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Chapter 1

Trigometric Functions

As mentioned in an earlier chapter, in a right triangle where one angle is θ, the sine of θ is the length of the side opposite θ divided by the length of the hypotenuse.

The sine function is defined for any real number. We treat that real number θ as an angle, we draw a ray from the origin out to the unit circle. The y value of that point is the sine. For example, the $\sin(\frac{4\pi}{3})$ is $-\sqrt{3}/2$ (see figure 1.1).

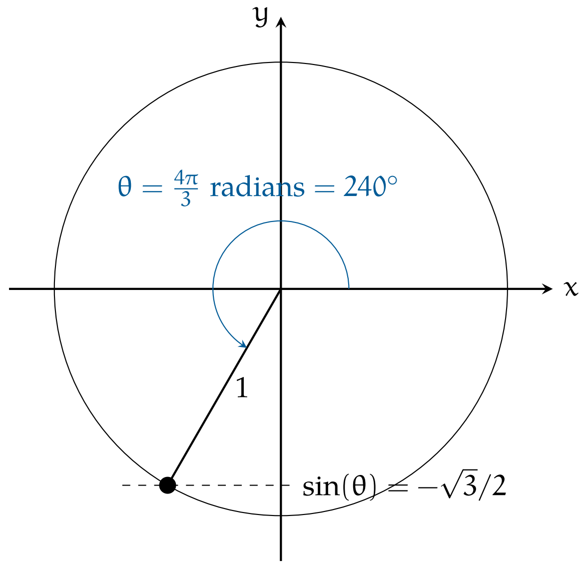


Figure 1.1: $\sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}$

(Note that in this section, we will be using radians instead of degrees unless otherwise noted. While degrees are more familiar to most people, engineers and mathematicians nearly always use radians when solving problems. Your calculator should have a radians mode and a degrees mode; you want to be in radians mode.)

Similarly, we define cosine using the unit circle. To find the cosine of θ, we draw a ray from the origin at the angle θ. The x component of the point where the ray intersects the unit circle is the cosine of θ (shown in figure 1.2).

From this description, it is easy to see why $\sin(\theta)^2 + \cos(\theta)^2 = 1$. They are the legs of a right triangle with a hypotenuse of length 1.

It should also be easy to see why $\sin(\theta) = \sin(\theta + 2\pi)$: Each time you go around the circle, you come back to where you started.

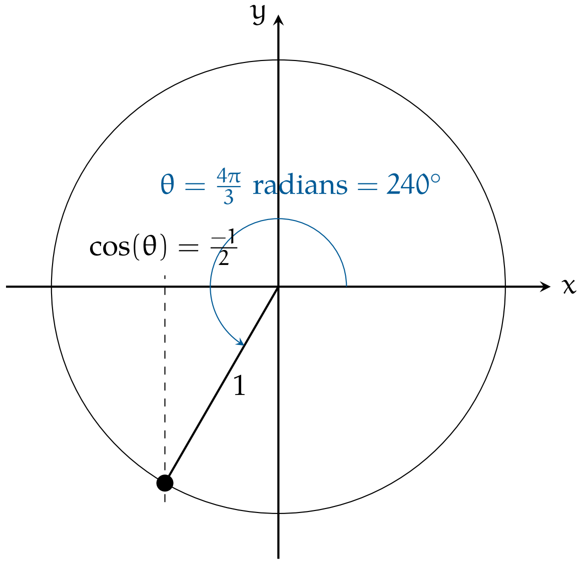
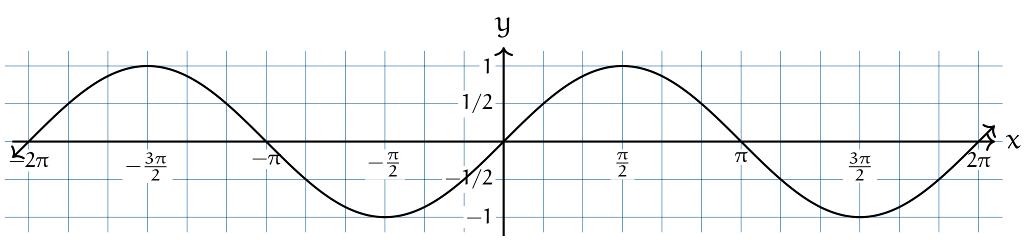


Figure 1.2: $\cos\frac{4\pi}{3} = \frac{-1}{2}$

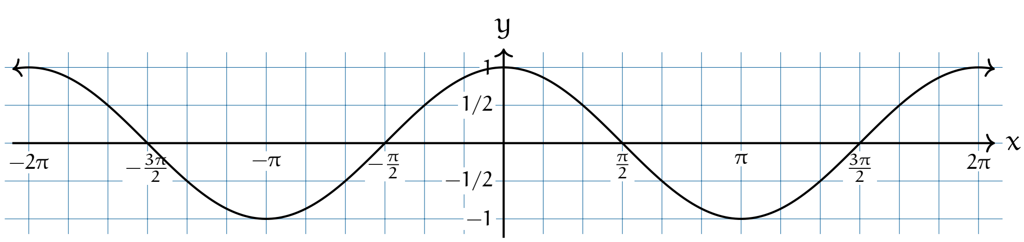
Can you see why $\cos(\theta) = \sin(\theta + \pi/2)$? Turn the picture sideways.

1.1 Graphs of sine and cosine

Here is a graph of $y = \sin(x)$:



It looks like waves, right? It goes forever to the left and right. Remembering that $\cos(\theta) = \sin(\theta + \pi/2)$, we can guess what the graph of $y = \cos(x)$ looks like:

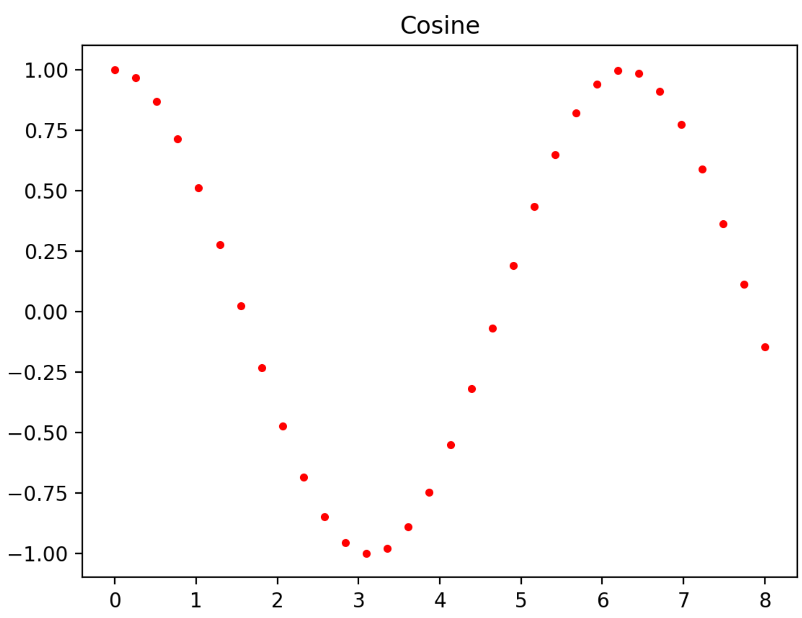


1.2 Plot cosine in Python

Create a file called `cos.py`:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(thetas, cosines, 'r.', label="Cosine")
ax.set_title("Cosine")
plt.show()
```

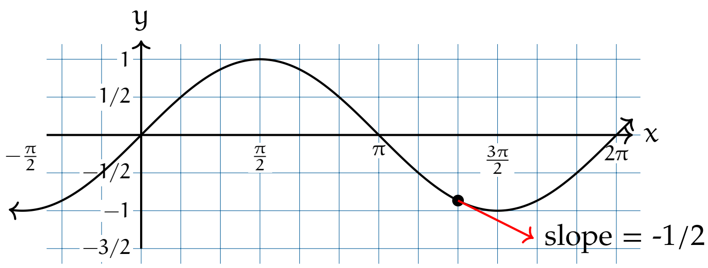
This will plot 32 points on the cosine wave between 0 and 8. When you run it, you should see something like this:



1.3 Derivatives of trigonometic functions

Here is a wonderful property of sine and cosine functions: At any point θ, the slope of the sine graph at θ equals $\cos(\theta)$.

For example, we know that $\sin(4\pi/3) = -(1/2)\sqrt{3}$ and $\cos(4\pi/3) = -1/2$. If we drew a line tangent to the sine curve at this point, it would have a slope of -1/2:



We say “The derivative of the sine function is the cosine function.”

Can you guess the derivative of the cosine function? For any θ, the slope of the graph of the $\cos(\theta)$ is $-\sin(\theta)$.

Exercise 1 Derivatives of Trig Functions Practice 1

Use the limit definition of a derivative to show that $\frac{d}{dx}\cos x = -\sin x$

Working Space

Answer on Page 45

The derivatives of all the trigonometric functions are presented below:

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\sec x \sec x \cdot \tan x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$

Example: Find the derivative of $f(x)$ if $f(x) = x^2 \sin x$ **Solution**: Using the product rule, we find that:

$$\frac{d}{dx}f(x) = (x^2)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)$$

Taking the derivatives:

$$= x^2(\cos x) + 2x(\sin x)$$

Exercise 2 Derivatives of Trig Functions 2

Find the derivative of the following functions:

1. $f(x) = \frac{\sec x}{1+\tan x}$
2. $y = \sec t \tan t$
3. $f(\theta) = \frac{\theta}{4-\tan\theta}$
4. $f(t) = 2\sec t - \csc t$
5. $f(\theta) = \frac{\sin\theta}{1+\cos\theta}$
6. $f(x) = \sin x \cos x$

Working Space

Answer on Page 46

1.4 A weight on a spring

Let's say you fill a rollerskate with heavy rocks and attach it to the wall with a stiff spring. If you push the skate toward the wall and release it, it will roll back and forth. Engineers would say "The skate will oscillate."

Intuitively, you can probably guess:

- If the spring is stronger, the skate will oscillate more times per minute.
- If the rocks are lighter, the skate will oscillate more times per minute.

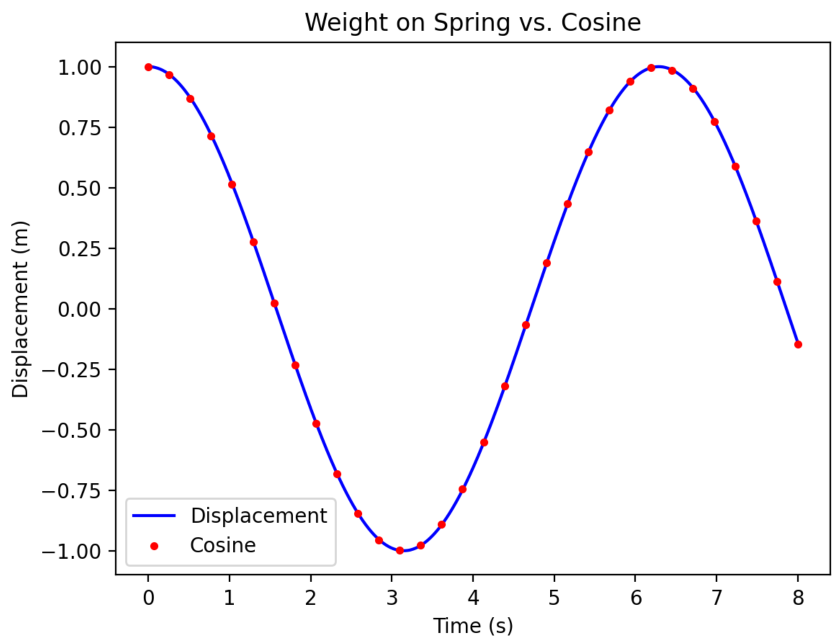
The force that the spring exerts on the skate is proportional to how far its length is from its relaxed length. When you buy a spring, the manufacturer advertises its "spring rate", which is in pounds per inch or newtons per meter. If a spring has a rate of 5 newtons per meter, that means that if you stretch or compress it 10 cm, it will push back with a force of 0.5 newtons. If you stretch or compress it 20 cm, it will push back with a force of 1 newton.

Let's write a simulation of the skate-on-a-spring. Duplicate cos.py, and name the new copy spring.py. Add code to implement the simulation:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Constants
mass = 100 # kg
spring_constant = -1 # newtons per meter displacement
time_step = 0.01 # s

# Initial state
displacement = 1.0 # height above equilibrium in meters
velocity = 0.0
time = 0.0 # seconds

# Lists to gather data
displacements = []
times = []

# Run it for a little while
while time <= until:
    # Record data
    displacements.append(displacement)
    times.append(time)

    # Calculate the next state
    time += time_step
    displacement += time_step * velocity
    force = spring_constant * displacement
    acceleration = force / mass
    velocity += acceleration

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(times, displacements, 'b', label="Displacement")
ax.plot(thetas, cosines, 'r.', label="Cosine")
```

```
ax.set_title("Weight on Spring vs. Cosine")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Displacement (m)")
ax.legend()
plt.show()
```

When you run it, you should get a plot of your spring and the cosine graph on the same plot.



The position of the skate is following a cosine curve. Why?

Because a sine or cosine waves happen whenever the acceleration of an object is proportional to -1 times its displacement. Or in symbols:

$$a \propto -p$$

where a is acceleration and p is the displacement from equilibrum.

Remember that if you take the derivative of the displacement, you get the velocity. And if you take the derivative of that, you get acceleration. So, the weight on the spring must follow a function f such that

$$f(t) \propto -f''(t)$$

Remember that the derivative of the $\sin(\theta)$ is $\cos(\theta)$.

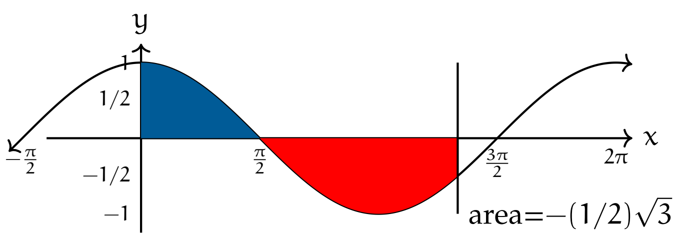
And the derivative of the $\cos(\theta)$ is $-\sin(\theta)$

These sorts of waves have an almost-magical power: Their acceleration is proportional to -1 times their displacement.

Thus, sine waves of various magnitudes and frequencies are ubiquitous in nature and technology.

1.5 Integral of sine and cosine

If we take the area between the graph and the x axis of the cosine function (and if the function is below the x axis, it counts as negative area), from 0 to $4\pi/3$, we find that it is equal to $-(1/2)\sqrt{3}$.



We say "The integral of the cosine function is the sine function."

1.5.1 Integrals of Trig Functions Practice

Exercise 3

Evaluate the following integrals:

1. $\int \sec x \tan x \, dx$

Working Space

Answer on Page 46

Chapter 2

Inverse Trigonometric Functions

Recall from the chapter on functions that an inverse of a function is a machine that turns y back into x. The inverses of trigonometric functions are essential to solving certain integrals (you will learn in a future chapter why integrals are useful — for now, trust us that they are!). Let's begin by discussing the sin function and its inverse, $\sin^{-1}$, also called arcsin.

Examine the graph of $\sin x$ in figure 2.1. See how the dashed horizontal line crosses the function at many points? This means the function $\sin x$ is not one-to-one. In other words, there is not a unique x-value for every y-value. This means that if we do not restrict the domain of $\arcsin x$, the result will not be a function (see figure 2.2). In figure 2.2, you can see that just reflecting the graph across $y = x$ fails the vertical line test: an x value has more than one y value.

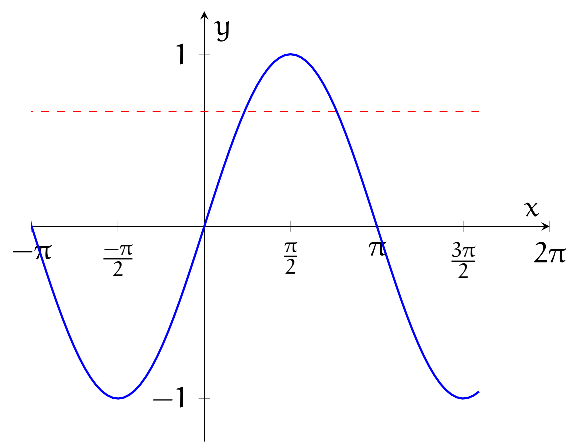


Figure 2.1: The horizontal line $y = \frac{2}{3}$ crosses $y = \sin x$ more than once

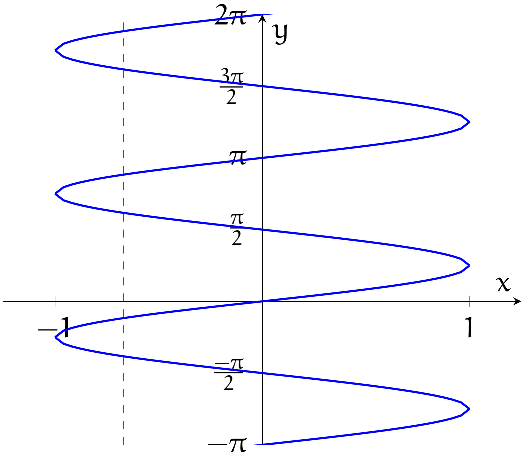


Figure 2.2: The inverse of an unrestricted sin function fails the vertical line test

2.1 Derivatives of Inverse Trigonometric Functions

f	f'
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\text{arccsc } x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\text{arcsec } x$	$\frac{1}{x\sqrt{x^2-1}}$
$\text{arccot } x$	$-\frac{1}{1+x^2}$

2.2 Practice

Exercise 4

Find the f'. Give your answer in a simplified form.

1. $f(x) = \arctan x^2$
2. $f(x) = x\text{arcsec } (x^3)$
3. $f(x) = \arcsin \frac{1}{x}$

Working Space

Answer on Page 46

Chapter 3

Trigonometric Identities

3.1 The Unit Circle

There are some values of $\sin\theta$ and $\cos\theta$ that will be useful to know off the top of your head. The Unit Circle will help you in this memorization process (see figure 3.1). When a circle of radius 1 is centered at the origin, the Cartesian coordinates of any point on the circle correspond to the values of cosine and sine of the angle above the horizontal (how far you've rotated from the positive x-axis).

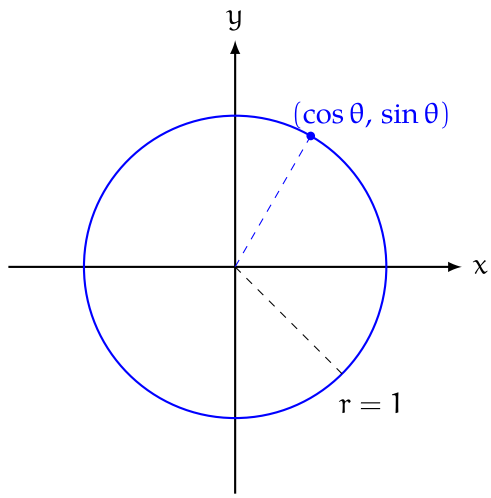


Figure 3.1: The Unit Circle is a circle with radius 1 centered at the origin

Let's take a closer look at a triangle in the first quadrant to see why this is true. Imagine some point on the circle, (x_o, y_o). Drawing a line from that point back to the origin creates an angle θ between the imaginary line and the positive x-axis (see figure 3.2). Extending an imaginary vertical down to $(x_o, 0)$, then an imaginary horizontal from $(x_o, 0)$ to the origin, creates a right triangle. What can we say about the legs of the triangle?

Recall SOH-CAH-TOA from a previous chapter. This acronym tells us that, for a right triangle, the sine of an angle is given by the ratio of the length of the leg opposite the angle to the hypotenuse. In our case, then, $\sin\theta = \frac{y_o}{1} = y_o$. [Remember: We are dealing with the Unit Circle, which has a radius of one. Examining figure 3.2 shows you that the hypotenuse of the imaginary triangle is the same as the circle's radius.] This means that the y-coordinate of any point on the Unit Circle is the sine of the angle of rotation from the horizontal.

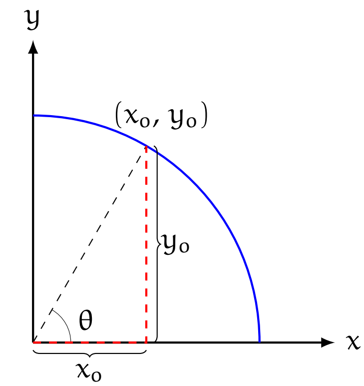


Figure 3.2: Drawing a line from any point on the circle to the origin creates an angle with the horizontal

Exercise 5

In a similar manner as we did with $\sin\theta$ above, prove the x-coordinate of any point on the unit circle is equal to $\cos\theta$, where θ is the angle of rotation from the horizontal.

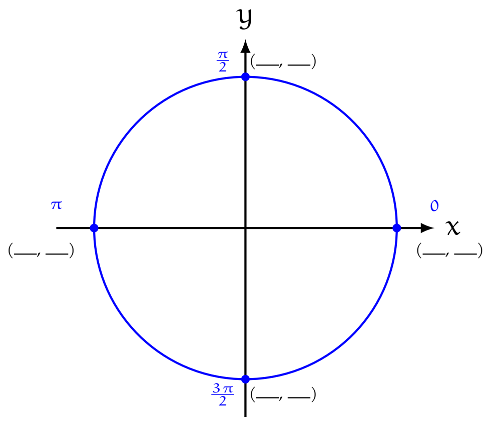
Working Space

Answer on Page 46

Exercise 6

Fill in the unit circle with the coordinates for $\theta = 0$, $\pi/2$, π, and $3\pi/2$. Use this to determine:

1. $\sin \frac{\pi}{2}$
2. $\cos \frac{3\pi}{2}$
3. $\sin \pi$
4. $\cos -\pi$



Working Space

Answer on *Page 46*

3.1.1 Exact Values of Key Angles

We will examine two triangles. First, a 30-60-90 triangle, then a 45-45-90 triangle. As shown in figure 3.3, you can get a 30-60-90 triangle with hypotenuse 1 by dividing an equilateral triangle in half. We will label the horizontal leg of the 30-60-90 triangle A and the vertical leg B.

From the figure, we see that the length of A is half that of the hypotenuse, which in this case is $\frac{1}{2}$. This means the $\cos 60° = \cos \frac{\pi}{3} = \frac{1}{2}$. To find the length of side B, we can use the Pythagorean theorem:

$$B^2 = C^2 - A^2, \text{ where C is the hypotenuse}$$

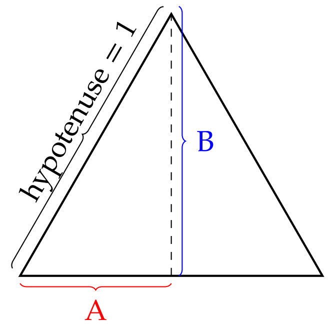


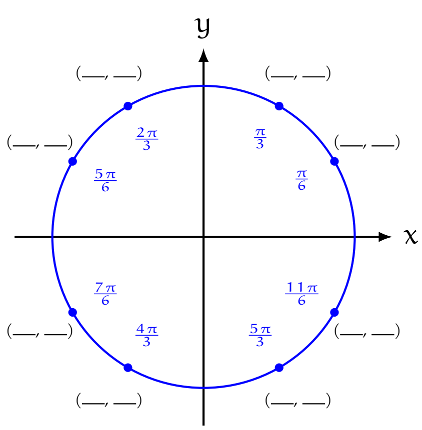
Figure 3.3: A 30-60-90 triangle is made by vertically bisecting an equilateral triangle

$$B^2 = 1^2 - \left(\frac{1}{2}\right)^2$$
$$B^2 = \frac{3}{4}$$
$$B = \frac{\sqrt{3}}{2}$$

Therefore, $\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Exercise 7

Use symmetry to complete the blank unit circle below. (Hint: We just showed that the (x, y) coordinate for $\frac{\pi}{3}$ is $(1/2, \sqrt{3}/2)$).



Working Space

Answer on Page 47

Now we will look at a 45-45-90 triangle (see figure 3.4), which will allow us to complete our Unit Circle. Recall that a 45-45-90 triangle is an isosceles triangle in addition to being a right triangle. This means both the legs are the same length. Using the Pythagorean theorem, we would say $A = B$. We also know that $C = 1$, since our triangle is inscribed in the unit circle. Let's find A:

$$A^2 + B^2 = C^2$$
$$A^2 + A^2 = 1^2$$
$$2A^2 = 1$$
$$A^2 = \frac{1}{2}$$
$$A = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, each leg has a length of $\sqrt{2}/2$, and the (x, y) coordinates for $\theta = 45° = \pi/4$ are $(\sqrt{2}/2, \sqrt{2}/2)$.

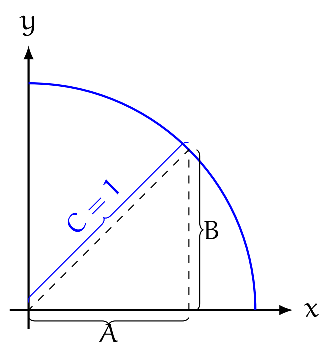
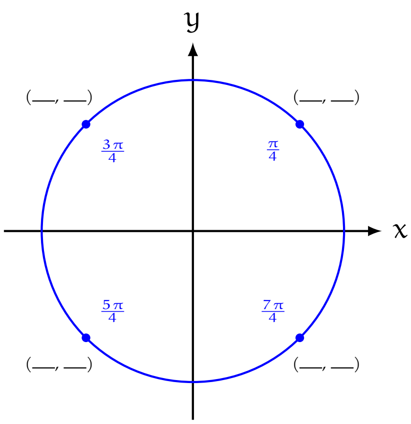


Figure 3.4: The two legs of a 45-45-90 triangle are the same length

Exercise 8

Use symmetry to complete the blank unit circle below.



Working Space

Answer on Page 48

Exercise 9

Without a calculator and using only your completed unit circles, determine the value requested (angles are given in radians unless otherwise indicated).

1. $\cos \frac{3\pi}{2}$
2. $\sin \frac{\pi}{4}$
3. $\sin -\frac{\pi}{6}$
4. $\cos \frac{4\pi}{3}$
5. $\sin \frac{3\pi}{4}$
6. $\cos -\frac{\pi}{3}$
7. $\sin 45°$
8. $\sin 270°$
9. $\sin -60°$
10. $\sin 150°$

Working Space

Answer on Page 48

3.2 Sum and Difference Formulas

Consider 4 points on the unit circle: B at $(1, 0)$, Q at some angle β, P at some angle α, and A at angle $\alpha - \beta$ (see figure 3.5).

The distance from P to Q is the same as the distance from A to B, since ΔPOQ is a rotation of ΔAOB. Because this is a Unit Circle, $P = (\cos\alpha, \sin\alpha)$, $Q = (\cos\beta, \sin\beta)$, and $A = (\cos\alpha - \beta, \sin\alpha - \beta)$. Let's use the distance formula to find the length of $\overline{PQ}$:

$$\overline{PQ} = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} =$$

$$\sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} =$$

$$\sqrt{\left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

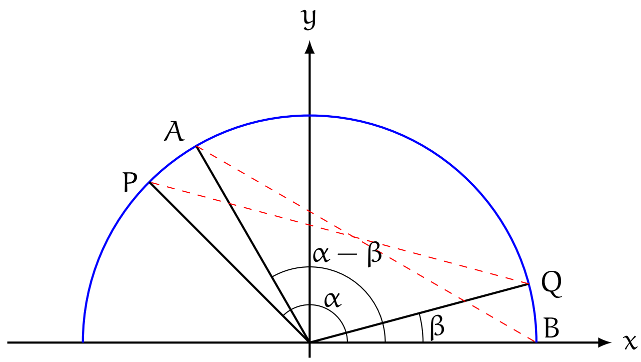


Figure 3.5: $\overline{AB} = \overline{PQ}$

Recall that for any angle, θ, $\sin^2\theta + \cos^2\theta = 1$. Substituting this identity, we see that:

$$\overline{PQ} = \sqrt{1 + 1 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta} = \sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta}$$

Let's leave this simplified equation for $\overline{PQ}$ alone for the moment and similarly find $\overline{AB}$:

$$\overline{AB} = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2} =$$

$$\sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)} =$$

$$\sqrt{\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) + 1 - 2\cos(\alpha - \beta)}$$

$$= \sqrt{2 - 2\cos(\alpha - \beta)} = \overline{AB}$$

Recall that we've established $\overline{AB} = \overline{PQ}$. We can set the statements equal to each other:

$$\sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta} = \sqrt{2 - 2\cos(\alpha - \beta)}$$

Squaring both sides and subtracting 2, we find:

$$-2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta = -2\cos(\alpha - \beta)$$

Finally, we can divide both sides by negative 2 to get the difference of angles formula for cosine:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

There are similar formulas for the sine and cosine of the sum of two angles, and for the sine of the difference of two angles, which we won't derive here.

Sum and Difference Formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Exercise 10

Without a calculator, find the exact value requested:

1. $\sin\frac{\pi}{12}$
2. $\cos\frac{7\pi}{12}$
3. $\tan\frac{13\pi}{12}$ (hint: $\tan\theta = \sin\theta/\cos\theta$)

Working Space

Answer on Page 48

3.3 Double and Half Angle Formulas

We can easily derive a formula for twice an angle by letting $\alpha = \beta$ for a sum formula.

Example: Derive a formula for $\cos 2\theta$ in terms of trigonometric functions of θ.

Solution: Using the sum formula for cosine, we see that:

$$\cos 2\theta = \cos(\theta+\theta)$$

$$= \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

Noting that $\sin^2\theta = 1 - \cos^2\theta$:

$$\cos 2\theta = 2\cos^2\theta - 1$$

Alternatively, we could note that $\cos^2\theta = 1 - \sin^2\theta$:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

Exercise 11

Derive a formula for $\sin 2\theta$ in terms of trigonometric functions of θ.

Working Space

Answer on Page 49

We can use these double-angle formulas for find half-angle formulas. Consider the double-angle formula for cosine:

$$\cos 2\theta = 2\cos^2\theta - 1$$

Let $\theta = \alpha/2$, then:

$$\cos\alpha = 2\cos^2(\alpha/2) - 1$$

Rearranging to solve for $\cos(\alpha/2)$:

$$2\cos^2(\alpha/2) = \cos\alpha + 1$$

$$\cos^2(\alpha/2) = \frac{\cos\alpha + 1}{2}$$

$$\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Exercise 12

Derive a formula for $\sin(\alpha/2)$.

Working Space

Answer on Page 49

There are two identities that will be very useful for integrals in a future chapter:

Squared Trigonometric Identities

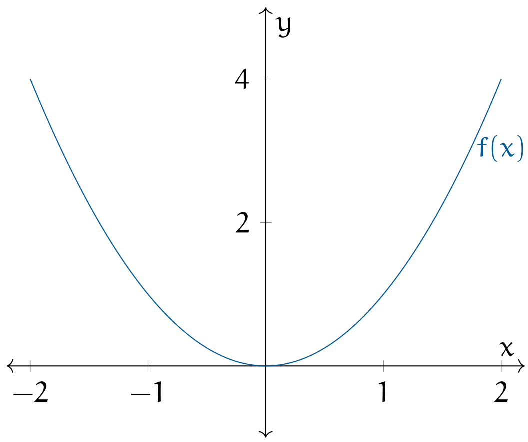
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

These are just specific re-writings of the half-angle identities.

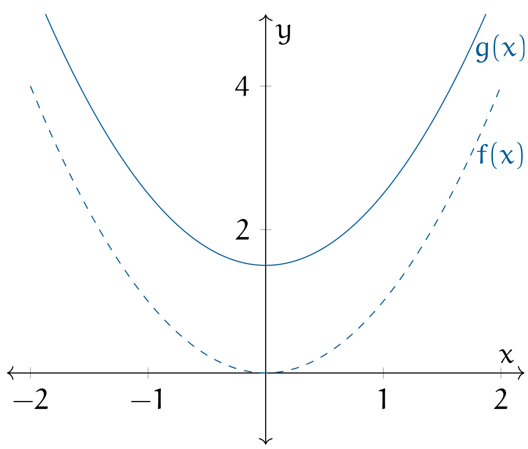
Chapter 4

Transforming Functions

Let's say we gave you the graph of a function f, like this:



We then tell you that the function is $g(x) = f(x) + 1.5$. Can you guess what the graph of g would look like? It is the same graph, just translated up 1.5:



There are four kinds of transformations that we do all the time:

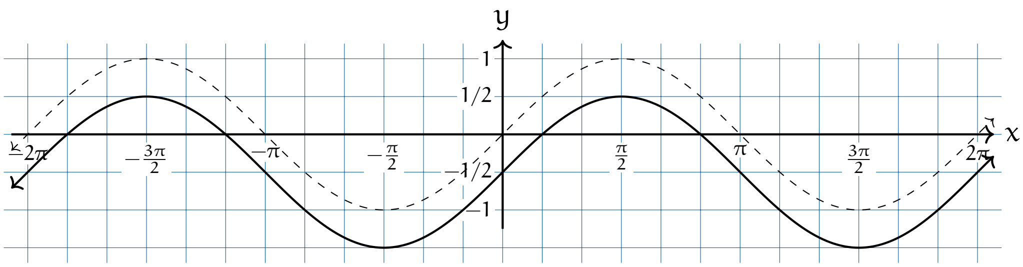
- Translation up and down in the direction of y axis (the one you just saw)
- Translation left and right in the direction of the x axis
- Scaling up and down along the y axis
- Scaling up and down along the x axis

Next, we will demonstrate each of the four using the graph of $\sin(x)$.

4.1 Translation up and down

When you add a positive constant to a function, you translate the whole graph up that much. A negative constant translates it down.

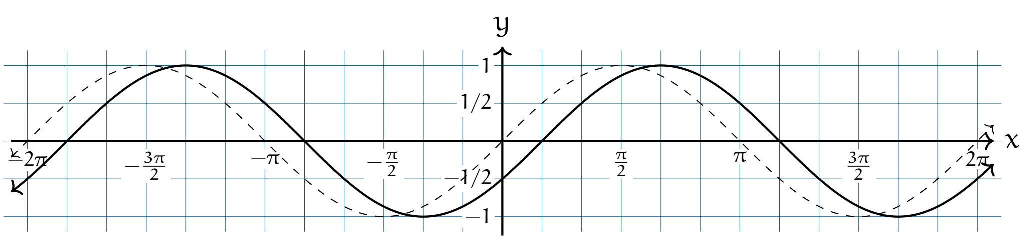
Here is the graph of $\sin(x) - 0.5$:



4.2 Translation left and right

When you add a positive number to x before running it through f, you translate the graph to the left by that amount. Adding a negative number translates the graph to the right.

Here is the graph of $\sin(x - \pi/6)$:



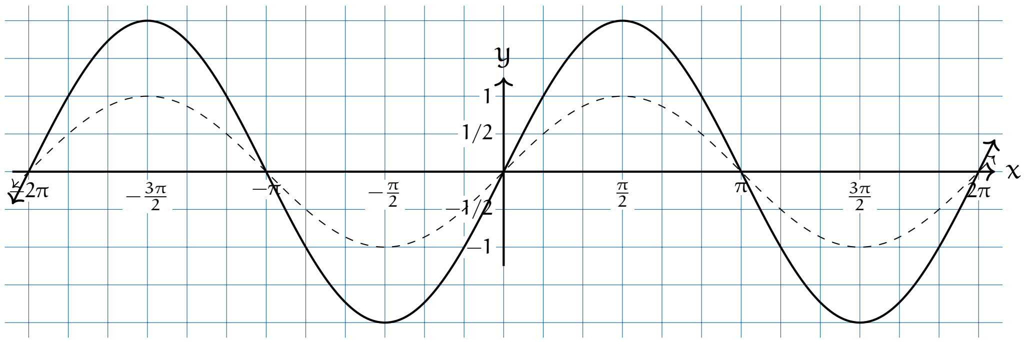
Notice the sign:

- Adding to x before processing with the function translates the graph to the *left*.
- Subtracting from x before processing with the function translates the graph to the *right*

4.3 Scaling up and down in the y direction

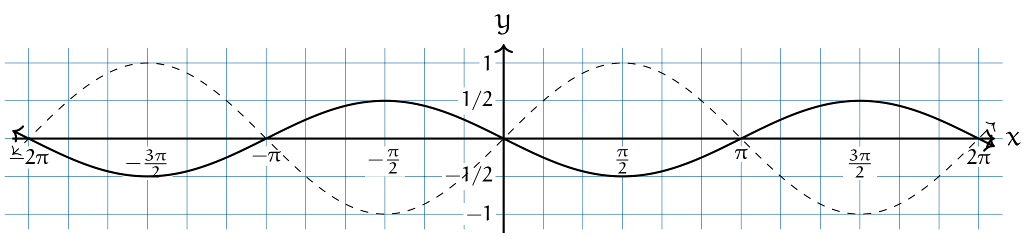
To scale the function up and down, you multiply the result of the function by a constant. If the constant is larger than 1, it stretches the function up and down.

Here is $y = 2\sin(x)$:



With a wave like this, we speak of its *Amplitude*, which you can think of as its height. The baseline that this wave oscillates around is zero. The maximum distance that it gets from that baseline is its amplitude. Thus, the amplitude here has been increased from 1 to 2.

If you multiply by a negative number, the function gets flipped. Here is $y = -0.5\sin(x)$:

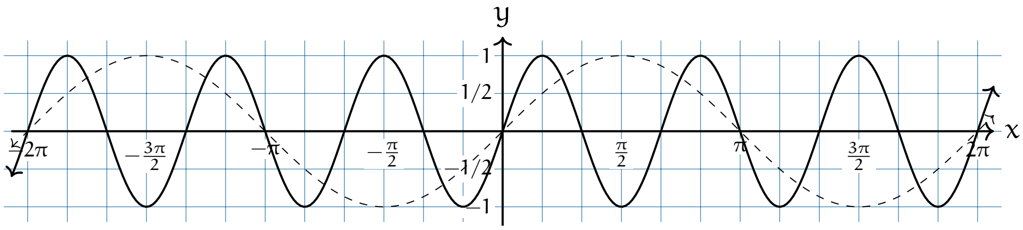


Amplitude is never negative. Thus, the amplitude of this wave is 0.5.

4.4 Scaling up and down in the x direction

If you multiply x by a number larger than 1 before running it through the function, the graph gets compressed toward zero.

Here is $y = \sin(3x)$:

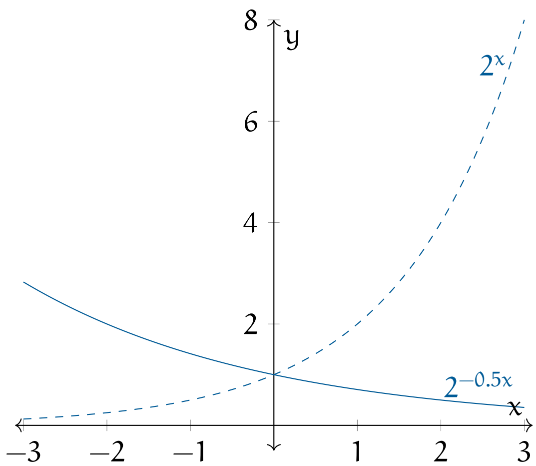


The distance between two peaks of a wave is known as its *wavelength*. The original wave had a wavelength of 2π. The compressed wave has a wavelength of $2\pi/3$.

If you multiply x by a number smaller than 1, it will stretch the function out, away from the y axis.

If you multiply x by a negative number, it will flip the function around the y axis.

Here is $y = 2^{(-0.5x)}$. Notice that it has flipped around the y axis and is stretched out along the x axis.

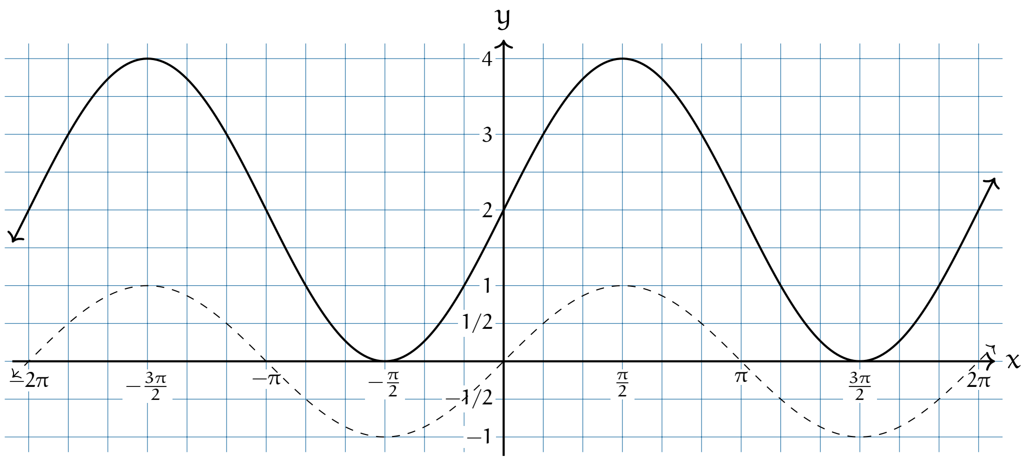


Reflection over *x*-axis	$(x, y) \to (x, -y)$
Reflection over *y*-axis	$(x, y) \to (-x, y)$
Translation	$(x, y) \to (x + a, y + b)$
Dilation	$(x, y) \to (kx, ky)$
Rotation 90° counterclockwise	$(x, y) \to (-y, x)$
Rotation 180°	$(x, y) \to (-x, -y)$

4.5 Order is important!

We can combine these transformations. This allows us, for example, to translate a function up 2, then scale along the y axis by 3.

Here is $y = 2.0(\sin(x) + 1)$:

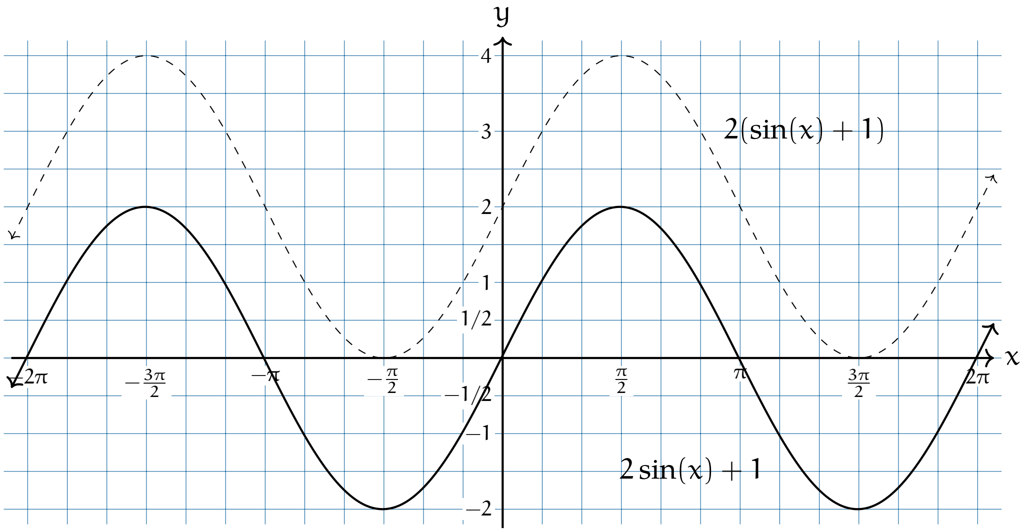


A function is often a series of steps. Here are the steps in $f(x) = 2(\sin(x) + 1)$:

1. Take the sine of x
2. Add 1 to that
3. Multiply that by 2

What if we change the order? Here are the steps in $g(x) = 2\sin(x) + 1$:

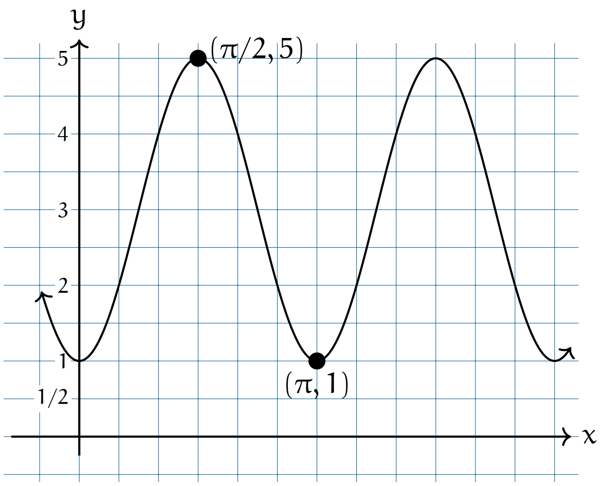
1. Take the sine of x
2. Multiply that by 2
3. Add 1 to that



The moral: You can do multiple transformations of your function, but the order in which you do them is important.

Exercise 13 Transforms

Find a function that creates a sine wave such that the top of the first crest is at the point $(\frac{\pi}{2}, 5)$ and the bottom of the trough that follows is at $(\pi, 1)$.



Working Space

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Chapter 5

Polar Coordinates

We have already seen how to plot a function with (x, y) coordinates. For every x that we put into a function, it returns a y. These pairs of coordinates tell us where on the xy-plane to graph the function. This coordinate system, where x and y are oriented horizontally and vertically, is called the *Cartesian* coordinate system. It can be used to describe 2D space, but it is not the only way.

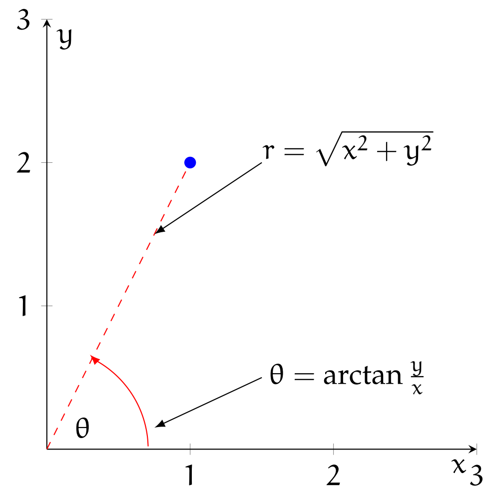


Figure 5.1: The point $(1, 2)$ is $\sqrt{5}$ units from the origin and approximately 1.107 radians counterclockwise from horizontal

Instead of thinking about the horizontal and vertical position, we could think about distance from the origin and rotation about the origin. Take the Cartesian coordinate point $(1, 2)$ (see figure 5.1). How far is $(1, 2)$ from the origin, $(0, 0)$? We can create a right triangle, where the legs are parallel to the x and y axes. This means the leg lengths are 1 and 2, and we can use the Pythagorean theorem to find the length of the hypotenuse (which is the distance from the origin to the point):

$$c^2 = a^2 + b^2$$

$$c^2 = 1^2 + 2^2 = 1 + 4 = 5$$

$$c = \sqrt{5}$$

Therefore, the Cartesian point $(1, 2)$ is $\sqrt{5}$ units from the origin. This is not enough to find our point: there are infinite points that are $\sqrt{5}$ from the origin (see 5.2). To identify a particular point that is a distance of $\sqrt{5}$ from the origin, we also need an *angle of rotation*. By convention, angles are measured from the positive x- axis. This means points on the positive x-axis have an angle of $\theta = 0$, points on the positive y-axis have an angle of $\theta = \frac{\pi}{2}$, and so on.

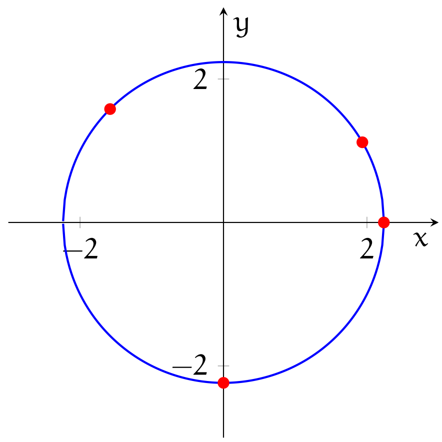


Figure 5.2: There are infinite points $\sqrt{5}$ from the origin, represented by the circle with a radius of $\sqrt{5}$ centered about the origin

We can use trigonometry to find the appropriate angle of rotation for our Cartesian point. There are many ways to do this, but using arctan is the most straightforward. Recall that:

$$\tan\theta = \frac{opposite}{adjacent}$$

That is, for a given angle in a right triangle, the tangent of that angle is given by the length of the opposite leg divided by the adjacent leg. In our case, the opposite leg is the vertical distance (y-value of the Cartesian point) and the adjacent leg is the horizontal distance (x-value of the Cartesian point), which means:

$$\tan\theta = \frac{2}{1}$$

$$\theta = \arctan 2 \approx 1.107 \text{ radians}$$

5.1 Plotting Polar Coordinate Points

How do we plot polar coordinate points? Begin by locating the angle given by the second coordinate (remember, the angle is measured counterclockwise from the horizontal). Your point will lie somewhere on this line. Next, move outwards along the angle by the radius given by the first coordinate.

Example: Plot the polar coordinate point $(2, \frac{\pi}{3})$.

Solution: Begin by locating $\theta = \frac{\pi}{3}$ (see figure 5.3)

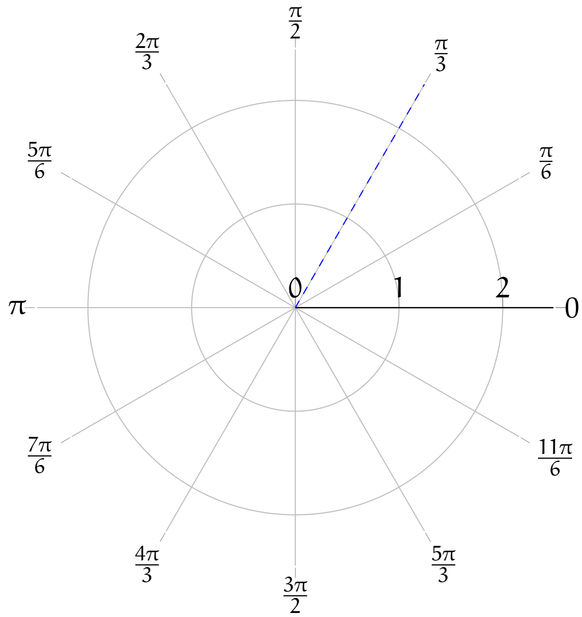


Figure 5.3: $\theta = \frac{\pi}{3}$

ThNexten, move your finger or pencil along the line $\theta = \frac{\pi}{3}$ until you reach $r = 2$ (see figure 5.4).

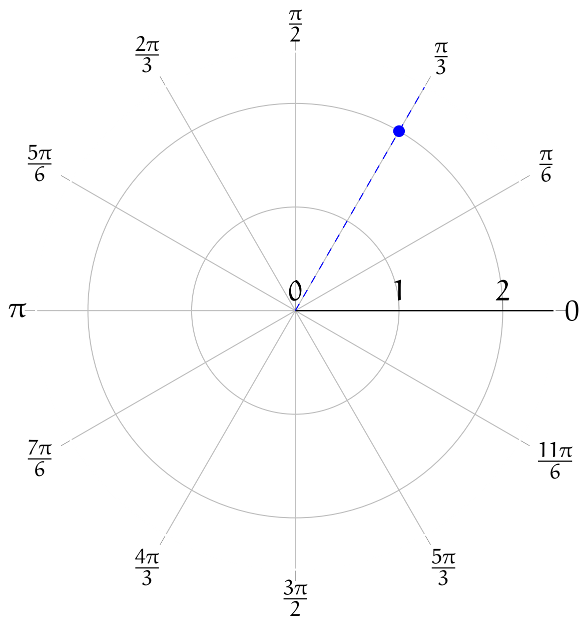
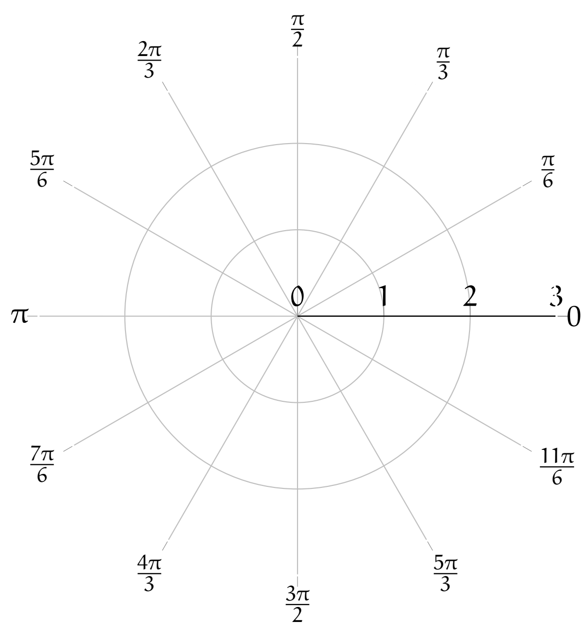


Figure 5.4: $(2, \frac{\pi}{3})$

Exercise 14

Plot the following polar coordinate points on the provided polar axis (hint: negative angles are taken counterclockwise):

1. $(1, \pi)$
2. $(1.5, \frac{\pi}{2})$
3. $(1.5, -\frac{\pi}{6})$
4. $(2, \frac{3\pi}{4})$



Working Space

Answer on Page 51

5.2 Equivalent Points

Unlike the Cartesian coordinate system, two different coordinates may lie at the same location. Consider the points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ (see figure 5.5). When a radius is negative, you move *backwards* back over the origin, like a mirror image.

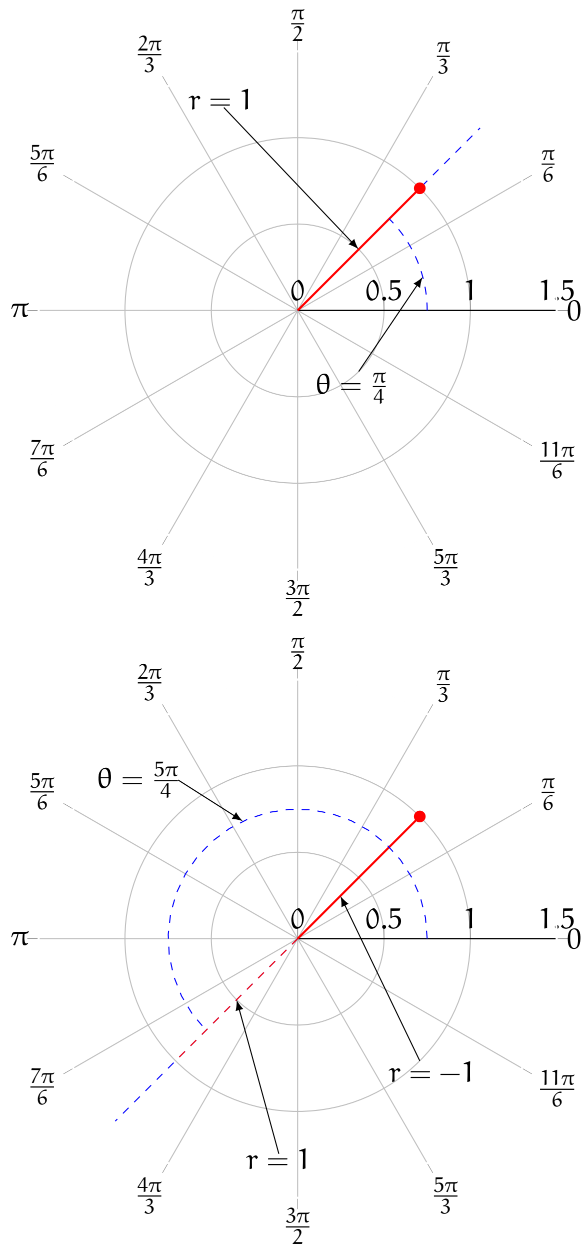


Figure 5.5: The polar coordinates points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ are the same location on a polar axis

5.3 Changing coordinate systems

5.3.1 Cartesian to Polar

From the example above, you should see that a given Cartesian coordinate, (x, y), can also be expressed as a polar coordinate, (r, θ), where r is the distance from the origin and θ is the angle of rotation from the horizontal. (Note: Polar functions are generally given as r defined in terms of θ, which means the *dependent* variable is listed first in the coordinate pair, unlike Cartesian coodinates.) Additionally,

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

Example: Express the Cartesian point $(-3, 4)$ in polar coordinates.

Solution: Taking $x = -3$ and $y = 4$, we find that:

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

We follow the convention of only taking the positive solution to the square root. Finding θ:

$$\theta = \arctan \frac{4}{-3}$$

When you evaluate the arctan with a calculator, you are likely to get back $\theta = -0.928$. Recall that $tan\theta = \tan\theta \pm n\pi$, where n is an integer. We know our Cartesian point, $(-3, 4)$, is in the II quadrant, while the angle -0.928 radians would fall in the IV quadrant. So, clearly, -0.928 radians is not correct. Most calculators restrict the output of arctan to angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because there are actually multiple angles where $\tan\theta = -\frac{4}{3}$. Since $\tan\theta = \tan\theta \pm n\pi$, we also know that:

$$\arctan -\frac{4}{3} = -0.928 \pm n\pi$$

Another possible θ is $-0.928 + \pi \approx 2.214$, which does fall in the appropriate quadrant. This means the polar coordinates $(5, 2.214)$ are the same as the Cartesian coordinates $(-3, 4)$. *Note:* It is standard practice to express angles in radians, and not degrees, when using polar coordinates.

5.3.2 Polar to Cartesian

We can also leverage our knowledge of right triangles to convert polar coordinates to Cartesian coordinates. Take the polar coordinate $(2, \frac{\pi}{4})$ (see figure 5.6). We can draw a right triangle with legs parallel to the x and y axes (not shown in the figure) and a hypotenuse that goes from the origin to the polar coordinate $(2, \frac{\pi}{4})$.

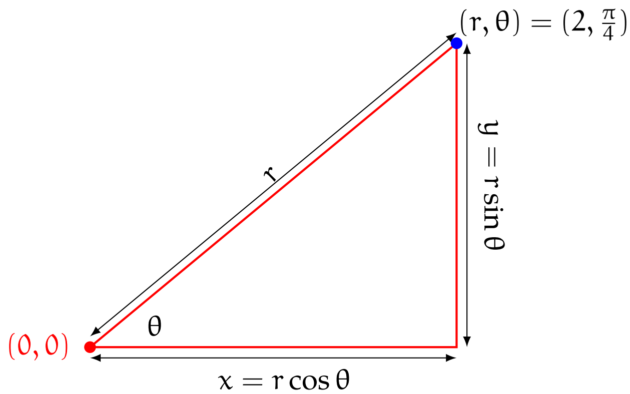


Figure 5.6: To convert from polar to Cartesian coordinates, use the identities $x = r\cos\theta$ and $y = r\sin\theta$

Recall from trigonometry that:

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

We know that the hypotenuse of this triangle has a length of r. The opposite leg is vertical and is the same length as the distance of the polar coordinate from the x-axis. Therefore, the length of the vertical leg represents the y value of that same polar coordinate if it were expressed in Cartesian coordinates. So, we can say that:

$$\sin\theta = \frac{y}{r}$$

And therefore:

$$y = r\sin\theta$$

By a similar process, we also see that:

$$x = r\cos\theta$$

This is easy to visualize and understand for $0 \leq \theta \leq \frac{\pi}{2}$, but it also holds for other values of θ.

Example: Express the polar coordinate $(\frac{3}{2}, \frac{2\pi}{3})$ in Cartesian coordinates.

Solution: From the polar coordinate, we see that $\theta = \frac{2\pi}{3}$ and $r = \frac{3}{2}$. Therefore:

$$x = r\cos\theta = \frac{3}{2} \cdot \cos\frac{2\pi}{3} = \frac{3}{2} \cdot -\frac{1}{2} = -\frac{3}{4}$$

$$y = r\sin\theta = \frac{3}{2} \cdot \sin\frac{2\pi}{3} = \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

The Cartesian coordinate $(-\frac{3}{4}, \frac{3\sqrt{3}}{4})$ has the same location as the given polar coordinate.

Exercise 15

Convert the following polar coordinates to Cartesian coordinates:

1. $(2, \frac{3\pi}{2})$
2. $(\sqrt{2}, \frac{3\pi}{4})$
3. $(3, -\frac{\pi}{4})$
4. $(-3, -\frac{\pi}{3})$
5. $(2, -\frac{\pi}{2})$

Working Space

Answer on Page 51

Exercise 16

Convert the following Cartesian coordinates to polar coordinates. Restrict θ to $0 \leq \theta < 2\pi$.

1. $(-4, 4)$
2. $(3, 3\sqrt{3})$
3. $(\sqrt{3}, -1)$
4. $(-6, 0)$
5. $(-2, -2)$

Working Space

Answer on Page 51

5.4 Circles in Polar Coordinates

Many conic sections, including circles, are simpler to express as polar functions than as Cartesian functions. Consider a circle with a radius of 2 centered about the origin. The polar function for this is $r = 2$ for all θ. Let's write a Cartesian function for the same circle.

We know that for every point on the circle, the distance to the origin is 2. This means that, by the Pythagorean theorem,

$$r^2 = x^2 + y^2$$

.

(see figure 5.7)

We can solve this equation for y, given that $r = 2$ (in this case):

$$y = \pm\sqrt{2^2 - x^2}$$

Notice that this is really two equations: $y = \sqrt{2^2 - x^2}$ and $y = -\sqrt{2^2 - x^2}$. This is more complex than the polar equation, $r = 2$.

As seen above, the equation of a circle with radius R centered on the origin is simply $r = R$

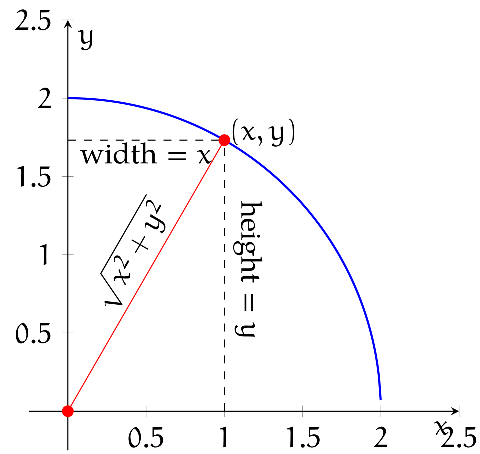


Figure 5.7: All (x, y) pairs on the circle are the same distance from the origin

in polar coordinates. What if we want a circle centered somewhere else? Polar coordinates are best when a circle is bisected by the x or y axis. Consider the polar equation $r = 3\sin\theta$. Let's use a table to find some points and plot the function:

θ	$r = 3\sin\theta$
0	0
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{\pi}{2}$	3
$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{3}{2}$
π	0

Here is how those points look plotted (see figures 5.8 and 5.9):

So, the polar equation $r = 3\sin\theta$ gives a circle with radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$.

Example: Describe the graph of $r = \cos\theta$. Feel free to make a rough plot on the blank polar axis below:

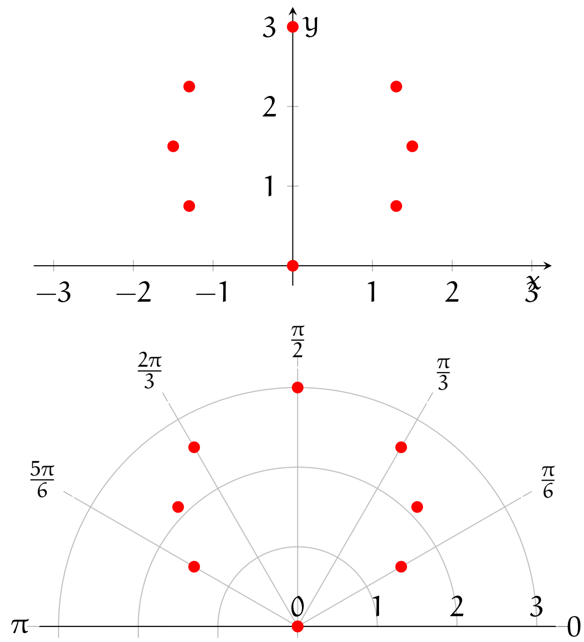


Figure 5.8: Several points for $r = 3 \sin \theta$ plotted on Cartesian and polar coordinate systems

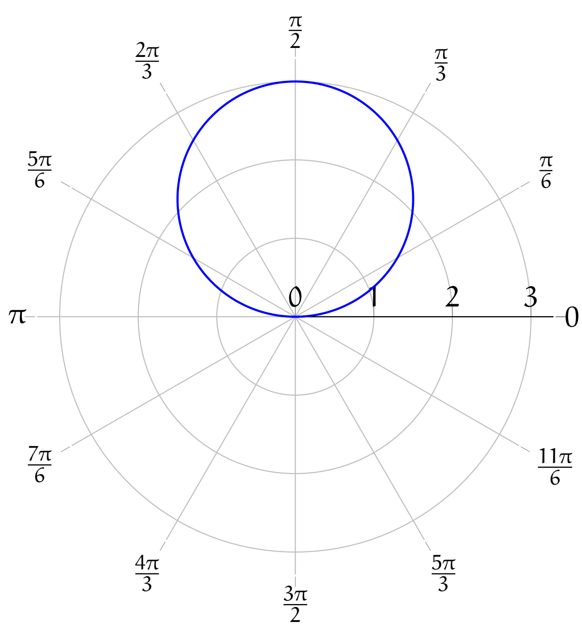
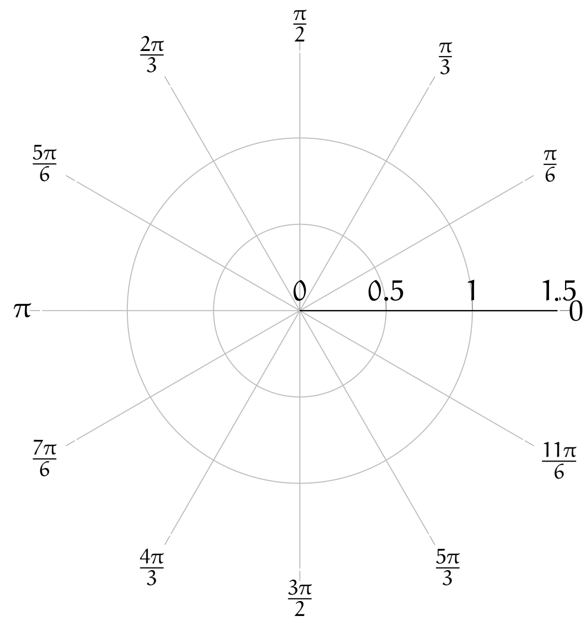
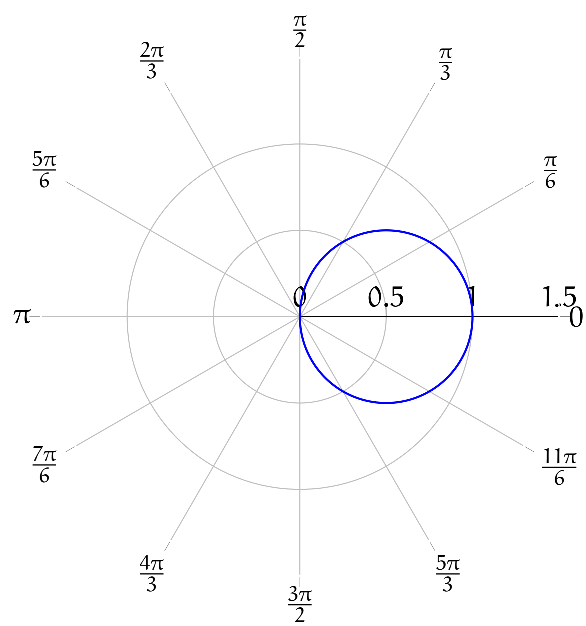


Figure 5.9: $r = 3 \sin \theta$ plotted on a polar coordinate system



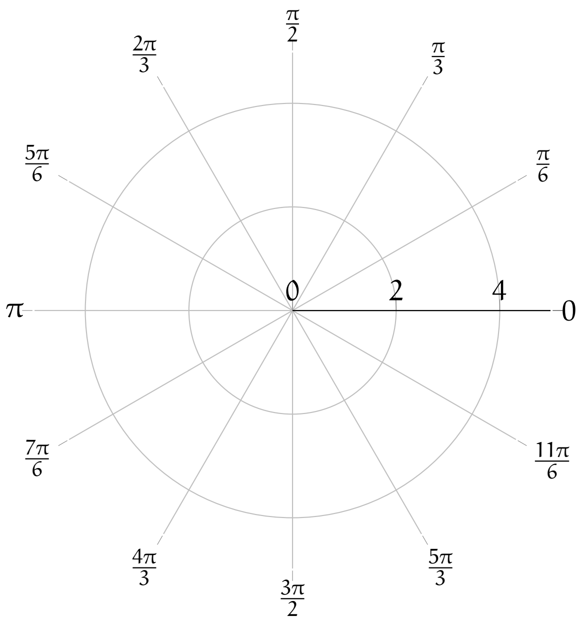
Solution: This plot will look like a circle of radius 0.5 centered at $(0.5, 0)$ (in polar coordinates).



Exercise 17

Sketch the following polar functions on the provided polar axis for $0 \le \theta < 2\pi$:

1. $r = 3$
2. $\theta = \pi$
3. $r = 2\cos\frac{\theta}{2}$
4. $r = -4\sin\theta$
5. $r = \theta$



Working Space

Answer on Page 52

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

We start by writing out the limit:

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos x + h - \cos x}{h}$$

Applying the sum formula for $\cos(x+h)$, we get:

$$= \lim_{h\to 0}\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

Rearranging to group the $\cos x$ and applying the Difference Rule:

$$= \lim_{h\to 0}\frac{\cos x \cos h - \cos x}{h} - \lim_{h\to 0}\frac{\sin x \sin h}{h}$$

Applying the Constant Multiple Rule:

$$= \cos x \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h}$$

Recalling that $\lim_{h\to 0}\frac{\sin h}{h} = 1$,

$$= \cos x \lim_{h\to a}\frac{\cos h - 1}{h} - \sin x \cdot 1$$

Recalling that $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$:

$$= \cos x \cdot 0 - \sin x = -\sin x$$

Therefore, $\frac{d}{dx}\cos x = -\sin x$

Answer to Exercise 2 (on page 7)

1. $\frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$
2. $\sec t[\sec^2 t + \tan^2 t]$
3. $\frac{4 - \tan\theta + \theta\sec^2\theta}{(4-\tan\theta)^2}$
4. $2\sec t\tan t + \csc t\cot t$
5. $\frac{2}{(1+\cos\theta)^2}$
6. $\cos^2 x - \sin^2 x$

Answer to Exercise 3 (on page 10)

1. $\sec x + C$

Answer to Exercise 4 (on page 12)

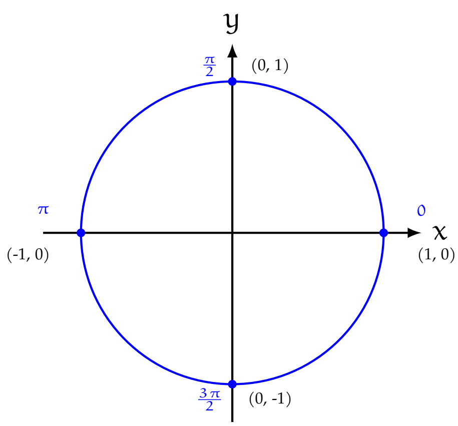
1. By the chain rule, $f'(x) = 2\arctan x \times \frac{d}{dx}\arctan x = 2\arctan x\frac{1}{1+x^2}$
2. By the Product rule, $f'(x) = x\frac{d}{dx}\text{arcsec}\,(x^3) + \text{arcsec}\,(x^3)$. Further, by the chain rule, $\frac{d}{dx}\text{arcsec}\,(x^3) = \frac{1}{(x^3)\sqrt{(x^3)^2-1}} \times \frac{d}{dx}(x^3) = \frac{3x^2}{x^3\sqrt{x^6-1}}$. Therefore, $f'(x) = \frac{3}{\sqrt{x^6-1}} + \text{arcsec}\,(x^3)$
3. By the chain rule, $f'(x) = \frac{1/x}{\sqrt{1-(1/x)^2}} \times -\frac{1}{x^2} = -\frac{1}{x^3\sqrt{1-\frac{1}{x^2}}}$

Answer to Exercise 5 (on page 14)

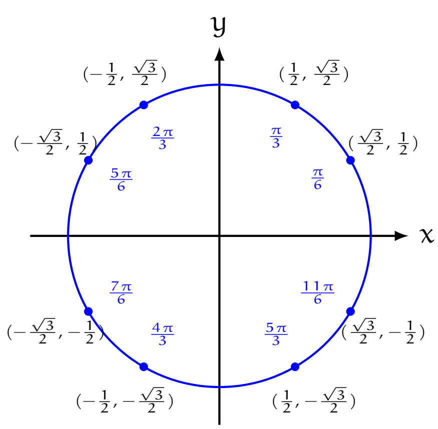
We know that for a right triangle, $\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$. For a right triangle inscribed in the Unit Circle, the adjacent leg is parallel to the x-axis and has the same length as the x-value of the coordinate point on the circle. Additionally, the length of the hypotenuse is 1. Therefore, $\cos\theta = \frac{x_o}{1} = x_o$.

Answer to Exercise 6 (on page 15)

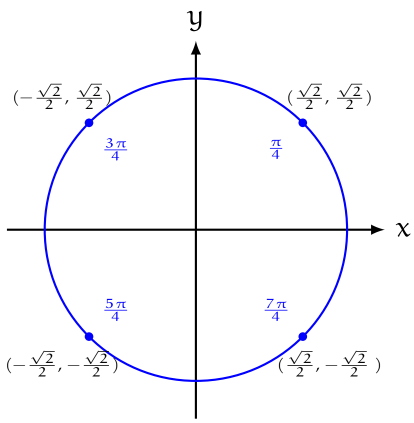
1. $\sin \frac{\pi}{2} = 1$
2. $\cos \frac{3\pi}{2} = 0$
3. $\sin \pi = 0$
4. $\cos -\pi = -1$ (Negative angles are measured clockwise from the x-axis, so $\theta = -\pi$ is at the same angle as $\theta = \pi$.)



Answer to Exercise 7 (on page 16)



Answer to Exercise 8 (on page 18)



Answer to Exercise 9 (on page 19)

1. 0
2. $\sqrt{2}/2$
3. $-1/2$
4. $-1/2$
5. $\sqrt{2}/2$
6. $1/2$
7. $\sqrt{2}/2$
8. -1
9. $-\sqrt{3}/2$
10. $1/2$

Answer to Exercise 10 (on page 21)

1. $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \left(\frac{\sqrt{3}}{3}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$
2. $\cos(7\pi/12) = \cos(4\pi/12 + 3\pi/12) = \cos(\pi/3 + \pi/4) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} =$

$\left(\frac{1}{2}\right) \cdot \left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$

3. $\tan(13\pi/12) = \frac{\sin(13\pi/12)}{\cos(13\pi/12)}$ First, we will find $\sin(13\pi/12)$: $\sin(13\pi/12) = \sin(3\pi/12 + 10\pi/12) = \sin(\pi/4 + 5\pi/6) = \sin\frac{\pi}{4}\cos\frac{5\pi}{6} + \cos\frac{\pi}{4}\sin\frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} = \frac{\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$. Next we find $\cos(13\pi/12) = \cos(\pi/4 + 5\pi/6) = \cos\frac{\pi}{4}\cos\frac{5\pi}{6} - \sin\frac{\pi}{4}\sin\frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{-\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-\sqrt{6}-\sqrt{2}}{4}$. And therefore $\tan(13\pi/12) = \frac{\sin 13\pi/12}{\cos 13\pi/12} = \frac{\sqrt{2}-\sqrt{6}}{4} \cdot \frac{4}{-\sqrt{6}-\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} \cdot \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{6-2\sqrt{12}+2}{6-2} = \frac{8-4\sqrt{3}}{4} = 2 - \sqrt{3}$

Answer to Exercise 11 (on page 22)

$\sin 2\theta = \sin\theta + \theta = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

Answer to Exercise 12 (on page 23)

Similar to $\cos(\alpha/2)$, we begin with the double angle formula for cosine, but another version:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

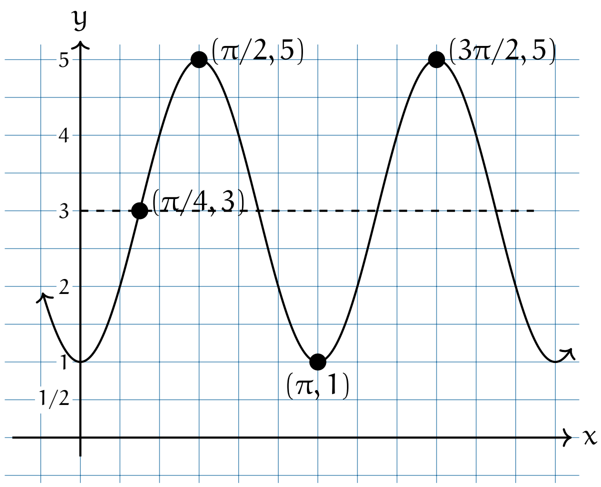
Substituting $\theta = \alpha/2$:

$$\cos\alpha = 1 - 2\sin^2(\alpha/2)$$

And rearranging to solve for $\sin(\alpha/2)$:

$$\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Answer to Exercise 13 (on page 30)



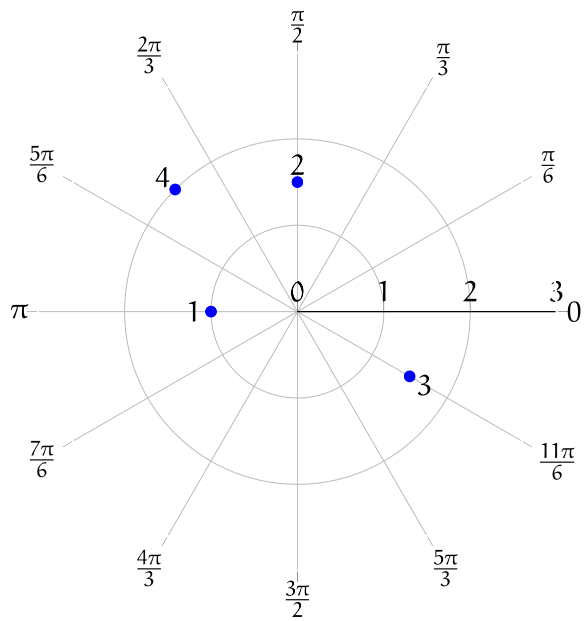
This wave has an amplitude of 2; its baseline has been translated up to 3.

This wave has wavelength of π. A sine wave usually has a wavelength of 2π, so we need to compress the x axis by a factor of 2.

The wave first crosses its baseline at $pi/4$. The sine wave starts by crossing its baseline, so we need to translate the curve right by $\pi/4$.

$$f(x) = 2\sin(2x - \frac{\pi}{4}) + 3$$

Answer to Exercise 14 (on page 34)



Answer to Exercise 15 (on page 39)

1. $(0, -2)$. $x = 2 \cdot \cos \frac{3\pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin \frac{3\pi}{2} = 2 \cdot -1 = -2$.
2. $(-1, 1)$. $x = \sqrt{2} \cdot \cos \frac{3\pi}{4} = \sqrt{2} \cdot -\frac{\sqrt{2}}{2} = \frac{2}{2} = -1$ and $y = \sqrt{2} \cdot \sin \frac{3\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{2} = 1$.
3. $(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2})$. $x = 3 \cdot \cos -\frac{\pi}{4} = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$ and $y = 3 \cdot \sin -\frac{\pi}{4} = 3 \cdot -\frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}$.
4. $(-\frac{3}{2}, -\frac{3\sqrt{3}}{2})$. $x = (-3) \cdot \cos \frac{\pi}{3} = (-3) \cdot \frac{1}{2} = -\frac{3}{2}$ and $y = (-3) \cdot \sin \frac{\pi}{3} = (-3) \cdot \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$.
5. $(0, -2)$. $x = 2 \cdot \cos -\frac{pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin -\frac{\pi}{2} = 2 \cdot -1 = -2$.

Answer to Exercise 16 (on page 40)

1. $(4\sqrt{2}, \frac{3\pi}{4})$. $r = \sqrt{x^2 + y^2} = \sqrt{32} = 4\sqrt{2}$. $\arctan \frac{y}{x} = \arctan \frac{4}{-4} = \arctan -1 = -\frac{\pi}{4} + n\pi$. We take $\theta = \frac{3\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.
2. $(6, \frac{\pi}{3})$. $r = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9 + 27} = \sqrt{36} = 6$. $\arctan \frac{3\sqrt{3}}{3} = \arctan \sqrt{3} = \frac{\pi}{3} + n\pi$. We take $\theta = \frac{\pi}{3}$ to satisfy the domain restriction and be in the correct quadrant.

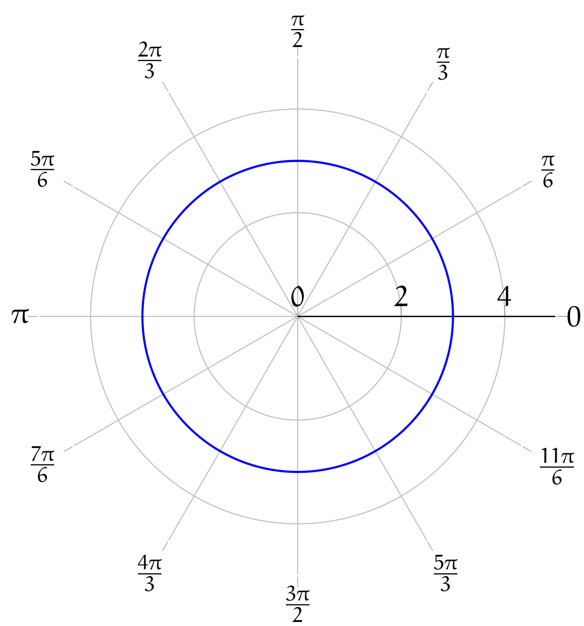
3. $(2, \frac{11\pi}{6})$. $r = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2$. $\arctan \frac{-1}{\sqrt{3}} = -\frac{\pi}{6} + n\pi$. We take $\theta = \frac{11\pi}{6}$ to satisfy the domain restriction and have the point in the correct quadrant.

4. $(6, \pi)$. $r = \sqrt{(-6)^2 + 0^2} = 6$. $\arctan \frac{0}{-6} = \pi + n\pi$. We take $\theta = \pi$ to satisfy the domain restriction.

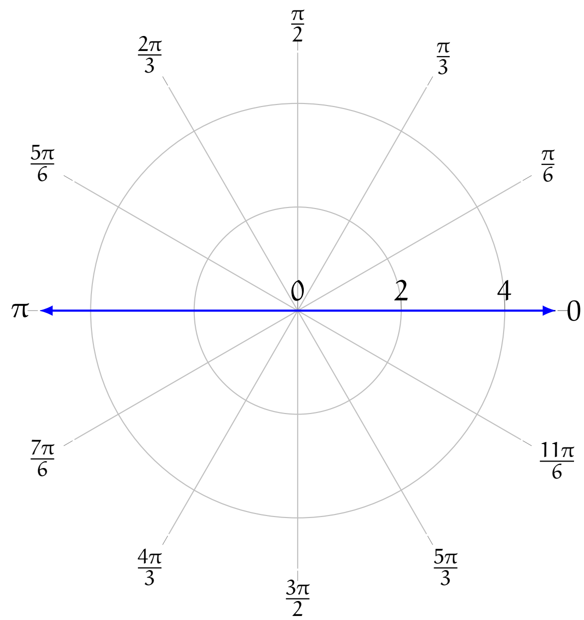
5. $(2\sqrt{2}, \frac{5\pi}{4})$. $r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$. $\arctan \frac{-2}{-2} = \arctan 1 = \frac{\pi}{4} + n\pi$. We take $\theta = \frac{5\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.

Answer to Exercise ?? (on page 44)

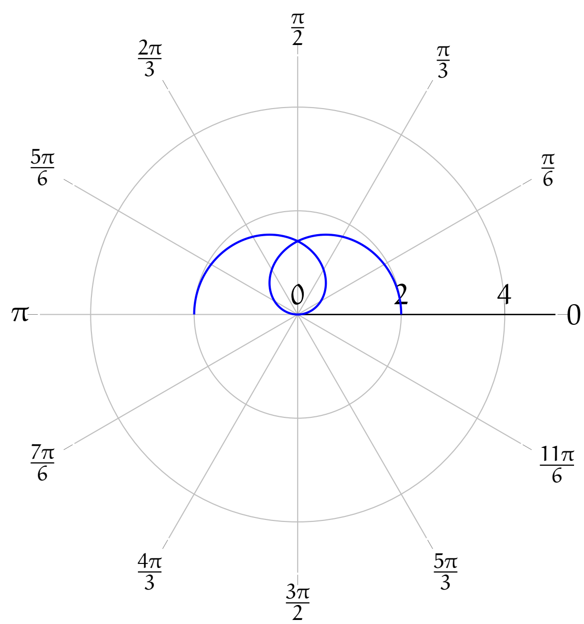
1. $r = 3$



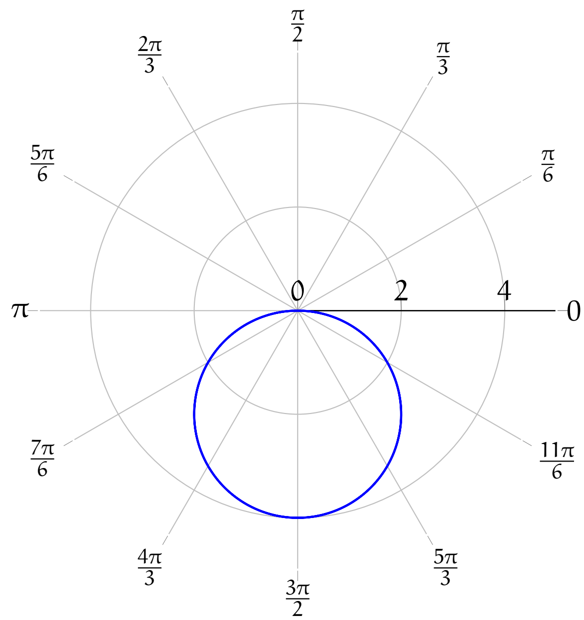
2. $\theta = \pi$ Because r includes all real numbers, negative r is possible and the line $\theta = \pi$ extends in both directions



3. $r = 2\cos\frac{\theta}{2}$



4. $r = -4\sin\theta$



5. $r = \theta$ (The spiral continues, but is beyond the boundary of the graph)