Contents

Chapter 1

Vectors

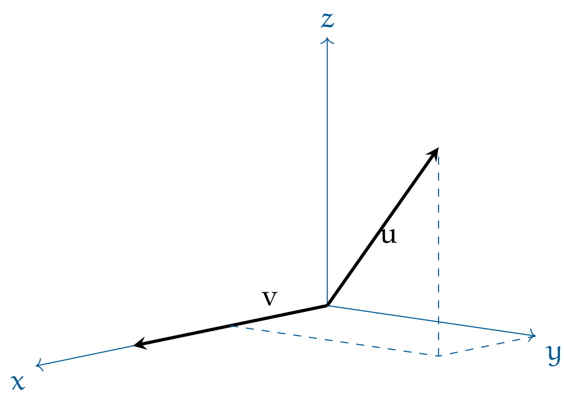
We have talked a some about forces, but in the calculations that we have done, we have only talked about the magnitude of a force. It is equally important to talk about its direction. To do the math on things with a magnitude and a direction (like forces), we need vectors.

For example, if you jump out of a plane (hopefully with a parachute), several forces with different magnitudes and directions will be acting upon you. Gravity will push you straight down. That force will be proportional to your weight. If there were a wind from the west, it would push you toward the east. That force will be proportional to the square of the speed of the wind and approximately proportional to your size. Once you are falling, there will be resistance from the air that you are pushing through – that force will point in the opposite direction from the direction you are moving and will be proportional to the square of your speed.

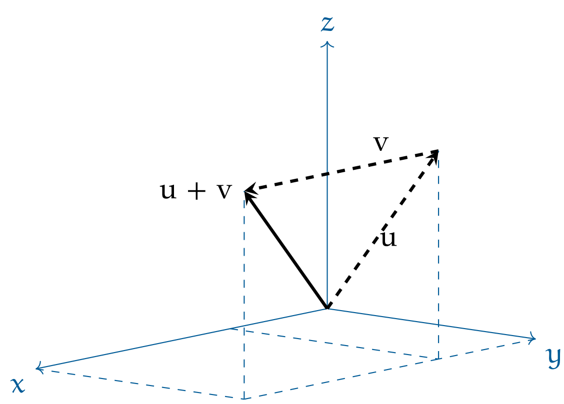
To figure out the net force (which will tell us how we will accelerate), we will need to add these forces together. So we need to learn to do math with vectors.

1.1 Adding Vectors

A vector is typically represented as a list of numbers, with each number representing a particular dimension. For example, if I am creating a 3-dimensional vector representing a force, it will have three numbers representing the amount of force in each of the three axes. For example, if a force of one newton is in the direction of the x-axis, I might represent the vector as $v = [1, 0, 0]$. Another vector might be $u = [0.5, 0.9, 0.7]$



Thinking visually, when we add to vectors, we put the starting point second vector at the ending point of the first vector.



If you know the vectors, you will just add them element-wise:

$$u + v = [0.5, 0.9, 0.7] + [1.0, 0.0, 0.0] = [1.5, 0.9.0.7]$$

These vectors have 3 components, so we say they are *3-dimensional*. Vectors can have any number of components. For example, the vector $[-12.2, 3, \pi, 10000]$ is 4-dimensional.

You can only add two vectors if they have the same dimension.

$$[12, -4] + [-1, 5] = [11, 1]$$

Addition is commutative: If you have two vectors a and b, then $a+b$ is the same as $b+a$.

Addition is also associative: If you have three vectors a, b, and c, it doesn't matter which order you add them in. That is, $a + (b + c) = (a + b) + c$.

A 1-dimensional vector is just a number. We say it is a *scalar*, not a vector.

Exercise 1 Adding vectors

Add the following vectors:

- $[1, 2, 3] + [4, 5, 6]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi]$

Working Space

Answer on Page 29

Exercise 2 Adding Forces

You are adrift in space. You are near two different stars. The gravity of one star is pulling you towards it with a force of $[4.2, 5.6, 9.0]$ newtons. The gravity of the other star is pulling you towards it with a force of $[-100.2, 30.2, -9.0]$ newtons. What is the net force?

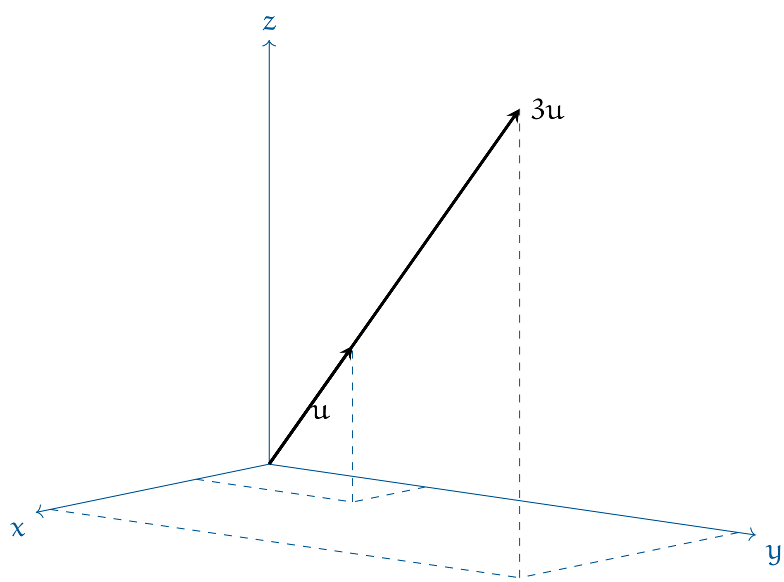
Working Space

Answer on Page 29

1.2 Multiplying a vector with a scalar

It is not uncommon to multiply a vector by a scalar. For example, a rocket engine might have a force vector v. If you fire 9 engines in the exact same direction, the resulting force vector would be $9v$.

Visually, when we multiply a vector u by a scalar a, we get a new vector that goes in the same direction as u but has a magnitude a times as long as u.



When you multiply a vector by a scalar, you just multiply each of the components by the scalar:

$$3 \times [0.5, 0.9, 0.7] = [1.5, 2.7, 3.6]$$

Exercise 3 Multiplying a vector and a scalar

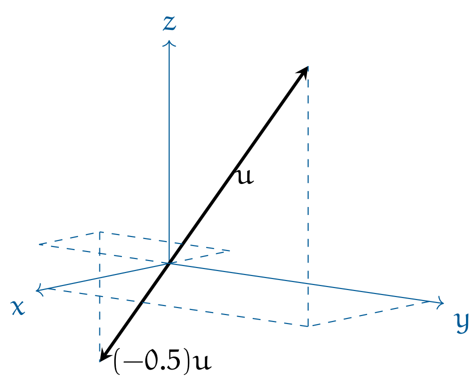
Simplify the following expressions:

- $2 \times [1, 2, 3]$
- $[-1, -2, -3, -4] \times -2$
- $\pi[\pi, 2\pi, 3\pi]$

Working Space

Answer on Page 30

Note that when you multiply a vector times a negative number, the new vector points in the opposite direction.



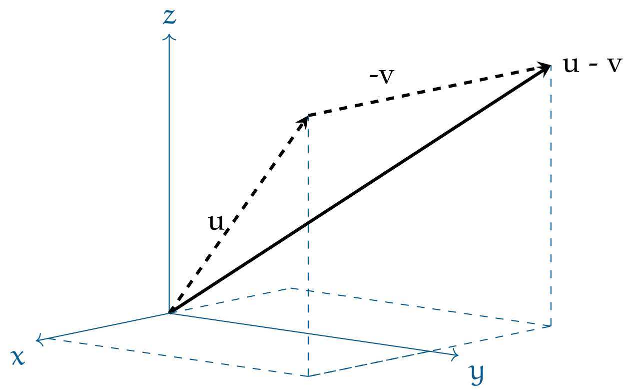
1.3 Vector Subtraction

As you might guess, when you subtract one vector from another, you just do element-wise subtraction:

$$[4, 2, 0] - [3, -2, 9] = [1, 4, -9]$$

So, $u - v = u + (-1v)$.

So visually, you reverse the one that is being subtracted:



1.4 Magnitude of a Vector

The *magnitude* of a vector is just its length. We write the magnitude of a vector v as $|v|$.

We compute the magnitude using the pythagorean theorem. If $v = [3, 4, 5]$, then

$$|v| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$$

(You might notice that the notation for the magnitude is exactly like the notation for absolute value. If you think of a scalar as a 1-dimensional vector, the absolute value and the magnitude are the same. For example, the absolute value of -5 is 5. If you take the magnitude of the one-dimenional vector $[-5]$, you get $\sqrt{25} = 5$.)

Notice that if you scale up a vector, its magnitude scales by the same amount. For example:

$$|7[3, 4, 5]| = 7\sqrt{50} \approx 7 \times 7.07$$

The rule then is: If you have any vector v and any scalar a:

$$|av| = |a||v|$$

Exercise 4 Magnitude of a Vector

Find the magnitude of the following vectors:

- $[1, 1, 1]$
- $[-5, -5, -5]$ (that is the same as $-5 \times [1, 1, 1]$)
- $[3, 4, -4] + [-2, -3, 5]$

Working Space

Answer on Page 30

1.5 Vectors in Python

NumPy is a library that allows you to work with vectors in Python. You might need to install it on your computer. This is done with `pip`. `pip3` installs things specifically for Python 3.

```
pip3 install NumPy
```

We can think of a vector as a list of numbers. There are also grids of numbers known as *matrices*. NumPy deals with both in the same way, so it refers to both of them as arrays.

The study of vectors and matrices is known as *Linear Algebra*. Some of the functions we need are in a sublibrary of NumPy called `linalg`.

As a convention, everyone who uses NumPy, imports it as *np*.

Create a file called `first_vectors.py`:

```
import NumPy as np

# Create two vectors
v = np.array([2,3,4])
u = np.array([-1,-2,3])
print(f"u = {u}, v = {v}")
```

```
# Add them
w = v + u
print(f"u + v = {w}")

# Multiply by a scalar
w = v * 3
print(f"v * 3 = {w}")

# Get the magnitude
# Get the magnitude
mv = np.linalg.norm(v)
mu = np.linalg.norm(u)
print(f"|v| = {mv}, |u| = {mu}")
```

When you run it, you should see:

```
> python3 first_vectors.py
u = [-1 -2  3], v = [2 3 4]
u + v = [1 1 7]
v * 3 = [ 6  9 12]
|v| = 5.385164807134504, |u| = 3.7416573867739413
```

1.5.1 Formatting Floats

The numbers 5.385164807134504 and 3.7416573867739413 are pretty long. You probably want it rounded off after a couple of decimal places.

Numbers with decimal places are called *floats*. In the placeholder for your float, you can specify how you want it formatted, including the number of decimal places.

Change the last line to look like this:

```
    print(f"|v| = {mv:.2f}, |u| = {mu:.2f}")
```

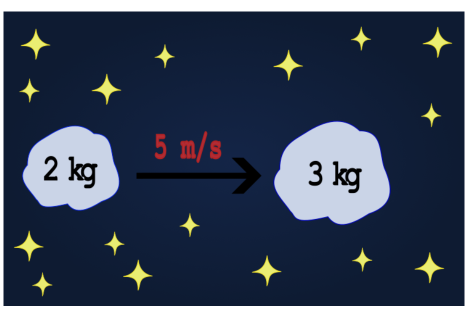
When you run the code, it will be neatly rounded off to two decimal places:

```
|v| = 5.39, |u| = 3.74
```

Chapter 2

Momentum

Let's say a 2 kg block of putty is flying through space at 5 meters per second, and it collides with a larger 3 kg block of putty that is not moving at all. When the two blocks deform and stick to each other, how fast will the resulting big block be moving?



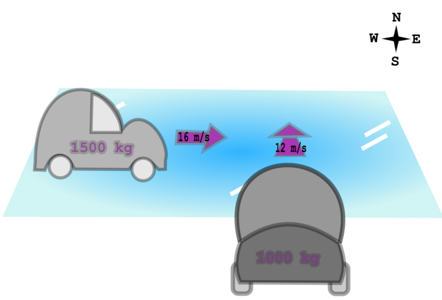
Every object has *momentum*. The momentum is a vector quantity: It points in the direction that the object is moving and has a magnitude equal to its mass times its speed.

Given a set of objects that are interacting, we can sum all their momentum vectors to get the total momentum. In such a set, the total momentum will stay constant.

So, in our example, one object has a momentum vector of magnitude of 10 kg m/s, the other has a momentum of magnitude 0. Once they have merged, they have a combined mass of 5 kg. Thus, the velocity vector must have magnitude 2 m/s and pointing in the same direction that the first mass was moving.

Exercise 5 Cars on Ice

A car weighing 1000 kg is going north at 12 m/s. Another car weighing 1500 kg is going east at 16 m/s. They both hit a patch of ice (with zero friction) and collide. Steel is bent and the two objects become one. How what is the velocity vector (direction and magnitude) of the new object sliding across the ice?



Working Space

Answer on Page 30

Notice that kinetic energy ($1/2mv^2$) is *not* conserved here. Before the collision, the moving putty block has $(1/2)(2)(5^2) = 25$ joules of kinetic energy. Afterward, the big block has $(1/2)(5)(2^2) = 10$ joules of kinetic energy. What happened to the energy that was lost? It was used up deforming the putty.

What if the blocks were marble instead of putty? Then there would be very little deforming, so kinetic energy *and* momentum would be conserved. The two blocks would end up having different velocity vectors.

Let's assume for a moment that they strike each other straight on, so there is motion in only one direction, both before and after the collision. Can we solve for the speeds of the first block (v_1) and the second block (v_2)?

We end up with two equations. Conservation of momentum says:

$$2v_1 + 3v_2 = 10$$

Conservation of kinetic energy says:

$$(1/2)(2)(v_1^2) + (1/2)(3)(v_2^2) = 25$$

Using the first equation, we can solve for v_1 in terms of v_2:

$$v_1 = \frac{10 - 3v_2}{2}$$

Substituting this into the second equation, we get:

$$\left(\frac{10 - 3v_2}{2}\right)^2 + \frac{3v_2^2}{2} = 25$$

Simplifying, we get:

$$v_2^2 - 4v_2 + 0 = 0$$

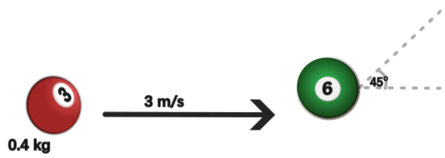
This quadratic has two solutions: $v_2 = 0$ and $v_2 = 4$. $v_2 = 0$ represents the situation before the collision. Substituting in $v_2 = 4$:

$$v_1 = \frac{10 - 3(4)}{2} = -1$$

Thus, if the blocks are hard enough that kinetic energy is conserved, after the collision, the smaller block will be heading in the opposite direction at 1 m/s. The larger block will be moving at 4 m/s in the direction of the original motion.

Exercise 6 Billiard Balls

A billiard ball weighing 0.4 kg and traveling at 3 m/s hits a billiard ball (same weight) at rest. It strikes obliquely so that the ball at rest starts to move at a 45 degree angle from the path of the ball that hit it.
Assuming all kinetic energy is conserved. How what is the velocity vector of each ball after the collision?



Working Space

Answer on Page 30

Chapter 3

The Dot Product

If you have two vectors $u = [u_1, u_2, \ldots, w_n]$ and $v = [v_1, v_2, \ldots, v_n]$, we define the *dot product* $u \cdot v$ as

$$u \cdot v = (u_1 \times v_1) + (u_2 \times v_2) + \cdots + (u_n \times v_n)$$

So, for example,

$$[2, 4, -3] \cdot [5, -1, 1] = 2 \times 5 + 4 \times -1 + -3 \times 1 = 3$$

This may not seem like a very powerful idea, but dot products are *incredibly* useful. The enormous GPUs(Graphics Processing Unit) that let video games render scenes so quickly? They primarily function by computing huge numbers of dot products at mind-boggling speeds.

Exercise 7 Basic dot products

Compute the dot product of each pair of vectors:

- $[1, 2, 3]$, $[4, 5, -6]$
- $[\pi, 2\pi]$, $[2, -1]$
- $[0, 0, 0, 0]$, $[10, 10, 10, 10]$

Working Space

Answer on Page 32

3.1 Properties of the dot product

Sometimes we need an easy way to say "The vector of appropriate length is filled with zeros." We use the notation $\vec{0}$ to represent this. Then, for any vector v, this is true:

$$v \cdot \vec{0} = 0$$

The dot product is commutative:

$$v \cdot u = u \cdot v$$

The dot product of a vector with itself is its magnitude squared:

$$v \cdot v = |v|^2$$

If you have a scalar a then:

$$(v) \cdot (au) = a(v \cdot u)$$

So, if v and w are vectors that go in the same direction,

$$v \cdot w = |v||w|$$

If v and w are vectors that go in opposite directions,

$$v \cdot w = -|v||w|$$

3.2 Cosines and dot products

Furthermore, dot products' interaction with cosine makes them even more useful is what makes them so useful: If you have two vectors v and u,

$$v \cdot u = |v||u| \cos \theta$$

where θ is the angle between them.

So, for example, if two vectors v and u are perpendicular, the angle between them is $\pi/2$. The cosine of $\pi/2$ is 0: The dot product of any two perpendicular vectors is always 0. In fact, if the dot product of two non-zero vectors is 0, the vectors *must be* perpendicular.

Exercise 8 Using dot products

What is the angle between these each pair of vectors:

- $[1, 0]$, $[0, 1]$
- $[3, 4]$, $[4, 3]$

Working Space

Answer on Page 32

If you have two non-zero vectors v and u, you can always compute the angle between them:

$$\theta = \arccos\left(\frac{v \cdot u}{|v||u|}\right)$$

3.3 Dot products in Python

NumPy will let you do dot products using the the symbol @. Open `first_vectors.py` and add the following to the end of the script:

```
# Take the dot product
d = v @ u
print("v @ u =", d)

# Get the angle between the vectors
a = np.arccos(d / (mv * mu))
print(f"The angle between u and v is {a * 180 / np.pi:.2f} degrees")
```

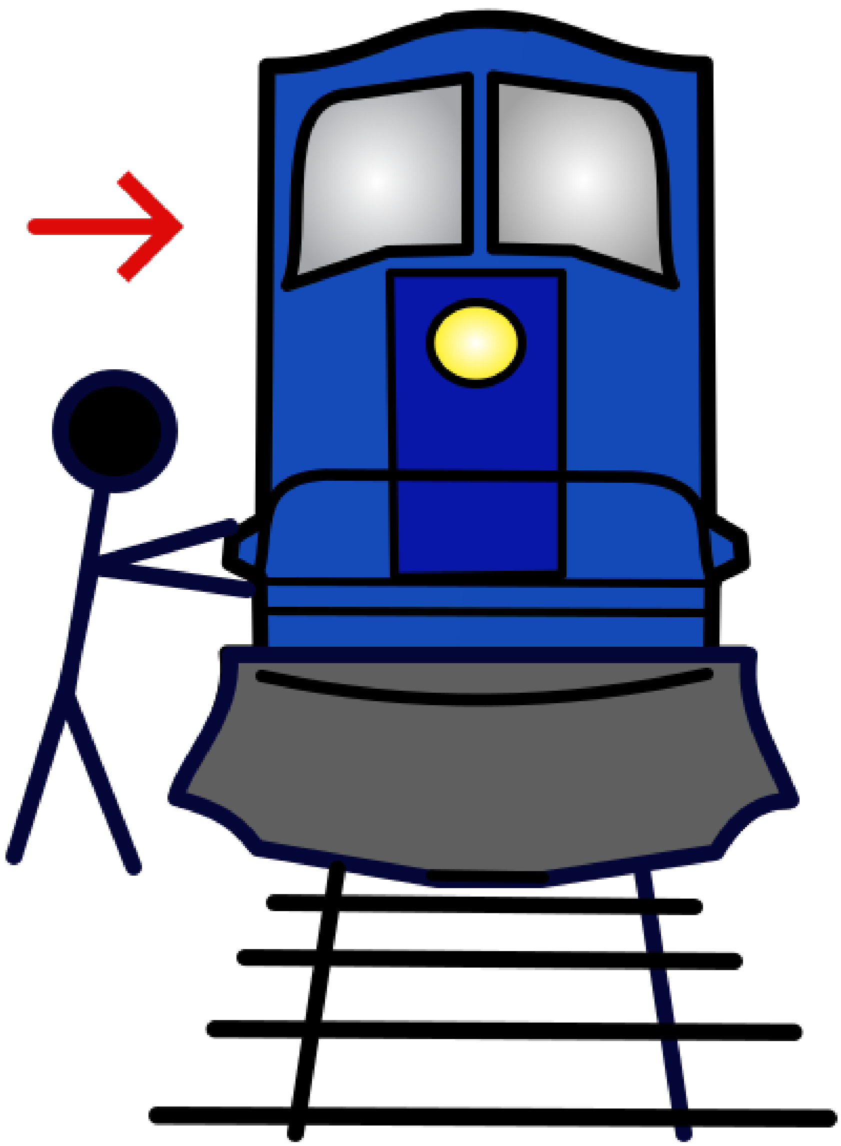
When you run it you should get:

```
v @ u = 4
The angle between u and v is 78.55 degrees
```

3.4 Work and Power

Earlier, we mentioned that mechanical work is the product of the force you apply to something and the amount it moves. For example, if you push a train with a force of 10 newtons as it moves 5 meters, you have done 50 joules of work.

What if you try to push the train sideways? That is, it moves down the track 5 meters, but you push it as if you were trying to derail it – perpendicular to its motion. You have done no work because the train didn't move at all in the direction you were pushing.



Now that you know about dot products: The work you do is the dot product of the force vector you apply and the displacement vector of the train. (The displacement vector is the vector that tells how the train moved while you pushed it.)

Similarly, we mentioned that power is the product of the force you apply and the velocity of the mass you are applying it to. It is actually the dot product of the force vector and the velocity vector.

For example, if you are pushing a sled with a force of 10 newtons and it is moving 2 meters per second, but your push is 20 degrees off, you aren't transferring 20 watts of power to the sled. You are transferring $10 \times 2 \times \cos(20 \text{ degrees}) \approx 18.8$ watts of power.

Chapter 4

Functions and Their Graphs

You can think of a function as a machine: you put something into the machine, it processes it, and out comes something else, a product. Just as we often use the variable x to stand in for a number, we often use the variable f to stand in for a function.

For example, we might ask, "Let the function f be defined like this:

$$f(x) = -5x^2 + 12x + 2$$

What is the value of $f(3)$?"

You would run the number 3 through "the machine": $-5(3^2)+12(3)+2 = -7$. The answer would be "$f(3)$ is 7".

However, Some functions are not defined for every possible input. For example:

$$f(x) = \frac{1}{x}$$

This is defined for any x except 0, because you can't divide 1 by 0. The set of values that a function can process is called its *domain*.

Exercise 9 Domain of a function

Let the function f be given by $f(x) = \sqrt{x-3}$. What is its domain?

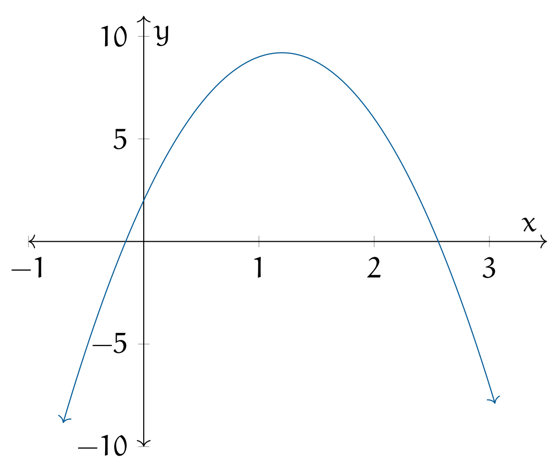
Working Space

Answer on Page 32

4.1 Graphs of Functions

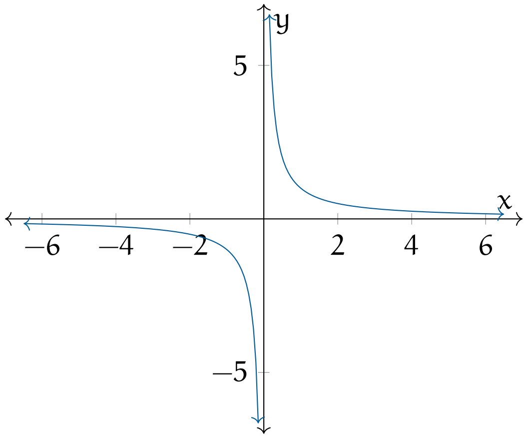
If you have a function, f, its graph is the set of pairs (x, y) such that $y = f(x)$. We usually draw a picture of this set, called a *graph*. The graph not only includes the picture, but also the values of x and y used to create it.

Here is the graph of the function $f(x) = -5x^2 + 12x + 2$:



(Note this is just part of the graph: it goes infinitely in both directions, remember your vectors.)

Here is the graph of the function $f(x) = \frac{1}{x}$:



Exercise 10 Draw a graph

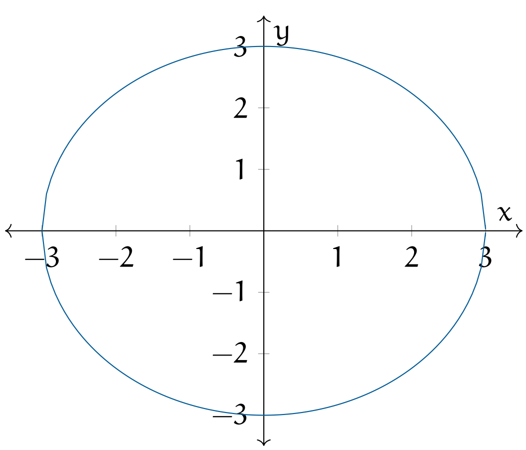
Let the function f be given by $f(x) = -3x + 3$. Sketch its graph.

Working Space

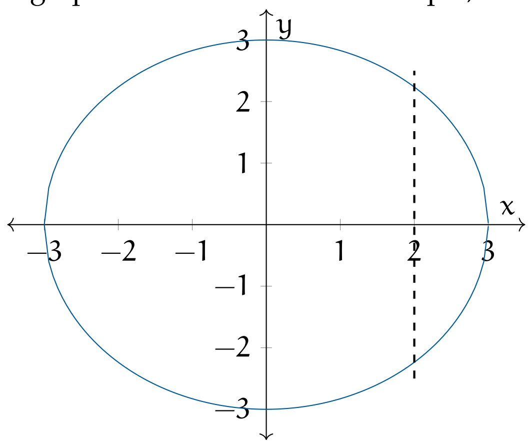
Answer on Page 32

4.2 Can this be expressed as a function?

Note that not all sets can be expressed as graphs of functions. For example, here is the set of points (x, y) such that $x^2 + y^2 = 9$:



This cannot be the graph of a function because what would $f(0)$ be? 3 or -3? This set fails what we call “the vertical line test”: If any vertical line contains more than one point from the set, it isn’t the graph of a function. For example, the vertical line $x = 2$ would cross the graph twice:

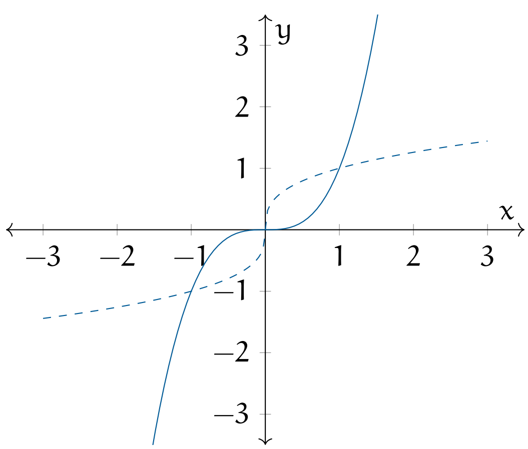


4.3 Inverses

Some functions have inverse functions. If a function f is a machine that turns number x into y, the inverse (usually denoted f^{-1}) is the machine that turns y back into x.

For example, let $f(x) = 5x + 1$. Its inverse is $f^{-1}(x) = (x - 1)/5$. (Spot check it: $f(3) = 16$ and $f^{-1}(16) = 3$)

Does the function $f(x) = x^3$ have an inverse? Yes, $f^{-1}(x) = \sqrt[3]{x}$. Let’s plot the function (solid line) and its inverse (dashed):

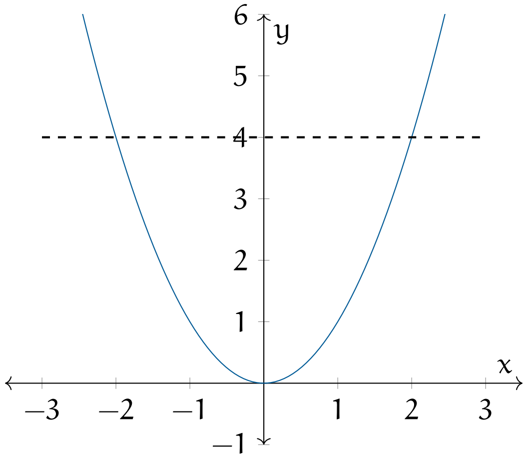


The inverse is the same as the function, just with its axes swapped. This tells us how to solve for an inverse: We swap x and y and solve for y.

For example, if you are given the function $f(x) = 5x + 1$, its graph is all (x, y) such that $y = 5x + 1$. The graph of its inverse is all (x, y) such that $x = 5y + 1$. So you solve for y:

$y = (x - 1)/5$.

Not every function has an inverse. For example, $f(x) = x^2$. Note that $f(2) = f(-2) = 4$. What would $f^{-1}(4)$ be? 2 or -2? This implies the "horizontal line test": If any horizontal line contains more than one point of a function's graph, that function has no inverse.



In some problems, you need an inverse and you don't need the whole domain, so you trim the domain to a set you can define an inverse on. This allow you to make claims such as "If we restrict the domain to the nonnegative numbers, the function $f(x) = x^2 - 5$ has an inverse: $f^{-1}(x) = \sqrt{x + 5}$.

This begs the question: What is the domain of the inverse function f^{-1}?

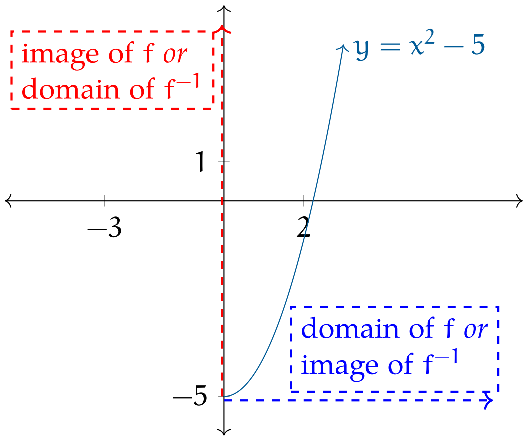
If we let X be the domain of f, we can run every member of X through "the machine" and gather them in a set on the other side. This set would be the *image* of the f "machine". (This is the *range* of f.)

What is the image of $f(x) = x^2 - 5$? It is the set of all real numbers greater than or equal to -5. We write this

$$\{x \in \mathbb{R} | x \geq -5\}$$

Now we can say: **The image of the function is the domain of the inverse function.**

In our example, we can use any number greater than or equal to -5 as input into the inverse function.



Exercise 11 Find the inverse

Let $f(x) = (x-3)^2 + 2$. Sketch the graph. Using all the real numbers as a domain, does this function have an inverse?
How would you restrict the domain to make the function invertible?
What is the inverse of that restricted function?
What is the domain of the inverse?

Working Space

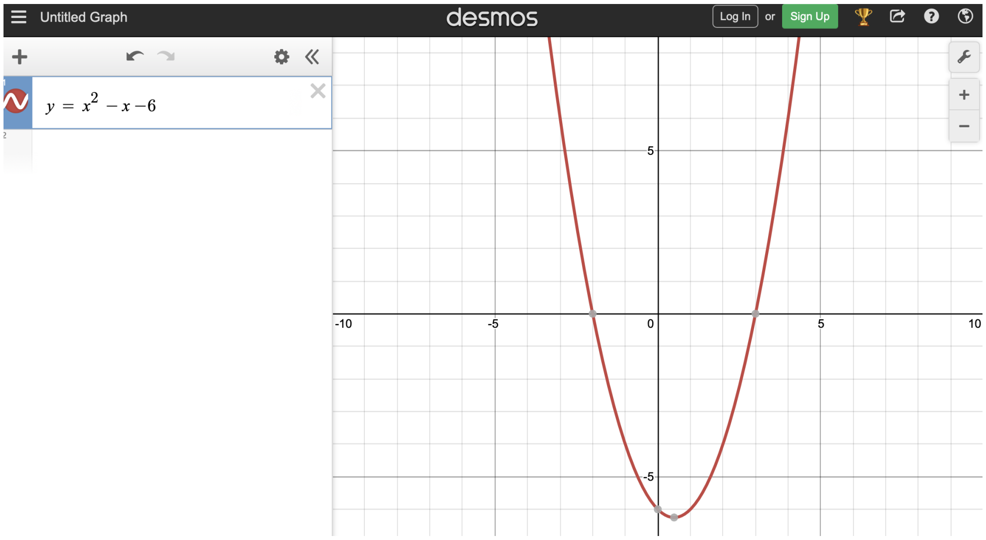
Answer on Page 33

4.4 Graphing Calculators

One really easy way to understand your function better is to use a graphing calculator. Desmos is a great, free online graphing calculator.

In a web browser, go to Desmos: `https://www.desmos.com/calculator`

In the field on the left, enter the function $y = x^2 - x - 6$. (For the exponent, just prefix it with a caret symbol: "x2".)


Untitled Graph
desmos
Log In
or
Sign Up
$y = x^2 - x - 6$
5
-10
-5
0
5
10
-5

APPENDIX A

Answers to Exercises

Answer to Exercise 1 (on page 5)

- $[1, 2, 3] + [4, 5, 6] = [5, 7, 9]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7] = [3, 3, 3, 3]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi] = [\pi, \pi, \pi]$

Answer to Exercise 2 (on page 5)

To get the net force, you add the two forces:

$$F = [4.2, 5.6, 9.0] + [-100.2, 30.2, -9.0] = [-96, 35.8, 0.0] \text{ newtons}$$

Answer to Exercise 3 (on page 7)

- $2 \times [1, 2, 3] = [2, 4, 6]$
- $[-1, -2, -3, -4] \times -3 = [3, 6, 9, 12]$
- $\pi[\pi, 2\pi, 3\pi] = \pi^2, 2\pi^2, 3\pi^2]$

Answer to Exercise 4 (on page 9)

- $|[1, 1, 1]| = \sqrt{3} \approx 1.73$
- $|[-5, -5, -5]| = | - 5 \times [1, 1, 1]| = 5\sqrt{3} \approx 8.66$
- $|[3, 4, 5] + [-2, -3, -4]| = |[1, 1, 1]| = \sqrt{3} \approx 1.73$

Answer to Exercise 5 (on page 12)

The momentum of the first car is 12,000 kg m/s in the north direction.

The momentum of the second car is 24,000 kg m/s in the east direction.

The new object will be moving northeast. What angle is the angle compared with the east?

$$\theta = \arctan \frac{12,000}{24,000} \approx 0.4636 \text{ radians } \approx 26.565 \text{ degrees north of east}$$

The magnitude of the momentum of the new object is $\sqrt{12,000^2 + 24,000^2} \approx 26,833$ kg m/s

Its new mass is 2,5000 kg. So the speed will be $26,833/2,500 = 10.73$ m/s.

Answer to Exercise 6 (on page 14)

The original forward momentum was 1.2 kg m/s. The original kinetic energy is $(1/2)(0.4)(3^2)$ $= 1.8$ joules.

Let s be the post-collision speed of the ball that had been at rest. Let x and y be the forward and sideways speeds (post-collision) of the other ball. Conservation of kinetic energy says

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(x^2 + y^2) = 1.8$$

Forward momentum is conserved:

$$0.4\frac{s}{\sqrt{2}} + 0.4x = 1.2$$

Which can be rewritten:

$$x = 3 - \frac{s}{\sqrt{2}}$$

Sideways momentum stays zero:

$$(0.4)\frac{s}{\sqrt{2}} - 0.4y = 0.0$$

Which can be rewritten:

$$y = \frac{s}{\sqrt{2}}$$

Substituting into to the conservation of kinetic energy equation above:

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(\left(3 - \frac{s}{\sqrt{2}}\right)^2 + \left(\frac{s}{\sqrt{2}}\right)^2 = 1.8$$

Which can be rewritten:

$$s^2 - \frac{3}{\sqrt{2}}s + 0 = 0$$

There are two solutions to this quadratic: $s = 0$ (before collision) and $s = \frac{3}{\sqrt{2}}$. Thus,

$$y = \frac{3}{2}$$

and

$$x = 3 - \frac{3}{2} = \frac{3}{2}$$

So both balls careen off at 45° angles at the exact same speed.

Answer to Exercise 7 (on page 16)

- $[1, 2, 3] \cdot [4, 5, -6] = 4 + 10 - 18 = -4$
- $[\pi, 2\pi] \cdot [2, -1] = 2\pi - 2\pi = 0$
- $[0, 0, 0, 0] \cdot [10, 10, 10, 10] = 0 + 0 + 0 + 0 = 0$

Answer to Exercise 8 (on page 17)

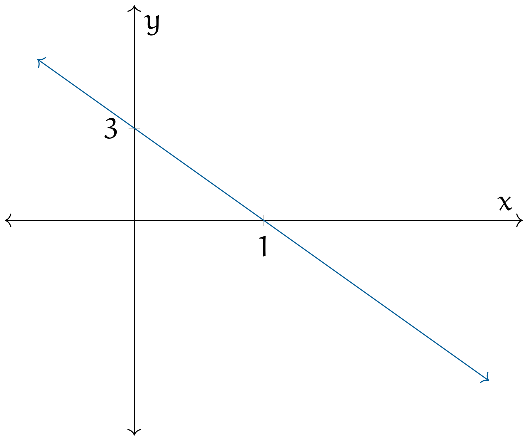
- $[1, 0] \cdot [0, 1] = 0$. The angle must be $\pi/2$.
- $[3, 4] \cdot [4, 3] = 24$. $|[3, 4]||[4, 3]| \cos(\theta) = 24$. $\cos(\theta) = \frac{24}{(5)(5)}$. $\theta = \arccos(\frac{24}{25}) \approx 0.284$ radians.

Answer to Exercise 9 (on page 22)

You can only take the square root of nonnegative numbers, so the function is only defined when $x - 3 \geq 0$. Thus the domain is all real numbers greater than or equal to 3.

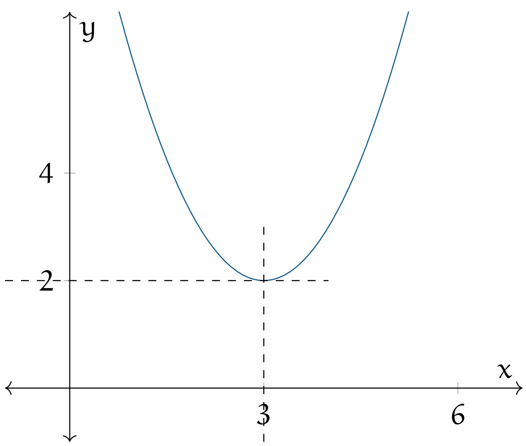
Answer to Exercise 10 (on page 23)

The graph of this function is a line. Its slope is -3. It intersects the y axis at $(0, 3)$



Answer to Exercise 11 (on page 26)

This graph is the graph of $y = x^2$ that has been moved to the right by three units and up two units:



To prevent any horizontal line from containing more than one point of the graph, you would need to use the left or the right side: Either $\{x \in \mathbb{R} | x \leq 3\}$ or $\{x \mathbb{R} | x \geq 3\}$. Most people will choose the right side; the rest of the solution will assume that you did too.

To find the inverse we swap x and y: $x = (y-3)^2 + 2$

The we solve for y to get the inverse: $y = \sqrt{x-2} + 3$

You can take the square root of nonnegative numbers. So the function $f^{-1}(x) = \sqrt{x-2} + 3$ is defined whenever x is greater than or equal to 2.

Index