Contents

Chapter 1

Implicit Differentiation

Implicit differentiation is a technique in calculus for finding the derivative of a relation defined implicitly, that is, a relation between variables x and y that is not explicitly solved for one variable in terms of the other.

1.1 Implicit Differentiation Procedure

Consider an equation that defines a relationship between x and y:

$$F(x, y) = 0$$

To find the derivative of y with respect to x, we differentiate both sides of this equation with respect to x, treating y as an implicit function of x:

$$\frac{d}{dx}F(x, y) = \frac{d}{dx}0$$

Applying the chain rule during the differentiation on the left side of the equation gives:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Finally, we solve for $\frac{dy}{dx}$ to find the derivative of y with respect to x:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

This result is obtained using the implicit differentiation method.

1.2 Example

Consider the equation of a circle with radius r:

$$x^2 + y^2 = r^2$$

First, we'll find $\frac{dy}{dx}$ the without implicit differentiation. Then, we'll apply implicit differentiation to get the same result.

1.2.1 Without Implicit Differentiation

First, we need to re-arrange the equation to solve for y:

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

We take the derivative of y by applying the Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2 \pm \sqrt{r^2 - x^2}} \cdot (-2x) = \frac{-x}{\pm\sqrt{r^2 - x^2}}$$

Notice the denominator of this fraction is the same as the solution we found for y, $y = \pm\sqrt{r^2 - x^2}$. So we can also represent this as:

$$\frac{dy}{dx} = \frac{-x}{y}$$

1.2.2 With Implicit Differentiation

With implicit differentiation, we assume y is a function of x and apply the Chain Rule.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[r^2]$$

For x^2 and r^2, we take the derivative as we normally would. For y^2, we apply the Chain Rule as outlined above.

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{-x}{y}$$

which is the same result as we found without implicit differentiation.

1.3 Folium of Descartes

It was relatively easy to rearrange the equation for a circle to solve for y, but that is not always the case. Consider the equation for the folium of Descartes (yes, that Descartes!):

$$x^3 + y^3 = 3xy$$

It is much more difficult to isolate y in this equation. In fact, were we to do so, we would need 3 separate equations to completely describe the original equation.

1.3.1 Example: Tangent to Folium of Descartes

In this example, we will use implicit differentiation to easily find the tangent line at a point on the folium.

(a) Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

(b) Find the tangent to the folium $x^3 + y^3 = 3xy$ at the point $(2, 2)$

(c) Is there any place in the first quadrant where the tangent line is horizontal? If so, state the point(s).

Solution:

(a) $\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

Rearranging to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

(b) We already have the coordinate point, $(2, 2)$, so to write an equation for the tangent line all we need is the slope. Substituting $x = 2$ and $y = 2$ into our result from part (a):

$$\frac{dy}{dx} = \frac{2 - 2^2}{2^2 - 2} = \frac{-2}{2} = -1$$

This is the slope, m. Using the point-slope form of a line, our tangent line is $y-2 = -(x-2)$.

(c) Recall that in the first quadrant, $x > 0$ and $y > 0$. We will set our solution for $\frac{dy}{dx}$ equal to 0:

$$\frac{y - x^2}{y + 2 - x} = 0$$

which implies that

$$y - x^2 = 0$$

Substituting $y = x^2$ into the original equation:

$$x^3 + (x^2)^3 = 3(x)(x^2)$$
$$x^3 + x^6 = 3x^3$$

Which simplifies to

$$x^6 = 2x^3$$

Since we have excluded $x = 0$ by restricting our search to the first quadrant, we can divide both sides by x^3:

$$x^3 = 2$$
$$x = \sqrt[3]{2} \approx 1.26$$

Substituting $x \approx 1.26$ into our equation for y:

$$y \approx 1.26^2 = 1.59$$

Therefore, the folium has a horizontal tangent line at the point $(1.26, 1.59)$.

1.4 Practice

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] If $\arcsin x = \ln y$, what is $\frac{dy}{dx}$?

Working Space

Answer on Page 15

Exercise 2

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] The points $(-1,-1)$ and $(1,-5)$ are on the graph of a function $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = x^2+y$. Use implicit differentiation to find $\frac{d^2y}{dx^2}$. Determine if each point is a local minimum, local maximum, or inflection point by substituting the x and y values of the coordinates into $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Working Space

Answer on Page 15

Chapter 2

Related Rates

In calculus, related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. The technique used to solve these problems is known as ”related rates” because one rate is related to another rate.

2.1 Steps to solve related rates problems

2.1.1 Step 1: Understand the problem

First, read the problem carefully. Understand what rates are given and what rate you need to find.

2.1.2 Step 2: Draw a diagram

For most problems, especially geometry problems, drawing a diagram can be very helpful.

2.1.3 Step 3: Write down what you know

Write down the rates that you know and the rate that you need to find.

2.1.4 Step 4: Write an equation

Write an equation that relates the quantities in the problem. This equation will be your main tool to solve the problem.

2.1.5 Step 5: Differentiate both sides of the equation

Now you can use calculus. Differentiate both sides of the equation with respect to time.

2.1.6 Step 6: Substitute the known rates and solve for the unknown

Now that you have an equation that relates the rates, substitute the known rates into the equation and solve for the unknown rate.

2.2 Example

Here is an example of a related rates problem:

A balloon is rising at a constant rate of 5 m/s. A boy is cycling towards the balloon along a straight path at 15 m/s. If the balloon is 100 m above the ground, find the rate at which the distance from the boy to the balloon is changing when the boy is 40 m from the point on the ground directly beneath the balloon.

The problem can be modeled with a right triangle where the vertical side is the height of the balloon, the horizontal side is the distance of the boy from the point on the ground directly beneath the balloon, and the hypotenuse is the distance from the boy to the balloon.

Let x be the distance of the boy from the point on the ground directly beneath the balloon, y the height of the balloon above the ground, and z the distance from the boy to the balloon. From the Pythagorean theorem, we have

$$z^2 = x^2 + y^2 \tag{2.1}$$

Differentiating both sides with respect to time t gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \tag{2.2}$$

Given that $\frac{dx}{dt} = -15$ m/s (the boy is moving towards the point beneath the balloon), $\frac{dy}{dt} = 5$ m/s (the balloon is rising), $x = 40$ m, $y = 100$ m, we can substitute these into the equation and solve for $\frac{dz}{dt}$.

Chapter 3

Multivariate Functions

A real-valued multivariate function is a function that takes multiple real variables as input and produces a single real output.

We generally denote such a function as $f : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is the domain and $\mathbb{R}$ is the co-domain.

For example, consider a function f that takes two variables x and y:

$$f(x, y) = x^2 + y^2$$

Here, $f : \mathbb{R}^2 \to \mathbb{R}$ takes an ordered pair (x, y) from the 2-dimensional real coordinate space, squares each, and adds them to produce a real number.

In a similar way, a function $g : \mathbb{R}^3 \to \mathbb{R}$ could take three variables x, y, and z, and might be defined as:

$$g(x, y, z) = x^2 + y^2 + z^2$$

Here, the function squares each of the input variables and then adds them to produce a real number.

These functions are "real-valued" because their outputs are real numbers, and "multivariate" because they take multiple variables as inputs.

The concepts of limits, continuity, differentiability, and integrability can all be extended to multivariate functions, although they become more complex because we now have to consider different directions in which we approach a point, not just from the left or right as in the univariate case. For example, the partial derivative is the derivative of the function with respect to one variable, holding the others constant. It is one of the basic concepts in the calculus of multivariate functions.

For example, given the function $f(x, y) = x^2 + y^2$, the partial derivatives of f are computed as:

$$\frac{\partial f}{\partial x}(x, y) = 2x$$

$$\frac{\partial f}{\partial y}(x, y) = 2y$$

Chapter 4

Partial Derivatives and Gradients

This chapter will introduce you to partial derivatives and gradients, equipping you with the tools to study functions of multiple variables. We will explore how these concepts provide valuable insights into optimization, vector calculus, and various fields of science and engineering.

Partial derivatives come into play when dealing with functions that depend on multiple variables. Unlike ordinary derivatives that consider changes along a single variable, partial derivatives focus on how a function changes concerning each individual variable while holding the others constant. In essence, partial derivatives measure the rate of change of a function with respect to one variable while keeping the other variables fixed.

The notation for a partial derivative of a function $f(x, y, \ldots)$ with respect to a specific variable, say x, is denoted as $\frac{\partial f}{\partial x}$. Similarly, $\frac{\partial f}{\partial y}$ represents the partial derivative with respect to y, and so on. It is essential to remember that when taking partial derivatives, we treat the other variables as constants during the differentiation process.

The gradient is a vector that combines the partial derivatives of a function. It provides a concise representation of the direction and magnitude of the steepest ascent or descent of the function. The gradient vector points in the direction of the greatest rate of increase of the function. By understanding the gradient, we gain insights into optimizing functions and finding critical points where the function reaches maximum or minimum values.

Throughout this chapter, we will explore the following key topics related to partial derivatives and gradients:

- Calculating partial derivatives: We will delve into the techniques and rules for computing partial derivatives of various functions, including polynomials, exponential functions, and trigonometric functions. We will also explore higher-order partial derivatives and mixed partial derivatives.
- Interpreting partial derivatives: Understanding the geometric and physical interpretations of partial derivatives is essential. We will discuss the notion of tangent planes, directional derivatives, and the relationship between partial derivatives and local linearity.
- Gradient vectors and their properties: We will introduce the gradient vector and its properties, such as its connection to the direction of steepest ascent, its relationship

with partial derivatives, and how it relates to level curves and level surfaces.

- Applications of partial derivatives and gradients: We will explore various applications of these concepts, including optimization problems, constrained optimization, tangent planes, linear approximations, and their relevance in fields like physics, economics, and engineering.

By grasping the concepts of partial derivatives and gradients, you will unlock a powerful mathematical framework for analyzing and optimizing functions of multiple variables. These tools will equip you to tackle advanced calculus problems and gain deeper insights into the behavior of functions in diverse fields.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

Using implicit differentiation, we see that:

$$\frac{d}{dx}\arcsin x = \frac{d}{dx}\ln y$$
$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{y}\frac{dy}{dx}$$

Multiplying both sides by y to isolate $\frac{dy}{dx}$, we find that:

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$$

Answer to Exercise 2 (on page 7)

First, we need to find $\frac{d^y}{dx^2}$:

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}y$$
$$= 2x + \frac{dy}{dx} = 2x + x^2 + y$$

At $(-1,-1)$, $\frac{dy}{dx} = (-1)^2 + (-1) = 0$ and $\frac{d^2y}{dx^2} = 2(-1) + (-1)^2 + (-1) = -2 < 0$. Since the slope of y is zero and the graph of y is concave down, $(-1,-1)$ is a local maximum. At $(1,-5)$, $\frac{dy}{dx} = 1^2 + -5 = -4 \neq 0$ and $\frac{d^2y}{dx^2} = 2(1) + 1^2 + (-5) = -2 \neq 0$. Since neither the first nor second derivative of y are zero, $(1,-5)$ is neither a local extrema nor an inflection point.

Index