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Chapter 1

Differentiating Polynomials

If you had a function that gave you the height of an object, it would be handy to be able to figure out a function that gave you the velocity at which it was rising or falling. The process of converting the position function into a velocity function is known as *differentiation* or *finding the derivative*.

There are a bunch of rules for finding a derivative, but differentiating polynomials only requires three:

- The derivative of a sum is equal to the sum of the derivatives.
- The derivative of a constant is zero.
- The derivative of a nonconstant monomial at^b (a and b are constant numbers, t is time) is abt^{b-1}

So, for example, if we tell you that the height in meters of a quadcopter at second t is given by $2t^3 - 5t^2 + 9t + 200$. You could tell us that its vertical velocity is $6t^2 - 10t + 9$.

We indicate the derivative of a function with an apostrophe (read as "prime") between the name of the function and the variable. For example, the derivative of $h(t)$ is $h'(t)$ (which is read out loud as "h prime of t").

Exercise 1 Differentiation of polynomials

Differentiate the following polynomials.

1. $f(t) = 2t^3 - 3t^2 - 4t$
2. $g(t) = 2t^{-3/4}$
3. $F(r) = \frac{5}{r^3}$
4. $H(u) = (3u - 1)(u + 2)$

Working Space

Answer on Page 31

Notice that the degree of the derivative is one less than the degree of the original polynomial. (Unless, of course, the degree of the original is already zero.)

Now, if you know that a position is given by a polynomial, you can differentiate it to find the object's velocity at any time.

The same trick works for acceleration: Let's say you know a function that gives an object's velocity. To find its acceleration at any time, you take the derivative of the velocity function (the second derivative).

Exercise 2 Differentiation of polynomials in Python

Working Space

Write a function that returns the derivative of a polynomial in `poly.py`. It should look like this:

```
def derivative_of_polynomial(pn):
  ...Your code here...
```

When you test it in `test.py`, it should look like this:

```
# 3x**3 + 2x + 5
p1 = [5.0, 2.0, 0.0, 3.0]
d1 = poly.derivative_of_polynomial(p1)
# d1 should be 9x**2 + 2
print("Derivative of", poly.polynomial_to_string(p1),"is", poly.polynomial_to_string(d1))

# Check constant polynomials
p2 = [-9.0]
d2 = poly.derivative_of_polynomial(p2)
# d2 should be 0.0
print("Derivative of", poly.polynomial_to_string(p2),"is", poly.polynomial_to_string(d2))
```

Answer on Page 31

1.1 Second order and higher derivatives

As seen from the example, with height, velocity, and acceleration, you can take the derivative of a derivative, which is called the second derivative and is indicated with two marks, like so:

$$\frac{d}{dx}f'(x) = f''(x)$$

When you have the height function (or position function, in the case of horizontal motion) of an object, the first derivative describes the velocity of the object, and the second derivative describes the acceleration. Suppose the motion of a particle is given by $s(t) = t^3 - 5t$, where s is in meters and t is in seconds. What is the acceleration when the velocity is 0? First, we find the velocity function, $s'(t)$, and the acceleration function, $s''(t)$:

$$s'(t) = 3t^2 - 5$$

$$s''(t) = 6t$$

To find where the velocity is 0, set $s'(t) = 0$:

$$3t^2 - 5 = 0$$

$$3t^2 = 5$$

$$t^2 = \frac{5}{3}$$

$$t = \sqrt{\frac{5}{3}} \approx 1.29s$$

(we ignore the other solution, $t = -\sqrt{\frac{5}{3}}$ because it is usual for time to start at zero.)

Next, we use $t \approx 1.29s$ in the acceleration function, $s''(t)$:

$$s''(\sqrt{\frac{5}{3}}) = 6\sqrt{\frac{5}{3}} \approx 7.75\frac{m}{s^2}$$

For higher order derivatives, you just keep taking the derivative! So a third derivative is found by taking the derivative of the second derivative, and so on.

Exercise 3 Using Derivatives to Describe Motion

The position of a particle is described by the equation $s(t) = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.

(a) Find the velocity and acceleration as functions of t.

(b) Find the velocity after 1.5 s.

(c) Find the acceleration after 1.5 s.

(d) Is the object speeding up or slowing down at $t = 1.5$? How do you know?

Working Space

Answer on Page 32

Chapter 2

Python Classes

The built-in types, such as strings, have functions associated with them. So, for example, if you needed a string converted to uppercase, you would call its `upper()` function: -

```
my_string = "houston, we have a problem!"
louder_string = my_string.upper()
```

This would set `louder_string` to "HOUSTON, WE HAVE A PROBLEM!" When a function is associated with a datatype like this, it called a *method*. A datatype with methods is known as a *class*. The data of that type is known as the *instance* of that class. For example, in the example, we would say "`my_string` is an instance of the class `str`. `str` has a method called `upper`"

The function `type` will tell you the type of any data:

```
  print(type(my_string))
```

This will output

```
<class 'str'>
```

A class can also define operators. `+`, for example, is redefined by `str` to concatenate strings together:

```
long_string = "I saw " + "15 people"
```

2.1 Making a Polynomial class

You have created a bunch of useful python functions for dealing with polynomials. Notice how each one has the word "polynomial" in the function name like `derivative_of_polynomial`. Wouldn't it be more elegant if you had a Polynomial class with a `derivative` method? Then you could use your polynomial like this:

```
a = Polynomial([9.0, 0.0, 2.3])
b = Polynomial([-2.0, 4.5, 0.0, 2.1])
```

```
print(a, "plus", b , "is", a+b)
print(a, "times", b , "is", a*b)
print(a, "times", 3 , "is", a*3)
print(a, "minus", b , "is", a-b)

c = b.derivative()

print("Derivative of", b ,"is", c)
```

And it would output:

```
2.30x^2 + 9.00 plus 2.10x^3 + 4.50x + -2.00 is 2.10x^3 + 2.30x^2 + 4.50x + 7.00
2.30x^2 + 9.00 times 2.10x^3 + 4.50x + -2.00 is 4.83x^5 + 29.25x^3 + -4.60x^2 + 40.50x + -18.00
2.30x^2 + 9.00 times 3 is 6.90x^2 + 27.00
2.30x^2 + 9.00 minus 2.10x^3 + 4.50x + -2.00 is -2.10x^3 + 2.30x^2 + -4.50x + 11.00
Derivative of 2.10x^3 + 4.50x + -2.00 is 6.30x^2 + 4.50
```

Create a file for your class definition called `Polynomial.py`. Enter the following:

```
class Polynomial:
    def __init__(self, coeffs):
        self.coefficients = coeffs.copy()

    def __repr__(self):
        # Make a list of the monomial strings
        monomial_strings = []

        # For standard form we start at the largest degree
        degree = len(self.coefficients) - 1

        # Go through the list backwards
        while degree >= 0:
            coefficient = self.coefficients[degree]

            if coefficient != 0.0:
                # Describe the monomial
                if degree == 0:
                    monomial_string = "{:.2f}".format(coefficient)
                elif degree == 1:
                    monomial_string = "{:.2f}x".format(coefficient)
                else:
                    monomial_string = "{:.2f}x^{}".format(coefficient, degree)

                # Add it to the list
                monomial_strings.append(monomial_string)
```

```
            # Move to the previous term
            degree = degree - 1

        # Deal with the zero polynomial
        if len(monomial_strings) == 0:
            monomial_strings.append("0.0")

        # Separate the terms with a plus sign
        return " + ".join(monomial_strings)

    def __call__(self, x):
        sum = 0.0
        for degree, coefficient in enumerate(self.coefficients):
            sum = sum + coefficient * x ** degree
        return sum

    def __add__(self, b):
        result_length = max(len(self.coefficients), len(b.coefficients))
        result = []
        for i in range(result_length):
            if i < len(self.coefficients):
                coefficient_a = self.coefficients[i]
            else:
                coefficient_a = 0.0

            if i < len(b.coefficients):
                coefficient_b = b.coefficients[i]
            else:
                coefficient_b = 0.0
            result.append(coefficient_a + coefficient_b)

        return Polynomial(result)

    def __mul__(self, other):

        # Not a polynomial?
        if not isinstance(other, Polynomial):
            # Try to make it a constant polynomial
            other = Polynomial([other])

        # What is the degree of the resulting polynomial?
        result_degree = (len(self.coefficients) - 1) + (len(other.coefficients) - 1)

        # Make a list of zeros to hold the coefficents
        result = [0.0] * (result_degree + 1)
```

```
        # Iterate over the indices and values of a
        for a_degree, a_coefficient in enumerate(self.coefficients):

            # Iterate over the indices and values of b
            for b_degree, b_coefficient in enumerate(other.coefficients):

                # Calculate the resulting monomial
                coefficient = a_coefficient * b_coefficient
                degree = a_degree + b_degree

                # Add it to the right bucket
                result[degree] = result[degree] + coefficient

        return Polynomial(result)

    __rmul__ = __mul__

    def __sub__(self, other):
        return self + other * -1.0

    def derivative(self):

        # What is the degree of the resulting polynomial?
        original_degree = len(self.coefficients) - 1
        if original_degree > 0:
            degree_of_derivative = original_degree - 1
        else:
            degree_of_derivative = 0

        # We can ignore the constant term (skip the first coefficient)
        current_degree = 1
        result = []

        # Differentiate each monomial
        while current_degree < len(self.coefficients):
            coefficient = self.coefficients[current_degree]
            result.append(coefficient * current_degree)
            current_degree = current_degree + 1

        # No terms? Make it the zero polynomial
        if len(result) == 0:
            result.append(0.0)

        return Polynomial(result)
```

Create a second file called `test_polynomial.py` to test it:

```
from Polynomial import Polynomial

a = Polynomial([9.0, 0.0, 2.3])
b = Polynomial([-2.0, 4.5, 0.0, 2.1])

print(a, "plus", b , "is", a+b)
print(a, "times", b , "is", a*b)
print(a, "times", 3 , "is", a*3)
print(a, "minus", b , "is", a-b)

c = b.derivative()

print("Derivative of", b ,"is", c)

slope = c(3)
print("Value of the derivative at 3 is", slope)
```

Run the test code:

```
python3 test_polynomial.py
```

Chapter 3

Common Polynomial Products

In math and physics, you will run into certain kinds of polynomials over and over again. In this chapter, We are going to cover some patterns that you will want to be able to recognize.

3.1 Difference of squares

Watch **Polynomial special products: difference of squares** from Khan Academy at `https://youtu.be/uNweU6I4Icw`.

If you are asked what $(3x-7)(3x+7)$ is, you would use the distributive property to expand that to $(3x)(3x)+(3x)(7)+(-7)(3x)+(-7)(7)$. Two of the terms cancel each other, so this is $(3x)^2-(7)^2$. This would simplify to $9x^2-49$

You will see this pattern often. Anytime you see $(a+b)(a-b)$, you should immediately recognize it equals a^2-b^2. (Note that the order doesn't matter: $(a-b)(a+b)$ also a^2-b^2.)

Working the other way is important too. Any time you see a^2-b^2, that you should recognize that you can change that into the product $(a+b)(a-b)$. Making something into a product like this is known as *factoring*. You probably have done prime factorization of numbers like $42 = 2 \times 3 \times 7$. In the next couple of chapters, you will learn to factorize polynomials.

Exercise 4 Difference of Squares

Working Space

Simplify the following products:

1. $(2x - 3)(2x + 3)$
2. $(7 + 5x^3)(7 - 5x^3)$
3. $(x - a)(x + a)$
4. $(3 - \pi)(3 + \pi)$
5. $(-4x^3 + 10)(-4x^3 - 10)$
6. $(x + \sqrt{7})(x - \sqrt{7})$ Factor the following polynomials:
7. $x^2 - 9$
8. $49 - 16x^6$
9. $\pi^2 - 25x^8$
10. $x^2 - 5$

Answer on Page 32

We are often interested in the roots of a polynomial. That is, we want to know "For what values of x does the polynomial evaluate to zer?" For example, when you deal with falling bodies, the first question you might ask would be "How many seconds before the hammer hits the ground?" Once you have factored a polynomial into binomials, you can easily find the roots.

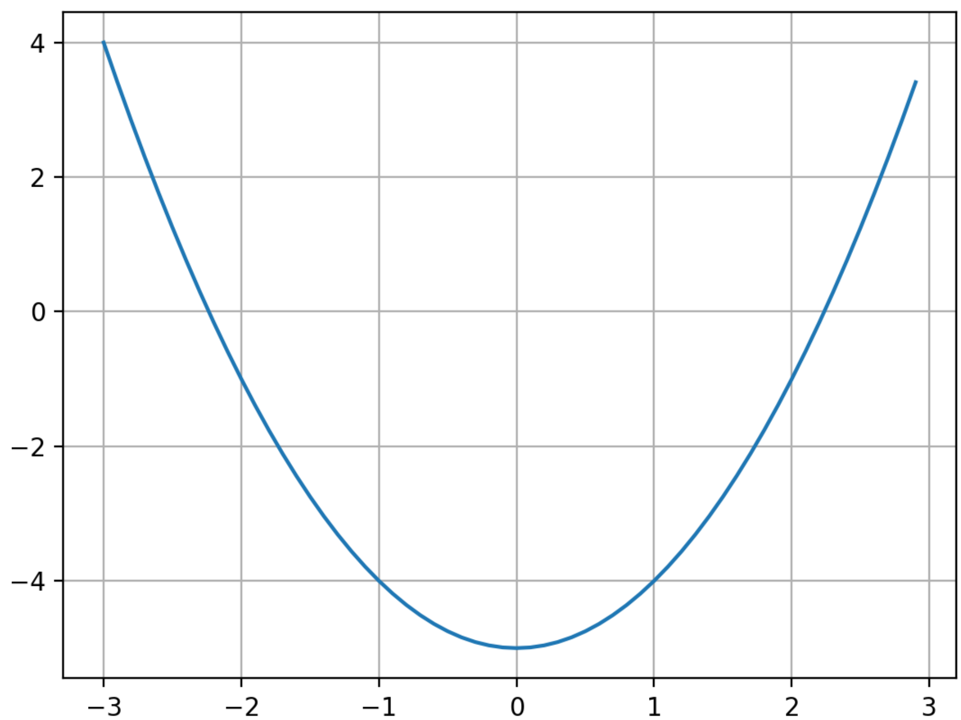
For example, what are the roots of $x^2 - 5$? You just factored it into $(x + \sqrt{5})(x - \sqrt{5})$ This product is zero if and only if one of the factors is zero. The first factor is only zero when x is $-\sqrt{5}$. The second factor is zero only when x is $\sqrt{5}$. Those are the only two roots of this polynomial.

Let's check that result. $\sqrt{5}$ is a little more than 2.2. Using your Python code, you can graph the polynomial:

```
import poly.py
import matplotlib.pyplot as plt

# x**2 - 5
pn = [-5.0, 0.0, 1.0]
```

```
# These lists will hold our x and y values
x_list = []
y_list = []

# Start at x=-3
current_x =-3.0

# End at x=3.0
while current_x < 3.0:
    current_y = poly.evaluate_polynomial(pn, current_x)

    # Add x and y to respective lists
    x_list.append(current_x)
    y_list.append(current_y)

    # Move x forward
    current_x += 0.1

# Plot the curve
plt.plot(x_list, y_list)
plt.grid(True)
plt.show()
```

You should get a plot like this:



It does, indeed, seem to cross the x-axis near -2.2 and 2.2.

3.2 Powers of binomials

You can raise whole polynomials to exponents. For example,

$$(3x^3+5)^2 = (3x^3+5)(3x^3+5)$$
$$= 9x^6 + 15x^3 + 15x^3 + 25 = 9x^6 + 30x^3 + 25$$

A polynomial with two terms is called a *binomial*. $5x^9 - 2x^4$, for example, is a binomial. In this section, we are going to develop some handy techniques for raising a binomial to some power.

Looking at the previous example, you can see that for any monomials a and b, $(a+b)^2 = a^2 + 2ab + b^2$. So, for example, $(7x^3+\pi)^2 = 49x^6 + 14\pi x^3 + \pi^2$

Exercise 5 Squaring binomials

Simply the following

1. $(x+1)^2$
2. $(3x^5+5)^2$
3. $(x^3-1)^2$
4. $(x-\sqrt{7})^2$

Working Space

Answer on Page 33

What about $(x+2)^3$? You can do it as two separate multiplications:

$$(x+2)^3 = (x+2)(x+2)(x+2)$$
$$= (x+2)(x^2+4x+4) = x^3+4x^2+4x+2x^2+8x+8$$
$$= x^3+6x^2+12x+8$$

In general, we can say that for any monomials a and b, $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

What about higher powers? $(a+b)^4$, for example? You could use the distributive property four times, but it starts to get pretty tedious.

Here is a trick. This is known as *Pascal's triangle*

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\ldots
\end{array}$$

Each entry is the sum of the two above it.

The coefficients of each term are given by the entries in Pascal's triangle:

$$(a+b)^4 = 1a^4+4a^3b+6a^2b^2+4ab^3+1b^4$$

Exercise 6 Using Pascal's Triangle

Working Space

1. What is $(x + \pi)^5$?

Answer on Page 33

Chapter 4

Factoring Polynomials

We factor a polynomial into two or more polynomials of lower degree. For example, let's say that you wanted to factor $5x^3 - 45x$. You would note that you can factor out $5x$ from every term. Thus,

$$5x^3 - 45x = (5x)(x^2 - 9)$$

You might notice that the second factor looks like the difference of squares, so

$$5x^3 - 45x = (5x)(x + 3)(x - 3)$$

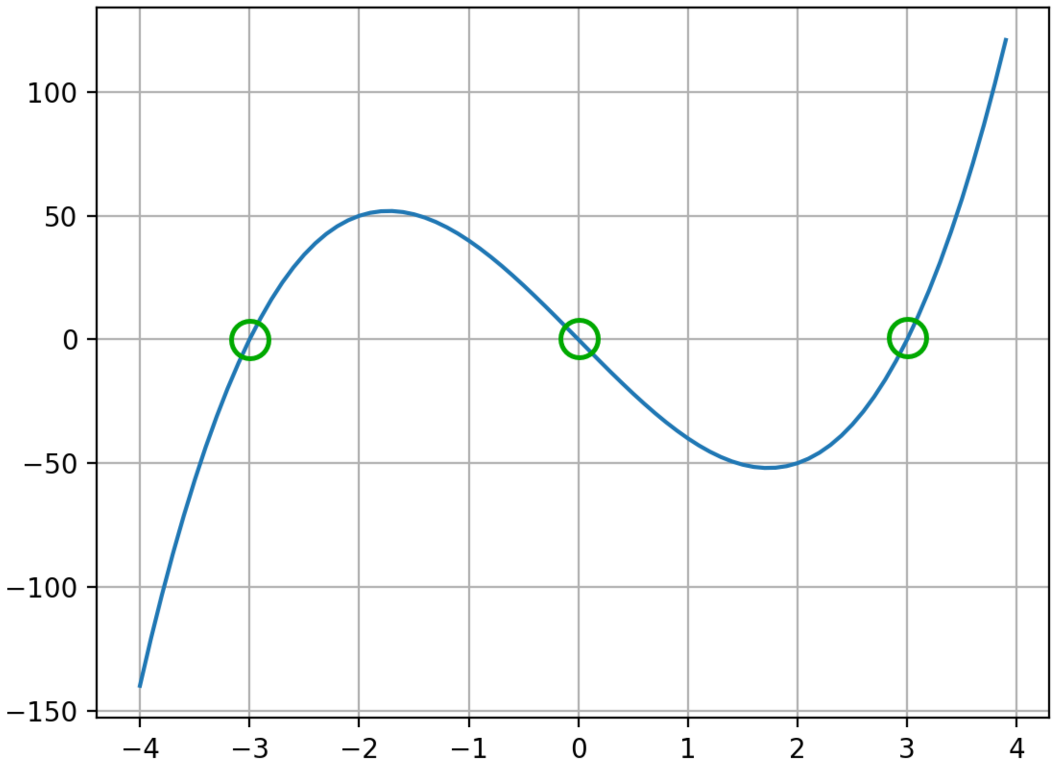
That is as far as we can factorize this polynomial.

Why do we care? The factors make it easy to find the roots of the polynomial. This polynomial evaluates to zero if and only if at least one of the factors is zero. Here, we see that

- The factor $(5x)$ is zero when x is zero.
- The factor $(x + 3)$ is zero when x is -3.
- The factor$(x - 3)$ is zero when x is 3.

So, looking at the factorization, you can see that $5x^3 - 45x$ is zero when x is 0, -3, or 3.

This is a graph of that polynomial with its roots circled:



4.1 How to factor polynomials

The first step when you are trying to factor a polynomial is to find the greatest common divisor for all the terms, then pull that out. In this case, the greatest common divisor will also be a monomial. Its degree is the least of the degrees of the terms, its coefficient will be the greatest common divisor of the coefficients of the terms.

For example, what can you pull out of this polynomial?

$$12x^100 + 30x^31 + 42x^17$$

The greatest common divisor of the coefficients (12, 30, and 42) is 6. The least of the degrees of terms (100, 31, and 17) is 17. So, you can pull out $6x^17$:

$$12x^100 + 30x^31 + 42x^17 = (6x^17)(2x^83 + 5x^14 + 7)$$

Exercise 7 Factoring out the GCD monomial

Working Space

Answer on Page 33

So, now you have the product of a monomial and a polynomial. If you are lucky, the polynomial part looks familiar, like the difference of squares or a row from Pascal's triangle.

Often, you are trying factor a quadratic like $x^2 + 5x + 6$ in a pair of binomials. In this case, the result would be $(x + 3)(x + 2)$. Let's check that:

$$(x + 3)(x + 2) = (x)(x) + (3)(x) + (2)(x) + (3)(2) = x^2 + 5x + 6$$

Notice that 3 and 2 multiply to 6 and add to 5. If you were trying to factor $x^2 + 5x + 6$, you would ask yourself"What are two numbers that when multiplied equal 6 and when added equal 5?" And you might guess wrong a couple of times. For example, you might say to youself, "Well, 6 times 1 is 6. Maybe those work. But 6 and 1 add 7. So those don't work."

Solving these sorts of problems are like solving a Sudoku puzzle. You try things and realize they are wrong, so you backtrack and try something else.

The numbers are sometimes negative. For example, $x^2 + 3x - 10$ factors into $(x + 5)(x - 2)$.

Exercise 8 Factoring quadratics

Working Space

Answer on Page 33

Chapter 5

Partial Fractions

How can you add fractions with different denominators, like $\frac{1}{x} + \frac{2}{x+3}$? You would need to make the denominators the same; after that, you could just add the numerators. You achieve this by multiplying the numerator and denominator of each fraction by the denominator of the other fraction:

$$\frac{1}{x} + \frac{2}{x+3} = \frac{1}{x}\left(\frac{x+3}{x+3}\right) + \frac{2}{x+3}\left(\frac{x}{x}\right)$$

Recall that when the numerator and denominator of a fraction are the same, the fraction is equal to one. So, we are not changing the *value* of each fraction, since we are just multiplying by one. Continuing, we can perform the multiplication and see that:

$$\frac{1}{x} + \frac{2}{x+3} = \frac{x+3}{x(x+3)} + \frac{2x}{x(x+3)}$$

$$= \frac{(x+3)+2x}{x(x+3)} = \frac{3x+3}{x^2+3x} = \frac{3(x+1)}{x^2+3x}$$

The inverse of this process is called **partial fraction decomposition** (or partial fraction expansion). This method has applications in many fields, but we will find it most useful as a tool to evaluate integrals in a later chapter.

Let $g(x)$ be a rational function such that

$$g(x) = \frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials. If $g(x)$ is proper (that is, the degree of P is less than the degree of Q) then we can express $g(x)$ as the sum of simpler rational fractions. If $g(x)$ is improper (that is, the degree of P is greater than or equal to the degree of Q), then we must first perform long division to obtain a remainder, $R(x)$, where the degree of R is less than the degree of Q:

$$g(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

5.0.1 Improper fractions

What is $\int \frac{x^3+x}{x-1}\,dx$. Using long division, we see that:

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

(see figure 5.1 for an explanation). Then we can also say that:

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

$$\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x - 1 & x^3 + 0x^2 + x \\
 & -(x^3 - x^2) \\
\cline{2-2}
 & x^2 + x \\
 & -(x^2 - x) \\
\cline{2-2}
 & 2x \\
 & -(2x - 2) \\
\cline{2-2}
 & 2
\end{array}$$

Figure 5.1: Evaluating $(x^3 + x) \div (x - 1)$ with the long division method

When you start with an improper fraction, use long division to reduce it to a term plus a proper fraction, then use the methods outlined below to further manipulate the proper fraction.

Exercise 9

Use long division to reduce the following improper rational functions to a term plus a proper rational fraction.

1. $\frac{x^4+x^3+2x^2+2x-3}{x^2-3x+2}$
2. $\frac{2x^3+5}{x^3-3x^2+2x-4}$
3. $\frac{3x^4-2x^3-x^2+1}{x^3-3x}$

Working Space

Answer on Page 33

5.0.2 Proper fractions

When the order of the numerator is less than or equal to the denominator, there are three further possibilities.

No repeated linear factors

In the first case, the denominator, $Q(x)$ is composed of distinct linear factors. In this case, we can say that $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$, where no factor is repeated (including constant multiples). Then, there exists $A, B, C, \cdots$, such that:

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x + b_1} + \frac{B}{a_2x + b_2} + \cdots$$

Let's see an example of this by decomposing $\frac{4x^2-7x-12}{x(x+2)(x-3)}$. We start by defining A, B, and C, such that:

$$\frac{4x^2 - 7x - 12}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$

Multiplying both sides by $x(x+2)(x-3)$ we get:

$$4x^2 - 7x - 12 = A(x+2)(x-3) + B(x)(x-3) + C(x)(x+2)$$

We have 3 unknowns and only one equation! Don't worry — remember this equation is true for all x, so we can choose a convenient value of x to isolate each unknown in turn. Starting, let $x = 0$. Then:

$$4(0)^2 - 7(0) - 12 = A(0+2)(0-3) + B(0)(x-3) + C(0)(x+2)$$

$$-12 = A(2)(-3) + 0 + 0$$

Notice that the B and C disappear, and we can solve for A:

$$A = \frac{-12}{-6} = 2$$

We can solve for B by setting $x = -2$ and for C by setting $x = 3$ (notice, we've used all three zeroes of the denominator polynomial):

$$4(-2)^2 - 7(-2) - 12 = A(-2+2)(-2-3) + B(-2)(-2-3) + C(-2)(-2+2)$$

$$4(4) + 14 - 12 = 0 + B(-2)(-5) + 0$$

$$16 + 2 = 10B$$

$$B = \frac{9}{5}$$

and

$$4(3)^2 - 7(3) - 12 = A(3+2)(3-3) + B(3)(3-3) + C(3)(3+2)$$
$$4(9) - 21 - 12 = 0 + 0 + C(3)(5)$$
$$36 - 33 = 15C$$
$$C = \frac{1}{5}$$

We can then decompose our original fraction:

$$\frac{4x^2 - 7x - 12}{x(x+2)(x-3)} = \frac{2}{x} + \frac{9}{5(x+2)} + \frac{1}{5(x-3)}$$

You can check your answer by cross-multiplying and adding. You should get the same rational function back.

Repeated linear factors

The second case is if $Q(x)$ has repeated factors (such as $x^2 + 8x + 16 = (x+4)^2$). Suppose the first linear factor, $(a_1x + b_1)$ is repeated r times (that is, $Q(x)$ contains the factor $(a_1x + b_1)^r$). Then, instead of $\frac{A}{a_1x+b_1}$, we should write:

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Let's look at a concrete example to see how this works:

Example: Decompose $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Solution: We start by defining:

$$\frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

Multiplying both sides by $(x+1)^2(x+2)$:

$$x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Since there are only 2 roots to $(x+1)^2(x+2)$, we will use another method called "equating the coefficients" to find A, B, and C. We start by expanding the right side of the equation:

$$x^2 + x + 1 = A(x^2 + 3x + 2) + B(x+2) + C(x^2 + 2x + 1)$$

Distributing and combining, we find that:

$$x^2 + x + 1 = Ax^2 + 3Ax + 2A + Bx + 2B + Cx^2 + 2Cx + C$$

$$x^2 + x + 1 = (A + C)x^2 + (3A + B + 2C)x + (2A + 2B + C)$$

For this equation to be true, we know that:

$$A + C = 1$$

$$3A + B + 2C = 1$$

$$2A + 2B + C = 1$$

(That is, the coefficient for x^2 on the left, 1, must be equal to the coefficient for x^2 on the right, (A + C), and so on.) We now have a system of 3 equations and 3 unknowns. When you solve for each, you should find that:

$$A = -2$$

$$B = 1$$

$$C = 3$$

Therefore,

$$\frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$

Irreducible quadratic factors

Sometimes, we cannot express a polynomial as the product of two linear statements (that is, terms in the form $ax + b$). Take $x^2 + 1$, which cannot be expressed as the product of real, linear terms. What do you do if something like $x^2 + 1$ is in the denominator? In this case, when we write an expression for $\frac{P(x)}{Q(x)}$, we include a term in the form:

$$\frac{Ax + B}{ax^2 + bx + c}$$

For example, we can write:

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}$$

Example: Decompose $\frac{2x^2 - x + 4}{x^3 + 4x}$

Solution: We begin by factoring the denominator:

$$x^3 + 4x = x(x^2 + 4)$$

Which cannot be factored further. Therefore, we define:

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx + C)x$$
$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

Which implies that:

$$2 = A + B$$
$$C = -1$$
$$4A = 4$$

Therefore, $A = 1$, $B = 1$, and $C = -1$ and we can say that:

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

Repeated irreducible quadratic factors

Lastly, the denominator might contain repeated irreducible quadratic factors. Similar to repeated linear factors, when setting up your partial fractions, instead of only writing

$$\frac{A}{ax^2 + bx + c}$$

For a quadratic factor that is repeated r times, your equation should include:

$$\frac{A_1}{ax^2 + bx + c} + \frac{A_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r}{(ax^2 + bx + c)^r}$$

Exercise 10

Decompose the following proper fractions

1. $\frac{x-4}{x^2+5x-6}$
2. $\frac{x^2+x+1}{(x^2+1)^2}$
3. $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Working Space

Answer on Page 33

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

1. $f'(t) = 3t^2 - 6t - 4$
2. $g'(t) = (\frac{-3}{4})2t^{-3/4-1} = \frac{-3}{2}t^{-7/4}$
3. $F'(r) = \frac{-15}{r^4}$
4. First, we expand the function by multiplying out the two binomials: $(3u-1)(u+2) = 3u^2 + 6u - u - 2$. Therefore, $H(u) = 3u^2 + 5u - 2$, and we can differentiate using what we have learned about differentiating polynomials. $H'(u) = 6u + 5$. In a later chapter, you will learn the Product rule, which will allow you to differentiate this function without multiplying out the binomials.

Answer to Exercise 2 (on page 4)

```
def derivative_of_polynomial(pn):

    # What is the degree of the resulting polynomial?
    original_degree = len(pn) - 1
    if original_degree > 0:
        degree_of_derivative = original_degree - 1
    else:
        degree_of_derivative = 0

    # We can ignore the constant term (skip the first coefficient)
    current_degree = 1
    result = []

    # Differentiate each monomial
    while current_degree < len(pn):
        coefficient = pn[current_degree]
        result.append(coefficient * current_degree)
        current_degree = current_degree + 1

    # No terms? Make it the zero polynomial
```

```
    if len(result) == 0:
        result.append(0.0)

    return result
```

Answer to Exercise 3 (on page 6)

(a) Velocity is the first derivative of the position function, $s'(t) = 4t^3 - 6t^2 + 2t - 1$. cceleration is the derivative of the velocity function, $s''(t) = 12t^2 - 12t + 2$.

(b) $s'(1.5) = 4(1.5)^3 - 6(1.5)^2 + 2(1.5) - 1 = 2$ We should note that this is a measurement and needs units to make sense. Since s is in meters and t is in seconds, our velocity should have units of $\frac{m}{s}$, so our final answer is $s'(1.5s) = 2\frac{m}{s}$.

(c) $s''(1.5) = 12(1.5)^2 - 12(1.5) + 2 = 11$. Similarly to part (b), our answer needs units. The units for acceleration are the units for velocity divided by the unit for time (because acceleration is a rate of change of velocity), and our final answer should be $s''(1.5s) = 11\frac{m}{s^2}$.

(d) When velocity and acceleration are occurring in the same direction (i.e. have the same sign), the speed (the absolute value of velocity) is increasing. Since $s'(1.5s)$ and $s''(1.5s)$ are both > 0, the speed of the object is increasing.

Answer to Exercise 4 (on page 14)

$(2x - 3)(2x + 3) = 4x^2 - 9$

$(7 + 5x^3)(7 - 5x^3) = 49 - 25x^6$

$(x - a)(x + a) = x^2 - a^2$

$(3 - \pi)(3 + \pi) = 9 - \pi^2$

$(-4x^3 + 10)(-4x^3 - 10) = 16x^6 - 100$

$(x + \sqrt{7})(x - \sqrt{7}) = x^2 - 7$

$x^2 - 9 = (x + 3)(x - 3)$

$49 - 16x^6 = (7 + 4x^3)(7 + 4^3)$

$\pi^2 - 25x^8 = (\pi + 5x^4)(\pi - 5x^4)$

$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$

Answer to Exercise 5 (on page 17)

$(x + 1)^2 = x^2 + 2x + 1$

$(3x^5 + 5)^2 = 9x^10 + 30x^5 + 25$

$(x^3 - 1)^2 = x^6 - 2x^3 + 1$

$(x - \sqrt{7})^2 = x^2 - 2x\sqrt{7} + 7$

Answer to Exercise 6 (on page 18)

$(x + \pi)^5 = x^5 + 5\pi x^4 + 10\pi^2 x^3 + 10\pi^3 + x^2 + 5\pi^2 x + \pi^5$

Answer to Exercise 7 (on page 21)

Answer to Exercise 8 (on page 21)

Answer to Exercise 9 (on page 24)

1. $x^2 + 4x + 12\frac{30x-27}{x^2-3x+2}$
2. $2 + \frac{6x^2-4x+13}{x^3-3x^2+2x-4}$
3. $3x - 2 + \frac{8x^2-6x+1}{x^3-3x}$

Answer to Exercise 10 (on page 29)

1. $\frac{10}{7(x+6)} + \frac{-3}{7(x-1)}$
2. $\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}$

3. $\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$

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