Contents

Chapter 1

Antiderivatives

In your study of calculus, you have learned about derivatives, which allow us to find the rate of change of a function at any given point. Derivatives are powerful tools that help us analyze the behavior of functions. Now, we will explore another concept called antiderivatives, which are closely related to derivatives.

An antiderivative, also known as an integral or primitive, is the reverse process of differentiation. It involves finding a function whose derivative is equal to a given function. In simple terms, if you have a function and you want to find another function that, when differentiated, gives you the original function back, you are looking for its antiderivative. Consider the graph of $f(x)$ below. We can sketch a possible antiderivative of f by noting the slope of the antiderivative is equal to the value of f. We will refer to the antiderivative of f as $F(x)$ (that is, $F'(x) = f(x)$).

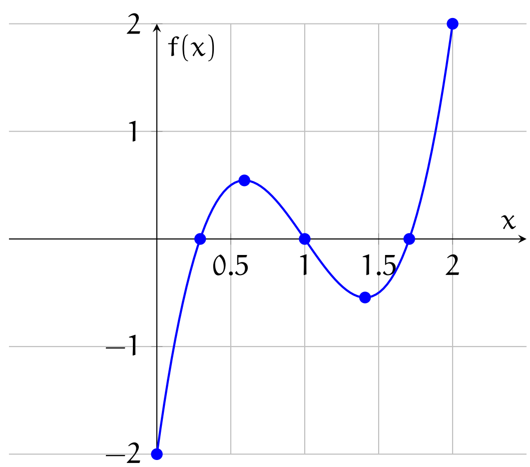

Figure 1.1: Plot of f with select points

If we are given a coordinate for $F(x)$, then we can use the graph of $f(x)$ to sketch $F(x)$. Suppose we know that $F(0) = 1$. From the graph of f, we also know that

x	$f(x) =$ slope of $F(x)$
0	-2
≈ 0.3	0
≈ 0.6	≈ 0.5
1	0
≈ 1.4	≈ -0.4
≈ 1.7	0
2	2

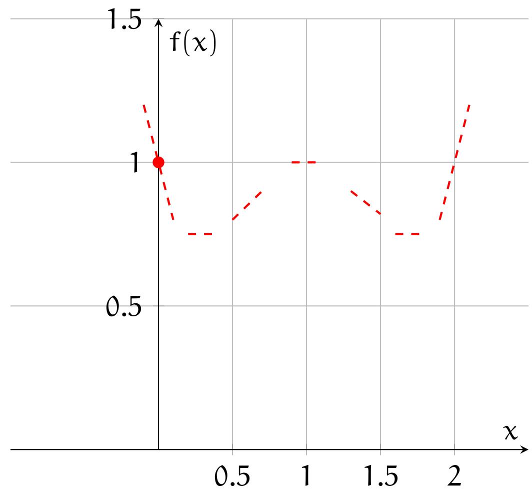

Figure 1.2: Beginning sketch of $F(x)$

Then we can connect these slopes to have an approximate sketch of $F(x)$:

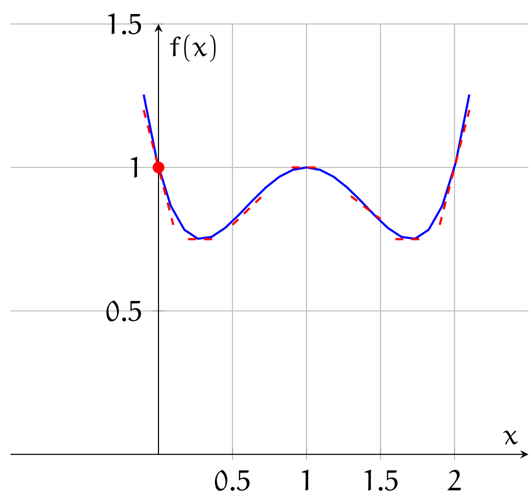

Figure 1.3: Sketch of $F(x)$

The symbol used to represent an antiderivative is $\int$. It is called the integral sign. For example, if $f(x)$ is a function, then the antiderivative of $f(x)$ with respect to x is denoted as $\int f(x)\,dx$. The dx at the end indicates that we are integrating with respect to x.

Another way to state this is that F is the antiderivative of f on an interval, I, if $F'(x) = f(x)$ over the interval. The relationship between f and F is discussed more in the chapter on the Fundamental Theorem of Calculus.

Finding antiderivatives requires using specific techniques and rules. Some common antiderivative rules include:

- The power rule: If $f(x) = x^n$, where n is any real number except -1, then the antiderivative of $f(x)$ is given by $\int f(x)\,dx = \frac{1}{n+1}x^{n+1} + C$, where C is the constant of integration.
- The constant rule: The antiderivative of a constant function is equal to the constant times x. For example, if $f(x) = 5$, then $\int f(x)\,dx = 5x + C$.
- The sum and difference rule: If $f(x)$ and $g(x)$ are functions, then $\int(f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$. Similarly, $\int(f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$.

Antiderivatives have various applications in mathematics and science. They allow us to calculate the total accumulation of a quantity over a given interval, compute areas under curves, and solve differential equations, among other things.

1.1 General Antiderivatives

It is important to note that an antiderivative is not a unique function. Since the derivative of a constant is zero, any constant added to an antiderivative will still be an antiderivative of the original function. This is why we include the constant of integration, denoted by C, in the antiderivative expression.

Stated formally, if F is an antiderivative of f on interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

A concrete example of this is $f(x) = x^2$. Let us define $F(x)$ such that $F'(x) = f(x)$. That is, there is some function F such that the derivative of F is x^2. One possible solution for F is $F(x) = \frac{1}{3}x^3$. You can check using the power rule that $\frac{d}{dx}F(x) = f(x)$. What if we added or subtracted a constant from F? Let us define $G(x) = \frac{1}{3}x^3 + 2$. Well, $G'(x) = f(x)$ also! Same for $H(x) = \frac{1}{3}x^3 - 7$. Several possible antiderivatives of $f(x) = x^2$ are shown in figure 1.4.

Since taking a derivative "erases" any constant, you must always add back in the unknown constant, C, when finding the general antiderivative.

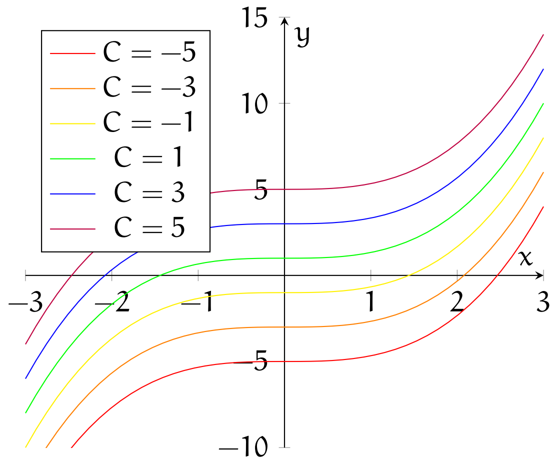

Figure 1.4: If $F'(x) = x^2$, then the general solution is $F(x) = \frac{1}{3}x^3 + C$

1.2 Specific Antiderivatives

If you are given a condition, you can often solve for C and find a specific antiderivative. For example, suppose that in addition to knowing that $F'(x) = x^2$, we also know that $F(3) = 2$. We can use the fact that F passes through $(3, 2)$ to find the value of C:

$$
\begin{aligned}
F(x) &= \frac{1}{3}x^3 + C \\
F(3) &= \frac{1}{3}(3)^2 + C = 2 \\
39 + C &= 2 \\
C &= -7
\end{aligned}
$$

Therefore, the specific solution to $F'(x) = x^2$ with the condition that $F(3) = 2$ is $F(x) = \frac{1}{3}x^3 - 7$.

1.3 Antiderivatives of Trig Functions

We already know that $\frac{d}{dx} \sin x = \cos x$. Taking $\sin x$ to be $F(x)$ and $\cos x$ to be $f(x)$, we see that $F'(x) = f(x)$ and therefore $\sin x$ is the antiderivative of $\cos x$.

Exercise 1

What is the antiderivative of $\sin x$? Explain your answer.

Working Space

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You should have found that the antiderivative of $\sin x$ is $-\cos x$. Other general antiderivatives of trigonometric functions are presented in the table below.

Function	Antiderivative
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
$-\csc^2 x$	$\cot x + C$
$-\csc x \cot x$	$\csc x + C$

Notice this is the flipped version of the derivatives of trigonometric functions presented in the Trigonometric Functions chapter. This hints at the relationship between derivatives and integrals: they are opposite processes.

1.4 Other Important Antidervatives

The Power Rule only applies when $n \neq -1$. Then what is the antiderivative of $f(x) = \frac{1}{x}$? Recall from the chapters on derivatives that $\frac{d}{dx} \ln x = \frac{1}{x}$ (see figure 1.5). Therefore, the general antiderivative of $\frac{1}{x}$ is $\ln |x| + C$. We have to take the absolute value because of the domain restrictions of $\ln x$. Notice that for $x < 0$, the slope of $\ln |x|$ is negative and decreasing (becoming more negative) and the value of $\frac{1}{x}$ is also negative and decreasing. Similarly, for $x > 0$, the slope of $\ln |x|$ is positive and decreasing (becoming less positive) and the value of $\frac{1}{x}$ is also positive and decreasing.

Since the derivative of e^x is e^x, it follows that the general antiderivative of e^x is $e^x + C$. What if there is a multiplying factor in the exponent, such as e^{kx}? Recall that $\frac{d}{dx} e^{kx} = ke^{kx}$. It follows that $\frac{d}{dx} \frac{1}{k} e^{kx} = e^{kx}$. Therefore, the general antiderivative of e^{kx} is $\frac{1}{k} e^{kx} + C$. (See figure 1.6 for an example where $k = 2$.)

Often, the base of an exponential function isn't e. We can also find the general antiderivative of b^x, where $b \neq e$. Recall that $\frac{d}{dx} b^x = \ln b b^x$. Therefore $\frac{d}{dx} \frac{1}{\ln b} b^x = b^x$, and the general

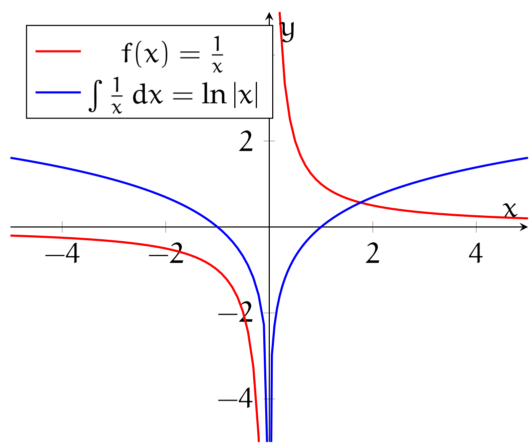

Figure 1.5: $\frac{1}{x}$ and its antiderivative, $\ln|x|$

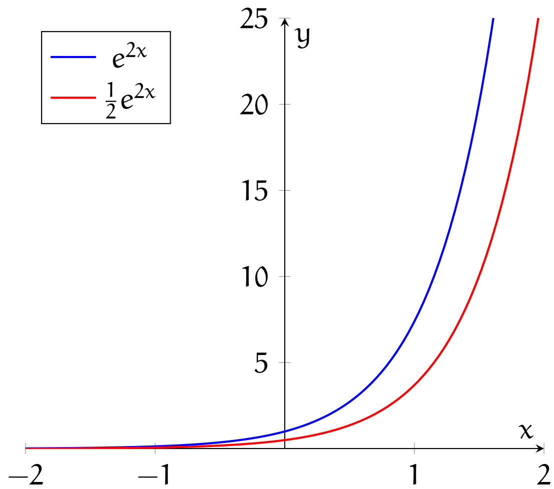

Figure 1.6: e^{2x} and its antiderivative $\frac{1}{2}e^{2x}$

antiderivative of b^x is $\frac{b^x}{\ln b}$.

1.5 Higher order antiderivatives

What if we are given the second order derivative, or a higher order? Take this example: $f''(x) = 2x + 3e^x$. The antiderivative of f'' is f'. Applying the Power Rule and knowing the antiderivative of e^x is e^x, we find that $f'(x) = x^2 + 3e^x + C_1$. We designate the constant as C_1 because we'll have to determine the antiderivative a second time and we don't want to confuse our constants with each other. To find f, we apply the Power Rule again, and we find that $f(x) = \frac{1}{3}x^3 + 3e^x + C_1x + C_2$. You can check if this is correct by taking the derivative of $f(x)$ twice, which should yield the $f''(x)$ originally given.

In summary, antiderivatives are the reverse process of differentiation. They help us find functions whose derivatives match a given function. Understanding antiderivatives is crucial for various advanced calculus concepts and real-world applications.

Now, let's explore different techniques and methods for finding antiderivatives and discover how they can be applied in solving problems.

1.6 Additional Practice

Exercise 2

A particle moving in a straight line has an acceleration given by $a(t) = 6t+4$ (in units of $\frac{cm}{s^s}$). If its initial velocity is $-6\frac{cm}{s}$ and its initial position is $9cm$, what is the function $s(t)$ that describes the particle's position in cm?

Working Space

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Exercise 3

Let $f'(x) = 2\sin x$. If $f(\pi) = 1$, write an expression for $f(x)$.

Working Space

Answer on Page 115

Exercise 4

Find the general antiderivatives of the following functions:

1. $f(x) = x^2 + 2x - 4$
2. $g(x) = \sqrt[3]{x^2} + x\sqrt{x}$
3. $h(x) = \frac{1}{5} - \frac{2}{x}$
4. $r(\theta) = 2\sin\theta - \sec^2\theta$

Working Space

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Exercise 5

Find the f that satisfies the given conditions:

1. $f'(\theta) = \sin\theta + \cos\theta$, $f(\pi) = 2$
2. $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$ and $f(1) = 1$

Working Space

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Chapter 2

Riemann Sums

2.1 The Meaning of the Area Under a Function

Let's look at the example of a hammer tossed in the air from a previous chapter. As you may recall, if a hammer is tossed up from the ground at 5 m/s, its velocity can be described as $v(t) = 5 - 9.8t$ (on Earth, where the acceleration due to gravity is approximately -9.8 $\frac{m}{s^2}$). The velocity function of our hammer from when it is tossed ($t = 0$) to when it hits the ground $t \approx 1.02$) is shown in figure 2.1.

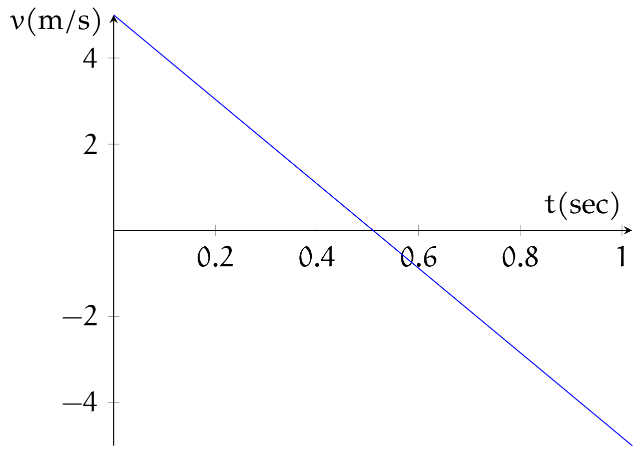


Figure 2.1: Velocity of a hammer thrown upwards at 5 m/s

Now, suppose we only have this velocity function and we want to know how high above its initial position the hammer is tossed. Examine the graph: at approximately what time does the hammer reach its peak height? (Hint: what should the hammer's velocity be when it reaches its peak?). At the highest point of its flight, the hammer's velocity will be 0 $\frac{m}{s}$, which occurs at approximately $t = 0.5s$ (it's actually $t = 0.5102s$ but we don't need to be that precise for this example).

Now that we know *when* the hammer reaches its peak, how can we determine *how high* that peak is? Recall that velocity is the slope of the position-time graph. Since slope is change in position divided by change in time (in this case, as time is on the x-axis and position on the y-axis), then the slope must have units of [position]/[time] which could be $\frac{m}{s}$, $\frac{\text{miles}}{\text{hr}}$, etc. These are units of velocity!

In figure 2.1, you can see that the units on the x-axis are seconds and on the y-axis the units are $\frac{m}{s}$. If we are looking for a *displacement* (that is, how far from its initial position the hammer has traveled), we are looking for a solution with units of meters. To yield an answer with those units, we wouldn't use the slope of the graph: this would yield an answer with units $\frac{m}{s^s}$, the units for acceleration. Instead, we need to *multiply*! The area between the velocity function and the x-axis (see figure 2.2) can be found this way:

$$Area = \frac{1}{2}bh$$

where b is the base of the triangle and h is the height.

$$Area = \frac{1}{2}(0.5s)(5\frac{m}{s})$$

$$Area = 1.25m$$

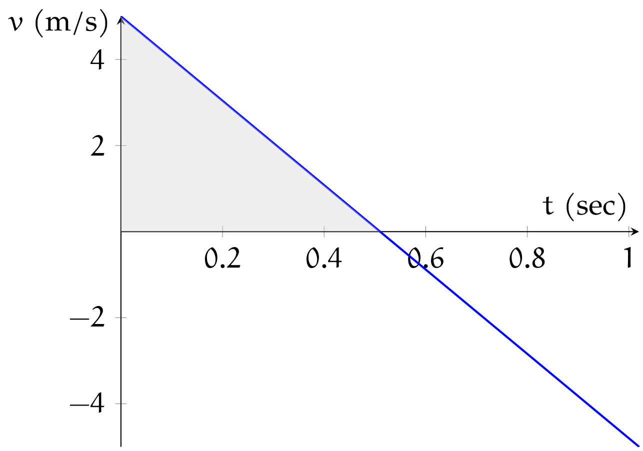


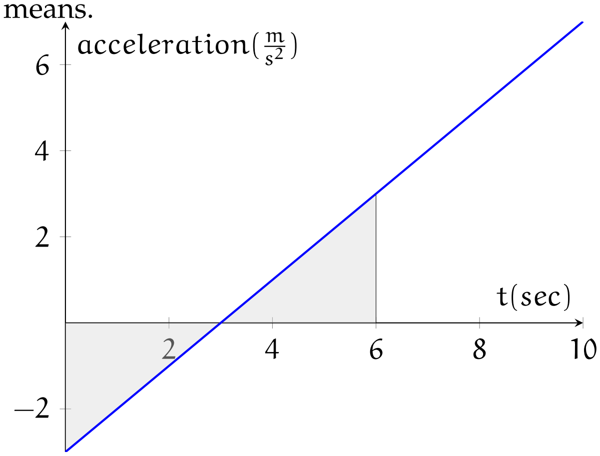
Figure 2.2: The area under $v(t)$ from $x = 0$ to $x = 0.5$ is equal to the displacement of the hammer

Notice that when multiplying the change in time $(0.5\ s)$ by the change in velocity (1.25 $\frac{m}{s}$), the seconds units cancel, yielding a result with units of meters. Therefore, the hammer reaches a peak height of $\approx 1.25\ m$, which you can confirm by examining the graph originally presented for the hammer toss in the chapter on graph shape.

2.1.1 Determining the Meaning of the Area with Units

Exercise 6

What units will the area shown in the graph have? Based on your answer, does the area represent a displacement, a net change in velocity, or a net change in acceleration? Calculate the shaded area [hint: areas below the x-axis are negative]. Write a sentence in plain English explaining what the are you calculated means.



Working Space

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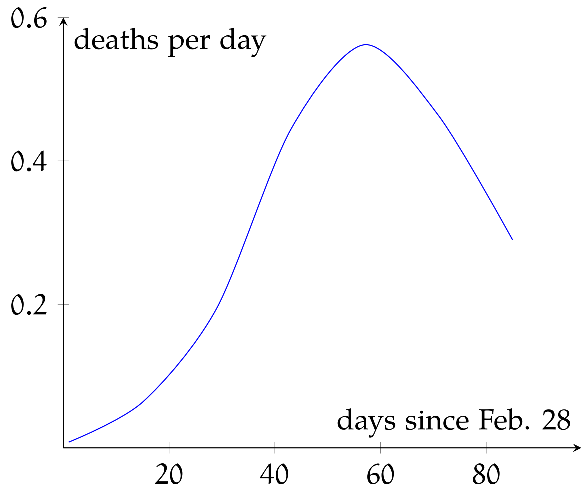
Exercise 7

The graph below shows historical data of the number of deaths due to SARS in Singapore over several months in 2003. What would the area under the curve represent?



Working Space

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Exercise 8

Oil leaked from a tank at a rate of $r(t)$ liters per hour. A site engineer recorded the leak rate over a period of 10 hours, shown in the table. Plot the data. How could you estimate the total volume of oil lost?

t(h)	0	2	4	6	8	10
$r(t)$(L/h)	8.7	7.6	6.8	6.2	5.7	5.3

Working Space

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2.2 Estimating the area under functions

In the hammer example above, it was easy to determine the area under the function, since the area took the shape of a triangle. But what about finding the area under a more complex function, such as $f(x) = sin x + x$ (shown in figure 2.3)?

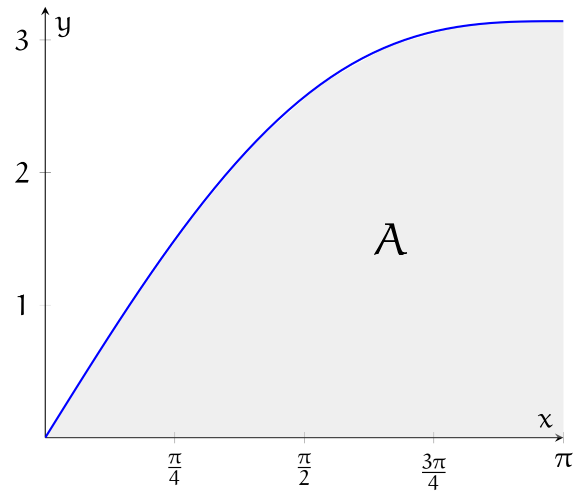

Figure 2.3: $f(x) = \sin x + x$

How can we determine the area under $f(x) = \sin x + x$ from $x = 0$ to $x = \pi$? We can *estimate* the area of that region by dividing the region into rectangles, finding the areas of the rectangles, and adding the areas. As an example, we will divide the region under $f(x) = \sin x + x$ into 4 intervals, shown in figure 2.4.

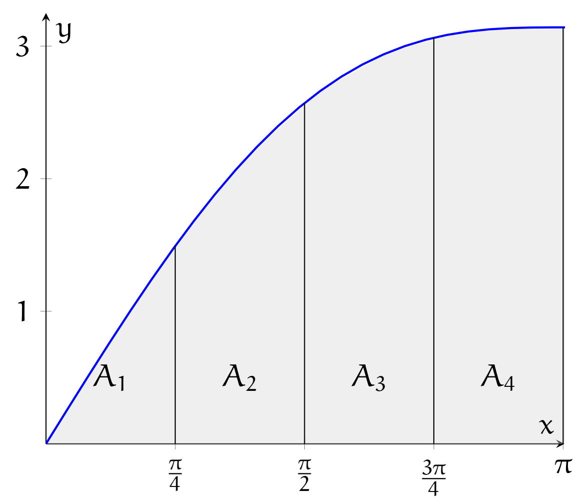

Figure 2.4: $f(x) = \sin x + x$ divided into 4 regions

As you can see in figure 2.4, each rectangle will have a width of $\frac{\pi}{4}$. But what about the height? One way is to use the value of the function at the rightmost value of each rectangle, as shown in figure 2.5.

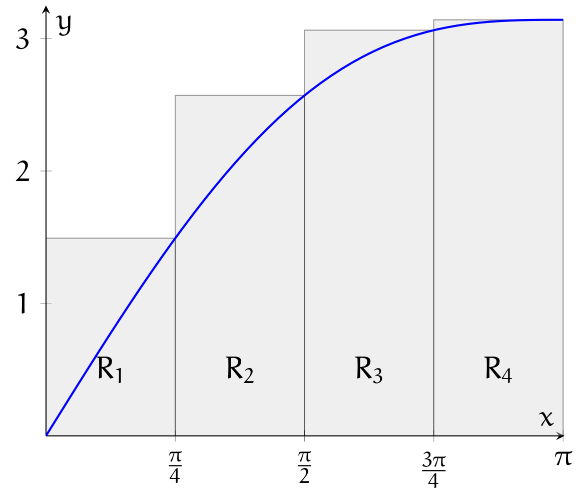

Figure 2.5: Four rectangle sections with heights determined by rightmost value of $f(x)$ on each interval

We can easily calculate the areas of each of these rectangles:

$$\frac{\pi}{4} \times f(\frac{\pi}{4}) + \frac{\pi}{4} \times f(\frac{\pi}{2}) + \frac{\pi}{4} \times f(\frac{3\pi}{4}) + \frac{\pi}{4} \times f(\pi)$$

$$\approx \frac{\pi}{4} \times (1.4925 + 2.5708 + 3.0633 + 3.1416) = 8.0646$$

Based on figure 2.5, will the calculated area be an overestimate or an underestimate? Each of the rectangles overshoots the function, so this will be an overestimate. What about using the leftmost value of $f(x)$ of each interval to determine the height of the rectangles? This is shown in figure 2.6.

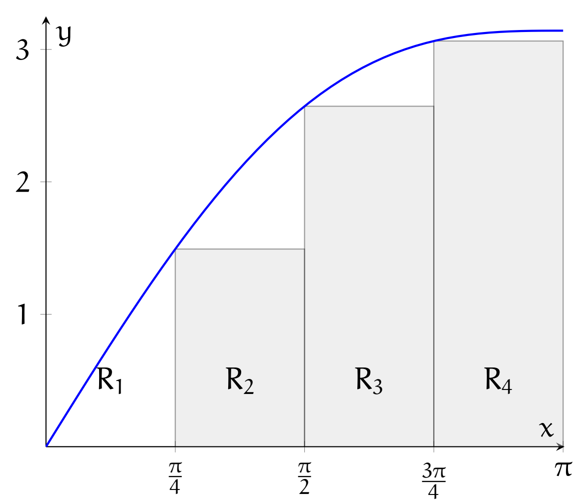

Figure 2.6: Four rectangle sections with heights determined by leftmost value of $f(x)$ on each interval

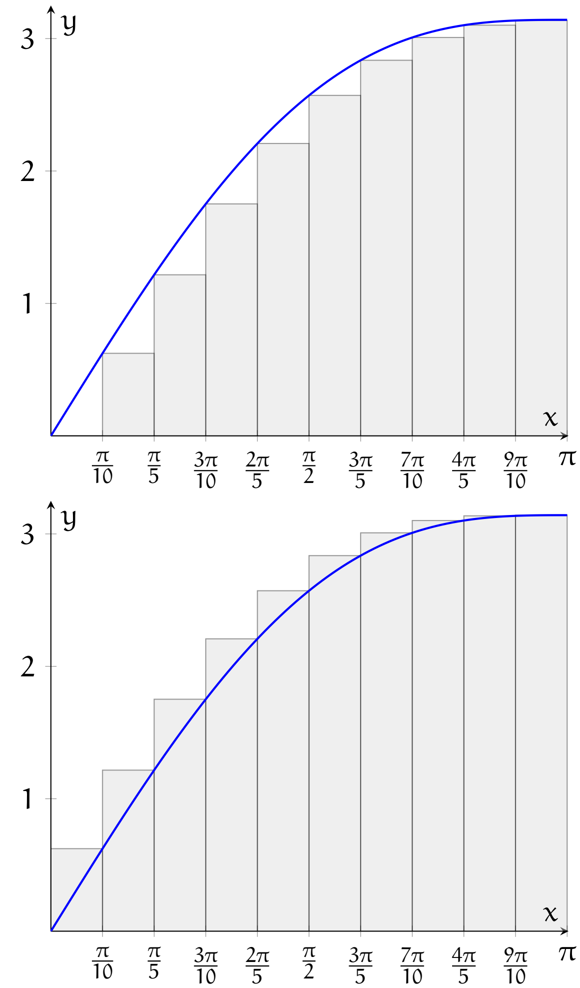

Figure 2.7: $\sin x + x$ broken into 10 intervals using either the left or right value to determine the height.

Notice that because $f(0) = 0$, the height of the first rectangle is zero, so we don't see it on the graph. To find the area of these rectangles:

$$\frac{\pi}{4} \times f(0) + \frac{\pi}{4} \times f(\frac{\pi}{4}) + \frac{\pi}{4} \times f(\frac{\pi}{2}) + \frac{\pi}{4} \times f(\frac{3\pi}{4})$$

$$\approx \frac{\pi}{4} \times (0 + 1.4925 + 2.5708 + 3.0633) = 5.5972$$

This is an underestimate. Therefore, the true value of the area under $f(x) = \sin x + x$ is between 5.5972 and 8.0646. This is an awfully wide window! We can narrow our estimate by increasing the number of intervals. Graphs of $f(x)$ with 10 intervals are shown in figure 2.7.

The total area for the left-determined rectangles is ≈ 6.4248 and for the right-determined is ≈ 7.4118. Therefore, we have narrowed the range for the true area under the curve to $6.4248 < A < 7.4118$. In general, as you increase the number of intervals, you get closer to the true area.

For a strictly increasing function, the right sum will be an overestimate and the left sum will be an underestimate of the true area under the curve. In the exercise below, you will examine a strictly decreasing function:

Exercise 9

Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$ using four rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an overestimate or an underestimate? Repeat using left endpoints.

Working Space

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You should have found that for the strictly decreasing function $f(x) = \frac{1}{x}$, the right-determined sum is an *underestimate* while the left-determined sum is an *overestimate*.

2.3 The Riemann Sum

In the previous section, we estimated the area under functions by dividing the area into approximating rectangles. This method is called a *Riemann Sum*. We will use a general example to formally define the Riemann sum. Consider a generic function divided into strips of equal width (shown in figure 2.8). The width of each strip is

$$\Delta x = \frac{b-a}{n}$$

where a is the left endpoint of the interval, b is the right endpoint of the interval, and n is the number of strips. Then the right endpoints of the sections are

$$x_1 = a + \Delta x$$

$$x_2 = a + 2\Delta x$$

$$\ldots \; x_n = a + n\Delta x$$

As above, we can use the value of the function to determine the height of a rectangle whose area approximates the area of the section. (E.g. for the i^{th} strip, the width is Δx and the height is $f(x_i)$, see figure 2.9). Then the total area approximated by the rectangles is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

This is the formal definition of the Right Riemann Sum. You can also take a Left Riemann Sum or a Midpoint Riemann Sum, as discussed below.

2.3.1 Right Riemann Sums

As seen above, a right Riemann sum uses the right-most value of $f(x)$ to determine the height of the rectangle (an example is shown in figure 2.10). We will refer to the right Riemann sum as R_n, where n is the number of intervals.

2.3.2 Left Riemann Sums

When taking a left Riemann sum, the height of the rectangle is determined by the value of the function at the lower (left-most) x-value. See figure 2.11. We will refer to left Riemann sums as L_n, where n is the number of intervals. Then the total area approximated by a

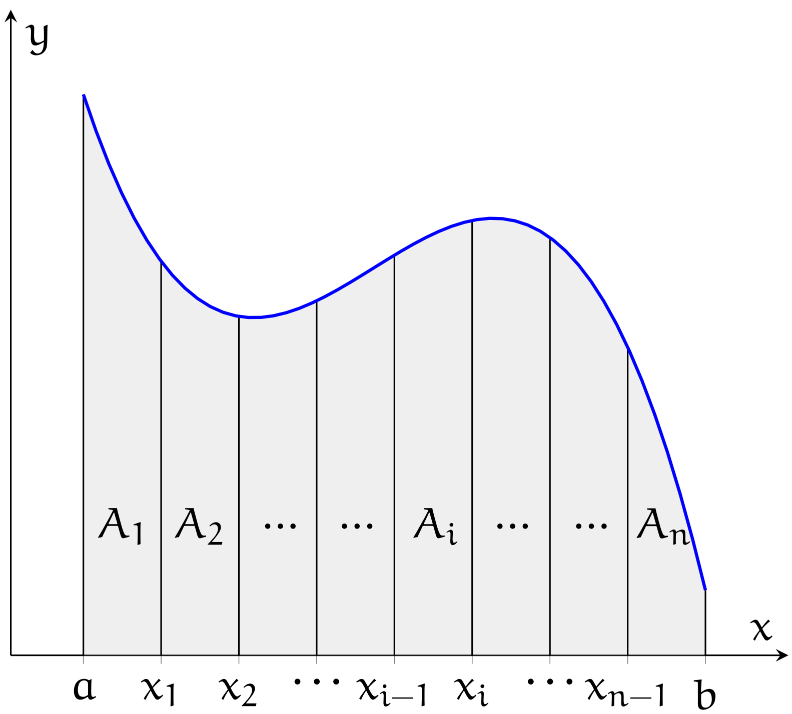


Figure 2.8: A representative function divided into n strips of equal width

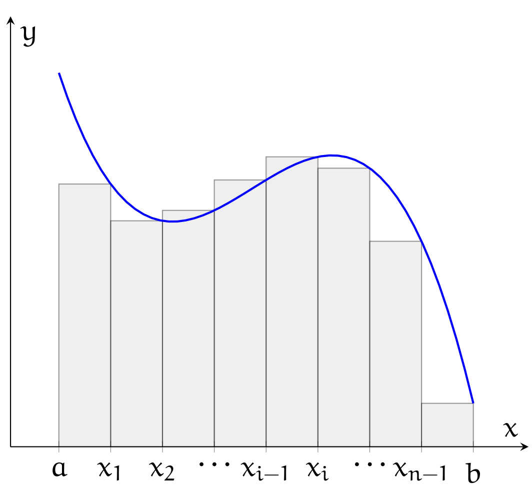


Figure 2.9: A representative function divided into n rectangles of equal width, with rectangle height determined by the right endpoint of the subinterval

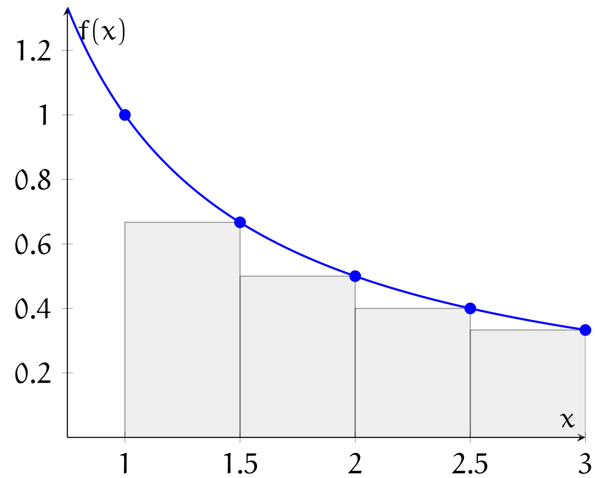


Figure 2.10: R_4 for $f(x) = \frac{1}{x}$

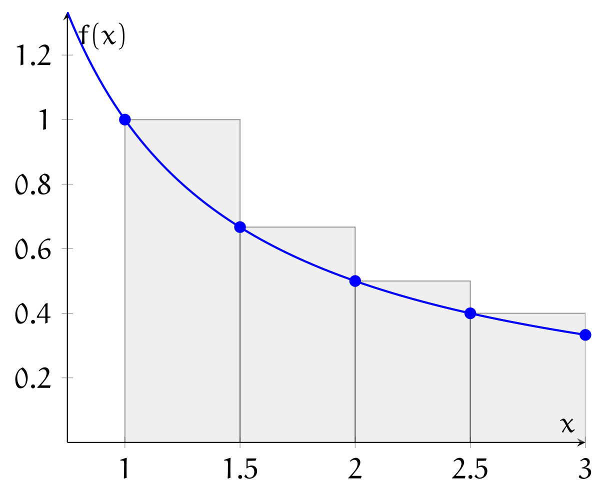


Figure 2.11: L_4 for $f(x) = \frac{1}{x}$

Left Riemann sum is is

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

.

2.3.3 Midpoint Riemann Sums

A midpoint Riemann sum uses the value of $f(x)$ at the midpoint of the division to determine the height of the rectangle, as shown in figure 2.12. We will refer to the midpoint Riemann sum as M_n, where n is the number of intervals. Then the total area approxi-

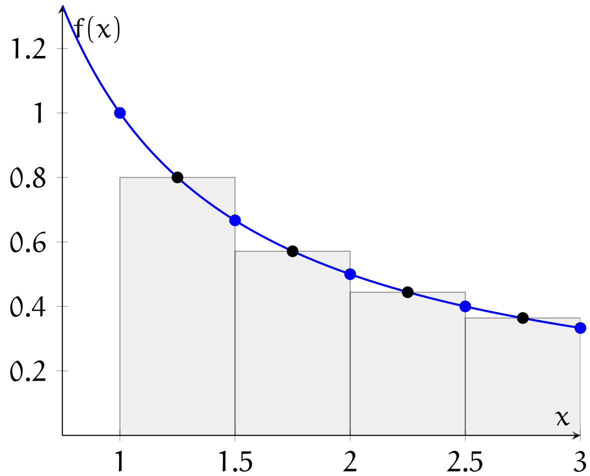


Figure 2.12: M_4 of $f(x) = \frac{1}{x}$

mated by the rectangles is

$$M_n = f(\frac{x_0 + x_1}{2})\Delta x + f(\frac{x_1 + x_2}{x})\Delta x + \ldots + f(\frac{x_{n-1} + x_n}{2})\Delta x$$

.

2.3.4 Riemann sum sigma notation

As you may recall, mathematicians use sigma notation to concisely express sums, such as Riemann sums. We can re-write the definition of a right Riemann sum in sigma notation:

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

where n is the number of subintervals. Then, the actual area under the curve is the limit as n approaches ∞ of the above sum. Let's apply this by writing a sum that represents the area, A, of the region that lies between the x-axis and the function $f(x) = e^{-x}$ from $x = 0$ to $x = 2$.

First, we find an expression for Δx:

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Recall that $x_i = a + i\Delta x$. Since a (the beginning of the interval) $= 0$, then the general expression for x_i in this case is $0 + i \times \frac{2}{n} = \frac{2i}{n}$. Substituting our expressions for Δx and x_i into the sum formula, we see that:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} e^{\frac{-2i}{n}} \frac{2}{n}$$

We can also interpret a sum as the area under a specific function. Take the expression:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin \frac{i\pi}{n}$$

There are two expressions in the sum: $\frac{\pi}{n}$ and $\sin \frac{i\pi}{n}$. It makes sense that $\Delta x = \frac{\pi}{n}$ and $f(x_i) = \sin \frac{i\pi}{n}$. Because $\Delta x = \frac{b-a}{n} = \frac{\pi}{n}$, it follows that the interval of the area has a width of π. We will need to examine the other expression, $\sin \frac{i\pi}{n}$ to determine an exact window.

Since $f(x_i) = \sin \frac{i\pi}{n}$, it follows that the function we are looking for is a sine function. Further, the expression for $x_i = \frac{i\pi}{n}$. Recall that $x_i = a + i\Delta x$, where a is the left-most boundary of the interval. Substituting what we have found already, we see that:

$$x_i = a + i\frac{\pi}{n} = \frac{i\pi}{n}$$

which implies that $a = 0$. Since we have established the interval is π wide, we can infer that $b = \pi$. Therefore, the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin \frac{i\pi}{n}$ is equal to the area under $f(x) = \sin x$ from $x = 0$ to $x = \pi$.

Exercise 10

Use the formal definition of a Right Riemann sum to write a limit of a sum that is equal to the total area under the graph of f on the specified interval. Do not evaluate the limit.

1. $f(x) = \frac{2x}{x^2+1}$, $1 \le x \le 3$
2. $f(x) = x^2 + \sqrt{1+2x}$, $4 \le x \le 7$
3. $f(x) = \sqrt{\sin x}$, $0 \le x \le \pi$

Working Space

Answer on Page 118

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

Figure 2.13: Speed of *Endeavour* from launch to booster separation

Exercise 11

Use the formal definition of a Right Riemann sum to find a region on a graph whose are is equal to the given limit. Do not evaluate the limit.

1. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$
2. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

Working Space

Answer on Page 118

2.3.5 Real-world Riemann Sums

Sometimes we are working from real data and the intervals aren't evenly spaced. That's ok! We can still use Riemann sums to make an estimate. Consider the velocity data from the 1992 launch of the space shuttle *Endeavour*, shown in tabular form in figure 2.13:

We can use a Riemann sum to estimate how far the space shuttle traveled in the first 62 seconds of flight. First, let's visualize our data (see figure 2.14). There are 7 time intervals from the data, but we only need the first 6. We can find a reasonable range for the distance the space shuttle travels by finding the left and right Riemann sums. Remember: because these data are strictly increasing, the left sum will be our lower bound and the right sum will be our upper bound.

First, we'll find L_6. The width of the first interval is 10 seconds $(10-0=10)$ and the height

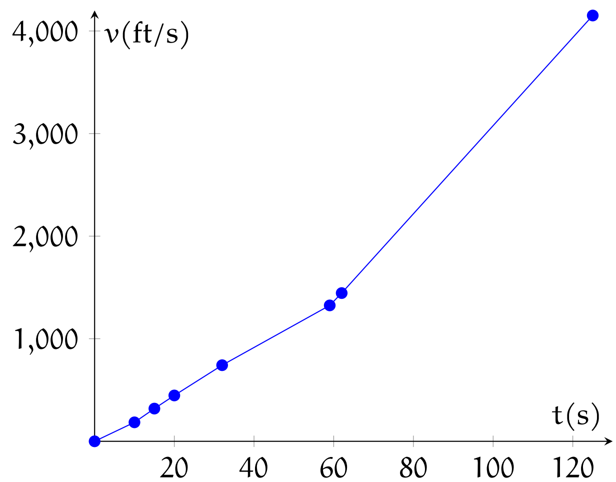


Figure 2.14: Plot of time, velocity data for the *Endeavour*

of the rectangle will be $v(0) = 0$. Calculations for the additional intervals are shown in the table:

Interval	Width(s)	Height(ft/s)	Area(ft)
1	10	0	0
2	5	185	925
3	5	319	1595
4	12	447	5364
5	27	742	20034
6	3	1325	3975

Adding the areas, we find the lower limit for the distance traveled is 31,893 feet. We can determine the upper bound, R_6, in a similar manner:

Interval	Width(s)	Height(ft/s)	Area(ft)
1	10	185	1850
2	5	319	1595
3	5	447	2235
4	12	742	8904
5	27	1325	35775
6	3	1445	4335

Adding the areas, we find the upper limit for the distance traveled is 54,694 feet. Therefore, the *Endeavour* traveled between 31,893 and 54,694 feet during the first 62 seconds of this flight.

2.4 Code for a Riemann Sum

You can create a program that automatically calculates a Riemann sum. Create a file called `riemann.py` and type the following into it:

```
import matplotlib.pyplot as plt
import sys
import math

from matplotlib.table import Rectangle

# Did the user supply two arguments?
if len(sys.argv) != 3:
    print(f"Usage: {sys.argv[0]} <stop> <divisions>")
    print(f"Numerically integrates 1/x from 1 to <stop>.")
    print(f"Calculates the value of 1/x at <divisions> spots in the range.")
    exit(1)

# Check to make sure the number of divisions is greater than zero?
divisions = int(sys.argv[2])
if divisions <= 0:
    print("ERROR: Divisions must be at least 1.")
    exit(1)

# Is the stopping point after 1.0?
stop = float(sys.argv[1])
if stop <= 1.0:
    print("ERROR: Stopping point must be greater than 1.0")
    exit(1)

start = 1.0
step_size = (stop - start)/divisions

print(f"Step size is {step_size:.5f}.")
x_values = []
y_values = []
sum = 0.0
for i in range(divisions):
    current_x = start + i * step_size
    current_y = 1.0/current_x
    area = current_y * step_size
    print(f"{i}: 1 / {current_x:.3f} = {current_y:4f}, area of rect = {area:8f} ")

    x_values.append(current_x)
    y_values.append(current_y)
```

```
    sum += area
    print(f"\tCumulative={sum:.3f}, ln({current_x:.3f})={math.log(current_x):.3f}")

print(f"Numerical integration of 1/x from 1.0 to {stop:.4f} is {sum:.4f}")
print(f"The natural log of {stop:.4f} is {math.log(stop):.4f}")

# Create data for the smooth 1/x line
SMOOTH_DIVISIONS = 200
smooth_start = start - 0.15
smooth_stop = stop + 1.0
smooth_step = (smooth_stop - smooth_start)/SMOOTH_DIVISIONS
smooth_x_values = []
smooth_y_values = []
for i in range(SMOOTH_DIVISIONS):
    current_x = smooth_start + i * smooth_step
    current_y = 1.0/current_x
    smooth_x_values.append(current_x)
    smooth_y_values.append(current_y)

# Put it on a plot
fig, ax = plt.subplots()
ax.set_xlim((smooth_x_values[0], smooth_x_values[-1]))
ax.set_ylim((0, smooth_y_values[0]))
ax.set_title("Riemann Sums for 1/x")

# Make the Riemann rects
for i in range(divisions):
    current_x = x_values[i]
    next_x = current_x + step_size
    current_y = y_values[i]
    rect = Rectangle((current_x, 0), step_size, current_y, edgecolor="green", facecolor=
    ax.add_patch(rect)

# Make the true 1/x curve
ax.plot(smooth_x_values, smooth_y_values, c="k", label="1/x")

# Show the user
plt.show()
```

This program will calculate and display a graph of the left Riemann sum of $\frac{1}{x}$ from 1 to the provided stop value with the indicated number of subintervals. When you run it, you'll see a graph in a new window and something like this in the terminal:

```
Step size is 0.40000.
```

```
0: 1 / 1.000 = 1.000000, area of rect = 0.400000
        Cumulative=0.400, ln(1.000)=0.000
1: 1 / 1.400 = 0.714286, area of rect = 0.285714
        Cumulative=0.686, ln(1.400)=0.336
2: 1 / 1.800 = 0.555556, area of rect = 0.222222
        Cumulative=0.908, ln(1.800)=0.588
3: 1 / 2.200 = 0.454545, area of rect = 0.181818
        Cumulative=1.090, ln(2.200)=0.788
4: 1 / 2.600 = 0.384615, area of rect = 0.153846
        Cumulative=1.244, ln(2.600)=0.956
5: 1 / 3.000 = 0.333333, area of rect = 0.133333
        Cumulative=1.377, ln(3.000)=1.099
6: 1 / 3.400 = 0.294118, area of rect = 0.117647
        Cumulative=1.495, ln(3.400)=1.224
7: 1 / 3.800 = 0.263158, area of rect = 0.105263
        Cumulative=1.600, ln(3.800)=1.335
8: 1 / 4.200 = 0.238095, area of rect = 0.095238
        Cumulative=1.695, ln(4.200)=1.435
9: 1 / 4.600 = 0.217391, area of rect = 0.086957
        Cumulative=1.782, ln(4.600)=1.526
Numerical integration of 1/x from 1.0 to 5.0000 is 1.7820
The natural log of 5.0000 is 1.6094
```

Exercise 12

Use the python program you created to find L_{10}, L_{50}, L_{100}, L_{500}, L_{1000}, and L_{5000} for the function $\frac{1}{x}$ from $x = 1$ to $x = 5$. What do you notice about the results?

Working Space

Answer on Page 119

2.5 Riemann Sum Practice

Exercise 13

t (hours)	4	7	12	15
$R(t)$ (L/hr)	6.5	6.2	5.9	5.6

A tank contains 50 liters of water after 4 hours of filling. Water is being added to the tank at rate $R(t)$. The value of $R(t)$ at select times is shown in the table. Using a right Riemann sum, estimate the amount of water in the tank after 15 hours of filling.

Working Space

Answer on Page 119

Exercise 14

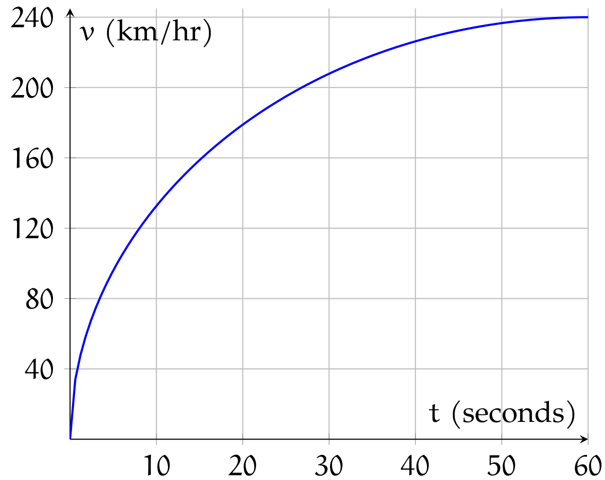
Let $f(x) = x - 2\ln x$. Estimate the area under f from $x = 1$ to $x = 5$ using four rectangles and the value of f at the midpoint of each interval. Sketch the curve and your approximating rectangles.

Working Space

Answer on Page 119

Exercise 15

A graph of a car's velocity over a period of 60 seconds is shown. Estimate the distance traveled during this period.



Working Space

Answer on Page 120

Chapter 3

Definite Integrals

Integrals are a fundamental concept in calculus, which are used to calculate areas, volumes, and many other things. A definite integral calculates the net area between the function and the x-axis over a given interval.

Recall that you can use a Riemann sum to estimate the area under a function, and that as we increase the number of subintervals, the estimated area approaches the actual area. In sigma notation we can express a Riemann sum as

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

3.1 Definition

The definite integral of a function $f(x)$ over an interval $[a, b]$ is defined as the limit of a Riemann sum as n approaches ∞:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{3.1}$$

where x_i^* is a sample point in the i^{th} subinterval of a partition of $[a, b]$, $\Delta x = \frac{b-a}{n}$ is the width of each subinterval, and the limit is taken as the number of subintervals n approaches infinity.

Exercise 16

Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$$

as an integral on the interval $[0, \pi]$.

Working Space

Answer on Page 121

3.2 Positive and Negative Areas

What if the function dips below the x-axis? We consider that area negative: that is, it represents a *decrease* as opposed to an increase. Consider an oscillating object where $v(t) = \sin \pi x$ (figure 3.1). From $t = 0$ to $t = 1$, the velocity is positive, which means the object is moving *away from* the starting position. The is a positive displacement. From $t = 1$ to $t = 2$, the velocity is negative. What does this tell you about the direction the object is moving and its displacement during this time period? A negative velocity means the object is moving *back towards* the starting position.

In general, areas above the x-axis are positive while areas below the x-axis are negative.

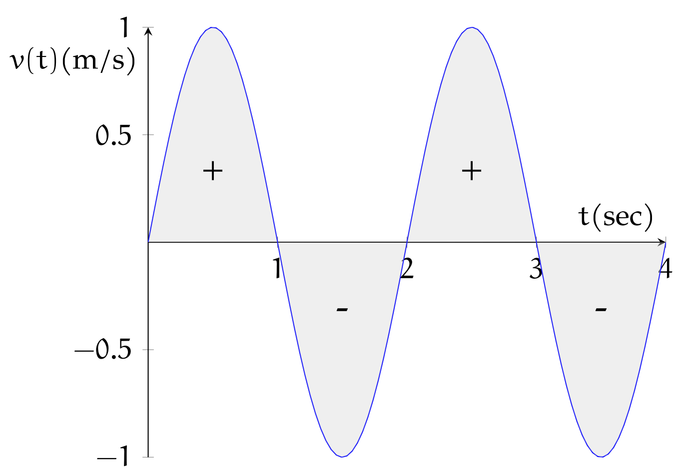


Figure 3.1: velocity of an oscillating object

3.3 Properties of Integrals

There are several important properties of integrals that will help us evaluate more complex integrals in the future. The following examples apply when $f(x)$ is continuous or has a finite number of jump discontinuities on the interval $a \leq x \leq b$:

3.3.1 What happens when $a = b$?

What if the endpoints of the integral are the same? Let's consider $\int_a^b x^2\,dx$ and take the limit as $b \to a$ (shown in figure 3.2). As you can see, as b approaches a, the calculated area decreases. Intuitively, we can guess that when $b = a$, then the width of the area (Δx) is zero, and therefore the area is also zero. Let's prove this formally.

Recall that $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. To evaluate the integral when $b = a$, we will take the limit of the limit:

$$\lim_{b\to a} \lim_{n\to\infty} f(x_i)\frac{b-a}{n}$$

This can be rewritten as

$$\lim_{b\to a}(b-a) \lim_{n\to\infty} \frac{f(x_i)}{n}$$

We know that $\lim_{b\to a}(b-a) = (a-a) = 0$, and therefore

$$\int_a^a f(x)\,dx = 0 \cdot \lim_{n\to\infty} \frac{f(x_i)}{n} = 0$$

This is true for any function.

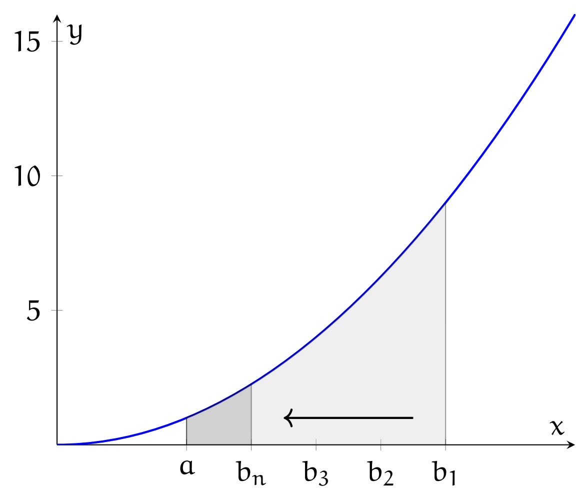


Figure 3.2: As b gets closer to a, the area represented by the integral decreases

3.3.2 The integral of a constant

When the function we are integrating is a constant (that is, it takes the form $f(x) = C$), the area is simply $(b - a) \cdot C$. This is shown graphically in figure 3.3.

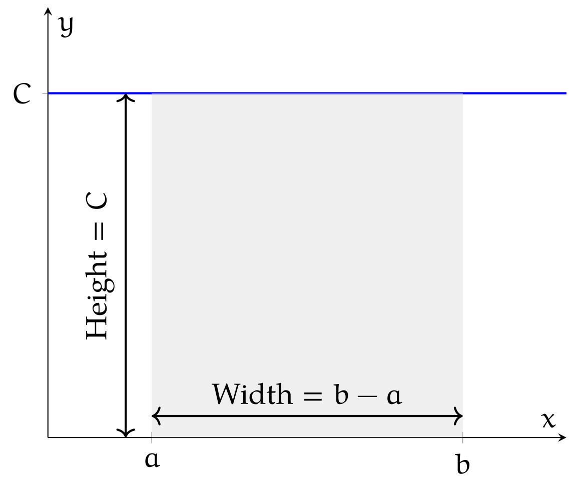


Figure 3.3: $\int_a^b f(x)\,dx = (b - a) \cdot C$

Since $f(x) = C$ is a horizontal line, the area under $f(x)$ is simply a rectangle. As you can see in figure 3.3, the width of the rectangle is $b - a$ and the height is C. To find the area of a rectangle, we multiply the width by the height, and therefore $\int_a^b C\,dx = (b - a) \cdot C$.

3.3.3 The integral of a function multiplied by a constant

How is $\int_a^b f(x)\,dx$ related to $\int_a^b C \cdot f(x)\,dx$? Intuitively, we know that multiplying a function by a constant, C, vertically stretches the graph by a factor of C. In turn, the area under the curve increases by a factor of C. Imagine a simple shape, like a triangle. If we keep the base of the triangle the same (analogous to the integral being over the same interval) and make the triangle three times taller (analogous to multiplying the function we're integrating by a factor of $C = 3$), then we would expect the total area of the triangle to be 3 times greater. Therefore, $\int_a^b C \cdot f(x)\,dx = C \int_a^b f(x)\,dx$.

3.3.4 Integrals of sums and differences of functions

If a function can be described as a sum of two other functions, then the integral of the original function is the same as the sum of the integrals of the two other functions. Concretely, we say $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. Figure 3.4 shows $f(x) = x + 2$, $g(x) = 4x^3 - 12x^2 + 10x$, and $f(x) + g(x)$. As you can see, the area under $f(x) + g(x)$ is equal to the area under $f(x)$ (the red area) plus the area under $g(x)$ (the diagonal lined area).

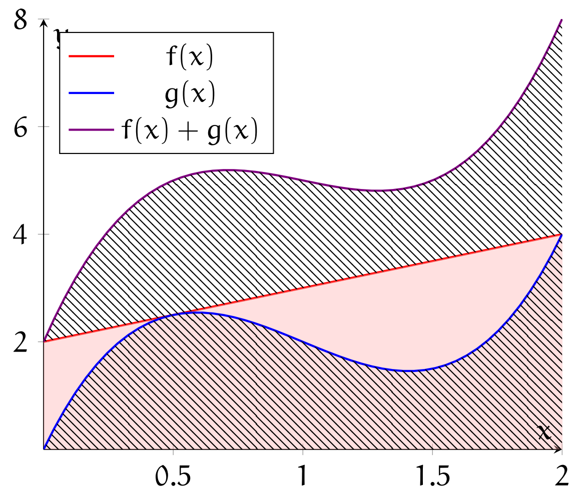


Figure 3.4: The intergral of $f(x) + g(x)$ is equal to the integral of $f(x)$ plus the integral of $g(x)$

Mathematically, we can prove this by recalling that the limit of a sum is the sum of the limits:

$$\int_a^b f(x) + g(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\Delta x$$

$$= \lim_{n\to\infty} [\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} g(x_i)\Delta x]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x$$

$$= \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Similar to the addition property, the integral of the difference between two function is equal to the difference of the integrals of two functions.

$$\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

This is more difficult to visualize than addition, but we can easily prove it by applying the constant multiple and addition properties. Let's define $f(x) - g(x) = f(x) + (-g(x))$:

$$\int_a^b f(x) - g(x)\, dx = \int_a^b f(x) + (-g(x))\, dx$$

By the addition property,

$$= \int_a^b f(x)\,dx + \int_a^b -g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b (-1)\cdot g(x)\,dx$$

And by the constant multiple property:

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

3.3.5 Integrals of adjacent areas

If c is some x-value between a and b, then $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$. This is shown graphically in figure 3.5. The total area from $x = a$ to $x = b$ is equal to the red area (the integral from a to c) plus the blue area (the integral from c to b).

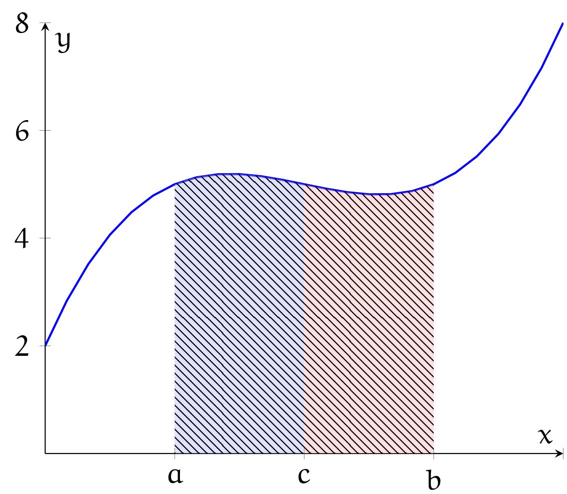


Figure 3.5: $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

3.3.6 Estimating the value of an integral

Suppose we need to know the area under a complex function. We can estimate a range for the value of the integral if we can bookend the function over the interval we are interested. Suppose there is some value m such that $f(x) \geq m$ and some other value M such that $f(x) \leq M$ on the interval we are interested in (see figure 3.6). The total area under $f(x)$ is the light blue plus the darker blue. The total area under $y = M$ is the darker blue, plus the light blue, plus the white area. The darker blue area under the curve has total area $m \cdot (b - a)$ and the rectangle under $y = M$ has total area $M \cdot (b - a)$ (since these are both integrals of a constant, which we learned about above). The actual area under our function is more than just the dark blue area, but less than the total area under $y = M$. Therefore, $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$ if $m \le f(x)$ and $M \ge f(x)$ on the interval $x \in [a, b]$.

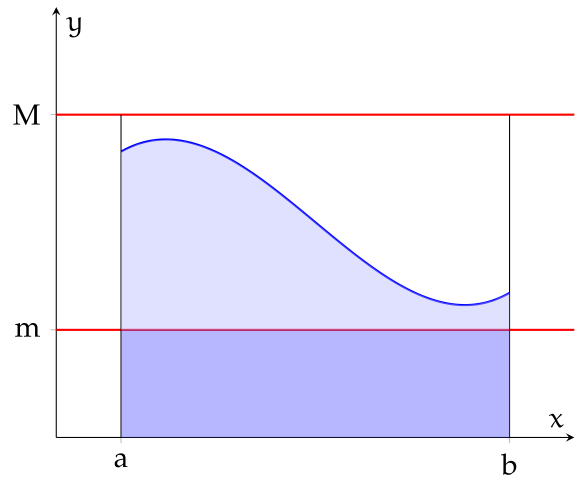


Figure 3.6: $m \le f(x) \le M$

3.3.7 Other Properties of Integrals

If $f(x) \ge 0$ over the for $a \le x \le b$, then $\int_a^b f(x)\,dx > 0$. We can make an intuitive, geometric argument to support this claim. Recall that areas above the x-axis are considered positive. If $f(x) \ge 0$, then all the area of the integral lies above the x-axis, and therefore the total area must be positive.

Similarly, if $f(x) \ge g(x)$ on the interval $a \le x \le b$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ (see figure 3.7). The entire area under $g(x)$ is contained in the area under $f(x)$. Therefore, $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Lastly, we see what happens when we switch a and b. While it is unusual to integrate from right to left (that is, in a case where $a > b$), this property will be useful. Recall that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n}$$

What is $\int_b^a f(x)\,dx$? Substituting, we see that

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(a-b)}{n}$$

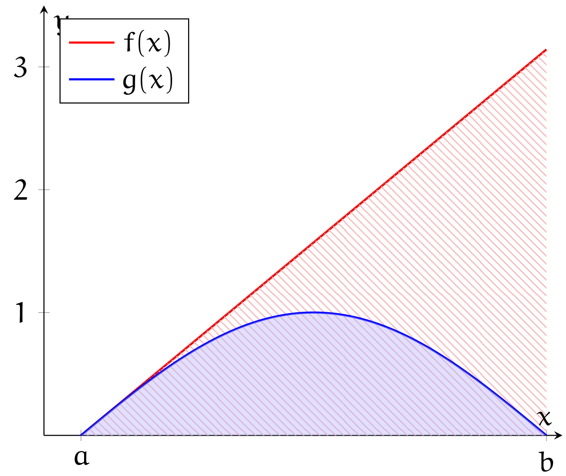


Figure 3.7: $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Noting that $(a-b) = -(b-a)$ we see:

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{-(b-a)}{n}$$

$$= (-1) \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n}$$

$$= (-1) \int_a^b f(x)\,dx$$

Therefore, it is true that $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

3.4 Applications in Physics

We've already seen that the area under a velocity function is displacement and the area under an acceleration function is change in velocity (Riemann Sums). We can use integrals to determine the change in position of an object over a given time frame. If we *also* know the object's starting position, then we can state the object's ending position. Consider the graph of an object's velocity in figure 3.8:

We can determine the net displacement of the object from $t = 0$ to $t = 9$ by evaluating $\int_0^9 v(t)\,dt$. Since the definite integral is equal to the area under the curve, we need to find the total area. As the function consists of straight lines, we will leave the explicit calculation of the area as an exercise for the student. You should find that the total positive area (above the x-axis) is 10 meters and the total negative area (below the x-axis) is 8 meters. Therefore, the object's displacement over the specified time interval is $10 - 8 = 2$ meters.

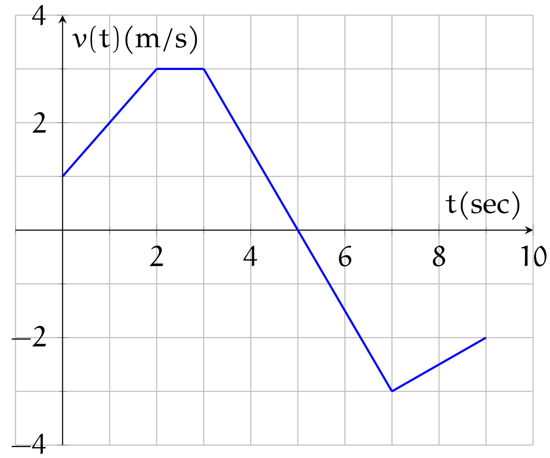


Figure 3.8: Velocity of an object from $t = 0$ to $t = 9$

When you push on something to move it, you are applying a force over a distance (assuming you are strong enough to move it!). The integral of force as a function of distance is the *work* done on that object. Work is the change in kinetic energy (KE) of an object. Mathematically, this is

$$\int_a^b F(x)\,dx = \Delta KE = \frac{1}{2}m(v_f^2 - v_i^2)$$

If you integrate the force as a function of time, that is *impulse*. Impulse is the change in momentum (p) of the object. Mathematically, this is

$$\int_a^b F(t)\,dt = \Delta p = m(v_f - v_i)$$

Example problem: You push a 3 kg box with force $F(x) = 0.5x$, where x is measured in meters and F is measured in Newtons. If the box was initially at rest, what is its speed when it reaches the 2 meter mark? (Hint: $KE = \frac{1}{2}mv^2$.)

Solution: Change in kinetic energy is the area under a force-distance curve. We can plot the force applied to the box from d=0 to d=2 (see figure 3.9):

Given that the box's initial velocity is $0\frac{m}{s}$, we know that the initial kinetic energy (KE) is $0J$. This implies that $KE_f = \Delta KE$. We can find ΔKE from the shaded area:

$$\Delta KE = \frac{1}{2}(2m)(1N) = 1J = KE_f$$

Solving for the final velocity:

$$KE_f = 1J = \frac{1}{2}(3kg)(v^2)$$

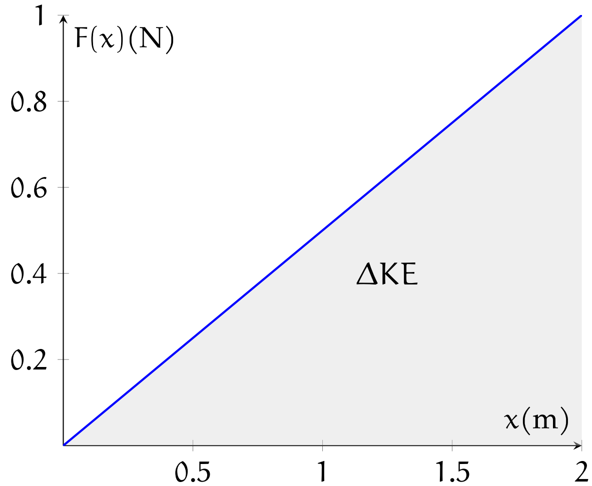


Figure 3.9: Force applied to a box over a distance; the shaded area represents the change in kinetic energy.

$$2\mathrm{J} = (3\mathrm{kg})(v^2)$$
$$\frac{2}{3}\frac{\mathrm{m}^2}{\mathrm{s}^2} = v^2$$
$$v = \sqrt{\frac{2}{3}\frac{\mathrm{m}^2}{\mathrm{s}^2}} \approx 0.816\frac{\mathrm{m}}{\mathrm{s}}$$

3.5 Practice Exercises

Exercise 17

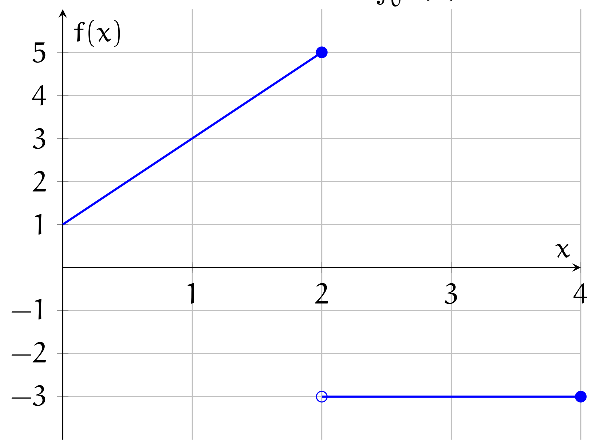
Given that $\int_0^1 x^2\,dx = \frac{1}{3}$, use the properties of integrals to evaluate $\int_0^1 (5-6x^2)\,dx$.

Working Space

Answer on Page 121

Exercise 18

This question was originally presented as a multiple-choice problem on the 2012 AP Calculus BC exam. The graph of f is shown. What is the value of $\int_0^4 f(x)\,dx$?

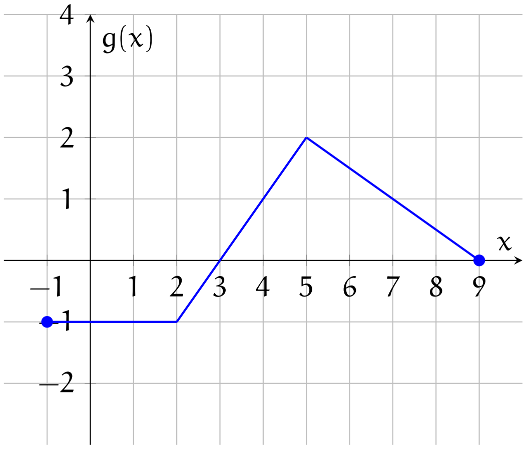


Working Space

Answer on Page 121

Exercise 19

This question was originally presented as a multiple-choice problem on the 2012 AP Calculus BC exam. The graph of the piecewise function $g(x)$ is shown. What is the value of $\int_{-1}^{9} 3g(x) + 2\,dx$?



Working Space

Answer on *Page 122*

Exercise 20

[This question was originally presented as a calculator-allowed, multiple- choice question on the 2012 AP Calculus BC exam.] If $f'(x) > 0$ for all real numbers and $\int_4^7 f(x)\,dx = 0$, which of the following could be a table of values for the function f?

(A)

x	$f(x)$
4	-4
5	-3
7	0

(B)

x	$f(x)$
4	-4
5	-2
7	5

(C)

x	$f(x)$
4	-4
5	6
7	3

(D)

x	$f(x)$
4	0
5	0
7	0

(E)

x	$f(x)$
4	0
5	4
7	6

Working Space

Answer on Page 122

Chapter 4

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) is a theorem that connects the concept of differentiating a function with the concept of integrating a function. This theorem is divided into two parts:

4.1 First Part

The first part of the Fundamental Theorem of Calculus states that if f is a continuous real-valued function defined on a closed interval $[a, b]$ and F is the function defined, for all x in $[a, b]$, by:

$$F(x) = \int_a^x f(t)\,dt \tag{4.1}$$

Then, F is uniformly continuous and differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all x in (a, b). (That is $F(x)$ is the antiderivative of $f(x)$.)

4.2 Second Part

The second part of the Fundamental Theorem of Calculus states that if f is a real-valued function defined on a closed interval $[a, b]$ that admits an antiderivative F on $[a, b]$, and f is integrable on $[a, b]$ (it need not be continuous), then

$$\int_a^b f(t)\,dt = F(b) - F(a). \tag{4.2}$$

We will also use shorthand as follows:

$$\int_a^b f(t)\,dt = F(t)\big|_a^b \tag{4.3}$$

Which means "$F(t)$ evaluated from $t = a$ to $t = b$".

4.3 FTC and Definite Integrals

Let f be a function that is continuous on the interval $x \in [a, b]$ and $g(x)$ is given by:

$$g(x) = \int_a^x f(t)\,dt$$

Then g is continuous on $[a, b]$ and differentiable on (a, b). Additionally,

$$g'(x) = f(x)$$

Proof: Let x and $x + h$ be in (a, b). Then,

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

Recall from the chapter on definite integrals that we can split the first integral, rewriting it as:

$$g(x+h) - g(x) = \left[\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right] - \int_a^x f(t)\,dt$$

$$g(x+h) - g(x) = \int_x^{x+h} f(t)\,dt$$

And for $h \neq 0$:

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Since f is continuous, there is some u in (a, b) such that $f(u) = m$, where m is the minimum value of f on the interval (a, b). Similarly, there is also some v such that $f(v) = M$, where M is the maximum value (see figure ??). Then we can state the true inequality that:

$$mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$$

And therefore (assuming $h > 0$):

$$f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v)$$

Substituting the equation above for the integral, we see that:

$$f(u) \leq \frac{g(x+h)-g(x)}{h} \leq f(v)$$

If we let h approach zero, then the window that u and v are in collapses and u and v both approach x. Therefore,

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x)$$

Recall also that

$$\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$$

Then taking the limit as $h \to 0$ of the whole inequality becomes the Squeeze Theorem:

$$\lim_{h \to 0} f(u) \leq \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \leq \lim_{h \to 0} f(v)$$

$$f(x) \leq g'(x) \leq f(x)$$

And therefore if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$. Notice it doesn't matter what a is!

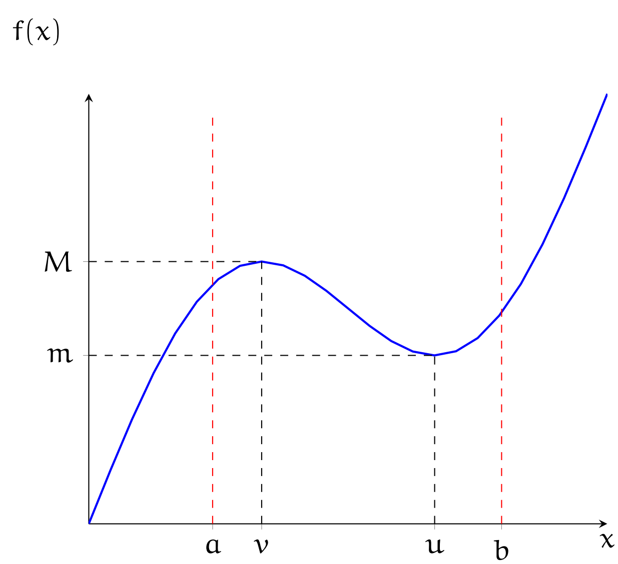


Figure 4.1: $f(v) = M$, the maximum value, and $f(u) = m$, the minimum value on the interval $x \in [a, b]$

Exercise 21

[This question was originally presented as a no-calculator, multiple-choice problem on the 2012 AP Calculus BC Exam.] Let g be a continuously differentiable function with $g(1) = 6$ and $g'(1) = 3$. What is the value of $\lim_{x \to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6}$?

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) 2
(E) The limit does not exist

Working Space

Answer on Page 122

4.4 The Meaning of the FTC

What the Fundamental Theorem of Calculus is really saying is that differentiation and integration are opposite processes. Mathematically, we can say

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

This may seem clunky, but many useful functions are defined this way. Consider the Fresnel function, $S(x) = \int_0^x \sin \frac{\pi t^2}{2}\,dt$. Originally used in optics, this equation is also used by civil engineers to design road and railway curves. According to FTC, then, $S'(x) = \sin \frac{\pi t^2}{2}$.

We can also apply the Chain Rule when taking derivatives of integrals. Let $f(x) =$

$\int_1^{x^4} \sec t\, dt$. What is $f'(x)$? First, let us define $u = x^4$. By the Chain Rule,

$$\frac{d}{dx}\int_0^{x^4} \sec t\, dt = \frac{d}{dx}\int_0^{u} \sec t\, dt$$
$$= \frac{d}{du}[\int_0^{u} \sec t\, dt]\frac{du}{dx}$$
$$= \sec u \frac{du}{dx}$$

Noting that $\frac{du}{dx} = \frac{d}{dx}x^4 = 4x^3$,

$$f'(x) = \sec x^4(4x^3)$$

4.4.1 FTC Practice

Exercise 22

Use the Fundamental Theorem of Calculus to find the derivative of the function.

1. $g(x) = \int_0^x \sqrt{t + t^3}\, dt$
2. $F(x) = \int_x^0 \sqrt{1 + \sec t}\, dt$
3. $h(x) = \int_1^{e^x} \ln t\, dt$
4. $y = \int_{\sqrt{x}}^{\frac{\pi}{4}} \theta \tan\theta\, d\theta$

Working Space

Answer on Page 122

4.5 Using Antiderivatives to Evaluate Definite Integrals

In everyday English, the FTC states that the integral from a to b of a function is the antiderivative of that function evaluated from a to b. In the previous chapter, the integrals presented were of linear functions where the area under the curve could be equally calculated by hand. The FTC connects integrals to antiderivatives, allowing us to evaluate more complex integrals. Consider the following example:

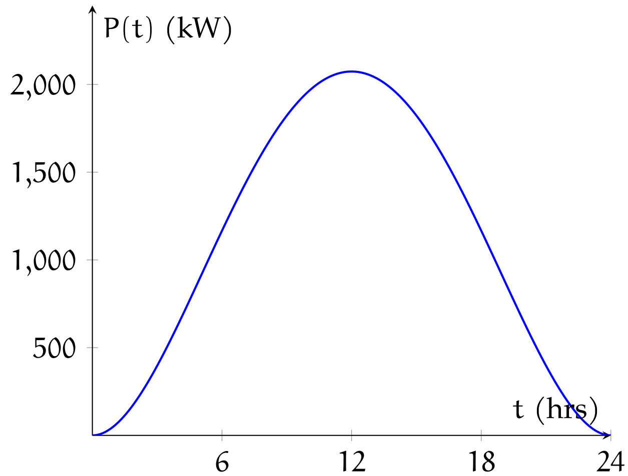


Figure 4.2: Power consumption of a household in a day

The power consumption of a household can be modeled as $P(t) = \frac{1}{10}t^2(t-24)^2$ from $t = 0$ to $t = 24$, where P is measured in watts and t is measured in hours ($t = 0$ is midnight). The total energy the household uses is given by $\int_0^{24} P(t)\,dt$. As you can see from the graph (see figure 4.2), we cannot simply use our geometry skills to determine the area under the curve.

To determine the total energy use, we need to evaluate $\int_0^{24} \frac{1}{10}t^2(t-24)^2\,dt$. First, we expand the polynomial:

$$\begin{aligned} E_{tot} &= \frac{1}{10}\int_0^{24} t^2(t^2 - 48t + 576)\,dt = \frac{1}{10}\int_0^{24} t^4 - 48t^3 + 576t^2\,dt \\ &= \frac{1}{10}\int_0^{24} t^4\,dt - \frac{24}{5}\int_0^{24} t^3\,dt + \frac{288}{5}\int_0^{24} t^2\,dt \end{aligned}$$

Using the Power Rule to determine the antiderivatives of t^4, t^3, and t^2, we see:

$$= \frac{1}{10}[\frac{1}{5}t^5]|_0^{24} - \frac{24}{5}[\frac{1}{4}t^4]|_0^{24} + \frac{288}{5}[\frac{1}{3}t^3]|_0^{24} = 26542.1Whr = 26.5421kWhr$$

4.5.1 Definite Integrals Practice

Exercise 23

Evaluate the following integrals:

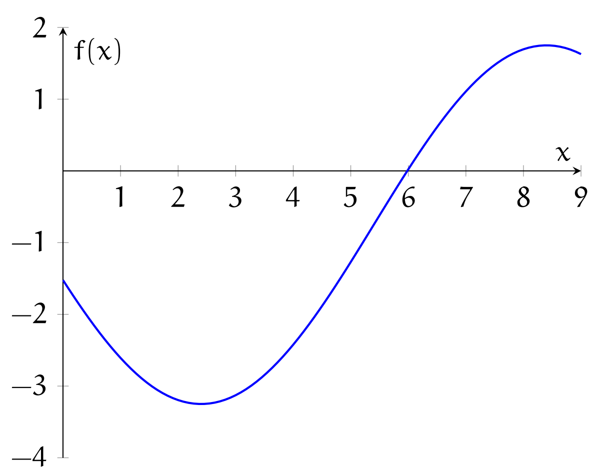
1. $\int_1^4 t^{-3/2}\, dt$

Working Space

Answer on Page 123

Exercise 24

[This question was originally presented as a multiple-choice, no-calculator problem on the 2012 Calculus BC exam.] The graph of a differentiable function f is shown in the graph. $h(x) = \int_0^x f(t)\, dt$. Rank the relative values of $h(6)$, $h'(6)$, and $h''(6)$ from lowest to highest.

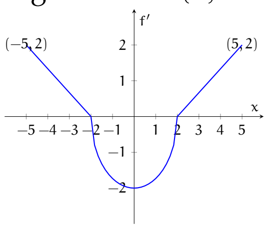


Working Space

Answer on Page 123

Exercise 25

The graph of f' is shown in the graph and consists of a semi-circle and two line segments. If $f(2) = 1$, then what is $f(-5)$?



Working Space

Answer on Page 123

Exercise 26

[This question was originally presented as a calculator-allowed, multiple- choice question on the 2012 AP Calculus BC exam.] A particle moves along a line so that its acceleration for $t \geq 0$ is given by $a(t) = \frac{t+3}{\sqrt{t^3+1}}$. If the particle's velocity at $t = 0$ is 5, what is the velocity of the particle at $t = 3$?

Working Space

Answer on Page 123

4.6 Average Value of a Function

The average value of a function, f, on an interval $[a, b]$ is given by:

Average Value of a Function

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example: Find the average value of $f(x) = 3 + x^2$ on the interval $[-2, 1]$.

Solution: Taking $a = -2$ and $b = 1$, we have:

$$f_{avg} = \frac{1}{1-(-2)} \int_{-2}^{1} \left[3 + x^2\right] \, dx$$

$$f_{avg} = \frac{1}{3} \int_{-2}^{1} \left[3 + x^2\right] \, dx$$

$$f_{avg} = \frac{1}{3} \left[3x + \frac{1}{3}x^3\right]_{x=-2}^{x=1}$$

$$f_{avg} = \frac{1}{3} \left[\left(3 \cdot 1 + \frac{1}{3} \cdot 1^3\right) - \left(3 \cdot (-2) + \frac{1}{3} \cdot (-2)^3\right)\right]$$

$$f_{avg} = \frac{1}{3} \left[\left(3 + \frac{1}{3}\right) - \left(-6 - \frac{8}{3}\right)\right]$$

$$f_{avg} = \frac{1}{3} \left[\frac{10}{3} + 6 + \frac{8}{3}\right] = \frac{1}{3}(12) = 4$$

Therefore, the average value of $f(x) = 3 + x^2$ on the interval $[-2, 1]$ is 4.

Exercise 27

[This question was originally presented as a multiple-choice, calculator- allowed problem on the 2012 AP Calculus BC Exam.] What is the average value of $y = \sqrt{\cos x}$ on the interval $0 \leq x \leq \frac{\pi}{2}$?

Working Space

Answer on Page 124

Chapter 5

Arc Lengths

5.1 Determining the Arc Length of a Curve

Another application of integrals is finding the length of a curve. In real-life, we could do this by laying a piece of string up against the curve, then straightening out the string and measuring its length with a ruler (you may have done this is elementary school when you were first learning about the relationship between the radius and circumference of a circle). Archimedes estimated the circumference of a circle by inscribing a circle with polygons of increasing numbers of sides. (Archimedes' proof that π is between $3\frac{10}{71}$ and $3\frac{1}{7}$ is more complicated but we won't dive into that here.) As we increase the number of sides of the inscribed polygon, the perimeter of the polygon (the sum of the lengths of the sides) gets closer to the circumference of the circle (see figure 5.1). Now, it's easy to find the length of a polygon: just add up the length of the line segments! Using this, we can find the length of a curve by approximating it as many short lines and adding up the lengths of those lines.

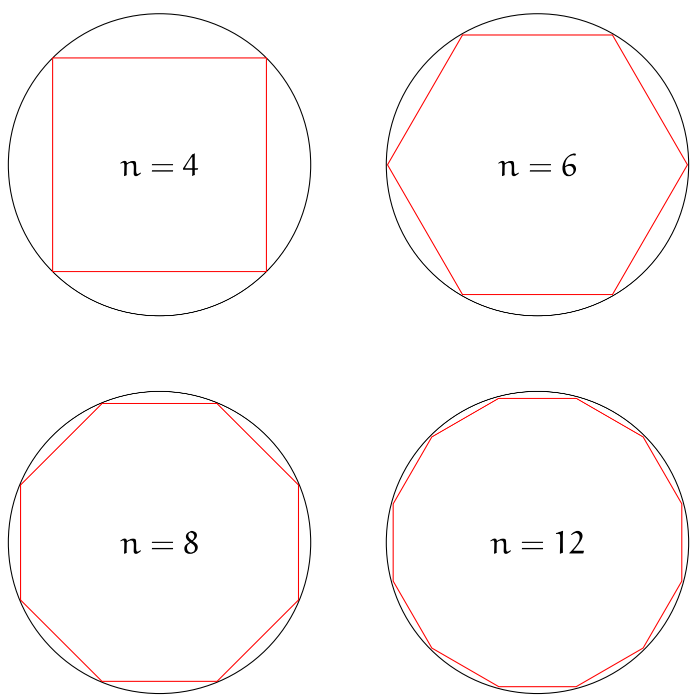


Figure 5.1: As n increases, the perimeter of the inscribed polygon approaches the circumference of the circle

We can choose n points along the graph of $f(x)$ and connect each point with a straight line (this is shown in figure 5.2). If we add up the length of the lines, we get an estimate of the length of the curve. We represent the length of the line between the i^{th} point, P_i and the previous point, P_{i-1} as $|P_{i-1}P_i|$ (recall that the absolute value sign can be used to signify the length of something). Therefore, the sum of the lengths of the lines approximating the curve is:

$$\sum_{i=1}^{n} |P_{i-1}P_i|$$

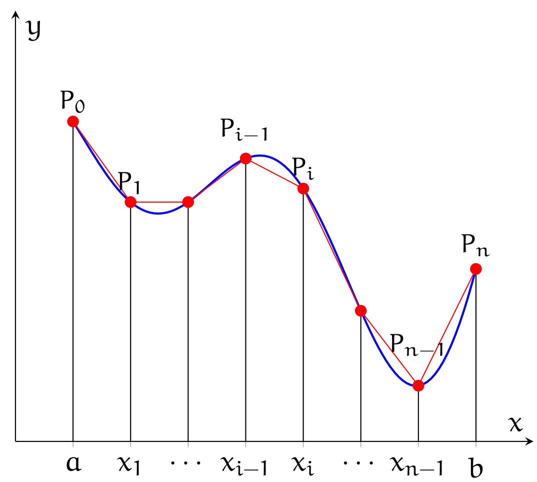


Figure 5.2: Polygon approximation of $f(x)$

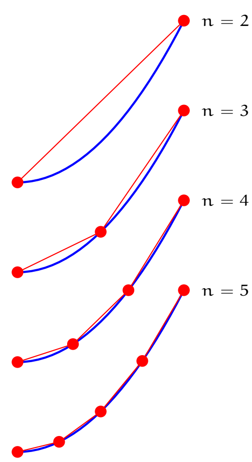


Figure 5.3: As the number of points increases, the total length of the lines segments approaches the true length of the curve

The more points we choose, the closer the lines lay to the actual curve (see figure 5.3), and the closer our estimate is to the true length. So, to find the true length, we will want to take n to ∞. Therefore, the actual curve length is the limit as $n \to \infty$ of that sum:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The length of each segment can be found using the Pythgorean theorem. Recall that the distance between two points on the xy-plane is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. (For a reminder of why this is, see figure 5.4.) The coordinates of P_{i-1} are $(x_{i-1}, f(x_{i-1}))$ and the coordinates of P_i are $(x_i, f(x_i))$. Substituting this into the above sum, we see that the total length of the segments is

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

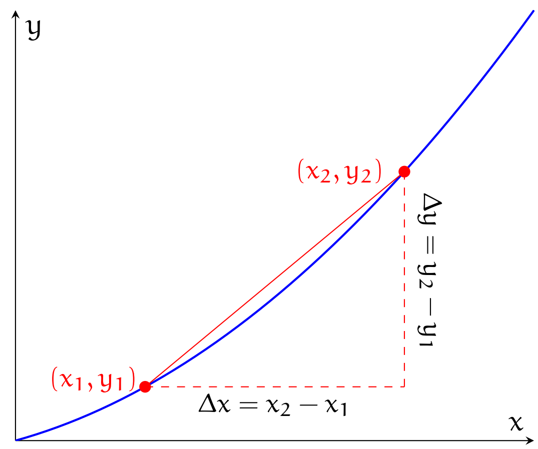


Figure 5.4: The distance between two points on the xy-plane

Recall the Mean Value Theorem, which states that there is some x_i^* such that $f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$. Substituting this into the above sum, we get:

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f'(x_i^*)(x_i - x_{i-1}))^2}$$

Recall from the chapter on Riemann sums and the integral that we defined $\Delta x = x_i - x_{i-1}$ and we can further re-write the sum as

$$\sum_{i=1}^{n} \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\sqrt{(\Delta x)^2} = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

Putting this all together, we see that that actual length of the curve is defined as

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

This is the definition of the integral of $\sqrt{1 + [f'(x)]^2}$ and therefore the length of some function $f(x)$ on the interval $a < x < b$ is $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. In another notation, this is equivalent to $\int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$.

5.2 Arc Length of Vector-valued Functions

Suppose you have a vector-valued function, $f(t) = [x(t), y(t)]$. A common example might be an artillery shell shot at an angle. For a shell shot with an initial velocity v_o at angle θ from the ground, its position can be described with the vector-valued function $f(t) = [v_o \cos\theta(t), v_o \sin\theta(t) - 4.9t^2]$. (A concrete example where $v_o = 12\frac{m}{s}$ and $\theta = 30^o$ is shown in figure 5.5.)

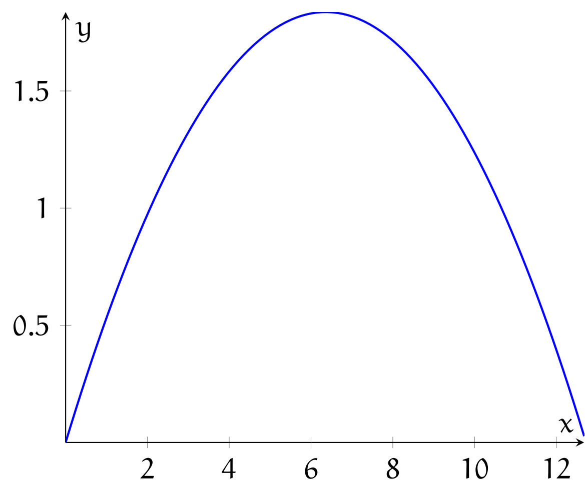


Figure 5.5: The path of an artillery shell with shot with initial velocity v_o at angle θ

How can we find the length of the flight path of the artillery shell? We can re-interpret the length integral for a vector-valued function, $f(t) = [x(t), y(t)]$.

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_a^b \sqrt{1 + \frac{(dy/dt)^2}{(dx/dt)^2}} \frac{dx}{dt}\, dt$$

Moving the $\frac{dx}{dt}$ under the square root, we see that

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$$

Which is equivalent to

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dx$$

5.3 Applications in Physics

When we take the integral of a velocity function, we get the *displacement*. For example, if you drove to school and home again, your displacement would be zero. However, the *distance* you traveled is not zero! We can use the arc length formula to find the total distance traveled. (Remember, that if $x(t)$ is the object's position, then its velocity is given by $x'(t)$.)

Suppose a block is attached to a spring on a frictionless horizontal surface. You pull on the block, initiating harmonic motion described by $v(t) = (-0.16)\sin 9t$, where v is in $\frac{m}{s}$ and t is in *sec*. (Note: we are working in radians, not degrees.) What is the block's displacement from $t = 0$ to $t = 3$? What is the total distance the block moves from $t = 0$ to $t = 3$?

To find the displacement, we integrate the velocity function over the specified interval:

$$\int_0^3 (-0.16)\sin 9t\, dt = \frac{-0.16}{9}(-\cos 9t)|_0^3$$

$$= \frac{0.16}{9}[\cos 27 - \cos 0] = \frac{(0.16)(-0.29-1)}{9} = -0.0229m$$

The position function and displacement are shown in figure 5.6. To find the total distance traveled, we need to find the length of the curve. Before we do so, take a minute to mentally predict: will the distance be more or less than the displacement?

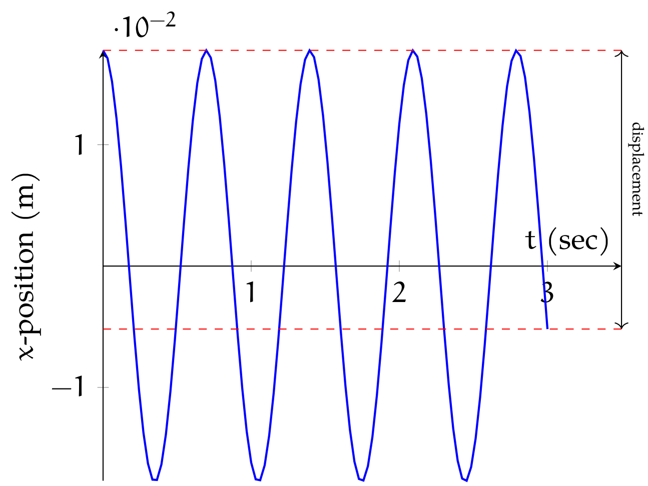


Figure 5.6: The position of the block with displacement shown. The distance traveled is the total length of the curve

Recalling that the distance traveled by an object is $\int_a^b \sqrt{1+[x'(t)]^2}\,dt = \int_a^b \sqrt{1+[v(t)]^2}\,dt$, we can write an integral to determine the total distance traveled by the block:

$$\int_0^3 \sqrt{1+[-0.16\sin 9t]^2}\,dt$$

Unfortunately, we do not know an antiderivative for this integral, and u-substitution won't help us. However, for definite integrals, calculators such as a TI-89 or Wolfram Alpha can easily use Riemann sums to determine the value of the integral to a high precision. Using such a tool, we find that the total distance traveled by the block is ≈ 3.019 meters. Did you predict that the distance would be greater than the displacement?

5.4 Practice

Exercise 28

Write an integral that gives the length of the requested curve.

1. $y = \ln x$ from $x = 1$ to $x = 3$
2. $y = \sin x$ from $x = 0$ to $x = \pi$
3. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 4$
4. $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{3}$

Working Space

Answer on Page 124

Exercise 29

The arc length function for a curve $f(x)$, where f is an increasing function, is given by $s(x) = \int_0^x \sqrt{3t+5}\,dt$. If f has y-intercept 2, find an equation for f. What point on f is three units from the y-intercept? Give your answer to the thousandths place.

Working Space

Answer on Page 124

Exercise 30

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] Write an integral that gives the length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

Working Space

Answer on Page 125

Exercise 31

An out of control rocket ship is spiraling out of control through space. Its velocity can be described with the vector-valued function $v(t) = [-1412 \sin t, 1412 \cos t, t]$ where v is in $\frac{m}{s}$ and t is in sec. How far does the ship travel in the first 60 seconds? In the second 60 seconds? [Hint: in three dimensions, the length of a vector-valued function is $\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt$]

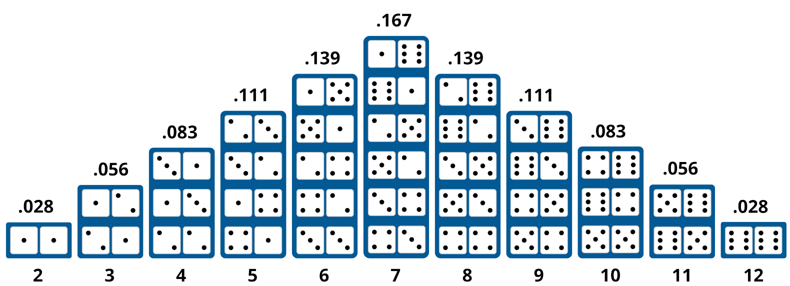
Working Space

Answer on Page 125

Chapter 6

Continuous Probability Distributions

When we talked about the probability distribution of the sum of two dice, we assigned a probability to each of the 11 possibilities:



The probabilities all added up to be 1.0. That is a way of saying "100% of the times you throw the dice, the sum will be an integer between 2 and 12."

Now we need to talk about probabilities of properties that are continuous, not discrete. For example, we might want to ask the question "If I randomly pick a cow from all the cows in the world, what is the probability that it will weigh less than 597.34 kg?" What does a probability distribution for a continuous variable look like?

6.1 Cumulative Distribution Function

Imagine that you live in ancient times. You buy, sell, and ship cows. A lot of people come into your office and brag about their cows: "Bessie is heavier than 99% of the cows in world!" So you need to develop some statistics on cow weights.

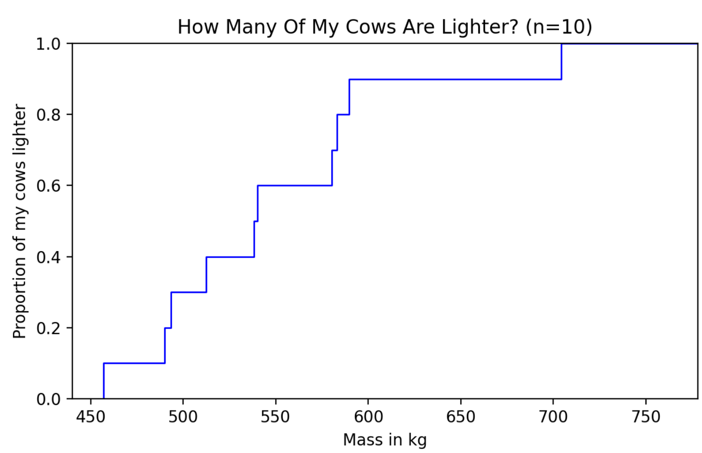
You have ten cows. You weight them:

Cow	Mass in kg
Cow 1	580.22
Cow 2	540.07
Cow 3	538.20
Cow 4	512.39
Cow 5	589.75
Cow 6	456.91
Cow 7	583.09
Cow 8	493.56
Cow 9	489.97
Cow 10	704.15

If someone comes into your shop and says "My Bessie is an astonishing 530 kg!" it would be cool to have a list on the wall that would let you yell back "Half my cows are heavier than that, Silly!" So you sort the cows by weight. For each weight, you say how many of your cows are lighter than that:

Proportion of cows lighter	Mass in kg
0.00	456.91
0.10	489.97
0.20	493.56
0.30	512.39
0.40	538.20
0.50	540.07
0.60	580.22
0.70	583.09
0.80	589.75
0.90	704.15

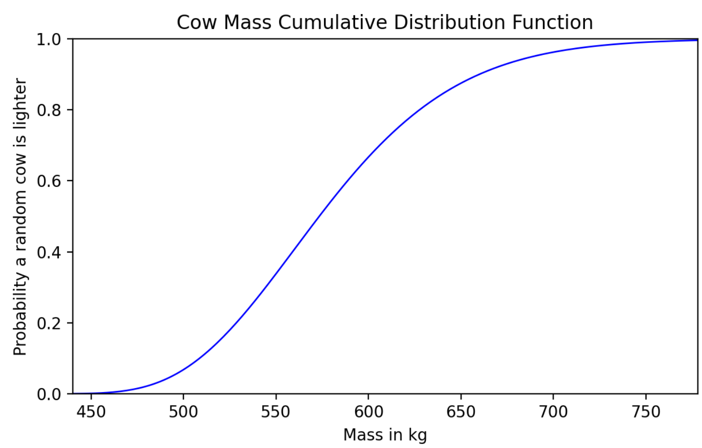
In fact, for easy reference, you make a plot:



Now for any weight you can quickly look up what proportion of your cows are lighter.

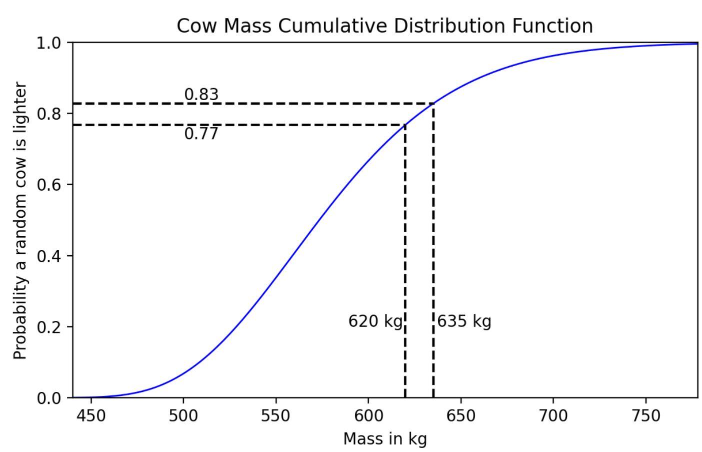
(And, if you subtract that from 1, what proportion of your cows are heavier.)

See how jagged that graph is? That is because you only have the data for 10 cows. However, as the years pass and you weight thousands of cows, the plot will become smoother. Because it always accumulates more cows as you move from left to right, this is known as a *cumulative distribution function* or CDF:



A cumulative distribution function always starts at 0 and ends at 1. On that journey, it never decreases.

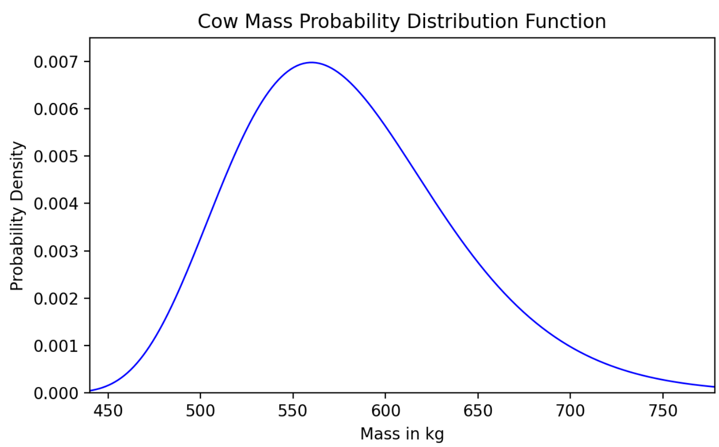
Let's say you want to know what proportion of cows weigh between 620 kg and 635 kg. Using the CDF, you could figure out that 77% of all cows weigh less than 620 kg and 83% of all cows weigh less than 635 kg. Thus 6% of all cows must weigh more than 620 kg and less than 635 kg.



6.2 Probability Density Function

The cumulative density function is handy, but some of its information can be hard to see. For example, how would you answer the question ”What is the most common weight of a cow?” You would squint at the CDF and try to determine where it was steepest.

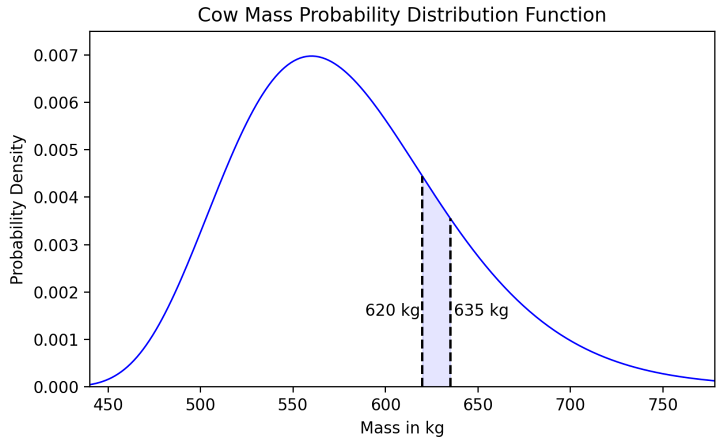
To make these sorts of questions easier to answer, we take the derivative of the CDF to get the *probability density function* (or PDF). For the cows, it would look like this:



Now you can easily see that the CDF was steepest at about 560 kg. We call the highest point on the PDF the *most likely estimator*. For example, you might say ”560 kg is the most likely estimator of cow mass.” Sometimes we just say ”the MLE”.

Note that the MLE is often different from the mean or the median. In this case, for example, the distribution is skewed right – there are more cows that are heavier than the MLE than there are cows that are lighter than the MLE. The MLE would be less than the mean or the median.

Once again, lets say you want to know what proportion of cows weight between 620 kg and 635 kg. This is more difficult with a PDF than it is with a CDF. With a PDF, you have to find the area under the curve between $x = 620$ and $x = 635$.

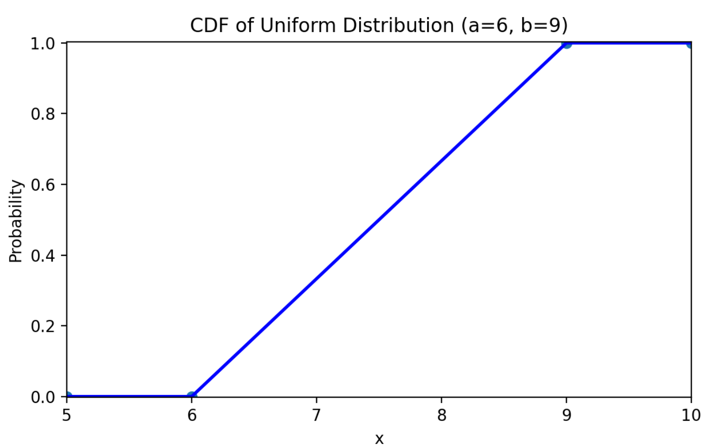


This is why it is called a "probability density" – to get a true probability you need to multiply the density by the width of the region.

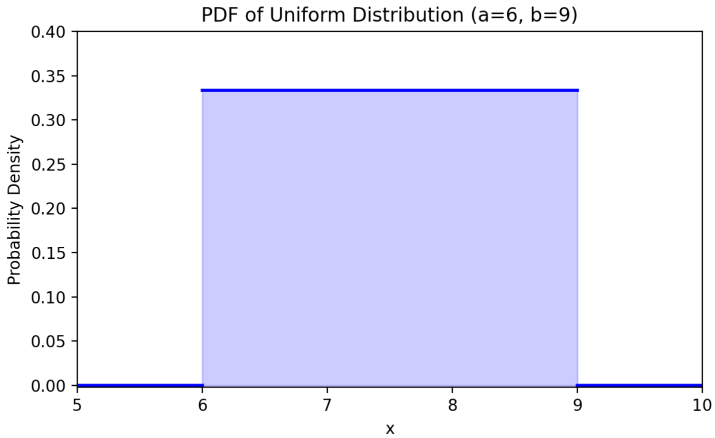
What is the area under the entire curve? If you integrated it, you would get the CDF. The CDF goes from 0 to 1.0. The area under a PDF is *always* 1.0.

6.3 The Continuous Uniform Distribution

The most simple continuous distribution is the uniform distribution between two numbers a and b. The CDF is a straight line from zero at a to 1 at b. For example, here is the CDF for the uniform distribution between 6 and 9.



That line goes from 0 to 1 over a distance of 3, so its slope is $\frac{1}{3}$ between 6 and 9 and zero everywhere else. Thus the PDF (its derivative), looks like this:



So we can write the probability distribution of a continuous uniform distribution between a and b as:

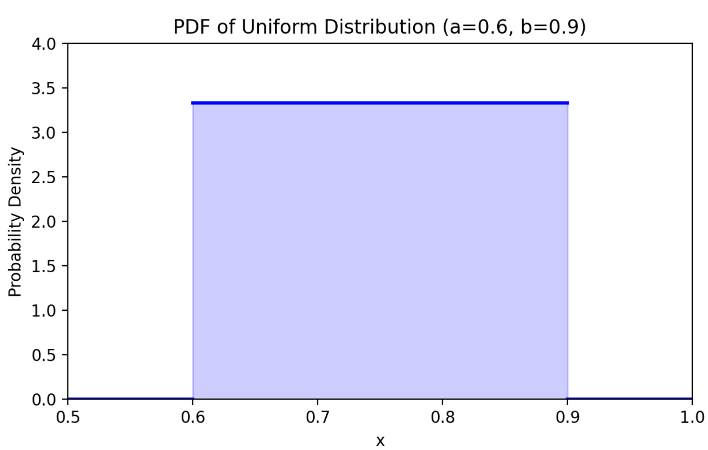
$$p(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{for } x < a \ \text{ or } \ x > b. \end{cases}$$

Notice that if a and b are less than 1 apart, the value of $p(x)$ will be greater than 1. This is a really important difference between a probability and a probability density:

- A probability will always be in the interval $[0, 1]$.
- A probability density will never be less than 0, but can be much larger than 1.

That said, the probability density will always integrate to 1.

Here is the PDF for a uniform distribution between 0.6 and 0.9:



The mean and median of a uniform distribution between a and b is its midpoint: $\frac{a+b}{2}$.

The variance (σ^2) is $\frac{(b-a)^2}{12}$.

6.4 Continuous Distributions In Python

The SciPy library has functions that let a programmer work with a large collection of different probability distributions.

For example, if you wanted to work with a continuous uniform distribution between 6 and 9, you would import the relevant functions like this:

```
from scipy.stats import uniform
```

Now if you wanted a numpy array containing a sample of 300 numbers generated randomly from that distribution:

```
samples = uniform.rvs(loc=6, scale=3, size=300)
```

The `loc` argument is a. The `scale` argument is $b - a$.

If you wanted to know the value of the probability density function at 8 and 10, you could use the `pdf` function:

```
x_values = np.array([8, 10])
p_values  = uniform.pdf(x_values, loc=6, scale=3)
```

Now `p_values` contains 0.33333 and 0.0.

To get the value of the cumulative distribution function at those points, you would use the `cdf` function:

```
cdf_values = uniform.cdf(x_values, loc=6, scale=3)
```

Now `cdf_values` contains 0.666667 and 1.0.

The inverse of the CDF is very useful. It answers questions like ”How heavy does a cow have to be to be in top 1%?”

```
bottom_top_percentiles = np.array([0.01, 0.99])
boundaries = uniform.ppf(bottom_top_percentiles, loc=6, scale=3)
```

Now `boundaries` contains 6.03 and 8.97.

The SciPy library supplies these functions (`rvs`, `pdf`, `cdf`, and `ppf`) for over a hundred common continuous probability distributions.

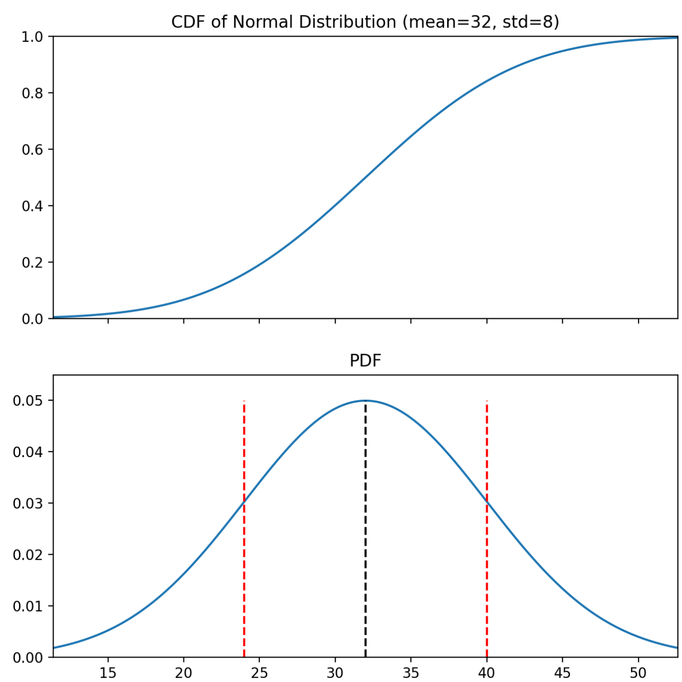
The common "bell curve" shaped distribution is called a Gaussian or Normal distribution. It is described by its mean (the midpoint of the bell) and its standard deviation. For the normal distribution, the standard deviation is the distance you have to go from the mean to reach 68% of the population. We will talk a lot more about the normal distribution in other chapters, but lets take this opportunity to plot the CDF and PDF of a normal distribution with a mean of 32 and a standard deviation of 8.

Create a file called `plot_norm.py` and add the following lines:

```
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt

# Constants
MEAN = 32
STD = 8

# Plottin from the 0.5 percentile to the 99.5 percentile
x_min = norm.ppf(0.005, loc=MEAN, scale=STD)
x_max = norm.ppf(0.995, loc=MEAN, scale=STD)

# Make 200 points between x_min and x_max
x_values = np.linspace(x_min, x_max, 200)

# Get CDF for each x value
cdf_values = norm.cdf(x_values, loc=MEAN, scale=STD)

# Get PDF for each x value
pdf_values = norm.pdf(x_values, loc=MEAN, scale=STD)

# What is the highest density?
max_density = norm.pdf(MEAN, loc=MEAN, scale=STD)

# Make a figure with two axes
fig, axs = plt.subplots(nrows=2, sharex=True, figsize=(8, 8), dpi=200)
axs[0].set_xlim(left=x_min, right=x_max)

# Draw the CDF on the first axis
axs[0].set_title("CDF of Normal Distribution (mean=32, std=8)")
axs[0].set_ylim(bottom=0.0, top=1.0)
axs[0].plot(x_values, cdf_values)

# Draw the PDF on the second axix
axs[1].set_title("PDF")
axs[1].set_ylim(bottom=0.0, top=max_density * 1.1)
```

```
axs[1].plot(x_values, pdf_values)

# Add lines for mean,  mean-std, and mean+std
axs[1].vlines(MEAN - STD, 0, max_density, "r", linestyle="dashed")
axs[1].vlines(MEAN + STD, 0, max_density, "r", linestyle="dashed")

# Save out the figure
fig.savefig("norm_32_8.png")
```

The resulting plot should look like this:



What do those vertical lines mean? An ornithologist might tell you "The wingspan of adult robins are normally distributed with a mean of 32 cm and a standard deviation of 8 cm." Then 68% of the population of adult robins would have wingspans between the two red lines.

Exercise 32 SciPy Stats

Working Space

Globally, the height of adult women is approximately normally distributed. The mean is 164.7 cm. The standard deviation is 7.1 cm.

Use python and SciPy stats to answer these questions:

- To be in the tallest decile (the top 10%) of adult women, how tall does one need to be?
- What percentage of adult women are between 160 cm and 165 cm?

(In case you are wondering: For men the mean is 178.4 cm and the standard deviation is 7.6 cm.)

Answer on Page 125

Chapter 7

The Physics of Gases

Now, let's say you start to heat the helium inside the balloon. As the temperature goes up, the molecules inside will start to move faster.

Remember that the kinetic energy of an object with mass m and velocity v is given by

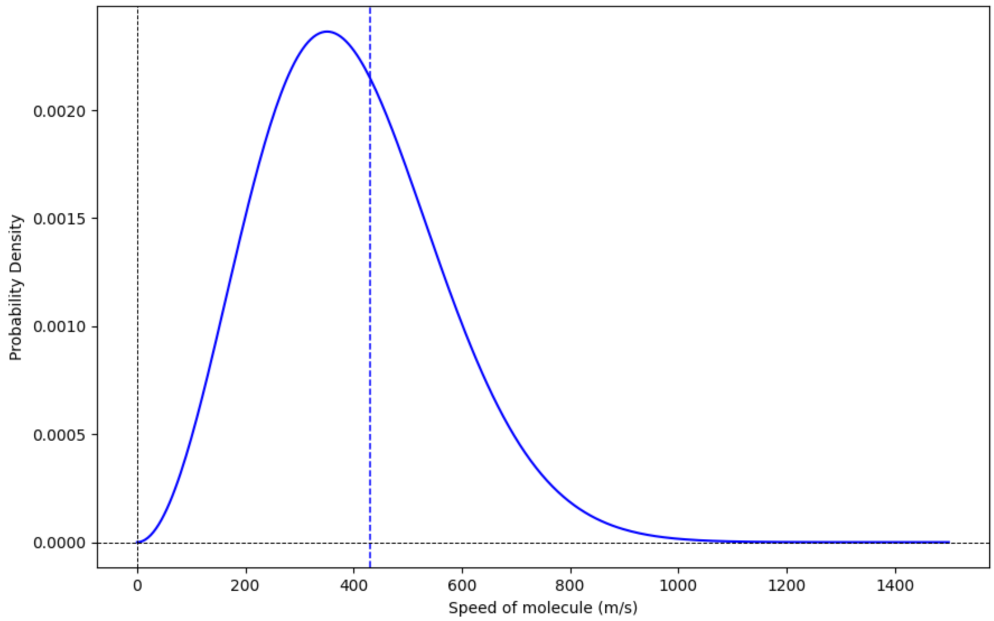
$$k = \frac{1}{2}mv^2$$

So, you could say "As the temperature of the gas increases, the kinetic energy of the molecules increases." But a physicist would say "The temperature of the gas is how we measure its kinetic energy."

7.0.1 A Statistical Look At Temperature

If you say "This jar of argon gas is 25 degrees Celsius," you have told me about the *average* kinetic energy of the molecules in the jar. However, some molecules are moving very slowly. Others are moving really, really fast.

We could plot the probability distribution of the speeds of the molecules. For argon at 25 degrees Celsius, it would look like this:



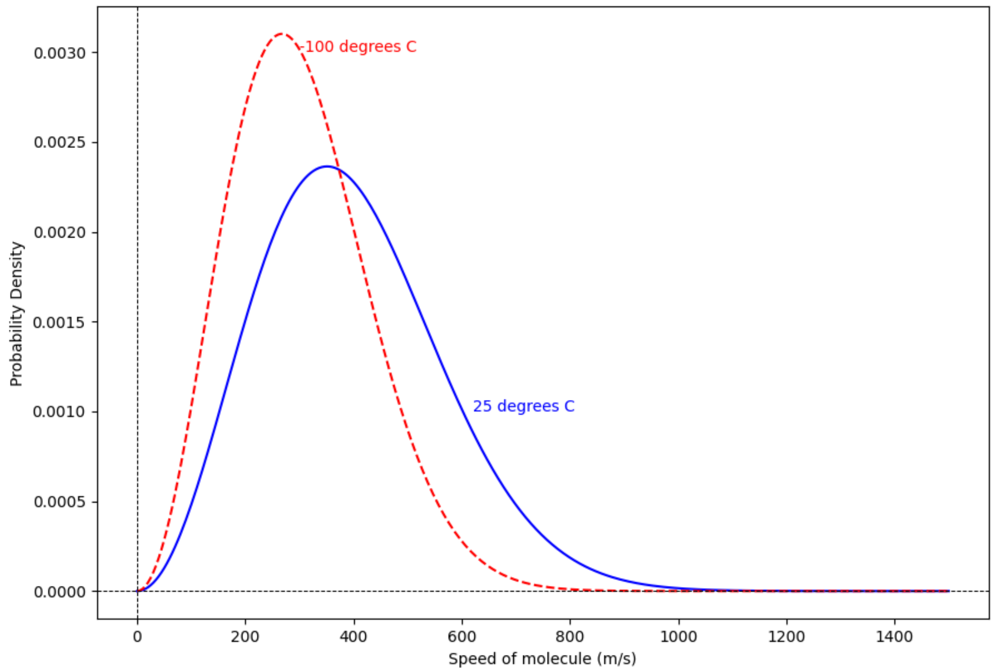
The temperature, remember, is determined by the average kinetic energy of the molecules. Some molecules are moving slowly and have less kinetic energy than the average. Some molecules are moving very quickly and have more kinetic energy. The dotted line is the divider between the two groups: molecules moving at speeds to the left of the line have less kinetic energy than average; those on the right have more kinetic energy than average.

Where is that line? That is the RMS of the speeds of the molecules. That is, if we measured all the speeds of all the molecules $s_1, s_2, s_3, \ldots, s_n$, that line would be given by the root of the mean of the squares:

$$v_{rms} = \sqrt{\frac{1}{n}\left(s_1^2 + s_2^2 + s_3^2 + \ldots s_n^2\right)}$$

If you have the same gas at a lower temperature, the distribution shifts toward zero:

Here is probability distribution of molecular speeds for argon gas at 25 degrees and -100 degrees Celsius.



7.0.2 Absolute Zero and Degrees Kelvin

If you keep lowering the temperature, eventually all the molecules stop moving. This is known as *absolute zero* – you can't make anything colder than absolute zero. Absolute zero is -273.15 degrees Celsius.

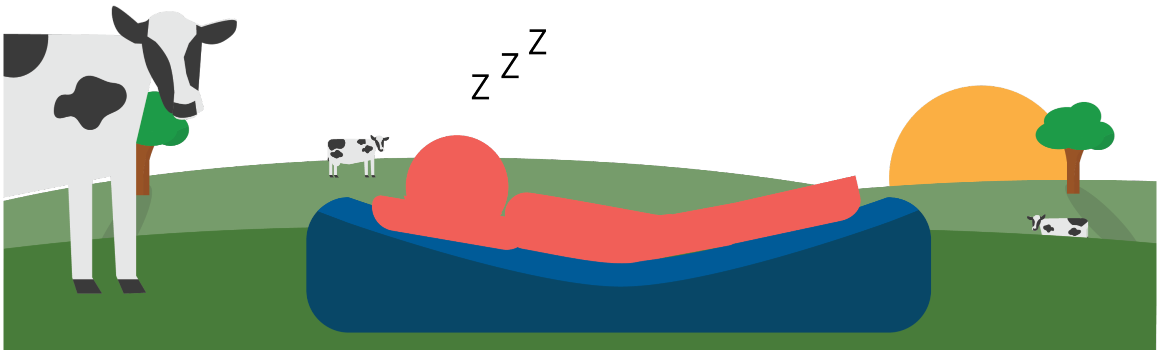
Besides Celsius and Fahrenheit, there is a third temperature system: Kelvin. The Kelvin has the same scale as Celsius, but it starts at absolute zero. So, 0 degrees Celsius is 273.15 degrees Kelvin. And 100 degrees Celsius is 373.15 degrees Kelvin.

Any time you are working with the physics of temperature, you will use Kelvin.

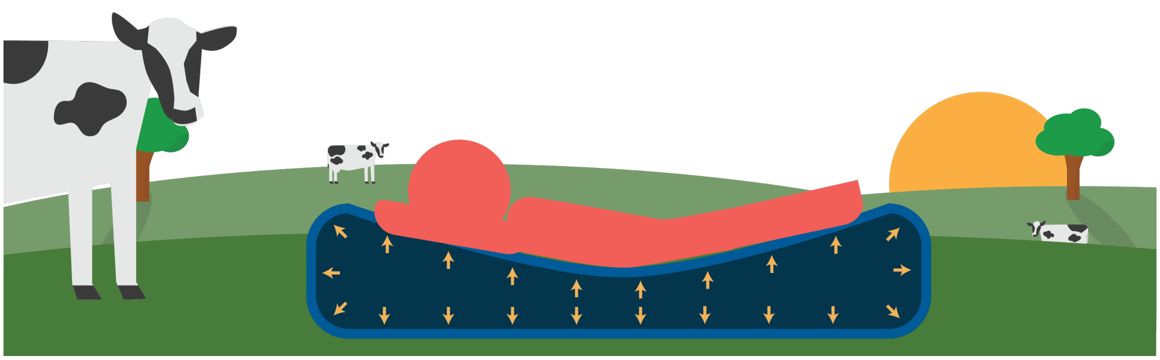
Sometimes, when reading about gases, you will see "STP" which stands for "Standard Temperature and Pressure." STP is defined to be 0° Celsius and 100 kPa.

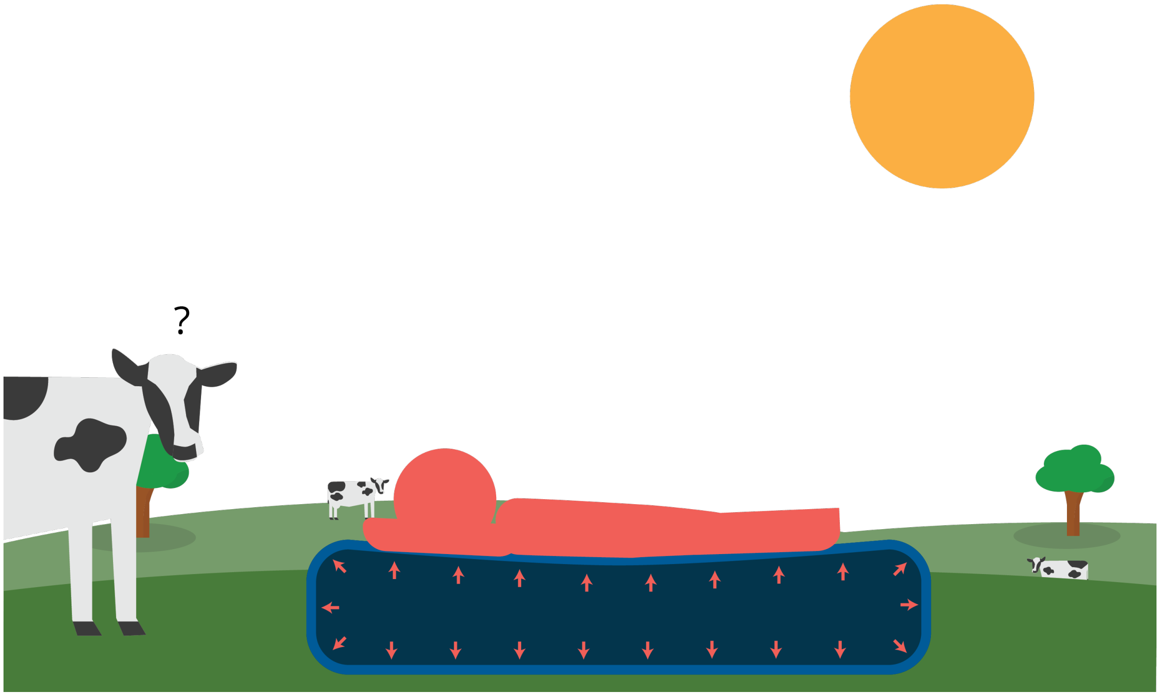
7.1 Temperature and Volume

Let's say you have a half-full air mattress in a field with a person lying on it around dawn. The weight of the person will keep the pressure of the air inside constant (or pretty close).



The molecules in the mattress are not entering or leaving that mattress. However, as the sun rises, the air inside will get warmer and expand. The person will be gently lifted by the expanding air. You might wonder: how much will the air expand?





If you have constant pressure and and a constant number of molecules, the volume of the gas is proportional to the temperature in Kelvin:

$$V \propto T$$

Exercise 33 Temperature and Volume

Working Space

At dawn, the air inside mattress at dawn has a volume of 1000 liters and a temperature of 12 degrees Celsius.

At noon, that same air has a temperature of 28 degrees Celsius. The pressure on the gas has not changed at all.

What is the volume of the gas at noon?

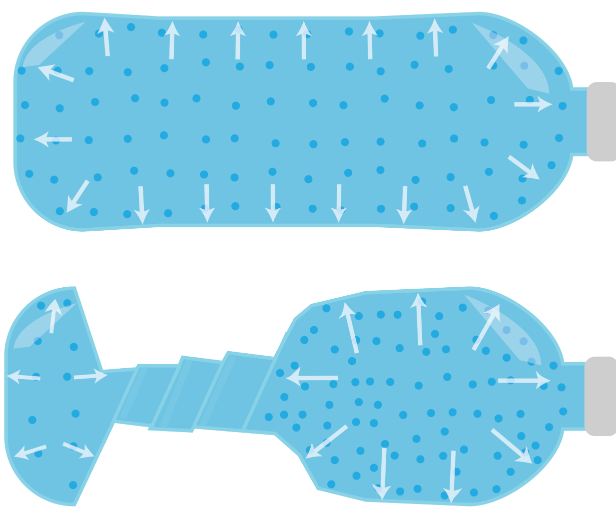
Answer on Page 126

Note: Volume and temperature are only proportional as long as the substance is a gas. We will talk about liquids and solids soon.

7.2 Pressure and Volume

As you increase the pressure on a gas, the molecules will get pushed closer together, and the volume will decrease.

For example, if you put the cap on an empty plastic bottle and squeeze it. As you put the gas inside the bottle under pressure, its volume will decrease.



If you keep the number of molecules and the temperature constant, the pressure of the gas and its volume are inversely proportional:

$$P \propto \frac{1}{V}$$

"But," you say with disbelief, "if I increase the pressure on my empty water bottle from 5 kPa to 10 kPa, the volume inside won't decrease by half!"

Don't forget that the air inside the bottle is under 101 kPa of atmospheric pressure before you even start to squeeze it.

Exercise 34 Temperature and Volume

Working Space

At an altitude where the atmospheric pressure is 100 kPa, you seal air in a 1 liter water bottle.

Squeezing the water bottle, you raise the internal pressure by 20 kPa What is the volume inside the bottle now?

Answer on Page 126

7.3 The Ideal Gas Law

You are gradually getting an intuition for the relationship between the number of molecules, the volume, the pressure, and the temperature of a gas. We can actually bring these together in one handy equation.

Ideal Gas Law

$$PV = nRT$$

where:

- P is the pressure in pascals
- V is the volume in cubic meters
- n is the number of molecules in moles
- R is the molar gas constant: 8.31446
- T is the temperature in Kelvin

(You can remember this as the ”Pivnert.”)

From the name, you might predict the following: The Idea Gas Law is not 100% accurate. But for most purposes, it works remarkably well.

Notice that the ideal gas law says nothing about what kind of gas it is; it works regardless.

Exercise 35 Ideal Gas Law

Working Space

You have a cylinder containing O_2. The chamber inside has a radius of 12 cm and a length of 50 cm. The temperature inside the cylinder is 20 degrees Celsius. The pressure inside the tank is 600 kPa.

How many moles of O_2 are inside?

Answer on Page 127

7.4 Molecules Like To Stay Close to Each Other

When two molecules get close to each other a few things can happen:

- They can under go a chemical reaction: electrons are exchanged or shared and a different molecule or molecules come into existence. This is the realm of chemistry, and we won't go into it in this course.
- One or both of them have so much kinetic energy that they just pass each other or bounce off each other. This is what happens in a gas.
- The two molecules can "stick" together. This is what happens in a liquid or a solid.

Why do they stick together if they aren't combined in a chemical reaction?

First, they don't get *too close*. If they get too close, their electron clouds repel each other with a strong force. This is what happens in a gas when two molecules bounce off of each other.

But if the molecules are quite close to each other, there are forces that will attract them toward each other. These intermoleculare forces are beyond the scope of this course, but they called Van der Waals forces and hydrogen bonds. The strength of these forces vary based on the two molecules involved.

Which is why some of the matter around you is in gas form (molecules that don't stick together at the temperature and pressure you are living in because they have weak attractive forces) and some is non-gas (gangs of molecules with stronger attraction that makes them clump together as a liquid or a solid at that same temperature and pressure).

But. What if we change the temperature and pressure, we can change if and how the molecules clump together. This is known a *phase change*; We will cover it soon.

Chapter 8

Kinetic Energy and Temperature of a Gas

As mentioned in the previous chapter, for a particular gas, the temperature (in Kelvin) is proportional to the average kinetic energy of the individual molecules.

Perhaps you want to warm 3 moles of helium gas (trapped in a metal cylinder) from 10 degrees Celsius to 30 degrees Celsius. How would you compute exactly how many Joules of energy this would require?

The amount of energy necessary to raise one mole of a molecule by one degree is known as *molar heat capacity*. (The molar heat capacity of liquid water, for example, is 75.38 J per mole-degree.)

With gases, are actually two different possible situations:

1. Constant volume: As you heat the gas, the pressure and the temperature increase. This molar heat capacity is usually denoted as $C_{V,m}$.
2. Constant pressure: As you heat the gas, the temperature and the volume increase. This molar heat capacity is usually denoted as $C_{P,m}$.

All gases made up of one atom (Helium, for example, is a monoatomic gas.) have the same values for $C_{V,m}$ and $C_{P,m}$:

$$C_{V,m} = \frac{3}{2}R \approx 12.47 \text{ Joules per mole-degree}$$

$$C_{P,m} = \frac{5}{2}R \approx 20.8 \text{ Joules per mole-degree}$$

(Remember from last chapter that R is the ideal gas constant ≈ 8.31446 Joules per mole-degree.)

Exercise 36 Warming Helium

Working Space

You have 3 moles of helium.

1. How many Joules would be required to warm 3 moles of helium gas by 20 degrees Celsius at constant volume?
2. How many Joules would be required to warm 3 moles of helium gas by 20 degrees Celsius at constant pressure?

Answer on Page 127

8.1 Molecule Shape and Molar Heat Capacity

We told you that gases made up of one atom have the same values for $C_{V,m}$ and $C_{P,m}$:

$C_{V,m} = \frac{3}{2}R \approx 12.47$ Joules per mole-degree

$C_{P,m} = \frac{5}{2}R \approx 20.8$ Joules per mole-degree

For any molecule, it is generally true that

$$C_{P,m} \approx C_{V,m} + R$$

It is also true that for any molecule, there is some integer d such that

$$C_{V,m} \approx \frac{d}{2}R$$

For example, for all monoatomic gases, $d = 3$. For diatomic gases (like N_2 and O_2, d is 5.

d is known as the *degree of freedom* of the molecule. When you study chemistry, they will teach you to predict d based on the shape of the molecule.

Here are the relevant numbers for some gases you are likely to work with:

Gas	type	$C_{V,m}$	$C_{P,m}$	d
He	monoatomic	12.4717	20.7862	3
Ar	monoatomic	12.4717	20.7862	3
O_2	diatomic	21.0	29.38	5
N_2	diatomic	20.8	29.12	5
HO_2 (water vapor)	3 atoms	28.03	37.47	7
CO_2	3 atoms	28.46	36.94	7

8.2 Kinetic Energy and Temperature

For a sample of a gas, we can calculate its kinetic energy based on its molar heat capacity, the number of molecules, and the temperature:

$$E_K = C_{V,m} n T$$

where

- E_K is the kinetic energy in Joules
- $C_{V,m}$ is the molar heat capacity of the gas at constant volume
- n is the number of molecules in moles
- T is the temperature in Kelvin

Exercise 37 Warming Helium Revisited

Working Space

How much kinetic energy does 3 moles of helium have at 10 degrees Celsius?

How much kinetic energy does 3 moles of helium have at 30 degrees Celsius?

What is the difference?

Answer on Page 127

8.3 Why is $C_{V,m}$ different from $C_{P,m}$?

What if, instead of keeping the volume constant while we heat the molecules in the helium tank, we keep the pressure constant and let the gas expand? The change in kinetic energy is the same: 748 Joules.

However, we know that the molar heat capacity if we keep pressure constant is $\frac{5}{2}R$, so heating will require $\frac{5}{2}R(3)(20) = 1247$ Joules.

What happened to the 499 missing Joules!? Thermodynamics tells us energy is neither created nor destroyed. So it must have gone somewhere.

That energy was used pushing against the pressure as the gas expanded. For example, maybe the sample was in a balloon in space – the extra energy stretched the surface of the balloon.

The 499 Joules were converted into potential energy.

8.4 Work of Creating Volume Against Constant Pressure

Let's imagine that you had a total vacuum (zero pressure) with a piston. As you pulled the piston out, you would be pulling against the atmospheric pressure. How much energy would that require?

If you increased the volume of the vacuum by V against a pressure of P, you would do

VP work.

Let's check to make sure the 499 Joules mentioned above makes sense with this in mind.

No initial pressure was given in the problem, so let's just make one up and see how things work out: 100 kPa. Using the ideal gas law, the initial volume would be:

$$V_1 = \frac{nRT}{P} = \frac{(3)(8.31446)(283.15)}{100,000} = 0.07063 \text{ cubic meters}$$

The volume after we heated the gas and let it expand against 100 kPa would be:

$$V_2 = \frac{nRT}{P} = \frac{(3)(8.31446)(303.15)}{100,000} = 0.07562 \text{ cubic meters}$$

So the volume increased by $0.07562 - 0.07063 = 0.00499$ cubic meters. Multiplying that by 100,000 pa, we get 499 Joules as we expected!

8.5 Why does a gas get hotter when you compress it?

Now imagine that there is gas inside the piston and you push on the piston to compress that air. The work that you do is converted into kinetic energy, and that kinetic energy raises the temperature of the gas.

So, for example, if you had two moles of argon gas in the piston. If you pushed the piston 0.1 meters with an average force of 50 newtons, you will have done 5 Joules of work.

How much would 5 Joules raise the temperature of 2 mole of a monoatomic gas?

$$\Delta T = \frac{5}{(2)(C_{V,m})} = 0.2^\circ \text{ Kelvin}$$

It works both ways: compression makes a gas hotter Decompression makes a gas colder. You can sometimes experience the heat of compression when you pump up a bicycle tire – as you pump the tire will get warmer.

If you compress or decompress a gas without letting any heat enter or depart, we say the compression or decompression was *adiabatic*. In order to solve any interesting problems about heating/cooling due to compression/decompression, you will need to assume the process was adiabatic.

When a spacecraft enters the atmosphere, it has to deal with a lot heat. Some people assume that heat is due to friction of the air rubbing against the spacecraft at over 7,000 meters per second. Actually, most of the heat is due to the compression of the air as it gets pushed out of the way of the spacecraft.

8.6 How much hotter?

Let's say you have a accordion-like container filled with helium at 100 kPa (about 1 atmosphere) and 300 degrees Kelvin. It holds 2 cubic meters. And then you put it in a vice and quickly compress it down to 0.5 cubic meters. Assuming it was adiabatic, how hot would the gas inside be after the compression?

Here is the challenging part: As you crush the container, the temperature and the pressure in the container are both increasing. So as you go, it gets harder and hard to crush. So each milliliter of volume that you eliminate requires a little more work than the milliliter before.

Let's simulate the process in python, and then I'll give you the formula.

In the simulation, you will start with an initial volume of 2 cubic meters and crush it down to 0.5 cubic meters in 40 steps. At each step you will recalculate the temperature and pressure.

Then you will plot the results. Make a file called `gas_crunch.py`:

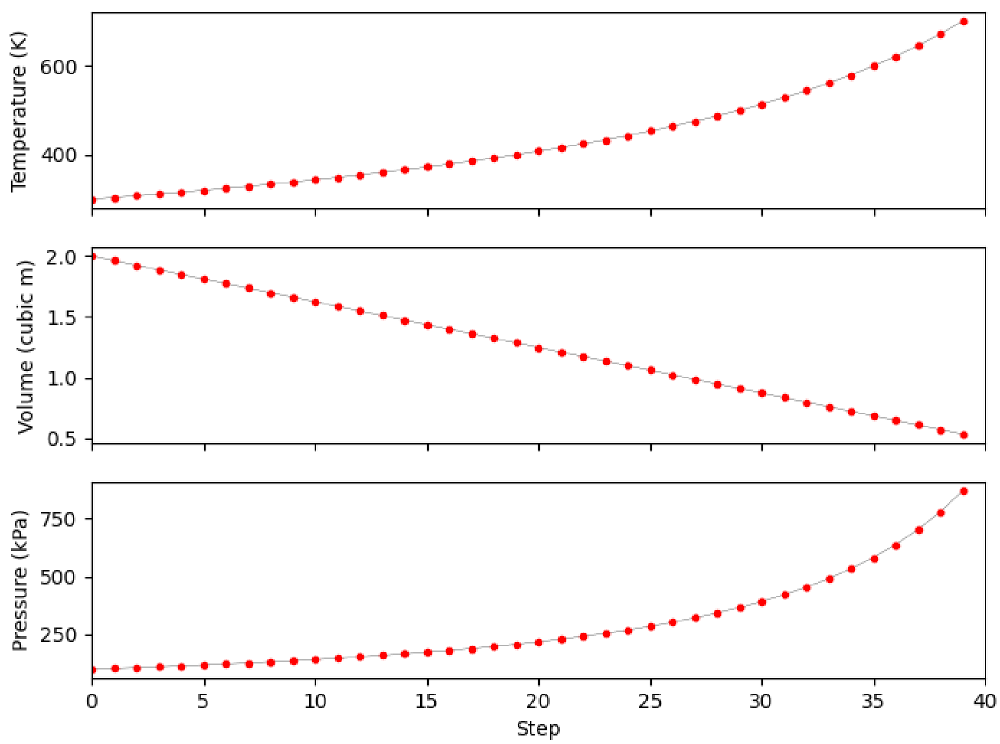
```
import numpy as np
import matplotlib.pyplot as plt

V_initial = 2.0 # cubic meters
V_final = 0.5 # cubic meters
step_count = 40 # steps

T_initial = 300.0 # kelvin
P_initial = 100000 # pascals

# Constants
R = 8.314462618 # ideal gas constnt
C_v =  3.0 * R / 2.0 # molar heat capacity (constant volume)

# Compute the number of moles
n = P_initial * V_initial/(R * T_initial)
print(f"The container holds {n:.2f} moles of helium")

# How much volume do we need to eliminate in each step?
# (in cubic meters)
```

```
step_size = (V_initial - V_final) / step_count

# For recording the state for each step
data_log = np.zeros((step_count, 3))

# Variables to update in the loop
V_current = V_initial
T_current = T_initial
P_current = P_initial

for i in range(step_count):
    # Record the current state
    data_log[i,:] = [T_current, V_current, P_current/1000.0]

    # Find how much energy to make the step at the current pressure
    E_step = step_size * P_current

    # Find how big the change in temperature will be from that energy
    delta_T = E_step / (n * C_v)

    # Update the current temperature, volume, and pressure
    T_current += delta_T
    V_current -= step_size
    P_current = n * R * T_current / V_current

print(f"Iterative:{T_current:0.3f} K, {V_current:0.3f} m3, {P_current/1000.0:0.3f} kPa")

fig, axs = plt.subplots(3,1,sharex=True, figsize=(8, 6))
axs[0].set_xlim((0,step_count))
axs[0].plot(data_log[:,0], 'k', lw=0.2)
axs[0].plot(data_log[:,0], 'r.')
axs[0].set_ylabel("Temperature (K)")

axs[1].plot(data_log[:,1],'k', lw=0.2)
axs[1].plot(data_log[:,1], 'r.')
axs[1].set_ylabel("Volume (cubic m)")

axs[2].plot(data_log[:,2], 'k', lw=0.2)
axs[2].plot(data_log[:,2], 'r.')
axs[2].set_ylabel("Pressure (kPa)")

axs[2].set_xlabel("Step")

fig.savefig('tvpplot.png')
```

When you run this, you will see how many moles of gas there are and reasonable estimates of the temperature, volume, and pressure:

```
> python3 gas_crunch.py
The container holds 80.18 moles of helium
Iterative:733.499 K, 0.500 m3, 977.999 kPa
```

And a good plot of the intermediate values:

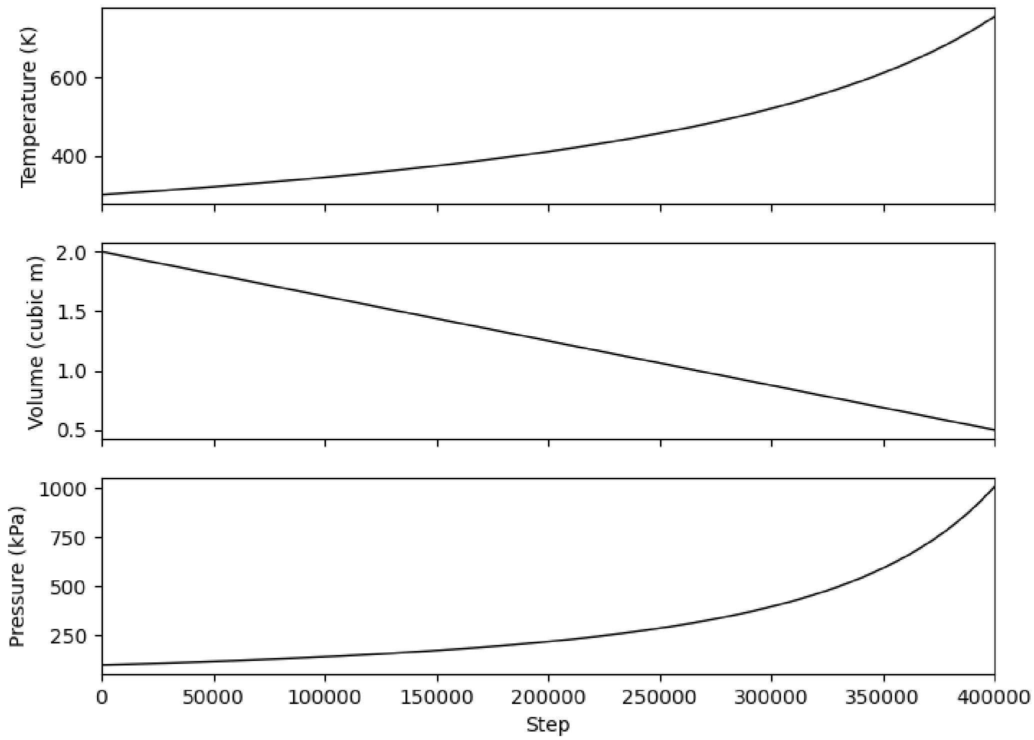


But, we will get better estimates if we break it up into 400 steps instead of 40. Change the line that defines the number of steps:

```
step_count = 400 # steps
```

Now the predicted temperature and pressure should be something like 753.603° K and 1004.803 kPa. (This is much closer to the correct result: :755.953° K and 1007.937 kPa.)

What if you break it into 400,000 steps? Now the result should be really, really close to correct. And the plot is quite accurate:



(You can comment out the lines that make the red dots on the graphs. No one wants to see 400,000 red dots.)

It is inefficient to have to do long simulations to guess the final temperature and pressure. Fortunately, there are two handy rules you can use to skip this:

Adiabatic Compression and Decompression

Let

$$\gamma = \frac{C_{P,m}}{C_{V,m}}$$

In an adiabatic compression or decompression, P and V change, but

$$P\left(V^{\gamma}\right)$$

stays constant.

Also

$$T\left(V^{(\gamma-1)}\right)$$

stays constant

For a monoatomic gas:

$$\gamma = \frac{C_{P,m}}{C_{V,m}} = \frac{5}{3}$$

$$\gamma - 1 = \frac{2}{3}$$

Before the compression:

$$T\left(V^{(\gamma-1)}\right) = 300\left(2^{0.6667}\right) = 476.22$$

After the compression it has to be the same:

$$T\left(V^{(\gamma-1)}\right) = T\left(0.5^{0.6667}\right) = 476.22$$

Thus

$$T = 755.95^\circ \text{ Kelvin}$$

We can then use the ideal gas law to solve for the final pressure:

$$P = \frac{nRT}{V} = \frac{(80.18)(8.31446)(755.95)}{0.5} = 1007937 \text{ pascals}$$

That's hot! As you let it cool back down to 300 degrees Kelvin, how much heat would be released?

$$E = C_{V,m} n \Delta T = (12.47)(80.2)(755.95 - 300) \approx 456 \text{ kJ}$$

8.7 How an Air Conditioner Works

Once again, imagine the accordion-like container filled with helium. Let's say you walked it outside and compressed it from 2 cubic meters to 0.5 cubic meters in a vise. The container would get to 755.95 degrees Kelvin. You keep it compressed, in the vise but let it cool down outside. When it gets back to 300 degrees Kelvin, you walk it back inside.

Now, without letting any molecules in or out of the container, you release the vise. The gas is decompressed and gets very cold – how cold? Cold enough to accept about 456 kJ of kinetic energy from your house. That is, it would absorb heat from your house until the gas inside was the same temperature as your house.

Now you walk outside with your accordion and your vise and repeat:

1. Compress the gas outside.
2. Let the hot gas cool down outside.
3. Walk the room-temperature compressed gas inside.
4. Decompress the gas inside.
5. Let the cold gas warm up inside.

You could keep your house cool on a hot day this way. And this is not unlike how an air conditioner works.

There is a hose filled with refrigerant that does a loop:

- Outside, the refrigerant is compressed and allowed to cool to the outside temperature. (Usually there is a big fan blowing on a coil of refrigerant to speed the process.) Inside, the refrigerant is decompressed and allowed to warm to the inside temperature. (Usually there is a big fan blowing the air of the home past a coil of refrigerant to speed the process.)

In each pass of the loop, the refrigerant absorbs some of the kinetic energy from inside the house, and releases it on the outside.

This same mechanism can be used to heat your house. (Units that both heat and cool are known as *heat pumps*.) The heat pump does the process backwards: The hot compressed refrigerant cools down inside. The cold decompressed refrigerant warms up outside.

Chapter 9

Phases of Matter

You have experienced H_2O in three phases of matter:

- Ice is H_2O in the solid phase. At standard pressure, when the temperature of H_2O is below 0° C, it is a solid.
- Water is H_2O in the liquid phase. At standard pressure, when the temperature of H_2O is between 0° C and 100° C, it is a liquid.
- Water vapor (or steam) is the gas phase. At standard pressure, when the temperature of H_2) is above 100° C, it is a gas.

Let's look at some of the properties of the three phases:

Gas	Liquid	Solid
Assumes the volume and shape of its container	Assumes the shape, but not the volume, of its container	Retains its shape and volume
Compressible	Not compressible	Not compressible

9.1 Thinking Microscopically About Phase

As mentioned in an early chapter, there are intermolecular forces that attract molecules to each other. A pair of molecules will have very strong intermolecular forces or very weak intermolecular forces depending on what atoms they are made of.

For example, two helium molecules are very weakly attracted to each other due to weak intermolecular forces. Two molecules of $NaCl$ (table salt) will experience very strong intermolecular attraction.

In a gas, the molecules have lots of room to roam and lots of kinetic energy: The intermolecular attraction has very little effect.

In a liquid, the molecules are sticking close together, but are still moving around, sort of like bees in a hive.

In a solid, the molecules are not changing their configuration, and the kinetic energy they have is just expressed as vibrations within that configuration. You can imagine them like eggs in a carton just vibrating.

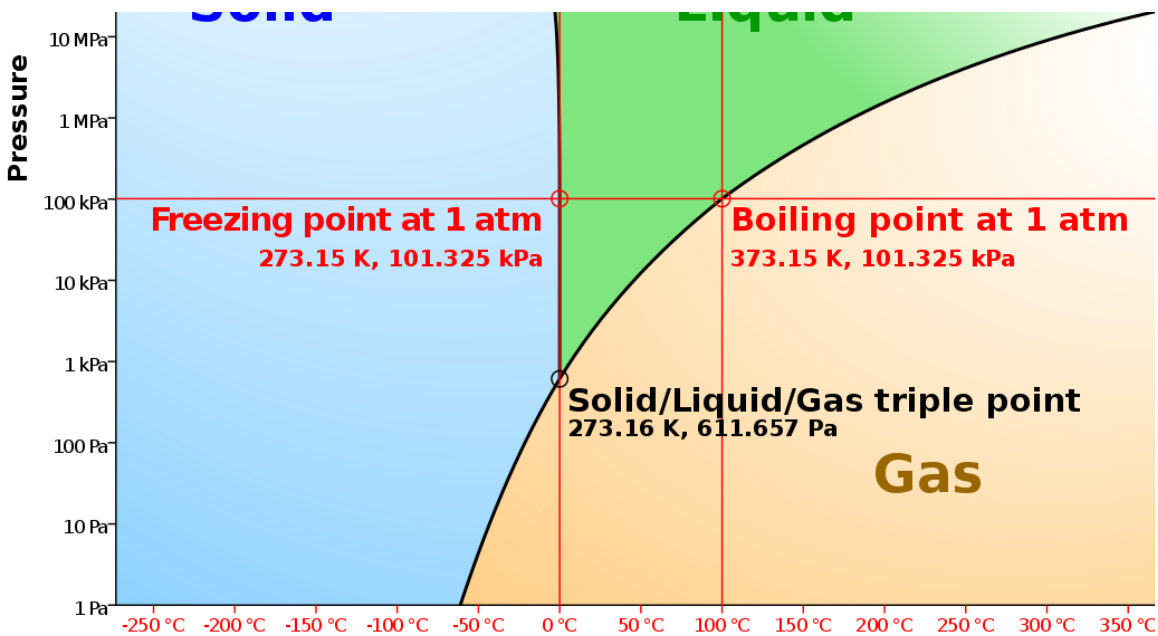
As you would expect, molecules with strong intermolecular attraction require more kinetic energy to change phases. For example, helium is a liquid below $-269\circ$ C. $NaCl$, on the other hand, is a liquid between 801° and $1,413^\circ$ C.

The temperatures I just gave you are at standard pressure (100 kPa or 1 atm). Pressure also has a role in phase change: In low pressure environments, it is much easier for the molecules to make the jump to being a gas.

For example, if you climb a mountain until the atmospheric pressure is 70 kPa, your water will boil at about 90° C.

If you rise in a balloon until the atmospheric pressure is 500 Pa, if your water is colder than -2° C, it will be ice. If it is warmer it will vaporize. There is no liquid water at 500 Pa!

For any molecule, we could observe its phase at a wide range of temperatures and pressures. This would let us create a phase diagram. Here is the phase diagram for H_2O:



(FIXME: This diagram needs to be recreated prettier.)

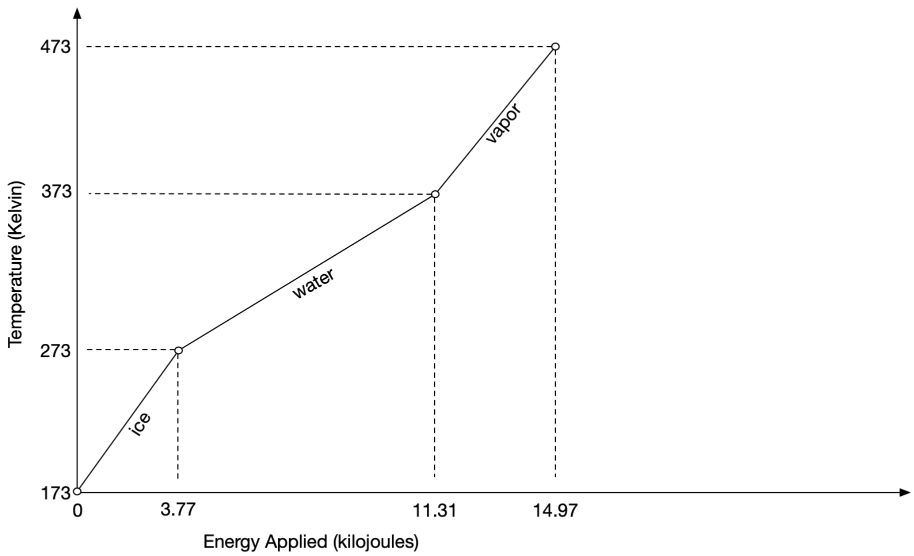
9.2 Phase Changes and Energy

The molar heat capacity of ice is about 37.7 J/mol-K. That is it takes about 37.7 Joules of energy to raise the temperature of one mole of ice by one degree kelvin.

The molar heat capacity of liquid water is about 75.4 J/mol-K. For water vapor, it is about

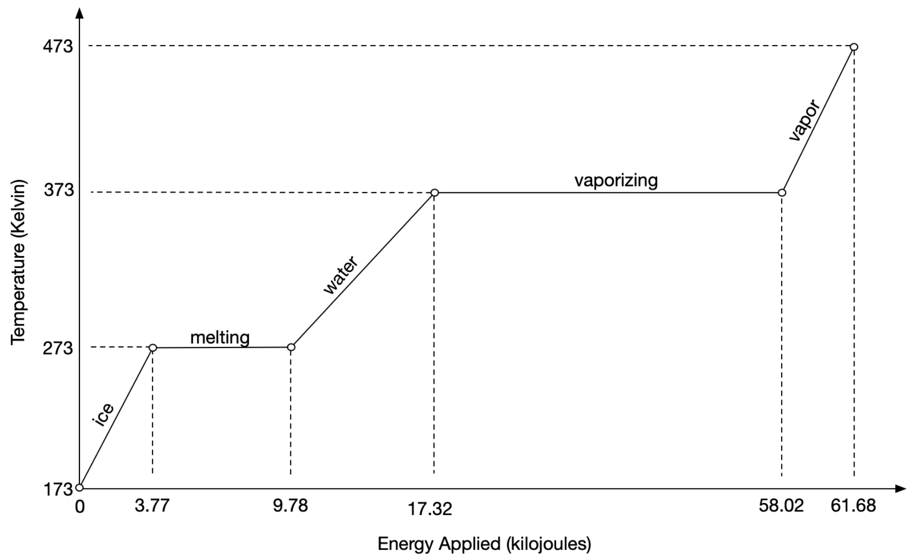
36.6 J/mol-K.

Imagine you have mole of ice at 173° K and you are going gradually add kinetic energy into it until you have steam at 473° K. You might guess (wrongly) that the temperature vs. energy applied would look like this:



However, once molecules are nestled into their solid state (like eggs in cartons), it take extra energy to make them move like a liquid. How much more energy? For water, it is 6.01 kilojoules per mole. This is known as *the latent heat of melting* or *the heat of fusion*.

Similarly, the transition from liquid to gas takes energy. At standard pressure, converting a mole of liquid water to vapor requires 40. 7 kilojoules per mole. This is known as *the latent heat of vaporization*. So the graph would actually look like this:



Note that just as melting and vaporizing require energy. Going the other way (freezing and condensing, respectively) give off energy. Thus, we can store energy using the phase change.

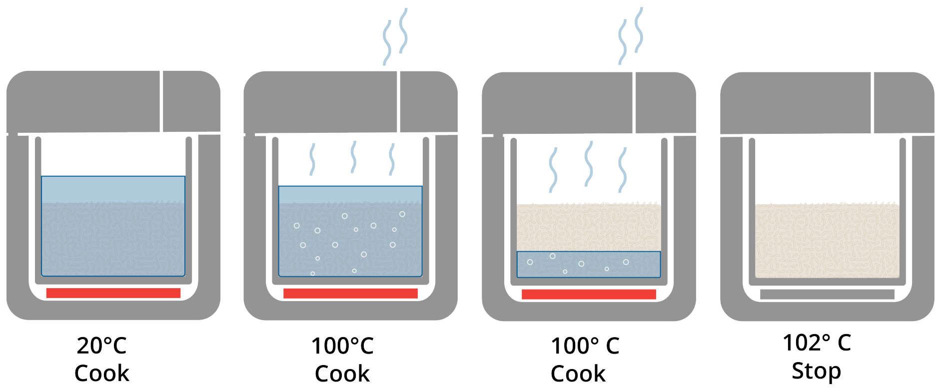
9.3 How a Rice Cooker Works

As you might imagine, a rice cooker is a bowl with a lid and an electric heating element. You put rice and water into the bowl and turn on the heating element. The heating element pushes kinetic energy into the water, which gets warmer and eventually starts to boil.

How does the rice cooker know when to turn off (or at least down to a low-heat ”keep warm” mode) before the rice starts to burn?

As long as there is a little liquid water in the bottom of the bowl, the rice won't burn, so the question really is ”How does it know when there is no more liquid water in the bottom of the bowl?”

There is a mechanism (and there have been a few different versions of this mechanism) that monitors the temperature of the surface of the bowl. As long as there is liquid water in the bowl, it *cannot* go above 100° C! When all the water has been absorbed by the rice or turned to steam, the temperature rises above 100° C, and the mechanism cuts off the heat.


20°C
Cook
100°C
Cook
100° C
Cook
102° C
Stop

Exercise 38 Using Water For Thermal Energy Storage

Working Space

Tom is building a passive solar house: the front of his house is a greenhouse. He also likes to eat dinner in the greenhouse. He will have barrels (painted black) that hold 159 liters of water. His plan is to let the sun heat the barrels to 33° C by the time the sun goes down. (Any warmer and it would be unpleasant to eat dinner near them.)

At night, he will circulate air past the barrels and through his house. He is OK with the house and the barrels dropping to 17° C.

Looking at the insulation on his house and the expected nighttime temperatures, Tom estimates that he needs to store 300,000 KJ of energy in the barrels.

A mole of water is about 0.018 liters.

The molar heat capacity of water is about 75.38 J/mole-K.

How many barrels of water does Tom need to install in his greenhouse?

Answer on Page 128

Exercise 39 Using Mirabilite For Thermal Energy Storage

Working Space

Water barrels are going to take up too much of Tom's greenhouse!

There is a substance known as mirabilite, or Glauber's salt, or sodium sulfate decahydrate. It is relatively cheap to produce, and it has a melting point of 32.4° C.

The molar heat capacity of mirabilite is 550 J/mole-K.

The latent heat of melting mirabilite is 82 KJ per mole.

Mirabilite comes in a powder form. Assume that a mole of mirabilite occupies about 0.22 liters.

If Tom fills his barrels with mirabilite, how many barrels will he need?

Answer on Page 128

9.4 Thinking Statistically About Phase Change

We like to say simple stuff like "At 100° C, water changes to vapor." However, remember what temperature is: Temperature tells you how much average kinetic energy the water molecules have. The key word here is *average*; Some molecules are going faster than average and some are going slower.

A puddle in the street on a warm night will evaporate. It isn't 100° C. Why would the puddle turn to vapor?

While the *average* molecule in the puddle doesn't have enough energy to escape the intermolecular forces, some of the molecules do. When a molecule on the surface has enough velocity (toward the sky!) to escape the intermolecular forces, it becomes vapor and leaves the puddle.

What happens to the temperature of the puddle during this sort of evaporation? Temperature is proportional to the average kinetic energy of the molecules. If a bunch of molecules with a lot of kinetic energy leave, the average kinetic energy (of the molecules that remain) will decrease.

The most obvious example of this process is sweating: When your body is in danger of getting too hot, sweat comes out of your pores and covers your body. The fastest moving molecules escape your body, taking the excess kinetic energy with them.

9.4.1 Evaporative Cooling Systems

An evaporative cooling system (also called a ”Swamp Cooler”) uses this idea to cool air. You can imagine a fan drawing warm air through a duct from the outside. Before the air is released inside, it passes very close to a cloth that is soaked with water. The warm air molecules (which has a lot of kinetic energy) slam into the water molecules, some of which get enough kinetic energy to become vapor. Then the cool air and the water vapor enter the room.

”Wait, wait, wait,” you say, ”The heat hasn’t gone away. It is just transferred into the water molecules.”

Remember that it takes 40. 7 kilojoules to change a mole of liquid water at 100° C to water vapor at 100°. Escaping those intermolecular bonds takes a lot of energy!

Thus, if a mole of water evaporates, it is because it has absorbed 40.7 kj of heat you can feel (*sensible heat*) and used it to liberate the molecules from their intermolecular bonds. For convenience, physicists call this *latent heat*.

9.4.2 Humidity and Condensation

When a puddle is evaporating on a warm day, there might be some water vapor already in the air. Even as the water molecules in the puddle are evaporating, some water molecules in the air are crashing into the puddle and become liquid again. (We say they *condensed*, thus the word *condensation* to describe the water that accumulates on a cold glass on a warm day.)

When there is a lot of water vapor in the air, the puddle will evaporate more slowly. In fact, if there is enough water vapor in the air, the puddle won’t evaporate at all. At this point, we say ”The relative humidity is 100%” That is, relative to the amount of water the air will hold, it already has 100% that amount.”

Neither sweating nor evaporative cooling systems work well when the relative humidity is high.

As the temperature goes up, the air can hold more water. We usually notice it when it goes the other way: the air cools and has more water vapor than it can hold. Some of the water vapor condenses into water droplets. If the droplets land on something, we call it ”dew”. If it is high in the sky, we call the droplets ”a cloud”. If it is near the ground, we call it ”fog.”

Chapter 10

The Piston Engine

Most cars, airplanes, and chainsaws get their power from burning hydrocarbons in a combustion chamber. We say they have *internal combustion engines*. There are many types of internal combustion engines: jet engines, rotary engines, diesel engines, etc. In this chapter, we are going to explain how one type, piston engines, work. Most cars have piston engines.

Most piston engines burn gasoline, which is a blend of liquid hydrocarbons. Hydrocarbons are molecules made of hydrogen and carbon (and maybe a little oxygen). In the presence of oxygen and heat, hydrocarbons burn – the carbon combines with oxygen to become CO_2 and the hydrogen combines with oxygen to become H_2O. In the process, heat is released, which causes the gases in the cylinder to create a lot of pressure on the piston.

10.1 Parts of the Engine

The engine block is a big hunk of metal. There are cylindrical holes bored into the engine block. A piston can slide up and down the cylinder. There are two valves in the wall of the cylinder:

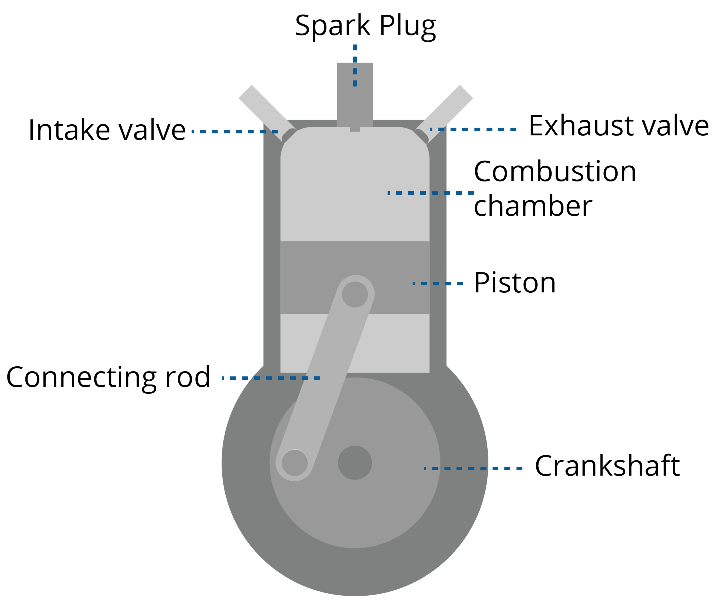
- Before the burn, one valve opens to let ethanol and air into the cylinder.
- After the burn, the other valve opens to let the exhaust out.

There is also a spark plug, which creates the spark that triggers the burn.

As you give the engine more gas, the cylinder does more frequent burns. When the engine is just idling, the cylinder fires about 9 times per second. When you depress the gas pedal all the way down, it is more like 40 times per second.

The cylinder has a rod that connects it to the crank shaft. As the pistons move back and forth, the crank shaft turns around and around. Sometimes a piston is pushing the crankshaft, and sometimes the crank shaft is pushing or pulling the piston. All the cylinders share one crank shaft.

How many cylinders does a car have? Nearly all car models have between 3 and 8 cylinders. The opening of the valve and the firing of the spark plugs are timed so that cylinders all do their burns at different times. This makes the total power delivered to the crank shaft smoother.



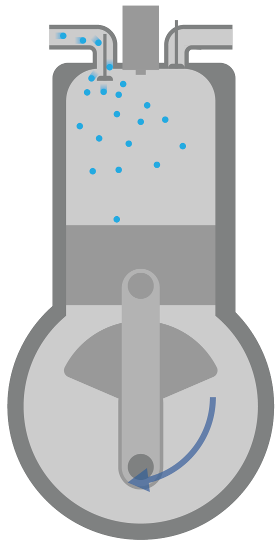
10.2 The Four-Stroke Process

Cars have four-stroke engines – this means for every two rotations of the crank shaft, each cylinder fires once. Smaller engines, like those in chainsaws, are often two-stroke engines – every cylinder fires every time the crank shaft rotates. For now, let's focus on four-stroke engines.

Here is the cycle of a single cylinder:

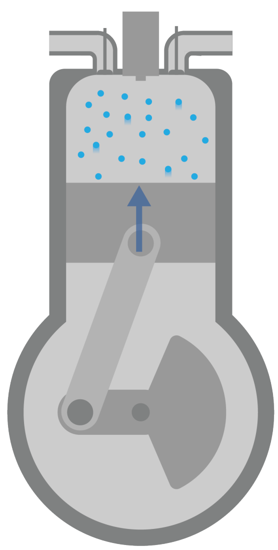
- As the drive shaft turns, it pulls the piston down. The intake valve opens and lets the gas/air mixture into the combustion chamber.

Intake



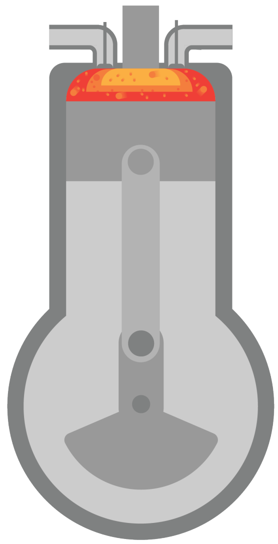
- As the piston reaches the bottom of the stroke, the intake valve closes.
- Now the crank shaft starts to push the piston up, compressing the gas and oxygen.

Compression



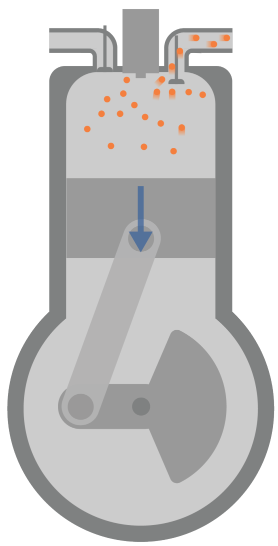
- As the piston reaches the top of its stroke, the spark plug creates a spark. The fuel and oxygen burn quickly. The cool liquid fuel becomes hot carbon dioxide and water vapor.

Combustion



- Now there is very high pressure inside the cylinder. It pushes hard on the piston which pushes the crank shaft.
- When the piston reaches the bottom of this stroke, the exhaust valve opens.
- As the crank shaft pushes the piston up, the carbon dioxide and water vapor is pushed out.

Exhaust



- When the piston reaches the top of this stroke, the exhaust valve is closed.

10.3 Dealing with Heat

Burning fuel inside a block of metal generates a lot of heat. If there is too much heat, parts of the engine will start to melt. So modern car engines are liquid cooled – there are arteries in the engine block carrying a liquid (called ”coolant”). The hot coolant is pumped through the radiator (where the air passing through takes away the heat) and then back into the engine.

Note that the heat that is carried away by the coolant is wasted energy. In fact, of the total energy created in burning the fuel, most car engines only transfer about 30% to turning the crank shaft. About 35% of the heat goes out with the exhaust. About 30% is carried away by the coolant. The remaining waste (usually about 5%) is lost to friction.

10.4 Dealing with Friction

From the description, it is clear that there is a lot of metal sliding against metal, which would grind the engine up quickly if there were no lubrication. In a modern car, the moving parts in the engine are constantly bathed in oil. There is an oil pump that causes it to get sprayed on the crankshaft, the connecting rod, and in the cylinder under the piston. (That is, not on the combustion side.)

The oil eventually falls through the oil into a pan at the bottom of the engine. The oil pump sucks the oil up, pushes it through a filter (so bits of metal are not pumped back into the engine), and then is sprayed on the moving parts again.

10.5 Challenges

With a piston engine, there are a lot of things that can go wrong. Let’s enumerate a few:

- *The seal around the piston leaks*. Mechanics say ”We aren’t getting any compression.” The cylinder doesn’t get much power to the drive train.
- *The valves open or the spark plug fires at the wrong time.* This is known as a timing problem.
- *The spark plug doesn’t make a spark*. The spark plug has two prongs of metal and electrons jump from one to the other. For a good spark, the prongs need to be a very precise distance apart. Sometimes you need to bend one of the prongs to get the right gap. This is known as *gapping*.

- *The mix of fuel and oxygen is wrong.* If there is too much fuel and not enough oxygen (so not all the fuel burns), we say the mix is too rich. If there is not enough fuel (so the pressure created by the burn is as high as possible), we say the mix is too lean.

10.6 How We Measure Engines

If you look up the specs on an engine, you will see the following:

- The number of cylinders
- The cylinder bore, which is the diameter of the cylinder
- The piston stroke, which is the distance the piston travels in the cylinder
- The compression ratio, which is the ratio between the maximum volume of the combustion change and the minimum volume of the combustion chamber.
- What fuel it runs on.

The difference between the minimum and maximum value of the cylinder is known as its *displacement*. The displacement represents the volume of air/fuel sucked into the intake valve before the compression begins.

We often talk about the displacement of the entire engine, which the cylinder's displacement times the number of cylinders. The displacement of an engine can give you a good idea of how much power it can produce.

For motorcycles, the displacement is often part of the name. For example, the Kawasaki Ninja 650 has about 650 cubic centimeters of displacement.

10.7 The Ford Model T and Ethanol

The Ford Model T was the first popular car. It came out in 1908 and remained in production until 1927. It had a four-cylinder engine that would run on ethanol, benzene, or kerosene. For the purposes of this exercise, let's assume you are running yours on ethanol.

A molecule of ethanol has 2 carbon atoms, 6 hydrogen atoms, and 1 oxygen atom. The oxygen in the atmosphere is O_2. When one molecule of ethanol combines with three molecules of O_2, 2 molecules of CO_2 and 3 molecules of H_2O are created. Also, a lot of heat is created: 1330 kilojoules for every mole of ethanol burned.

The engine block is usually very hot once the engine has been running. That is important because the ethanol will be completely vaporized at that temperature.

In any sample of air, 21 percent of the molecules will be O_2.

Exercise 40 Fuel Mix for the Model T

Working Space

On the Model T, a carburetor mixed the fuel and air before it went into the cylinder. The question to answer in this exercise is: How rich should the mix be at sea level (100 kPa)?

On the Wikipedia page for the Model T, we see the following facts:

- Cylinder bore: 9.525 cm
- Piston stroke: 10.16 cm

You can assume that the air/fuel mixture is 80° C before the pre-burn compression starts. (Thus the ethanol, which boils at 78° C, is in its vapor phase.)

The questions, then, are:

- What is the displacement of a single cylinder?
- How many moles of gas (80° C and 100 kPa) will get sucked through the intake valve?
- How many moles of vaporized ethanol should be part of that?
- How many moles of CO_2 and H_2O are created in each burn?
- How much heat is created in each burn?

(This exercise is a lot of steps, but nothing you don't know. You will need the ideal gas law to figure out how much many moles of air gets dragged into the cylinder.)

Answer on Page 128

10.8 Compression Ratio

Most of the inefficiency of a motor is heat that escapes through the exhaust valve. If your piston stroke were long enough, you could keep increasing the volume (which would cool the gases inside) until the gases inside were the same temperature as the outside world. Then there would be no wasted heat in the exhaust.

For this reason, generally, engines with a higher compression ratios tend to waste less energy through the heat of the exhaust. The Model T had a compression ratio close to 4:1. Modern car engines typically have compression ratios between to 8:1 and 12:1.

Cars with really high compression ratios often require fuels with a lot of kilojoules per mole – we say *high octane*. If the fuel doesn't have enough energy, the engine makes loud knocking noises as the pistons don't have enough energy to push through their entire stroke.

It turned out that an easy way to boost the octane of the gasoline was to add a chemical called tetraethyl lead. Gasoline containing tetraethyl lead was known at "Leaded Gasoline" and was intended to prevent the knocking. It is difficult to overstate the damages caused by putting large amounts of lead in the air. Gradually, starting in with Japan in 1986, every country in the world has banned leaded gasoline.

10.9 The Choke and Direct Fuel Injection

Most cars built before 1990 will have a carburetor, which ensures that the ratio between fuel and oxygen is constant regardless of the amount of fuel released by the throttle.

If you go to start an old car on a cold morning, the cold engine will not properly vaporize the fuel and the engine may not have enough power to start. For this reason, most carburetors have a *choke value* that makes the mix richer. (If you pull the choke valve, be sure to push it back after the engine warms up.)

The carburetor was a common source of engine problems and inefficiencies. Starting in the 1990s, car engines started using direct fuel injection: Air still came in through the intake manifold, but fuel was sprayed directly into the cylinder by a fuel injection system.

In modern cars, the fuel injection system is controlled by a computer (an *Engine Control Module* or ECM) which delivers the fuel at the perfect time with the perfect amount based on environmental variables like the temperature of the engine and the barometric pressure (usually related to that altitude at which the engine is operating).

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 7)

Since we are finding the antiderivative of $\sin x$, we will define $f(x) = \sin x$. We are looking for a F such that $F'(x) = \sin x$. The derivative of $\cos x$ is $-\sin x \neq f(x)$. But the derivative of $-\cos x = \sin x = f(x)$. Since $\frac{d}{dx}[-\cos x] = \sin x$, the antiderivative of $\sin x$ is $-\cos x$.

Answer to Exercise 2 (on page 9)

First, we will find $v(t)$ by taking the antiderivative of $a(t)$ and using the initial condition $v(0) = -6$

$$\int 6t + 4 \, , dt = 3t^2 + 4t + C = v(t)$$

$$v(0) = 3(0)^2 + 4(0) + C = -6$$

$$C = -6$$

Therefore, the velocity function is $v(t) = 3t^2 + 4t - 6$. Now we repeat the process to find $s(t)$:

$$\int 3t^2 + 4t - 6 \, , dt = t^3 + 2t^2 - 6t + C = s(t)$$

$$s(0) = (0)^3 + 2(0)^2 - 6(0) + C = 9$$

$$C = 9$$

Therefore, the position function is $s(t) = t^3 + 2t^2 - 6t + 9$.

Answer to Exercise 3 (on page 10)

The antiderivative of $\sin x$ is $-\cos x$ and therefore the general solution is $f(x) = -2\cos x + C$. We use the given condition, $f(\pi) = 1$ to find C:

$$f(\pi) = -2\cos\pi + C = 1$$

$$C = 1 + 2\cos\pi = 1 + 2(-1) = -1$$

Therefore, the specific solution is $f(x) = -2\cos x - 1$

Answer to Exercise 4 (on page 10)

1. By the power rule, the antiderivative of x^2 is $\frac{1}{3}x^3$, the antiderivative of $2x$ is x_2, and the antiderivative of 4 is $4x$. So the general antiderivative of $f(x)$ is $\frac{1}{3}x^3 + x^2 - 4x + C$
2. We can rewrite $g(x)$ to more clearly see the powers of x. $g(x) = x^{\frac{2}{3}} + x^{\frac{3}{2}}$. Applying the Power rule, we find the general antiderivative of $g(x)$ is $\frac{3}{5}x^{\frac{5}{3}} + \frac{2}{5}x^{\frac{5}{2}} + C$.
3. Recalling that the antiderivative of $\frac{1}{x}$ is $\ln|x|$, the general antiderivative of $h(x)$ is $\frac{1}{5}x - 2\ln|x| + C$
4. The antiderivative of $\sin\theta$ is $-\cos\theta$ and the antiderivative of $\sec^2\theta$ is $\tan x$. Therefore, the general antiderivative of $r(\theta)$ is $-2\cos\theta - \tan\theta + C$

Answer to Exercise 5 (on page 10)

1. The antiderivative of $\sin\theta$ is $-\cos\theta$ and the antiderivative of $\cos\theta$ is $\sin\theta$. The general form of f is $f(\theta) = -\cos\theta + \sin\theta + C$. Substituting $\theta = \pi$, we find that $f(\pi) = -\cos\pi + \sin\pi + C = 1 + 0 + C = 2$, which implies $C = 1$. Therefore $f(\theta) = -\cos\theta + \sin\theta + 1$.
2. The general antiderivative of f'' is $f'(x) = 4x^3 + 3x^2 - 4x + C_1$. We don't have a condition for f', so we continue to find f. The antiderivative of f' is $f(x) = x^4 + x^3 - 2x^2 + C_1x + C_2$. We can find C_2 with the condition $f(0) = 4$. $f(0) = C_2 = 4$, so we know $f(x) = x^4 + x^3 - 2x^2 + C_1x + 4$. Using the condition $f(1) = 1$, we find that $C_1 = 3$. Therefore, the specific solution is $f(x) = x^4 + x^3 - 2x^2 + 3x - 4$.

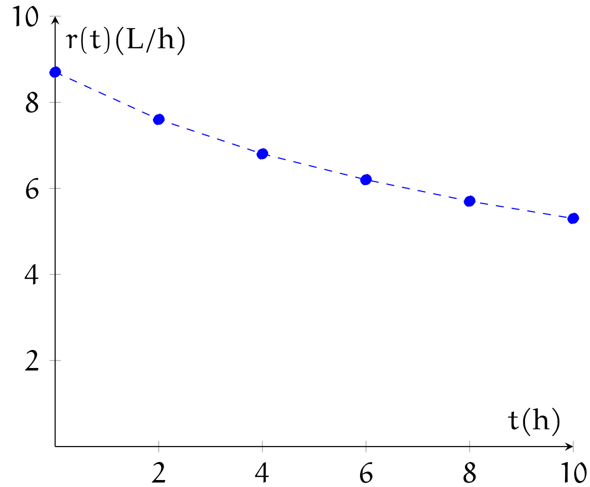
Answer to Exercise 6 (on page 13)

The units on the x-axis are s and the units on the y-axis are $\frac{m}{s^2}$. The area then would have units of $s \times \frac{m}{s^2} = \frac{m}{s}$. Based on the units, the area represents a net change in velocity. The area above and below the axis are equal $(4.5\frac{m}{s})$, therefore the total area is 0. This means the object's starting and ending velocity are the same.

Answer to Exercise 7 (on page 14)

The units of the area will be $days \times \frac{deaths}{day} = deaths$. The area under the curve represents the total number of people who died of SARS in Singapore during the time period represented [from March 1 to May 24 (if you took the time to do the math for the dates)].

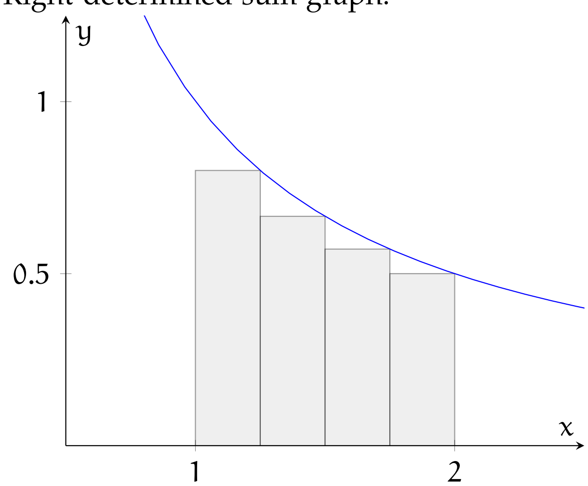
Answer to Exercise 8 (on page 14)



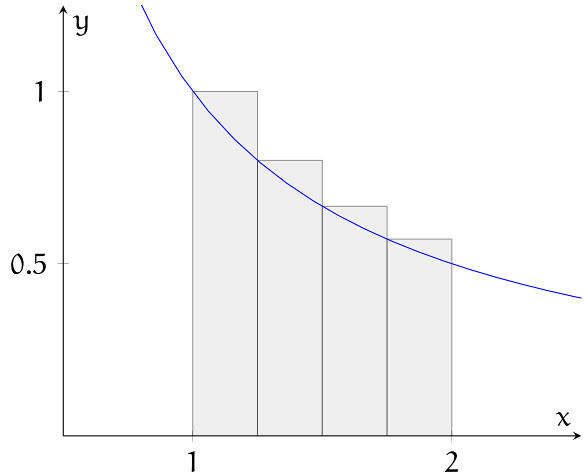
Based on the units, the area under the data would represent the total oil lost. One way to estimate this area would be to create rectangles, but there are other valid methods.

Answer to Exercise 9 (on page 18)

Right-determined sum graph:



The area of the right-determined sum is $0.25 \times (0.8 + 0.6667 + 0.5714 + 0.5) = 0.4202$. This is an underestimate of the actual area.
Left-determined sum graph:



The area of the left-determined sum is $0.25 \times (1 + 0.8 + 0.6667 + 0.5714) = 0.7595$. This is an overestimate of the actual area.

Answer to Exercise 10 (on page 23)

1. $\Delta x = \frac{3-1}{n} = \frac{2}{n}$ and $x_i = 1 + i\frac{2}{n} = 1 + \frac{2i}{n}$. Substituting, we get $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2(1+\frac{2i}{n})}{(1+\frac{2i}{n})^2+1} \cdot \frac{2}{n}$

2. $\Delta x = \frac{7-4}{n} = \frac{3}{n}$ and $x_i = 4 + \frac{3i}{n}$. Substituting, we get $\lim_{n\to\infty} \sum_{i=1}^{n} [(4 + \frac{3i}{n})^2 + \sqrt{1 + 2(4 + \frac{3i}{n})}]\frac{3i}{n}$

3. $\Delta x = \frac{\pi - 0}{n} = \frac{\pi}{n}$ and $x_i = \frac{i\pi}{n}$. Substituting, we get $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin \frac{i\pi}{n}} \frac{\pi}{n}$

Answer to Exercise 11 (on page 24)

1. $\Delta x = \frac{3}{n}$, which implies $b - a = 3$. We could interpret $\sqrt{1 + \frac{3i}{n}}$ two ways: either $f(x) = \sqrt{1+x}$ and $x_i = \frac{3i}{n}$ or $f(x) = \sqrt{x}$ and $x_i = 1 + \frac{3i}{n}$. In the first case, we can find that $a = 0$ and $b = 3$, so the limit of the sum represents the area under $f(x) = \sqrt{1+x}$ from $x = 0$ to $x = 3$. For the second case, we can find that $a = 1$ and $b = 4$, so the limit of the sum represents the area under $f(x) = \sqrt{x}$ from $x = 1$ to $x = 4$.

2. $\Delta x = \frac{\pi}{4n}$ which implies $b - a = \frac{\pi}{4}$. We can see that $x_i = \frac{i\pi}{4n}$, which implies $a = 0$ and therefore also that $b = \frac{\pi}{4}$. Therefore, the limit of the sum represents the total area under $f(x) = \tan x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Answer to Exercise 12 (on page 28)

Number of Intervals	Calculated Area
10	1.7820
50	1.6419
100	1.6256
500	1.6126
1000	1.6110
5000	1.6098

The area approaches the natural log of the endpoint, $\ln 5 \approx 1.6094$.

Answer to Exercise 13 (on page 29)

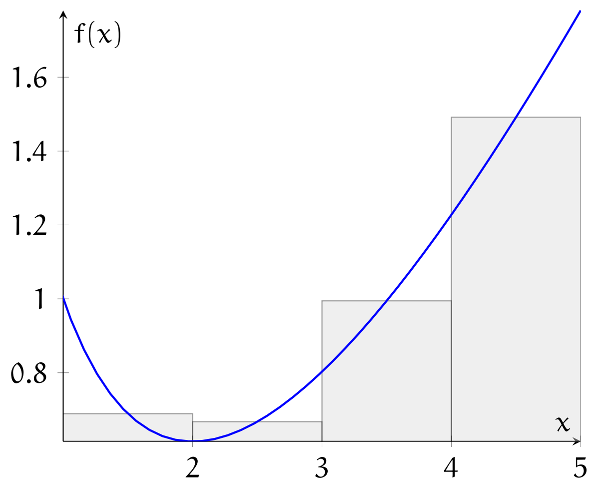
The volume of water will be the amount of water at 4 hours (50 liters) plus the area under the graph of $R(t)$ from $t = 4$ to $t = 15$. We will estimate this area with a right Riemann sum. The approximate volume added from $t = 4$ to $t = 7$ is $(7 - 4) * (6.2) = 18.6$ liters. The approximate volume added from $t = 7$ to $t = 12$ is $(12 - 7) * (5.9) = 29.5$ liters. The approximate volume added from $t = 12$ to $t = 15$ is $(15-12)*(5.6) = 16.8$ liters. Therefore, the approximate total volume of water in the tank at $t = 15$ is $50+18.6+29.5+16.8 = 114.9$ liters.

Answer to Exercise 14 (on page 29)

We will divide the area from $x = 1$ to $x = 5$ into four intervals at $x = 2$, $x = 3$, and $x = 4$. Then we will find the value of $f(x)$ at the midpoint of each interval:

Interval	Midpoint	Value of $f(x)$ at midpoint
1	1.5	≈ 0.68907
2	2.5	≈ 0.66742
3	3.5	≈ 0.99447
4	4.5	≈ 1.49185

Using the values in the table, we can make a possible sketch of $f(x)$:

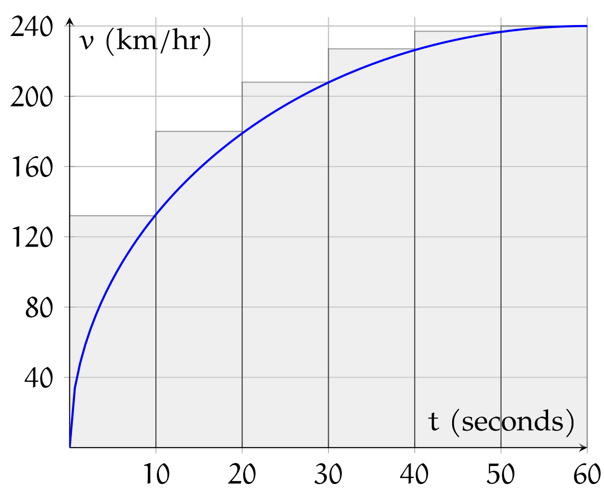


And we calculate the total area in the rectangles:

$$1 \times (0.68907 + 0.66742 + 0.99447 + 1.49185 = 3.84281$$

Answer to Exercise 15 (on page 30)

The question allows the student to choose the type of sum (left, right, or midpoint) and the number of intervals. A possible solution is given, but there are many ways to answer the question.
The tricky part here is noticing the units! In order to have a solution in kilometers, we'll need to convert km/hr to m/s when we calculate the areas. A possible solution is to divide the graph into 6 intervals (one every 10 seconds) and use a right Riemann sum.



We can use the graph to *estimate* the height of each rectangle. Some reasonable estimates are $f(10) = 130\frac{km}{hr} \approx 36.1\frac{ms}{s}$, $f(20) = 180\frac{km}{hr} = 50\frac{m}{s}$, $f(30) = 210\frac{km}{hr} \approx 58.3\frac{m}{s}$, $f(40) = 230\frac{km}{hr} \approx 63.9\frac{m}{s}$, $f(50) = 235\frac{km}{hr} \approx 65.3\frac{m}{s}$, and $f(60) = 240\frac{km}{hr} \approx 66.7\frac{m}{s}$. [Any values within

± 5 of the listed values are reasonable.] Noting that each interval is 10 sec wide and usingß the estimates of $f(x)$ listed, we can estimate that the distance traveled is 10 $sec \times (36.1\frac{m}{s} + 50\frac{m}{s} + 58.3\frac{m}{s} + 63.9\frac{m}{s} + 65.3\frac{m}{s} + 66.7\frac{m}{s}) = 3403$ meters.

Answer to Exercise 16 (on page 32)

Following the structure shown in the formal definition of a definite integral, we can set $f(x) = x^3 + x\sin x$ and re-write the limit of the sum as $\lim_{n\to\infty} \Sigma_{i=1}^{n} f(x)\Delta x = \int_0^\pi f(x)\,dx$. Therefore, the full definite integral would be written as $\int_0^\pi (x^3 + x\sin x)\,dx$.

Answer to Exercise 17 (on page 40)

By property 6, we know that

$$\int_0^1 (5 - 6x^2)\,dx = \int_0^1 5\,dx - \int_0^1 6x^2\,dx$$

By property 5, we know that

$$\int_0^1 5\,dx - \int_0^1 6x^2\,dx = \int_0^1 5\,dx - 6\int_0^1 x^2\,dx$$

By property 3, we know that

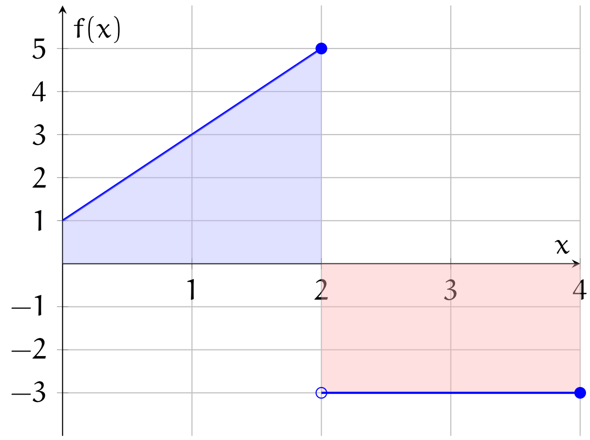
$$\int_0^1 5\,dx = 5(1 - 0) = 5$$

Putting it all together, we see that

$$\int_0^1 (5 - 6x^2)\,dx = 5 - 6(\frac{1}{3}) = 5 - 2 = 3$$

Answer to Exercise 18 (on page 41)

We can break the integral into two parts: from $x = 0$ to $x = 2$ (shaded in blue), and from $x = 2$ to $x = 4$ (shaded in red). The blue portion is a trapezoid and so has a total area of $\frac{1}{2}(b_1 + b_2)(h) = \frac{1}{2}(1 + 5)(2) = 6$. Because it is above the x-axis, the area is positive. The red portion is a rectangle and has a total area of $2 \times 3 = 6$ and is *negative* because it lies below the x-axis. Therefore, the total area is $6 + -6 = 0$.



Answer to Exercise 19 (on page 42)

Using the properties of integrals, we can rewrite $\int_{-1}^{9} 3g(x) + 2\,dx$ as $3\int_{-1}^{9} g(x)\,dx + 2(9 - (-1))$. From the graph, we can determine $\int_{-1}^{9} g(x)\,dx = 2.5$. Therefore, $\int_{-1}^{9} 3g(x) + 2\,dx = 3(2.5) + 2(10) = 27.5$.

Answer to Exercise 21 (on page 48)

(B). If $f'(x) > 0$ for all x, then f must be increasing for $4 < x < 7$. Since (C) decreases from $x = 5$ to $x = 7$, we can eliminate it. We can also eliminate (D), since $f(4) = f(5) = f(7)$, which implies either the slope of f is zero or changes from positive to negative. If $\int_4^7 f(x)\,dx =$, then some portion of $f(x)$ lies above the $x-axis$, while some other portion lies below (we must have positive and negative areas for the sum to be zero). This eliminates (A) and (E), since the integral of (A) would have a negative value and the integral of (E) would have a positive value. This leaves (B).

Answer to Exercise 21 (on page 48)

(D) 2. First, we try to compute the limit directly: $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6} = \frac{\int_1^1 g(t)\,dt}{6-6} = \frac{0}{0}$, which is undefined. Because g is continuous and differentiable, we can apply L'Hospital's rule. $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6} = \lim_{x\to 1} \frac{d}{dx}\left[\frac{\int_1^x g(t)\,dt}{g(x)-6}\right] = \lim_{x\to 1} \frac{g(x)}{g'(x)} = \frac{g(6)}{g'(6)} = \frac{6}{3} = 2.$

Answer to Exercise 22 (on page 49)

1. $g'(x) = \sqrt{x + x^3}$
2. $F(x) = -\int_0^x \sqrt{1 + \sec t}\, dt$ and therefore $F'(x) = -\sqrt{1 + \sec x}$
3. setting $u = e^x$ and noting $\frac{du}{dx} = e^x$, then $h'(x) = \frac{d}{du}\int_1^u \ln t\, dt(\frac{du}{dx})$ Taking the derivative and substituting for $\frac{du}{dx}$, we find $h'(x) = \ln u \cdot e^x = \ln(e^x) \cdot e^x = x \cdot e^x$
4. $y = -\int_{\frac{\pi}{4}}^{\sqrt{x}} \theta \tan\theta\, d\theta$. Setting $u = \sqrt{x}$ and noting that $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, we see that $y' = -\frac{d}{du}[\int_{\frac{\pi}{4}}^{u} \theta \tan\theta\, d\theta]\frac{du}{dx} = u \tan u \cdot \frac{1}{2\sqrt{x}} = -\sqrt{x}\tan\sqrt{x} \cdot \frac{1}{2\sqrt{x}} = \frac{-\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}} = -\frac{1}{2}\tan\sqrt{x}$

Answer to Exercise 23 (on page 51)

1. The antiderivative of $t^{-3/2}$ is $\frac{-2}{\sqrt{t}}$. Therefore, the integral is equal to $[\frac{-2}{\sqrt{t}}]_1^4 = \frac{-2}{\sqrt{4}} - \frac{-2}{\sqrt{1}} = -1 + 2 = 1$.

Answer to Exercise 24 (on page 51)

According to FTC, $h'(x) = f(x)$ and $h''(x) = f'(x)$. Examining the graph, we see that the curve lies below the x-axis for $0 < x < 6$, which means that $h(6) = \int_0^6 f(t)\, dt < 0$. $h'(6) = f(6) = 0$ and $h''(6) = f'(6) > 0$. Therefore, $h(6) < h'(6) < h''(6)$.

Answer to Exercise 25 (on page 52)

We know that $f(2) = \int_{-5}^{2} f'(x)\, dx + f(-5)$. Examining the graph, we know that $\int_{-5}^{2} f'(x)\, dx = frac12(3)(2) - \frac{1}{2}\pi(2^2)$ (the area of the triangle above the x-axis less the area of the semi-circle below the axis). Therefore, $f(-5) = f(2) - \int_{-5}^{2} f'(x)\, dx = 1 - (3 - 2\pi) = 2\pi - 2$

Answer to Exercise 26 (on page 52)

11.71. The particle's velocity will be given by its initial velocity plus the integral of its acceleration over the time period. Therefore, $v(3) = v(0) + \int_0^3 a(t)\, dt = 5 + \int_0^3 \frac{t+3}{\sqrt{t^3+1}}\, dt \approx 5 + 6.71 = 11.71$.

Answer to Exercise 27 (on page 53)

We begin by setting up the integral for average value of a function with $a = 0$, $b = \frac{\pi}{2}$, and $f(x) = \sqrt{\cos x}$:

$$\frac{1}{\pi/2 - 0} \int_0^{\pi/2} \sqrt{\cos x}\, dx$$
$$= \frac{2}{\pi} \int_0^{\pi/2} \sqrt{\cos x}\, dx$$

There's not an obvious way to evaluate this integral by hand. Luckily, this question allows for the use of a calculator. Entering this integral into a calculator (such as a TI-89 or Wolfram Alpha), we find that:

$$\frac{2}{\pi} \int_0^{\pi/2} \sqrt{\cos x}\, dx \approx 0.763$$

Answer to Exercise 28 (on page 60)

1. $L = \int_1^3 \sqrt{1 + \frac{1}{x^2}}\, dx$
2. $L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$
3. $L = \int_1^4 \sqrt{1 + (x^2 - \frac{1}{4x^2})^2}\, dx$
4. $L = \int_0^{\frac{\pi}{3}} \sqrt{1 + (\frac{1}{\cos x} \times - \sin x)^2}\, dx = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\, dx$

Answer to Exercise 29 (on page 60)

From looking at the structure of the given arc length integral, we can see that $f'(t) = \sqrt{3t + 4}$. Taking the antiderivative, we find that $f(x) = \frac{2}{9}(3x + 4)^{3/2} + C$. Substituting $f(0) = 2$, we can solve for C.

$$2 = \frac{2}{9}(3(0) + 4)^{3/2} + C$$
$$2 = \frac{2}{9}(4)^{3/2} + C$$
$$2 = \frac{2}{9}(2)^3 + C$$
$$2 = \frac{16}{9} + C$$

$$\frac{18}{9} = \frac{16+9C}{9}$$
$$18 = 16+9C$$
$$2 = 9C$$
$$C = \frac{2}{9}$$

Therefore, $f(x) = \frac{2}{3}(3x+4)^{3/2} + \frac{2}{9}$. To find the coordinate point where $s(x) = 3$, we first note that the antiderivative of $\sqrt{3t+5}$ is $\frac{2}{9}(3t+5)^{3/2}+C$. Therefore, $s(x) = \frac{2}{9}(3x+5)^{3/2} - \frac{2}{9}(5)^{3/2}$. Setting $s(x) = 3$ and solving for x, we find that $x = 1.159$

Answer to Exercise 30 (on page 61)

Recall that arc length is given by $\int_a^b \sqrt{1+[f'(x)]^2}\,dx$. Since $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. Taking $a = 1$, $b = 2$, and $f'(x) = \frac{1}{x}$, the integral that gives the length of the curve $y = \ln x$ on the specified interval is $\int_1^2 \sqrt{1+\frac{1}{x^2}}\,dx$.

Answer to Exercise 31 (on page 61)

Since we are told the vector-valued velocity of the ship, we know that $\frac{dx}{dt} = -1412 \sin t$, $\frac{dy}{dt} = 1412 \cos t$, and $\frac{dz}{dt} = t$. The distance traveled in the first 60 seconds is given by $\int_0^{60} \sqrt{(-1412 \sin t)^2 + (1412 \cos t)^2 + t^2}\,dt$. Using a calculator, the integral evaluates to 84745 meters. The distance traveled in the second 60 seconds is given by $\int_{60}^{120} \sqrt{(-1412 \sin t)^2 + (1412 \cos t)^2 + t^2}\,dt$. Using a calculator, this integral evaluates to 84898 meters.

Answer to Exercise 32 (on page 72)

```
from scipy.stats import norm

# Constants
MEAN = 164.7
STD = 7.1

# What is the cutoff for the top decile?
cutoff = norm.ppf(0.9, loc=MEAN, scale=STD);
print(f"To be in the top 10 percent, you must be at least {cutoff:.2f} cm")

# What proportion of women are between 160cm and 165cm?
shorter_than_160 = norm.cdf(160, loc=MEAN, scale=STD)
```

```
shorter_than_165 = norm.cdf(165, loc=MEAN, scale=STD)
between = shorter_than_165 - shorter_than_160
print(f"{between * 100.0:.2f}% of adult women are between 160 and 165 cm.")
```

When run, this will give you:

```
> python3 women.py
To be in the top 10 percent, you must be at least 173.80 cm
26.29% of adult women are between 160 and 165 cm.
```

Answer to Exercise 34 (on page 79)

First, we convert the temperatures into Kelvin:

- Dawn: $12 + 273.15 = 285.15$
- Noon: $28 + 273.15 = 301.15$

So, the temperature T has increased by a factor of $\frac{301.15}{285.15} \approx 1.056$

Thus the volume of the air mattress has also increased by a factor of 1.056.

So the air mattress that had a volume of 1000 liters at dawn, will have a volume 1056 liters at noon.

Answer to Exercise 34 (on page 79)

What is the pressure in kPA?

- Before squeezing: 100 kPa
- While squeezing: 120 kPa

So, the pressure P has increased by a factor of $\frac{120}{100} = 1.2$

$1/1.2 \approx 0.833$

The air in the bottle had a volume of 1 liter before squeezing, so it has a volume of 833 milliliters while being squeezed.

Answer to Exercise 35 (on page 80)

First, let's convert the known values to the right unit:

- Radius $= 0.12$ m
- Length $= 0.5$ m
- $T = 20 + 273.15 = 293.15$ degrees Kelvin
- $P = 600$ kPa $= 600,000$ Pa

The volume of the cylindrical chamber is $V = \pi r^2 h = \pi (0.12)^2 0 (0.5) \approx 0.0226$.

The Ideal Gas Law tell us that $PV = nRT$. We are solving for n.

$$n = \frac{PV}{RT} = \frac{(600,000)(0.0226)}{(8.31446)(293.15)} \approx 5.68 \text{ moles of } O_2$$

Answer to Exercise 36 (on page 84)

$E = C_{V,m}(3 \text{ moles })(20 degrees Celsius) = (12.47)(3)(20) = 748$ Joules

$E = C_{V,m}(3 \text{ moles })(20 degrees Celsius) = (20.8)(3)(20) = 1247$ Joules

Answer to Exercise 37 (on page 86)

10 degrees Celsius is 283.15 degrees Kelvin. 30 degrees Celsius is 303.15.

For any gas:

$$E_K = C_{V,m} n T$$

And $C_{V,m} = 12.47$ for all monoatomic gases.

So the energy at 10 degrees Celsius:

$$E_1 = (12.47)(3)(283.15) = 10,594 \text{ Joules}$$

The energy at 30 degrees Celsius:

$$E_2 = (12.47)(3)(303.15) = 11,342 \text{ Joules}$$

The difference?

$$E_2 - E_1 = 11,342 - 10,594 = 748 \text{ Joules}$$

Which is consistent with your earlier exercise.

Answer to Exercise 38 (on page 100)

When one mole of water goes from 33° to 17°, it will give off $(75.38)(33 - 17) = 1,206$ Joules or 1.206 kJ.

Tom needs 300,000 kJ, so he needs $300,000/1.206 = 248,739.72$ moles of water.

How many liters is that? $248,739.72 * 0.018 = 4,477.31$ liters.

How many barrels is that? $4,477.31/159 = 28.16$ barrels. He will need 29 barrels.

Answer to Exercise 39 (on page 101)

When one mole of mirabilite goes from 33° to 17°, it will give off $(550)(33-17)+82,000 = 90,800$ Joules or 90.8 kJ.

Tom needs 300,000 kJ, so he needs $300,000/90.8 = 3,304$ moles of mirabilite.

How many liters is that? $3,304 * 0.22 = 726.9$ liters.

How many barrels is that? $726.9/159 = 4.57$. He will need 5 barrels.

Answer to Exercise 40 (on page 112)

The pre-compression temperature is $80irc\ C\ + 273.15 = 353.15°$ K.

The radius of the cylinder is $9.535/2 = 4.7625$ cm.

The area of a cross section of the cylinder is $\pi r^2 = \pi(4.7625)^2 \approx 71.26$ ml.

So the change in volume between the minimum and maximum volume is $(71.26)(10.16) =$ 724 ml, or 0.724 liters.

(With four cylinders, the total displacement of a Model T is thus $(4)(724) = 2,896$ cc.)

Now we use the ideal gas to figure out how many moles of gas will fit into 0.724 liters at 100 kPa and 353.15° K.

$$n = \frac{PV}{rT} = \frac{(100)(0.724)}{(8.314)(353.15)} = 0.02466 \text{ moles of air+fuel}$$

So, if we suck n_a moles of air and n_e moles of vaporized ethanol in to the cylinder:

$$n_a + n_e = 0.02466$$

So

$$n_a = 0.02466 - n_e$$

21% of n_a is O_2:

$$n_{O_2} = 0.21n_a = 0.21(0.02466 - n_e) = 0.005178 - 0.21n_e$$

For a clean burn, we need 3 times as many O_2 molecules as ethanol molecules. Thus:

$$3n_e = n_{O_2} = 0.005178 - 0.21n_e$$

Solving for n_e:

$$n_e = \frac{0.005178}{3.21} = 0.001613 \text{ moles of ethanol}$$

For every molecule of ethanol that burns we get 2 molecules of CO_2: 0.003226 moles.

For every molecule of ethanl that burns we get 3 molecules of H_2OI: 0.004839 moles.

How much heat? $(0.001613)(1330) = 2.145$ kilojoules from each burn.

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