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Chapter 1

Multiple Integrals

In this chapter, we extend this powerful idea into higher dimensions using the tools of multiple integration. While single integration enables us to calculate the area under a curve or the volume under a surface, multiple integration allows us to calculate volumes in three dimensions, and even hypervolumes in higher dimensions.

We start by discussing double integration, which allows us to find the volume under a surface in three dimensions. This method involves slicing the solid into infinitesimally thin disks, and summing the volumes of these disks.

Next, we'll cover triple integration, a tool that lets us find the volume of more complicated solids in three-dimensional space. The idea is similar to double integration, but instead of slicing the solid into disks, we slice it into infinitesimally small cubes.

To properly implement these techniques, we'll also discuss the different coordinate systems that can be used in multiple integration, such as rectangular, cylindrical, and spherical coordinates, and when it's advantageous to use one system over another.

By the end of this chapter, you will have a deeper understanding of the techniques of multiple integration and how to apply them to find the volumes of various types of solids. The methods we study here will serve as a foundation for many topics in higher mathematics and physics, including electromagnetism, fluid dynamics, and quantum mechanics.

Exercise 1 Using Polar Coordinates in Multiple Integration

1. Use double integration to find the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy-plane.

Working Space

Answer on Page 9

Chapter 2

Multivariate Distributions

The world of probability and statistics doesn't limit itself to the study of single variables. Often, we are interested in the interconnections, relationships, and associations among several variables. In such a scenario, the univariate distributions that we have studied so far become inadequate. To comprehend the joint behavior of these variables and to uncover the underlying patterns of dependency, we must turn to the realm of multivariate distributions.

This chapter aims to introduce the reader to the concept of multivariate probability distributions. These are probability distributions that take into account and describe the behavior of more than one random variable. We will start our exploration with a discussion on the joint probability mass and density functions. These functions extend the concepts of probability mass and density functions for one variable to the situation where we have multiple variables.

Next, we will explore important properties of joint distributions, including the concept of marginal distribution and conditional distribution, which allow us to explore the probability of a subset of variables while conditioning on, or ignoring, other variables. We will also introduce the idea of independence of random variables in the multivariate context.

Subsequently, we will discuss some commonly used multivariate distributions such as the multivariate normal distribution, and the multivariate Bernoulli and binomial distributions. These specific distributions will provide us with practical tools for modelling multivariate data.

Finally, we will delve into covariance and correlation, two key measures that give us a sense of how two variables change together. Understanding these measures is critical for capturing the relationships in multivariate data.

Chapter 3

The Multivariate Normal Distribution

The multivariate normal distribution is a generalization of the one-dimensional (univariate) normal distribution to higher dimensions. It is used in statistics to describe any set of correlated real-valued random variables.

3.1 Multivariate Normal Distribution

A random vector $X = [X_1, X_2, ..., X_n]^T$ follows a multivariate normal distribution if every linear combination of its components has a univariate normal distribution. The distribution is parameterized by a mean vector and a covariance matrix.

The probability density function (pdf) of an n-dimensional multivariate normal distribution is given by:

$$f(\mathbf{x}|\boldsymbol{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^n|\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

where:

- $\mathbf{x} = [x_1, x_2, ..., x_n]^T$ is the point up to which the function is integrated,
- $\boldsymbol{\mu} = [\mu_1, \mu_2, ..., \mu_n]^T$ is the mean vector,
- Σ is the covariance matrix,
- $|\Sigma|$ denotes the determinant of the covariance matrix,
- T denotes the matrix transpose.

3.2 Covariance Matrix

The covariance matrix, Σ, is a symmetric matrix that contains information about the variance of each variable and the covariance between every pair of variables in the distribution.

The element Σ_{ij} is the covariance between the i-th and the j-th random variable, and Σ_{ii} is the variance of the i-th random variable.

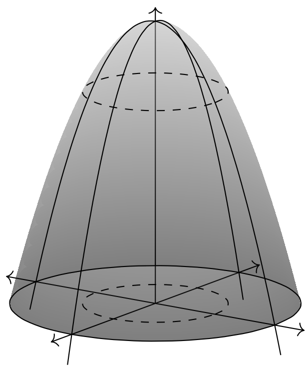
The covariance matrix provides a measure of how much each of the dimensions varies from the mean with respect to each other. A positive covariance between two variables indicates that the variables increase or decrease together, whereas a negative covariance indicates that one variable increases when the other decreases.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

We are finding the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy-plane.



We can use polar coordinates to simplify the double integral. In polar coordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so $x^2 + y^2 = r^2$. The volume under the surface and above the xy-plane is given by

$$V = \int\int (4 - r^2) r \, dr \, d\theta, \tag{1.1}$$

where r ranges from 0 to 2 (since $4 - r^2 \geq 0$ if $0 \leq r \leq 2$) and θ ranges from 0 to 2π.

Hence,

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 (4r - r^3)\, dr\, d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{1}{4} r^4 \right]_0^2 d\theta \\
&= \int_0^{2\pi} (8 - 4)\, d\theta \\
&= \int_0^{2\pi} 4\, d\theta \\
&= [4\theta]_0^{2\pi} \\
&= 8\pi.
\end{aligned}
$$

So the volume of the solid is 8π cubic units.

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