Contents

Chapter 1

Fabrication

Fabrication technologies include 3D Printers, CNC machines, and even some types of molding.

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1.1 3D Printing

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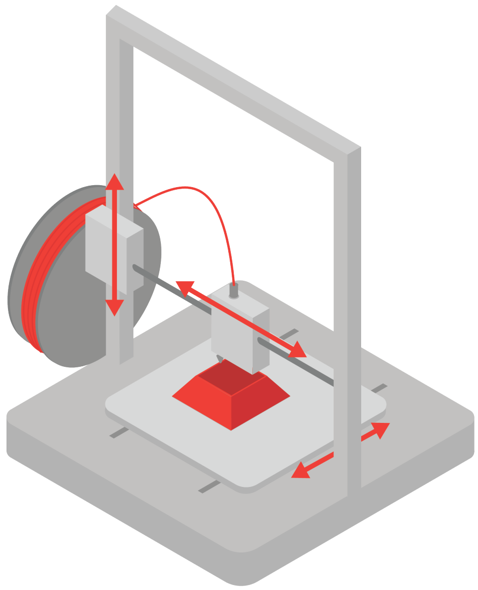
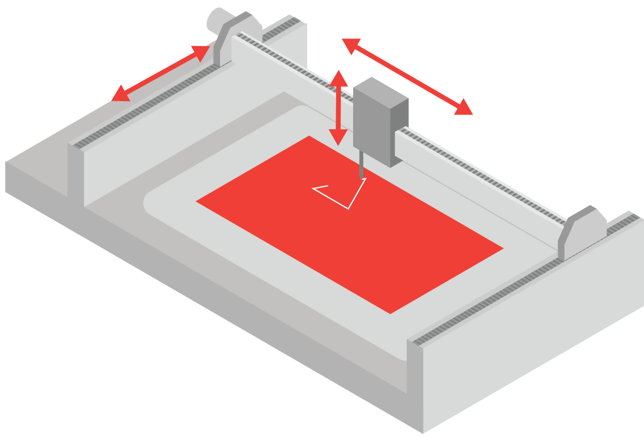


1.2 CNCs

Computer numerical control (CNC) is a manufacturing method that automates the control, movement, and precision of machine tools through the use of preprogrammed computer software, which is embedded inside the tools



We will explore three common types of CNC machines.

Chapter 2

Limits

The asymptotic behavior we see in rational functions suggests that we need to expand our vocabulary of function characteristics. We examined vertical asymptotes and end behavior through graphs and tables, and discussed them in English. The language of limits enables us to discuss these attributes mathematically and with greater efficiency.

Let's revisit an example from the previous chapter. This function has a hole at $x = 1$, a vertical asymptote at $x = 3$, and a horizontal asymptote of $y = 1$.

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$

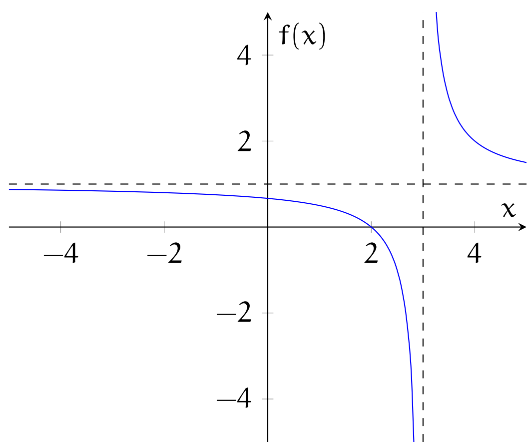


Figure 2.1: Graph of $f(x) = \frac{x^2-3x+2}{x^2-4x+3}$ with asymptotes

First, consider the vertical asymptote. We see that the graph goes down as it hugs the left side of the vertical asymptote, and goes up as it hugs the right side. We can describe these behaviors as the left- and right-hand limits, respectively. We say that the left-hand limit of f at $x = 3$ is negative infinity. Another way of communicating this is to say that as x approaches 3 from the left, the function approaches negative infinity. Symbolically, we summarize this as

$$\lim_{x \to 3^-} f(x) = -\infty$$

The little negative sign in $x \to 3^-$ indicates we are approaching $x = 3$ from the left (the negative side of the axis).

Similarly, the right-hand limit of f at $x = 3$ is positive infinity. In other words, as x

approaches 3 from the right, the function approaches positive infinity. Symbolically, we write

$$\lim_{x\to 3^+} f(x) = \infty$$

This time, the little $^+$ indicates we are approaching the x-value from the right (positive) side of the axis.

The limit of a function at a particular x-value is the y-value that the function approaches as it approaches the given x-value. In the previous example, we could only specify the left- and right-hand limits, because they were different. In cases where the left- and right-hand limits are equal, we can say that the function has a limit there. The hole in our function f is one such value. We see that as we approach the hole from both the left and right, the function takes on values near $\frac{1}{2}$. This is more apparent numerically:

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	0.5238	0.5025	0.5003	undefined	0.4998	0.4975	0.4737

We can also see this by zooming in on the graph (see figure ??):

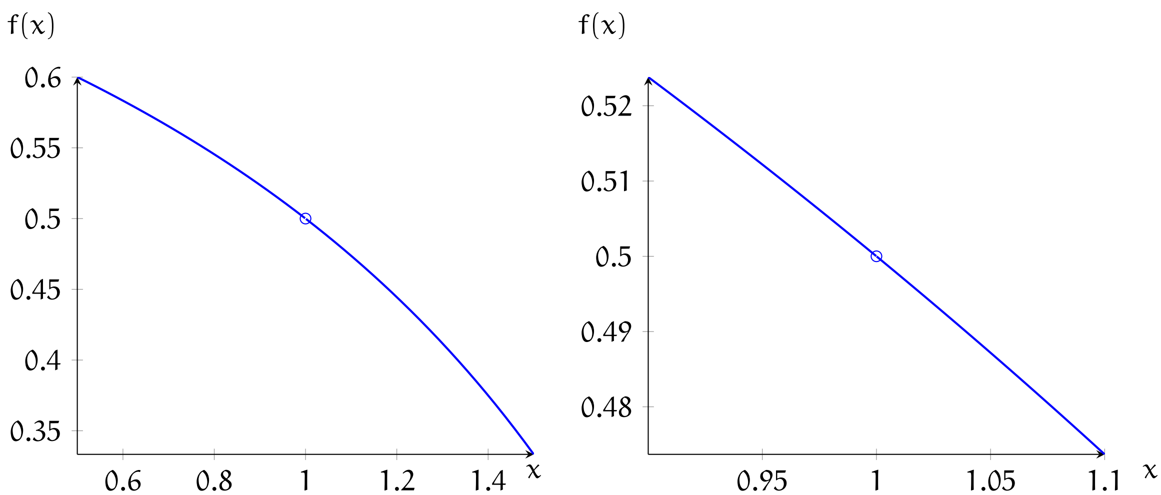


Figure 2.2: Two graphs of $f(x) = \frac{x^2-3x+2}{x^2-4x+3}$ zoomed in about $x = 1$

The left-hand and right-hand limits of f at 1 are both $\frac{1}{2}$. Since they are equal, we can also say that the limit of f at 1 is $\frac{1}{2}$. This allows us to efficiently discuss the behavior of f at 1, even though the function is not defined there, as substituting 1 into the function gives division by zero.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = \frac{1}{2}$$

We can also talk about limits at x-values where nothing weird is happening (that is, no hole or vertical asymptote). For example, as x approaches 4 from the left and right, y approaches 2.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	2.1111	2.0101	2.0010	2	1.9990	1.9901	1.9091

In this case, since nothing weird is happening, the limit is equal to the function value. This is an example of continuity, which we will discuss in more detail in the next chapter. By contrast, at the vertical asymptote $x = 1$, since the left- and right-hand limits are not equal, we say the function does not have a limit, or the limit does not exist.

Finally, let's consider the horizontal asymptote of f. The graph hugs the line $y = 1$ as x goes far to the left and far to the right. We say that as x approaches negative infinity, f approaches 1; likewise, that as x approaches positive infinity, f approaches 1. We write these symbolically as $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 1$.

Exercise 1 Limits Practice 1

Determine the left- and right-hand limits of the function as x approaches the given values. At x-values where the limit exists, determine it.

1. $p(x) = \frac{x+3}{x^2+9x+18}, x = -6, -5, -3, \infty$

Working Space

Answer on Page 39

We have seen two weird behaviors of rational functions at certain x-values: holes and vertical asymptotes. Now, we will examine another type of weird behavior: jumps. This is a characteristic of some piecewise-defined functions. In piecewise-defined functions, the domain is divided into two or more pieces, and a different expression is used to give the y-value depending on which piece contains the x- value. One common piecewise-defined function is the floor function (shown in figure 2.3), sometimes denoted $\lfloor x \rfloor$. The standard floor function rounds any real number down to the nearest integer. So, for a price quoted in dollars and cents, the floor would just be the number of dollars.

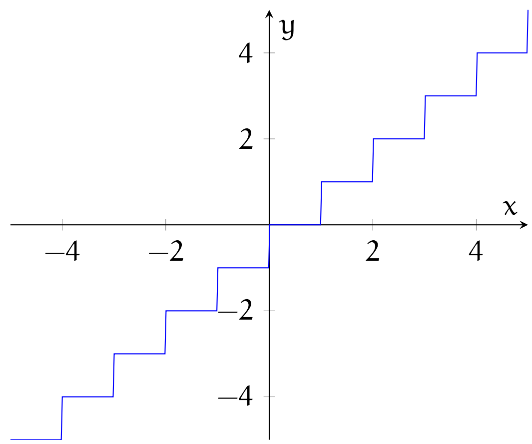


Figure 2.3: Graph of $y = \lfloor x \rfloor$

When x is exactly 1, the function value is 1: the number of dollars in a price of \$1.00. When x is any number greater than 1 but less than 2, the function value is still 1. Also, $\lfloor 1.01 \rfloor, \lfloor 1.5 \rfloor, \text{and} \lfloor 1.99999 \rfloor$ are all 1. As we continue to look to the right, once x equals exactly 2, h jumps up to the value 2. So, $\lim_{x \to 2^-} \lfloor x \rfloor = 1$, while $\lim_{x \to 2^+} \lfloor x \rfloor = 2$.

Besides rational and piecewise defined functions, there are other functions with interesting limits. Consider the standard exponential function, $y = e^x$ (shown in figure 2.4).

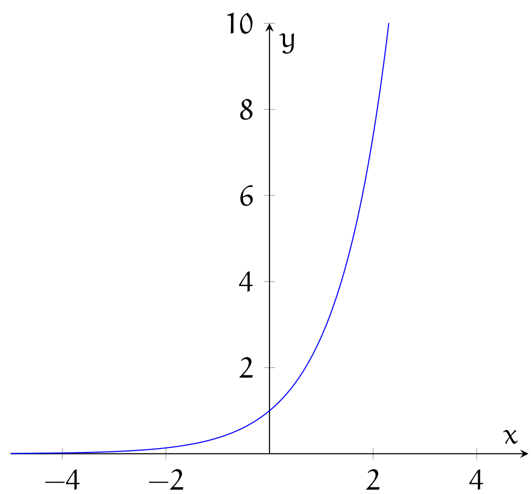


Figure 2.4: Graph of $y = e^x$

As x increases, y increases without bound; that is, $\lim_{x \to \infty} e^x = \infty$. However, looking far to the left, we see that y hugs the x-axis. This is because raising e to a large negative exponent is the same as 1 divided by e raised to a large positive exponent; that is, 1 divided by a very large number, which yields a very small positive number. In limit notation, $\lim_{x \to -\infty} e^x = 0$. This example illustrates that horizontal asymptotes need not model end behavior in both directions. Note that this reasoning holds for $y = b^x$ for any $b > 1$, so all such functions have the same horizontal asymptote, $y = 0$.

We know that the natural logarithm function, $y = \ln x$, is the inverse of $y = e^x$. Since inverse functions swap the role of x and y, it stands to reason that a horizontal asymptote in one function corresponds with a vertical asymptote in the other function, and that is indeed the case (see figure 2.5).

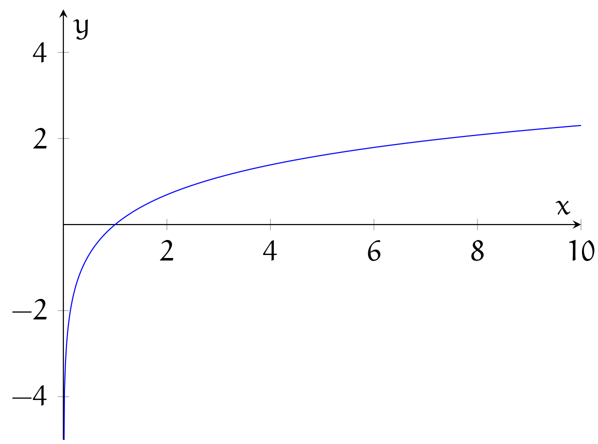


Figure 2.5: Graph of $y = \ln x$

An untransformed logarithm function is defined only for positive inputs. That is because it is not possible to find an exponent of a positive number that will yield a negative or zero result. What type of exponent on a positive number yields a number near zero? That would be a large-magnitude negative number. So, on the logarithm graph, large negative y-values correspond with x-values only slightly greater than zero. So, $\ln x$ (and $\log_2 x$, and indeed $\log_b x$ for any $b > 1$) approaches negative infinity as x approaches 0 from the right. There is no left-hand limit at 0, however. In limit notation, $\lim_{x \to 0^+} \ln x = -\infty$.

Exercise 2 Limits Practice 2

State the asymptotes of the following transformed exponential and logarithmic functions. Give the limit statement which describes the behavior of the function along the asymptote.

1. $y = 3^x + 1$
2. $y = \log_2(x - 4)$
3. $y = 2^{1-x}$
4. $y = \log_{10}(-2x)$

Working Space

Answer on Page 39

We next consider two functions that each have two horizontal asymptotes. These two seemingly obscure functions are quite important in data science.

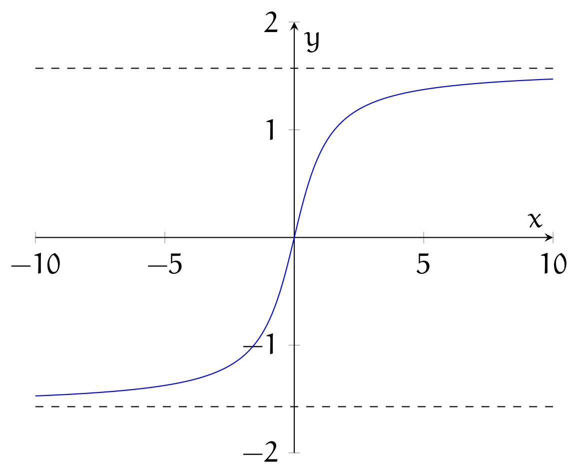


Figure 2.6: Graph of $y = \arctan x$

We know that the arctangent, or inverse tangent, function is the inverse of the piece of the tangent function which passes through the origin. The vertical asymptotes bounding this piece become horizontal asymptotes when the function is inverted.

Here are the equation and graph of the logistic function:

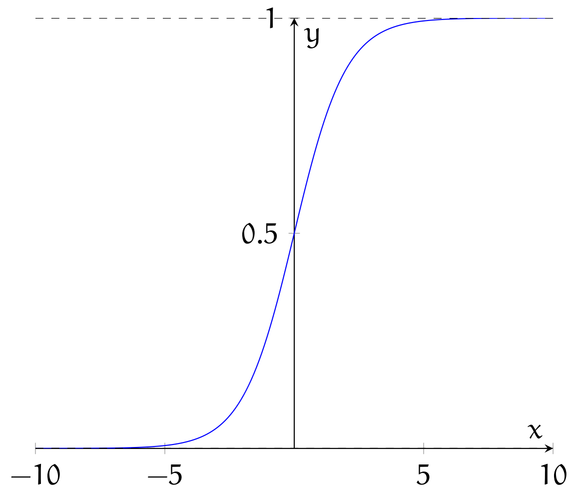


Figure 2.7: Graph of the logistic function, $y = \frac{1}{1+e^{-x}}$

For large magnitude negative values of x, the exponential term in the denominator becomes a very large positive value. The fraction thus becomes a positive number very close to zero. For large magnitude positive values of x, that exponential term becomes a very small positive number. Adding it to 1 yields a denominator just barely greater than 1. Dividing 1 by this number therefore yields a function value just barely less than 1. So, the logistic function yields values between 0 and 1, though never equaling either of these values exactly. It is precisely this characteristic which makes the logistic function so useful.

Exercise 3 Limits Practice 3

Using limit notation, state the limits as x approaches negative and positive infinity for the inverse tangent and logistic functions given above.

Working Space

Answer on Page 39

As seen above, the limit of a function from the left may be different from the limit of the function from the right. Additionally, the actual *value* of the function may be different from the limit. Consider the piecewise function $h(x)$:

$$h(x) = \begin{cases} -x^2 + 3, & \text{if } x < 0 \\ 2, x = 0 \\ -x + 3, & \text{if } x > 0 \end{cases}$$

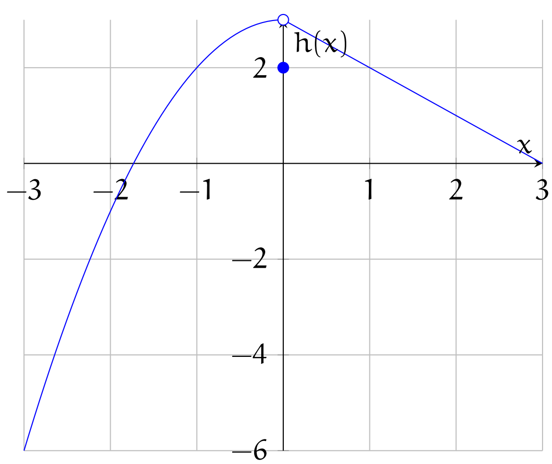


Figure 2.8: Graph of the piecewise function, $h(x)$

From examining the graph, we see that

$$\lim_{x \to 0_-} h(x) = \lim_{x \to 0_+} h(x) = 3$$

However, $h(0) = 2 \neq 3$. So, does this limit exist? It does! The limit of a function describes the *behavior* of the function around a particular value, not the value of the function itself. In order for a limit to exist, the limits from the left and right must be equal to each other, but not necessarily the actual value of the function.

Use the graph of $h(x)$ above and the graphs of $f(x)$ and $g(x)$ below to complete the following exercise.

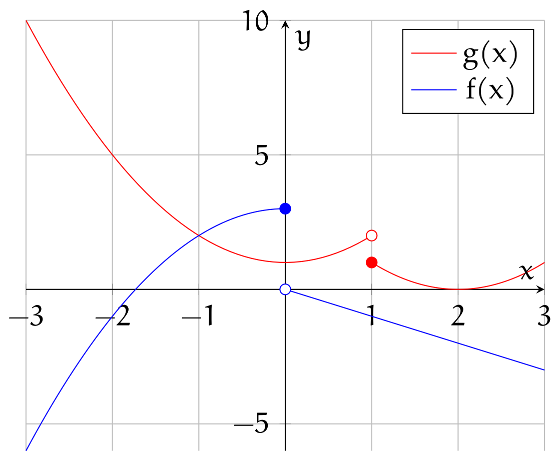


Figure 2.9: Piecewise functions $f(x)$ and $g(x)$

Exercise 4 Limits Practice 4

Determine the limit from the left and the right for each function at the given value(s). State the limit at that value, if it exists.

1. $h(x), x = -1, 0, 1$
2. $f(x), x = -1, 0, 2$
3. $g(x), x = -2, 0, 1, 2$

Working Space

Answer on Page 39

2.1 Continuity

A note about continuity:

In order to be able to talk more about limits and know when we can apply certain rules and theorems, we first must discuss continuity. A function is continuous if there are no "jumps" or "gaps" in the graph of the function. For example, the function $f(x) = x^2$ is continuous for all real values of x. On the other hand, the function $g(x) = \tan(x)$ has many discontinuities, including at $x = \frac{\pi}{2}$. Let's examine the graph of each of these functions:

If you wanted, you could trace your finger along the graph of $f(x)$ from $x = -3$ to $x = 3$ without ever picking up your finger. This means the function is continuous in the domain from $-3 \leq x \leq 3$. In this case, the domain of continuity *includes* the end points ($x = 3$ and

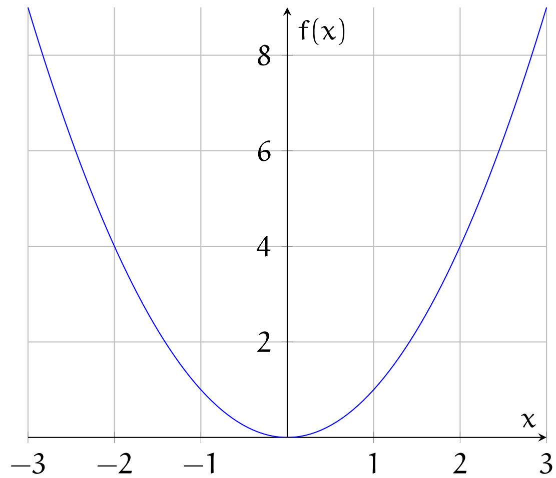


Figure 2.10: Graph of $f(x) = x^2$

$x = -3$). This is called a closed interval. In other cases, the function will be continuous right up to, but not including, the endpoints, as with the domains of continuity for our other example, $g(x) = \tan x$. This is called an open interval. Let's learn more about intervals of continuity by examining $g(x) = \tan x$.

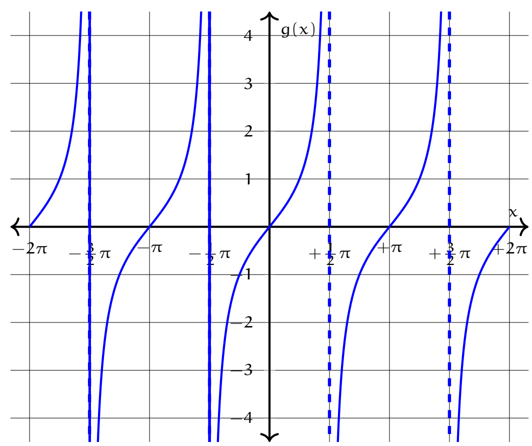


Figure 2.11: Graph of $g(x) = \tan x$

As you can see, if you trace your finger along the graph of the function starting at $x = 0$, you can continue without lifting your finger to $x = \frac{\pi}{2}$. As you approach $x = \frac{\pi}{2}$ from the left, the value of $g(x)$ approaches ∞. In order to continue tracing the function PAST $x = \frac{\pi}{2}$, you have to lift your finger and bring it down to $-\infty$. The function then continues continuously again until $x = \frac{3\pi}{2}$.

In the case of $g(x) = \tan x$, the function is continuous on *open intervals*, including the open interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

There is a shorter way to represent open and closed domain intervals. We can represent that $f(x) = x^2$ is continuous on the closed interval $-3 \leq x \leq 3$ in the following way:

$$x \in [3, -3]$$

This reads as "x contained in the domain -3 to 3, inclusive". That is, all the values from -3 to 3, including the endpoints. The inclusion of the endpoints is implied by the use of *brackets*. For open intervals, we use parentheses to communicate that the interval goes up to, but does not include, the endpoints. For $g(x) = \tan x$, we can use parentheses:

$$x \in \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$$

because the $g(x) = \tan x$ is not continuous at $x = \frac{-3\pi}{2}$ or at $x = \frac{-\pi}{2}$.

Formally, a function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x)$ exists *and* $\lim_{x\to a} f(x) = f(a)$. In other words, the limit is equal to the actual value of the function. Re-examine the graph of $h(x)$. We have already seen that $\lim_{x\to 0} h(x)$ exists and is equal to 3. However, $h(0) = 2 \neq \lim_{x\to 0} h(x)$. So $h(x)$ is not continuous at $x = 0$. Because $-x^2+3$ is evaluable all the way to $-\infty$ and $-x+3$ is evaluable all the way to ∞, the function $h(x)$ is continuous everywhere *except* $x = 0$. We can represent this mathematically by saying $h(x)$ is continuous on the domain $x \in (-\infty, 0) \cup (0, \infty)$. We use parentheses for $\pm\infty$ because we can never actually reach ∞. Additionally, the function is continuous up to, but not including 0, and the use of parentheses excludes $x = 0$ from the domain of continuity.

2.1.1 Continuity Practice

Exercise 5

Working Space

[This problem was originally presented as a calculator-allowed, multiple- choice question on the 2012 AP Calculus BC exam.] Suppose a function f is continuous at $x = 3$. Classify the following statements as always true, sometimes true, or never true. Explain your answers.

1. $f(3) < \lim_{x \to 3} f(x)$
2. $\lim x \to 3^{+} f(x) \neq \lim_{x \to 3^-} f(x)$
3. $f(3) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x)$
4. The derivative of f at $x = 3$ exists.
5. The derivative of f is positive for $x < 3$ and negative for $x > 3$.

Answer on Page 40

Exercise 6

Limits Practice 5

Working Space

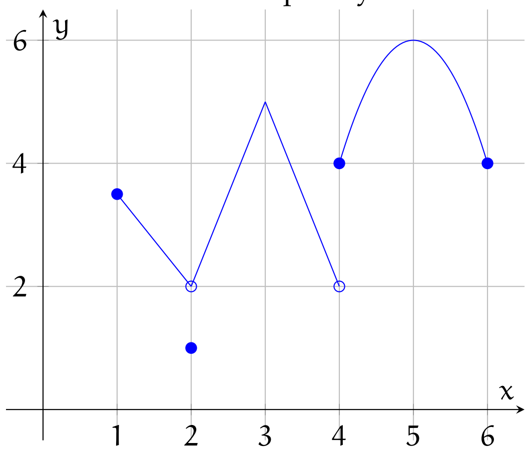
State the location of discontinuities (if any) and explain why the function is discontinuous at that location:

1. $f(x) = \frac{3x^2-8x-3}{x-3}$
2. $f(x) = \begin{cases} \frac{2}{x^4}, & \text{if } x <\neq 0 \\ 2, & \text{if } x = 0 \end{cases}$
3. $f(x) = \begin{cases} \frac{3x^2-8x-3}{x-3}, & \text{if } x \neq 3 \\ 1, & \text{if } , x = 3 \end{cases}$

Answer on Page 40

Exercise 7

The graph of a function, $h(x)$, is shown. Classify each of the following statements as true or false and explain your answer.



1. $\lim_{x\to 2} h(x)$ exists
2. $\lim_{x\to 3} h(x)$ does not exist
3. $\lim_{x\to 4} h(x)$ exists
4. $h(x)$ is continuous at $x = 5$
5. $h(x)$ is not continuous at $x = 4$
6. $\lim_{x\to 2} h(x) = h(2)$

Working Space

Answer on Page 41

2.2 Limits Rules

There are some mathematical properties of limits which allow us to determine the limit of complex functions without seeing a graph or using a calculator to generate a table.

The following laws are true given that c is a constant, $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} g(x)$ exists.

1. Sum Law $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. Difference Law $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. Constant Multiple Law $\lim_{x\to a}[cf(x)] = c \cdot \lim_{x\to a} f(x)$
4. Product Law $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. Quotient Law $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ given that $\lim_{x\to a} g(x) \neq 0$

These laws are fairly obvious — the limit of the sum of two functions is equal to the sum of the limits of each function individually. The only tricky one is the last: The limit of the quotient of two functions is equal to the quotient of the limits if and only if the limit of the function in the denominator does not equal zero. This makes sense, as we know dividing by zero yields an undefined result.

Let's practice applying these laws to evaluate the limits of the functions $f(x)$, shown in blue below, and $g(x)$, shown in red below:

$$f(x) = \begin{cases} -x^2+3, & \text{if } x \leq 0 \\ -x, & \text{if } x > 0 \end{cases}$$

$$g(x) = \begin{cases} x^2+1, & \text{if } x < 1 \\ (x-2)^2, & \text{if } x \geq 1 \end{cases}$$

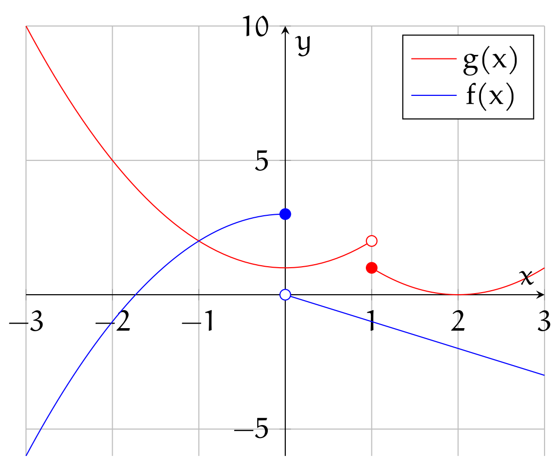


Figure 2.12: Graphs of the piecewise functions $f(x)$ and $g(x)$

We can use these laws to evaluate limits involving $f(x)$ and $g(x)$ (shown on the graph above). Here are some examples: Use the graphs of $f(x)$ and $g(x)$ given above to evaluate each limit, if it exists. If the limit does not exist, explain why. Two examples are given first:

Example 1: Evaluate $\lim_{x\to 0} f(x) \cdot g(x)$

Solution 1: From the Product Law, we know that:

$$\lim_{x\to 0} f(x) \cdot g(x) = \lim_{x\to 0} f(x) \cdot \lim_{x\to 0} g(x)$$

Looking at the graph, we can see that

$$\lim_{x\to 0} g(x) = 1$$

and there is a discontinuity in $f(x)$ at $x = 0$. Therefore,

$$\lim_{x\to 0} f(x) = \text{undef}$$

Substituting this, we get:

$$\lim_{x\to 0} f(x) \cdot \lim_{x\to 0} g(x) = \text{undef} \cdot 1 = \text{undef}$$

Therefore, *the limit does not exist*.

Example 2: Evaluate $\lim_{x\to 2} f(x) - g(x)$

Solution 2: Applying the Difference Law, we see that:

$$\lim_{x\to 2}[f(x) - g(x)] = \lim_{x\to 2} f(x) - \lim_{x\to 2} g(x)$$

Examining the graph, we see that

$$\lim_{x\to 2} f(x) = -2$$

and

$$\lim_{x\to 2} g(x) = 0$$

Substituting these values, we get:

$$\lim_{x\to 2} f(x) - g(x) = -2 - 0 = -2$$

Exercise 8 Limits Practice 6

Working Space

1. $\lim_{x \to -3} \frac{f(x)}{g(x)}$
2. $\lim_{x \to 2} [f(x) + 5g(x)]$
3. $\lim_{x \to -1} \frac{3g(x)}{f(x)}$
4. $\lim_{x \to 0} f(x) \cdot 5g(x)$
5. $\lim_{x \to -1} f(x) - 3g(x)$

Answer on Page 41

Recall that exponents represent repeated multiplication. Therefore, if we apply the Product Law multiple times, we obtain the Power Law for limits:

6. Power Law $\lim_{x \to \infty} [f(x)]^n = [\lim_{x \to \infty} f(x)]^n$ where n is a positive integer

There are two special limits that will be useful to us and are intuitively obvious, but we won't formally prove here.

7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$

Combining Law 8 with the Power Law, we find that:

9. $\lim_{x \to a} x^n = a^n$

And similarly, for square roots:

10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (if n is even, we assume $a > 0$)

Direct substitution property: If f is a polynomial or rational function and a is in the domain for f, then

$$\lim_{x\to a} f(x) = f(a)$$

Often, rational functions can be simplified. In an above example, we computed the limit by simplifying $f(x) = \frac{3x^2-8x-3}{x-3}$ to the simpler $g(x) = 3x+1$. This is a valid strategy because $\frac{3x^2-8x-3}{x-3} = 3x+1$ when $x \neq 3$. Remember: a limit describes how a function behaves *as it approaches* a, not its value/behavior when x *actually equals* a. This reveals the following useful rule:

If $f(x) = g(x)$ when $x \neq a$, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$, provided the limit exists.

2.3 Squeeze Theorem

The Squeeze Theorem states that if $f(x) \leq g(x) \leq h(x)$ when x is near a (except at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

, then

$$\lim_{x\to a} g(x) = L$$

In other words, if $g(x)$ is between $f(x)$ and $h(x)$ near a, and f and h have the same limit, L, then the limit of g must also be L.

Example: Let's examine the graph of $g(x) = x^2 \sin\frac{1}{x}$ and determine $\lim_{x\to 0} g(x)$:

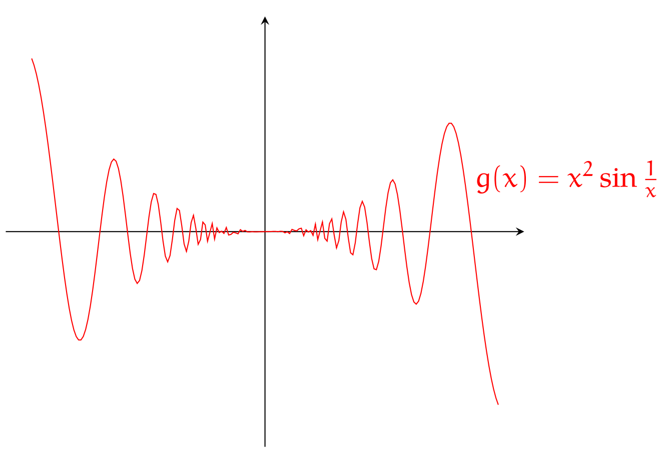


Figure 2.13: Graph of $g(x) = x^2 \sin\frac{1}{x}$

Solution: Because $\sin\frac{1}{x}$ is undefined at $x = 0$, we cannot compute the limit directly. However, from examining the graph, we can guess that $\lim_{x\to 0} g(x) = 0$. Feel free to confirm this with your calculator. We need to choose two functions: one that is larger than $g(x)$ near $x = 0$ and one that is smaller. Since $|\sin\frac{1}{x}| \leq 1$ (when $x \neq 0$), then

$$|x^2 \sin\frac{1}{x}| \leq x^2$$

and

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

Let's confirm this by plotting $f(x) = -x^2$, $g(x)$, and $h(x) = x^2$ on the same graph:

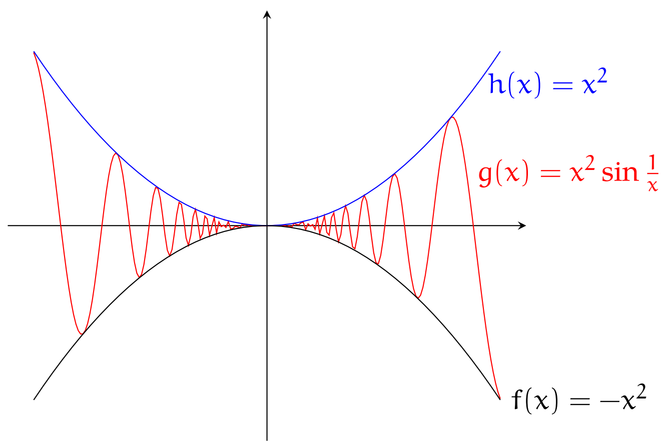


Figure 2.14: Squeeze Theorem example

As you can see, when x is near 0, $f(x) \leq g(x) \leq h(x)$. Because $f(x)$ and $h(x)$ are both polynomials, their limits are straightforward:

$$\lim_{x\to 0} -x^2 = 0 \text{ and } \lim_{x\to 0} x^2 = 0$$

Then, by the Squeeze Theorem, we can say that:

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

2.3.1 Squeeze Theorem Practice

Exercise 9 Squeeze Theorem 1

Use the Squeeze Theorem to show that $\lim_{x \to 0} \sqrt{x^3 + x^2} \cos \frac{1}{x} = 0$. Illustrate by graphing the functions you define as f, g, and h on the same plot.

Working Space

Answer on Page 43

Exercise 10 Squeeze Theorem 2

If $2x + 3 \leq f(x) \leq x^2 - 2x + 7$ for $x \geq 0$, find $\lim_{x \to 2} f(x)$

Working Space

Answer on Page 43

Exercise 11 Squeeze Theorem 3

Prove that $\lim_{x \to 0^+} \sqrt{x} e^{\sin \frac{\pi}{x}} = 0$

Working Space

Answer on Page 44

2.4 Intermediate Value Theorem

When considering functions that are continuous on a closed interval, the Intermediate Value Theorem can help us: Given a function, $f(x)$, that is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$, there is at least one number c such that $f(c) = N$, where N is any number between $f(a)$ and $f(b)$. The theorem is illustrated in figures 2.15 and 2.16:

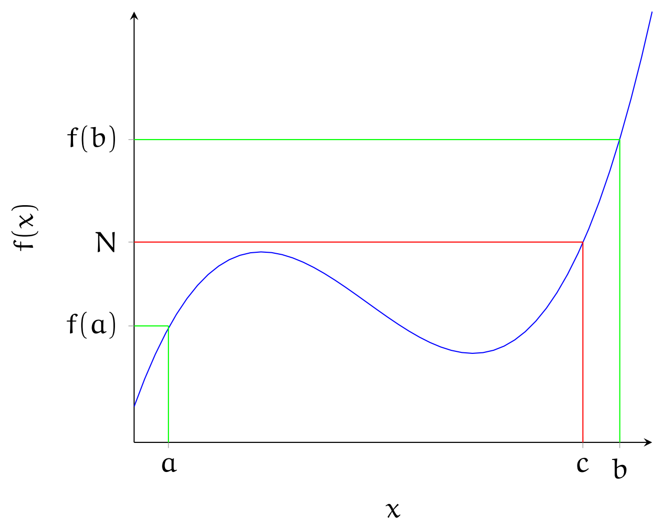


Figure 2.15: An example where one solution satisfies the Intermediate Value Theorem

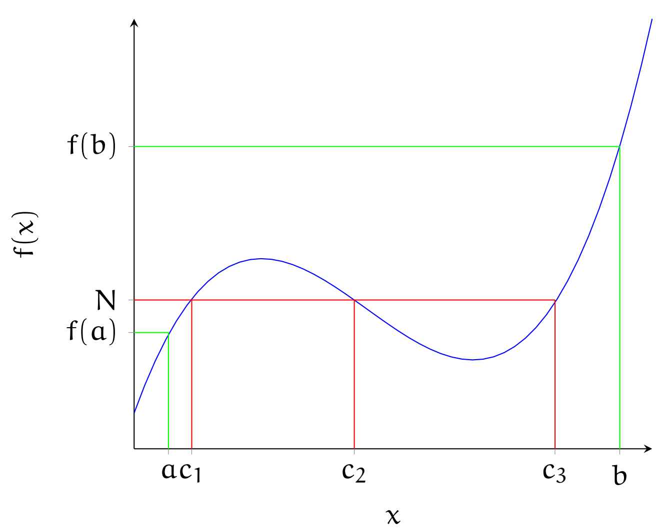


Figure 2.16: An example where more than one solution satisfies the Intermediate Value Theorem

Logically, we can think of the IVT this way: If a function is continuous on a closed interval, there are no gaps or breaks. If there are no gaps or breaks, then the function must pass through the line $y = N$, since it cannot jump over the line. Your graphing calculator uses IVT to find roots of functions:

Example: Show that there is at least one root of the equation $2x^3 - 6x^2 + 3x - 1 = 0$ between $x = 2$ and $x = 3$.

Solution: For IVT to apply, we must first check that the function is continuous on the closed interval $x \in [2, 3]$. We define $f(x) = 2x^2 - 6x^2 + 3x - 1$, which is continuous everywhere, because it is a polynomial function. *For more complex functions, always be sure to check the endpoints of an interval, since IVT only applies on closed intervals of continuity*. We will take $a = 2$, $b = 3$, and $N = 0$. We find the values of $f(x)$ at the endpoints:

$$f(2) = 2(2)^3 - 6(2)^2 + 3(2) - 1 = 16 - 24 + 6 - 1 = -3$$
$$f(3) = 2(3)^3 - 6(3)^2 + 3(3) - 1 = 54 - 54 + 9 - 1 = 8$$

Therefore, $f(2) < 0 < f(3)$ and according to IVT, there must exist some c such that $f(c) = 0$ and there is a root to the equation in the interval $x \in (2, 3)$.

Exercise 12 Intermediate Value Theorem Practice

Use the IVT to show there is a solution the given equation on the stated interval:

1. $2x^4 + x - 12 = 0$, $(1, 2)$
2. $\ln(x) = 3x - 4\sqrt{x}$, $(2, 3)$
3. $2\sin x = 3x^2 - 2x$, $(1, 2)$

Working Space

Answer on Page 44

Chapter 3

Rational Functions

We have discussed addition, subtraction, and multiplication of polynomials. What about division?

A quotient of polynomials is called a rational expression. When the polynomials are factored and the stars align, we can simplify the rational expression to a single polynomial, just like we might reduce a fraction to lowest terms.

Example

$$\begin{aligned}\frac{(x+1)(x+5)}{x+5} &= (x+1) * \frac{x+5}{x+5} \\ &= x+1\end{aligned} \tag{3.1}$$

What if the polynomials are not factored? Factor them first.

Example

$$\frac{x^2+6x+5}{x+5} = \frac{(x+1)(x+5)}{x+5}$$

and simplify as in the previous example.

Now, let us consider a rational expression which can be simplified to a single polynomial — but in the denominator.

Example

$$\begin{aligned}\frac{x+5}{x^2+6x+5} &= \frac{x+5}{(x+1)(x+5)} \\ &= \frac{1}{x+1} * \frac{x+5}{x+5} \\ &= \frac{1}{x+1}\end{aligned} \tag{3.2}$$

Consider this expression as a function: $f(x) = \frac{1}{x+1}$. As you might have guessed, this is called a rational function. We did not bother looking at the result of the previous example as a function, because we already know that function type: it is a line with slope 1 and y-intercept 1. However, this rational function is another animal entirely. Let us examine our first rational function with a familiar concept: the y-intercept.

y-intercept: $f(0) = \frac{1}{0+1} = \frac{1}{1} = 1$. The graph contains the point $(0, 1)$.

Does f have an x-intercept? That would be an x-value where $f(x) = 0$. But a fraction equals 0 only when its numerator equals 0; since the numerator of this expression is always 1, f has no x-intercept.

Knowing the y-intercept, and that there is no x-intercept, is a comforting start. But things get weird when we consider a concept that has previously seemed quite simple: domain. Recall that the domain of a function is the set of all values which can be used as inputs. In this case, the domain includes all real numbers, with one exception. The number -1 is not a valid input because $f(-1) = \frac{1}{-1+1} = \frac{1}{0}$, which is undefined. So, we say that the domain is all real numbers except -1. This means the graph contains a point corresponding to every x-value except -1.

There is no point at $x = -1$, but there is a point at every other x-value, such as, for example, -1.1, or -0.99999. So, what is happening near $x = -1$?

x	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9
$f(x)$	-10	-100	-1000	1000	100	10

The function is going haywire. As we choose x-values closer and closer to -1, the resulting function values are larger and larger in magnitude. Also, they are negative on one side, but positive on the other. So, how does a graph go from y-values of -10, to -100, to -1000, all in a space of less than 0.1 on the x-axis? Then, from there, suddenly to big positive numbers on the other side of $x = -1$? All without ever crossing the x-axis (since there is no x-intercept)? Let's look at the graph.

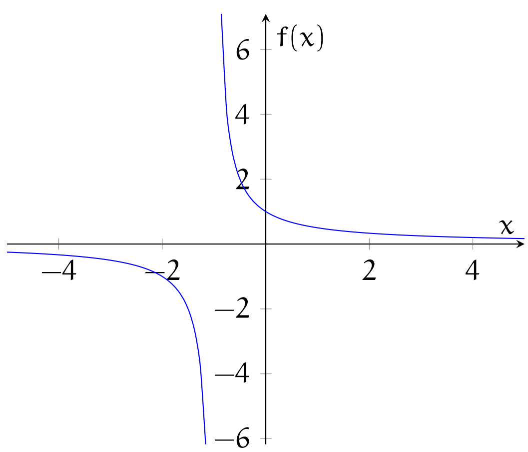


Figure 3.1: Graph of $f(x) = \frac{1}{x+1}$

We can see the y-intercept we found above. We can also see that the graph has no x-intercept, as expected. The phenomenon occurring at $x = -1$ is called a vertical asymptote. One other interesting feature of this graph is how it hugs the x-axis toward the left and right edges of the window. This makes the line $y = 0$ (the x-axis) a horizontal asymptote for this function. We can see why this is happening numerically by considering what happens for x-values far from 0. In this function, the result is a fraction with a numerator of 1 and a denominator that is large in size: a fraction that is close to 0.

x	-1000	-100	-10	10	100	1000
$f(x)$	-0.001	-0.01	-0.1	0.1	0.01	0.001

Let's examine another rational function. Begin by factoring to see if the function can be simplified.

$$g(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$

Consider the domain of g before continuing. Which values of x are valid inputs? Since substituting $x = 1$ or $x = 3$ would result in division by 0, these are not valid inputs. The domain of g is all real numbers except 1 and 3.

Now, for any x-value except 1, $\frac{x-1}{x-1} = 1$. This means that, for all x-values but 1, we can cancel those factors, leaving $g(x) = \frac{x-2}{x-3}$. (We will talk more about what is happening at $x = 1$ in a moment.)

This function has both x- and y-intercepts: y-intercept: $g(0) = \frac{0-2}{0-3} = \frac{2}{3}$. The graph contains the point $(0, \frac{2}{3})$. x-intercept: $g(x) = 0$ where the numerator equals 0 and the denominator does not equal 0. Since $x - 2 = 0$ when $x = 2$, the x-intercept is 2 and the graph contains the point $(2, 0)$.

The graph of g has a vertical asymptote at any x-value where substitution would result in dividing a nonzero number by zero. Thus, g has a vertical asymptote at $x = 3$.

Does g have a horizontal asymptote? Let us see what happens when we substitute x-values far from 0.

x	-1000	-100	-10	10	100	1000
$g(x)$	0.999	0.990	0.923	1.143	1.010	1.001

As we move further away from the y-axis, the y-values become closer to 1. The horizontal asymptote describes the end behavior of the function, or what the graph looks like far from the y-axis. In this case, if we ignore the portion close to the y-axis, the graph begins to look like the line $y = 1$, making this the horizontal asymptote of g.

So, what is happening at $x = 1$? The value is not in the domain of the function, but there is no vertical asymptote there. That is because substituting any other value for x, even values very close to 1, into $\frac{(x-1)(x-2)}{(x-1)(x-3)}$ gives the exact same number as substituting into $\frac{x-2}{x-3}$. So, there is a hole in the graph at $x = 1$, but nothing strange is happening on either side of 1. (Depending on the graphing software, the hole may not be visible.)

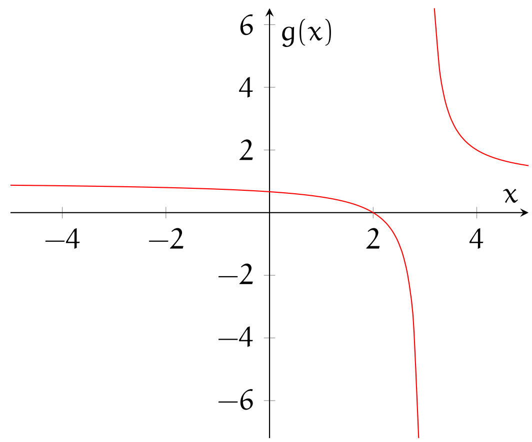


Figure 3.2: Graph of $g(x) = \frac{x^2-3x+2}{x^2-4x+3}$

Exercise 13 Rational Functions Practice 1

Working Space

Determine the x- and y-intercepts and horizontal and vertical asymptotes of the rational function:

1. $\frac{2x+5}{x+4}$

Answer on Page 45

Exercise 14

Working Space

[This question was originally presented in the no-calculator section of the 2012 AP Calculus BC exam.] The line $y = 5$ is a horizontal asymptote of which of the following functions? Explain.

A. $y = \frac{\sin 5x}{x}$
B. $y = 5x$
C. $y = \frac{1}{x-5}$
D. $y = \frac{5x}{1-x}$
E. $y = \frac{20x^2-x}{1+4x^2}$

Answer on Page 45

In those examples, common factors cancel, leaving one polynomial. Of course, there is no guarantee that any two polynomials will have common factors, or even be factorable at all. Now, we consider an example that cannot be simplified. We will focus on just the asymptotes here.

$$h(x) = \frac{x^2}{x-1}$$

We see that the x-value 1 gives division of a non-zero number by zero, giving a vertical asymptote at $x = 1$. How about a horizontal asymptote? We examine values of h for values of x far from 0.

x	-1000	-100	-10	10	100	1000
$h(x)$	-999	-99	-9	11	101	1001

Rather than seeing function values leveling off as in the previous examples, we see function values that grow in size along with x. The function h has no horizontal asymptote. Let's examine the graph:

This function exhibits a different type of end behavior: that of a line with slope 1. To see that, cover up the portion of the graph near the y-axis and focus on the left and right. The rather dull and time-consuming technique of polynomial long division can be used to rewrite the function as a quotient and a remainder. We encourage you to watch the Khan Academy video on the topic, but for now, let us instead use our knowledge of factoring techniques and a clever little trick.

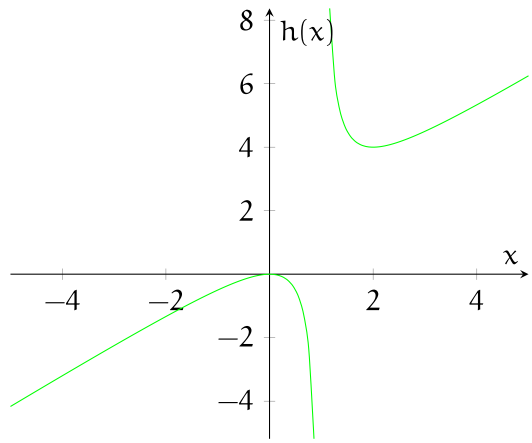


Figure 3.3: Graph of $h(x) = \frac{x^2}{x-1}$

$$
\begin{aligned}
h(x) &= \frac{x^2}{x-1} \\
&= \frac{x^2 - 1 + 1}{x-1} \\
&= \frac{x^2 - 1}{x-1} + \frac{1}{x-1} \\
&= \frac{(x-1)(x+1)}{x-1} + \frac{1}{x-1} \\
&= x + 1 + \frac{1}{x-1}
\end{aligned}
\tag{3.3}
$$

We obtain a quotient of $x + 1$ and a remainder of 1. It is the quotient that determines the end behavior of the graph. Why? Substituting x-values far from zero makes the remainder term very small, since it becomes a fraction with a large denominator but a numerator of only 1. So for x-values far from zero, the y-value is x plus 1 plus a very small number (so small that we can justifiably ignore it). This means that far from the y-axis, the function acts like the quotient: the line $y = x+1$. We call this line an oblique asymptote. See below how the graph of $h(x)$ hugs that line.

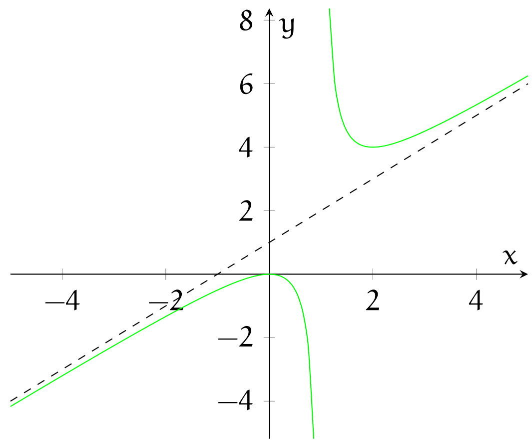


Figure 3.4: Graph of $h(x) = \frac{x^2}{x-1}$ and its oblique asymptote $y = x + 1$

Exercise 15 Rational Functions Practice 2

Factor and simplify the rational function, then determine any holes and vertical and oblique asymptotes of the rational function.

1. $\frac{x^3+2x^2}{x^2+x}$

Working Space

Answer on Page 45

We have seen lines act as end behaviors. Are there other possibilities? Sure! Here is an example with parabolic end behavior.

$$k(x) = \frac{x^3}{x-2}$$

We use our add-subtract trick to reveal the quotient, which describes the end behavior.

$$\begin{aligned} h(x) &= \frac{x^3}{x-2} \\ &= \frac{x^3-8+8}{x-2} \\ &= \frac{x^3-8}{x-2} + \frac{8}{x-2} \\ &= \frac{(x-2)(x^2+2x+4)}{x-2} + \frac{8}{x-2} \\ &= x^2+2x+4+\frac{8}{x-2} \end{aligned} \tag{3.4}$$

The quotient, $x2+2x+4$, should describe the end behavior. We confirm by graphing both k and the quotient - the parabolic asymptote.

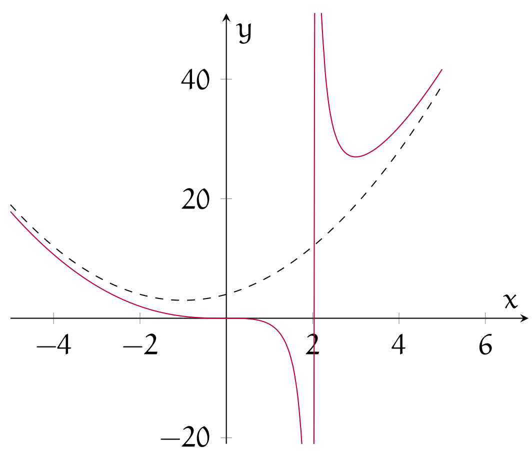


Figure 3.5: Graph of $k(x) = \frac{x^3}{x-2}$ and its parabolic asymptote $y = x^2+2x+4$

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 9)

$$\lim_{x \to -6^-} p(x) = -\infty, \lim_{x \to -6^+} p(x) = \infty$$

$$\lim_{x \to -5^-} p(x) = \lim_{x \to -5^+} p(x) = \lim_{x \to -5} p(x) = 1$$

$$\lim_{x \to -3^-} p(x) = \lim_{x \to -3^+} p(x) = \lim_{x \to -3} p(x) = \frac{1}{3}$$

$$\lim_{x \to \infty} p(x) = 0 \text{ called simply a limit, although it is a left-hand limit}$$

Answer to Exercise 2 (on page 12)

$$\lim_{x \to -\infty} 3^x + 1 = 1; \lim_{x \to 4^+} \log_2(x-4) = -\infty; \lim_{x \to \infty} 2^{1-x} = 0; \lim_{x \to 0^-} \log_{10}(-2x) = -\infty$$

Answer to Exercise 3 (on page 14)

$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}, \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}; \lim_{x \to -\infty} \frac{1}{1+e^{-x}} = 0, \lim_{x \to \infty} \frac{1}{1+e^{-x}} = 1$

Answer to Exercise 4 (on page 16)

1. $\lim_{x \to -1_-} h(x) = 2$ and $\lim_{x \to -1_+} h(x) = 2$, therefore the limit exists and $\lim_{x \to -1} h(x) = 2$

   $\lim_{x \to 0_-} h(x) = 3$ and $\lim_{x \to 0_+} h(x) = 3$, therefore the limit exists and $\lim_{x \to 0} h(x) = 3$

   $\lim_{x \to 1_-} h(x) = 2$ and $\lim_{x \to 1_+} h(x) = 2$, therefore the limit exists and $\lim_{x \to 1} h(x) = 2$

2. $\lim_{x \to -1_-} f(x) = 2$ and $\lim_{x \to -1_+} f(x) = 2$, therefore the limit exists and $\lim_{x \to -1} f(x) = 2$.

$\lim_{x\to 0_-} f(x) = 3$ and $\lim_{x\to 0_+} f(x) = 0$, and because $\lim_{x\to 0_-} f(x) \neq \lim_{x\to 0_+} f(x)$, the limit does not exist.

$\lim_{x\to 2_-} f(x) = -2$ and $\lim_{x\to 2_+} f(x) = -2$, therefore the limit exists and $\lim_{x\to 2} f(x) = -2$.

3. $\lim_{x\to -2_-} g(x) = -1$ and $\lim_{x\to -2_+} g(x) = -1$, therefore the limit exists and $\lim_{x\to -2} g(x) = -1$.

   $\lim_{x\to 0_-} g(x) = 1$ and $\lim_{x\to 0_+} g(x) = 1$, therefore the limit exists and $\lim_{x\to 0} g(x) = 1$

   $\lim_{x\to 1_-} g(x) = 2$ and $\lim_{x\to 0_+} g(x) = 1$, and because $\lim_{x\to 1_-} g(x) = 2 \neq \lim_{x\to 0_+} g(x)$, the limit does not exist.

   $\lim_{x\to 2_-} g(x) = 0$ and $\lim_{x\to 2_+} g(x) = 0$, therefore the limit exists and $\lim_{x\to 2} g(x) = 0$

Answer to Exercise 5 (on page 19)

1. Never true. If a function is continuous at a, then $f(a) = \lim_{x\to a} f(x)$.
2. Never true. If a function is continuous at a, then $\lim x \to a^+ f(x) = \lim x \to a^- f(x)$
3. Always true. This is the definition of continuity.
4. Sometimes true. The derivative of f at $x = 3$ exists for $f(x) = x^2$ but not for $f(x) = |x-3|$.
5. Sometimes true. This statement is true for $f(x) = -(x-3)^2$ but not for $f(x) = 4x$.

Answer to Exercise 6 (on page 19)

1. $f(x)$ is not defined at $x = 3$. Therefore, it is also discontinuous at $x = 3$. As we learn about the continuity of polynomials, we will see why $f(x)$ is continuous everywhere else.
2. Here, $f(0)$ is defined, so we need to check if $\lim_{x\to 0} f(x) = f(0)$. The left and right limits as x approaches 0 are the same (∞), so the limit exists. However, $f(0) = 1 \neq \lim_{x\to 0} f(x)$. Therefore, the function is discontinuous at $x = 0$.
3. In this function, $f(3)$ is defined, so we need to check if the limit equals the function value. The limit of $f(x)$ as x approaches 3 is:

   $$\lim_{x\to 3} \frac{3x^2 - 8x - 3}{x-3} = \lim_{x\to 3} \frac{(3x+1)(x-3)}{x-3} = \lim_{x\to 3} 3x + 1 = 10$$

   So the limit exists, but $\lim x \to 3 f(x) \neq f(3)$, and we see that the function is discontinuous at $x = 3$.

Answer to Exercise 8 (on page 23)

1. True, $h(x)$ approaches 2 from the left and right, therefore the limit exists
2. False, $h(x)$ approaches 5 from the left and right, therefore the limit exists
3. False, $h(x)$ approaches 2 from the left and 4 from the right, therefore the limit does not exist
4. True, $\lim_{x\to 5} h(x) = h(5)$, therefore $h(x)$ is continuous at $x = 5$
5. True, $\lim_{x\to 4} h(x)$ does not exist, therefore $h(x)$ is discontinuous at $x = 4$
6. False, $h(2) = 1 \neq 2 = \lim_{x\to 2} h(x)$

Answer to Exercise 8 (on page 23)

1. From the quotient law, we know that:
$$\lim_{x\to 3} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 3} f(x)}{\lim_{x\to 3} g(x)}$$
From the graph, we see that:
$$\lim_{x\to 3} f(x) = -3$$
and that:
$$\lim_{x\to 3} g(x) = 1$$
Substituting these values, we get:
$$\lim_{x\to 3} \frac{f(x)}{g(x)} = \frac{-3}{1} = -3$$
2. From the Sum Law, we know that:
$$\lim_{x\to 2} [f(x) + 5g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2} 5g(x)$$
and applying the Constant Multiple Law, we see that:
$$\lim_{x\to 2} [f(x) + 5g(x)] = \lim_{x\to 2} f(x) + 5 \lim_{x\to 2} g(x)$$

Examining the graph of $f(x)$ and $g(x)$, we can determine that

$$\lim_{x \to 2} f(x) = -2$$

and

$$\lim_{x/to2} g(x) = 0$$

Substituting these values, we get:

$$\lim_{x \to 2} [f(x) + 5g(x)] = -2 + 5 \cdot 0 = -2$$

3. From the quotient law, we see that:

$$\lim_{x \to -1} \frac{3g(x)}{f(x)} = \frac{\lim_{x \to -1} 3g(x)}{\lim_{x \to -1} f(x)}$$

Applying the Constant Multiple Law, we get:

$$\lim_{x \to -1} \left[\frac{3g(x)}{f(x)}\right] = \frac{3 \lim_{x \to -1} g(x)}{\lim_{x \to -1} f(x)}$$

From the graph, we see that:

$$\lim_{x \to -1} f(x) = 2$$

and

$$\lim_{x \to -1} g(x) = 2$$

Substituting, we get:

$$\lim_{x \to -1} \left[\frac{3g(x)}{f(x)}\right] = \frac{3 \cdot 2}{2} = 3$$

4. Applying the Product and Constant Multiple Laws, we get:

$$\lim_{x \to 0} [f(x) \cdot 5g(x)] = \lim_{x \to 0} f(x) \cdot 5 \cdot \lim_{x \to 0} g(x)$$

Examining the graphs, we see that $\lim_{x \to 0} f(x)$ does not exist and $\lim_{x \to 0} g(x) = 1$. Because $\lim_{x \to 0} f(x)$ does not exist, $\lim_{x \to 0} f(x) \cdot 5 \cdot \lim_{x \to 0} g(x)$ also does not exist.

5. Applying the Difference and Constant Multiple Laws, we see that:

$$\lim_{x \to -1} [f(x) - 3g(x)] = \lim_{x \to -1} f(x) - 3 \cdot \lim_{x \to -1} g(x)$$

Examining the graphs, we see that:

$$\lim_{x \to -1} f(x) = 2$$

and

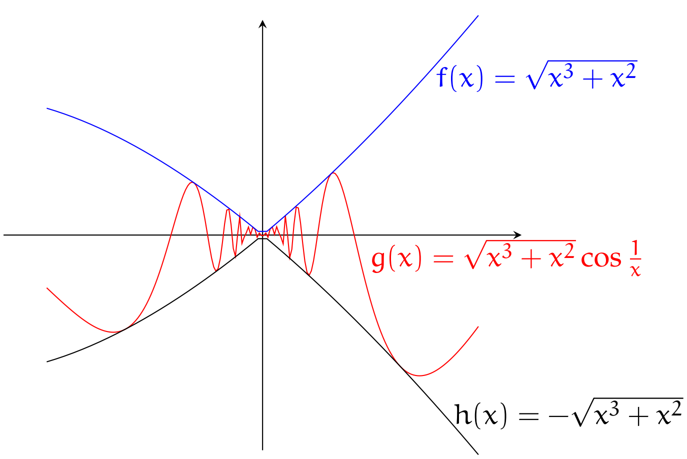
$$\lim_{x \to -1} g(x) = 2$$

Substituting, we get that:

$$\lim_{x \to -1} [f(x) - 3g(x)] = 2 - 3 \cdot 2 = 2 - 6 = -4$$

Answer to Exercise 9 (on page 26)

Let $f(x) = -\sqrt{x^3 + x^2}$ and $h(x) = \sqrt{x^3 + x^2}$. Near 0, $f(x) \leq \sqrt{x^3 + x^2} \cos \frac{1}{x} \leq h(x)$. Additionally, $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$. Therefore, by the Squeeze Theorem, we can state that $\lim_{x \to 0} \sqrt{x^3 + x^2} \cos \frac{1}{x} = 0$. Plotting all three functions, we can confirm our answer:



Answer to Exercise 10 (on page 27)

The question tells us that the function in question, $f(x)$, is between two other functions.

$\lim_{x\to 2} 2x+3 = 2(2)+3 = 7$ and $\lim_{x\to 2} x^2-2x+7 = 2^2-2(2)+7 = 7$. Since the limits are equal, by Squeeze Theorem we can also know that $\lim_{x\to 2} f(x) = 7$.

Answer to Exercise 11 (on page 27)

Note that we can only evaluate the limit from the right, as the domain for this function is $x \geq 0$. Since the range of the sine function is $[-1, 1]$, we can state that

$$-1 \leq \sin\frac{\pi}{x} \leq 1$$

and therefore

$$\frac{1}{e} \leq e^{\sin\frac{\pi}{x}} \leq e$$

. Because we assume the positive root, it is also true that

$$\frac{\sqrt{x}}{e} \leq \sqrt{x}e^{\sin\frac{\pi}{x}} \leq \sqrt{x}e$$

. Taking the limits of the border functions, we see that

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{e} = \lim_{x\to 0^+} \sqrt{x}e = 0$$

Therefore,

$$\lim_{x\to 0^+} \sqrt{x}e^{\sin\frac{\pi}{x}} = 0$$

Answer to Exercise 12 (on page 30)

1. define $f(x) = 2x^4 + x - 12$, $a = 1$, $b = 2$, and $N = 0$. Calculate $f(a)$ and $f(b)$:

   $$f(1) = -9 \text{ and } f(2) = 22$$

   Since $f(x)$ is a polynomial, it is continuous on the interval $x \in [1, 2]$ and we see that $f(a) < 0 < f(b)$. Therefore, there exists some $c \in [1, 2]$ such that $f(c) = 0$.

2. First, we can rearrange the equation we are considering and define $f(x) = \ln x - 3x + 4\sqrt{x}$, and realize we are looking for values where $f(x) = 0$. Both $\ln x$ and $\sqrt{x}$ are only continuous for $x > 0$. The interval we are interested in, $x \in [2, 3]$, is in the domain of continuity for both $\ln x$ and $\sqrt{x}$. Defining $a = 2$, $b = 3$, and $N = 0$, we find that $f(a) = 0.35 > N > -0.973 = f(b)$. Since $N \in [f(b), f(a)]$, there must exist some c such that $f(c) = N = 0$ and there is a solution to the equation $\ln x = 3x - 4\sqrt{x}$ on the interval $x \in (2, 3)$.

3. Similar to above, define $f(x) = 2\sin x - 3x^2 + 2x$, $a = 1$, $b = 2$, and $N = 0$. Calculate that $f(a) = 0.683$ and $f(b) = -6.181$. Since $f(x)$ is continuous on the interval $x \in [1, 2]$ and $f(b) < N < f(a)$, there exists some $c \in (1, 2)$ such that $f(c) = 0$. Therefore, there is a solution to $\sin x = 3x^2 - 2x$ on the interval $x \in (1, 2)$.

Answer to Exercise 13 (on page 34)

x-intercept: $(-5/2, 0)$; y-intercept: $(0, 5/4)$; horizontal asymptote: $y = 2$; vertical asymptote: $x = -4$

Answer to Exercise 15 (on page 37)

The correct answer is E. To explain, we examine the behavior of each function as $x \to \infty$.
A. $\lim_{x\to\infty} \frac{\sin 5x}{x} = \pm\infty \neq 5$
B. $\lim_{x\to\infty} 5x = \infty \neq 5$
C. $\lim_{x\to\infty} \frac{1}{x-5} = 0 \neq 5$ (this function does have a *vertical* asymptote at $x = 5$).
D. $\lim_{x\to\infty} \frac{5x}{1-x} = \frac{5}{-1} = -5 \neq 5$ (this function has a horizontal asymptote at $x = -5$).
E. $\lim_{x\to\infty} \frac{20x^2-x}{1+4x^2} = \frac{20}{4} = 5$.

Answer to Exercise 15 (on page 37)

Factored form: $\frac{x^2(x+2)}{x(x+1)}$; hole: $(0, 0)$; vertical asymptote: $x = -1$; oblique asymptote: $y = x + 1$

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