Contents

Chapter 1

Functions and Their Graphs

You can think of a function as a machine: you put something into the machine, it processes it, and out comes something else, a product. Just as we often use the variable x to stand in for a number, we often use the variable f to stand in for a function.

For example, we might ask, "Let the function f be defined like this:

$$f(x) = -5x^2 + 12x + 2$$

What is the value of $f(3)$?"

You would run the number 3 through "the machine": $-5(3^2)+12(3)+2 = -7$. The answer would be "$f(3)$ is 7".

However, Some functions are not defined for every possible input. For example:

$$f(x) = \frac{1}{x}$$

This is defined for any x except 0, because you can't divide 1 by 0. The set of values that a function can process is called its *domain*.

Exercise 1 Domain of a function

Working Space

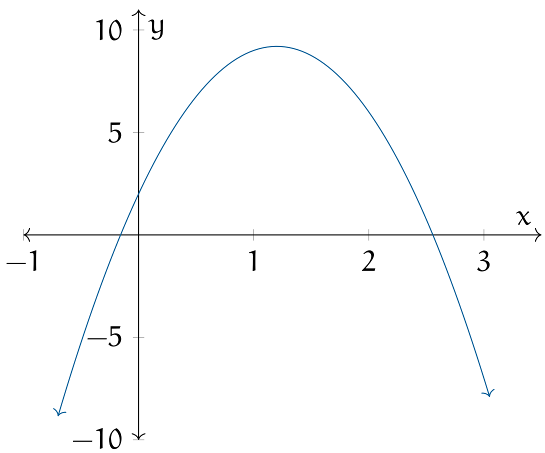
Let the function f be given by $f(x) = \sqrt{x-3}$. What is its domain?

Answer on Page 27

1.1 Graphs of Functions

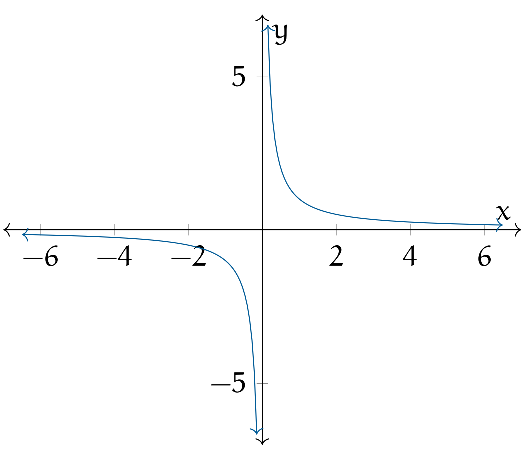
If you have a function, f, its graph is the set of pairs (x, y) such that $y = f(x)$. We usually draw a picture of this set, called a *graph*. The graph not only includes the picture, but also the values of x and y used to create it.

Here is the graph of the function $f(x) = -5x^2 + 12x + 2$:



(Note this is just part of the graph: it goes infinitely in both directions, remember your vectors.)

Here is the graph of the function $f(x) = \frac{1}{x}$:



Exercise 2 Draw a graph

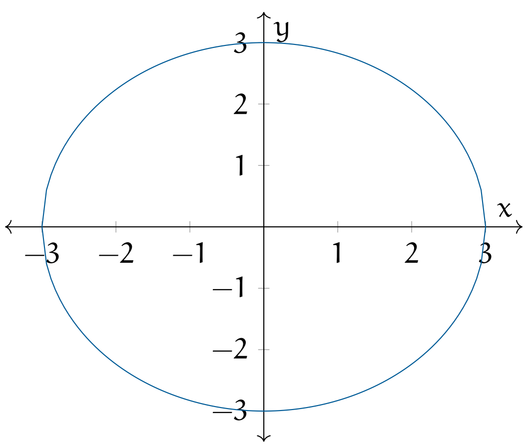
Let the function f be given by $f(x) = -3x + 3$. Sketch its graph.

Working Space

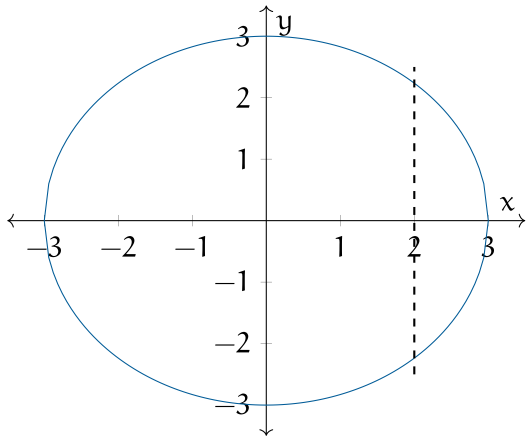
Answer on Page 27

1.2 Can this be expressed as a function?

Note that not all sets can be expressed as graphs of functions. For example, here is the set of points (x, y) such that $x^2 + y^2 = 9$:



This cannot be the graph of a function because what would $f(0)$ be? 3 or -3? This set fails what we call "the vertical line test": If any vertical line contains more than one point from the set, it isn't the graph of a function. For example, the vertical line $x = 2$ would cross the graph twice:

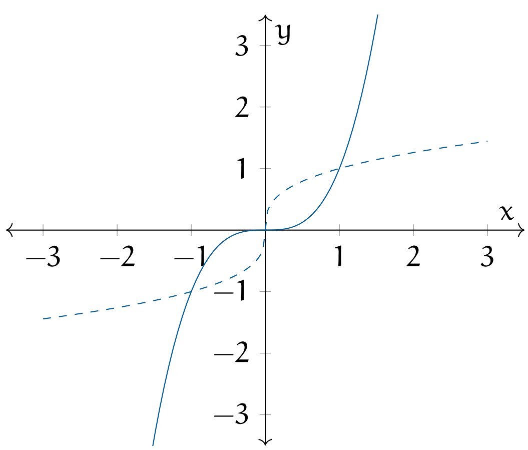


1.3 Inverses

Some functions have inverse functions. If a function f is a machine that turns number x into y, the inverse (usually denoted f^{-1}) is the machine that turns y back into x.

For example, let $f(x) = 5x + 1$. Its inverse is $f^{-1}(x) = (x - 1)/5$. (Spot check it: $f(3) = 16$ and $f^{-1}(16) = 3$)

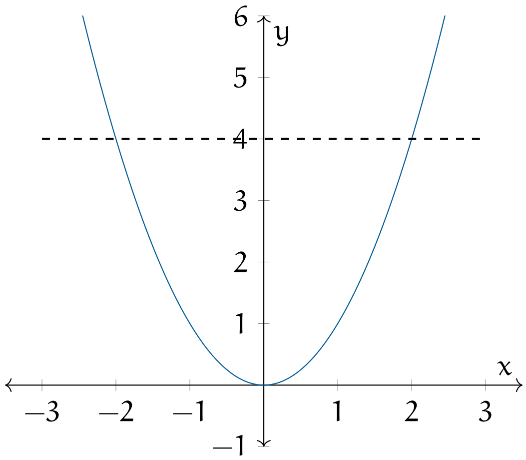
Does the function $f(x) = x^3$ have an inverse? Yes, $f^{-1}(x) = \sqrt[3]{x}$. Let's plot the function (solid line) and its inverse (dashed):



The inverse is the same as the function, just with its axes swapped. This tells us how to solve for an inverse: We swap x and y and solve for y.

For example, if you are given the function $f(x) = 5x + 1$, its graph is all (x, y) such that $y = 5x + 1$. The graph of its inverse is all (x, y) such that $x = 5y + 1$. So you solve for y: $y = (x - 1)/5$.

Not every function has an inverse. For example, $f(x) = x^2$. Note that $f(2) = f(-2) = 4$. What would $f^{-1}(4)$ be? 2 or -2? This implies the "horizontal line test": If any horizontal line contains more than one point of a function's graph, that function has no inverse. If a function passes the horizontal line test, it is called "one-to-one", meaning there is exactly one x that gives each y.



In some problems, you need an inverse and you don't need the whole domain, so you trim the domain to a set you can define an inverse on. This allow you to make claims such as "If we restrict the domain to the nonnegative numbers, the function $f(x) = x^2 - 5$ has an inverse: $f^{-1}(x) = \sqrt{x+5}$.

This begs the question: What is the domain of the inverse function f^{-1}?

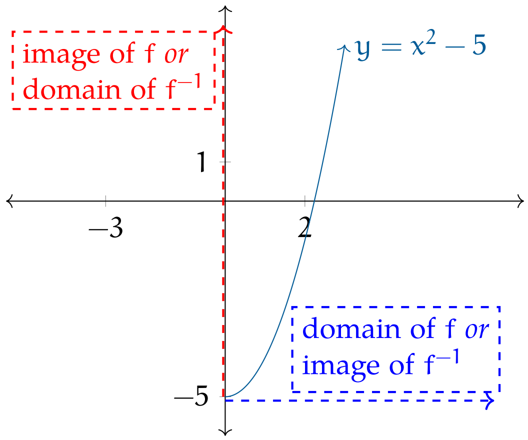
If we let X be the domain of f, we can run every member of X through "the machine" and gather them in a set on the other side. This set would be the *image* of the f "machine". (This is the *range* of f.)

What is the image of $f(x) = x^2 - 5$? It is the set of all real numbers greater than or equal to -5. We write this

$$\{x \in \mathbb{R} | x \geq -5\}$$

Now we can say: **The image of the function is the domain of the inverse function.**

In our example, we can use any number greater than or equal to -5 as input into the inverse function.



Exercise 3 Find the inverse

Let $f(x) = (x-3)^2 + 2$. Sketch the graph.

Using all the real numbers as a domain, does this function have an inverse?

How would you restrict the domain to make the function invertible?

What is the inverse of that restricted function?

What is the domain of the inverse?

Working Space

Answer on Page 27

Exercise 4

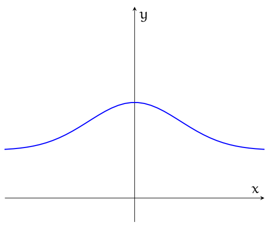
A function is given by a table of values, a graph, or a written description. Determine whether it is one-to-one.

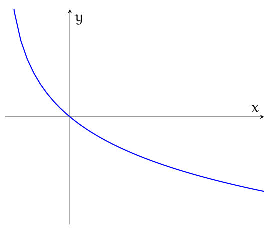
1.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

2.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

3. 

4. 

5. $f(t)$ is the height of a football t seconds after kickoff

6. $v(t)$ is the velocity of a dropped object

Working Space

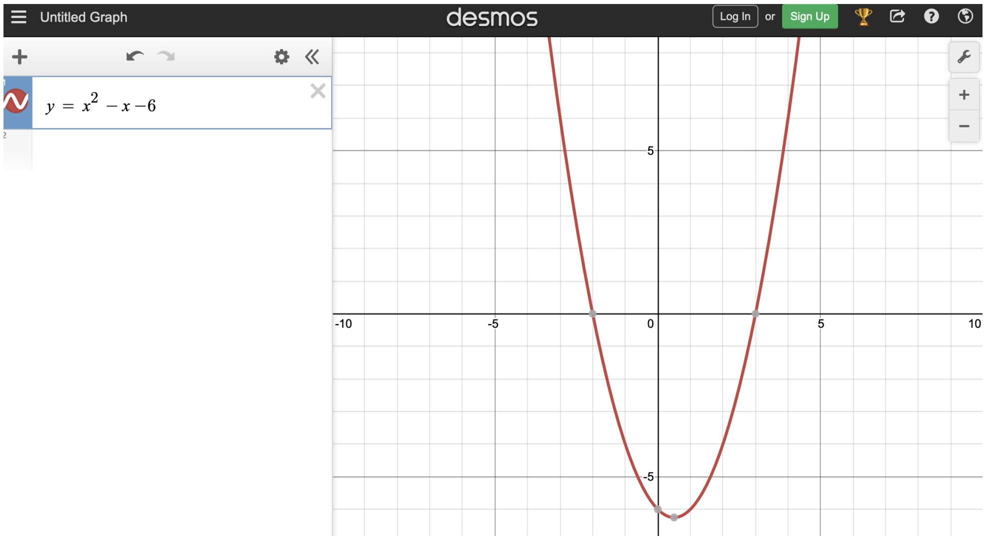
Answer on Page 28

1.4 Graphing Calculators

One really easy way to understand your function better is to use a graphing calculator. Desmos is a great, free online graphing calculator.

In a web browser, go to Desmos: https://www.desmos.com/calculator

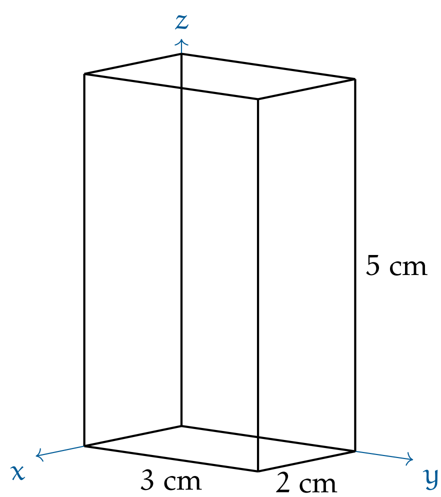
In the field on the left, enter the function $y = x^2 - x - 6$. (For the exponent, just prefix it with a caret symbol: "x2".)



Chapter 2

Volumes of Common Solids

The volume of a rectangular solid is the product of its three dimensions. So if a block of ice is 5 cm tall, 3 cm wide and, 2 cm deep, it's volume is $5 \times 3 \times 2 = 30$ cubic centimeters.



A cubic centimeter is the same as a milliliter. A milliliter of ice weighs about 0.92 grams. So the block of ice would have a mass of $30 \times 0.92 = 27.6$ grams.

Volume of a Sphere

A sphere with a radius of r has a volume of

$$v = \frac{4}{3}\pi r^3$$

(For completeness, the surface area of that sphere would be

$$a = 4\pi r^2$$

Note that a circle of radius r is one quarter of ths: πr^2.)

Exercise 5 Flying Sphere

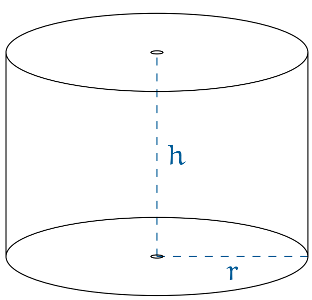
An iron sphere is traveling at 5 m/s. (It is not spinning.) The sphere has a radius of 1.5 m. Iron has a density of 7,800 kg per cubic meter. How much kinetic energy does the sphere have?

Working Space

Answer on Page 29

2.1 Cylinders

The base and the top of a right cylinder are identical circles. The circles are on parallel planes. The sides are perpendicular to those planes.



Volume of a cylinder

The volume of the a right cylinder of radius r and height h is given by:

$$v = \pi r^2 h$$

That is, it is the area of the base times the height.

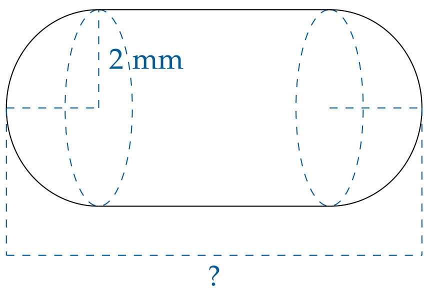
Exercise 6 Tablet

Working Space

A drug company has to create a tablet with volume of 90 cubic millimeters.

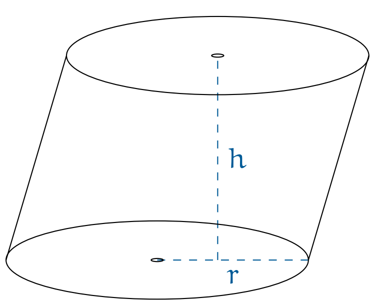
The tablet will be a cylinder with half spheres on each end. The radius will be 2mm.

How long do they need to make the tablet to be?



Answer on Page 29

What if the base and top are identical, but the sides aren't perpendicular to the base? This is called *oblique cylinder*.

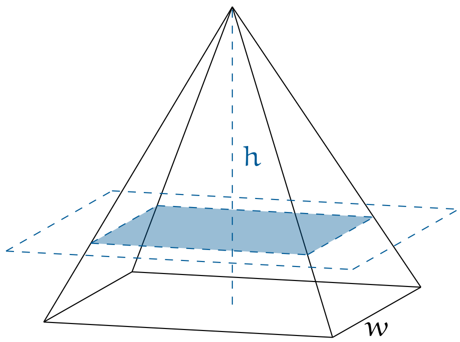


The volume is still the height times the area of the base. Note, however, that the height is measured perpendicular to the bottom and top.

Why?

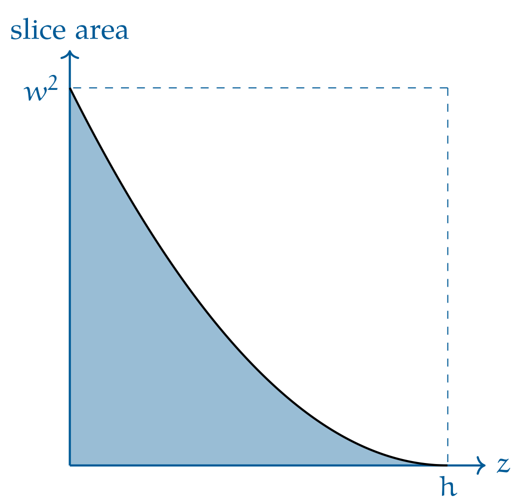
2.2 Volume, Area, and Height

On a solid with a flat base, the line that we use to measure height is always perpendicular to the plane of the base. We can take slices through the solid that are parallel to that base plane. For example, if we have a pyramid with a square base, each slice will be a square – small squares near the top, larger squares near the bottom.



We can figure out the area of the slice at every height z. For example, at $z = 0$ the slice would have area w^2. At $z = h$, the slice would have zero area. What about an arbitrary z in between? The edge of the square would be $w(1 - \frac{z}{h})$. So the area of the slice would be $w^2(1 - \frac{z}{h})^2$

The graph of this would look like this:



The volume is given by the area under the curve and above the axis. Once you learn integration, you will be really good at finding the area under the curve. In this case, I will just tell you that in the picture, the colored region is one third of the rectangle.

Thus, the area of a square-based pyramid is $\frac{1}{3}hw^2$.

In fact:

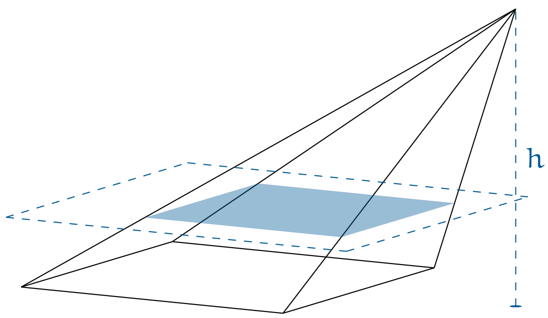
Volume of a pyramid

The volume of pyramid whose base has an area of b and height h is given by:

$$V = \frac{1}{3}hb$$

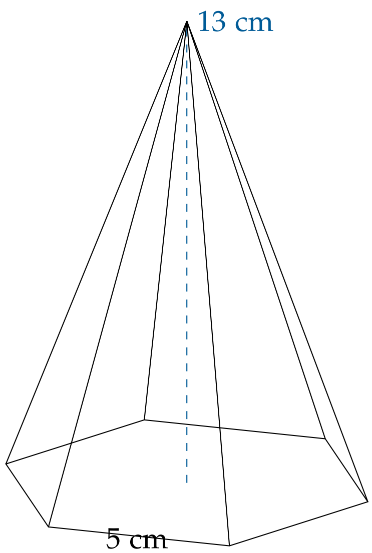
Regardless of the shape of the base.

Note that this is true even for oblique pyramids:



Exercise 7 Hexagon-based Pyramid

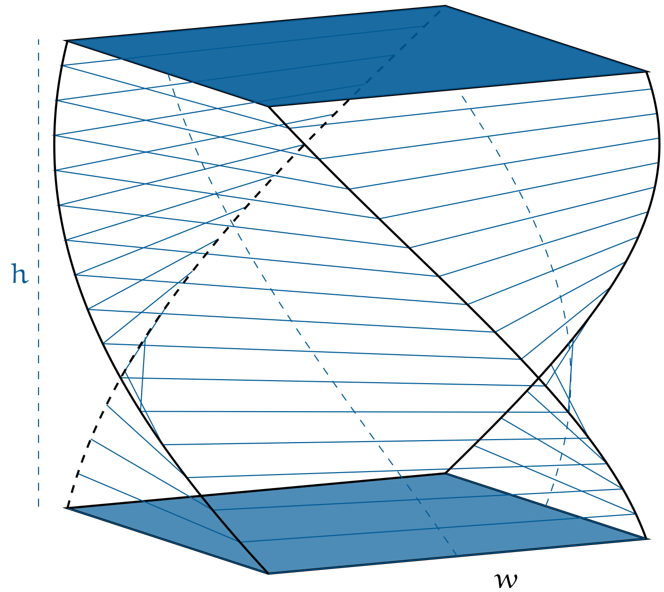
There is a pyramid with a regular hexagon for a base. Each edge is 5 cm long. The pyramid is 13 cm tall. What is its volume?



Working Space

Answer on Page 30

Note that plotting the area of each slice and finding the area under the curve will let you find the area of many things. For example, let's say that you have a four-sided spiral, where each face has the same width w:



Every slice still has an area of w^2, thus this figure has a volume of hw^2.

Exercise 8 Volume of a building

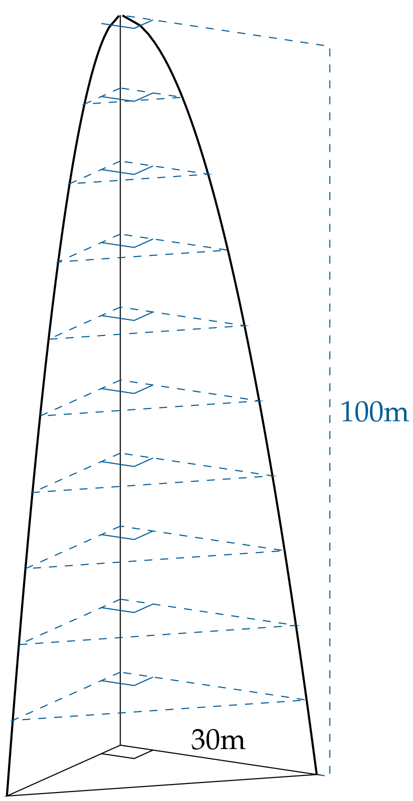
Working Space

An architect is designing a hotel with a right triangular base; the base is 30 meters on each leg. The building gets narrower as you get closer to the top, and finally shrinks to a point. The spine of the building is where the right angle is. That spine is straight and perpendicular to the ground.

Each floor has a right isosceles triangle as its floor plan. The length of each leg is given by this formula:

$$w = 30\sqrt{1 - \frac{z}{100}}$$

So the width of the building is 30 meters at height $z = 0$. At 100 meters, the building comes to a point. It will like this:



What is the volume of the building in cubic meters?

Chapter 3

Conic Sections

In mathematics, conic sections (or simply conics) are curves obtained as the intersection of the surface of a cone with a plane. The three types of conic section are the hyperbola, the parabola, and the ellipse; the circle is a special case of the ellipse, though historically it was sometimes called a fourth type.

3.1 Definitions

Each type of conic sections can be defined as follows:

3.1.1 Circle

A circle is the set of all points in a plane that are at a given distance (the radius) from a given point (the center). The standard equation for a circle with center (h, k) and radius r is:

$$(x-h)^2 + (y-k)^2 = r^2 \tag{3.1}$$

3.1.2 Ellipse

An ellipse is the set of all points such that the sum of the distances from two fixed points (the foci) is constant. The standard equation for an ellipse centered at the origin with semi-major axis a and semi-minor axis b is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3.2}$$

3.1.3 Hyperbola

A hyperbola is the set of all points such that the absolute difference of the distances from two fixed points (the foci) is constant. The standard equation for a hyperbola centered at the origin is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{3.3}$$

or

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{3.4}$$

depending on the orientation of the hyperbola.

3.1.4 Parabola

A parabola is the set of all points that are equidistant from a fixed point (the focus) and a fixed line (the directrix). The standard equation for a parabola that opens upwards or downwards is:

$$y = a(x - h)^2 + k \tag{3.5}$$

and that opens leftwards or rightwards is:

$$x = a(y - k)^2 + h \tag{3.6}$$

where (h, k) is the vertex of the parabola.

Chapter 4

Manufacturing

If you try to think of any man-made object, whether it was made from woods, metals, or plastics, chances are it was produced through a manufacturing process.

Over time, these processes have been refined to be more efficient, cost-effective, and faster at producing the goods that we use on a daily basis.

New methods are also constantly being developed by engineers and scientists, and today the range of options available means that choosing the most appropriate manufacturing method involves finding the sweet spot between cost-effectiveness, yield, and time needed.

4.1 Woods and Metals Processes

Manufacturing methods that are able to process woods and metals are typically the processes that are used to construct the vast majority of the built world around us.

Infrastructure, transportation methods, and buildings would not exist without the advent of processes that allow us to accurately machine raw woods and metals into our desired forms.

The following sections will cover methods used to process both material types.

4.2 Milling

Mills are powerful tools that allows us to carve out complex shapes from blocks of raw material. A tool bit follows a path to remove the desired material, which makes it a *subtractive* manufacturing process. The tool bit itself rotates at a very high speed, which allows for it to process harder materials such as woods and metals.

There are also various different kinds of mills, ranging from a 3-axis mill which can cut out simple shapes in the X, Y, and Z axes, all the way to 6-axis mills that can also rotate about those axes to create more complex curvatures.

In manufacturing settings where speed and repeatability is paramount, mills are often computer controlled. This functionality is referred to as *Computer Numerical Control*, or CNC. CNC mills are able to repeatedly follow a tool path, resulting in consistent and accurate parts.

4.3 Lathing

Lathes are tools that allow us to carve out complex cylindrical shapes from raw material. Like a mill it also is a subtractive manufacturing process, however this time it is the material that rotates at a very high speed instead of the tool bit. As the material rotates, the tool bit can be used to extract material layer by layer.

In the manufacturing industry lathes are also often computer controlled. Along with CNC mills, these CNC lathes are responsible for a majority of the objects that we interact with everyday.

4.4 Metal-specific Processes

While mills and lathes are already able to cover the vast majority of manufacturing needs for woods and metals, there are certain processes that are specifically enabled by the unique properties of metals. More often than not, these processes leverage the malleability of metals at room temperature and at higher temperatures as well.

4.5 Sheet Metal Processes

While milling allows us to process blocks of metal to great effect, sometimes the application we need doesn't require material of such thickness.

This is where sheet metal comes in, and the methods we use to process it. One of the most common in manufacturing is rolling, where a raw sheet is gradually rolled into the desired shape. This method is used to create many objects you may recognize, such as metal roofings, aircraft frames, and more.

Another method is stamping, where a raw sheet is stamped into the desired shape. This allows us to create surfaces with complex geometries in an instant, and in large quantities. This method is often used for applications such as the exterior panels of a car, where parts with compound curves are needed.

Lastly, there are also separate processes used to increase the strength of sheet metal parts. This falls under the category of sheet metal forming, and involves bending the edges of a part to form a reinforced flap. Almost all sheet metal parts are reinforced in this manner, as it is a relatively simple process and also helps to create a clean edge for a more finished look.

4.6 Casting

The last kind of metal-specific process we'll cover is casting. Casting involves pouring molten metal into a mold, and then letting the metal cool and set inside the mold. Smaller components with complex geometries and limited structural requirements such as toys are often cast, as it is an accurate and high-volume manufacturing method.

Casting also results in minimal material wastage, as it is not a subtractive manufacturing method where excess material is machined away, but rather only the specific amount of material required is poured in each time.

4.7 Wood-specific Processes

Similar to metals, there are also certain manufacturing processes that are enabled by the unique qualities of wood. These make use of the water content inherent in wood, and the flexibility it enables.

4.8 Bending

Turning raw wood into flat, workable pieces involves a variety of tools that you probably know of already, such as saws and drills. But there are specific processes that help us create curved shapes with wood, besides the mill and the lathe mentioned earlier.

This is where bent lamination comes in. Bent lamination involves layering multiple thin veneers or strips of wood with adhesive, and clamping it to create the desired form while letting the glue dry. This method is often used for furniture production, enabling continuous curves in wood with tight radii.

Steam can also be used for bending, by helping soften the wood fibers to increase flexibility. Once the desired form is reached, the part can then cool down and harden. Unlike bent lamination, steam bending can be done without adhesives.

4.9 Plastic-specific Processes

Although plastics only came into prominence in the mid 20th century, they have changed manufacturing and by extension the world as we know it. Easily manufacturable, durable, and cost-effective, they've come to permeate almost everything we use on a daily basis.

It must also be noted that these exact qualities have also resulted in the proliferation of plastics in our environment, and as such usage of plastics should be well considered and limited. Alternative biodegradable materials are currently being trialled by scientists and would look to replicate many of the same qualities we expect from plastics, including its manufacturability.

4.10 Injection Molding

Injection molding is responsible for the vast majority of plastic products that you interact with on a daily basis. It's extremely quick, highly accurate, and has minimal material wastage, making it a popular and cost effective method of manufacturing plastic goods.

Similar to casting, injection molding involves injecting molten plastic into a mold, and then allowing the part to cool and set inside the cavity.

You can often tell when a part was produced through injection molding, with telltale signs such as the parting line. This is where the parts of the mold meet, forming a visible line on the surface of a part.

4.11 3D Printing

Injection molding has traditionally been the go-to technique for manufacturing plastic goods, however new technologies result in the advent of new manufacturing methods. 3D printing is one of them, having come to prominence in the last few decades as a way to quickly prototype parts without having to create the molds needed for injection molding.

3D printing is an *additive* manufacturing process, where molten material is applied layer by layer to form the desired geometry. It allows for complex geometries, and whilst the accuracy may currently lag behind traditional injection molding, it is also improving rapidly.

4.12 Laser Cutting

Similarly, another manufacturing method enabled by new technologies is laser cutting. Like 3D printing it has come into prominence as a method to quickly prototype parts, however it is not an additive manufacturing process.

Instead laser cutting uses a laser beam to cut and etch through sheets of plastic, however it can also be used for other materials such as fabrics and card stocks. Laser cutting is mostly limited by material thickness, and as such can only cut through thinner sheets of material.

Appendix A

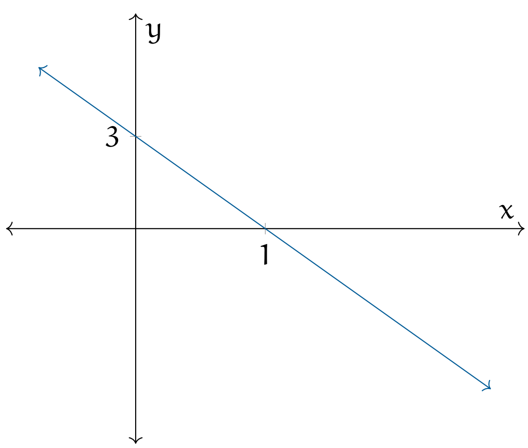
Answers to Exercises

Answer to Exercise 1 (on page 3)

You can only take the square root of nonnegative numbers, so the function is only defined when $x - 3 \geq 0$. Thus the domain is all real numbers greater than or equal to 3.

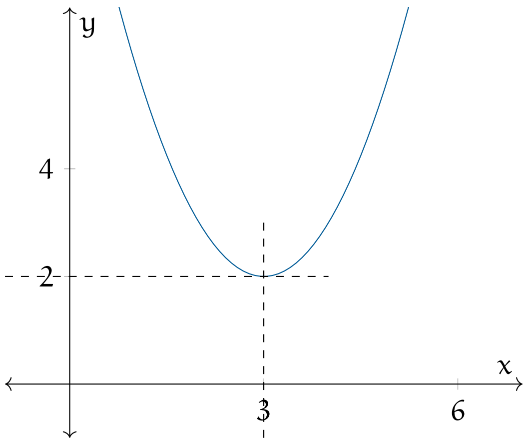
Answer to Exercise 2 (on page 5)

The graph of this function is a line. Its slope is -3. It intersects the y axis at $(0, 3)$



Answer to Exercise 3 (on page 8)

This graph is the graph of $y = x^2$ that has been moved to the right by three units and up two units:



To prevent any horizontal line from containing more than one point of the graph, you would need to use the left or the right side: Either $\{x \in \mathbb{R} | x \leq 3\}$ or $\{x \mathbb{R} | x \geq 3\}$. Most people will choose the right side; the rest of the solution will assume that you did too.

To find the inverse we swap x and y: $x = (y-3)^2 + 2$

The we solve for y to get the inverse: $y = \sqrt{x-2} + 3$

You can take the square root of nonnegative numbers. So the function $f^{-1}(x) = \sqrt{x-2}+3$ is defined whenever x is greater than or equal to 2.

Answer to Exercise 4 (on page 9)

1. This function is not one-to-one. From $x = 3$ to $x = 4$, the function increases from 3.6 to 5.3, which means it must pass through $f(x_1) = 4.0$. From $x = 4$ to $x = 5$, the function decreases from 5.3 to 2.8, which means it must pass through $f(x_2) = 4.0$ again.
2. This function is not one-to-one by a similar argument in the above solution
3. This function is not one-to-one because it fails the horizontal line test
4. This function is one-to-one because it passes the horizontal line test
5. $f(t)$ would not be one-to-one because the football must pass through each height (except the peak height) both on the way up and on the way back down
6. $v(t)$ would be one-to-one because a falling object only speeds up. Therefore, every time has a unique speed.

Answer to Exercise 5 (on page 12)

The volume of the sphere (in cubic meters) is

$$\frac{4}{3}\pi(1.5)^3 = 4.5\pi \approx 14.14$$

The mass (in kg) is $14.14 \times 7800 \approx 110,269$

The kinetic energy (in joules) is

$$k = \frac{110269 \times 5^2}{2} = 1,378,373$$

About 1.4 million joules.

Answer to Exercise 6 (on page 13)

In your mind, you can dissemble the tablet into a sphere (made up of the two ends) and a cylinder (between the two ends)

The volume of the sphere (in cubic millimeters) is

$$\frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi \approx 33.5$$

Thus the cylinder part has to be $90 - 33.5 = 56.5$ cubic mm. The cylinder part has a radius of 2 mm. If the length of the cylinder part is x, then

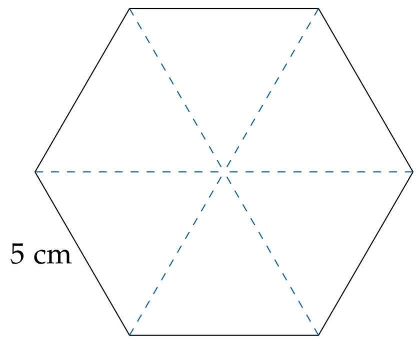
$$\pi 2^2 x = 56.5$$

Thus $x = \frac{56.5}{4\pi} \approx 4.5$ mm.

The cylinder part of the table needs to be 4.5mm. Thus the entire tablet is 8.5mm long.

Answer to Exercise 7 (on page 16)

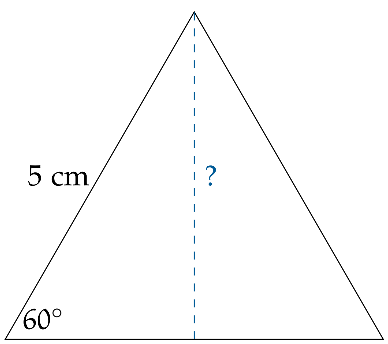
First, you need to find the area of the base, which is a regular hexagon:



All the angles in this picture are 60° or $\frac{\pi}{3}$ radians. Thus, each line is 5 cm long.

Thus, we need to find the area of one of these triangles and multiply that by six.

Every triangle has a base of 5cm. How tall are they?



$$5 \sin 60^\circ = 5\frac{\sqrt{3}}{2}$$

Which is about 4.33 cm.

Thus, the area of single triangle is

$$\frac{1}{2}(5)\left(5\frac{\sqrt{3}}{2}\right) = 25\frac{\sqrt{3}}{4}$$

And the area of the whole hexagon is six times that:

$$75\frac{\sqrt{3}}{2}$$

Thus, the volume of the pyramid is:

$$\frac{1}{3}hb = \frac{1}{3}13\left(75\frac{\sqrt{3}}{2}\right)$$

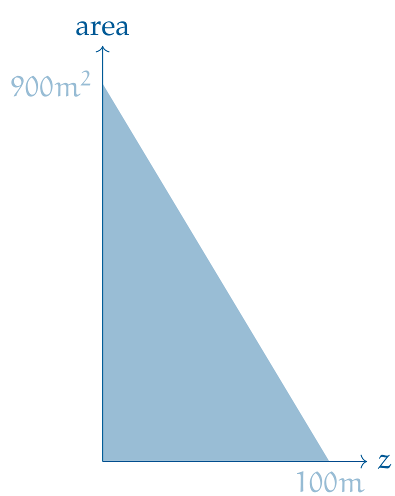
About 281.46 cubic centimeters.

Answer to Exercise 8 (on page 18)

The area at height z is given by:

$$a = \frac{1}{2}w^2 = \frac{1}{2}\left(30\sqrt{1-\frac{z}{100}}\right)^2 = \frac{1}{2}900\left(1-\frac{z}{100}\right)$$

If we plot that, it looks like this:



What is the area of the blue region? $\frac{1}{2}(900)(100) = 45,000$

The building will be 45 thousand cubic meters.

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