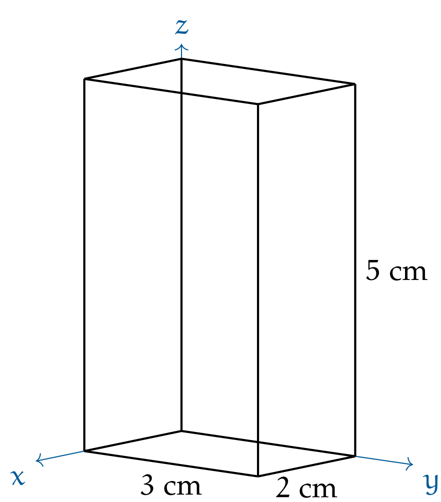
Contents

Chapter 1

Volumes of Common Solids

The volume of a rectangular solid is the product of its three dimensions. So if a block of ice is 5 cm tall, 3 cm wide and, 2 cm deep, it's volume is $5 \times 3 \times 2 = 30$ cubic centimeters.



A cubic centimeter is the same as a milliliter. A milliliter of ice weighs about 0.92 grams.

So the block of ice would have a mass of $30 \times 0.92 = 27.6$ grams.

Volume of a Sphere

A sphere with a radius of r has a volume of

$$v = \frac{4}{3}\pi r^3$$

(For completeness, the surface area of that sphere would be

$$a = 4\pi r^2$$

Note that a circle of radius r is one quarter of ths: πr^2.)

Exercise 1 Flying Sphere

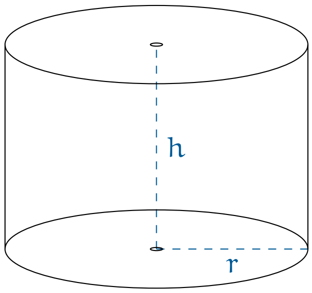
An iron sphere is traveling at 5 m/s. (It is not spinning.) The sphere has a radius of 1.5 m. Iron has a density of 7,800 kg per cubic meter. How much kinetic energy does the sphere have?

Working Space

Answer on Page 17

1.1 Cylinders

The base and the top of a right cylinder are identical circles. The circles are on parallel planes. The sides are perpendicular to those planes.



Volume of a cylinder

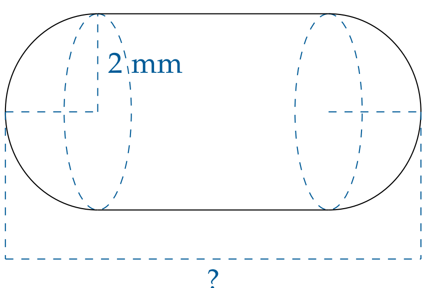
The volume of the a right cylinder of radius r and height h is given by:

$$v = \pi r^2 h$$

That is, it is the area of the base times the height.

Exercise 2 Tablet

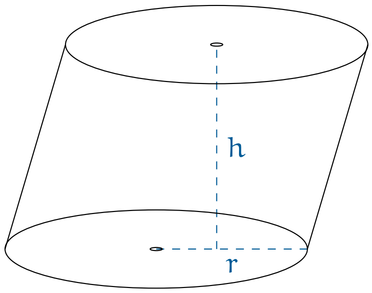
A drug company has to create a tablet with volume of 90 cubic millimeters. The tablet will be a cylinder with half spheres on each end. The radius will be 2mm.
How long do they need to make the tablet to be?



Working Space

Answer on Page 18

What if the base and top are identical, but the sides aren't perpendicular to the base? This is called *oblique cylinder*.

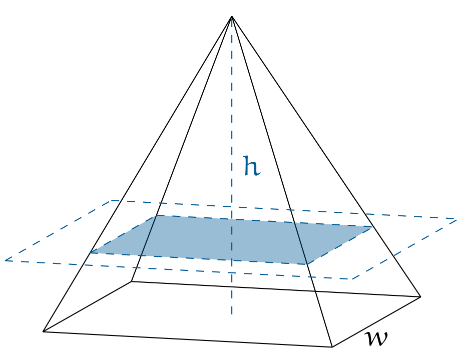


The volume is still the height times the area of the base. Note, however, that the height is measured perpendicular to the bottom and top.

Why?

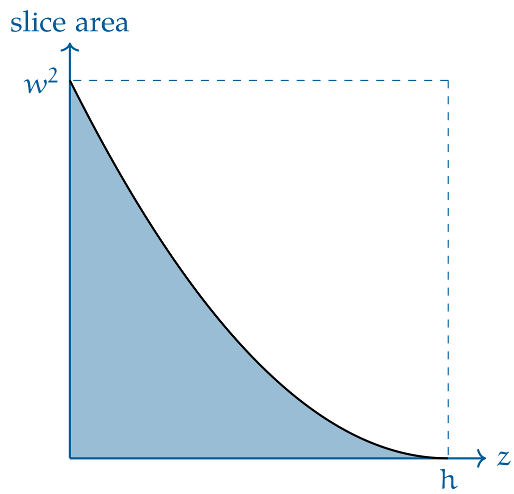
1.2 Volume, Area, and Height

On a solid with a flat base, the line that we use to measure height is always perpendicular to the plane of the base. We can take slices through the solid that are parallel to that base plane. For example, if we have a pyramid with a square base, each slice will be a square – small squares near the top, larger squares near the bottom.



We can figure out the area of the slice at every height z. For example, at $z = 0$ the slice would have area w^2. At $z = h$, the slice would have zero area. What about an arbitrary z in between? The edge of the square would be $w(1 - \frac{z}{h})$. So the area of the slice would be $w^2(1 - \frac{z}{h})^2$

The graph of this would look like this:



The volume is given by the area under the curve and above the axis. Once you learn integration, you will be really good at finding the area under the curve. In this case, I will just tell you that in the picture, the colored region is one third of the rectangle.

Thus, the area of a square-based pyramid is $\frac{1}{3}hw^2$.

In fact:

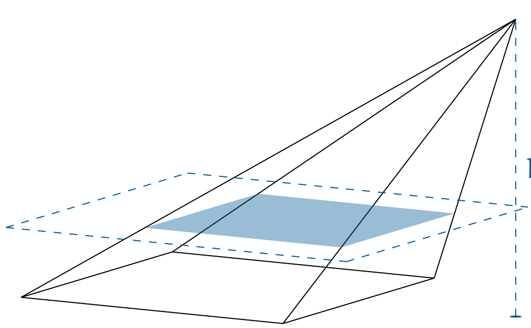
Volume of a pyramid

The volume of pyramid whose base has an area of b and height h is given by:

$$V = \frac{1}{3}hb$$

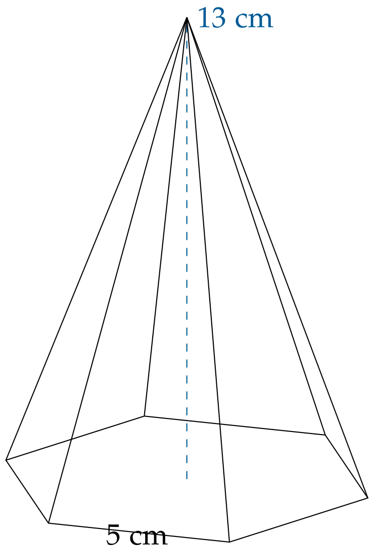
Regardless of the shape of the base.

Note that this is true even for oblique pyramids:



Exercise 3 Hexagon-based Pyramid

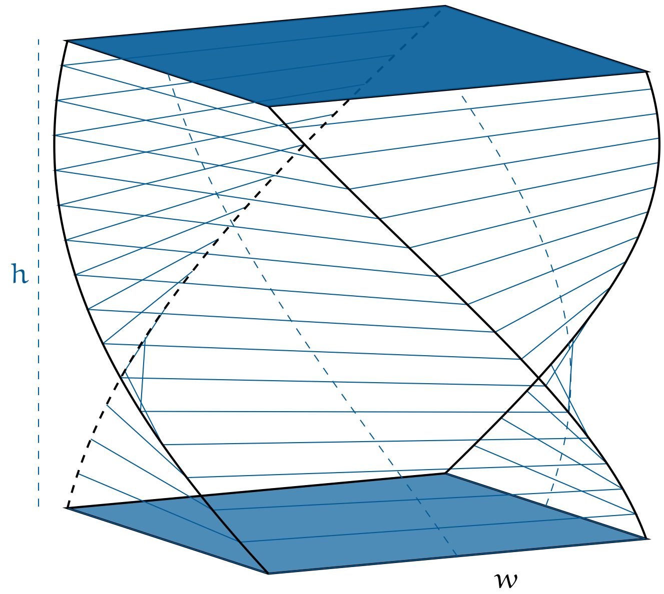
There is a pyramid with a regular hexagon for a base. Each edge is 5 cm long. The pyramid is 13 cm tall. What is its volume?



Working Space

Answer on Page 18

Note that plotting the area of each slice and finding the area under the curve will let you find the area of many things. For example, let's say that you have a four-sided spiral, where each face has the same width w:



Every slice still has an area of w^2, thus this figure has a volume of hw^2.

Exercise 4 Volume of a building

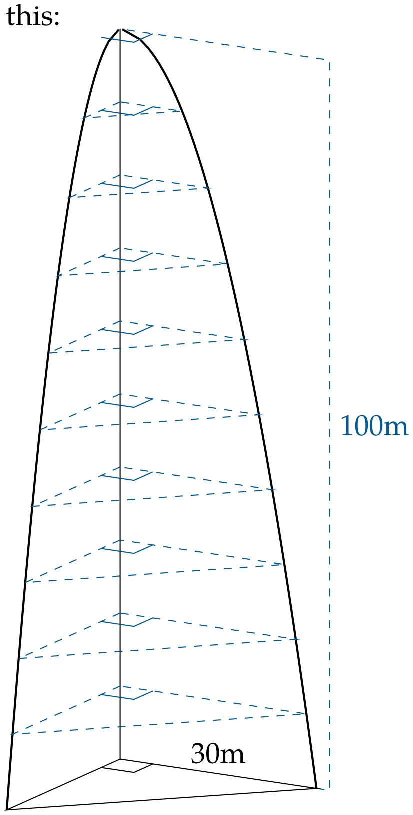
Working Space

An architect is designing a hotel with a right triangular base; the base is 30 meters on each leg. The building gets narrower as you get closer to the top, and finally shrinks to a point. The spine of the building is where the right angle is. That spine is straight and perpendicular to the ground.

Each floor has a right isosceles triangle as its floor plan. The length of each leg is given by this formula:

$$w = 30\sqrt{1 - \frac{z}{100}}$$

So the width of the building is 30 meters at height $z = 0$. At 100 meters, the building comes to a point. It will like this:



What is the volume of the building in cubic meters?

Answer on Page 19

Chapter 2

Conic Sections

In mathematics, conic sections (or simply conics) are curves obtained as the intersection of the surface of a cone with a plane. The three types of conic section are the hyperbola, the parabola, and the ellipse; the circle is a special case of the ellipse, though historically it was sometimes called a fourth type.

2.1 Definitions

Each type of conic sections can be defined as follows:

2.1.1 Circle

A circle is the set of all points in a plane that are at a given distance (the radius) from a given point (the center). The standard equation for a circle with center (h, k) and radius r is:

$$(x-h)^2 + (y-k)^2 = r^2 \tag{2.1}$$

2.1.2 Ellipse

An ellipse is the set of all points such that the sum of the distances from two fixed points (the foci) is constant. The standard equation for an ellipse centered at the origin with semi-major axis a and semi-minor axis b is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.2}$$

2.1.3 Hyperbola

A hyperbola is the set of all points such that the absolute difference of the distances from two fixed points (the foci) is constant. The standard equation for a hyperbola centered at the origin is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2.3}$$

or

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{2.4}$$

depending on the orientation of the hyperbola.

2.1.4 Parabola

A parabola is the set of all points that are equidistant from a fixed point (the focus) and a fixed line (the directrix). The standard equation for a parabola that opens upwards or downwards is:

$$y = a(x-h)^2 + k \tag{2.5}$$

and that opens leftwards or rightwards is:

$$x = a(y - k)^2 + h \tag{2.6}$$

where (h, k) is the vertex of the parabola.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

The volume of the sphere (in cubic meters) is

$$\frac{4}{3}\pi(1.5)^3 = 4.5\pi \approx 14.14$$

The mass (in kg) is $14.14 \times 7800 \approx 110,269$

The kinetic energy (in joules) is

$$k = \frac{110269 \times 5^2}{2} = 1,378,373$$

About 1.4 million joules.

Answer to Exercise 2 (on page 5)

In your mind, you can dissemble the tablet into a sphere (made up of the two ends) and a cylinder (between the two ends)

The volume of the sphere (in cubic millimeters) is

$$\frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi \approx 33.5$$

Thus the cylinder part has to be $90 - 33.5 = 56.5$ cubic mm. The cylinder part has a radius of 2 mm. If the length of the cylinder part is x, then

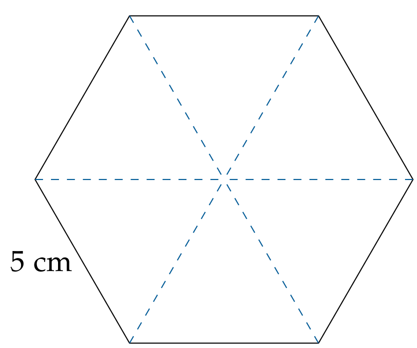
$$\pi 2^2 x = 56.5$$

Thus $x = \frac{56.5}{4\pi} \approx 4.5$ mm.

The cylinder part of the table needs to be 4.5mm. Thus the entire tablet is 8.5mm long.

Answer to Exercise 3 (on page 8)

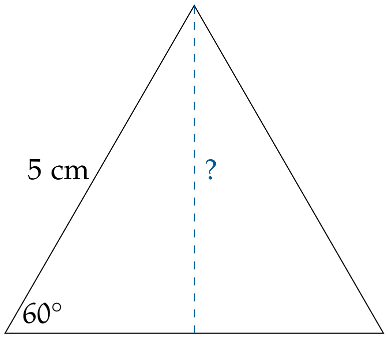
First, you need to find the area of the base, which is a regular hexagon:



All the angles in this picture are $60°$ or $\frac{\pi}{3}$ radians. Thus, each line is 5 cm long.

Thus, we need to find the area of one of these triangles and multiply that by six.

Every triangle has a base of 5cm. How tall are they?



$$5 \sin 60^\circ = 5\frac{\sqrt{3}}{2}$$

Which is about 4.33 cm.

Thus, the area of single triangle is

$$\frac{1}{2}(5)\left(5\frac{\sqrt{3}}{2}\right) = 25\frac{\sqrt{3}}{4}$$

And the area of the whole hexagon is six times that:

$$75\frac{\sqrt{3}}{2}$$

Thus, the volume of the pyramid is:

$$\frac{1}{3}hb = \frac{1}{3}13\left(75\frac{\sqrt{3}}{2}\right)$$

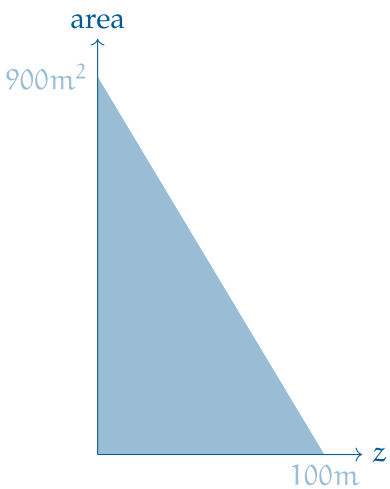
About 281.46 cubic centimeters.

Answer to Exercise 4 (on page 10)

The area at height z is given by:

$$a = \frac{1}{2}w^2 = \frac{1}{2}\left(30\sqrt{1 - \frac{z}{100}}\right)^2 = \frac{1}{2}900\left(1 - \frac{z}{100}\right)$$

If we plot that, it looks like this:



What is the area of the blue region? $\frac{1}{2}(900)(100) = 45,000$

The building will be 45 thousand cubic meters.

Index