partial derivative with respect to y, f_y, treat x as a constant and differentiate with respect to y.

Let's check this by predicting f_y and then using the limit definition to confirm our prediction. Applying the polynomial rule, we predict that f_y is:

$$f_y(x, y) = 3y^2 + 4x$$

Which we found by treating x as a constant and taking the derivative of each term with respect to y. Let's see if we get the same result using the limit definition of the derivative with respect to y:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$
$$= \lim_{h \to 0} \frac{\left[3x^2 + (y + h)^3 + 4x(y + h)\right] - \left[3x^2 + y^3 + 4xy\right]}{h}$$
$$= \lim_{h \to 0} \frac{3x^2 + y^3 + 3y^2h + 3yh^2 + h^3 + 4xy + 4xh - 3x^2 - y^3 - 4xy}{h}$$
$$= \lim_{h \to 0} \frac{3y^2h + 3yh^2 + h^3 + 4xh}{h} = \lim_{h \to 0} 3y^2 + 3yh + h^2 + 4x = 3y^2 + 4x$$

Which is our expected result. In summary, you find the partial derivative with respect to a particular variable by treating all the other variables as constants and differentiating with respect to the particular variable, applying the rules of differentiation you've already learned.

4.1.1 Partial Derivative Notation

There are many ways to denote a partial derivative. We've already seen one way, f_x and f_y. Another common notation uses a lowercase Greek letter delta, and a further uses capital D. They are shown below:

Partial Derivative Notations

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = D_y f$$

Exercise 3 First Partial Derivatives

Find f_x and f_y for the following functions.

1. $f(x, y) = 3x^4 + 4x^2y^3$
2. $f(x, y) = xe^{-y}$
3. $f(x, y) = \sqrt{3x + 4y^2}$
4. $f(x, y) = \sin x^2 y$
5. $f(x, y) = \ln(x^y)$

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4.1.2 Partial Derivatives of Functions of More than Two Variables

The above method of determining partial derivatives applies to functions with three, four, or any number of variables.

Example: Find all the first derivatives of the function $f(x, y, z) = y\cos\left(x^2 + 3z\right)$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[y\cos\left(x^2 + 3z\right)\right] = -y\sin\left(x^2 + 3z\right)\left(\frac{\partial}{\partial x}\left(x^2 + 3z\right)\right)$$

$$\frac{\partial f}{\partial x} = -2xy\sin\left(x^2 + 3z\right)$$

And

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[y\cos\left(x^2+3z\right)\right]$$
$$\frac{\partial f}{\partial y} = \cos\left(x^2+3z\right)$$

And

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[y\cos\left(x^2+3z\right)\right] = -y\sin\left(x^2+3z\right)\left(\frac{\partial}{\partial z}\left(x^2+3z\right)\right)$$
$$\frac{\partial f}{\partial z} = -3y\sin\left(x^2+3z\right)$$

Exercise 4 Partial Derivatives with 3 or More Variables

Find all first partial derivatives of the following functions.

1. $f = \sin\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)$
2. $q = \sqrt[3]{t^3 + u^3 \sin(5v)}$
3. $w = x^z y^x$

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4.1.3 Higher Order Partial Derivatives

Just like with single-variable equations, we can take the partial derivative more than once. There are also several notations for second partial derivatives.

Second Partial Derivative Notation

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Notice that for $\left(\partial^2 f/\partial y \partial x\right)$, we first take the derivative with respect to x, then with respect to y.

Example: Find all the second order partial derivatives of $f(x, y) = 2x^2 - x^3y^2 + y^3$.

Solution: We begin by finding f_x and f_y:

$$f_x(x, y) = 4x - 3x^2y^2$$

$$f_y(x, y) = -2x^3y + 3y^2$$

We then take another partial derivative to find all the second order partial derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial}{\partial x}\left(4x - 3x^2y^2\right) = 4 - 6xy^2$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial y}\left(4x - 3x^2y^2\right) = -6x^2y$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial}{\partial x}\left(-2x^3y + 3y^2\right) = -6x^2y$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial}{\partial y}\left(-2x^3y + 3y^2\right) = -2x^3 + 6y$$

What do you notice about f_{xy} and f_{yx}? They are the same! This is not a coincidence of the particular function used in the example. For most functions, $f_{xy} = f_{yx}$, as stated by Clairaut's theorem.

Clairaut's Theorem

If f is defined on a disk D and f_{xy} and f_{yx} are both continuous on D, then $f_{xy} = f_{yx}$ on D.

This is also true for third, fourth, and higher-order derivatives.

Exercise 5 Clairaut's Theorem

Show that Clairaut's theorem holds for the following functions (show that $f_{xy} = f_{yx}$).

1. $f(x, y) = e^{2xy} \sin x$
2. $f(x, y) = \frac{x^2}{x+y}$
3. $f(x, y) = \ln(2x + 3y)$

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Exercise 6 Second Order Partial Derivatives

Find all second order partial derivatives of the function.

1. $f(x, y) = x^5y^2 - 3x^3y^2$
2. $v = \sin\left(p^3 + q^2\right)$
3. $T = e^{-3r}\cos\theta^2$

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4.2 Interpreting Partial Derivatives

What is the meaning of a partial derivative? Recall that $z = f(x, y)$ plots a surface, S. Consider the function $z = \cos y - x^2$, shown in figure 4.1.

We can see that $f(1, \pi/3) = -1/2$, therefore the point $(1, \pi/3, -1/2)$ lies on the surface $z = \cos y - x^2$ (the black dot shown in figure **??**). If we fix y such that $y = \pi/3$, we are looking at the intersection between the surface and the plane $y = \pi/3$ (see figure **??**).

We can describe this intersection as $g(x) = f(x, \pi/3)$, and therefore the slope of a tangent line to this intersection is given by $g'(x) = f_x(x, \pi/3)$. This means, geometrically, $f_x(1, \pi/3)$ is the slope of the line that lies tangent to $z = f(x, y)$ at the point $(1, \pi/3, -1/2)$ and in

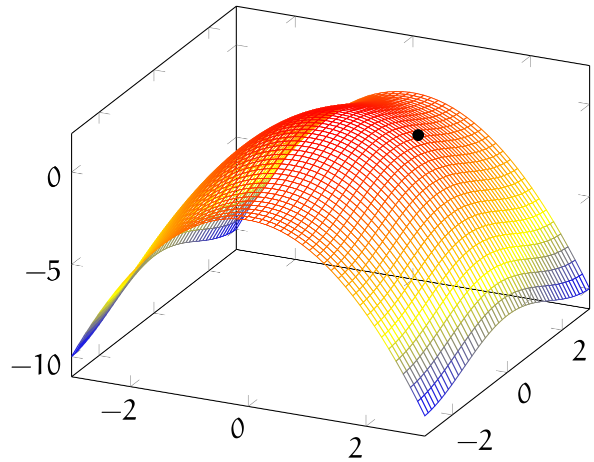

Figure 4.1: The surface $z = \cos y - x^2$

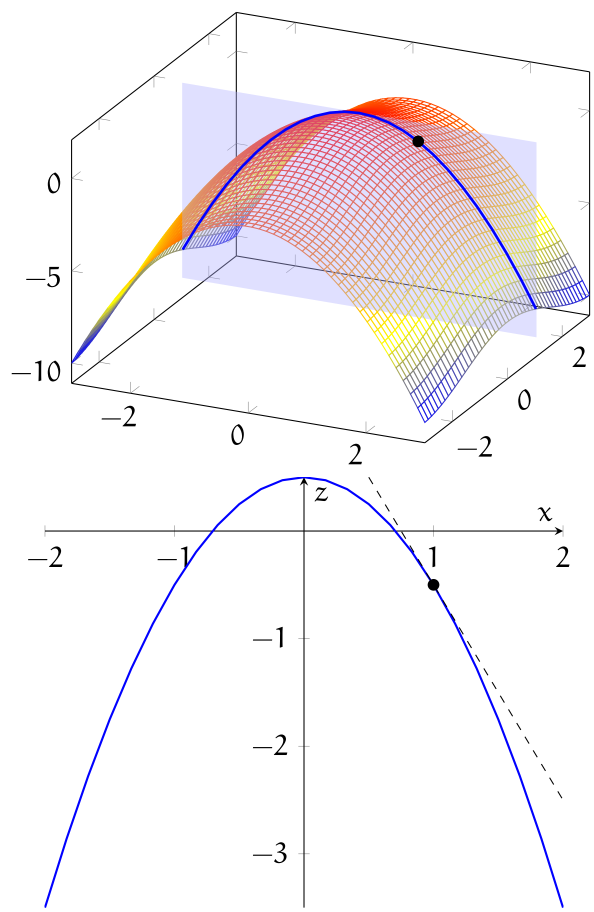

Figure 4.2: The intersection between the surface $z = \cos y - x^2$ and $y = \pi/3$ is the parabola $z(x) = 1/2 - x^2$

the plane $y = \pi/3$ (see figure 4.2). Alternatively, you could think of f_x as the slope of the tangent line to the surface that is parallel to the x-axis.

Similarly, we can fix $x = 1$ and look at the intersection between the surface $z = \cos y - x^2$ and the plane $x = 1$ (see figure 4.3. And just like before, we can describe this intersection as $h(y) = f(1, y)$, which means the slope of a line tangent to the intersection is given by $h'(y) = f_y(1, y)$. Therefore, as with f_x, $f_y(a, b)$ gives the slope of a line tangent to the point $(a, b, f(a, b))$ and parallel to the y-axis.

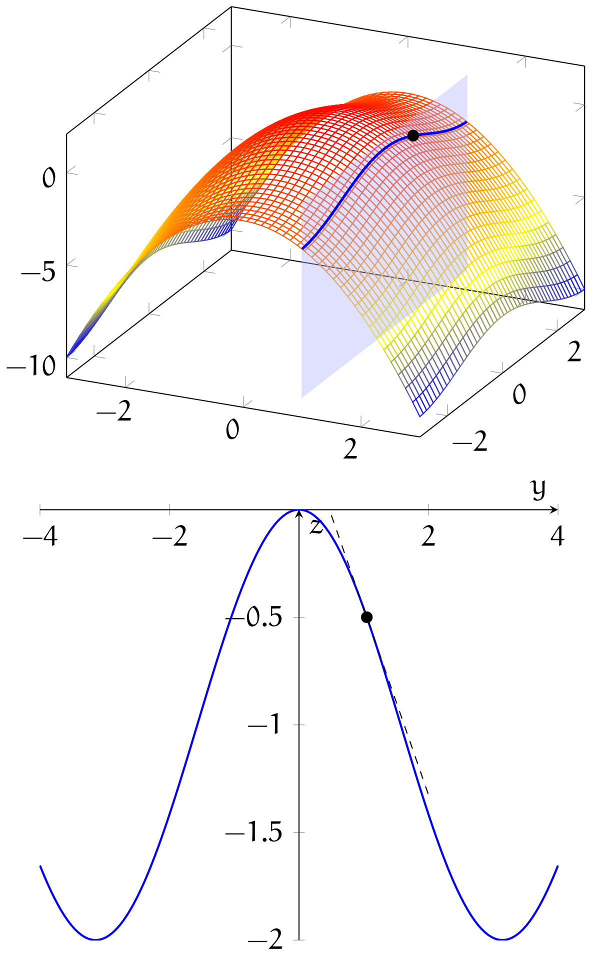

Figure 4.3: The intersection between the surface $z = \cos y - x^2$ and $x = 1$ is the trigonometric function $z = \cos y - 1$

Example: The density of bacterial growth at a point (x, y) on a flat agar plate is given by $D = 45/\left(2 + x^2 + y^2\right)$. Find the rate of change of bacterial density at the point $(1, 3)$ (a) in the x-direction and (b) in the y-direction. Interpret the meaning of your results.

Solution: The rate of change of a two-variable function in the x-direction is given by the partial derivative with respect to x:

$$
\begin{aligned}
D_x = \frac{\partial}{\partial x}\frac{45}{2+x^2+y^2} &= \frac{-45\,(\partial/\partial x)\left(2+x^2+y^2\right)}{(2+x^2+y^2)^2} \\
&= \frac{-90x}{(2+x^2+y^2)^2}
\end{aligned}
$$

And the rate of change in the x-direction at $(x, y) = (1, 3)$ is given by:

$$
D_x(1,3) = \frac{-90(1)}{(2+1^2+3^2)^2} = \frac{-90}{(12)^2} = \frac{-90}{144} = -\frac{5}{8}
$$

This means that at $(1, 3)$ the density of bacteria is decreasing as you move away $x = 0$ along the line $y = 3$.

Similarly, the rate of change in the y-direction is given by the partial derivative with respect to y:

$$
\begin{aligned}
D_y = \frac{\partial}{\partial y}\frac{45}{2+x^2+y^2} &= \frac{-45\,(\partial/\partial y)\left(2+x^2+y^2\right)}{(2+x^2+y^2)^2} \\
&= \frac{-90y}{(2+x^2+y^2)^2}
\end{aligned}
$$

And the rate of change in the y-direction at $(x, y) = (1, 3)$ is given by:

$$
D_y(1,3) = \frac{-90(3)}{(2+1^2+3^2)^2} = \frac{-270}{144} = -\frac{15}{8}
$$

This means that at $(1, 3)$ the density of bacteria is decreasing faster along the y-direction than along the x-direction.

Exercise 7 Using partial derivatives to find tangent lines

Find equations for tangent lines to the surface at the given xy-coordinate. In which direction is the function changing the fastest?

1. $z = x^2 e^{y/x}$, $(1, -1)$
2. $z = \cos x + y \sin y$, $(\pi, \pi/2)$
3. $z = x^2 y - 3xy^2$, $(3, 2)$

Working Space

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4.3 Gradient Vectors

The gradient vector is used to find the direction of the maximum rate of change of a surface (for example, the steepest part of a mountain). In order to understand the gradient, we must first discuss directional derivatives. Recall that the partial derivatives, f_x and f_y, can be used to define a plane tangent to the surface $z = f(x, y)$ (see figure 4.4). Directional derivatives allow us to find the slope of the tangent plane in directions other than the x- and y-directions.

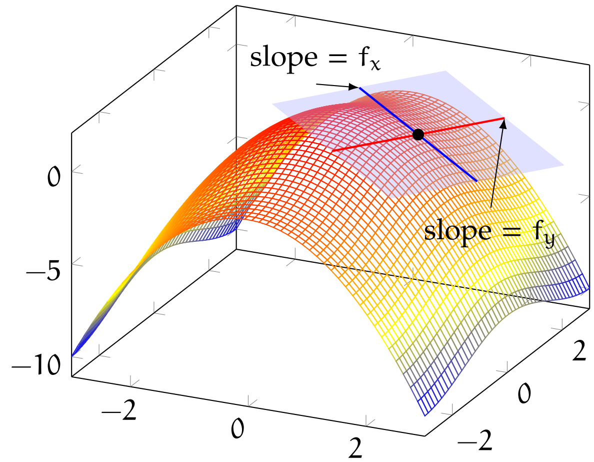

Figure 4.4: The directional derivatives, f_x and f_y define a tangent plane

4.3.1 Directional Derivatives

The contour map in figure 4.5 shows the elevation, $H(x, y)$ for a mountain. You already know that you can use the partial derivatives, H_x and H_y to find the rate of change in elevation going east-west or north-south. But what about other directions? Suppose the hiking path you're on goes north-east. How can you predict the steepness (i.e. the rate of elevation change) along this path? The directional derivative allows us to find the rate of change in any direction.

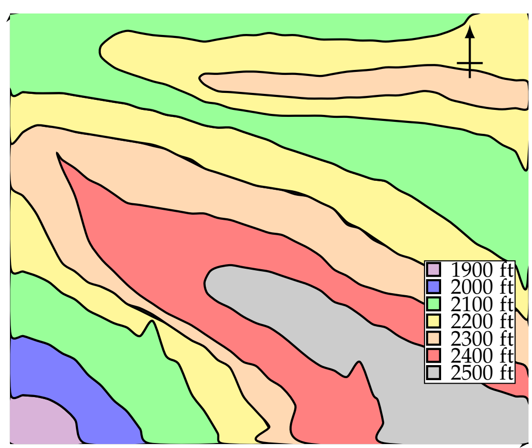

Figure 4.5: The contour plot shows the elevation of a mountain. H_x gives the slope going east, while H_y gives the slope going north

At some point, (x_o, y_o), the partial derivatives $H_x(x_o, y_o)$ and $H_y(x_o, y_o)$ give the rate of change of elevation in the east-west and north-south directions, respectively (see figure 4.6). To find the rate of change at (x_o, y_o), in the direction of some arbitrary unit vector, $\mathbf{u} = [a, b] = a\mathbf{i} + b\mathbf{j}$, we first note that the point (x_o, y_o, z_o), where $z_o = H(x_o, y_o)$, lies on the surface defined by $z = H(x, y)$. There is a vertical plane, P, that passes through

(x_o, y_o, z_o) and points in the direction of $\mathbf{u}$. This intersection defines curve C, which lies on the surface, and the slope of this curve at (x_o, y_o, z_o) is the directional derivative of H in the direction of $\mathbf{u}$ (see figure 4.7).

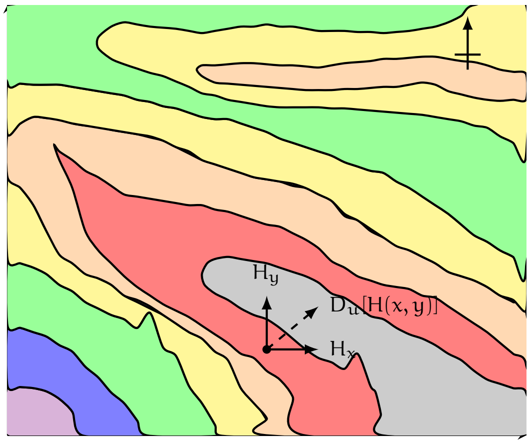

Figure 4.6: If $\mathbf{u}$ points north-east, then the directional derivative of $H(x, y)$, $D_u[H(x, y)]$, tells the rate of change going north-east

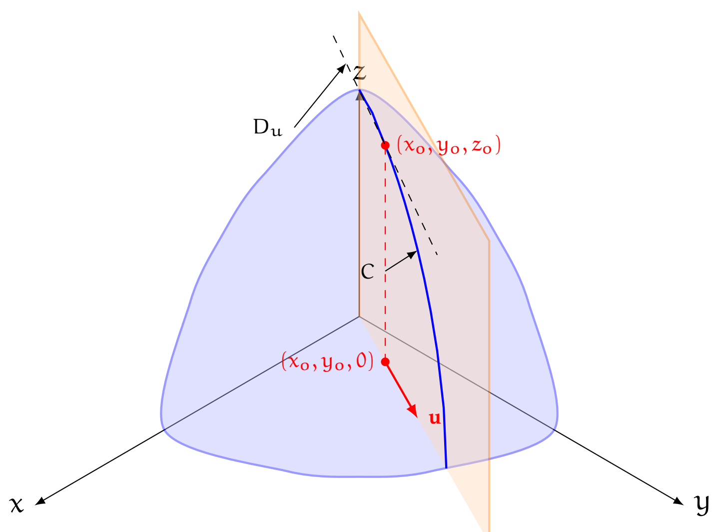

Figure 4.7: The slope of the curve formed between the plane parallel to $\mathbf{u}$ and the surface $z = H(x, y)$ is the directional derivative, D_u

The Directional Derivative

Let f be a differentiable function and $\mathbf{u}$ be a unit vector, $\mathbf{u} = [a, b]$. Then the directional derivative in the direction of $\mathbf{u}$ is:

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b = \mathbf{u}_x \left[\frac{\partial}{\partial x} f(x, y)\right] + \mathbf{u}_y \left[\frac{\partial}{\partial y} f(x, y)\right]$$

Where $\mathbf{u}_x$ and $\mathbf{u}_y$ are the x- and y-components of $\mathbf{u}$, respectively.

Example: Find the directional derivative $D_u f(x, y)$ if $f(x, y) = y^3 - 3xy + 4x^2$ and **u** is the unit vector given by the angle $\theta = \pi/3$. What is the rate of change in the direction of **u** at $(1, 2)$?

Solution: We can describe **u** thusly:

$$\mathbf{u} = \left[\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right] = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$

And therefore:

$$D_u f(x, y) = f_x(x, y)\left(\frac{1}{2}\right) + f_y(x, y)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\partial}{\partial x}\left(y^3 - 3xy + 4x^2\right)\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left(y^3 - 3xy + 4x^2\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}(-3y + 8x) + \frac{\sqrt{3}}{2}\left(3y^2 - 3x\right)$$

$$= \frac{-3}{2}y + 4x + \frac{3\sqrt{3}}{2}y^2 - \frac{3\sqrt{3}}{2}x = \frac{3\sqrt{3}}{2}y^2 + \frac{8 - 3\sqrt{3}}{2}x - \frac{3}{2}y$$

And therefore $D_u f(1, 2)$ is:

$$= \frac{3\sqrt{3}}{2}(2)^2 + \frac{8 - 3\sqrt{3}}{2}(1) - \frac{3}{2}(2) = 6\sqrt{3} + 4 - \frac{3\sqrt{3}}{2} - 3$$

$$= 1 + \frac{9\sqrt{3}}{2}$$

4.3.2 Unit Vectors in Two Dimensions

What if the given vector is not a unit vector? We can scale the given vector to find a unit vector in the same direction:

Example: Find the directional derivative of $f(x, y) = 3x\sqrt{y}$ at $(1, 4)$ in the direction of $\mathbf{v} = [2, 1]$.

Solution: First, we need to find a unit vector in the same direction as **v**. There are several ways to do this. In two dimensions, a unit vector in the same direction as **v** can be found using trigonometry (see figure 4.8 for an illustration).

We know that $\theta = \arctan(\mathbf{v}_y/\mathbf{v}_x)$. Therefore, the x-component of the unit vector, **u**, is

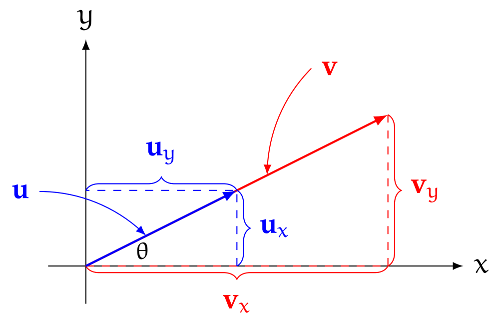

Figure 4.8: $\mathbf{u}$ is a unit vector in the same direction as $\mathbf{v}$

given by:

$$\mathbf{u}_x = |\mathbf{u}| \cos\theta = \cos\left(\arctan \frac{\mathbf{v}_y}{\mathbf{v}_x}\right)$$

And similarly, we know that:

$$\mathbf{u}_y = |\mathbf{u}| \sin\theta = \sin\left(\arctan \frac{\mathbf{v}_y}{\mathbf{v}_x}\right)$$

(Recall that since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = 1$).

Let's use this method to find a unit vector, $\mathbf{u}$, in the same direction as $\mathbf{v} = [2, 1]$:

$$\mathbf{u}_x = \cos\left(\arctan \frac{1}{2}\right) \approx \cos(0.464) = \frac{2}{\sqrt{5}}$$

$$\mathbf{u}_y = \sin\left(\arctan \frac{1}{2}\right) \approx \sin(0.464) = \frac{1}{\sqrt{5}}$$

Therefore, a unit vector in the same direction as $\mathbf{v}$ is $\mathbf{u} = \left[2/\sqrt{5}, 1/\sqrt{5}\right]$.

And we can find the directional derivative:

$$D_u(x, y) = \mathbf{u}_x \left[\frac{\partial}{\partial x} f(x, y)\right] + \mathbf{u}_y \left[\frac{\partial}{\partial y} f(x, y)\right]$$

$$D_u(x, y) = \left(\frac{2}{\sqrt{5}}\right)\left[\frac{\partial}{\partial x}(3x\sqrt{y})\right] + \left(\frac{1}{\sqrt{5}}\right)\left[\frac{\partial}{\partial y}(3x\sqrt{y})\right]$$

$$D_u(x, y) = \left(\frac{2}{\sqrt{5}}\right)(3\sqrt{y}) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3x}{2\sqrt{y}}\right)$$

$$D_u(x, y) = \frac{12y + 3x}{2\sqrt{5y}}$$

And to find the magnitude of the directional derivative at $(1, 4)$, we substitute for x and y:

$$D_u(1,4) = \frac{12(4) + 3(1)}{2\sqrt{5(4)}} = \frac{51}{4\sqrt{5}} \approx 5.702$$

4.3.3 Unit Vectors in Higher Dimensions

The trigonometric explanation for finding unit vectors is more difficult to visualize in higher dimensions. However, there is another method that works well in 2, 3, and higher dimensions. Recall that the magnitude of a vector, $\mathbf{v} = [\mathbf{v}_x, \mathbf{v}_y]$ is given by $|\mathbf{v}| = \sqrt{(\mathbf{v}_x)^2 + (\mathbf{v}_y)^2}$. For a vector with n dimensions, $\mathbf{v} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n]$, the magnitude is given by $|\mathbf{v}| = \sqrt{(\mathbf{v}_1)^2 + (\mathbf{v}_2)^2 + \cdots + (\mathbf{v}_n)^2}$.

To find a unit vector, $\mathbf{u}$, in the same direction as $\mathbf{v}$, we can scale $\mathbf{v}$ up or down so that its magnitude is 1. We can do this by dividing by $\mathbf{v}$'s magnitude. Consider the two-dimensional vector used in the last example, $\mathbf{v} = [2, 1]$. Its magnitude is:

$$|\mathbf{v}| = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$

Let's check if $\mathbf{v}/|\mathbf{v}|$ is a unit vector:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{5}}\right)[2,1] = \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$$

And the magnitude of this scaled vector is:

$$\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{1} = 1$$

Notice our unit vector is the same as we found using the trigonometric method above.

Another way to think of the question is: what factor, k, can we multiply $\mathbf{v}$ by to yield a vector with a magnitude of 1? Let's see this method for the 3-dimensional vector $\mathbf{v} = [3, 2, 1]$. We are looking for a k such that:

$$|k\mathbf{v}| = 1$$

$$|k\mathbf{v}| = |[3k, 2k, 1k]| = \sqrt{(3k)^2 + (2k)^2 + (1k)^2}$$
$$= \sqrt{9k^2 + 4k^2 + k^2} = k\sqrt{14} = 1$$

Which implies that $k = 1/\sqrt{14}$, which is $1/|\mathbf{v}|$. And therefore a unit vector in the same direction as $\mathbf{v} = [3, 2, 1]$ is:

$$\mathbf{u} = \frac{1}{\sqrt{14}}[3, 2, 1] = \left[\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right]$$

Exercise 8 Finding Directional Derivatives

Find the directional derivative of the function at the given point in the direction of the given vector.

1. $f(x, y) = e^{3x} \sin 2y$, $(0, \pi/6)$, $\mathbf{v} = [-3, 4]$
2. $f(x, y) = x^2y + xy^3$, $(2, 4)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$
3. $f(x, y, z) = \ln\left(x^2 + 3y - z\right)$, $(2, 2, 1)$, $\mathbf{v} = [1, 1, 1]$

Working Space

Answer on Page 41

4.3.4 Maximizing the Gradient

The directional derivative can be written as the dot product of two vectors:

$$D_u f(x, y) = a f_x(x, y) + b f_y(x, y) = [f_x(x, y), f_y(x, y)] \cdot \mathbf{u}$$

The first vector, $[f_x(x, y), f_y(x, y)]$ is called *the gradient of f* and is noted as ∇f. The gradient operator, ∇, is the derivative of a scalar function that results in a vector which shows the magnitude and direction of the greatest rate of change.

The Gradient

For a two-variable function, $f(x, y)$, the gradient of f is the vector:

$$\nabla f = [f_x(x, y), f_y(x, y)] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x- and y-directions, respectively.

Think back to the elevation example we opened the chapter with. What if we wanted to complete our ascent as quickly as possible? Then we would want to know the direction in which the elevation is changing the fastest. This occurs when the direction we are going is the same direction as the gradient vector, ∇f.

Recall that the dot product is defined as:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

Where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Applying this to the directional derivative, we see that:

$$D_u f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

Which is at its maximum when ∇f and $\mathbf{u}$ point in the same direction (because $\cos(0) = 1$). Therefore, the gradient vector points in the direction of maximum change and the magnitude of that vector is the rate of maximum change.

Example: Find the maximum rate of change of $f(x, y) = 4y\sqrt{x}$ at $(4, 1)$. In what direction does the maximum change occur?

Solution: We begin by finding ∇f:

$$\nabla f = \left[\frac{\partial}{\partial x}\left(4y\sqrt{x}\right), \frac{\partial}{\partial y}\left(4y\sqrt{x}\right)\right]$$

$$\nabla f = \left[\frac{2y}{\sqrt{x}}, 4\sqrt{x}\right]$$

And thus,

$$\nabla f(4, 1) = \left[\frac{2(1)}{\sqrt{4}}, 4\sqrt{4}\right] = [1, 8]$$

Therefore, the maximum value of ∇f at $(4, 1)$ is:

$$|\nabla f| = \sqrt{1^2 + 8^2} = \sqrt{65}$$

in the direction of the vector $[1, 8]$.

Exercise 9 Using the Gradient to find Maximum Change

Suppose you are climbing a mountain whose elevation is described by $z = 3000 - 0.01x^2 - 0.02y^2$. Take the positive x-direction to be east and the positive y-direction to be north.

1. If you are at $(x, y) = (50, 50)$, what is your elevation?
2. If you walk south, will you ascend or descend?
3. If you walk northwest, will you ascend or descend? Will the rate of elevation change be greater or less than if you walked south?
4. In what direction should you walk for the steepest ascent? What will your ascension rate be?

Working Space

Answer on Page 43

4.4 Applications of Partial Derivatives and Gradients

4.4.1 Laplace's Equation

A partial differential equation that has applications in fluid dynamics and electronics is Laplace's equation. Solutions to Laplace's equation are called *harmonic functions*.

Laplace's Equation

Consider a twice-differntiable function, f. In two dimensions, Laplace's Equation is given by:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

And in three dimensions,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Another way to represent Laplace's equation is:

$$\delta f = \nabla^2 f = \nabla \cdot \nabla f = 0$$

Where $\nabla^2 = \delta$ is called the *Laplace operator*.

Example:

4.4.2 The Wave Equation

4.4.3 Cobb-Douglas Production Function

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

Using implicit differentiation, we see that:

$$\frac{d}{dx}\arcsin x = \frac{d}{dx}\ln y$$
$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{y}\frac{dy}{dx}$$

Multiplying both sides by y to isolate $\frac{dy}{dx}$, we find that:

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$$

Answer to Exercise 2 (on page 7)

First, we need to find $\frac{d^y}{dx^2}$:

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}y$$
$$= 2x + \frac{dy}{dx} = 2x + x^2 + y$$

At $(-1,-1)$, $\frac{dy}{dx} = (-1)^2 + (-1) = 0$ and $\frac{d^2y}{dx^2} = 2(-1) + (-1)^2 + (-1) = -2 < 0$. Since the slope of y is zero and the graph of y is concave down, $(-1,-1)$ is a local maximum. At $(1,-5)$, $\frac{dy}{dx} = 1^2 + -5 = -4 \neq 0$ and $\frac{d^2y}{dx^2} = 2(1) + 1^2 + (-5) = -2 \neq 0$. Since neither the first nor second derivative of y are zero, $(1,-5)$ is neither a local extrema nor an inflection point.

Answer to Exercise 3 (on page 17)

1. $f_x(x,y) = \frac{\partial}{\partial x}\left[3x^4 + 4x^2y^3\right] = 12x^3 + 8y^3$ and $f_y(x,y) = \frac{\partial}{\partial y}\left[3x^4 + 4x^2y^3\right] = 12x^2y^2$
2. $f_x(x,y) = \frac{\partial}{\partial x}(xe^{-y}) = e^{-y}$ and $f_y(x,y) = \frac{\partial}{\partial y}(xe^{-y}) = -xe^{-y}$

3. $f_x(x,y) = \frac{\partial}{\partial x}\sqrt{3x+4y^2} = \left(\frac{1}{2\sqrt{3x+4y^2}}\right)\left(\frac{\partial}{\partial x}\left(3x+4y^2\right)\right) = \frac{3}{2\sqrt{3x+4y^2}}$ and f)y(x, y) $= \frac{\partial}{\partial y}\sqrt{3x+4y^2} = \frac{1}{2\sqrt{3x+4y^2}}\left(\frac{\partial}{\partial y}\left(3x+4y^2\right)\right) = \frac{8y}{2\sqrt{3x+4y^2}} = \frac{4y}{\sqrt{3x+4y^2}}$
4. $f_x(x,y) = \frac{\partial}{\partial x}\sin\left(x^2y\right) = \cos\left(x^2y\right)\left(\frac{\partial}{\partial x}\left(x^2y\right)\right) = 2xy\cos\left(x^2y\right)$ and $f_y(x,y) = \frac{\partial}{\partial y}\sin\left(x^2y\right) = \cos\left(x^2y\right)\left(\frac{\partial}{\partial y}\left(x^2y\right)\right) = x^2\cos\left(x^2y\right)$
5. $f_x(x,y) = \frac{\partial}{\partial x}\ln\left(x^y\right) = \frac{\partial}{\partial x}\left(y\ln x\right) = \frac{y}{x}$ and $f_y(x,y) = \frac{\partial}{\partial y}\left(y\ln x\right) = \ln x$

Answer to Exercise 4 (on page 19)

1. Finding f_x:

$$f_x = \frac{\partial}{\partial x}\left[\sin\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\right] = \cos\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\left[\frac{\partial}{\partial x}\left(x^2-y^2\right)\right]$$

$$f_x = 2x\cos\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)$$

Finding f_y:

$$f_y = \frac{\partial}{\partial y}\left[\sin\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\right] = \cos\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\left[\frac{\partial}{\partial y}\left(x^2-y^2\right)\right]$$

$$f_y = -2y\cos\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)$$

Finding f_z:

$$f_z = \frac{\partial}{\partial z}\left[\sin\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\right] = \sin\left(x^2-y^2\right)\left(-\sin\sqrt{z}\right)\cdot\left(\frac{\partial}{\partial z}\sqrt{z}\right)$$

$$f_z = \frac{-\sin\left(x^2-y^2\right)\sin\left(\sqrt{z}\right)}{2\sqrt{z}}$$

2. Finding q_t:

$$q_t = \frac{\partial}{\partial t}\sqrt[3]{t^3+u^3\sin(5v)} = \frac{1}{3\left(t^3+u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial t}\left(t^3+u^3\sin(5v)\right)\right)$$

$$q_t = \frac{t^2}{\left(t^3+u^3\sin(5v)\right)^{2/3}}$$

Finding q_u:

$$q_u = \frac{\partial}{\partial u}\sqrt[3]{t^3+u^3\sin(5v)} = \frac{1}{3\left(t^3+u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial u}\left(t^3+u^3\sin(5v)\right)\right)$$

$$q_u = \frac{u^2 \sin(5v)}{\left(t^3 + u^3 \sin(5v)\right)^{2/3}}$$

Finding q_v:

$$q_v = \frac{\partial}{\partial v} \sqrt[3]{t^3 + u^3 \sin(5v)} = \frac{1}{3\left(t^3 + u^3 \sin(5v)\right)^{2/3}} \left(\frac{\partial}{\partial v}\left(t^3 + u^3 \sin(5v)\right)\right)$$

$$q_v = \frac{u^3 \cos(5v)}{3\left(t^3 + u^3 \sin(5v)\right)^{2/3}} \left(\frac{\partial}{\partial v}(5v)\right) = \frac{5u^3 \cos(5v)}{3\left(t^3 + u^3 \sin(5v)\right)^{2/3}}$$

3. Finding w_x:

$$w_x = \frac{\partial}{\partial x}\left(x^z y^x\right) = \left(x^z\right) \cdot \left(\frac{\partial}{\partial x} y^x\right) + \left(y^x\right) \cdot \left(\frac{\partial}{\partial x} x^z\right)$$

$$w_x = \left(x^z\right)\left(\ln(y) y^x\right) + \left(y^x\right)\left(z x^{z-1}\right) = \left(x^{z-1} y^x\right)\left(x \ln(y) + z\right)$$

Finding w_y:

$$w_y = \frac{\partial}{\partial y}\left(x^z y^x\right) = \left(x^z\right)\left(\frac{\partial}{\partial y} y^x\right) = x^z\left(x y^{x-1}\right)$$

$$w_y = x^{z+1} y^{x-1}$$

Finding w_z:

$$w_z = \frac{\partial}{\partial z}\left(x^z y^x\right) = \left(y^x\right)\left(\frac{\partial}{\partial z} x^z\right) = \left(y^x\right)\left(\ln(x) x^z\right)$$

$$w_z = \ln(x) y^x x^z$$

Answer to Exercise 5 (on page 21)

1. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(e^{2xy}\sin x\right)\right] = \frac{\partial}{\partial y}\left[\left(e^{2xy}\right)\left(\frac{\partial}{\partial x}\sin x\right) + (\sin x)\left(\frac{\partial}{\partial x}e^{2xy}\right)\right] = \frac{\partial}{\partial y}\left[e^{2xy}\cos x + 2y e^{2xy}\sin x\right] = \frac{\partial}{\partial y}\left(e^{2xy}\cos x\right) + \frac{\partial}{\partial y}\left(2y e^{2xy}\sin x\right) = 2x e^{2xy}\cos x + (2y)\left(\frac{\partial}{\partial y}e^{2xy}\sin x\right) + \left(e^{2xy}\sin x\right)\left(\frac{\partial}{\partial y}2y\right) = 2x e^{2xy}\cos x + 4xy e^{2xy}\sin x + 2e^{2xy}\sin x$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(e^{2xy}\sin x\right)\right] = \frac{\partial}{\partial x}\left(2x e^{2xy}\sin x\right) = (2x)\left[\frac{\partial}{\partial x}\left(e^{2xy}\sin x\right)\right] + \left(e^{2xy}\sin x\right)\left(\frac{\partial}{\partial x}2x\right) = (2x)\left[\left(e^{2xy}\right)\left(\frac{\partial}{\partial x}\sin x\right) + (\sin x)\left(\frac{\partial}{\partial x}e^{2xy}\right)\right] + 2e^{2xy}\sin x = 2x e^{2xy}\cos x + 4xy e^{2xy}\sin x + 2e^{2xy}\sin x = f_{xy}$

2. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{x^2}{x+y}\right)\right] = \frac{\partial}{\partial y}\left[\frac{(x+y)(2x) - x^2(1)}{(x+y)^2}\right] = \frac{\partial}{\partial y}\left[\frac{x^2+2xy}{(x+y)^2}\right] = \frac{(x+y)^2(2x) - \left(x^2+2xy\right)(2(x+y))}{(x+y)^4} = \frac{\left(x^2+2xy+y^2\right)(2x) - \left(x^2+2xy\right)(2x+2y)}{(x+y)^4} = \frac{2x^3+4x^2y+2xy^2-2x^3-2x^2y-4x^2y-4xy^2}{(x+y)^4} = \frac{-2x^2y-2xy^2}{(x+y)^4}$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{x^2}{x+y}\right)\right] = \frac{\partial}{\partial x}\left[\frac{-x^2}{(x+y)^2}\right] = \frac{(x+y)^2(-2x) - \left(-x^2\right)(2(x+y))}{(x+y)^4} = \frac{\left(x^2+2xy+y^2\right)(-2x) + x^2(2x+2y)}{(x+y)^4} = \frac{-2x^3-4x^2y-2xy^2+2x^3+2x^2y}{(x+y)^4} = \frac{-2x^2y-2xy^2}{(x+y)^4} = f_{xy}$

3. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\ln\left(2x+3y\right)\right)\right] = \frac{\partial}{\partial y}\left[\frac{2}{2x+3y}\right] = \frac{-2(3)}{(2x+3y)^2} = \frac{-6}{(2x+3y)^2}$
$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\ln\left(2x+3y\right)\right)\right] = \frac{\partial}{\partial x}\left(\frac{3}{2x+3y}\right) = \frac{-3(2)}{(2x+3y)^2} = \frac{-6}{(2x+3y)} = f_{xy}$

Answer to Exercise 6 (on page 22)

1. $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial x}\left(5x^4y^2 - 9x^2y^2\right) = 20x^3y^2 - 18xy^2$.
$f_{xy} = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial y}\left(5x^4y^2 - 9x^2y^2\right) = 10x^4y - 18x^2y$.
$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial y}\left(2x^5y - 6x^3y\right) = 2x^5 - 6x^3$.

2. $v_{pp} = \frac{\partial}{\partial p}\left(\frac{\partial}{\partial p} v(p,q)\right) = \frac{\partial}{\partial p}\left[\frac{\partial}{\partial p}\left(\sin\left(p^3+q^2\right)\right)\right] = \frac{\partial}{\partial p}\left(\cos\left(p^3+q^2\right)\left(3p^2\right)\right) = \cos\left(p^3+q^2\right) \cdot \frac{\partial}{\partial p}\left(3p^2\right) + 3p^2 \cdot \frac{\partial}{\partial p}\left(\cos\left(p^3+q^2\right)\right)$
$v_{pq} = v_{qp} = \frac{\partial}{\partial q}\left(\frac{\partial}{\partial p} v(p,q)\right) = \frac{\partial}{\partial q}\left[\frac{\partial}{\partial p}\left(\sin\left(p^3+q^2\right)\right)\right] = \frac{\partial}{\partial q}\left(\cos\left(p^3+q^2\right)\left(3p^2\right)\right) = \cos\left(p^3+q^2\right)\frac{\partial}{\partial q}\left(3p^2\right) + 3p^2\frac{\partial}{\partial q}\cos\left(p^3+q^2\right) = 0 + 3p^2\left(-\sin\left(p^3+q^2\right)\right)\left(\frac{\partial}{\partial q}\left(p^3+q^2\right)\right) = -6p^2q\sin\left(p^3+q^2\right)$
$v_{qq} = \frac{\partial}{\partial q}\left(\frac{\partial}{\partial q} v(p,q)\right) = \frac{\partial}{\partial q}\left[\frac{\partial}{\partial q}\left(\sin\left(p^3+q^2\right)\right)\right] = \frac{\partial}{\partial q}\left[2q\cos\left(p^3+q^2\right)\right] = 2q\left[\frac{\partial}{\partial q}\cos\left(p^3+q^2\right)\right] + \cos\left(p^3+q^2\right)\left[\frac{\partial}{\partial q}\left(2q\right)\right] = (2q)\cdot\left[-2q\sin\left(p^3+q^2\right)\right] + 2\cos\left(p^3+q^2\right) = 2\cos\left(p^3+q^2\right) - 4q^2\sin\left(p^3+q^2\right)$

3. $T_{rr} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} T(r,\theta)\right) = \frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left(e^{-3r}\cos\theta^2\right)\right] = \frac{\partial}{\partial r}\left(-3e^{-3r}\cos\theta^2\right) = 9e^{-3r}\cos\theta^2$
$T_{\theta r} = T_{r\theta} = \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial r} T(r,\theta)\right) = \frac{\partial}{\partial \theta}\left[-3e^{-3r}\cos\theta^2\right] = 3re^{-3r}\sin\theta^2\left(\frac{\partial}{\partial \theta}\theta^2\right) = 6r\theta e^{-3r}\sin\theta^2$
$T_{\theta\theta} = \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial \theta} T(r,\theta)\right) = \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial \theta}\left(e^{-3r}\cos\theta^2\right)\right] = \frac{\partial}{\partial \theta}\left[-e^{-3r}\sin\theta^2\left(\frac{\partial}{\partial \theta}\theta^2\right)\right] = \frac{\partial}{\partial \theta}\left(-2\theta e^{-3r}\sin\theta^2\right) = \left(-2\theta e^{-3r}\right)\left(\frac{\partial}{\partial \theta}\sin\theta^2\right) + \left(\sin\theta^2\right)\left[\frac{\partial}{\partial \theta}\left(-2\theta e^{-3r}\right)\right] = \left(-2\theta e^{-3r}\right)\left(\cos\theta^2\right)\left(\frac{\partial}{\partial \theta}\theta^2\right) + \left(\sin\theta^2\right)\left(-2e^{-3r}\right) = -4\theta^2e^{-3r}\cos\theta^2 - 2e^{-3r}\sin\theta^2$

Answer to Exercise 7 (on page 26)

1. $z(1,-1) = (1)^2e(-1/1 = 1/e$. Therefore, we are looking for tangent lines through the point $(1,-1,1/e)$. Finding a tangent line parallel to the x-axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2e^{y/x}\right) = x^2\left(\frac{\partial}{\partial x}e^{y/x}\right) + e^{y/x}\left(\frac{\partial}{\partial x}x^2\right) = x^2e^{y/x}\left(\frac{\partial}{\partial x}\frac{y}{x}\right) + 2xe^{y/x} = x^2e^{y/x}\left(\frac{-y}{x^2}\right) + 2xe^{y/x} = (2x-y)\,e^{y/x}$ and $z_x(1,-1) = (2(1)-(-1))\,e^{-1/1} = (3)\,e^{-1} = 3/e$. So the slope of a line tangent to the surface at $(1,-1,1/e)$ parallel to the x-axis is $3/e$ and an equation for that line is $z = 3/e\,(x-1) - 1/e$.
Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2e^{y/x}\right) = xe^{y/x}$ and $z_y(1,-1) =$

$(1)e^{-1/1} = 1/e$. So the slope of a line tangent to the surface at $(1, -1, 1/e)$ parallel to the y-axis is $1/e$ and an equation for that line is $z = 1/e\,(y+1) - 1/e$.

The function is changing faster in the x-direction.

2. $z(\pi, \pi/2) = \cos(\pi) + \frac{\pi}{2}\sin(\pi/2) = \frac{\pi}{2} - 1$. Therefore, we are looking for tangent lines through the point $(\pi, \pi/2, \pi/2 - 1)$. Finding a tangent line parallel to the x-axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\cos x + y \sin y) = -\sin x$ and $z_x(\pi, \pi/2) = -\sin \pi = 0$. So the slope of a line tangent to the surface at $(\pi, \pi/2, \pi/2 - 1)$ parallel to the x-axis is 0 and an equation for that line is $z = \pi/2 - 1$.

   Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\cos x + y \sin y) = y\left(\frac{\partial}{\partial y}\sin y\right) + \sin y\left(\frac{\partial}{\partial y} y\right) = y\cos y + \sin y$ and $z_y(\pi, \pi/2) = \left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1$. So the slope of a line tangent to the surface at $(\pi, \pi/2, \pi/2 - 1)$ parallel to the y-axis is 1 and an equation for that line is $z = (y - \pi/2) - (\pi/2 - 1) = y - \pi + 1$.

   The function is changing faster in the y-direction.

3. $z(3, 2) = 3^2(2) - 3(3)(2^2) = 18 - 36 = -18$. Therefore, we are looking for tangent lines through the point $(3, 2, -18)$. Finding a tangent line parallel to the x-axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 y - 3xy^2) = 2xy - 3y^2$ and $z_x(3, 2) = 2(3)(2) - 3(2)^2 = 0$. So the slope of a line tangent to the surface at $(3, 2, -18)$ is 0 and an equation for that line is $z = -18$

   Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 y - 3xy^2) = x^2 - 6xy$ and $z_y(3, 2) = 3^2 - 6(3)(2) = 9 - 36 = -27$. So the slope of a line tangent to the surface at $(3, 2, -18)$ is -27 and an equation for that line is $z = -27(y-2) + -18 = -27y + 54 - 18 = 36 - 27y$.

   The function is changing faster in the y-direction.

Answer to Exercise 8 (on page 32)

1. First, we define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [-3k, -4k]$$

$$\sqrt{(-3k)^2 + (4k)^2} = 1$$

$$\sqrt{9k^2 + 16k^2} = \sqrt{25k^2} = 5k = 1$$

$$k = \frac{1}{5}$$

   Therefore, we define $\mathbf{u} = [-3/5, 4/5]$ and the directional derivative is given by:

$$D_u(x, y) = \left(\frac{-3}{5}\right)\frac{\partial}{\partial x}f(x, y) + \left(\frac{4}{5}\right)\frac{\partial}{\partial y}f(x, y)$$

$$= \left(\frac{-3}{5}\right) \frac{\partial}{\partial x} \left[e^{3x} \sin 2y\right] + \left(\frac{4}{5}\right) \frac{\partial}{\partial y} \left[e^{3x} \sin 2y\right]$$

$$= \left(\frac{-3}{5}\right) \left(3e^{3x} \sin 2y\right) + \left(\frac{4}{5}\right) \left(2e^{3x} \cos 2y\right)$$

And substituting for $(x, y) = (0, \pi/6)$:

$$D_u(0, \pi/6) = \left(\frac{-3}{5}\right) \cdot \left[3e^{3\cdot 0} \sin\left(\frac{\pi}{3}\right)\right] + \left(\frac{4}{5}\right) \cdot \left[2e^{3\cdot 0} \cos\left(\frac{\pi}{3}\right)\right]$$

$$D_u(0, \pi/6) = \left(\frac{-3}{5}\right) \cdot \left[3 \cdot \frac{\sqrt{3}}{2}\right] + \left(\frac{4}{5}\right) \cdot \left[2 \cdot \frac{1}{2}\right]$$

$$d_u(0, \pi/6) = \left(\frac{-3}{5}\right) \cdot \left(\frac{3\sqrt{3}}{2}\right) + \left(\frac{4}{5}\right) \cdot (1)$$

$$D_u(0, \pi/6) = \frac{-9\sqrt{3}}{10} + \frac{8}{10} = \frac{8 - 9\sqrt{3}}{10} \approx -0.759$$

2. We can express $\mathbf{v}$ as $\mathbf{v} = [2, -1]$. And we define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [2k, -k]$$

$$\sqrt{(2k)^2 + (-k)^2} = 1$$

$$\sqrt{4k^2 + k^2} = \sqrt{5}k = 1$$

$$k = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Therefore, we define $\mathbf{u} = \left[2\sqrt{5}/5, -\sqrt{5}/5\right]$ and the directional derivative is given by:

$$D_u(x, y) = \left(\frac{2\sqrt{5}}{5}\right) \frac{\partial}{\partial x} f(x, y) + \left(\frac{-\sqrt{5}}{5}\right) \frac{\partial}{\partial y} f(x, y)$$

$$= \left(\frac{2\sqrt{5}}{5}\right) \frac{\partial}{\partial x} \left[x^2 y + xy^3\right] + \left(\frac{-\sqrt{5}}{5}\right) \frac{\partial}{\partial y} \left[x^2 y + xy^3\right]$$

$$= \left(\frac{2\sqrt{5}}{5}\right) \left[2xy + y^3\right] + \left(\frac{-\sqrt{5}}{5}\right) \left[x^2 + 3xy^2\right]$$

And substituting $(x, y) = (2, 4)$:

$$D_u(2, 4) = \left(\frac{2\sqrt{5}}{5}\right) \left[2(2)(4) + 4^3\right] + \left(\frac{-\sqrt{5}}{5}\right) \left[2^2 + 3(2)(4^2)\right]$$

$$D_u(2,4) = \left(\frac{2\sqrt{5}}{5}\right)[80] + \left(\frac{-\sqrt{5}}{5}\right)[100]$$

$$D_u(2,4) = 32\sqrt{5} - 20\sqrt{5} = 12\sqrt{5} \approx 26.833$$

3. We define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [k, k, k]$$

$$\sqrt{k^2 + k^2 + k^2} = 1$$

$$\sqrt{3}k = 1$$

$$k = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Therefore, we let $\mathbf{u} = \left[\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3\right]$ and the directional derivative is given by:

$$D_u(x,y,z) = \left(\frac{\sqrt{3}}{3}\right)\frac{\partial}{\partial x}f(x,y,z) + \left(\frac{\sqrt{3}}{3}\right)\frac{\partial}{\partial y}f(x,y,z) + \left(\frac{\sqrt{3}}{3}\right)\frac{\partial}{\partial z}f(x,y,z)$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{\partial}{\partial x}\ln\left(x^2 + 3y - z\right) + \frac{\partial}{\partial y}\ln\left(x^2 + 3y - z\right) + \frac{\partial}{\partial z}\ln\left(x^2 + 3y - z\right)\right]$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{2x}{x^2 + 3y - z} + \frac{3}{x^2 + 3y - z} + \frac{-1}{x^2 + 3y - z}\right]$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{2x + 2}{x^2 + 3y - z}\right] = \frac{\sqrt{3}\,(2x + 2)}{3\,(x^2 + 3y - z)}$$

And substituting $(x, y, z) = (2, 2, 1)$:

$$D_u(2,2,1) = \frac{\sqrt{3}\,(2(2) + 2)}{3\,(2^2 + 3(2) - 1)} = \frac{\sqrt{3}\,(6)}{3\,(9)} = \frac{2\sqrt{3}}{9} \approx 0.385$$

Answer to Exercise 9 (on page 35)

1. $z = f(50, 50) = 3000 - 0.01(50)^2 - 0.02(50)^2 = 2925$
2. A south-pointing unit vector is $\mathbf{u} = [0, -1]$. To find the rate of change, we find the directional derivative in the direction of $\mathbf{u}$ at $(50, 50)$:

$$D_u f(x,y) = (-1)\left[\frac{\partial}{\partial y}\left(3000 - 0.01x^2 - 0.02y^2\right)\right]$$

$$D_u f(x, y) = (-1)(-0.04y) = 0.04y$$

And at $(50, 50)$, $D_u f(50, 50) = 0.04(50) = 2 > 0$. Therefore, if you walk south, you will ascend.

3. A northwest-pointing unit vector is $\mathbf{u} = \left[-\sqrt{2}/2, \sqrt{2}/2\right]$. To find the rate of change, we find the directional derivative at $(50, 50)$ in the direction of $\mathbf{u}$:

$$D_u f(x, y) = \left(\frac{-\sqrt{2}}{2}\right)\left[\frac{\partial}{\partial x} f(x, y)\right] + \left(\frac{\sqrt{2}}{2}\right)\left[\frac{\partial}{\partial y} f(x, y)\right]$$

$$D_u f(x, y) = \left(\frac{-\sqrt{2}}{2}\right)[-0.02x] + \left(\frac{\sqrt{2}}{2}\right)[-0.04y]$$

$$D_u f(x, y) = 0.01\sqrt{2}x - 0.02\sqrt{2}y$$

$$D_u f(50, 50) = 0.01\sqrt{2}\,(50) - 0.02\sqrt{2}\,(50) = \frac{-\sqrt{2}}{2} \approx -0.707$$

The rate of elevation change walking northwest is approximately -0.707, so you will descend and your rate of elevation change would be less than if you walked south.

4. To find the direction of maximum elevation gain, we find the direction the gradient vector points in:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

$$\nabla f = [-0.02x, -0.04y]$$

And at $(50, 50)$,

$$\nabla f(50, 50) = [-0.02(50), -0.04(50)] = [-1, -2]$$

Therefore the rate of greatest elevation change is in a south-by-southwest direction indicated by the vector $[-1, -2]$ and the rate of elevation change is $|\nabla f(50, 50)| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$. Notice this is greater than the other two rates of change we have found.

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