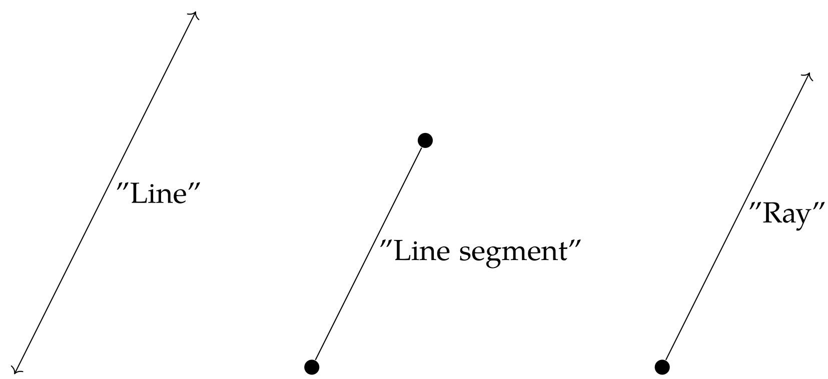
Contents

CHAPTER 1

Angles

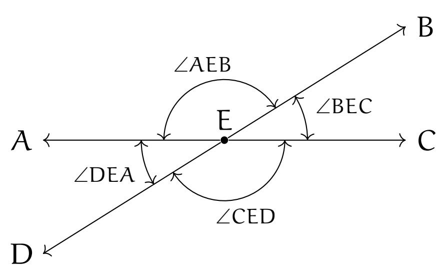
In the following recommend videos, the narrator talks about lines, line segments, and rays. When mathematicians talk about *lines*, they mean a straight line that goes forever in two directions. If you pick any two points on that line, the space between them is a *line segment*. If you take any line, pick a point on it, and discard all the points on one side of the point, that is a *ray*. All three have no width.



Watch the following videos from Khan Academy:

- Introduction to angles: `https://youtu.be/H-de6Tkxej8`
- Measuring angles in degrees: `https://youtu.be/92aLiyeQj0w`

When two lines cross, they form four angles:



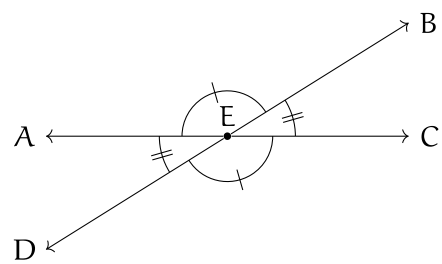
What do we know about those angles?

- The sum of any two adjacent angles adds to be 180°. So, for example, $m\angle AEB +$

$m\angle BEC = 180°$. We use the phrase "adds to be 180°" so often that we have a special word for it: *supplementary*.

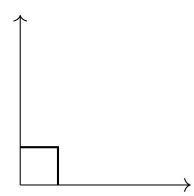
- The sum of all four angles is 360°.
- Angles opposite each other are equal. So, for example, $m\angle AEB = m\angle CED$.

In a diagram, to indicate that two angles are equal we often put hash marks in the angle:

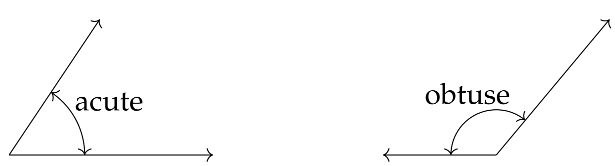


Here, the two angles with a single hash mark are equal, and the two angles with double hash marks are equal.

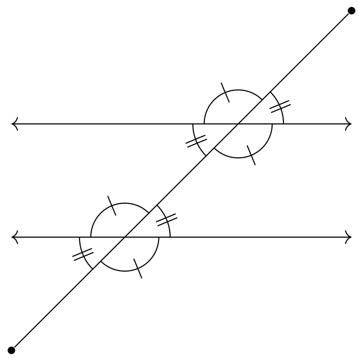
When two lines are perpendicular, the angle between them is 90°, and we say they meet at a *right angle*. When drawing diagrams, we indicate right angles with an elbow:



When an angle is less than 90°, it is said to be *acute*. When an angle is more than 90°, it is said to be *obtuse*.



If two lines are parallel, line segments that intersect both lines form the same angles with each line:



1.1 Radians

As you've seen above, angles can be measured in degrees. Just like you can measure length in more than one unit (inches, meters, etc.), there is more than one unit to measure angles in. Angles can also be measured in *radians*. Radians are unitless (that is, you don't have to put a letter after the number) and there are π radians across a straight line. This means $180°$ is the same as π radians.

Example: An angle is measured to be $\frac{\pi}{2}$ radians. What is the angle in degrees?

Solution: Since we know that π radians is the same as $180°$, we can set up the unit conversion:

$$\frac{\pi}{2} \cdot \frac{180°}{\pi} = 90^{o}$$

Therefore, a $\frac{\pi}{2}$ angle is $90°$.

Exercise 1

Convert the following angles from degrees to radians, or from radians to degrees.

1. $360°$
2. $\frac{\pi}{3}$
3. $225°$
4. $\frac{3\pi}{4}$
5. $30°$
6. $45°$

Working Space

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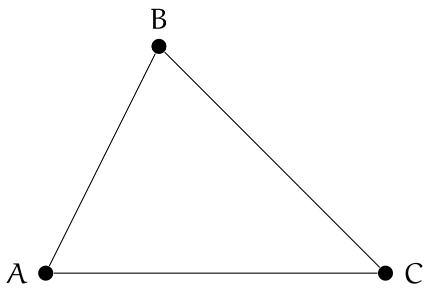
Chapter 2

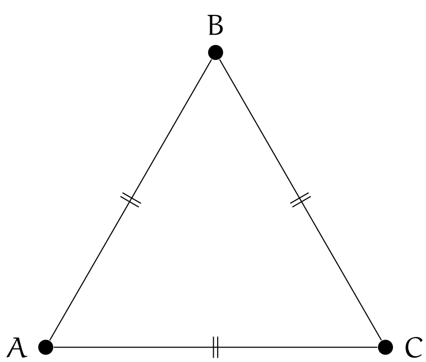
Introduction to Triangles

Connecting any three points with three line segments will get you a triangle. Here is the triangle ABC, which was created by connecting three points A, B, and C:

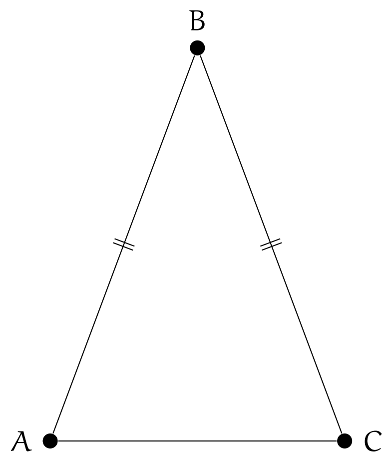


2.1 Equilateral and Isosceles Triangles

We talk a great deal about the length of the sides of triangles. If all three sides of the triangle are the same length, we say it is an *equilateral triangle*:

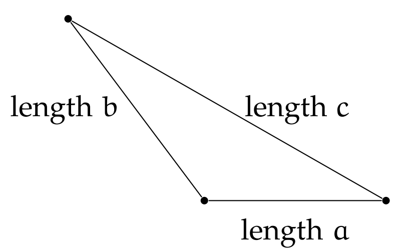


If only two sides of the triangle are the same length, we say it is an *isosceles triangle*:



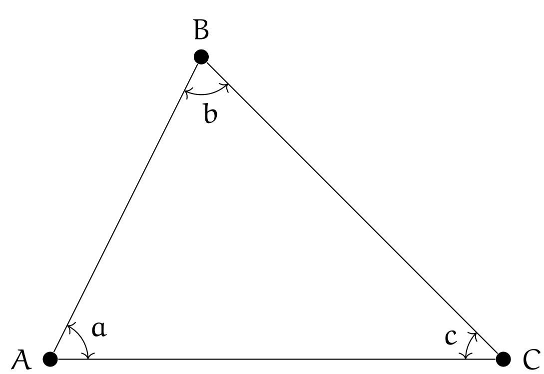
The shortest distance between two points is always the straight line between them. This means you can be certain that the length of one side will *always* be less than the sum of the lengths of the remaining two sides. This is known as the *triangle inequality*.

For example, in this diagram, c must be less than $a + b$.

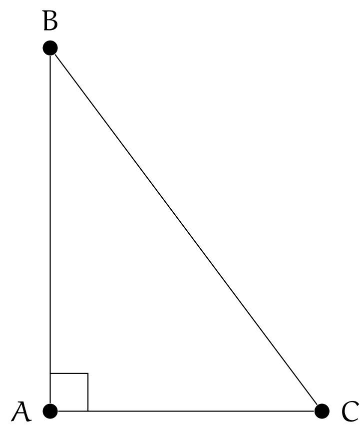


2.2 Interior Angles of a Triangle

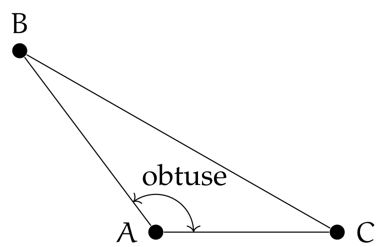
We also talk a lot about the interior angles of a triangle:



A triangle where one of the interior angles is a right angle is said to be a *right triangle*:



If a triangle has an obtuse interior angle, it is said to be an *obtuse triangle*:



If all three interior angles of a triangle are less than 90°, it is said to be an *acute triangle*.

The measures of the interior angles of a triangle always add up to 180°. For example, if we know that a triangle has interior angles of 37° and 56°, we know that the third interior angle is 87°.

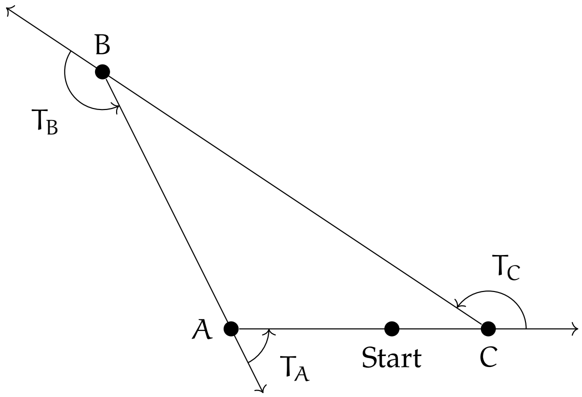
Exercise 2 Missing Angle

One interior angle of a triangle is 92°. The second angle is 42°. What is the measure of the third interior angle?

Working Space

Answer on Page 33

How can you know that the sum of the interior angles is 180°? Imagine that you started on the edge of a triangle and walked all the way around to where you started. (going counter-clockwise.) You would turn three times to the left:



After these three turns, you would be facing the same direction that you started in. Thus, $T_A + T_B + T_C = 360°$. The measures of the interior angles are a, b, and c. Notice that a and T_A are supplementary. So we know that:

- $T_A = 180 - a$
- $T_B = 180 - b$
- $T_C = 180 - c$

So we can rewrite the equation above as

$$(180 - a) + (180 - b) + (180 - c) = 360°$$

Which is equivalent to

$$a + b + c = 360°$$

Exercise 3 Interior Angles of a Quadrilateral

Any four-sided polygon is a *quadrilateral*. Using the same "walk around the edge" logic, what is the sum of the interior angles of any quadrilateral?

Working Space

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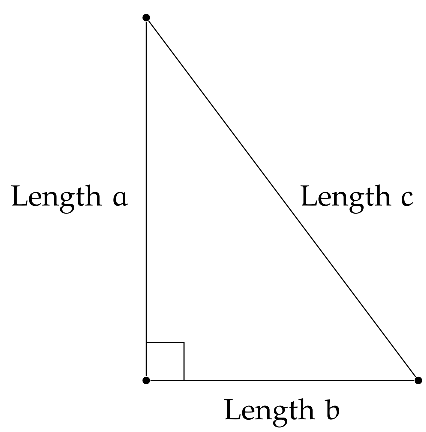
Chapter 3

Pythagorean Theorem

Watch's Khan Academy's Intro to the Pythagorean Theorem video at https://youtu.be/AA6RfgP-AHU.

If you have a right triangle, the edges that touch the right angle are called *the legs*. The third edge, which is always the longest, is known as *the hypotenuse*. The Pythagorean Theorem gives us the relationship between the length of the legs and the length of the hypotenuse.



The Pythagorean Theorem tells us that $a^2 + b^2 = c^2$.

For example, if one leg has a length of 3 and the other has a length of 4, then $a^2 + b^2 = 3^2 + 4^2 = 25$. Thus, c^2 must equal 25. This means you know the hypotenuse must be of length 5.

(In reality, it rarely works out to be such a tidy number. For example, what is the length of the hypotenuse if the two legs are 3 and 6? $a^2 + b^2 = 3^2 + 6^2 = 45$. The length of the hypotenuse is the square root of that: $\sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$, which is approximately 6.708203932499369.)

Exercise 4 Find the Missing Length

What is the missing measure?

Leg 1 = 6, Leg 2 = 8, Hypotenuse = ?
(It should be a whole number.)

Leg 1 = 5, Leg 2 = ?, Hypotenuse = 13
(It should be a whole number.)

Leg 1 = ?, Leg 2 = 15, Hypotenuse = 17
(It should be a whole number.)

Leg 1 = 3, Leg 2 = 3, Hypotenuse = ?
(It is an irrational number. Give the exact answer and then use a calculator to get an approximation.)

Working Space

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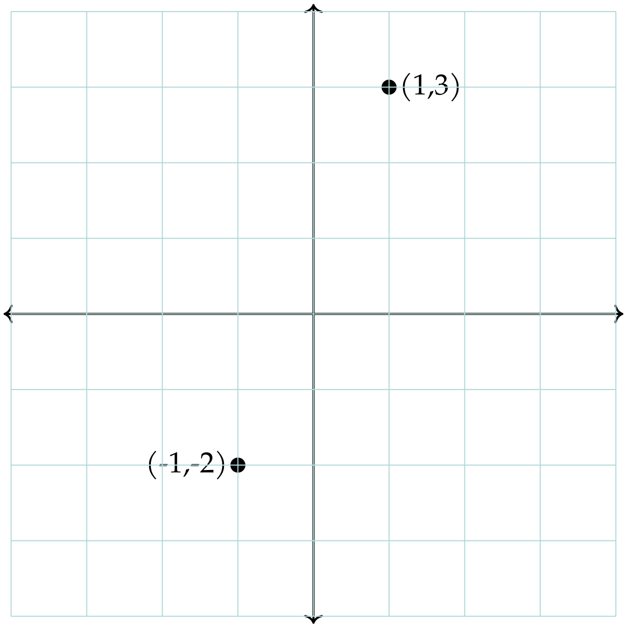
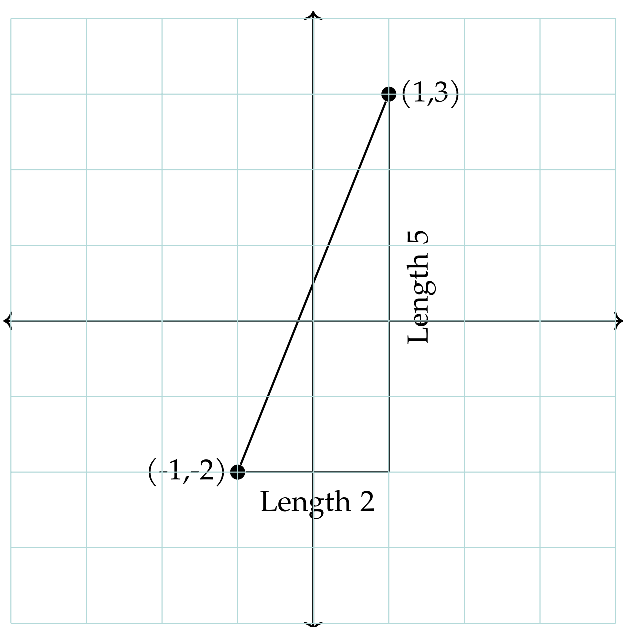
3.1 Distance between Points

What is the distance between these two points?



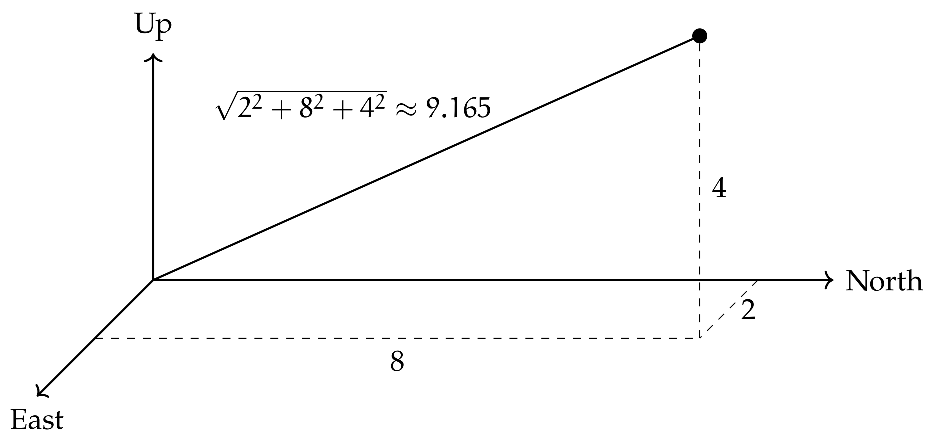
We can draw a right triangle and use the Pythagorean Theorem:



The distance between the two points is $\sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.385165$. In other words, you square the change in x and add it to the square of the change in y. The distance is the square root of that sum.

3.2 Distance in 3 Dimensions

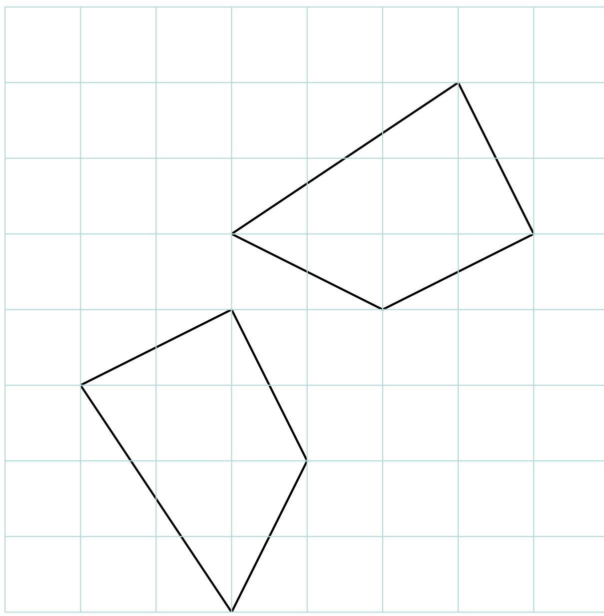
What if the point is in three-dimensional space? For example, you move 2 meters East, 8 meters North, and 4 meters up in the air. How far are you from where you started? You just square each, sum them, and take the square root: $\sqrt{2^2 + 8^2 + 4^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$ meters.



Chapter 4

Congruence

Look at this picture of two geometric figures.



They are the same shape, right? If you cut one out with scissors, it would lay perfectly on top of the other. In geometry, we say they are *congruent*.

What is the official definition of "congruent"? Two geometric figures are congruent if you can transform one into the other using only rigid transformations.

You might be wondering now, what are rigid transformations? A transformation is *Rigid* if it doesn't change the distances between the points or the measure of the angles between the lines they form. The following are all rigid transformations:

- Translations
- Rotations
- Reflections

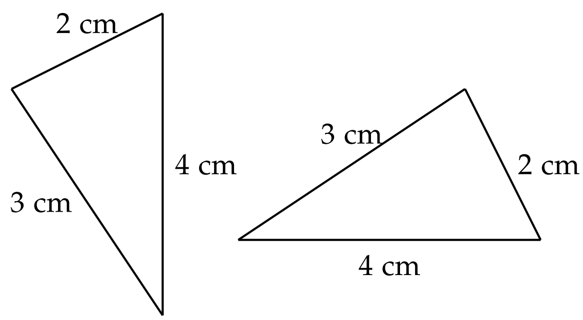
Once again, imagine cutting out one figure with scissors and trying to match it with the second figure; your actions are rigid transformations:

- Translations - Sliding the cutout left and right and up and down
- Rotations - Rotating the cutout clockwise and counterclockwise
- Reflection - Flipping the piece of paper over

A transformation is rigid if it is some combination of translations, rotations, or reflections.

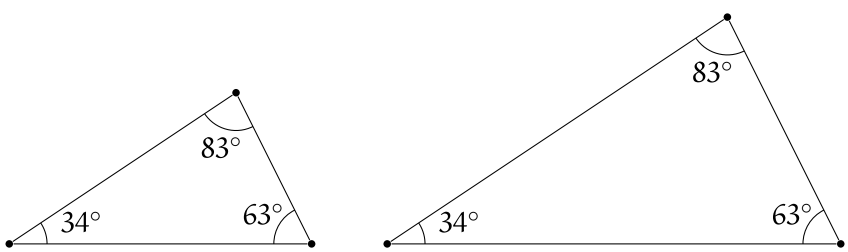
4.1 Triangle Congruency

If the sides of two triangles have the same length, the triangles must be congruent:



To be precise, the Side-Side-Side Congruency Test says that two triangles are congruent if three sides in one triangle are the same length as the corresponding sides in the other. We usually refer to this as the SSS test.

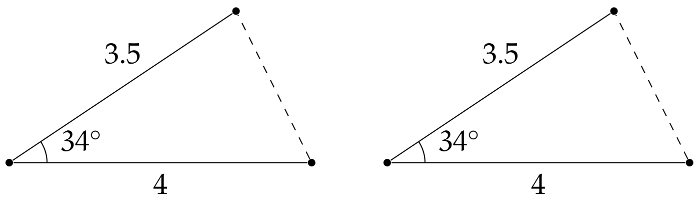
Note that two triangles with all three angles equal are not necessarily congruent. For example, here are two triangles with the same interior angles, but they are different sizes:



These triangles are not congruent, but they are *similar*. Meaning they have the same shape, but are not necessarily the same size.

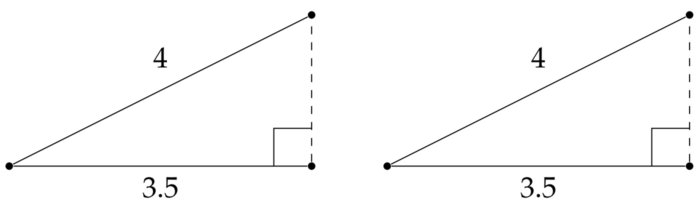
Therefore, if you know two angles of a triangle, you can calculate the third. This is why it makes sense to say "If two triangles have two angles that are equal, they are similar triangles." And if two similar triangles have one side that is equal in length, they must be the same size — so they are congruent. Thus, the Side-Angle-Angle Congruency Test says that two triangles are congruent if two angles and one side match.

What if you know that two triangles have two sides that are the same length, and that the angle between them is also equal?



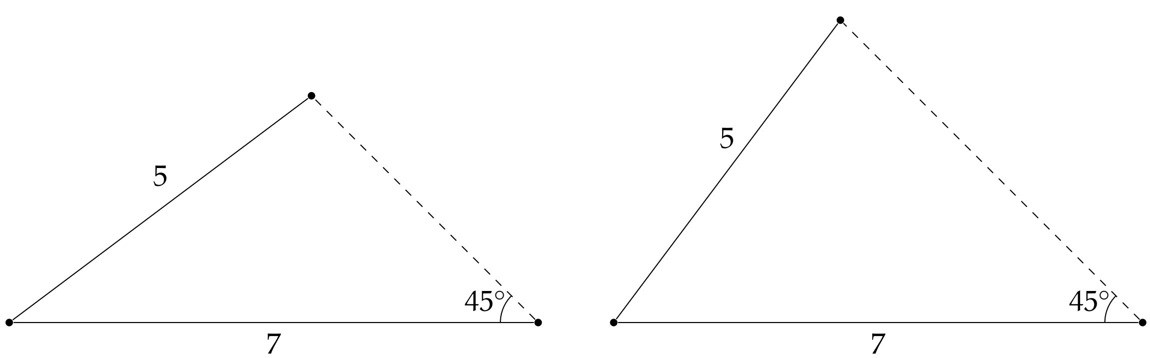
Yes, they must be congruent. This is the Side-Angle-Size Congruency Test.

What if the angle isn't the one between the two known sides? If it is a right angle, you can be certain the two triangles are congruent. (How do we know? Because the Pythagorean Theorem tells us that we can calculate the length of the third side. There is only one possibility, so all three sides must be the same length.)

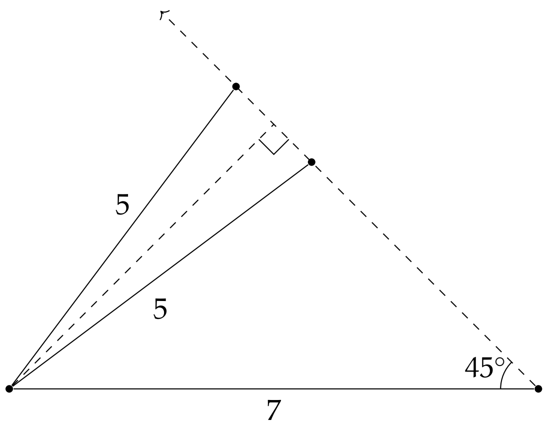


In this case, the third side of each triangle must be $\sqrt{4^2 - 3.5^2} \approx 1.9$.

What if the know angle is less than 90°? *The triangles are not necessarily congruent.* For example, let's say that there are two triangles with sides of length 5 and 7 and that the corresponding angle (at the end of the side of length 7) on each is 45°. Two different triangles satisfy this:



Let's look at this another way by laying one triangle on top of the other:



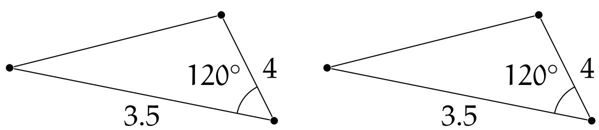
This is why there is *not* a general Side-Side-Angle Congruency Test.

Here, then, is the list of common congruency tests:

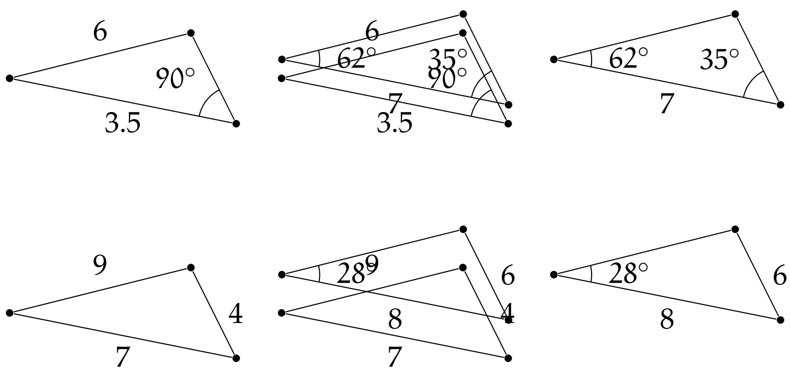
- Side-Side-Side: All three sides have the same measure
- Side-Angle-Angle: Two angles and one side have the same measure
- Side-Angle-Side: Two sides and the angle between them have the same measure
- Side-Side-Right: They are right triangles and have two sides have the same measure

Exercise 5 Congruent Triangles

Ted is terrible at drawing triangles; he always draws them exactly the same. Fortunately, he has marked these diagrams with the sides and angles that he measured. For each pair of triangles, write whether you know them to be congruent and which congruency test proves it. For example:



(These drawings are clearly not accurate, but you are told the measurements are.) The answer is "Congruent by the Side-Angle-Side test."



Working Space

Answer on Page 34

Chapter 5

Parallel and Perpendicular

Two vectors are said to be parallel if they have the same or opposite direction. In simpler terms, if two vectors are pointing in the same direction (even if their magnitudes differ), they are considered parallel. For example, imagine you have a vector representing the direction and speed of a car moving north. If you have another vector representing the direction and speed of a different car also moving north, these vectors are parallel.

On the other hand, if two vectors point are in completely opposite directions, they are still considered parallel. For example, if one vector represents a car moving north and the other represents a car moving south, these vectors are parallel, but in opposite directions.

Perpendicular vectors, as the name suggests, are vectors that intersect each other at a right angle, forming a 90-degree angle. If we imagine a sheet of paper, drawing a horizontal vector and a vertical vector on that paper would create perpendicular vectors. In this case, the horizontal vector represents left-right direction, while the vertical vector represents up-down direction. Perpendicular vectors are often seen in geometric shapes, such as squares and rectangles, where their sides intersect at right angles.

A fundamental property of perpendicular vectors is that their dot product is zero. The dot product is a mathematical operation that measures the extent to which two vectors align with each other. When two vectors are perpendicular, their dot product is always zero. This property provides a useful tool for determining whether two given vectors are perpendicular.

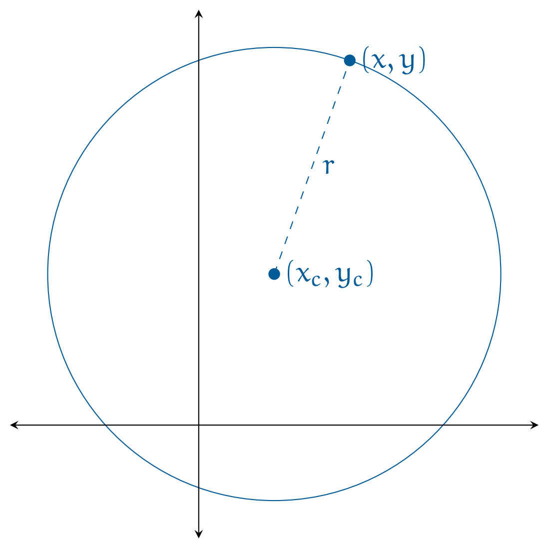
Understanding parallel and perpendicular vectors is essential in various areas of mathematics and physics. For example, in geometry, knowledge of perpendicular vectors helps us determine whether lines are perpendicular or parallel. In physics, vectors can represent forces, velocities, or displacements, and identifying parallel or perpendicular vectors aids in analyzing motion and forces acting on objects.

In summary, parallel vectors have the same or opposite direction, while perpendicular vectors intersect at a right angle. Recognizing these relationships between vectors enables us to solve problems involving geometry, physics, and many other fields. As you delve deeper into the exciting world of vectors, keep an eye out for parallel and perpendicular relationships, as they often hold valuable insights and solutions.

Chapter 6

Circles

A circle is the set of points (x, y) that are a particular distance r from a particular point (x_c, y_c). We say that r is the *radius* and (x_c, y_c) is the *center*.



Area and Radius

If the radius of a circle is r, the area of its interior (a) is given by

$$a = \pi r^2$$

Exercise 6 Area of a Circle

The paint you have says "One liter covers 6 square meters."

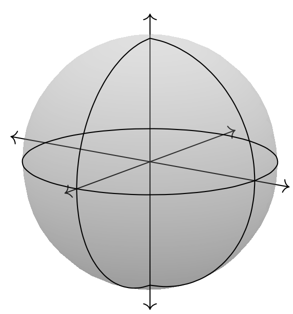
You are painting the top of a circular table with a radius of 3 meters.

How much paint will you need?

Working Space

Answer on Page 34

Note that a circle lives in a particular plane. The points (x, y, z) that are a particular distance r from a particular point (x_c, y_c, z_c) are a sphere:



The distance all the way across the middle of a circle (or a sphere) is its *diameter*. The diameter is always twice the radius.

For the rest of the chapter, we will be talking about circles, points, and lines *in a plane*.

Circumference and Diameter

The circumference (c) of a circle is the distance around the circle. If the diameter is d,

$$c = \pi d$$

Exercise 7 Circumference

Working Space

Using a tape measure, you figure out that the circumference of a tree in your yard is 64 cm.

Assuming the trunk is basically circular, what is its diameter?

Answer on Page 34

Exercise 8 Splitting a Pie

Working Space

A pie has a radius of 13 cm. 7 friends all want equal sized wedges. You have a tape measure to assist you.

How many centimeters will each outer crust be?

Answer on Page 35

6.1 Arc Length

Previously, you learned that angles can be measured in degrees and radians. A circle is 360° (see figure 6.1).

This means a circle is also 2π radians:

$$360^\circ \cdot \frac{\pi}{180^\circ} = 2\pi$$

You may be wondering: why is it that there are π radians in a 180° angle? A radian is

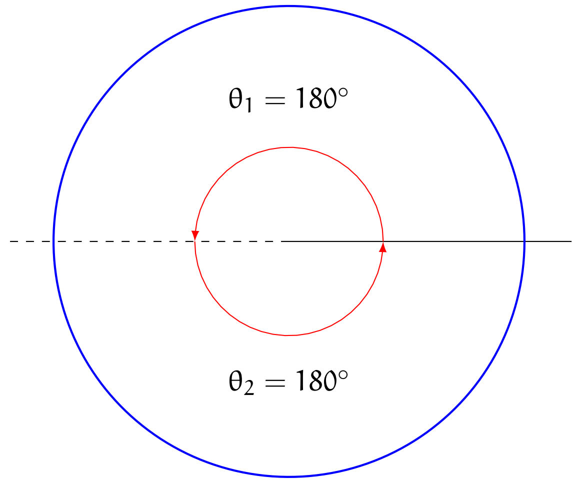


Figure 6.1: The total internal angle of a circle is $\theta_1 + \theta_2 = 360°$

defined such that one radian is the angle at the center of a circle which defines an arc of the circumference equal to the radius of the circle (see figure 6.2).

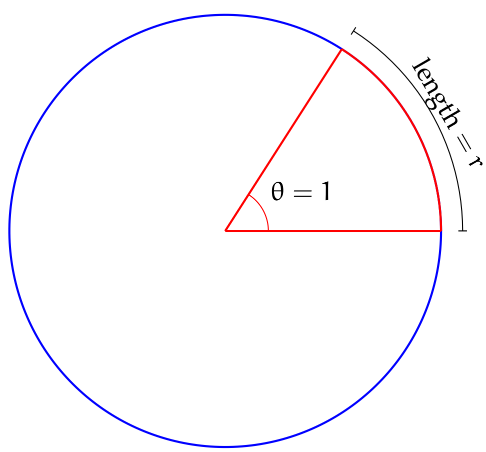
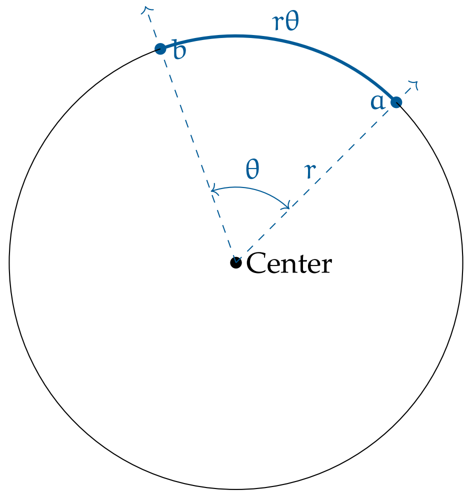


Figure 6.2: When the center angle is 1 radian, the length of the arc is equal to the radius of the circle

This makes it very straightforward to find the lengths of arcs if we know the center angle in radians. The arc length is just $r\theta$, where θ is the center angle in radians.

Length of an Arc

If you have two points a and b on a circle, the ray from the center through a and the ray from the center through b form an angle. If θ is the angle in radians and r is the radius of the circle, the distance from a to b on the circle is $r\theta$.



This shows us why π radians $= 180°$. Recall the formula for circumference: $c = \pi d$, where d is the diameter of the circle. Since the diameter is twice the radius, we can also say that $c = 2\pi r$, where r is the radius of the circle. The circumference of the circle is just an arc where the central angle is the entirety of the circle. Since we know that the length of an arc is $r\theta$, we can find the total internal angle of a circle in radians:

$$2\pi r = r\theta$$

$$\theta = 2\pi$$

This is how we know $360° = 2\pi$ radians.

Exercise 9 Angle of Rotation

A car tire has a radius of approximately 25 centimeters. If you roll your car forward 10 cm, by how many radians has your tire rotated?

Working Space

Answer on Page 35

Exercise 10 Arc Length Ranking

Rank the following arc lengths from longest to shortest (the central angle that defines the arc and the radius of the circle are provided:

1. central angle of $\frac{\pi}{4}$ and a radius of 2 cm
2. central angle of π and a radius of 1 cm
3. central angle of $\frac{\pi}{10}$ and a radius of 5 cm
4. central angle of $\frac{3\pi}{4}$ and a radius of 3 cm

Working Space

Answer on Page 35

Exercise 11 Arc Length

Working Space

You have been asked to find the radius of a very large cylindrical tank. You have a tape measure, but it is only 15 meters long and doesn't reach all the way around the tank.

However, you have a compass. So you stick one end of the tape measure to the side of the tank and measure the orientation of the wall at that point. You then walk the 15 meters and measure the orientation of the wall there.

You find that 15 meters represents 72 degrees of arc.

What is the radius of the tank in meters?

Answer on Page 36

6.2 Sector Area

We already know the area of a circle is given by $A = \pi r^2$. What about a piece of a circle? Let's start with a straightforward example:

Example: A pizza with a radius of 15 cm is divided into 6 equal pieces. What is the area of each piece?

Solution: First, we find the area of the entire pizza:

$$A = \pi r^2$$

$$A = \pi (18 \text{ cm})^2$$

$$A = 324\pi \text{ cm}^2 \approx 1018 \text{ cm}^2$$

Then, we divide by 6, since the pieces of equal sizes:

$$A_{\text{piece}} = \frac{A}{6} = \frac{324\pi\ \text{cm}^2}{6} = 54\pi\ \text{cm}^2$$

Let's use this to write a general formula for the area of a sector defined by a central angle θ (see figure 6.3). We know that when a circle is divided into 6 equal sectors, the central angle of each wedge is $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$. Additionally, we know the area of each wedge is the total area divided by 6: $A_{\text{sector}} = \frac{\pi r^2}{6} = \frac{\pi}{6} r^2 = \frac{\theta}{2} r^2$.

Area of a Wedge

For a sector whose corner is at the center of a circle, the area is given by $A_{\text{sector}} = \frac{\theta}{2} r^2$, where θ is the central angle and r is the radius.

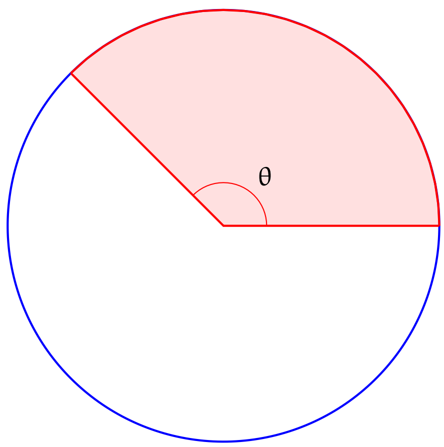


Figure 6.3: The area of a sector with central angle θ is $\frac{\theta}{2} r^2$

Exercise 12 Area of a Wedge

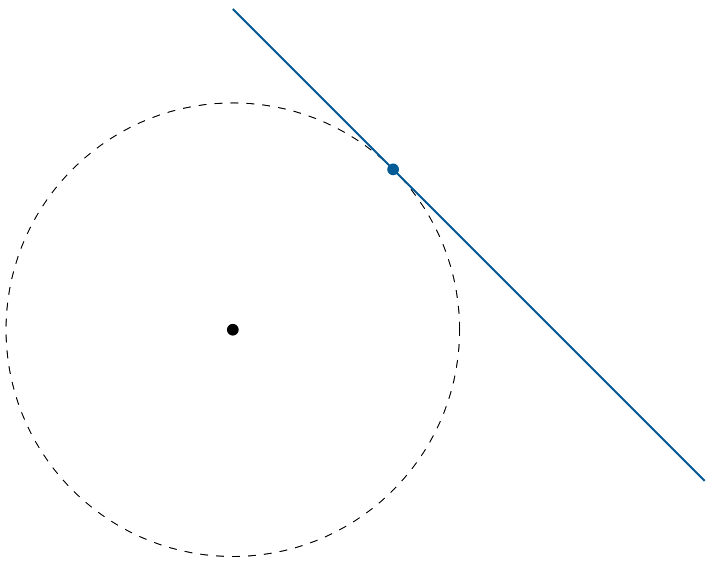
You are tasked with painting a large, circular logo on the side of a building. If a liter of paint covers covers 6 square meters and the logo is 5 meters wide, how many liters of red paint will you need to paint a wedge whose central angle is $\frac{3\pi}{4}$ radians?

Working Space

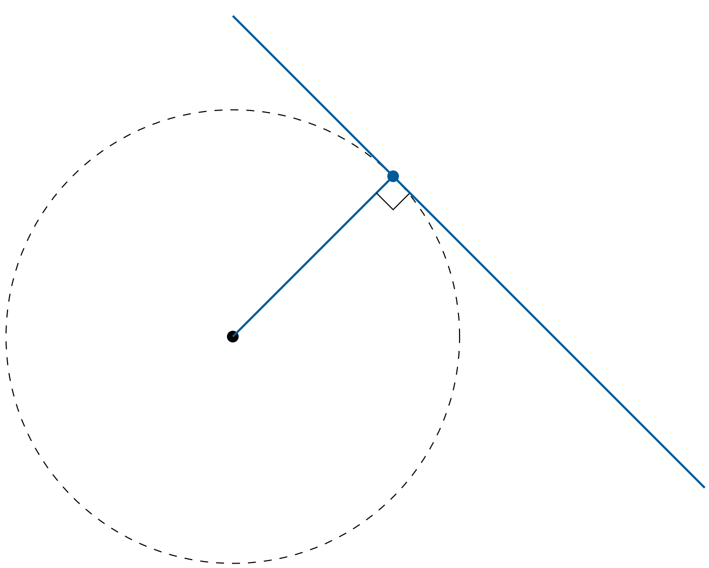
Answer on Page 36

6.3 Tangents

A line that is *tangent* to a circle touches it at exactly one point:



The tangent line is always perpendicular to the radius to the point of tangency:



Exercise 13 Painting a Comet

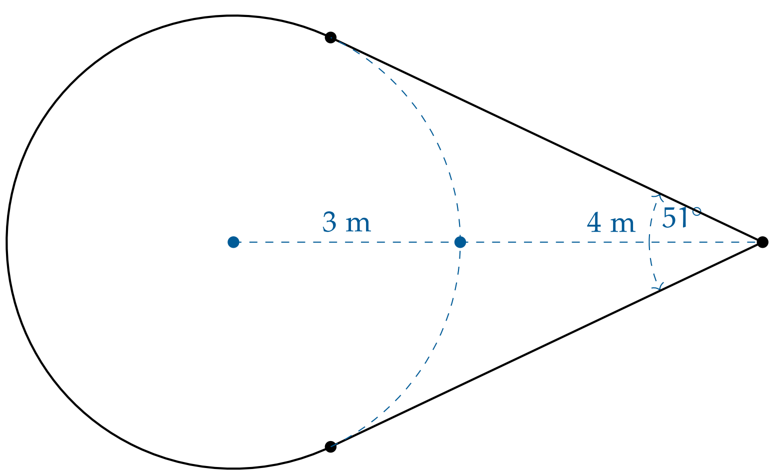
Working Space

You have been asked to paint a comet and its tail in yellow on the floor of a gymnasium.

A liter of yellow paint covers 6 square meters.

First you draw a circle with a radius of 3 meters. You then mark a point D on the floor 7 meters from the center of the circle. Then you draw two tangent lines that pass through D.

You use a protractor to measure the angle at which the tangent lines meet: about 51°



Before you paint the area contained by the circle and the two tangent lines, how much paint will you need?

Answer on Page 36

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

1. 2π
2. $60°$
3. $\frac{5\pi}{4}$
4. $135°$
5. $\frac{\pi}{6}$
6. $\frac{\pi}{4}$

Answer to Exercise 2 (on page 9)

$180° - (92° + 42°) = 46°$

Answer to Exercise 3 (on page 10)

$360°$

Answer to Exercise 4 (on page 12)

10 because $6^2 + 8^2 = 10^2$

12 because $5^2 + 12^2 = 13^2$

8 because $8^2 + 15^2 = 17^2$

$3\sqrt{2} \approx 4.24$ because $3^2 + 3^2 = \left(3\sqrt{2}\right)^2$

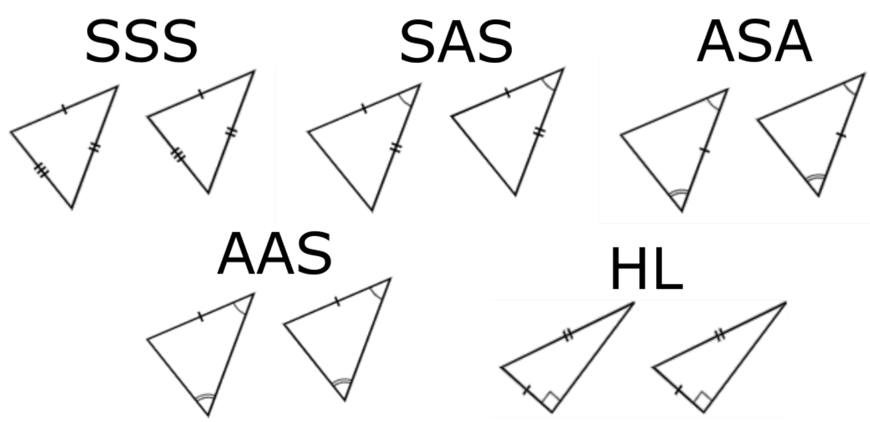
Answer to Exercise 5 (on page 19)

Congruent by the Side-Side-Right Congruency Test.

Congruent by the Side-Side-Side Congruency Test.

Congruent by the Side-Angle-Angle Congruency Test.

We don't know if they are congruent. The measured angle is not between the measured sides.



Answer to Exercise 6 (on page 24)

The table has a radius of 3 meters.

So the area of its top is $3^2\pi \approx 28.27$.

$$28.27 \text{ square meters} \left(\frac{1 \text{ liter}}{6 \text{ square meters}}\right) = 4.72 \text{ liters}$$

Answer to Exercise 7 (on page 25)

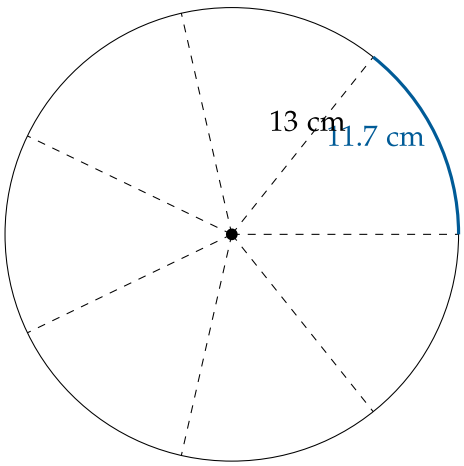
The diameter is

$$\frac{c}{\pi} = \frac{64}{\pi} \approx 20.37 \text{ centimeters}$$

Answer to Exercise 8 (on page 25)

The circumference of the pie is $26\pi \approx 81.7$ centimenters.

The length of the crust for each piece would be about $\frac{81.7}{7} = 11.7$ cm.



Answer to Exercise 9 (on page 27)

If you roll forward by 10 cm, that means you move along the edge of your tire such that the arc length is 10 cm. So, we are looking for a central angle such that $r\theta = 10$ cm. Substituting $r = 25$ cm and solving for θ: $\theta = \frac{10 \text{ cm}}{25 \text{ cm}} = 0.4$ radians.

Answer to Exercise 10 (on page 28)

1. $\frac{\pi}{4} \cdot 2 \text{ cm} = \frac{\pi}{2} \text{ cm}$
2. $\pi \cdot 1 \text{ cm} = \pi \text{ cm}$
3. $\frac{\pi}{10} \cdot 5 \text{ cm} = \frac{\pi}{2} \text{ cm}$
4. $\frac{3\pi}{4} \cdot 3 \text{ cm} = \frac{9\pi}{4} \text{ cm}$

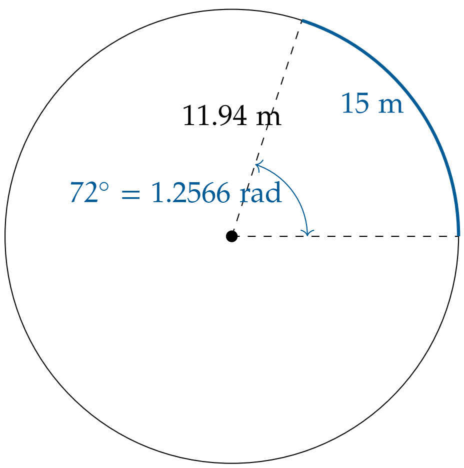
Therefore, from longest to shortest are 4, (1,3), 2 (1 and 3 are the same length).

Answer to Exercise 11 (on page 29)

$$72 \text{ degrees} \left(\frac{2\pi \text{ radians}}{360 \text{ degrees}}\right) \approx 1.2566 \text{ radians}$$

$$15 = 1.2566r$$

$$r = 11.94 \text{ meters}$$

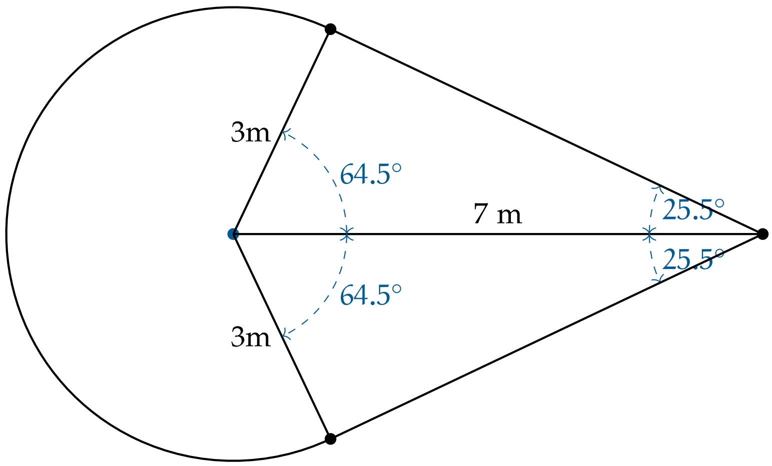


Answer to Exercise 12 (on page 30)

If the logo is 5 meters wide, the diameter is 5 meters and the radius is 2.5 meters. Using the formula for the area of a wedge: $A_{wedge} = \frac{1}{2}\frac{3\pi}{4}(21.5 \text{ m})^2 \approx 7.363 \text{ m}^2$. Since a liter covers 6 m^2, you will need $\frac{7.363}{6} \approx 1.227$ L of paint.

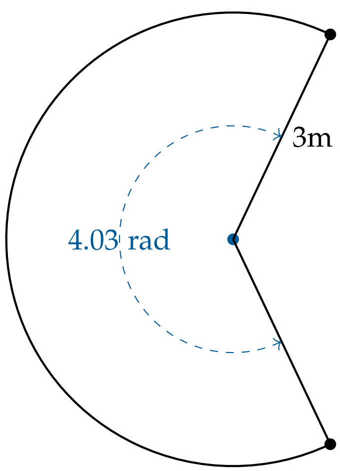
Answer to Exercise 13 (on page 32)

The trick here is to take advantage of the fact that the tangent is perpendicular to the radius to make right triangles:



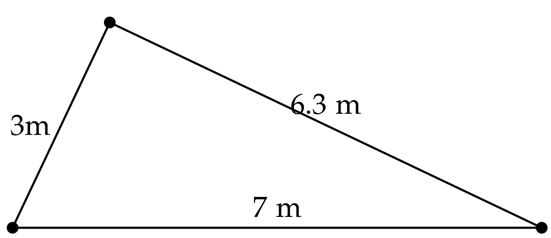
The wedge has radius 3 and represents $360 - 2(64.5) = 231° \approx 4.03$ radians.

We are finding the area of this piece:



The area of this piece is $(4.03)(3^2) = 36.27$ square meters.

If a right triangle has a hypotenuse of 7m and one leg is 3m, the other leg is $\sqrt{7^2 - 3^2} = 2\sqrt{10} \approx 6.3$ m.



A right triangle with legs of 3m and 6.3m has an area of 9.45 square meters.

There are two of them, so the total area is $36.27 + 2(18.9) = 74.07$ square meters.

Six square meters per liter, so you need $\frac{74.07}{6} = 12.35$ liters of paint.

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