Contents

Chapter 1

u-Substitution

U-Substitution, also known as the method of substitution, is a technique used to simplify the process of finding antiderivatives and integrals of complicated functions. The method is similar to the chain rule for differentiation in reverse.

Suppose we have an integral of the form:

$$\int f(g(x)) \cdot g'(x)\, dx \tag{1.1}$$

The u-substitution method suggests letting a new variable u equal to the inside function $g(x)$, i.e.,

$$u = g(x) \tag{1.2}$$

Then, the differential of u, du, is given by:

$$du = g'(x)\, dx \tag{1.3}$$

Substituting u and du back into the integral gives us a simpler integral:

$$\int f(u)\, du \tag{1.4}$$

This new integral can often be simpler to evaluate. Once the antiderivative of $f(u)$ is found, we can substitute $u = g(x)$ back into the antiderivative to get the antiderivative of the original function in terms of x.

The method of u-substitution is a powerful tool for evaluating integrals, especially when combined with other techniques like integration by parts, partial fractions, and trigonometric substitutions.

CHAPTER 2

Differential Equations

Differential equations are equations involving an unknown function and its derivatives. They play a crucial role in mathematics, physics, engineering, economics, and other disciplines due to their ability to describe change over time or in response to changing conditions.

2.1 Ordinary Differential Equations

An ordinary differential equation (ODE) involves a function of a single independent variable and its derivatives. The order of an ODE is determined by the order of the highest derivative present in the equation. An example of a first-order ODE is:

$$\frac{dy}{dx} + y = x \tag{2.1}$$

Here, y is the function of the independent variable x, and $\frac{dy}{dx}$ represents its first derivative.

A real-world example of the application of differential equations is an oscillating spring (or any harmonic motion). When a spring is stretched, the restoring force (the force pulling or pushing it back to its neutral position) is proportional to the distance by which the spring has been stretched (see figure — fixme image of spring at equilibrium position and displaced with label). Mathematically, we say that

$$\text{restoring force} = -kx$$

where k is the positive spring constant (the stiffer a spring, the greater k). Recall that Newton's Second Law tells us that force is equal to mass times acceleration, and that acceleration is the second derivative of position. We can then write the differential equation:

$$m\frac{d^2x}{dt^2} = -kx$$

This is called a **second-order differential equation** because it involves second-order derivatives. The order of a differential equation is the same as the highest order of derivative in the equation. We can further re-write the equation to isolate the second derivative:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

In everyday language, this is saying that the second derivative is proportional to the original function, just negative. There are two trigonometric functions that have this property, take a second to see if you remember and write down your guess.

The sine and cosine functions both have the property $\frac{d^2x}{dt^2} \propto -x(t)$ (recall that $\propto$ means "proportional to").

Example: Assuming $x(t)$ is a sine function, solve the second-order differential equation $\frac{d^2x}{dt^2} = \frac{-k}{m}x$.

Solution: Let $x(t) = \sin Ct$. Then $\frac{dx}{dt} = C\cos Ct$ and $\frac{d^2x}{dt^2} = -C^2 \sin t$. This implies that $C^2 = \frac{k}{m}$ and $C = \pm\sqrt{\frac{k}{m}}$. So a solution to the differential equation $\frac{d^2x}{dt^2} = \frac{-k}{m}x$ is $x(t) = \sin\sqrt{\frac{k}{m}}t$.

2.1.1 Population Growth

Another real-world application of differential equations is modeling population growth. Under ideal conditions (unlimited food, no predators, disease-free, etc.), the population of a species grows at a rate proportional to the current population size. We can identify 2 variables:

$$t = \text{ time (the independent variable)}$$

$$P = \text{ the number pf individuals in the population (the dependent variable)}$$

Then what is the rate of growth? Recall that a rate is change over time. In that case, the rate of growth is given by $\frac{dP}{dt}$. If the rate of growth is proportional to the population, then we can write a first-order differential equation:

$$\frac{dP}{dt} = kP$$

Where k is a proportionality constant. This is called **natural growth** or **logarithmic growth**. To find a solution, we must answer the question: what function's derivative is a constant multiple of itself? Recall that we've seen that the derivative of the exponential function e^{kt} is ke^{kt}. Setting $P(t) = Ce^{kt}$ (where C is some constant), we see that the derivative is $\frac{dP}{dt} = kCe^{kt} = kP(t)$ (see figure 2.1). You can determine C from initial conditions.

Example: Suppose a population of bacteria has an initial population of 100 bacteria. If the bacteria's growth rate is given by $\frac{dP}{dt} = 2P$, (where t is in hours) how many bacteria are present after 4 hours?

Solution: We have seen that the solution to $\frac{dP}{dt} = 2P$ is $P(t) = Ce^{2t}$. We can then use the

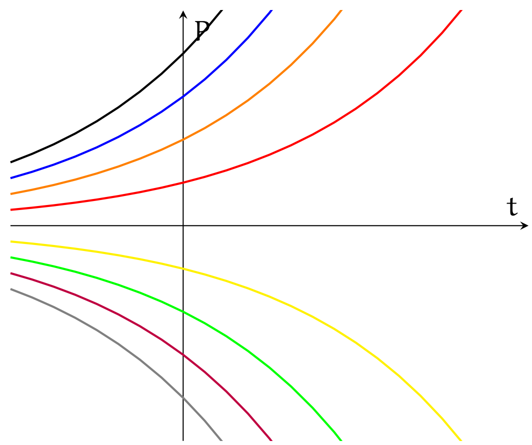


Figure 2.1: Several solutions to $\frac{dP}{dt} = kP$

given initial condition to find C:

$$P(0) = 100 = Ce^{2 \cdot 0} = C \cdot 1 = C$$

Which means that the complete solution is:

$$P(t) = 100e^{2t}$$

To answer the question, we need to find $P(4)$:

$$P(4) = 100e^{2 \cdot 4} = 100e^{8} \approx 298096$$

As stated above, this model works well for populations under specific, ideal conditions. However, there are very few environments in which these conditions are met. Real animals suffer from disease, are hunted by predators, and have limited food supplies. Most environments have a maximum number of animals they can support, which ecologists call a **carrying capacity**. Let us call the carrying capacity of an environment M. Then the population growth can be modeled by the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

This is called a **logistic differential growth model**. Notice that if P is small, then $\frac{dP}{dt} \approx kP$. This makes sense: if the population is very small compared to the carrying capacity, the conditions are nearly ideal, and so growth should be nearly ideal too. On the other hand, if the population ever goes *above* the carrying capacity, the $\frac{dP}{dt} < 0$ and the population will decrease back below the carrying capacity (see figure 2.2). Notice that if the initial population is $P_0 = M$, then $\frac{dP}{dt} = kP(1-1) = 0$ and the population is stable at $P(t) = M$. We call this an **equilibrium solution**. Can you logically find the other equilibrium

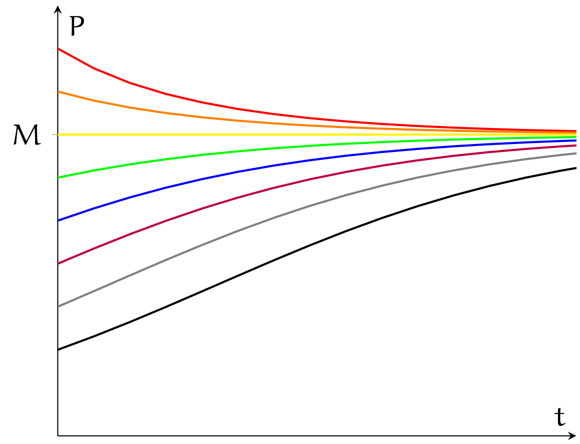


Figure 2.2: Several solutions to $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

solution?

If there are no animals to begin with, then there are none to reproduce, and $P(t) = 0$. This is the other equilibrium solution. Notice that when the population is in equilibrium, then the rate of change is zero. Mathematically, to find equilibrium solutions, we can set $\frac{dP}{dt} = 0$ and solve for P.

Exercise 1

A population is modeled by the differential equation $\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$.

1. What is the carrying capacity of the environment?
2. For what values of P is the population increasing?
3. For what values of P is the population decreasing?
4. What are the equilibrium solutions?

Working Space

Answer on Page 85

Exercise 2

[This problem was originally presented as a calculator-allowed, free response question on the 2012 AP Calculus BC exam.] Let k be a positive constant. Which of the following is a logistic differential equation? (a) $\frac{dy}{dt} = kt$ (b) $\frac{dy}{dt} = ky$ (c) $\frac{dy}{dt} = kt(1-t)$ (d) $\frac{dy}{dt} = ky(1-t)$ (e) $\frac{dy}{dt} = ky(1-y)$

Working Space

Answer on Page 85

2.1.2 Separable Differential Equations

Sometimes, differential equations can be explicitly solved. A first-order differential equation is separable if $\frac{dy}{dx}$ can be written as a function of x times a function of y. Symbolically, a differential equation is separable if it takes the form

$$\frac{dy}{dx} = g(x)f(y)$$

The equations may be solvable by separating the x from the y and integrating each side. For our generic form, we can separate the variables thusly if $f(y) \neq 0$:

$$\frac{dy}{dx}\frac{1}{f(y)} = g(x)$$

$$\frac{1}{f(y)}dy = g(x)dx$$

Integrating both sides:

$$\int \frac{1}{f(y)}\, dy = \int g(x)\, dx$$

.

Let's look at the example $\frac{dy}{dx} = \frac{x^2}{y}$. We can separate the variables by multiplying both sides by $y dx$:

$$y dy = x^2 dx$$

Integrating both sides:

$$\int y\,dy = \int x^2\,dx$$
$$\frac{1}{2}y^2 + C_1 = \frac{1}{3}x^3 + C_2$$

We can combine the constants by defining $C = C_2 - C_1$. Making this substitution and solving for y, we find:

$$y^2 = \frac{2}{3}x^3 + 2C$$
$$y = \sqrt{\frac{2}{3}x^3 + 2C}$$

Noting that $2C$ is also a constant (which we'll call K for convenience), we find the general solution is

$$y = \sqrt{\frac{2}{3}x^3 + K}$$

A graph showing the solution for several values of K is in figure 2.3.

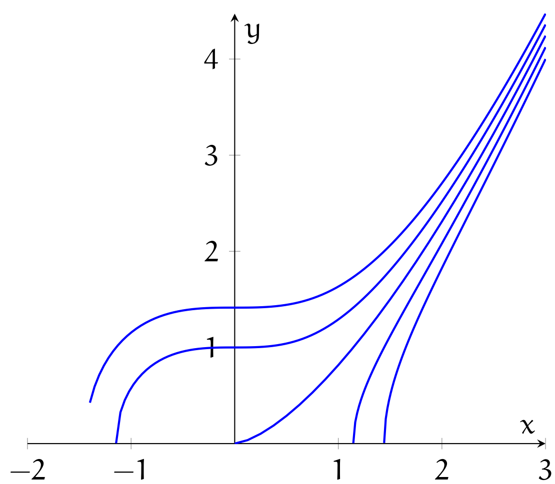


Figure 2.3: Several possible solutions to $\frac{dy}{dx} = \frac{x^2}{y}$

It is not always possible to solve for y explicitly in terms of x. The practice problem below is an example of this.

Exercise 3

Solve the differential equation $\frac{dy}{dx} = \frac{3x^2}{2y+\sin y}$.

Working Space

Answer on Page 85

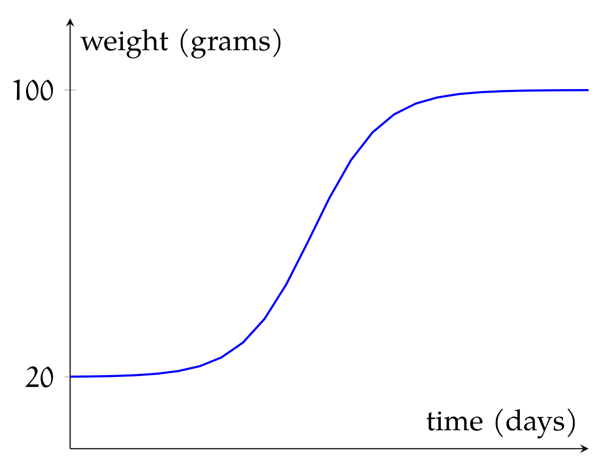
Exercise 4

[This problem was originally presented as a calculator-allowed, free response question on the 2012 AP Calculus BC exam.] The rate at which a baby bird gains mass is proportional to the difference between its adult mass and its current mass. At time $t = 0$, when the bird is first weighed, its mass is 20 grams. If $B(t)$ is the mass of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

Let $y = B(t)$ be the solution to the differential equation with initial condition $B(0) = 20$.

1. Is the bird gaining mass faster when it masses 40 grams or when it masses 70 grams? Explain your reasoning.
2. Find $\frac{d^2B}{dt^2}$ in terms of B. Use it to explain why the graph of B cannot resemble the graph shown below.
3. Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



Working Space

Answer on Page 86

Exercise 5

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] If $P(t)$ is the size of a population at time t, which of the following differential equations describes *linear* growth in the size of the population? (a) $\frac{dP}{dt} = 200$ (b) $\frac{dP}{dt} = 200t$ (c) $\frac{dP}{dt} = 100t^2$ (d) $\frac{dP}{dt} = 200P$ (e) $\frac{dP}{dt} = 100P^2$

Working Space

Answer on Page 86

2.2 Partial Differential Equations

Partial differential equations (PDEs), on the other hand, involve a function of multiple independent variables and their partial derivatives. An example of a PDE is the heat equation, a second-order PDE:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{2.2}$$

In this equation, $u = u(x, t)$ is a function of the two independent variables x and t, $\frac{\partial u}{\partial t}$ is the first partial derivative of u with respect to t, and $\frac{\partial^2 u}{\partial x^2}$ is the second partial derivative of u with respect to x.

Chapter 3

Slope Fields

While separable differential equations are solvable, most differential equations are not separable. And, in fact, it's impossible to obtain an explicit formula as a solution to most differential equations. How do computers solve these, then? They start with a given quantity (usually initial conditions) and perform many small calculations to estimate the behavior of the solution. We can do this graphically with slope fields (also called direction fields), which allow us to visualize the family of solutions to the differential equation.

3.1 Drawing Slope Fields

When a differential equation is in the form

$$y' = f(x, y)$$

we can use the coordinates (x, y) to determine the slope of a solution to the differential equation at that coordinate. Take $y' = x + y$ as an example. According to this differential equation, a solution that passes through the point $(1, 1)$ would have a slope of 2. We can represent this with a small tick of slope 2 at the $(1, 1)$ (see figure 3.1).

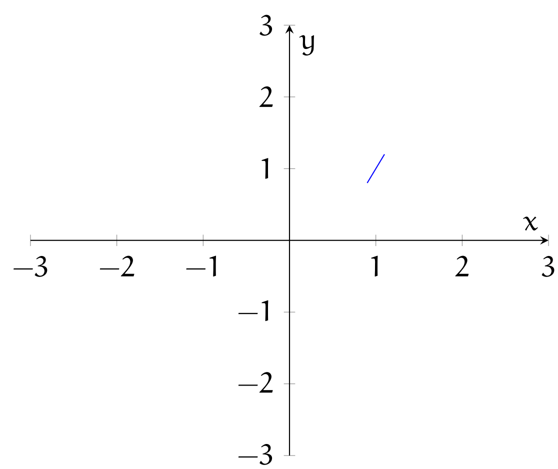


Figure 3.1: A solution to $y' = x + y$ that passes through $(1, 1)$ will have a slope of 2 at that point

Continuing, we want to choose coordinates that are easy to determine the slope. Notice that $y' = 0$ when $-x = y$, so let's go ahead and fill those ticks in (see figure 3.2):

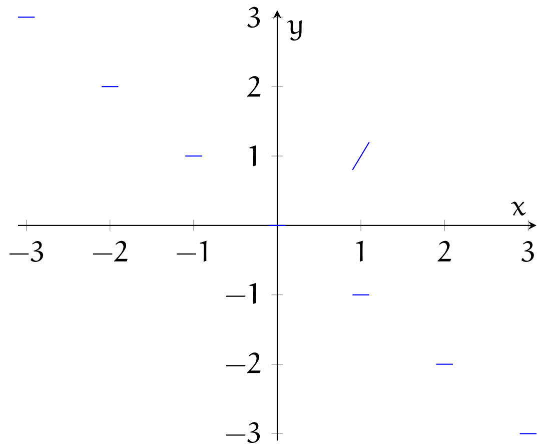


Figure 3.2: Solutions to $y' = x + y$ that lie on the line $y = -x$ will have a slope of 0.

We can repeat this process for all the coordinates shown, resulting in a slope field (see figure 3.3).

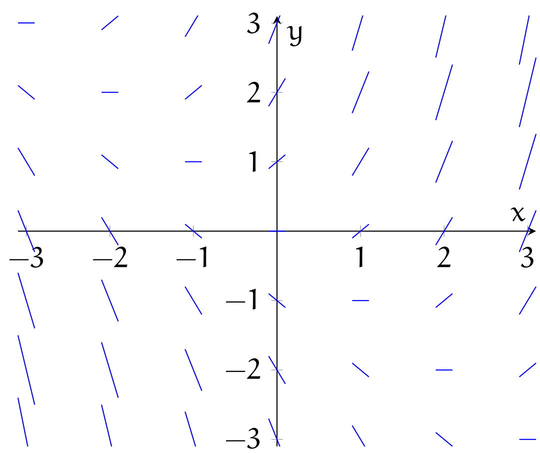


Figure 3.3: Slope field of $y' = x + y$

3.2 Sketching solutions on slope fields

If you are given an initial condition or a known point in the solution to the differential equation, you can begin sketching a curve on the slope field. Start at the given point and draw parallel to the nearby slopes. For example, suppose we know that particular solution to $y' = x + y$ passes through the point $(1, 0)$. Begin by extending the dash at $(1, 0)$ (see figure 3.4), changing the slope of your sketched solution to be approximately parallel to the nearby slopes (see figure 3.5).

While this method doesn't yield an exact, formulaic solution to the differential equation, it

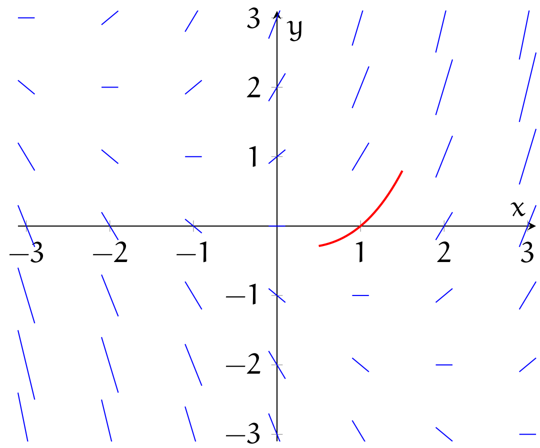


Figure 3.4: To begin sketching a solution to the differential equation, start at the point given as part of the solution

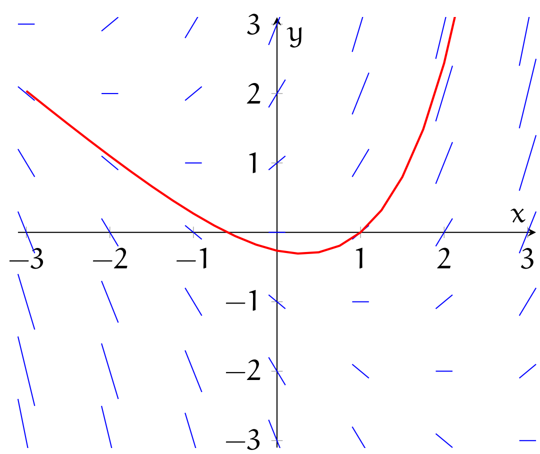


Figure 3.5: To sketch a solution to the differential equation, draw a function parallel to the nearby slopes that passes through the given point in the particular solution

does allow us to visualize solutions and generally describe the behavior of any solutions. Sketching solutions in this way is logically similar to Euler's method for finding numerical approximations of solutions to differential equations, which we will discuss more in the next chapter.

3.3 Example: Application of Differential Equations to Electronics

Think back to the chapter on DC circuits. You learned that Ohm's Law relates voltage (electromotive force), current, and resistance for simple DC circuits:

$$V = IR$$

Simple resistors have a constant resistance, so once the voltage source (battery) is connected, the current is constant. There are other electronic components, such as inductors and capacitors, that behave differently. When current changes in an inductor, a voltage drop is induced across the inductor. This is described by the differential equation:

$$V = -L\frac{dI}{dt}$$

Where L is inductance, measured in henries (H), of the inductor. Consider, then, a circuit consisting of a constant-voltage battery, a fixed resistor, and an inductor (shown in figure 3.6). Since Kirchoff's Law states that the sum of the voltage drops across each component must equal the voltage supplied by the battery, we can write a differential equation to describe the circuit:

$$V = L\frac{dI}{dt} + RI$$

Where the current, I, is a function of time, t.

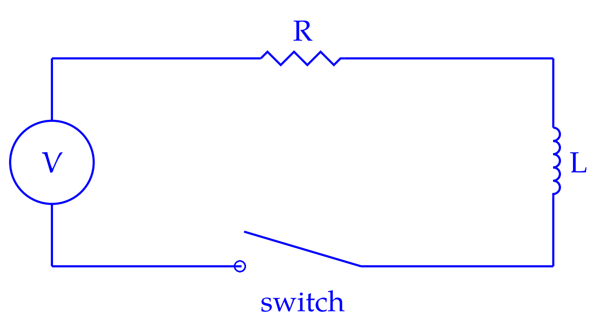


Figure 3.6: A simple circuit with a battery, resistor, inductor, and switch

Example: If the resistor is 12Ω, the inductance is 4H, and the battery supplies a constant voltage of 60V:

1. Draw a slope field for the differential equation describing the current in the circuit.
2. Describe the expected behavior of the current over a long period of time.
3. Identify any equilibrium solutions.
4. If the initial current at $t = 0$ is $I(0) = 0$, sketch the particular solution to the differential equation on the slope field.

Solution: Substituting the given values into the differential equation and rearranging to isolate $\frac{dI}{dt}$, we get $\frac{dI}{dt} = 15 - 3I$. Notice that the current is not dependent on time. When the slope is only dependent on the value of the function (as in this case), we call this an **autonomous differential equation**. This means that the slope will be the same of all values of t for a given I. The slope field is shown in figure 3.7.

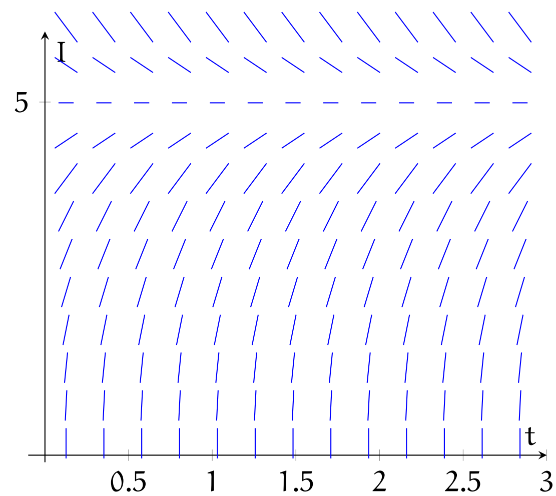


Figure 3.7: Slope field for the differential equation $\frac{dI}{dt} = 15 - 3I$

Examining the slope field, we see that the solutions tend towards $I(t) = 5$, which suggests that over an extended period of time, the current will approach 5 amperes. Similarly, if the initial current were 5 amperes, then the current would be constant at 5 amperes. Therefore, $I(t) = 5$ is an equilibrium solution. A sketch of the solution with $I(0) = 0$ is shown in figure 3.8.

3.4 Practice

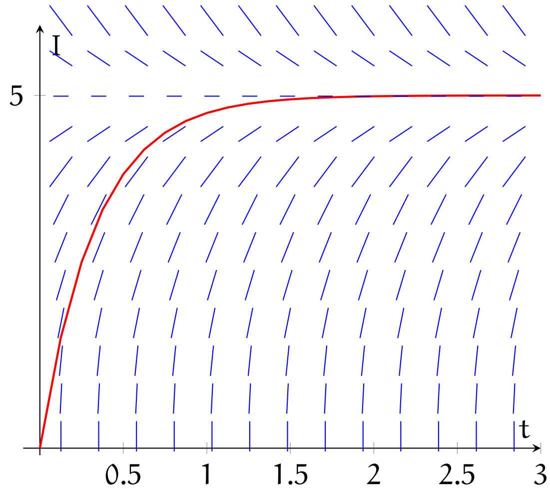


Figure 3.8: Slope field for the differential equation $\frac{dI}{dt} = 15 - 3I$

Exercise 6

Sketch the slope field for the differential equation $y' = x + y^2$. Use your slope field to sketch a solution that passes through the point $(0, 0)$.

Working Space

Answer on Page 86

Chapter 4

Euler's Method

How do computers approximate the solution to a differential equation that cannot be explicitly solved? Let's consider the differential equation

$$\frac{dy}{dx} = x + y \text{ with initial condition } y(0) = 1$$

This means the solution passes through the point (0, 1). Additionally, the slope of the solution is $\frac{dy}{dx} = 0 + 1 = 1$ at that point. This means we can approximate the solution with the linear function $L(x) = x + 1$ (see figure 4.1). As you can see, near $(0, 1)$ the approximation is good, but as x increases, the divergence between the actual solution and the approximation grows.

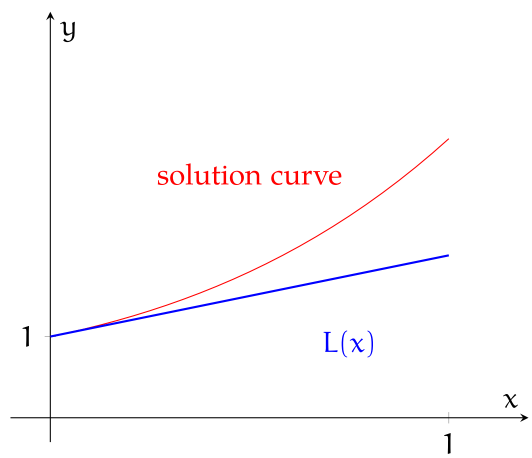


Figure 4.1: A first Euler approximation

How can we make a better approximation? Suppose we stop the first approximation at $x = 1.5$, re-evaluate $\frac{dy}{dx}$, and use that to make a second linear approximation. When $x = 0.5$, $L(x) = 0.5 + 1 = 1.5$. Taking the point $(0.5, 1.5)$, then $\frac{dy}{dx} = 0.5 + 1.5 = 2$. Then we can write a second linear approximation, $L_2(x) = 2(x - 0.5) + 1.5 = 2x - 1 + 1.5 = 2x + 0.5$ As you can see (figure 4.2), this new approximation is closer than our first approximation. We call this an approximation with a step size of 0.5.

We can improve this further by taking a step size of 0.25 (see figure 4.3). As the step size decreases and the step number increases, the approximation gets closer and closer to the true solution.

In general, Euler's method is a numerical process similar to sketching a solution on a

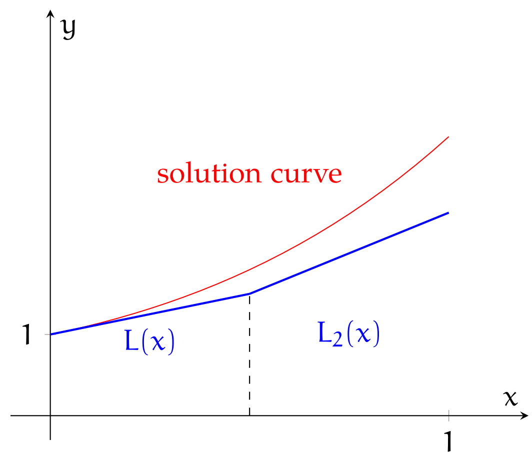


Figure 4.2: An Euler approximation with step size 0.5

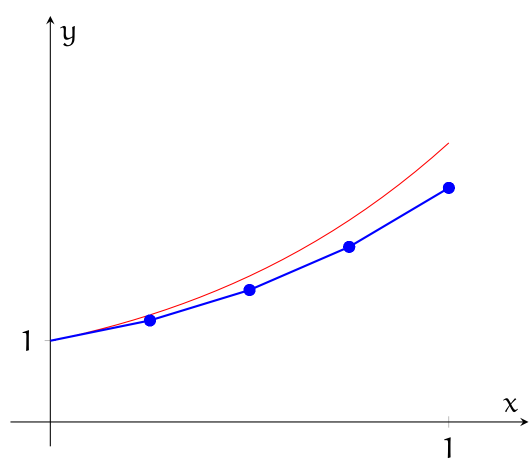


Figure 4.3: An Euler approximation with step size 0.25

slope field. One begins at the given initial value, proceeds for a short step in the direction indicated by the slope field. You adjust the slope of your approximation based on the value of the slope field at the end of each step.

For a first-order differential equation, let $\frac{dy}{dx} = F(x, y)$ and $y(x_0) = y_0$. If we have step size h, then our successive x-values are $x_1 = x_0 + h$, $x_2 = x_1 + h$, etc. The differential equation tells us that the slope at x_0 is $F(x_0, y_0)$. Then $y_1 = y_0 + hF(x_0, y_0)$ (see figure 4.4).

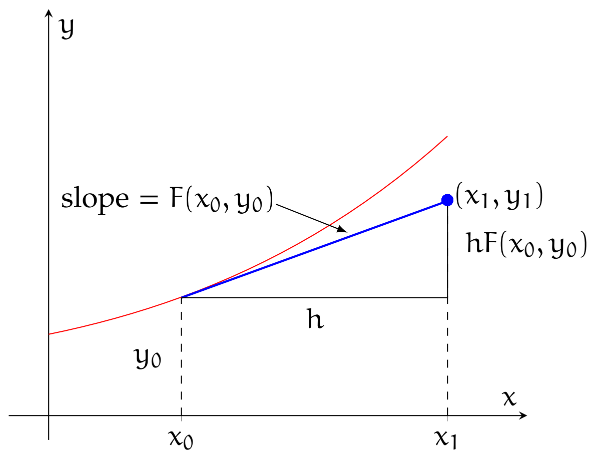


Figure 4.4: Visualization of Euler's method

Continuing, once we've found y_1, we can then define $x_2 = x_1 + h$ and $y_2 = y_1 + hF(x_1, y_1)$. And in general, for an initial-value problem when $\frac{dy}{dx} = F(x, y)$ and $y(x_0) = y_0$, we can make an approximation with step size h where:

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

where $n = 1, 2, 3, \cdots$.

Example: Use Euler's method with a step size of 0.2 to approximate the value of $y(1)$ if $\frac{dy}{dx} = 2x + y$ and $y(0) = 1$.

Solution: We are given $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = 2x + y$. This means we will need 5 steps to reach $x_5 = 1$. So we know that:

$$y_1 = 1 + 0.2[2(0) + 1] = 1 + 0.2[1] = 1.2$$

$$y_2 = 1.2 + 0.2[2(0.2) + 1.2] = 1.2 + 0.2(1.6) = 1.52$$

$$y_3 = 1.52 + 0.2[2(0.4) + 1.52] = 1.984$$

We can continue in this manner. The results are shown in the table:

n	x_n	y_n	$F(x_n, y_n)$
0	0	1	1
1	0.2	1.2	1.6
2	0.4	1.52	2.32
3	0.6	1.984	3.184
4	0.8	2.6208	4.2208
5	1	3.46496	–

Therefore, $y(1) \approx 3.4696$.

Example: This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam. Let $y = f(x)$ be the solution to $\frac{dy}{dx} = x - y$ with initial condition $f(1) = 3$. What is the approximation of $f(2)$ obtained using Euler's method with two steps of equal length starting at $x = 1$?

Solution: The question asks that we use Euler's method two steps. The step size should be $h = \frac{x_2 - x_0}{2} = \frac{2-1}{2} = \frac{1}{2}$. Taking $x_0 = 1$ and $y_0 = 3$, we find that:

$$y_1 = y_0 + h\,[x_0 - y_0]$$

$$y_1 = 3 + \frac{1}{2}\,[1 - 3]$$

$$y_1 = 3 + \frac{1}{2}\,[-2] = 3 - 1 = 2$$

So our intermediate point is $(x_1, y_1) = (\frac{3}{2}, 2)$. Finding y_2:

$$y_2 = y_1 + h\,[x_1 - y_1]$$

$$y_2 = 2 + \frac{1}{2}\left[\frac{3}{2} - 2\right]$$

$$y_2 = 2 + \frac{1}{2}\left[\frac{-1}{2}\right] = 2 - \frac{1}{4} = \frac{7}{4}$$

So the approximate value of $f(2)$ is $\frac{7}{4}$.

Exercise 7

Working Space

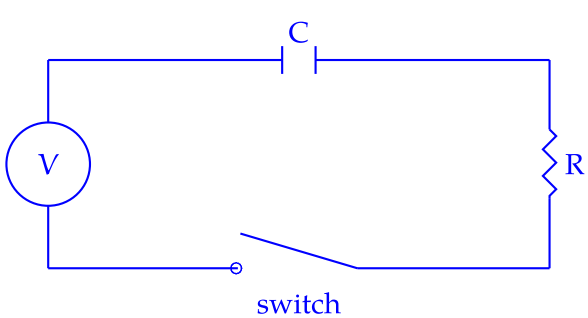
In the previous chapter on slope fields, we discussed the behavior of inductors in electronic circuits. As you may recall, capacitors also exhibit more complex behavior than regular resistors. Consider a circuit with a resistor and capacitor (see figure below). Let the resistor have resistance R ohms and the capacitor have capacitance C farads. Then by Kirchhoff's law, we know that:

$$RI + \frac{Q}{C} = V$$

Where Q is the charge on each side of the capacitor and $\frac{Q}{C}$ is the voltage drop across the capacitor. Recall that current is the change in charge over time. Therefore, $I = \frac{dQ}{dt}$ and we can write the differential equation:

$$R\frac{dQ}{dt} + \frac{1}{C}Q = V$$

When the switch is first closed, there is no charge (that is, $Q(0) = 0$). If the resistor is 5Ω, the battery is $60V$, and the capacitor is $0.05F$, use Euler's method with a step size of 0.1 to estimate the charge after half a second.



Answer on Page 87

Exercise 8

[This problem was originally presented as a calculator-allowed, free-response question on the 2012 AP Calculus BC exam.] The function f is twice- differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table below. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

Working Space

Answer on Page 90

Chapter 5

Sequences in Calculus

We have introduced sequences in a previous chapter. Now, we will examine them in more detail in a calculus context. You already know about arithmetic and geometric sequences, but not all sequences can be classified as arithmetic or geometric. Take the famous Fibonacci sequence, {1, 1, 2, 3, 5, 8, ...}, which can be explicitly defined as $a_n = a_{n-1} + a_{n-2}$, with $a_1 = a_2 = 1$. There is no common difference or common ratio, so the Fibonacci sequence is not arithmetic or geometric. Another example is $a_n = \sin \frac{n\pi}{6}$, which will cycle through a set of values.

Sequences have many real-world applications, including compound interest and modeling population growth. In later chapters, you will learn that the sum of all the values in a sequence is a series and how to use series to describe functions. In order to be able to do all that, first we need to talk in-depth about sequences.

Some sequences are defined explicitly, like $a_n = \sin \frac{n\pi}{6}$, while others are defined recursively, like $a_n = a_{n-1} + a_{n-2}$.

Example: Write the first 5 terms for the explicitly defined sequence $a_n = \frac{n}{n+1}$.

Solution: We can construct a table to keep track of our work:

n	work	a_n
1	$\frac{1}{1+1}$	$\frac{1}{2}$
2	$\frac{2}{2+1}$	$\frac{2}{3}$
3	$\frac{3}{3+1}$	$\frac{3}{4}$
4	$\frac{4}{4+1}$	$\frac{4}{5}$
5	$\frac{5}{5+1}$	$\frac{5}{6}$

So the first five terms are $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$.

Exercise 9

Write the first 5 terms for each sequence.

1. $a_n = \frac{2^n}{2n+1}$
2. $a_n = \cos \frac{n\pi}{2}$
3. $a_1 = 1$, $a_{n+1} = 5a_n - 3$
4. $a_1 = 6$, $a_{n+1} = \frac{a_n}{n+1}$

Working Space

Answer on Page 87

5.1 Convergence and Divergence

You can visualize a sequence on an xy-plane or a number line. Figures 5.1 and 5.2 show visualizations of the sequence $a_n = \frac{n}{n+1}$. To visualize this on the xy-plane, we take points such that $x = n$ and $y = a_n$, where n is a positive integer. What do you notice about this sequence? As n increases, a_n gets closer and closer to 1.

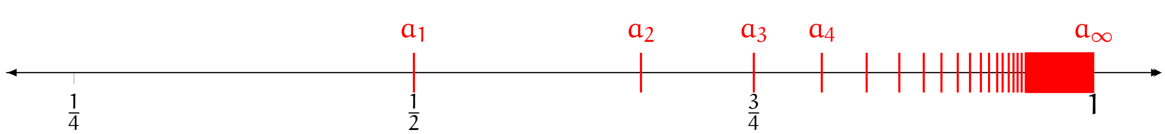


Figure 5.1: $a_n = \frac{n}{n+1}$ on a number line

Because a_n approaches a specific number as $n \to \infty$, we call the series $a_n = \frac{n}{n+1}$ *convergent*. We prove a sequence is convergent by taking the limit as n approaches ∞. If the limit exists and approaches a specific number, the sequence is convergent. If the limit does not exist or approaches $\pm\infty$, the sequence is divergent.

We can see graphically that $\lim_{n\to\infty} \frac{n}{n+1} = 1$, so that sequence is convergent. What about $b_n = \frac{n}{\sqrt{10+n}}$? Is b_n convergent or divergent?

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{n/n}{\sqrt{\frac{10}{n^2} + \frac{n}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

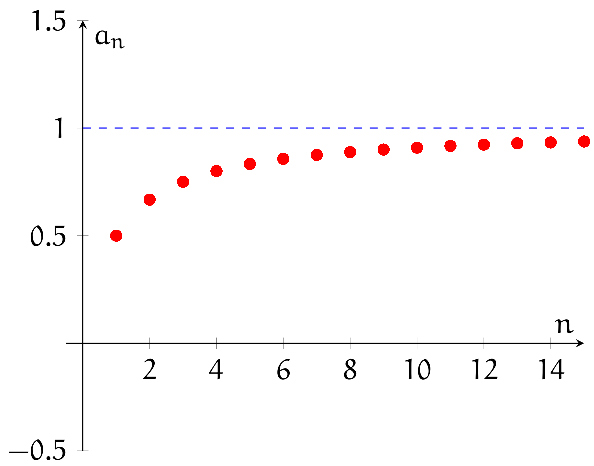


Figure 5.2: $a_n = \frac{n}{n+1}$ on an xy-plane

Therefore, the sequence $b_n = \frac{n}{\sqrt{10+n}}$ is divergent.

Here is another example of a divergent sequence: $c_n = \sin \frac{n\pi}{2}$. The graph is shown in figure 5.3. As you can see, the value of c_n oscillates between 1, 0, and -1 without approaching a specific number. This means that c_n does not approach a particular number as $n \to \infty$ and the sequence is divergent.

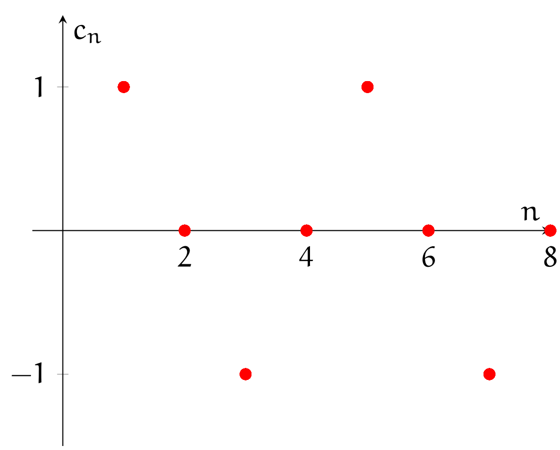


Figure 5.3: $c_n = \sin \frac{n\pi}{2}$ on an xy-plane

Exercise 10

Classify each sequence as convergent or divergent. If the sequence is convergent, find the limit as $n \to \infty$.

1. $a_n = \frac{3+5n^2}{n+n^2}$
2. $a_n = \frac{n^4}{n^3-2n}$
3. $a_n = 2 + (0.86)^n$
4. $a_n = \cos\frac{n\pi}{n+1}$
5. $a_n = \sin n$

Working Space

Answer on Page 87

5.2 Evaluating limits of sequences

Recall that a sequence can be considered a function where the domain is restricted to positive integers. If there is some $f(x)$ such that $a_n = f(n)$ when n is an integer, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$ (see figure 5.4). This means that all the rules that apply to the limits of functions also apply to the limits of sequences, including the Squeeze Theorem and l'Hospital's rule.

Example: What is $\lim_{n\to\infty} \frac{\ln n}{n}$?

Solution: First, we will try to compute the limit directly:

$$\lim_{n\to\infty} \frac{\ln n}{n} =$$

$$\frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n} = \frac{\infty}{\infty}$$

This is undefined, but fits the criteria for l'Hospital's rule:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln n}{\frac{d}{dn} n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

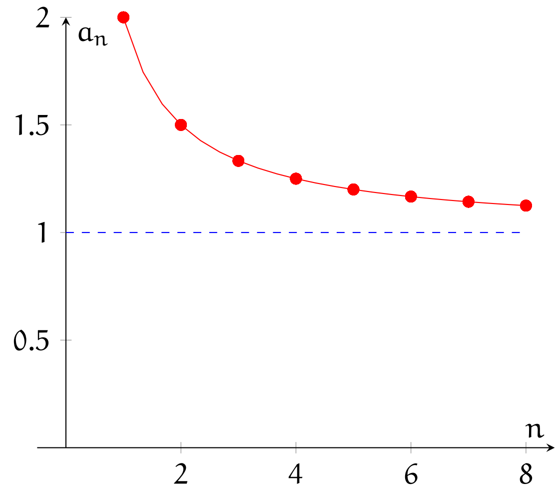


Figure 5.4: The limit of the function is the same as the limit of the sequence

Example: is the sequence $a_n = \frac{n!}{n^n}$ convergent or divergent?

Solution: First trying to take the limit directly, we see that:

$$\lim_{n\to\infty} \frac{n!}{n^n} = \frac{\infty}{\infty}$$

which is undefined. Because the factorial cannot be described as a continuous function, we can't use l'Hospital's rule. We can examine this sequence graphically (see figure 5.5) and mathematically. We examine it mathematically by writing out a few terms to get an idea of what happens to a_n as n gets large:

$$a_1 = \frac{1!}{1^1} = 1$$

$$a_2 = \frac{2!}{2^2} = \frac{1\cdot 2}{2\cdot 2}$$
$$a_3 = \frac{3!}{3^3} = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}$$
$$\cdots$$
$$a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots\cdot n}{n\cdot n\cdot n\cdots n}$$

From examining the graph in figure 5.5, we can guess that $\lim_{n\to\infty} a_n = 0$. Let's prove that mathematically. We can rewrite our expression for a_n as n gets large:

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n}\right)$$

The expression inside the parentheses is less than 1, therefore $0 < a_n < \frac{1}{n}$. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, by Squeeze Theorem we know that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. Therefore, the sequence $a_n = \frac{n!}{n^n}$ is convergent.

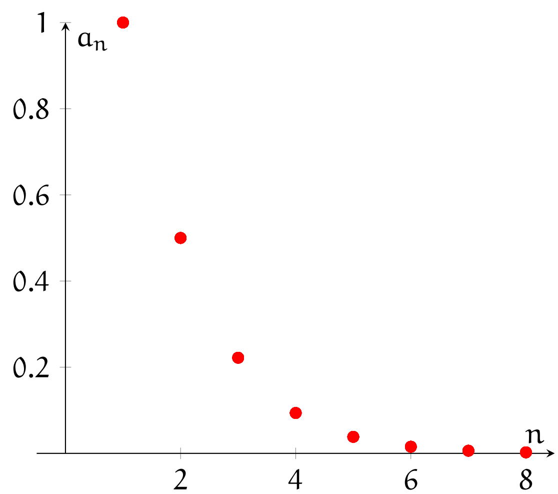


Figure 5.5: $a_n = \frac{n!}{n^n}$

[[FIX ME intro]] If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. For example, what is $\lim_{n\to\infty} \sin\frac{\pi}{n}$? Well, we know that $\lim_{n\to\infty} \frac{\pi}{n} = 0$ and that the sine function is continuous at 0. Therefore, $\lim_{n\to\infty} \sin\frac{\pi}{n} = \sin \lim_{n\to\infty} \frac{\pi}{n} = \sin 0 = 0$.

5.3 Monotonic and Bounded sequences

Just like functions, sequences can be increasing or decreasing. A sequence is increasing if $a_n < a_{n+1}$ for $n \geq 1$. Similarly, a sequence is decreasing if $a_n > a_{n+1}$ for $n \geq 1$. If a sequence is strictly increasing or decreasing, it is called *monotonic*.

The sequence $a_n = \frac{1}{n+6}$ is decreasing. We prove this formally by comparing a_n to a_{n+1}:

$$\frac{1}{n+6} > \frac{1}{(n+1)+6} = \frac{1}{n+7}$$

Example: Is the sequence $a_n = \frac{n}{n^2+1}$ increasing or decreasing?

Solution: First, we find an expression for a_{n+1}:

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$$

Since the degree of n is greater in the denominator, we have a guess that the sequence is decreasing. To prove this, we check if $a_n > a_{n+1}$ is true:

$$\frac{n}{n^2+1} > \frac{n+1}{n^2+2n+2}$$

We can cross-multiply, because $n > 0$ and the denominators are positive:

$$(n)(n^2+2n+2) > (n+1)(n^2+1)$$

$$n^3+2n^2+2n > n^3+n^2+n+1$$

Subtracting (n^3+n^2+n) from both sides we see that:

$$n^2+n > 1$$

Which is true for all $n \geq 1$. Therefore, $a_n > a_{n+1}$ for all $n \geq 1$ and the sequence is decreasing.

A sequence is *bounded above* if there is some number M such that $a_n \leq M$ for all $n \geq 1$. And a sequence is *bounded below* if there is some other number m such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above and below, then it is a *bounded sequence*.

Not all bounded sequences are convergent. Take our earlier example of $a_n = \sin\frac{n\pi}{6}$. This sequence is bounded, since we can say that $-1 \leq a_n \leq 1$ for all n. However, $a_n = \sin\frac{n\pi}{6}$ is divergent because $\lim_{n\to\infty}\sin\frac{n\pi}{6}$ does not exist (see figure 5.6). Additionally, not all monotonic sequences are convergent. Consider $b_n = 2^n$ (shown in figure 5.7). This is monotonically increasing (that is, $b_n > b_{n-1}$ for all n), but $\lim_{n\to\infty} 2^n = \infty$ and the sequence is divergent.

A sequence must be convergent if it is **both** monotonic and bounded. Why is this? Recall that to be bounded, then a sequence is bounded above and below, which means there is some m and some M such that $m \leq a_n \leq M$ for all n. If the sequence is increasing, the terms must get close to but not exceed M. Likewise, if the sequence is decreasing, the terms must get close to, but not be less than m.

Example: is the sequence given by $a_n = 4$ and $a_{n+1} = \frac{1}{2}(a_n + 7)$ bounded above, below, both, or neither?

Solution: We start by calculating the first several terms:

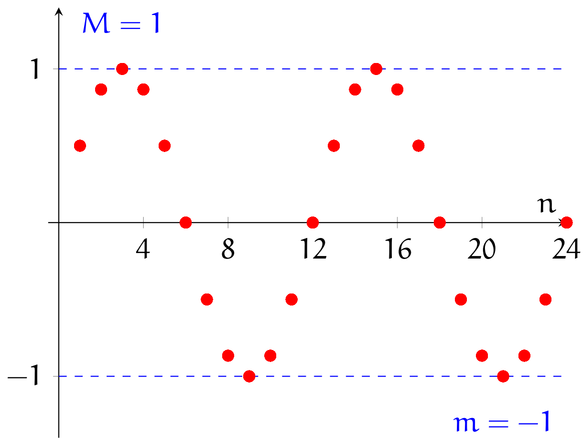


Figure 5.6: The sequence $a_n = \sin \frac{n\pi}{6}$ is bounded and divergent

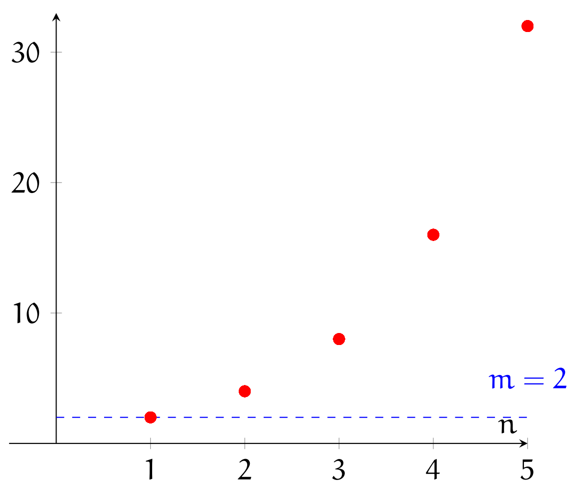


Figure 5.7: The sequence $b_n = 2^n$ is bounded below, monotonically increasing, and divergent

Term	Work	Value
a_1	$a_1 = 4$	4
a_2	$= \frac{1}{2}(4+7)$	5.5
a_3	$= \frac{1}{2}(5.5+7)$	6.25
a_4	$= \frac{1}{2}(6.25+7)$	6.625
a_5	$= \frac{1}{2}(6.625+7)$	6.8125
a_6	$= \frac{1}{2}(6.8125+7)$	6.90625
a_7	$= \frac{1}{2}(6.90625+7)$	6.953125
a_8	$= \frac{1}{2}(6.953125+7)$	6.9765625

The sequence is increasing, so it is bounded below by the initial term, $a_1 = 4$, and we can state that $a_n \geq 4$. Examining the computed terms, we see that $a_n \to 7$ as n grows larger. We can guess that this sequence is bounded above, with $a_n \leq 7$. We can prove this by induction. Suppose that there is some k such that $a_k < 7$ (which is true for a_1, etc.). Then,

$$\begin{gathered} a_k < 7 \\ a_k + 7 < 14 \\ \frac{1}{2}(a_k + 7) < \frac{1}{2}(14) \\ a_{k+1} < 7 \end{gathered}$$

Therefore, $a_n < 7$ for all n and the sequence is bounded above. Because the sequence is monotonic and bounded, we know the sequence is convergent and, therefore, that the limit of a_n as $n \to \infty$ exists.

5.4 Applications of Sequences

5.4.1 Compound Interest

You previously learned about compound interest and modeled the accumulation of compound interest by $P_n = P_0(1+r)^n$, where P_0 is the principal investment, r is the yearly interest rate, and n is the number of elapsed years. This sequence describes the value of an investment accumulating interest, but most people add to their savings on a regular schedule. We can write a sequence to model the value of a savings account that the owner makes regular deposits into.

Example: Suppose you open a savings account with an initial deposit of $3,000 and you plan to deposit an additional $1,200 at the end of every year. If your savings account has an annual interest rate of 3.25%, how long will it take you to save $10,000?

Solution: We can write a recursive definition for the sequence. At the end of each year, the account will gain the interest on the entirety of the previous year's balance plus $1200:

$$P_n = P_{n-1}(1 + 0.0325) + \$1200$$

With an initial investment $P_0 = \$3000$. We can write out the first few terms to find how many years it will take to save $10,000:

Year	Savings
0	$3,000
1	$4,297.50
2	$5,637.17
3	$7,020.38
4	$8,448.54
5	$9.923.12
6	$11,445.62

The accumulation of interest with deposits is better described by a sequence than a function. That's because the deposits are happening at discrete times, not continuously.

Exercise 11

You invest $1500 at 5%, compounded annually. Write an explicit formula that describes the value of your investment every year. What will your investment be worth after 10 years? Is the sequence convergent or divergent? Explain.

Working Space

Answer on Page 88

5.4.2 Population Growth

Sequences can be used to model a reproducing population that is being occasionally culled from or added to. Similar to compound interest, a population of living things (plants, animals, fungi, etc.) reproducing at a rate *r* can be modeled with an exponential function:

$$P_n = P_0(1 + r)^n$$

Where P_0 is the initial population, r is the yearly reproductive rate, and n is the number of years elapsed.

Example: Suppose the population of deer in a national park is estimated to be 6,500. If the deer reproduce at a rate of 8% per year and wolves hunt and kill 500 deer per year, how many deer will be in the park in 5 years?

Solution: We can write a recursive sequence:

$$P_n = P_{n-1}(1 + 0.08) - 500$$

$$P_0 = 6500$$

And calculate P_5 (we round to the nearest whole number because half of a deer is not a living deer):

Year		Population
1	$6500(1.08) - 500$	6520
2	$6520(1.08) - 500$	6542
3	$6542(1.08) - 500$	6565
4	$6565(1.08) - 500$	6590
5	$6590(1.08) - 500$	6617

There will be 6617 deer in the park after 5 years.

Exercise 12

A farmer keeps his pond stocked with fish. If the fish are eaten by predators at a rate of 5% per month and the farmer can afford to restock the pond with 10 fish every 6 months. If the farmer starts with 100 fish, how many total fish will he have lost to predation after 4 years?

Working Space

Answer on Page 88

Chapter 6

Series

When writing a number with an infinite decimal, such as the Golden Ratio (also known as the Golden Number):

$$\phi = 1.618033988\cdots$$

The decimal system means we can re-write the Golden Ratio (or any irrational number) as an infinite sum:

$$\phi = 1 + \frac{6}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{0}{10^4} + \frac{3}{10^5} + \cdots$$

You might recall from the chapter on Riemann Sums that we can represent the addition of many (or infinite) with big sigma notation:

$$\sum_{i=1}^{n} a_i$$

Where i is the index as discussed in Sequences and n is the number of terms. For infinite sums, $n = \infty$.

6.1 Partial Sums

Let us quickly define a *partial sum*. A partial sum is where we only look at the first n terms of a series. For the general series, $\sum_{i=1}^{n} a_i$, the partial sums are:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\cdots$$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Example: A series is given by $\sum_{i=1}^{\infty} (\frac{-3}{4})^i$. What is the value of the partial sum s_4?

Solution: s_4 is the sum of the first 4 terms:

$$(\frac{-3}{4})^1 + (\frac{-3}{4})^2 + (\frac{-3}{4})^3 + (\frac{-3}{4})^4$$
$$= \frac{-3}{4} + \frac{9}{16} + \frac{-27}{64} + \frac{81}{256} = \frac{-75}{256}$$

6.2 Reindexing

Sometimes it is necessary to re-index series. This means changing what n the series starts at . In general,

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+1}^{\infty} a_{n-1} \text{ and } \sum_{n=i}^{\infty} a_n = \sum_{n=i-1}^{\infty} a_{n+1}$$

That is, to increase the index by 1, you need to replace n with $(n-1)$ and do decrease the index by 1, you need to replace n with $(n+1)$. Let's visualize why this is true (see figure 6.1). Notice that for each series, the terms are the same. This is similar to shifting functions: to move the function to the left on the x-axis, you plot $f(x+1)$, and to move it to the right, $f(x-1)$.

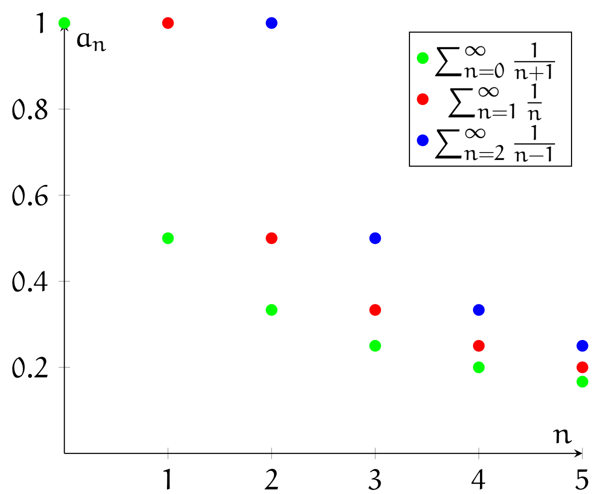


Figure 6.1: $\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

We can also prove each reindexing rule mathematically. Recall that

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

We also know that

$$\sum_{n=2}^{\infty} a_{n-1} = a_{2-1} + a_{3-1} + a_{4-1} + \cdots = a_1 + a_2 + a_3 + \cdots$$

Therefore, $\sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} a_{n-1}$.

Similarly,

$$\sum_{n=0}^{\infty} a_{n+1} = a_{0+1} + a_{1+1} + a_{2+1} + \cdots = a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

Example: Reindex the series $\sum_{n=3}^{\infty} \frac{n+1}{n^2-2}$ to begin with $n = 1$.

Solution: We are decreasing the index, so we will use $\sum_{n=i-1}^{\infty} a_{n+1} = \sum_{n=i}^{\infty} a_n$. We will apply this rule twice, to decrease the index from 3 to 1:

$$\sum_{n=2}^{\infty} \frac{(n+1)+1}{(n+1)^2-2} = \sum_{n=2}^{\infty} \frac{n+2}{(n+1)^2-2}$$

$$\sum_{n=1}^{\infty} \frac{(n+1)+2}{[(n+1)+1]^2-2} = \sum_{n=1}^{\infty} \frac{n+3}{(n+2)^2-2}$$

It is easier and faster to be able to reindex a series by more than one step at a time. Using the example above, we can write an even more general rule for reindexing:

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+j}^{\infty} a_{n-j}$$

where i and j are integers. (Then to decrease the index, you would choose a j such that $j < 0$.)

6.3 Convergent and Divergent Series

Just like sequences, series can also be convergent or divergent. Consider the series $\sum_{i=1}^{\infty} i$. Given what you already know about the meaning of "convergent" and "divergent", guess whether $\sum_{i=1}^{\infty} i$ is convergent or divergent.

Let's determine the first few partial sums of the series (shown graphically in figure 6.2):

n	Terms	Partial Sum
1	1	1
2	1+2	3
3	1+2+3	6
4	1+2+3+4	10

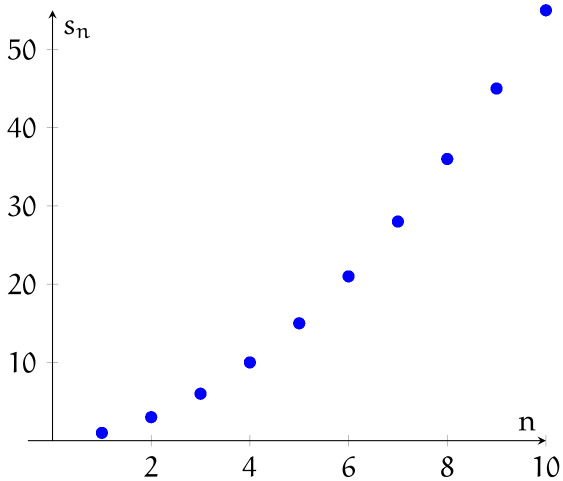


Figure 6.2: For the divergent series $\sum_{i=1}^{n} i$, the value of the partial sum increases to infinity as n increases

As you can see, as n increases, the value of the partial sum increases without approaching a particular value. We can also see that the value of the first n terms summed together is $\frac{n(n+1)}{2}$. This means that as n approaches ∞, the sum also approaches ∞ and the series is divergent.

Obviously, for a series to not become huge, the values of the terms should decrease as i increases (that is, each subsequent term is smaller than the one before it). Take the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. As i increases, $\frac{1}{2^i}$ decreases. Let's look at the first few partial sums of this series (shown graphically in figure 6.3):

n	Terms	Partial Sum
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2} + \frac{1}{4}$	$\frac{3}{4}$
3	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$\frac{7}{8}$
4	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	$\frac{15}{16}$

Do you see the pattern? The n^{th} partial sum is equal to $\frac{2^n-1}{2^n} = 1-\frac{1}{2^n}$. And as n approaches ∞, the partial sum approaches 1. The series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is convergent.

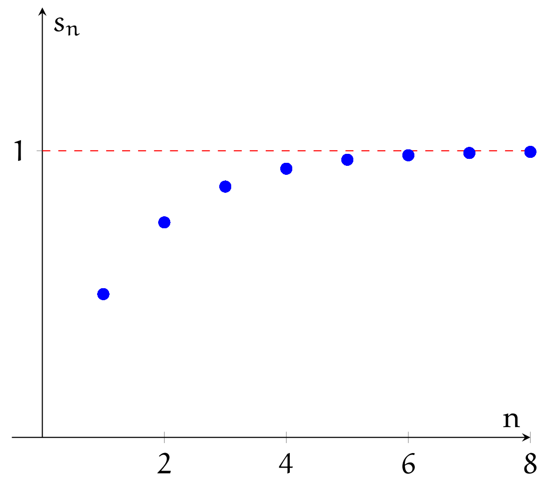


Figure 6.3: For the convergent series $\sum_{i=1}^{n} \frac{1}{2^i}$, the value of the partial sum approaches 1 as n increases

Let us define the sequence $\{s_n\}$ where s_n is the n^{th} partial sum of a series:

$$s_n = \sum_{i=1}^{n} a_i$$

.

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n$ exists, then the series $\sum_{i=1}^{\infty} a_i$ is also convergent. And if the sequence $\{s_n\}$ is divergent, then the series $\sum_{i=1}^{\infty} a_i$ is also divergent.

Example: is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent or divergent?

Solution: You may think that the series is convergent, since $\lim_{n\to\infty} \frac{1}{n} = 0$. Let's see if we can confirm this. We begin by looking at the partial sums s_2, s_4, s_8, and s_{16}:

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) >$$

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) = 1 + \frac{4}{2}$$

Notice that, in general, $s_{2^n} > 1 + \frac{n}{2}$ for $n > 1$. Taking the limit as $n \to \infty$, we see that

$\lim_{n\to\infty} s_{2^n} > \lim_{n\to\infty} 1 + \frac{n}{2} = \infty$. Therefore, s_{2^n} also approaches ∞ as n gets larger and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

This example shows a very important point: a series whose terms decrease to zero as n gets large is not necessarily convergent. What we can say, though, is that if the limit as n approaches infinity of the terms of a series does not exist or is not zero, then the series is divergent (i.e. not convergent). This is called the **Test for Divergence**, and we will explore it further in the next chapter.

6.3.1 Properties of Convergent Series

We just saw that if $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges. The contrapositive statement gives a property of convergent series:

$$\text{If the series } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n\to\infty} = 0$$

If a series is made of other convergent series, it may be convergent. Recall, if a series is convergent, this means the $\lim_{n\to\infty} \sum_{i=1}^{n} a_i = L$. By the properties of limits, then we can also say that the series multiplied by a constant is convergent:

$$\sum_{n=1}^{\infty} ca_n = c \cdot L = c \sum_{n=1}^{\infty} a_n$$

Suppose there is another convergent series such that $\lim_{n\to\infty} \sum_{i=1}^{n} b_i = M$. Then the sum of those series is also convergent. That is:

$$\sum_{n=1}^{\infty} (a_n + b_n) = L + M = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Similarly, the difference of the series is convergent:

$$\sum_{n=1}^{\infty} (a_n - b_n) = L - M = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

6.4 Geometric Series

A geometric series is the sum of a geometric sequence, and has the form:

$$\sum_{n=1}^{\infty} ar^n \text{ or } \sum_{n=1}^{\infty} ar^{n-1}$$

Where a is some constant and r is the common ratio. For $\sum_{n=1}^{\infty} ar^{n-1}$, a is also the first term.

Example: Write the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ in sigma notation.

Solution: We see that the first term is $a = 1$ and the common ratio is $\frac{1}{2}$, so we can write the series:

$$\sum_{n=1}^{\infty} 1(\frac{1}{2})^{n-1} = \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$$

When are geometric series convergent? First, let's consider the case where $r = 1$. If this is true, then $s_n = a + a + a + \cdots + a = na$. As n approaches ∞, the sum will approach $\pm\infty$ (depending on whether a is positive or negative), and the series is divergent.

When $r \neq 1$, we can write s_n and rs_n:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Subtracting rs_n from s_n, we get:

$$\begin{aligned} s_n - rs_n &= (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) \\ &= a - ar^n \end{aligned}$$

Solving for s_n, we find:

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

We take the limit as $n \to \infty$ to determine for what values of r the series converges:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r}$$

$$= \lim_{n\to\infty} \left[\frac{a}{1-r} - \frac{ar^n}{1-r}\right] = \frac{a}{1-r} - \left(\frac{a}{1-r}\right) \lim_{n\to\infty} r^n$$

This begs the question: when is $\lim_{n\to\infty} r^n$ convergent? From the sequences chapter, we know this limit converges if $|r| < 1$ (that is, $-1 < r < 1$). If this is true, then $\lim_{n\to\infty} r^n = 0$ and

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

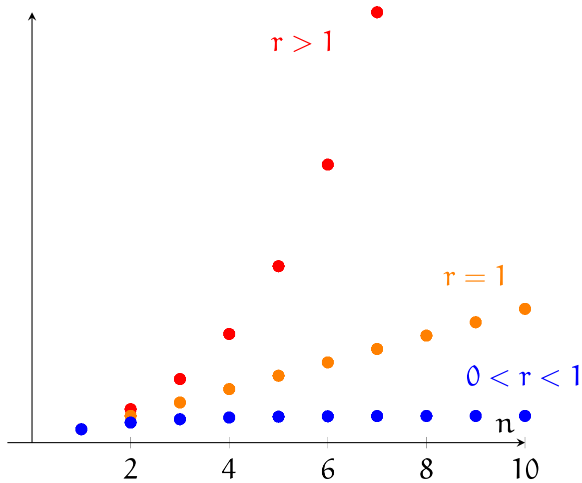


Figure 6.4: Geometric sequences are divergent if $r \geq 1$

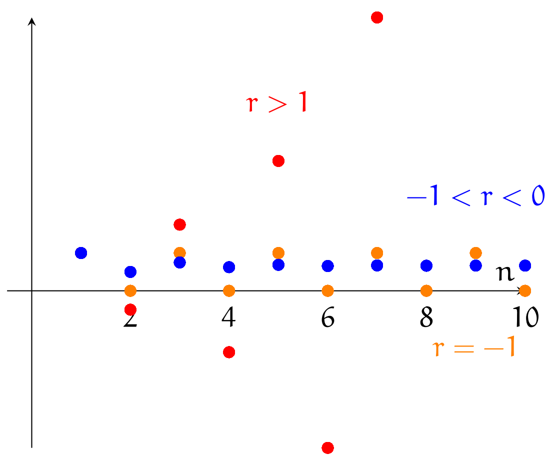


Figure 6.5: Geometric sequences are divergent if $r \leq 1$. Notice that for $r = -1$, the partial sums alternate between the initial term and zero.

(see figures 6.4 and 6.5 for a visual)

Example: Find the sum of the geometric series given by $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \cdots$.

Solution: The first term is $a = 2$ and each the common ratio is $r = \frac{-1}{3}$. Since $|r| < 1$, we know that the series converges. We can calculate the value of the sum using the geometric series formula:

$$\sum_{i=1}^{\infty} a(r)^{i-1} = \frac{a}{1-r}$$

$$\sum_{i=1}^{\infty} 2\left(\frac{-1}{3}\right)^{i-1} = \frac{2}{1 - \frac{-1}{3}} = \frac{2}{\frac{4}{3}} = \frac{6}{4} = 1.5$$

We can confirm this graphically (see figure 6.6). You can also write out the first several partial sequences: you should find the sums approach 1.5 as n increases.

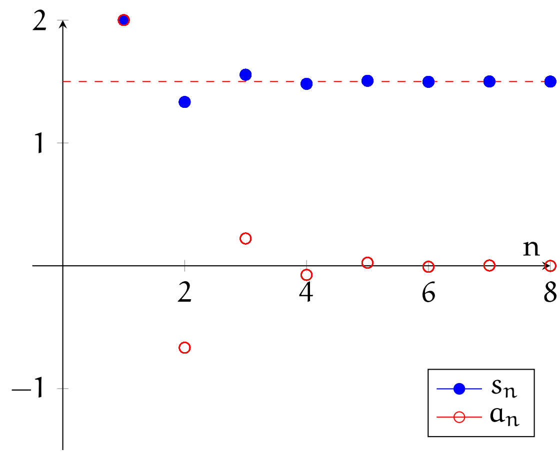


Figure 6.6: the n^{th} term and partial sums of $\sum_{i=1}^{n} 2(\frac{-1}{3})^{i-1}$

Example: What is the value of $\sum_{n=1}^{\infty} 2^{2n}5^{1-n}$

Solution: The key here is to re-write the series in the form $\sum_{n=1}^{\infty} ar^{n-1}$ so we can use the fact that convergent geometric series sum to $\frac{a}{1-r}$.

$$\sum_{n=1}^{\infty} 2^{2n}5^{1-n} = \sum_{n=1}^{\infty} \left(2^2\right)^n \left(\frac{1}{5}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 4 \cdot (4)^{n-1} \left(\frac{1}{5}\right)^{n-1} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^{n-1}$$

Which is in the form $\sum_{n=1}^{\infty} ar^{n-1}$ with $a = 4$ and $r = \frac{4}{5}$. Since $|r| < 1$, the series converges to

$$\frac{a}{1-r} = \frac{4}{1-\frac{4}{5}} = \frac{4}{\frac{1}{5}} = 20$$

Exercise 13

Determine whether the geometric series is convergent or divergent. If it convergent, find its sum.

1. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$
2. $2 + 0.5 + 0.125 + 0.03125 + \cdots$
3. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$
4. $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

Working Space

Answer on Page 89

Exercise 14

Find a value of c such that $\sum_{n=0}^{\infty} (1+c)^{-n} = \frac{5}{3}$

Working Space

Answer on Page 89

Exercise 15

For what values of p does the series $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converge?

Working Space

Answer on Page 89

6.5 p-series

A p-series takes the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and converges if $p > 1$ and diverges if $p \leq 1$. We won't prove this here, since it requires the application of a test you will learn about in the next chapter.

Example Write the series $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$. Is it convergent or divergent?

Solution: We see that $a_n = \frac{1}{\sqrt[3]{n}}$ and so the infinite series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

We see that this is a p-series with $p = \frac{1}{3}$. Since $p < 1$, the series is divergent.

Exercise 16

Euler found that the exact sum of the p-series where $p = 2$ is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

And that the exact sum of the p-series where $p = 4$ is:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use this and the properties of convergent series to find the sum of each of the following series:

1. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$
2. $\sum_{n=2}^{\infty} \frac{1}{n^2}$
3. $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$
4. $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$
5. $\sum_{n=1}^{\infty} \left(\frac{4}{n^2} + \frac{3}{n^4}\right)$

Working Space

Answer on Page 90

Exercise 17

For what values of k does the series $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ converge?

Working Space

Answer on Page 90

6.6 Alternating Series

An alternating series is one in which the terms alternate between positive and negative . Here is an example:

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

Alternating series are generally of the form

$$a_n = (-1)^n b_n \text{ or } a_n = (-1)^{n-1} b_n$$

Where b_n is positive (and therefore, $|a_n| = b_n$).

An alternating series is convergent if (i)$b_{n+1} \leq b_n$ and (ii)$\lim_{n\to\infty} b_n = 0$. In words, we say that if the absolute value of the terms of a series decrease towards zero, then the series converges. This is called the **Alternating Series Test**.

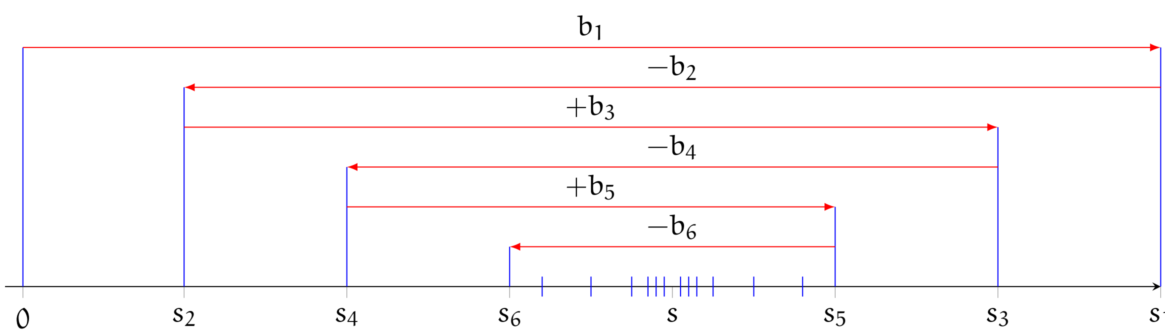


Figure 6.7: As n increases, s_n approaches s

Example: Is the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ convergent?

Solution: The Alternating series test states that an alternating series is convergent if

$|a_{n+1}| < |a_n|$:

$$\left|\frac{(-1)^{n-1+1}}{n+1}\right| < \left|\frac{(-1)^{n-1}}{n}\right|$$
$$\frac{1}{n+1} < \frac{1}{n}$$

Since $|a_{n+1}| < |a_n|$ and the series is alternating, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Exercise 18

Test the following alternating series for convergence:

1. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$
2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

Working Space

Answer on Page 90

Chapter 7

Convergence Tests for Series

7.1 Test for Divergence

Recall from the previous chapter that if the terms of a series do not approach zero as n approaches infinity, then the series is divergent. This is the Test for Divergence, and there are two possible outcomes. For a series $\sum_{n=1}^{\infty} a_n$:

$$\text{If } \lim_{n\to\infty} a_n \neq 0\text{, then the series diverges}$$

$$\text{If } \lim_{n\to\infty} a_n = 0\text{, then the test is inconclusive}$$

It is important to remember that the Test for Divergence cannot tell us conclusively that a series converges. Rather, it only identifies series that are divergent.

Example: Apply the Test for Divergence to the series $\sum_{n=1}^{\infty} \sqrt{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution: $\lim_{n\to\infty} \sqrt{n} = \infty \neq 0$. Therefore, the series $\sum_{n=1}^{\infty} \sqrt{n}$ is divergent.

$\lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ may be divergent or convergent. This is the harmonic series, which we proved to be divergent in the previous chapter. This is a good example which demonstrates that just because $\lim_{n\to\infty} a_n = 0$ does not mean the series is convergent.

7.2 The Integral Test

We were able to determine the exact value of some infinite series because it was possible to write the n^{th} partial sum, s_n, in terms of n. For example, we determined that the n^{th} partial sum of $\sum_{i=1}^{n} \frac{1}{2^i}$ is $s_n = 1 - \frac{1}{2^n}$. However, it is not always possible to do this. How can we estimate the value of an infinite series in cases where we can't explicitly write s_n in terms of n?

Consider the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$. The first few terms are:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

The series is decreasing, but is it convergent? Let's plot this series on an xy-plane (see figure 7.1).

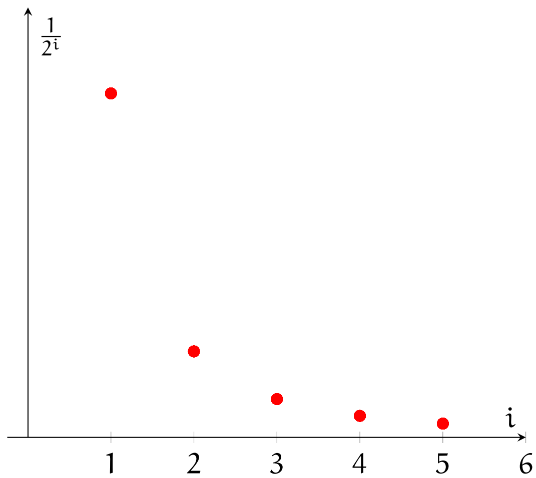


Figure 7.1: The first 5 terms of $\sum_{i=1}^{\infty} \frac{1}{2^i}$

We can overlay the function $y = \frac{1}{2^x}$ (figure 7.2). We can draw rectangles of width 1 and height $\frac{1}{x^2}$ (see figure 7.3). The area of the first n rectangles is equal to the n^{th} partial sum.

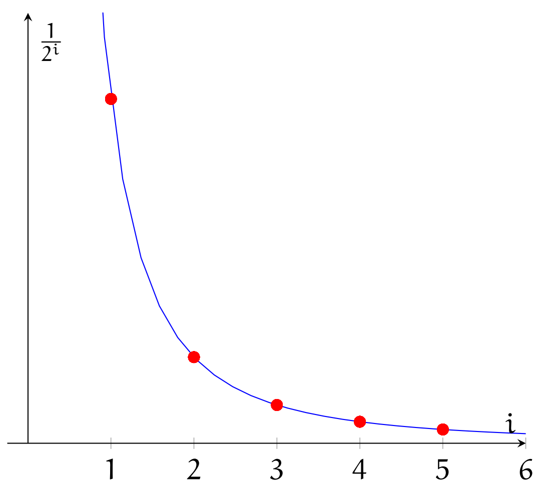


Figure 7.2: The first 5 terms of $\sum_{i=1}^{\infty} \frac{1}{2^i}$ lie on the curve $y = \frac{1}{x^2}$

This should remind you of a Riemann sum. Since the total area of the rectangles is less than the area under the curve, we can state:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < \int_0^{\infty} \frac{1}{x^2}\, dx$$

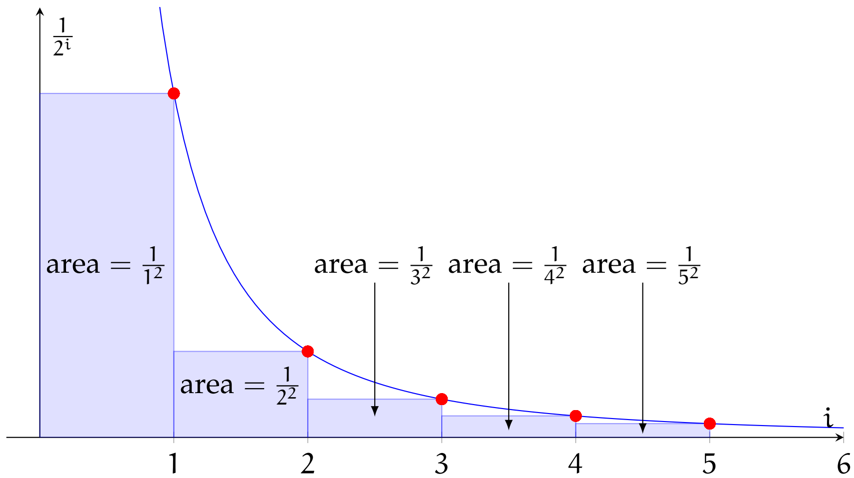


Figure 7.3: The partial sum $\sum_{i=1}^{n=5} \frac{1}{2^i}$ is equal to the area of the rectangles

We can exclude the first rectangle and also state that:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < 1 + \int_1^{\infty} \frac{1}{x^2}\, dx$$

We can evaluate this integral:

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{t\to\infty} \left[\int_1^t \frac{1}{x^2}\, dx \right]$$

$$= \lim_{t\to\infty} \frac{-1}{x}\Big|_{x=1}^{t} = \lim_{t\to\infty} \left(\frac{-1}{t} \right) - \frac{-1}{1} = 0 - (-1) = 1$$

And therefore:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < 1 + 1 = 2$$

This means the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is bounded above. Since the series is also monotonic (each term is positive, so the value of the sum increases as n increases), we can state that the sum is convergent!

Let's look at a divergent example: $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}}$. Again, we will make a visual, but this time we will draw rectangles that lie above the curve $y = \frac{1}{\sqrt{x}}$ (see figure 7.4). In this case, $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$. Let's evaluate the integral:

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx = \lim_{t\to\infty} \left[\int_1^t \frac{1}{\sqrt{x}}\, dx \right]$$

$$= \lim_{t\to\infty} \left[2\sqrt{x}\right]_{x=1}^{t} = \lim_{t\to\infty} \left(2\sqrt{t}\right) - 2\sqrt{1} = \infty - 2 \to \text{divergent}$$

Since the integral diverges to infinity and the series is greater than the integral, the series must also diverge to infinity. This is another case where a monotonic decreasing series is not convergent!

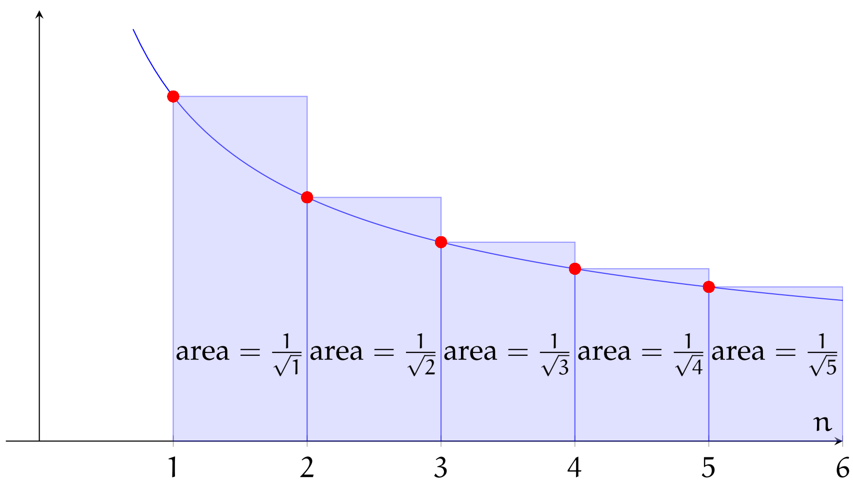


Figure 7.4: $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$

This leads us to the **Integral Test**. If f is a continuous, positive, decreasing function on the interval $x \in [1, \infty)$ and $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ is convergent. Subsequently, if $\int_1^{\infty} f(x)\,dx$ is divergent, then the series is also divergent.

Example: Is the series $\sum_{i=1}^{\infty} \frac{1}{n^2+1}$ convergent or divergent?

Solution: To apply the integral test, we define $f(x) = \frac{1}{x^2+1}$, which is a positive, decreasing function on the interval $x \in [1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} [\arctan x]_{x=1}^{t} = \lim_{t\to\infty} (\arctan t) - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Because the integral $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ converges, so does the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Exercise 19

Use the integral test to determine if the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} 2n^{-3}$
2. $\sum_{n=1}^{\infty} \frac{5}{3n-1}$
3. $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$

Working Space

Answer on Page 91

Exercise 20

Apply the Integral Test to show that p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ are convergent only when $p > 1$ (hint: consider the cases $p \leq 0$, $0 < p < 1$, $p = 1$ and $p > 1$).

Working Space

Answer on Page 91

7.2.1 Using Integrals to Estimate the Value of a Series

Recall that $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots = s$ and that the n^{th} partial sum, often represented as s_n, is $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$. Then we can define the n^{th} remainder $R_n = s - s_n$. Expanding s and s_n, we see that:

$$R_n = [a_1 + a_2 + \cdots + a_{n-1} + a_n + a_{n+1} + \cdots] - [a_1 + a_2 + \cdots + a_{n-1} + a_n]$$

$$R_n = [a_1 - a_1] + [a_2 - a_2] + \cdots + [a_{n-1} - a_{n-1}] + [a_n - a_n] + a_{n-1} + a_{n-2} + \cdots$$

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Just like the integral test, suppose there is some continuous, positive, decreasing function such that $a_n = f(n)$. The we can represent R_n as the right Riemann sum with width $\Delta x = 1$ from $x = n$ to ∞. Since the rectangles are below the curve (see figure 7.5), we can state that $R_n \leq \int_n^{\infty} f(x)\,dx$.

Similarly, we can represent R_n as the left Riemann sum with width $\Delta x = 1$ from $x = n + 1$ to ∞. This time the rectangles are above the curve (see figure 7.6), and we can state that $R_n \geq \int_{n+1}^{\infty} f(x)\,dx$. Putting this all together, we have an estimate for the remainder, R_n, from the integral test:

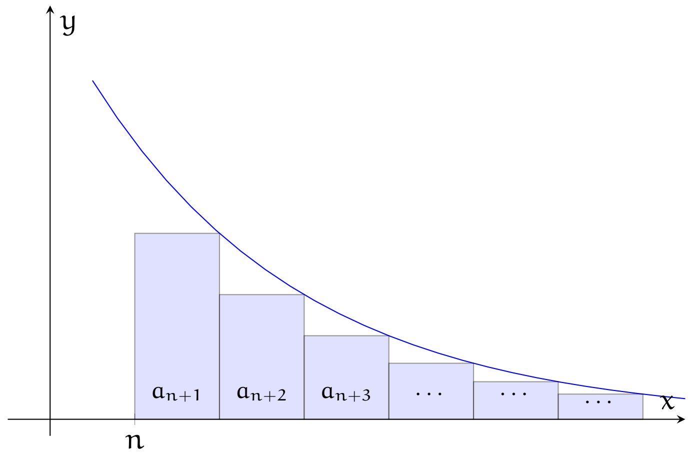


Figure 7.5: $R_n \le \int_n^\infty f(x)\,dx$

Suppose there is a function such that $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \ge n$ and $\sum a_n$ is convergent. Then, $\int_{n+1}^\infty f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx$, where R_n is $s - s_n$.

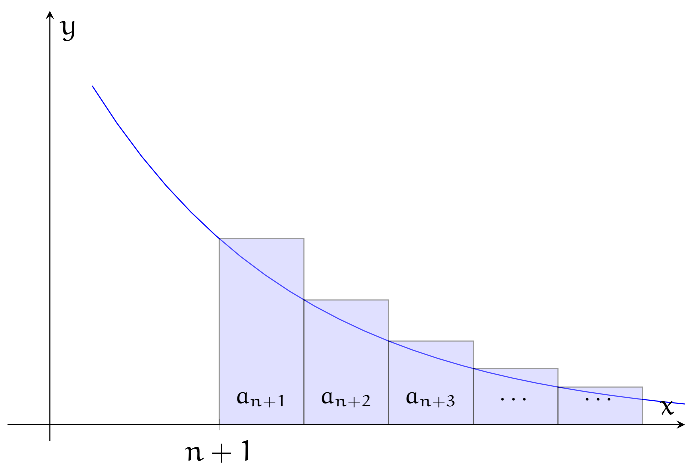


Figure 7.6: $R_n \ge \int_{n+1}^\infty f(x)\,dx$

Example: Approximate the sum of the series $\sum_{n=1}^\infty \frac{3}{n^3}$ by finding the 10^{th} partial sum. Estimate the error of this approximation.

Solution: Using a calculator, you can find the 10^{th} partial sum:

$$\sum_{n=1}^{10} \frac{3}{n^3} = \frac{3}{1^3} + \frac{3}{2^3} + \frac{3}{3^3} + \cdots + \frac{3}{10^3} \approx 3.593 = s_{10}$$

Recall that the remainder, R_{10} is the difference between the actual sum, s, and the partial sum, s_{10}. Using the integral test to estimate the remainder, we can state that:

$$R_{10} \leq \int_{10}^{\infty} \frac{3}{x^3}\, dx = \frac{3}{2(10)^2} = \frac{3}{200} = 0.015$$

Therefore, the size of the error is at most 0.015.

Example: How many terms are required for the error to be less than 0.0001 for the sum presented above?

Solution: We are looking for an n such that $R_n \leq 0.0001$. Recalling that $R_n \leq \int_n^{\infty} \frac{3}{x^3}\, dx$, we need to find an n such that $\int_n^{\infty} \frac{3}{x^3}\, dx \leq 0.0001$.

$$\begin{gathered}
\int_n^{\infty} \frac{3}{x^3}\, dx \leq 0.0001 \\
\frac{-1}{6x^2}\Big|_{x=n}^{\infty} \leq 0.0001 \\
\lim_{x\to\infty} \frac{-1}{6x^2} - \frac{-1}{6n^2} \leq 0.0001 \\
0 + \frac{1}{6n^2} = \frac{1}{6n^2} \leq 0.0001 \\
1 \leq 0.0006n^2 \\
1667 \leq n^2 \\
40.8 \leq n \to n = 41
\end{gathered}$$

Therefore, $s - s_{41} \leq 0.0001$ and the partial sum $\Sigma_{n=1}^{41} \frac{3}{n^3}$ is less than 0.0001 from the value of the infinite sum $\sum_{n=1}^{\infty} \frac{3}{n^3}$.

Exercise 21

Working Space

1. Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
2. Estimate the error from using s_{10} as an approximation of the series.
3. Use $s_n + \int_{n+1}^{\infty} \frac{1}{x^4}\,dx \leq s \leq s_n + \int_{n}^{\infty} \frac{1}{x^4}\,dx$ to give an improved estimate of the sum.
4. The actual value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is $\frac{\pi^4}{90}$. Compare your estimate with the actual value.
5. Find a value of n such that s_n is within 0.00001 of the sum.

Answer on Page 91

7.3 Comparison Tests

In comparison tests, we compare a series to a known convergent or divergent series. Take the series $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$. This is similar to $\sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a geometric series that converges to $\frac{1}{2}$. Notice that:

$$\frac{1}{3^n+3} < \frac{1}{3^n}$$

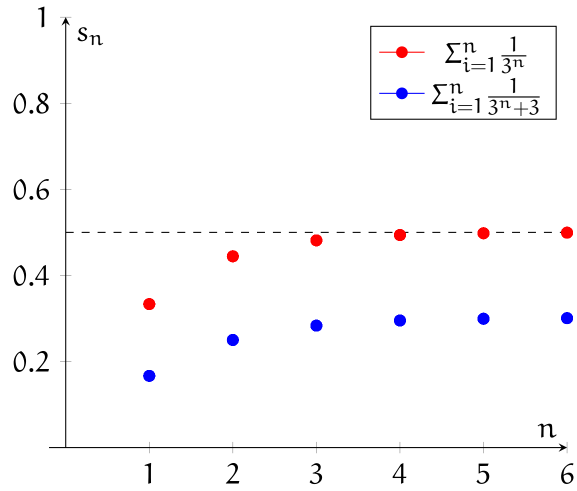


Figure 7.7: $\sum_{i=1}^{n} \frac{1}{3^n+3} < \Sigma_{i=1}^{n} \frac{1}{3^n}$ for all n

Which implies that

$$\sum_{n=1}^{\infty} \frac{1}{3^n+3} < \Sigma_{n=1}^{\infty} \frac{1}{3^n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent, it follows that $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$ is also convergent (see figure 7.7). As you can see, since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ approaches $\frac{1}{2}$, $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$ must be $\leq \frac{1}{2}$ and therefore convergent.

7.3.1 The Direct Comparison Test

F For the **Direct Comparison Test**, we compare the terms a_n to b_n directly. Take $\sum a_n$ and $\sum b_n$ to be series with positive terms. Then,

1. If $a_n \leq b_n$ and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
2. If $a_n \geq b_n$ and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

We already discussed why the first part is true above. The second part follows a similar argument: if a_n is greater than b_n, then you can imagine that as $\sum b_n$ grows and diverges, it is pushing upwards on $\sum a_n$, meaning that $\sum a_n$ must also diverge. Consider the series $\sum_{n=1}^{\infty} \frac{2\ln n}{n}$. For $n \geq 2$, $2\ln n > 1$, and therefore if $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\sum_{n=1}^{\infty} \frac{2\ln n}{n}$ must also diverge. We recognize the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore, $\sum_{n=1}^{\infty} \frac{2\ln n}{n}$ is also divergent (see figure 7.8).

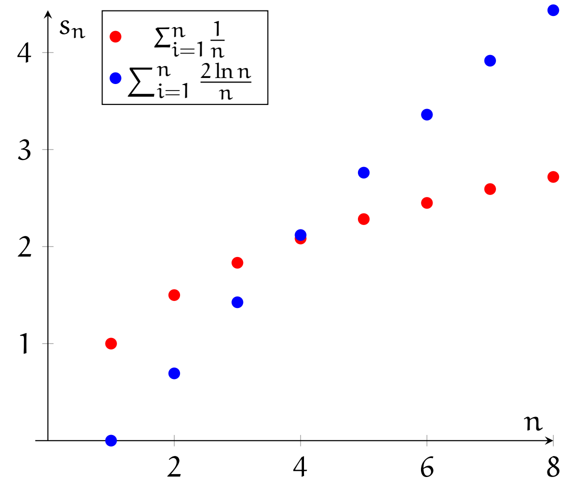


Figure 7.8: $\sum_{i=1}^{n} \frac{2\ln n}{n} > \sum_{i=1}^{n} \frac{1}{n}$ for $n \geq 4$

7.3.2 The Limit Comparison Test

Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. We may want to compare this to the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. The direct comparison test isn't helpful here, since $\frac{1}{2^n-1} > \frac{1}{2^n}$, so $\sum_{n=1}^{\infty} \frac{1}{2^n}$ doesn't put a cap on $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ like our earlier example (see figure 7.7). In a case such as this, we can use the **Limit Comparison Test**, which states that:
If $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both series diverge.

Let's apply this to the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. We know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, since it is a geometric series with $r < 1$.

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{2^n-1} \cdot \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = \frac{1}{1-0} = 1 > 0$$

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

In general, comparison tests are most useful for series resembling geometric or p-series. When choosing a p-series to compare the unknown series to, choose p such that the order of your p series is the same as the order of the unknown series.

Example: What p-series should one compare the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ to?

Solution: We can determine the order of $\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ by looking at the highest-order terms in the numerator and denominator:

$$\frac{\sqrt{n^3}}{n^3} = \frac{n^{3/2}}{n^3} = \frac{1}{n^{3/2}}$$

So we should compare $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

Example: Is $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ convergent or divergent?

Solution: We have already determine that we should compare this series to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. To apply the limit test, we need to evaluate

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n^3+1}}{3n^3+4n^2+2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^6+n^3}}{3n^3+4n^2+2} = \frac{1}{3} > 0$$

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ is convergent because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

Exercise 22

Use the Comparison Test or the Limit Comparison Test to determine if the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$
2. $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$
3. $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$

Working Space

Answer on Page 92

7.4 Ratio and Root Tests for Convergence

7.4.1 Absolute Convergence

Suppose there is a series $\sum_{n=1}^{\infty} a_n$, then there is a corresponding series $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent**.

Example: Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Is this series absolutely convergent?

Solution: We examine the corresponding series where we take the absolute value of each term:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We can identify $\sum_{n=1}^{\infty} \frac{1}{n^2}$ as a convergent p-series. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can state that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

Example Is the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ absolutely convergent?

Solution We consider the sum of the absolute values of the terms:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

You should recognize this as the harmonic series, which is divergent. When a series is convergent but the corresponding series of absolute values is not, we call it **conditionally convergent**.

We won't prove the theorem here, but it is useful to know that if a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent. You can prove this yourself using the Comparison Test.

Exercise 23

Is the series given by

$$\frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^3} + \cdots$$

convergent or divergent?

Working Space

Answer on Page 93

Exercise 24

Determine whether each of the following series is absolutely or conditionally convergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$
2. $\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{n^2+4}$

Working Space

Answer on Page 93

7.4.2 The Ratio Test

The ratio test compares the $(n+1)^{\text{th}}$ term of a series to the n^{th} term and takes the limit as $n \to \infty$ of the absolute value of this ratio:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

There are three possible outcomes of the ratio test:

1. if $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
2. if $L = 1$, then the ratio test is inconclusive and we cannot draw any conclusions about whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.
3. if $L > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Apply the ratio test to determine if $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ is convergent or divergent.

Solution:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{3^{n+1}}\cdot\frac{3^n}{n^3}$$

$$= \lim_{n\to\infty}\frac{(n+1)^3}{n^3}\cdot\frac{3^n}{3\cdot 3^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\cdot\frac{1}{3} = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

Since $L < 1$, the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ is absolutely convergent.

The ratio test is most useful for series that contain factorials, constants raised to the n^{th} power, or other products.

Exercise 25

[This question was originally presented as a multiple-choice, no-calculator problem on the 2012 AP Calculus BC exam.] Which of the following series are convergent?

1. $\sum_{n=1}^{\infty}\frac{8^n}{n!}$
2. $\sum_{n=1}^{\infty}\frac{n!}{n^{100}}$
3. $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)}$

Working Space

Answer on Page 93

Exercise 26

[This question was originally presented as a multiple-choice, calculator-allowed problem on the 2012 AP Calculus BC exam.]If the series $\Sigma_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n, which of the following statements must be true? Explain.

1. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$
2. $|a_n| < 1$ for all n
3. $\sum_{n=1}^{\infty} a_n = 0$
4. $\sum_{n=1}^{\infty} n a_n$ diverges
5. $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges

Working Space

Answer on Page 94

7.4.3 Root Test

The root test examines the behavior of the n^{th} root of a_n as $n \to \infty$. Similar to the ratio test, there are three possible outcomes:

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the Root Test is inconclusive.

The root test is best when there is a term or terms raised to the n^{th} power. Consider the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$:

Example: Is the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ convergent or divergent?

Solution: Since a_n consists of terms raised to the n^{th} power, we will apply the root test for convergence:

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

Therefore, by the root test, the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent.

Exercise 27

Use the Root Test to determine whether the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{n^2-4}\right)^n$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\ln n)^n}$
3. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

Working Space

Answer on Page 94

7.5 Strategies for Testing Series

When testing series for convergence, we want to choose a test based on the form of the series. While you may by tempted to try each test one-by-one until you find an answer, this quickly becomes cumbersome and time-consuming. Additionally, if you plan to take an AP Calculus exam, you need to be able to quickly choose an appropriate test as to conserve the time you have available for the exam. Here are some tips:

1. Check if the series is a p-series ($\sum_{n=1}^{\infty} \frac{1}{n^p}$). If so, then if $p > 1$ the series converges.

Otherwise, the series diverges.

2. If the series is not a p-series, check to see if you can write it as a geometric series ($\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^{n}$). Recall that geometric series are convergent if $|r| < 1$ and divergent otherwise.

3. If the series can't be written as a p-series or geometric series, but has a similar form, consider the comparison tests (the Direct Comparison Test and the Limit Comparison Test). When choosing a p-series to compare your series to, follow the guidelines outlined in the Comparison Tests section above.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then apply the Test for Divergence to show the series is divergent. REMEMBER: $\lim_{n\to\infty} a_n \neq 0$ implies the series $\sum_{n=1}^{\infty} a_n$ is divergent, but $\lim_{n\to\infty} a_n = 0$ DOES NOT necessarily imply the series $\sum_{n=1}^{\infty} a_n$ is convergent.

5. If the series is alternating (has $(-1)^n$ or $(-1)^{n-1}$ in the term), the Alternating Series test may provide an answer.

6. The Ratio Test is excellent for series with factorials, other products, or constants to the n^{th} power. Remember that the Ratio Test will be inconclusive for p-series, rational functions of n, and algebraic functions of n.

7. If a_n is of the form $(b_n)^n$, use the Root Test.

8. If $a_n = f(n)$ where $f(n)$ is continuous, positive, and decreasing and you can evaluate $\int_1^{\infty} f(x)\,dx$, use the Integral Test.

You don't need to treat this as a checklist, where you check for every condition. Rather, you should use this as a guide to quickly determine the convergence test most likely to be useful.

Exercise 28

Choose an appropriate test to determine if the series is convergent of divergent. Apply the test and classify the series as convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$
2. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
3. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
4. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
5. $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$

Working Space

Answer on Page 95

Chapter 8

Power Series

Consider the function $f(x) = \frac{1}{1-x}$. This looks similar to the value of convergent geometric series, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If we let $a = 1$ and $r = x$, then we see that $\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$. We can reindex to begin at $n = 0$ and see that:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is a **power series**. We use power series in place of functions for many applications: integrals where an explicit antiderivative don't exist, solving differential equations, and computer scientists representing functions on computers are a few examples. Consider $f(x) = \frac{1}{1-x^2}$. What is $\int f(x)\,dx$? We can't directly use u-substitution, and this isn't a derivative of any inverse trigonometric function. [You may have realized we could integrate this explicitly by using partial fractions, but this is not true for other functions, and we are using this as a demonstration anyway.] One way to evaluate this integral would be to represent $f(x)$ as a power series, then integrate the series. This is easier, since we know how to take the integral of any polynomial ($\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$). First, we discuss what power series are further.

8.1 Power Series

Power series are series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 = c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

for some fixed x. Depending on x, the series may converge or diverge. For example, the power series $\sum_{n=0}^{\infty} x^n$ converges for $-1 < x < 1$ and diverges for all other values of x. This is because $\sum_{n=0}^{\infty} x^n$ is essentially a geometric series with $r = x$, which we already know converges for $|r| < 1$.

The form given above is for a power series centered on 0, but a power series can be centered on any value, a. In that case, it looks like this:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

Which we say is a *power series in* $(x-a)$, or a *power series centered at* a, or a *power series about* a.

Example: Find a power series representation for $f(x) = \frac{2x-4}{x^2-4x+3}$.

Solution: Since we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we will use partial fractions to decompose the function into two fractions. (The process is left as an exercise for the student.) We find that:

$$\frac{2x-4}{x^2-4x+3} = \frac{1}{x-1} + \frac{1}{x-3}$$

Noting that $\frac{1}{x-1} = (-1) \cdot \frac{1}{1-x}$, we can say that:

$$\frac{1}{x-1} = (-1) \cdot \sum_{n=0}^{\infty} x^n$$

Now let's look at $\frac{1}{x-3}$. We can show that:

$$\frac{1}{x-3} = \frac{\frac{1}{3}}{\frac{x}{3}-1} = \frac{1}{3}\frac{1}{\frac{x}{3}-1} = \frac{-1}{3}\frac{1}{1-\frac{x}{3}}$$

Substituting $\frac{x}{3}$ for x into $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we see that:

$$\frac{1}{1-\frac{x}{3}} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

And therefore

$$\frac{1}{x-3} = \left(\frac{-1}{3}\right) \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Adding the terms, we see that:

$$\frac{1}{x-1} + \frac{1}{x-3} = (-1) \sum_{n=0}^{\infty} x^n + \left(\frac{-1}{3}\right) \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

8.2 Power Series Convergence

Sometimes, you'll be asked to find the value(s) of x for which a power series converges. To do this, choose a test to apply and then find x such that the test is passed.

Example: For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution: We will apply the Ratio Test (since there is a factorial in the series) and find x

such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$
$$= \lim_{n\to\infty}\frac{(n+1)n!x\cdot x^n}{n!x^n} = \lim_{n\to\infty}\frac{(n+1)\cdot x}{1}$$

Which converges to 0 when $x = 0$ and diverges for all other values of x. Therefore, $\sum_{n=0}^{\infty} n!x^n$ converges if $x = 0$.

Example: For what values of x does the series $\sum_{n=1}^{\infty}\frac{(x-4)^n}{2n}$ converge?

Solution: We will use the Ratio Test again. We are looking for an x such that:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{\frac{(x-4)^{n+1}}{2(n+1)}}{\frac{(x-4)^n}{2n}}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{(x-4)(x-4)^n}{2n+2}\cdot\frac{2n}{(x-4)^n}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{(x-4)(x-4)^n(2n)}{(x-4)^n(2n+2)}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{(x-4)(2n)}{2n+2}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{2(x-4)}{2+\frac{2}{n}}\right| < 1$$
$$2\cdot\lim_{n\to\infty}\left|\frac{x-4}{1+\frac{1}{n}}\right| < 1$$
$$2\cdot|x-4| < 1$$
$$|x-4| < \frac{1}{2}$$

Which is true when

$$-\frac{1}{2} < x-4 < \frac{1}{2}$$
$$3.5 < x < 4.5$$

We aren't done yet, though! We know the series converges for $3.5 < x < 4.5$ and diverges for $x < 3.5$ and $x > 4.5$. What about when $x = 3.5$ and $x = 4.5$? (These are the cases where the Ratio Test is indeterminate, because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.) We need to test each case. Substituting $x = 3.5$ into the series yields:

$$\sum_{n=1}^{\infty}\frac{(3.5-4)^n}{2n} = \sum_{n=1}^{\infty}\frac{\left(\frac{-1}{2}\right)^n}{2n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^{n+1}}$$

This is an alternating series, so we apply the alternating series test. First, we check that $|a_{n+1}| < |a_n|$:

$$\frac{1}{(n+1)\cdot 2^{n+2}} < \frac{1}{n\cdot 2^{n+1}}$$

Which is true for all $n > 0$. Now we check if $\lim_{n\to\infty}|a_n| = 0$:

$$\lim_{n\to\infty}\frac{1}{n\cdot 2^{n+1}} = \frac{1}{\infty} = 0$$

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^{n+1}}$ is convergent and $\sum_{n=1}^{\infty}\frac{(x-4)^n}{2n}$ is convergent for $x = 3.5$. Now we test $x = 4.5$ for convergence:

$$\sum_{n=1}^{\infty}\frac{(4.5-4)^n}{2n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\cdot\frac{1}{2n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n+1}\frac{1}{n}$$

This series is less than the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ for all n. We know the harmonic series diverges, therefore by the direct comparison test, $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n+1}\frac{1}{n}$ must also diverge. So our final answer to the original question is that the series is convergent fort $3.5 \le x < 4.5$.

8.2.1 Radius of Convergence

There are three possible outcomes when testing a power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ for convergence:

1. The series only converges for $x = a$
2. The series converges for all x
3. The series converges if $|x-a| < R$ and diverges for $|x-a| > R$, where R is some positive number

We call R the **radius of convergence** . If we rearrange $|x-a| < R$, we can see why this is called a radius (see figure 8.1):

$$a - R < x < a + R$$

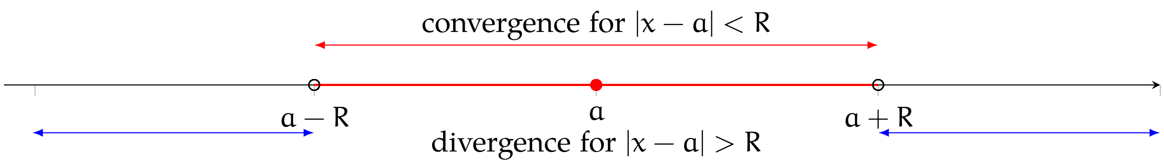


Figure 8.1: R is called the radius of convergence because it is half the width of the window of convergence

When $x = a \pm R$, the series could be convergent or divergent. You will need to test the endpoints of the windown of convergence to determine if the interval is open or closed. Thus, there are four possiblities for the interval of convergence:

1. $(a-R, a+R)$
2. $[a-R, a+R)$
3. $(a-R, a+R]$
4. $[a-R, a+r]$

In the example of $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n}$ (shown above), $a = 4$ and $R = 0.5$ and we found that the power series is convergent for $x \in [3.5, 4.5)$.

Example: For what values of x is the Bessel function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ convergent?

Solution: Because there is a factorial, we will apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}}{frac(-1)^n x^{2n}2^{2n}(n!)^2}\right| < 1$$

$$\lim_{n\to\infty}\frac{x^{2n}x^2 2^{2n} n!n!}{2^{2n}2^2(n=1)!(n+1)!x^{2n}} < 1$$

$$\lim_{n\to\infty}\frac{x^2 n!n!}{2^2(n+1)n!(n+1)n!} < 1$$

$$\lim_{n\to\infty}\frac{x^2}{4(n+1)^2} = 0 < 1$$

Because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ for all x, the Bessel function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ is convergent for all real values of x and the interval of convergence is $(-\infty, \infty)$.

Example: Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Solution: Again, we apply the ratio test to find values of x such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$:

$$\lim_{n\to\infty} \left| \frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+1+1}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{(-3)x}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{1} \right| < 1$$

$$\lim_{n\to\infty} \left| (-3)x\sqrt{\frac{n+2}{n+1}} \right| < 1$$

$$3|x| \lim_{n\to\infty} \sqrt{\frac{n+2}{n+1}} < 1$$

$$3|x| \lim_{n\to\infty} \sqrt{\frac{1+2/n}{1+1/n}} = 3|x|(1) < 1$$

$$3|x| < 1$$

$$|x| < \frac{1}{3}$$

Therefore, the radius of convergence is $\frac{1}{3}$. We need to test the endpoints, $x = \frac{-1}{3}$ and $x = \frac{1}{3}$ to determine the interval of convergence. First, we will test if $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ when $x = \frac{-1}{3}$:

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{-1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

This is a p-series such that $p < 1$, so it is divergent and our original series does not converge for $x = \frac{-1}{3}$. Next we test $x = \frac{1}{3}$:

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

Which is an alternating series that converges by the alternating series test. Therefore, the interval of convergence for $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ is $x \in \left(\frac{-1}{3}, \frac{1}{3}\right]$.

Exercise 29

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] What is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-4)^{2n}}{3^n}$?

Working Space

Answer on Page 95

Exercise 30

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] A power series is given by $\frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \cdots$. Write the series in sigma notation and use the Ratio Test to determine the interval of convergence.

Working Space

Answer on Page 96

8.3 Calculus with Power Series

You can integrate and differentiate power series. Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. Recall that $f(x)$ is just a very long polynomial and that the derivative of a polynomial x^n is $n \cdot x^{n-1}$. Then we can state that:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots + c_n(x-a)^n\right]$$

$$f'(x) = 0 + c_1 + 2c_2(x-a)^1 + \cdots + nc_n(x-a)^{n-1}$$

$$f'(x) = \sum_{n=1}^{\infty} c_n(x-a)^{n-1}$$

Which is true when x is in the interval of convergence for the series.

Similarly, we know $\int x^n\,dx = \frac{1}{n+1}x^{n+1}$. Then we can say that:

$$\int f(x)\,dx = \int \left[c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots + c_n(x-a)^n\right], dx$$

$$\int f(x)\,dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots \frac{c_n}{n+1}(x-a)^{n+1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

Where C is the integration constant. Again, this is true when x is in the interval of convergence for the series.

Example: Express $\frac{1}{(1-x)^2}$ as a power series by differentiating $\frac{1}{1-x}$.

Solution: Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$.

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

Differentiating both sides:

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$(-1)\cdot\frac{1}{(1-x)^2}\cdot\frac{d}{dx}(1-x) = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Reindexing to begin at $n = 0$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

Because $\sum_{n=0}^{\infty}$ has a radius of convergence of 1, so does $\sum_{n=0}^{\infty}(n+1)x^n$. We can confirm our series makes sense by plotting the partials sums for $n = 3, 5, and 7$ with the original function (see figure 8.2).

Example: Find a power series representing $\ln(1+x)$.

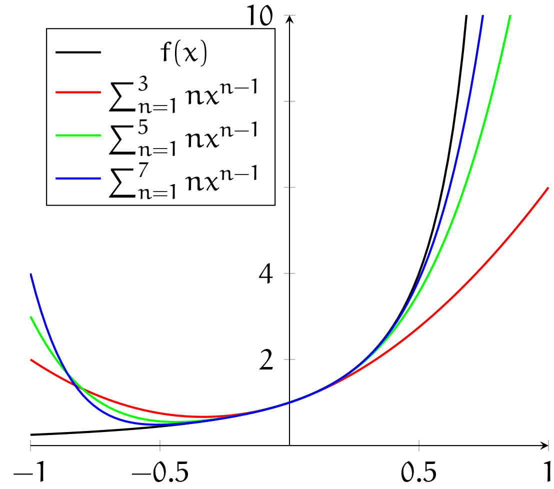


Figure 8.2: The function $f(x) = \frac{1}{(1-x)^2}$ is equal to the power series $\sum_{n=1}^{\infty} nx^{n-1}$

Solution: We know that $\frac{d}{dx}\frac{1}{1-x} = \ln(1-x)$. Replacing x with $-x$ we see that:

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$$

Which converges for $|x| < 1$. Then we can integrate both sides:

$$\int \frac{1}{1+x}\,dx = \int \left[\sum_{n=0}^{\infty}(-x)^n\right] dx$$

$$\ln(1+x) = \int \left(1 - x + x^2 - x^3 + \cdots\right) dx$$

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + C$$

When $|x| < 1$. To find C, substitute $x = 0$ and solve:

$$\ln(1+0) = C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{0^n}{n} = C + 0$$

$$C = \ln 1 = 0$$

So our final answer is $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$.

Exercise 31

Find a power series representation for $f(x) = \arctan x$.

Working Space

Answer on Page 97

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 8)

1. 4200
2. Logically, we can say that the population will increase if it is below the carrying capacity (that is, $P < 4200$), but we can also prove it mathematically: $\frac{dP}{dt} < 0 \rightarrow 1.2P\left(1 - \frac{P}{4200}\right) < 0 \rightarrow P\left(1 - \frac{P}{4200}\right) < 0$. Since we are talking about population, we can assume that $P > 0$ and continue: $1 - \frac{P}{4200} < 0 \rightarrow 1 < \frac{P}{4200} \rightarrow 4200 < P$, which is the result we expected.
3. Similarly, we know the population should be decreasing when P is greater than the carrying capacity of 4200.
4. The equilibrium solutions can be found by setting $\frac{dP}{dt} = 0$ and solving. The solutions are $P(t) = 0$ and $P(t) = 4200$.

Answer to Exercise 2 (on page 9)

Recall that logistic differential equations are of the form $\frac{dy}{dt} = ky(1 - \frac{y}{m})$ where y is a function and t is the independent variable. (e) is the only logistic differential equation, with $m = 1$.

Answer to Exercise 3 (on page 11)

$$\frac{dy}{dx}dx = \frac{3x^2}{2y + \sin y}dx$$

$$(2y + \sin y)(dy) = \frac{3x^2}{2y + \sin y}(2y + \sin y)(dx)$$

$$(2y + \sin y)dy = (3x^2)dx$$

$$\int 2y\,dy + \int \sin y\,dy = \int 3x^2\,dx$$

$$y^2 - \cos y = x^3 + C$$

Answer to Exercise 4 (on page 12)

1. Since $\frac{dB}{dt}$ depends only on B, we can use the given masses to find the rate of growth for each mass. $\frac{dB}{dt}(40) = \frac{1}{5}(100-40) = \frac{1}{5}(60) = 12$ and $\frac{dB}{dt}(70) = \frac{1}{5}(100-70) = \frac{1}{5}(30) = 6$. Since $\frac{dB}{dt}$ is greater when $B = 40$, the baby bird is gaining mass faster when it has a mass of 40 grams.

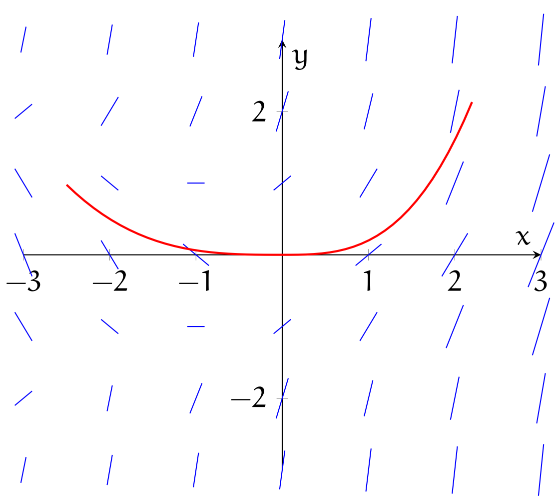
2. $\frac{d^2B}{dt^2} = \frac{d}{dt}\left(\frac{dB}{dt}\right) = \frac{d}{dt}\left[\frac{1}{5}(100-B)\right] = \frac{1}{5}\left(-\frac{dB}{dt}\right) = \frac{-1}{5}\left[\frac{1}{5}(100-B)\right] = -\frac{1}{25}(100-B)$. For $20 < B < 100$, $\frac{d^2B}{dt^2} < 0$ and the graph of B should be concave down. The graph shown has a concave up portion, so it cannot represent $B(t)$.

3. $\frac{dB}{dt} = \frac{1}{5}(100-B) \to \frac{dB}{100-B} = \frac{1}{5}dt \to \int(100-B)\ dB = \int \frac{1}{5}\,dt \to -\ln 100 - B = \frac{t}{5} + C \to e^{\frac{-t}{5}+C} = 100 - B \to ke^{\frac{-t}{5}} = 100 - B \to B(t) = 100 - ke^{\frac{-t}{5}}$. Setting $B(0) = 20$ to find k: $20 = 100 - ke^0 \to 20 = 100 - k \to k = 80$. So the particular solution is $B(t) = 100 - 80e^{\frac{-t}{5}}$

Answer to Exercise 5 (on page 14)

(A). (a), (b), and (c) are all separable equations. But only the solution to A is linear ($P(t) = 200t + C$). (d) is logarithmic, or natural growth and (e) is also not linear.

Answer to Exercise 6 (on page 20)



Answer to Exercise 7 (on page 25)

Substituting the given values, we find that $(5)\frac{dQ}{dT} + \frac{1}{0.05}Q = 60$. Solving for $\frac{dQ}{dt}$:

$$(5)\frac{dQ}{dt} + (20)Q = 60$$

$$\frac{dQ}{dt} + 4Q = 12$$

$$\frac{dQ}{dt} = 12 - 4Q$$

We also know that $Q(0) = 0$. Using Euler's method with step size $h = 0.1$, $Q(0.1) \approx Q(0) + h\,[12 - 4Q(0)] = 0 + 0.1\,[12 - 4(0)] = 1.2$. And $Q(0.2) \approx Q(0.1) + h\,[12 - 4Q(0.1)] = 1.2 + 0.1\,[12 - 4(1.2)] = 1.92$. And $Q(0.3) \approx Q(0.2) + h\,[12 - 4Q(0.2)] = 1.92 + 0.1\,[12 - 4(1.92)] = 2.352$. And $Q(0.4) \approx Q(0.3) + h\,[12 - 4Q(0.3)] = 2.352 + 0.1\,[12 - 4(2.352)] = 2.6112$. And finally, $Q(0.5) \approx Q(0.4) + h\,[12 - 4Q(0.4)] = 2.6112 + 0.1\,[12 - 4(2.6112)] = 2.76672$. Because we are finding a charge, the unit it Coulombs (C), so our final answer is $Q(0.5) \approx 2.77C$.

Answer to Exercise 16 (on page 52)

We are given $x_0 = 1$ and $x_2 = 1.4$. Therefore we will use step size $h = \frac{1.4-1}{2} = \frac{0.4}{2} = 0.2$. Taking $x_0 = 1$ and $y_0 = f(1) = 15$, we find y_1: $y_1 = y_0 + h \cdot f'(x_0) = 15 + 0.2 \cdot f'(1) = 15 + 0.2(8) = 15 + 1.6 = 16.6$. And then $y_2 = y_1 + h \cdot f'(x_1) = 16.6 + 0.2 \cdot f'(1.2) = 16.6 + 0.2(12) = 16.6 + 2.4 = 19$. Therefore, $f(1.4) \approx 19$.

Answer to Exercise 9 (on page 30)

1. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$
2. 0, -1, 0, 1, 0
3. 1, 2, 7, 32, 157
4. 6, 3, 1, $\frac{1}{4}, \frac{1}{20}$

Answer to Exercise 10 (on page 32)

1. convergent, 5

2. divergent
3. convergent, 2
4. convergent, -1
5. divergent

Answer to Exercise 11 (on page 38)

Out principal is $P = 1500$ and the interest rate is $r = 0.06$. After n years, your investment will be worth $a_n = 1500(1.06)^n$. For $n = 10$, your investment will be valued at $a_{10} = \$1500(1.06)^{10} = \$2686.27$ (that's over \$1000 in interest!). To determine if the sequence is convergent or divergent, we examine the limit as $n \to \infty$:

$$\lim_{n\to\infty} 1500(1.06)^n = 1500 \cdot \lim_{n\to\infty}(1.06)^n = 1500 \cdot \infty = \infty$$

The sequence is divergent.

Answer to Exercise 12 (on page 39)

The number of fish in the pond is:

$$P_n = P_{n-1}(0.95)^6 + 50$$

$$P_0 = 100$$

where n is the number of 6-month periods that have passed. The four year period is given by $1 \leq n \leq 8$. The amount lost to predation every 6 months is given by $P_{n-1}(1 - 0.95^6)$.

n	Fish Population	Lost to Predators
0	100	
1	84	26
2	71	22
3	62	19
4	56	17
5	51	15
6	48	14
7	45	13
8	43	12

Adding up all the fish lost to predators, we find that over 4 years the farmer loses 138 fish.

Answer to Exercise 13 (on page 50)

1. We need to identify a and r. If we use the form $\sum_{n=1}^{\infty} ar^{n-1}$, then $a = 3$. To find the common ratio, we can evaluate $\frac{a_{n+1}}{a_n} = \frac{-4}{3}$. Then we can write the series as $\sum_{n=1}^{\infty} 3\left(\frac{-4}{3}\right)^{n-1}$. In this case, $r = \frac{-4}{3}$ and $|r| \geq 1$, and therefore the series is divergent.

2. Following the process outlined above, we see that $a = 2$ and $r = \frac{1}{4}$. Therefore the series is $\sum_{n=1}^{\infty} 2\left(\frac{1}{4}\right)^{n-1}$. Since $|r| < 1$, the series converges to $\frac{a}{1-r} = \frac{2}{1-1/4} = \frac{2 \cdot 4}{3} = \frac{8}{3}$

3. We need to rewrite the series into a standard from in order to identify a and r:
$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4(4)^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{-3}{4}\right)^{n-1}$$
So $r = \frac{-3}{4}$ and $|r| < 1$. Therefore, the series converges to $\frac{1/4}{1-(-3/4)} = \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}$

4. We need to rewrite the series into a standard from in order to identify a and r:
$$\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)^n}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)(e^2)^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} e^2\left(\frac{e^2}{6}\right)^{n-1}$$
Therefore, $r = \frac{e^2}{6} \approx 1.232$. Since $|r| > 1$, the series diverges.

Answer to Exercise 14 (on page 50)

We want to rewrite this as a geometric series of the form $\sum_{n=i}^{\infty} ar^{n-1}$ so we can use the fact that the sum of a convergent geometric series is $\frac{a}{1-r}$. $\sum_{n=0}^{\infty}(1+c)^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{1+c}\right)^n = \sum_{n=1}^{\infty}\left(\frac{1}{1+c}\right)^{n-1}$. This is a geometric series with $a = 1$ and $r = \frac{1}{1+c}$. So the value of the series is $\frac{1}{1-\frac{1}{1+c}} = \frac{1}{\frac{c}{c+1}} = \frac{c+1}{c}$. Setting this equal to $\frac{5}{3}$ and solving for c, we find that $c = \frac{3}{2}$.

Answer to Exercise 15 (on page 50)

$-2 < p < 2$ Let's re-write this geometric series into standard form: $\sum_{n=1}^{\infty}\left(\frac{p}{2}\right)^n = \sum_{n=1}^{\infty}\frac{p}{2}\left(\frac{p}{2}\right)^n$ which means $a = \frac{p}{2}$ and $r = \frac{p}{2}$. We know that geometric series converge if $|r| < 1$, so we set up an inequality and solve for p:
$$\left|\frac{p}{2}\right| < 1$$
$$-1 < \frac{p}{2} < 1$$
$$-2 < p < 2$$

Answer to Exercise 16 (on page 52)

1. Separating the terms, we see that $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4} = \sum_{n=1}^{\infty} \left(\frac{n^2}{n^4} + \frac{1}{n^4}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{6} + \frac{\pi^4}{90}$
2. Notice that this series starts at $n = 2$. By the properties of series, we know that $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$. Therefore, $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) - \frac{1}{1^2} = \frac{\pi^2}{6} - 1$
3. We can begin by reindexing this series: $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2}$. Similar to the previous problem, we also know that $\sum_{n=4}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) = \frac{\pi^2}{6} - \frac{49}{36}$
4. We can re-write this series as $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4 = \sum_{n=1}^{\infty} (3^4) \frac{1}{n^4} = 81 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{81\pi^4}{90} = \frac{9\pi^4}{10}$
5. We can re-write the series as $\sum_{n=1}^{\infty} \left(\frac{4}{n^2} + \frac{3}{n^4}\right) = \sum_{n=1}^{\infty} \frac{4}{n^2} + \sum_{n=1}^{\infty} \frac{3}{n^4} = 4\sum_{n=1}^{\infty} \frac{1}{n^2} + 3\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{4\pi^2}{6} + \frac{3\pi^4}{90} = \frac{2\pi^2}{3} + \frac{\pi^4}{30}$

Answer to Exercise 17 (on page 53)

This is a p-series where $p = 2k$. We know that p-series converge for $p > 1$: $2k > 1 \to k > \frac{1}{2}$.

Answer to Exercise 18 (on page 54)

1. The series is convergent if $\left|\frac{(-1)^{n+1}3(n+1)}{4(n+1)-1}\right| < \left|\frac{(-1)^n 3n}{4n-1}\right|$ if $\frac{3n+3}{4n+4-1} < \frac{3n}{4n-1}$ and if $\frac{3n+3}{4n+3} < \frac{3n}{4n-1}$ if $(3n+3)(4n-1) < (3n)(4n+3)$ if $12n^2 + 12n - 3n - 3 < 12n^2 + 9n$ if $-3 < 0$ which is true. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is convergent.
2. The series is convergent if $\left|(-1)^{n+1+1} \frac{(n+1)^2}{(n+1)^3+1}\right| < \left|(-1)^{n+1} \frac{n^2}{n^3+1}\right|$ which is true if $\frac{(n+1)^2}{(n+1)^3+1} < \frac{n^2}{n^3+1}$ if $(n+1)^2(n^3+1) < (n^2)((n+1)^3+1)$ if $(n^2+2n+1)(n^3) < (n^2)(n^3+3n^2+3n+1+1)$ if $n^5 + 2n^4 + n^3 < n^5 + 3n^4 + 3n^3 + 2n^2$ which is true for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is convergent.
3. The series is convergent if $\left|(-1)^{n-1+1} e^{2/(n+1)}\right| < \left|(-1)^{n-1} e^{2/n}\right|$ which is true if $e^{2/(n+1)} < e^{2/n}$ which is true if $\frac{2}{n+1} < \frac{2}{n}$ which is true for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$ is convergent.

Answer to Exercise 19 (on page 59)

1. The function $2x^{-3}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^\infty 2x^{-3}\,dx = \lim_{t\to\infty} \int_1^t 2x^{-3}\,dx = \lim_{t\to\infty} \left[-x^{-2}\right]_{x=1}^{t} = \lim_{t\to\infty}(-t^{-2}) - -(1)^{-2} = 0+1 = 1$. Since the integral $\int_1^\infty 2x^{-3}\,dx$ converges, the series $\sum_{n=1}^\infty 2n^{-3}$ is also convergent.

2. The function $\frac{5}{3x+1}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^\infty \frac{5}{3x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{5}{3x-1}\,dx$ Using u-substitution to evaluate the integral, we set $u = 3x - 1$ and find that $du = 3dx \to dx = \frac{du}{3}$. Substituting, $\int_1^t \frac{5}{3x-1}\,dx = \int_{x=1}^{x=t} \frac{5}{3}\frac{1}{u}\,du$. Evaluating the integral, $\int_{x=1}^{x=t} \frac{5}{3}\frac{1}{u}\,du = \frac{5}{3}\ln u|_{x=1}^{x=t} = \frac{5}{3}\ln 3x + 1|_1^t$. Substituting this back into the limit, $\int_1^\infty \frac{5}{3x-1}\,dx = \lim_{t\to\infty} \frac{5}{3}\ln 3x + 1|_1^t = \lim_{t\to\infty}[\frac{5}{3}\ln 3t + 1] - \frac{5}{3}\ln 4 = \infty - \frac{5}{3}\ln 4 = \infty$. Therefore, the integral $\int_1^\infty \frac{5}{3x-1}\,dx$ is divergent and so is the series $\sum_{n=1}^\infty \frac{5}{3n-1}$

3. The function $\frac{x}{3x^2+1}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^\infty \frac{x}{3x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{3x^2+1}\,dx$. Applying the substitution $u = 3x^2+1$ and $\frac{du}{6} = x\,dx$, we see that $\lim_{t\to\infty} \int_1^t \frac{x}{3x^2+1}\,dx = \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{6u}\,du = \lim_{t\to\infty} \frac{1}{6}\ln u|_{x=1}^{x=t} = \lim_{t\to\infty} \frac{1}{6}\ln 3x^2 + 1|_1^t = \lim_{t\to\infty} \left[\frac{1}{6}\ln 3t^2 + 1\right] - \frac{1}{6}\ln 4 = \infty$. Therefore the integral $\int_1^\infty \frac{x}{3x^2+1}\,dx$ is divergent and so is the series $\sum_{n=1}^\infty \frac{n}{3n^2+1}$.

Answer to Exercise 20 (on page 60)

1. If $p \le 0$, then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, and the series fails the Test for Divergence. Therefore, a p-series is divergent if $p \le 0$.

2. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on the interval $x \in [1, \infty)$ and we can apply the integral test. So we want to know, when is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent? When $p = 1$, $\int_1^\infty \frac{1}{x^p}\,dx = \ln x|_{x=1}^{x=\infty} = \lim_{t\to\infty} \ln t - \ln 1 = \infty$ and the integral and p-series are both divergent.

3. What about when $0 < p < 1$? Then the integral $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}x^{1-p}|_{x=1}^{x=t} = \lim_{t\to\infty} \frac{1}{1-p}\frac{1}{x^{p-1}} = \left(\frac{1}{1-p}\right)\left[\lim_{t\to\infty}\left(\frac{1}{t^{p-1}}\right) - 1\right]$. When $0 < p < 1$, then $1 - p > 0$ is positive and $\lim_{t\to\infty} \frac{1}{t^{p-1}} = \lim_{t\to\infty} t^{1-p} = \infty$ and the integral diverges. Therefore, p-series are divergent for $0 < p < 1$.

4. When $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx = \left(\frac{1}{1-p}\right)\left[\lim_{t\to\infty}\left(\frac{1}{t^{p-1}}\right) - 1\right]$. When $p > 1$, $p-1 > 0$ and $\lim_{t\to\infty} \frac{1}{t^{p-1}} = 0$. Therefore, $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$ when $p > 1$, and therefore the p-series is convergent when $p > 1$.

Answer to Exercise 21 (on page 63)

1. $s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \cdots + \frac{1}{10^4} \approx 1.082037$.
2. $R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4}\,dx = \frac{-1}{3x^3}\Big|_{x=10}^{\infty} = \lim_{x\to\infty} \frac{-1}{3x^3} - \frac{-1}{3\cdot 10^3} = \frac{1}{3000} = 0.000333$. Therefore, the error is less than 0.000333.
3. Given $s_{10} \approx 1.082037$, we can say that $1.082037 + \int_{n+1}^{\infty} \frac{1}{x^4}\,dx \leq s \leq 1.082037 + \int_{n}^{\infty} \frac{1}{x^4}\,dx$. Using a calculator to evaluate each integral, we see that: $1.082037 + 0.000250 \leq s \leq 1.082037 + 0.000333$ and therefore the sum is between 1.082287 and 1.082370.
4. Writing the actual value as a decimal, $\frac{\pi^4}{90} \approx 1.082323$, which is in the estimate window from the previous part.
5. We are looking for an n such that $\int_{n}^{\infty} \frac{1}{x^4}\,dx \leq 0.00001$. $\lim_{x\to\infty} \frac{-1}{3x^3} - \frac{-1}{3n^3} = \frac{1}{3n^3} \leq 0.00001$. $100,000 \leq 3n^3$. $33,333.33 \leq n^3$. $32.183 \leq n$. Since n must be an integer, $n = 33$ gives $R_n \leq 0.00001$.

Answer to Exercise 22 (on page 67)

1. This is similar to $\sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent. Unfortunately, $\frac{1}{n} > \frac{1}{\sqrt{n^2+1}}$, so we can't use the direct comparison test. We will try the limit comparison test:

$$\lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} \right) = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} \right) = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = \frac{1}{1+0} = 1 > 0$$

   Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

2. This series is similar to the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$. Given that:

$$\left(\frac{9}{10}\right) = \frac{9^n}{10^n} < \frac{9^n}{3+10^n}$$

   Since $\frac{9^n}{3+10^n} < \left(\frac{9}{10}\right)^n$ and $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is convergent, by the direct comparison test, $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ is also convergent.

3. We can compare this to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Noting that $\sin^2 n \leq 1$:

$$\frac{n \sin^2 n}{1+n^3} < \frac{n \sin^2 n}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

   Because $\frac{n\sin^2 n}{1+n^3} \leq \frac{1}{n^2}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can state by the direct comparison test that $\sum_{n=1}^{\infty} \frac{n\sin^2 n}{1+n^3}$ is also convergent.

Answer to Exercise 23 (on page 68)

We can write the series as $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$. Since n is real, we know that $n^2 > 0$ and we can say that $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$. Additionally, $|\cos n| \leq 1$ for all n, and therefore $\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$. We know the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. And since we have shown that $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$, by the comparison test $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent. Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and therefore convergent.

Answer to Exercise 24 (on page 69)

1. Conditionally Convergent. $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{3n+2}\right| = \sum_{n=1}^{\infty} \frac{1}{3n+2}$ Applying the integral test to this sum: $\int_1^{\infty} \frac{1}{3x+2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{3x+2}\,dx = \left[\frac{1}{3}\ln 3x+2\right]_{x=1}^{t} = \lim_{t\to\infty} [\ln 3x+2] - \ln 3(1) - 2 = \infty - 0 = \infty$. Since $\int_1^{\infty} \frac{1}{3x+2}\,dx$ is divergent, $\sum_{n=1}^{\infty} \frac{1}{3n+2}$ is divergent and $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$ is conditionally convergent.

2. Absolutely Convergent. $\sum_{n=1}^{\infty} \left|\frac{\sin n}{4^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{4^n}$. Applying the integral test to $\sum_{n=1}^{\infty} \frac{1}{4^n}$: $\int_1^{\infty} \frac{1}{4^x}\,dx = \lim_{t\to\infty} \frac{-1}{4^x \ln 4}\Big|_{x=1}^{t} = \lim_{t\to\infty} \left[\frac{-1}{4^t \ln 4}\right] - \frac{-1}{4^1 \ln 4} = 0 + \frac{1}{4\ln 4} = \frac{1}{4\ln 4}$. Since $\int_1^{\infty} \frac{1}{4^x}\,dx$ is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is also convergent. And since $\sum_{n=1}^{\infty} \left|\frac{\sin n}{4^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{4^n}$, $\sum_{n=1}^{\infty} \left|\frac{\sin n}{4^n}\right|$ is also convergent, which shows that $\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$ is absolutely convergent.

3. Conditionally Convergent. We are asking if the series $\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{2n}{n^2+4}\right|$ is convergent. $\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{2n}{n^2+4}\right| = \sum_{n=1}^{\infty} \frac{2n}{n^2+4}$ We will apply the Limit Comparison test and compare this series to the known, divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n\to\infty} \frac{\frac{2n}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2}{n^2+4} = 2 > 0$. Therefore, by the Limit Comparison test, $\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{2n}{n^2+4}\right|$ is divergent AND $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{2n}{n^2+4}$ is conditionally convergent.

Answer to Exercise 25 (on page 70)

Series 1 and 3 converge

1. We apply the ratio test: $\lim_{n\to\infty} \left|\frac{\frac{8^{n+1}}{(n+1)!}}{\frac{8^n}{n!}}\right| = \lim_{n\to\infty} \frac{8\cdot 8^n}{(n+1)(n!)} \cdot \frac{n!}{8^n} = \lim_{n\to\infty} \frac{8}{n+1} = 0$. Therefore, the series converges.

2. We apply the ratio test: $\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{100}}}{\frac{n!}{n^{100}}}\right| = \lim_{n\to\infty}\frac{(n+1)n!}{(n+1)^{100}}\cdot\frac{n^{100}}{n!} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{100}\cdot (n+1) = \lim_{n\to\infty}\frac{n^{100}}{(n+1)^{99}} = \infty$. Therefore, the series diverges.

3. We apply the comparison test: $\frac{n+1}{(n)(n+2)(n+3)} = \frac{n}{(n)(n+2)(n+3)} + \frac{1}{(n)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} + \frac{1}{(n)(n+2)(n+3)} = \frac{1}{n^2+5n+6} + \frac{1}{n^3+5n^2+6n} \leq \frac{1}{n^2} + \frac{1}{n^3}$. The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^3}$ are both convergent, because they are p-series with $p > 1$. Having established that $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{n^3}$ and that $\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{n^3}$ converges, by the comparison test we can state that $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)}$ converges.

Answer to Exercise 26 (on page 71)

1. This is not necessarily true. For a convergent series, the result of the ratio test is $L < 1$, so the limit could be $\neq 0$.

2. This is not necessarily true. Consider the geometric series $\sum_{n=1}^{\infty} 2(\frac{1}{2})^{n-1}$. This series is convergent because the common ratio is less than one, but the first term is $2(\frac{1}{2})^0 = 2 > 1$.

3. This is not necessarily true. Again, consider the geometric series $\sum_{n=1}^{\infty} 2(\frac{1}{2})^{n-1}$, which converges to $4 \neq 0$.

4. This is not necessarily true. Consider the p-series $\sum_{n=1}^{\infty}\frac{1}{n^4}$. Then the series $\sum_{n=1}^{\infty} n\frac{1}{n^4} = \sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent.

5. This must be true. By the comparison test, $\sum_{n=1}^{\infty}\frac{a_n}{n} \leq \sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} a_n$ converges, so much $\sum_{n=1}^{\infty} n a_n$.

Answer to Exercise 27 (on page 72)

1. $\lim_{n=1}^{\infty}\sqrt[n]{\left|\left(\frac{3n^2+1}{n^2-4}\right)^n\right|} = \lim_{n=1}^{\infty}\frac{3n^2+1}{n^2-4} = 3 > 1$. Therefore, the series $\sum_{n=1}^{\infty}\left(\frac{3n^2+1}{n^2-4}\right)^n$ is divergent.

2. $\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty}\frac{1}{\ln n} = \frac{1}{\infty} = 0 < 1$. Therefore, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{(\ln n)^n}$ is convergent.

3. $\lim_{n\to\infty}\sqrt[n]{\left|(1+\frac{1}{n})^{n^2}\right|} = \lim_{n\to\infty}(1+\frac{1}{n})^n = e > 1$. Therefore, $\sum_{n=1}^{\infty}(1+\frac{1}{n})^{n^2}$ is divergent.

Answer to Exercise 28 (on page 74)

1. Divergent. Since there is a constant to the n^{th} power and an algebraic function of n, we will try the Ratio Test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{e^{n+1}}{(n+1)^2}\cdot\frac{n^2}{e^n} = \lim_{n\to\infty}\frac{e^n\cdot e}{e^n}\cdot\left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} e\cdot\left(\frac{n}{n+1}\right)^2 = e\cdot 1^2 = e > 1$. Therefore, $\sum_{n=1}^{\infty}\frac{e^n}{n^2}$ is divergent.

2. Convergent. Since there is a factorial, we will try the Ratio Test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2} = \lim_{n\to\infty}\frac{3\cdot 3^n}{3^n}\cdot\frac{n!}{(n+1)n!}\cdot\left(\frac{n+1}{n}\right)^2 = \lim_{n\to\infty}\frac{3(n+1)^2}{(n+1)n^2} = \lim_{n\to\infty}\frac{3(n+1)}{n^2} = 0 < 1$. Therefore, the series $\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$ is convergent.

3. Divergent. Since $\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx$ can be integrated, we will apply the integral test. $\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty}\int_2^t\frac{1}{x\sqrt{\ln x}}\,dx$. Setting $u = \ln x$, then $du = \frac{dx}{x}$ and $\frac{1}{x\sqrt{\ln x}}dx = \frac{1}{\sqrt{u}}du$. Then we can say that $\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty}\int_{x=2}^{x=t}\frac{1}{\sqrt{u}}\,du = \lim_{t\to\infty}\left(\frac{-1}{2}\right)\sqrt{u}|_{x=2}^{x=t} = \lim_{t\to\infty}\left(\frac{-1}{2}\right)\sqrt{\ln x}_2^t = \left(\frac{-1}{2}\right)\lim_{t\to\infty}\sqrt{\ln t} - \left(\frac{-1}{2}\right)\sqrt{\ln 2} = \infty$. Since the integral diverges, so does the series.

4. Convergent. Since this series has terms to the n^{th} power, we will try the Root Test. $\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n}{n+1}\right)^{n^2}\right|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n+2}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$ Therefore, by the root test, the series is convergent.

5. Convergent. This series also has terms raised to the n^{th} power, we will try the Root Test again. $\lim_{n\to\infty}\sqrt[n]{\left|\left(\sqrt[n]{2}-1\right)^n\right|} = \lim_{n\to\infty}\sqrt[n]{\left(\sqrt[n]{2}-1\right)^n} = \lim_{n\to\infty}\left(\sqrt[n]{2}-1\right)^{n/n} = \lim_{n\to\infty}\left(\sqrt[n]{2}-1\right) = \lim_{n\to\infty}2^{1/n}-1 = 1-1 = 0 < 1$. Therefore, the series converges.

Answer to Exercise 29 (on page 81)

Since this sum has terms to the n^{th} power, we will apply the Root Test, which states a series is convergent if $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$.

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(x-4)^{2n}}{3^n}\right|} < 1$$

$$\lim_{n\to\infty}\left|\frac{(x-4)^{2n/n}}{3^{n/n}}\right| < 1$$

$$\lim_{n\to\infty} \frac{\left|(x-4)^2\right|}{3} < 1$$

$$(x-4)^2 < 3$$

$$|x-4| < \sqrt{3}$$

Therefore, the radius of convergence is $\sqrt{3}$.

Answer to Exercise 30 (on page 81)

We see that the series is alternating, so we know it involves $(-1)^n$ (we will begin indexing at $n=0$). The powers of x are given by x^{2n+1} and the denominators are given by $2n+3$. Therefore, the sum in sigma notation is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3}$. Applying the ratio test, the series is convergent when:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2(n+1)+1}}{2(n+1)+3} \frac{2n+3}{(-1)^n x^{2n+1}} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+3}}{2n+5} \frac{2n+3}{x^{2n+1}} \right| < 1$$

$$\left|x^2\right| \lim_{n\to\infty} \frac{2n+3}{2n+5} < 1$$

$$\left|x^2\right| < 1$$

$$|x| < 1$$

So we know that the series is convergent on the open interval $x \in (-1,1)$. We check the endpoints, $x = -1, 1$ for convergence.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+3} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+3}$$

When $x = -1$, the series is an alternating series such that $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$. Therefore, the series converges for $x = -1$.

$$\sum_{n=1}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3}$$

which is also an alternating series such that $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$. Therefore, the series converges for $x = 1$ and the interval of convergence is $x \in [-1, 1]$, which can also be written as $-1 \le x \le 1$.

Answer to Exercise 31 (on page 84)

Recall that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$. Replacing x with $-x^2$, we see that $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left[-x^2\right]^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. Then we can also say that $\arctan x = \int \frac{1}{1+x^2}\, dx = \int \left[sum_{n=0}^{\infty}(-1)^n x^{2n}\right]\, dx$. Evaluating the integral, $\int \left[sum_{n=0}^{\infty}(-1)^n x^{2n}\right]\, dx = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Knowing that $\arctan 0 = 0$, we find that $C = 0$ and $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

Index