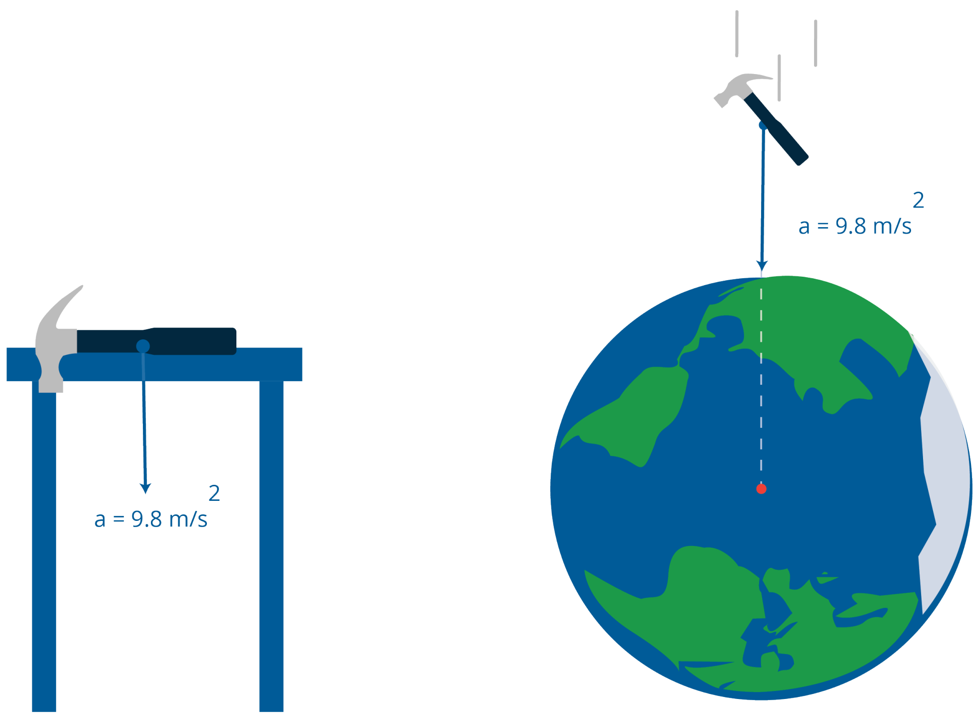
Contents

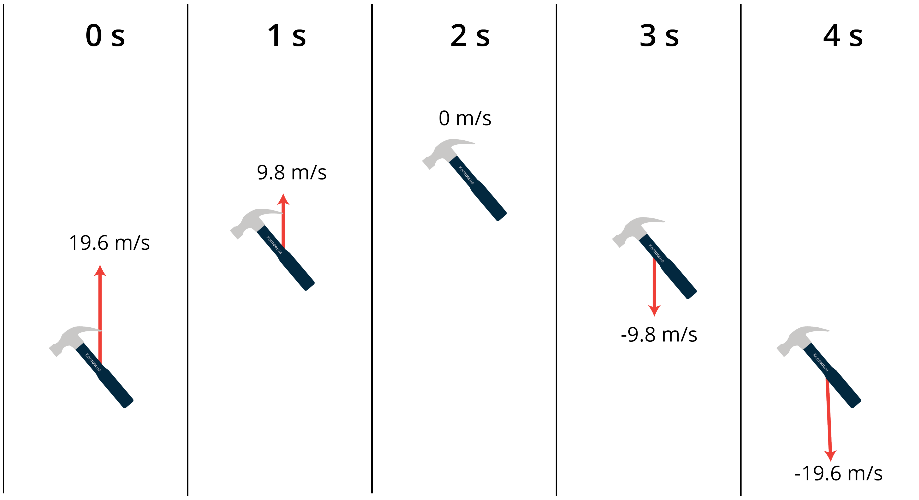
Chapter 1

Falling Bodies

Because of gravity, if you throw a hammer straight up in the air, from the moment it leaves your hand until it hits the ground, it is accelerating toward the center of the earth at a constant rate.



Acceleration can be defined as change in velocity. If the hammer leaves your hand with a velocity of 12 meters per second upward, one second later it will be rising, and its velocity will have slowed to 2.2 meters per second. One second after that, the hammer will be falling at a rate of 7.6 meters per second. Every second the hammer's velocity is changing by 9.8 meters per second, and that change is always toward the center of the earth. When the hammer is going up, gravity is slowing it down by 9.8 meters per second, each second it is in the air. When the hammer is coming down, gravity is speeding it up by 9.8 meters per second.



Acceleration due to gravity on earth is a constant negative 9.8 meters per second per second:

$$a = -9.8$$

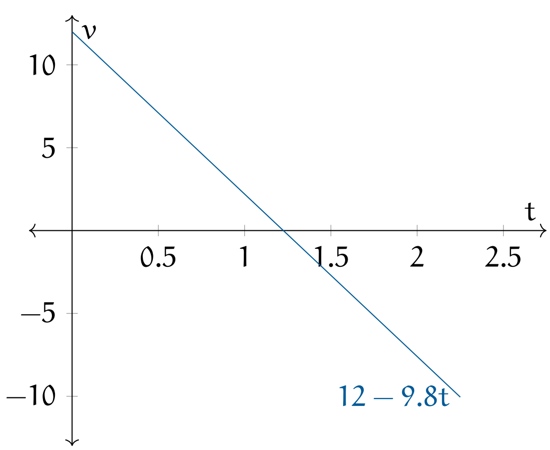
(Why is it negative? We are talking about height, which increases as you go away from the center of the earth. Acceleration is changing the velocity in the opposite direction.)

1.1 Calculating the Velocity

Given that the acceleration is constant, it makes sense that the velocity is a straight line. Assuming once again that the hammer leaves your hand at 12 meters per second, then the upwards velocity at time t is given by:

$$v = 12 - 9.8t$$

Note that the velocity of the hammer is being given as a function. Here is its graph:



Exercise 1 When is the apex of flight?

Given the hammer's velocity is given by $12-9.8t$, at what time (in seconds) does it stop rising and begin to fall?

Working Space

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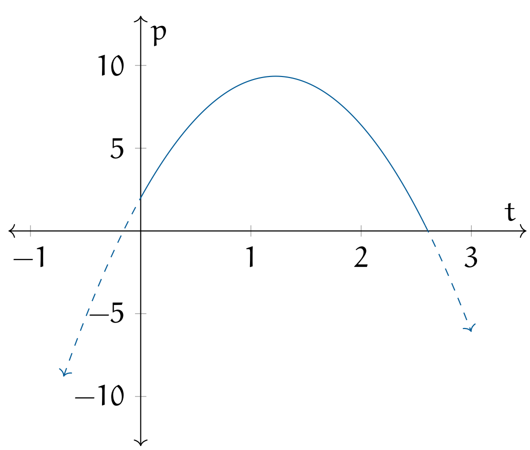
At this point, we need to acknowledge air resistance. Gravity is not the only force on the hammer; as it travels through the air, the air tries to slow it down. This force is called *air resistance*, and for a large, fast-moving object (like an airplane) it is GIGANTIC force. For a dense object (like a hammer) moving at a slow speed (what you generate with your hand), air resistance doesn't significantly affect acceleration.

1.2 Calculating Position

If you let go of the hammer when it is 2 meters above the ground, the height of the hammer is given by:

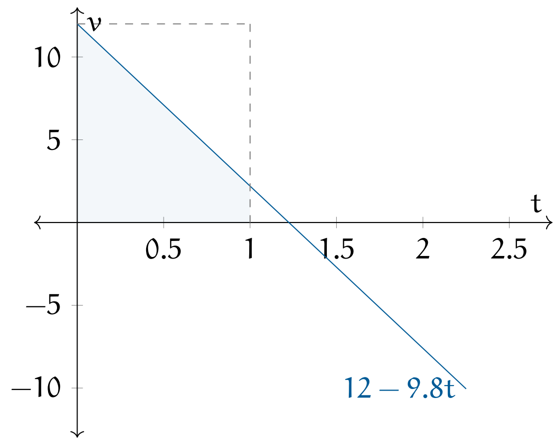
$$p = -\frac{9.8}{2}t^2 + 12t + 2$$

Here is a graph of this function:



How do we know? **The change in position between time 0 and any time t is equal to the area under the velocity graph between $x = 0$ and $x = t$.**

Let's use the velocity graph to figure out how much the position has changed in the first second of the hammer's flight. Here's the velocity graph with the area under the graph for the first second filled in:



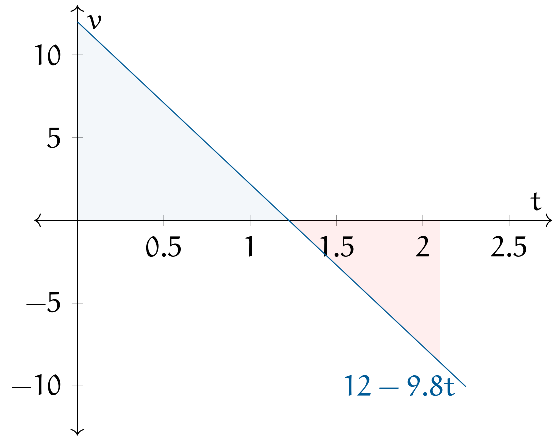
The blue filled region is the area of the dashed rectangle minus that empty triangle in its upper left. The height of the rectangle is twelve and its width is the amount of time the hammer has been in flight (t). The triangle is t wide and $9.8t$ tall. Thus, the area of the blue region is given by $12t - \frac{1}{2}9.8t^2$.

That's the change in position. Where was it originally? 2 meters off the ground. So the height is given by $p = 2 + 12t - \frac{1}{2}9.8t^2$. We usually write terms so that the exponent decreases, so:

$$p = -\frac{1}{2}9.8t^2 + 12t + 2$$

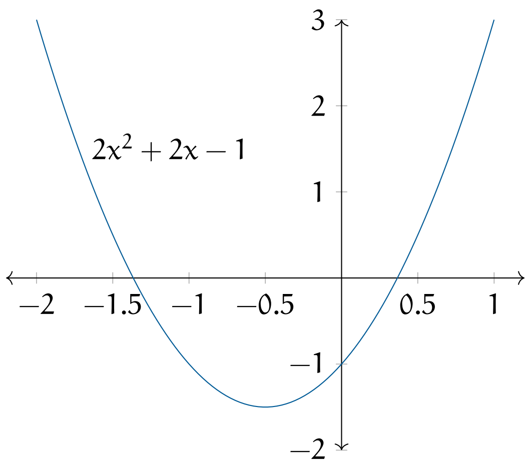
Finding the area under the curve like this is called *integration*. We say "To find a function that gives the change in position, we just integrate the velocity function." A lot of the study of calculus is learning to integrate different sorts of functions.

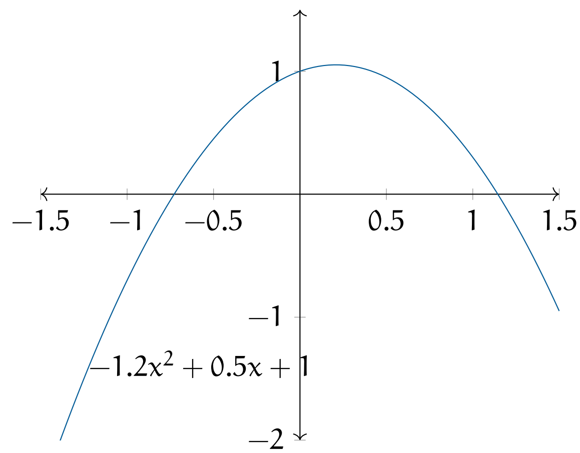
One important note about integration: Any time the curve drops under the x-axis, the area is considered negative. (Which makes sense, right? If the velocity is negative, the hammer's position is decreasing.)



1.3 Quadratic functions

Functions of the form $f(x) = ax^2 + bx + c$ are called *quadratic functions*. If $a > 0$, the ends go up. If $a < 0$, the ends go down.





The graph of a quadratic function is a *parabola*.

1.4 Simulating a falling body in Python

Now you are going to write some Python code that simulates the flying hammer. First, we are just going to print out the position, speed, and acceleration of the hammer for every 1/100th of a second after it leaves your hand. (Later we will make a graph.)

Create a file called `falling.py` and type this into it:

```
# Acceleration on earth
acceleration = -9.8 # m/s/s

# Size of time step
time_step = 0.01 # seconds

# Initial values
speed = 12  # m/s upward
height = 2  # m above the ground
current_time = 0.0  # seconds after release

# Is the hammer still aloft?
while height > 0.0:

    # Show the values
    print(f"{current_time:.2f} s:")
    print(f"\tacceleration: {acceleration:.2f} m/s/s")
    print(f"\tspeed: {speed:.2f} m/s")
    print(f"\theight: {height:.2f} m")

    # Update height
    height = height + time_step * speed

    # Update speed
    speed = speed + time_step * acceleration

    # Update time
    current_time = current_time + time_step


print(f"Hit the ground: Complete")
```

When you run it, you will see something like this:

```
0.00 s:
        acceleration: -9.80 m/s/s
        speed: 12.00 m/s
        height: 2.00 m
0.01 s:
        acceleration: -9.80 m/s/s
        speed: 11.90 m/s
        height: 2.12 m
0.02 s:
        acceleration: -9.80 m/s/s
        speed: 11.80 m/s
```

```
        height: 2.24 m
0.03 s:
        acceleration: -9.80 m/s/s
        speed: 11.71 m/s
        height: 2.36 m
...
2.60 s:
        acceleration: -9.80 m/s/s
        speed: -13.48 m/s
        height: 0.20 m
2.61 s:
        acceleration: -9.80 m/s/s
        speed: -13.58 m/s
        height: 0.07 m
Hit the ground: Complete
```

Note that the acceleration isn't changing at all, but it is changing the speed, and the speed is changing the height.

We can see that the hammer in our simulation hits the ground just after 2.61 seconds.

1.4.1 Graphs and Lists

Now, we are going to graph the acceleration, speed, and height using a library called matplotlib. However, to make the graphs, we need to gather all the data into lists.

For example, we will have a list of speeds, and the first three entries will be 12.0, 11.9, and 11.8.

We create an empty list and assign it to a variable like this:

```
x = []
```

Then we can add items like this:

```
x.append(3.14)
```

To get the first time back, we can ask for the object at index 0.

```
y = x[0]
```

Note that the list starts at 0. So if you have 32 items in the list, the first item is at index 0.

The last item is at index 31.

Duplicate the file `falling.py` and name the new copy `falling_graph.py`

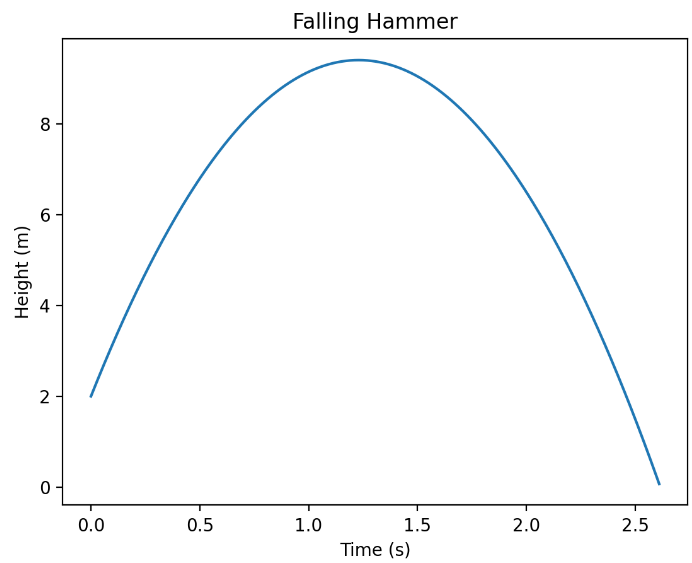
We are going to make a plot of the height over time. At the start of the program, you will import the matplotlib library. At the end of the program, you will create a plot and show it to the user.

In `falling_graph.py`, add the bold code:

```
import matplotlib.pyplot as plt

# Acceleration on earth
acceleration = -9.8 # m/s/s

# Size of time step
time_step = 0.01 # seconds

# Initial values
speed = 12  # m/s upward
height = 2  # m above the ground
current_time = 0.0  # seconds after release

# Create empty lists
accelerations = []
speeds = []
heights = []
times = []

# Is the hammer still aloft?
while height > 0.0:

    # Add the data to the lists
    times.append(current_time)
    accelerations.append(acceleration)
    speeds.append(speed)
    heights.append(height)

    # Update height
    height = height + time_step * speed

    # Update speed
    speed = speed + time_step * acceleration

    # Update time
    current_time = current_time + time_step
```

```
# Make a plot
fig, ax = plt.subplots()
fig.suptitle("Falling Hammer")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Height (m)")
ax.plot(times, heights)
plt.show()
```

When you run the program, you should see a graph of the height over time.

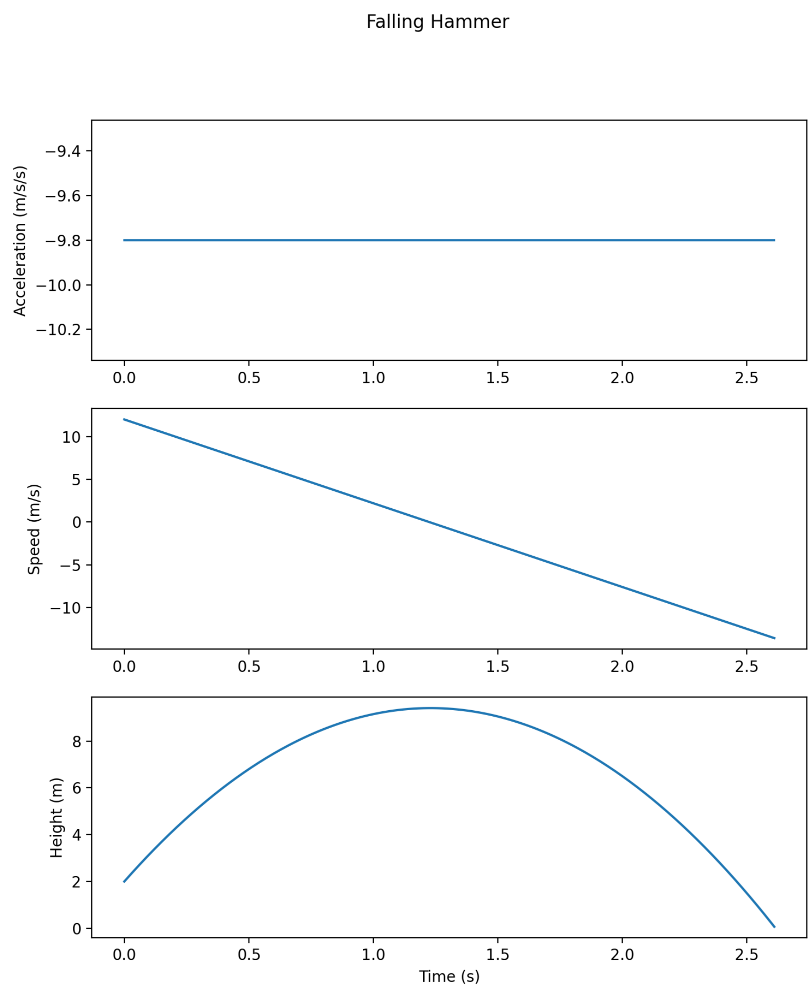


It is more interesting if we can see all three: acceleration, speed, and height. So lets make three stacked plots. Change the plotting code in `falling_graph.py` to:

```
# Make a plot with three subplots
fig, ax = plt.subplots(3,1)
fig.suptitle("Falling Hammer")

# The first subplot is acceleration
ax[0].set_ylabel("Acceleration (m/s/s)")
ax[0].plot(times, accelerations)

# Second subplot is speed
ax[1].set_ylabel("Speed (m/s)")
ax[1].plot(times, speeds)

# Third subplot is height
```

```
ax[2].set_xlabel("Time (s)")
ax[2].set_ylabel("Height (m)")
ax[2].plot(times, heights)
plt.show()
```

Now you will get plots of all three variables:



This is what we expected, right? The acceleration is a constant negative number. The speed is a straight line with a negative slope. The height is a parabola.

A natural question at this point is "When exactly will the hammer hit the ground?" That is, when does $\mathtt{height} = 0$? The values of t where a function is zero are known as its *roots*. Height is given by a quadratic function. In the next chapter, you will get the trick for finding the roots of any quadratic function.

Chapter 2

Solving Quadratics

A quadratic function has three terms: ax^2+bx+c. a, b, and c are known as the *coeffcients*. The coefficients can be any constant, except that a can never be zero. (If a is zero, it is a linear function, not a quadratic.)

When you have an equation with a quadratic function on one side and a zero on the other, you have a quadratic equation. For example:

$$72x^2 - 12x + 1.2 = 0$$

How can you find the values of x that will make this equation true?

You can always reduce a quadratic equation so that the first coefficient is 1, so that your equation looks like this:

$$x^2 + bx + c = 0$$

For example, if you are asked to solve $4x^2 + 8x - 19 = -2x^2 - 7$

$$4x^2 + 8x - 19 = -2x^2 - 7$$

$$6x^2 + 8x - 12 = 0$$

$$x^2 + \frac{4}{3}x - 2 = 0$$

Here, $b = \frac{4}{3}$ and $c = -2$.

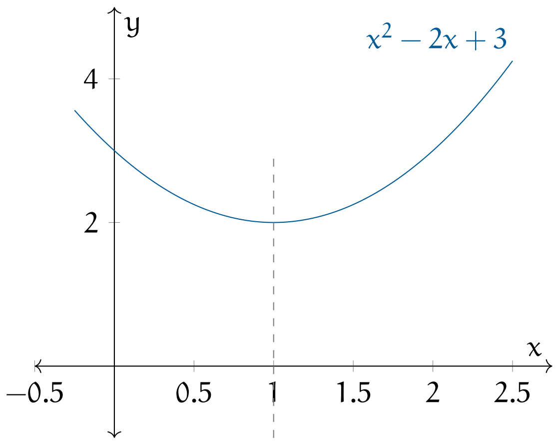
$x^2 + bx + c = 0$ when

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

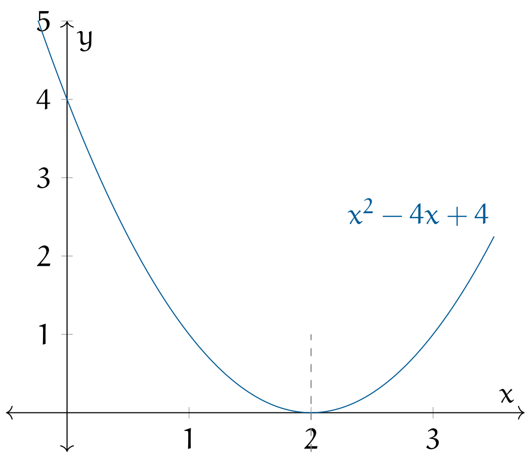
What does this mean?

For any b and c, the graph of $x^2 + bx + c$ is a parabola that goes up on each end. Its low point is at $x = -\frac{b}{2}$.

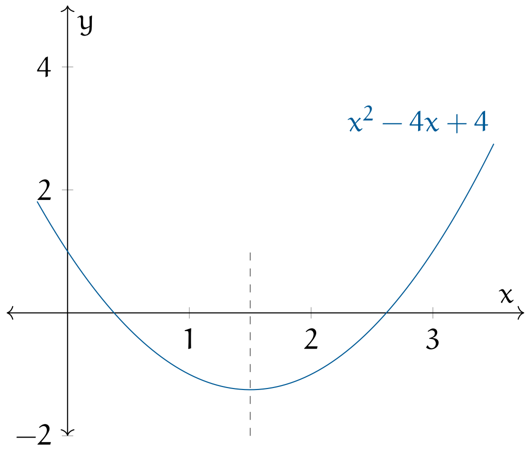
If there are no real roots ($b^2 - 4c < 0$), which means the parabola never gets low enough to cross the x-axis:



If there is one real root $(b^2 - 4c = 0)$, it means that the parabola just touches the x-axis.



If there are two real roots $(b^2 - 4c > 0)$, it means that the parabola crosses the x-axis twice as it dips below and then returns:



Exercise 2 Roots of a Quadratic

Working Space

In the last chapter, you found that the function for the height of your flying hammer is:

$$p = -\frac{1}{2}9.8t^2 + 12t + 2$$

At what time will the hammer hit the ground?

Answer on Page 27

2.1 The Traditional Quadratic Formula

If the last explanation was a little tricky to understand the quadratic formula is a nifty tool.

The Quadratic Formula

$ax^2 + bx + c = 0$ when

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Chapter 3

Complex Numbers

Complex numbers are an extension of the real numbers, which in turn are an extension of the rational numbers. In mathematics, the set of complex numbers is a number system that extends the real number line to a full two dimensions, using the imaginary unit which is denoted by i, with the property that $i^2 = -1$.

3.1 Definition

A complex number is a number of the form $a + bi$, where a and b are real numbers, and i is the imaginary unit, with the property that $i^2 = -1$. The real part of the complex number is a, and the imaginary part is b.

3.2 Why Are Complex Numbers Necessary?

Complex numbers are essential to many fields of science and engineering. Here are a few reasons why:

3.2.1 Roots of Negative Numbers

In the real number system, the square root of a negative number does not exist because there is no real number that you can square to get a negative number. The introduction of the imaginary unit i, which has the property that $i^2 = -1$, allows us to take square roots of negative numbers and gives rise to complex numbers.

3.2.2 Polynomial Equations

The fundamental theorem of algebra states that every non-constant polynomial equation with complex coefficients has a complex root. This theorem guarantees that polynomial equations of degree n always have n roots in the complex plane.

3.2.3 Physics and Engineering

In physics and engineering, complex numbers are used to represent waveforms, in control systems, in quantum mechanics, and many other areas. Their properties make many mathematical manipulations more convenient.

3.3 Adding Complex Numbers

The addition of complex numbers is straightforward. If we have two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, their sum is defined as:

$$z_1 + z_2 = (a + c) + (b + d)i \tag{3.1}$$

In other words, you add the real parts to get the real part of the sum, and add the imaginary parts to get the imaginary part of the sum.

3.4 Multiplying Complex Numbers

The multiplication of complex numbers is a bit more involved. If we have two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, their product is defined as:

$$z_1 \cdot z_2 = (a + bi) \cdot (c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i \tag{3.2}$$

Note the last term comes from $i^2 = -1$. You multiply the real parts and the imaginary parts just as you would in a binomial multiplication, and remember to replace i^2 with -1.

Chapter 4

Introduction to Sequences

A sequence is a list of numbers in a particular order. {1, 3, 5, 7, 9} is a sequence. So is $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$. There are many types of sequences. We will present two of the most common types in this chapter: arithmetic and geometric sequences.

Sequences are generally represented like this:

$$a_1, a_2, a_3, a_4, ..., a_n, ...$$

The first number, a_1, is called the *first term*, a_2 is the *second term*, and a_n is the *nth term*. A sequence can be finite or infinite. If the sequence is infinite, we represent that with ellipses ($\cdots$) at the end of the list, to indicate that there are more numbers.

We can also write formulas to represent a sequence. Take the first example, the finite sequence {1, 3, 5, 7, 9}. Notice that each term is two more than the previous term. We can define the sequence *recursively* by defining the n^{th} term as a function of the $(n-1)^{th}$ term. In our example, we see that $a_n = a_{n-1} + 2$ with $a_1 = 1$ for $1 \leq n \leq 5$. This is called a recursive formula, because you have to already know the $(n-1)^{th}$ term to find the n^{th} term.

Another way to write a formula for a sequence is to find a rule for the n^{th} term. In our example sequence, the first term is 1 plus 0 times 2, the second term is 1 plus 1 times 2, the third term is 1 plus 2 times 2, and so one. Did you notice the pattern? The n^{th} term is 1 plus (n-1) times 2. We can write this mathematically:

$$a_n = 1 + 2(n-1) \text{ for } 1 \leq n \leq 5$$

This is called the *explicit* formula because each term is explicitly defined. Notice, for the second way of writing a formula, we don't have to state what the first term is: the formula tells us.

4.1 Arithemtic sequences

Our first example sequence, {1, 3, 5, 7, 9} is a *finite*, *arithmetic* sequence. We know it is finite because there is a limited number of terms in the sequence (in this case, 5). How do we know it is arithmetic?

An arithmetic sequence is one where you add the same number every time to get the next term. Our example is an arithmetic sequence because you add 2 to get the next term every time. That number that you add is called the *common difference*, so we can say the sequence {1, 3, 5, 7, 9} has a common difference of 2. The common difference can be positive (in the case of an increasing arithmetic sequence) or negative (in the case of a decreasing arithmetic sequence). Formally, we can find the common difference of an arithmetic sequence by subtracting the $(n-1)^{th}$ term from the n^{th} term:

$$d = a_n - a_{n-1}$$

Exercise 3

Which of the following are arithmetic sequences? For the arithmetic sequences, find the common difference.

1. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, ...\}$
2. {5, 8, 11, 14, 17, ...}
3. {3, -1, -5, -9, ...}
4. {-1, 2, -3, 4, -5, 6, ...}

Working Space

Answer on Page 27

4.1.1 Formulas for arithmetic sequences

If you are given an arithmetic sequence, you can write an explicit or recursive formula. You can think of the formula as a function where the domain (input) is restricted to integers greater than or equal to one. Let's write explicit and recursive formulas for the sequence {3, -1, -5, -9, ...}.

For either type of formula, we need to identify the common difference. Since each term is 4 less than the previous term, the common difference is -4 (see figure 4.1). This means the n^{th} term is the $(n-1)^{th}$ term minus 4. The general form of a recursive formula is $a_n = a_{n-1} + d$, where d is the common difference. For our example, the common difference is -4, so we can write a recursive formula:

$$a_n = a_{n-1} - 4$$

However, this formula doesn't tell us what a_1 is! For recursive formulas, you have to specify the first term in the sequence. So the *complete* recursive formula for the sequence is:

$$a_n = a_{n-1} - 4$$

$$a_1 = 3$$

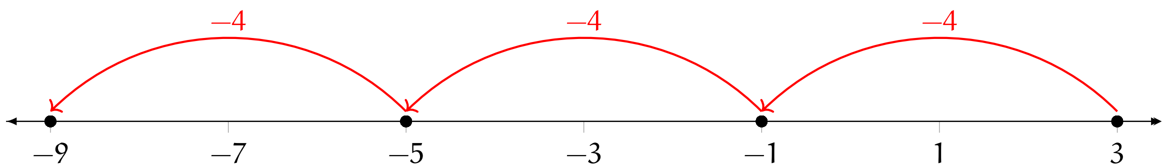


Figure 4.1: The common difference in the sequence {3, -1, -5, -9,...} is -4

Recursive formulas make it easy to see how each term is related to the next term. However, it is difficult to use recursive formulas to find a specific term. Say we wanted to know the 7^{th} term in the sequence. Well, from the formula, we know that

$$a_7 = a_6 - 4$$

What is a_6? Again, we see that

$$a_6 = a_5 - 4$$

Now we have to find a_5! If we keep going we see that:

$$a_5 = a_4 - 4$$

$$a_4 = a_3 - 4$$

$$a_3 = a_2 - 4$$

$$a_2 = a_1 - 4$$

Since we were told a_1, we can find a_2 and propagate our terms back up the chain to find a_7:

$$a_2 = 3 - 4 = -1$$

$$a_3 = a_2 - 4 = -1 - 4 = -5$$

$$a_4 = a_3 - 4 = -5 - 4 = -9$$

$$a_5 = a_4 - 4 = -9 - 4 = -13$$

$$a_6 = a_5 - 4 = -13 - 4 = -17$$

$$a_7 = a_6 - 4 = -17 - 4 = -21$$

So finally, we see that $a_7 = -21$. That was a lot of work! You can imagine that for higher n

terms, such as the 100^{th} or 1000^{th} term, this method becomes cumbersome. This is where the explicit formula is more useful.

The general form of an explicit formula for an arithmetic sequence is

$$a_n = a_1 + d \times (n - 1)$$

where d is the common difference. For our example sequence, {3, -1, -5, -9, ...}, the common difference is -4. So the explicit formula is

$$a_n = 3 + (-4)(n - 1) = 3 - 4(n - 1)$$

You may be tempted to distribute and simplify, which is fine and yields an equivalent formula:

$$a_n = 7 - 4n$$

Now, to find the 7^{th} term, all we have to do is substitute $n = 7$:

$$a_7 = 3 - 4(7 - 1) = 3 - 4(6) = 3 - 24 = -21$$

We get the same answer with much less effort!

Exercise 4

An arithmetic sequence is defined by the recursive formula $a_n = a_{n-1} + 5$ with $a_1 = -4$. Write the first 5 terms of the sequence and determine an explicit formula for the same sequence.

Working Space

Answer on Page 28

Exercise 5

The first four terms of an arithmetic sequence are $\{\pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}\}$. What is the common difference? Write explicit and recursive formulas for the infinite sequence.

Working Space

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4.2 Geometric sequences

Let's look at the other sequence given as an example at the beginning of the chapter: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$. How is each term related to the previous term? Well, $\frac{1}{4}$ is half of $\frac{1}{2}$, and $\frac{1}{8}$ is half of $\frac{1}{4}$, so each term is the previous term multiplied by $\frac{1}{2}$. When each term in a sequence is a multiple of the previous term, this is a *geometric* sequence. The number we multiply by each time (in our example, this is $\frac{1}{2}$) is called the *common ratio*. The common ratio can be positive or negative, but not zero.

An easy way to determine the common ratio (r) is to divide the n^{th} term by the $(n-1)^{th}$ term. In our example sequence, the first term is $\frac{1}{2}$ and the second is $\frac{1}{4}$.

$$r = \frac{a_2}{a_1} = \frac{1/4}{1/2} = \frac{1}{2}$$

which returns the common ratio we already identified, $r = \frac{1}{2}$.

If the common ratio is negative, then the sequence will "flip" back and forth from positive to negative. For example suppose there is a geometric sequence such that $a_1 = 1$ and $r = -2$. Then the first 5 terms are {1, -2, 4, -8, 16}. Whenever you see a sequence going back and forth from positive to negative, that means the common ratio is negative.

For positive common ratios, if $r > 1$, then the sequence is increasing. And if $r < 1$, the sequence is decreasing.

4.2.1 Formulas for geometric sequences

Like arithmetic sequences, we can write recursive and explicit formulas. For geometric sequences, the recursive formula has the general form:

$$a_n = r(a_{n-1})$$

where r is the common ratio and a_1 is specified. For our example sequence, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$, the recursive formula is:

$$a_n = \frac{1}{2}a_{n-1}$$
$$a_1 = \frac{1}{2}$$

In a geometric sequence, each term is the first term, a_1, multiplied by the common ratio, r, $n-1$ times. Therefore, the general form of an explicit formula for a geometric function is:

$$a_n = (a_1)r^{n-1}$$

Again, for our example sequence, $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$, so the explicit formula is:

$$a_n = (\frac{1}{2})(\frac{1}{2})^{(n-1)}$$

Exercise 6

Which of the following are geometric sequences? For each geometric sequence, determine the common ratio.

1. {2, -4, 6, -8,...}
2. {4, 2, 1, $\frac{1}{2}$, ...}
3. {-5, 25, -125, 525, ...}
4. {2, 0, -2, -4, ...}

Working Space

Answer on Page 28

Exercise 7

A geometric sequence is defined by the recursive formula $a_n = a_{n-1} \times \frac{3}{2}$ with $a_1 = 1$. Write the first 5 terms of the sequence and determine an explicit formula for the same sequence.

Working Space

Answer on Page 28

Exercise 8

The first four terms of a geometric sequence are {-4, 2, -1, $\frac{1}{2}$}. What is the common ratio? Write recursive and explicit formulas for the infinite sequence.

Working Space

Answer on Page 28

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

Solve for when the velocity is zero.

$t = \frac{12}{9.8} = 1.22$ seconds after release.

Answer to Exercise 2 (on page 15)

For what t is $-4.9t^2 + 12t + 2 = 0$? Start by dividing both sides of the equation by -4.9.

$$t^2 - 2.45t - 0.408 = 0$$

The roots of this are at

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = -\frac{-2.45}{2} \pm \frac{\sqrt{(-2.45)^2 - 4(-0.408)}}{2} = 1.22 \pm 1.36$$

We only care about the root after we release the hammer $(t > 0)$.

$1.22 + 1.36 = 2.58$ seconds after releasing the hammer, it will hit the ground.

Answer to Exercise 3 (on page 20)

1. not arithmetic
2. arithmetic, common difference is 3
3. arithmetic, common difference is -4
4. not arithmetic

Answer to Exercise 4 (on page 22)

The first five terms are {-4, 1, 6, 11, 16} and an explicit formula is $a_n = -4 + 5(n-1)$.

Answer to Exercise 5 (on page 23)

The common difference is $\frac{3\pi}{2} - \pi = \frac{\pi}{2}$. The recursive formula is $a_n = a_{n-1} + \frac{\pi}{2}$ with $a_1 = \pi$. The explicit formula is $a_n = \pi + \frac{\pi}{2}(n-1)$.

Answer to Exercise 6 (on page 24)

1. not geometric
2. geometric sequence with common ratio $r = \frac{1}{2}$
3. geometric sequence with common ration $r = -5$
4. not geometric

Answer to Exercise 7 (on page 25)

The first 5 terms are $\{1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}\}$. An explicit formula for this sequence is $a_n = 1(\frac{3}{2})^{(n-1)}$.

Answer to Exercise 8 (on page 25)

The common ratio is $\frac{a_n}{a_{n-1}} = \frac{2}{-4} = -\frac{1}{2}$. A recursive formula would be $a_n = a_{n-1} \times -\frac{1}{2}$ with $a_1 = -4$. An explicit formula would be $a_n = (-4)(-\frac{1}{2})^{(n-1)}$.

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