CONTENTS

Chapter 1

Trigometric Functions

As mentioned earlier, in a right triangle where one angle is θ, the sine of θ is the length of the side opposite θ divided by the length of the hypotenuse.

The sine function is defined for any real number. We treat that real number θ as an angle, we draw a ray from the origin out to the unit circle. The y value of that point is the sine. So, for example, the $\sin(\frac{4\pi}{3})$ is $-\sqrt{3}/2$ (see figure 1.1).

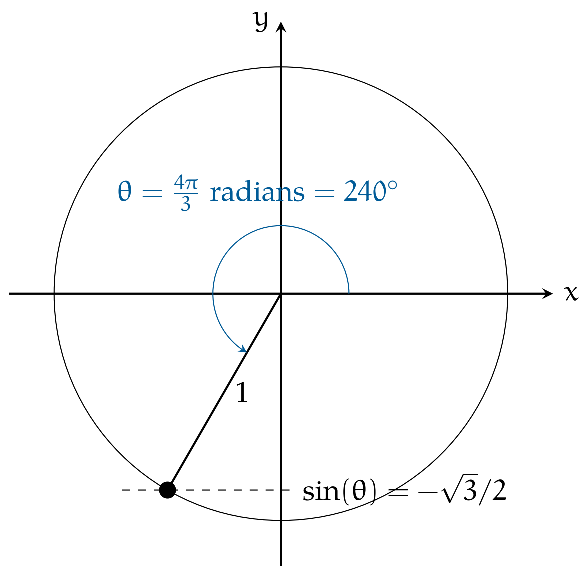


Figure 1.1: $\sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}$

(Note that in this section, we will be using radians instead of degrees unless otherwise noted. While degrees are more familiar to most people, engineers and mathematicians nearly always use radians when solving problems. Your calculator should have a radians mode and a degrees mode. You want to be in radians mode.)

Similarly, we define cosine using the unit circle: to find the cosine of θ, we draw a ray from the origin at the angle θ. The x component of the point where the ray intersects the unit circle is the cosine of θ (shown in figure 1.2).

From this description, it is easy to see why $\sin(\theta)^2 + \cos(\theta)^2 = 1$. They are the legs of a right triangle with a hypotenuse of length 1.

It should also be easy to see why $\sin(\theta) = \sin(\theta + 2\pi)$: Each time you go around the circle, you come back to where you started.

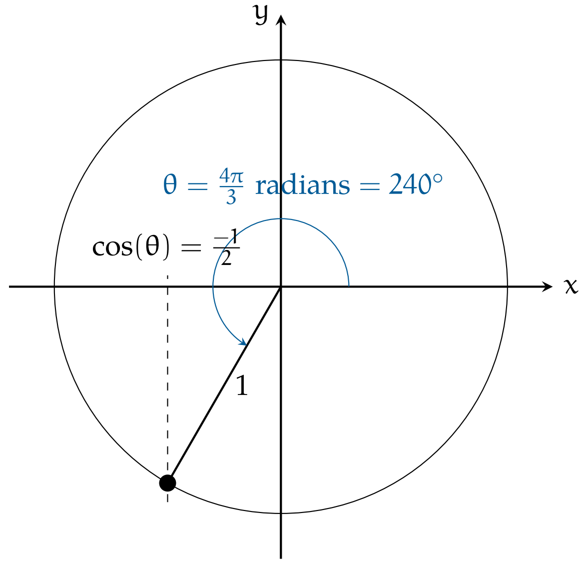
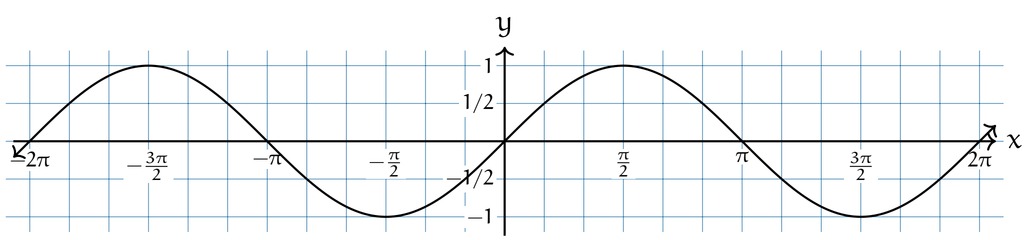


Figure 1.2: $\cos \frac{4\pi}{3} = \frac{-1}{2}$

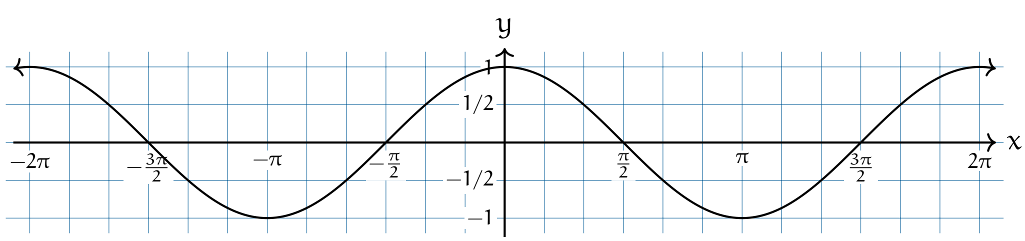
Can you see why $\cos(\theta) = \sin(\theta + \pi/2)$? Turn the picture sideways.

1.1 Graphs of sine and cosine

Here is a graph of $y = \sin(x)$:



It looks like waves, right? It goes forever to the left and right. Remembering that $\cos(\theta) = \sin(\theta + \pi/2)$, we can guess what the graph of $y = \cos(x)$ looks like:



1.2 Plot cosine in Python

Create a file called `cos.py`:

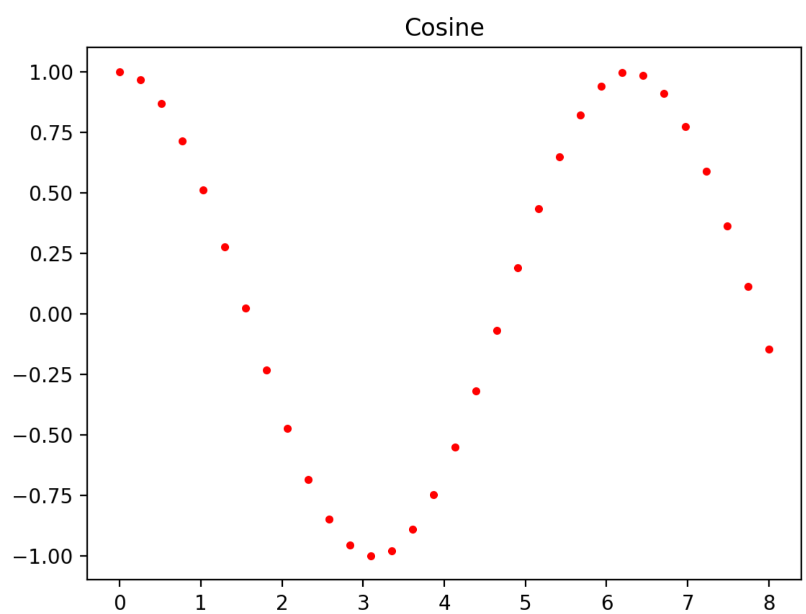
```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(thetas, cosines, 'r.', label="Cosine")
ax.set_title("Cosine")
plt.show()
```

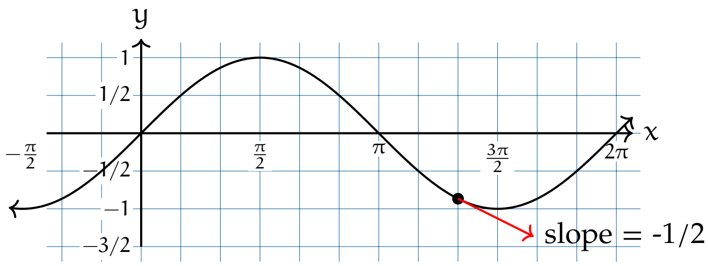
This will plot 32 points on the cosine wave between 0 and 8. When you run it, you should see something like this:



1.3 Derivatives of trigonometic functions

Here is a wonderful property of sine and cosine functions: At any point θ, the slope of the sine graph at θ equals $\cos(\theta)$.

For example, we know that $\sin(4\pi/3) = -(1/2)\sqrt{3}$ and $\cos(4\pi/3) = -1/2$. If we drew a line tangent to the sine curve at this point, it would have a slope of -1/2:



We say "The derivative of the sine function is the cosine function."

Can you guess the derivative of the cosine function? For any θ, the slope of the graph of the $\cos(\theta)$ is $-\sin(\theta)$.

Exercise 1 Derivatives of Trig Functions Practice 1

Use the limit definition of a derivative to show that $\frac{d}{dx}\cos x = -\sin x$

Working Space

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The derivatives of all the trigonometric functions are presented below:

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\sec x \sec x \cdot \tan x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$

Example: Find the derivative of $f(x)$ if $f(x) = x^2 \sin x$ **Solution**: Using the product rule, we find that:

$$\frac{d}{dx}f(x) = (x^2)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)$$

Taking the derivatives:

$$= x^2(\cos x) + 2x(\sin x)$$

Exercise 2 Derivatives of Trig Functions 2

Find the derivative of the following functions:

1. $f(x) = \frac{\sec x}{1+\tan x}$
2. $y = \sec t \tan t$
3. $f(\theta) = \frac{\theta}{4-\tan\theta}$
4. $f(t) = 2\sec t - \csc t$
5. $f(\theta) = \frac{\sin\theta}{1+\cos\theta}$
6. $f(x) = \sin x \cos x$

Working Space

Answer on Page 22

1.4 A weight on a spring

Let's say you fill a rollerskate with heavy rocks and attach it to the wall with a stiff spring. If you push the skate toward the wall a release it, it will roll back and forth. Engineers would say "The skate will oscillate."

Intuitively, you can probably guess:

- If the spring is stronger, the skate will oscillate more times per minute.
- If the rocks are lighter, the skate will oscillate more times per minute.

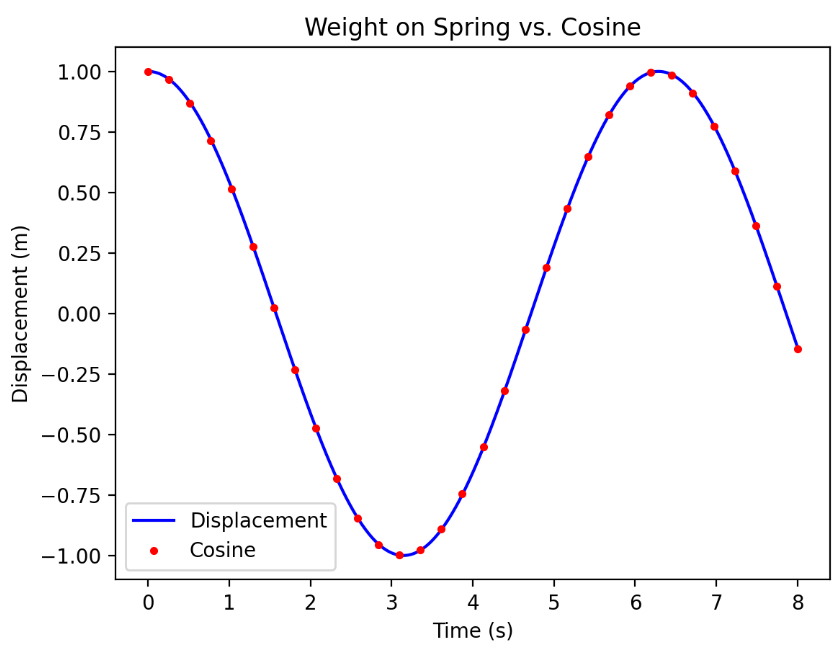
The force that the spring exerts on the skate is proportional to how far its length is from its relaxed length. When you buy a spring, the manufacturer advertises its "spring rate", which is in pounds per inch or newtons per meter. If a spring has a rate of 5 newtons per meter, which means that if stretch or compress it 10 cm, it will push back with a force of 0.5 newtons. If you stretch or compress it 20 cm, it will push back with a force of 1 newton.

Let's write a simulation of the skate-on-a-spring. Duplicate `cos.py`, and name the new copy `spring.py`. Add code to implement the simulation:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Constants
mass = 100 # kg
spring_constant = -1 # newtons per meter displacement
time_step = 0.01 # s

# Initial state
displacement = 1.0 # height above equilibrium in meters
velocity = 0.0
time = 0.0 # seconds

# Lists to gather data
displacements = []
times = []

# Run it for a little while
while time <= until:
    # Record data
    displacements.append(displacement)
    times.append(time)

    # Calculate the next state
    time += time_step
    displacement += time_step * velocity
    force = spring_constant * displacement
    acceleration = force / mass
    velocity += acceleration

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(times, displacements, 'b', label="Displacement")
ax.plot(thetas, cosines, 'r.', label="Cosine")
```

```
ax.set_title("Weight on Spring vs. Cosine")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Displacement (m)")
ax.legend()
plt.show()
```

When you run it, you should get a plot of your spring and the cosine graph on the same plot.



The position of the skate is following a cosine curve. Why?

Because a sine or cosine waves happen whenever the acceleration of an object is proportional to -1 times its displacement. Or in symbols:

$$a \propto -p$$

where a is acceleration and p is the displacement from equilibrum.

Remember that if you take the derivative of the displacement, you get the velocity. And if you take the derivative of that, you get acceleration. So, the weight on the spring must follow a function f such that

$$f(t) \propto -f''(t)$$

Remember that the derivative of the $\sin(\theta)$ is $\cos(\theta)$.

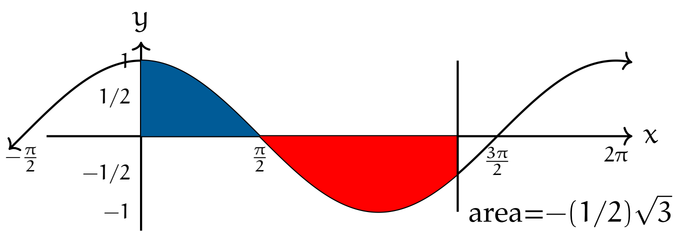
And the derivative of the $\cos(\theta)$ is $-\sin(\theta)$

Thus these sorts of waves have an almost-magical power: their acceleration is proportional to -1 times their displacement.

Thus sine waves of various magnitudes and frequencies are ubiquitous in nature and technology.

1.5 Integral of sine and cosine

If we take the area between the graph and the x axis of the cosine function (and if the function is below the x axis, it counts as negative area), from 0 to $4\pi/3$, we find that it is equal to $-(1/2)\sqrt{3}$



We say "The integral of the cosine function is the sine function."

1.5.1 Integrals of Trig Functions Practice

Exercise 3

Evaluate the following integrals:

1. $\int \sec x \tan x \, dx$

Working Space

Answer on Page 22

Chapter 2

Inverse Trigonometric Functions

Recall from the chapter on Functions that an inverse of a function is a machine that turns y back into x. The inverses of trigonometric functions are essential to solving certain integrals (you will learn in a future chapter why integrals are useful - for now, trust us that they are!). Let's begin by discussing the sin function and its inverse, $\sin^{-1}$, also called arcsin.

Examine the graph of $\sin x$ in figure 2.1. See how the dashed horizontal line crosses the function at many points? This means the function $\sin x$ is not one-to-one. That is: there is not a unique x-value for every y-value. This means that if we do not restrict the domain of $\arcsin x$, the result will not be a function (see figure 2.2). In figure 2.2, you can see that just reflecting the graph across $y = x$ fails the vertical line test: an x value has more than one y value.

2.1 Derivatives of Inverse Trigonometric Functions

f	f'
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\text{arccsc } x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\text{arcsec } x$	$\frac{1}{x\sqrt{x^2-1}}$
$\text{arccot } x$	$-\frac{1}{1+x^2}$

2.2 Practice

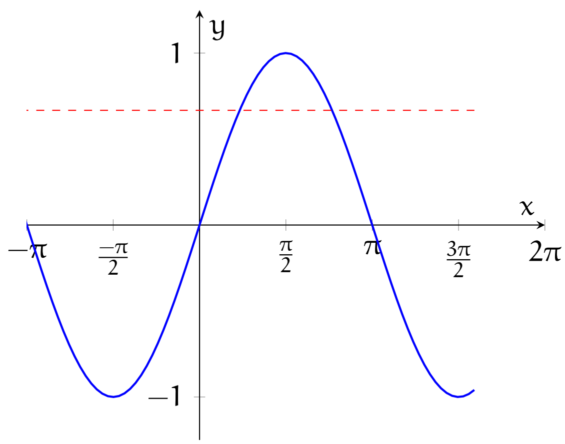


Figure 2.1: The horizontal line $y = \frac{2}{3}$ crosses $y = \sin x$ more than once

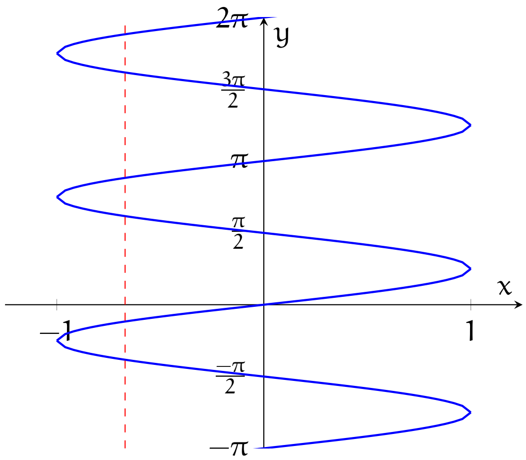


Figure 2.2: The inverse of an unrestricted sin function fails the vertical line test

Exercise 4

Find the f'. Give your answer in a simplified form.

1. $f(x) = \arctan x^2$
2. $f(x) = x\operatorname{arcsec}(x^3)$
3. $f(x) = \arcsin \frac{1}{x}$

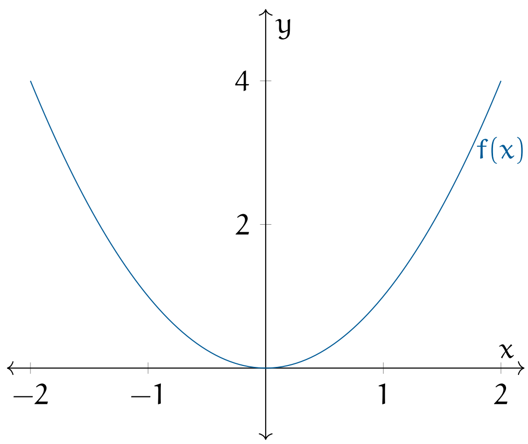
Working Space

Answer on Page 22

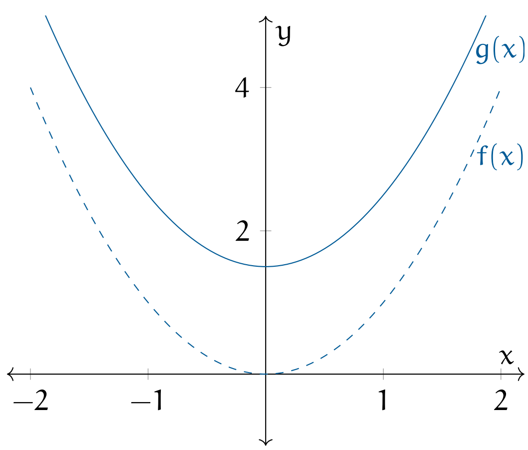
Chapter 3

Transforming Functions

Let's say I gave you the graph of a function f, like this:



And then I tell you that the function $g(x) = f(x) + 1.5$. Can you guess what the graph of g would look like? It is the same graph, just translated up 1.5:



There are four kinds of transformations that we do all the time:

- Translation up and down in the direction of y axis (the one you just saw)
- Translation left and right in the direction of the x axis
- Scaling up and down along the y axis

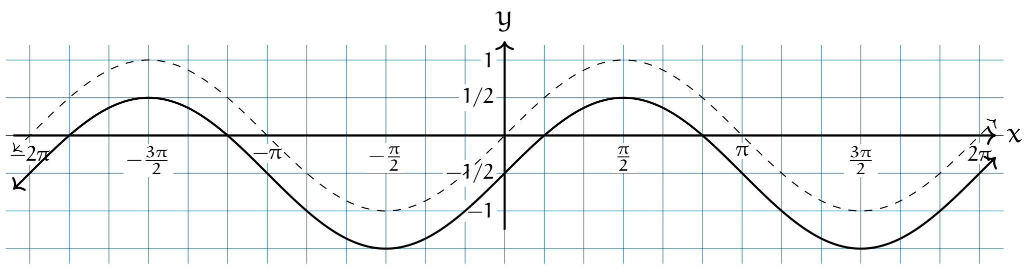
- Scaling up and down along the x axis

Now I will demonstrate each of the four using the graph of $\sin(x)$.

3.1 Translation up and down

When you add a positive constant to a function, you translate the whole graph up that much. A negative constant translates it down.

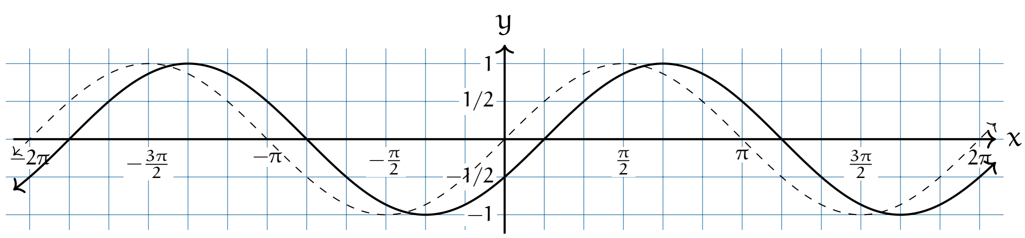
Here is the graph of $\sin(x) - 0.5$:



3.2 Translation left and right

When you add a positive number to x before running it through f, you translate the graph to the left that much. Adding a negative number translates the graph to the right.

Here is the graph of $\sin(x - \pi/6)$:



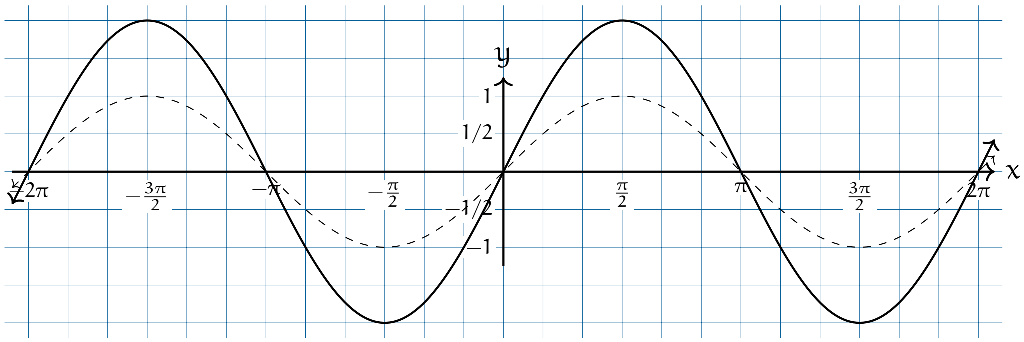
Notice the sign:

- Add to x before processing with the function translates the graph to the *left*.
- Subtract from x before processing with the function translates the graph to the *right*

3.3 Scaling up and down in the y direction

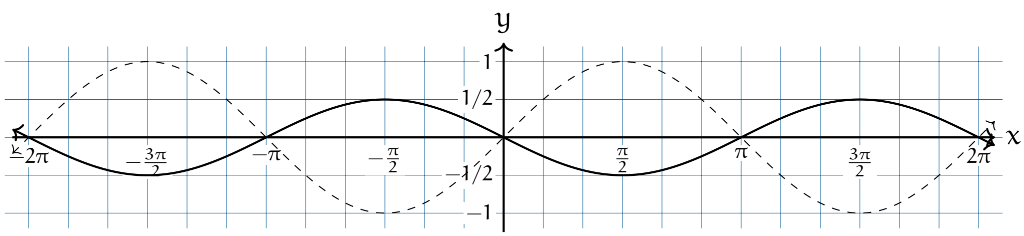
To scale the function up and down, you multiply the result of the function by a constant. If the constant is larger than 1, it stretches the function up and down.

Here is $y = 2\sin(x)$:



With a wave like this, we speak of its *Amplitude*, which you can think of as its height. The baseline that this wave oscillates around is zero. The maximum distance that it gets from that baseline is its amplitude. Thus, the amplitude here has been increased from 1 to 2.

If you multiply by a negative number, the function gets flipped. Here is $y = -0.5\sin(x)$

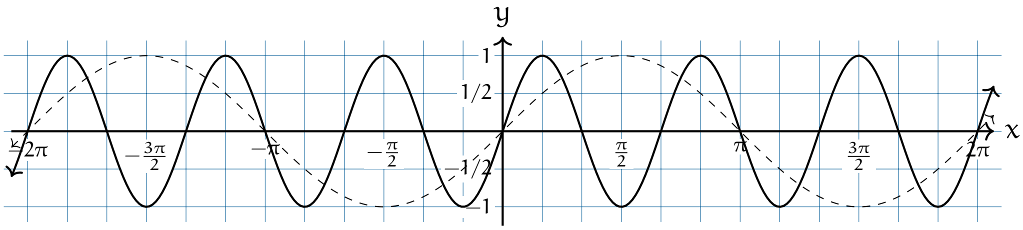


Amplitude is never negative. Thus, the amplitude of this wave is 0.5.

3.4 Scaling up and down in the x direction

If you multiply x by a number larger than 1 before running it through the function, the graph gets compressed toward zero.

Here is $y = \sin(3x)$:

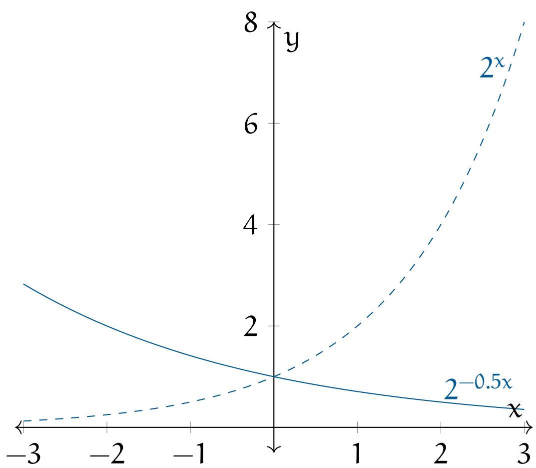


The distance between two peaks of a wave is known as its *wavelength*. The original wave had a wavelength of 2π. The compressed wave has a wavelength of $2\pi/3$.

If you multiply x by a number smaller than 1, it will stretch the function out, away from the y axis.

If you multiply x by a negative number, it will flip the function around the y axis.

Here is $y = 2^{(-0.5x)}$. Notice that it has flipped around the y axis and is stretched out along the x axis.

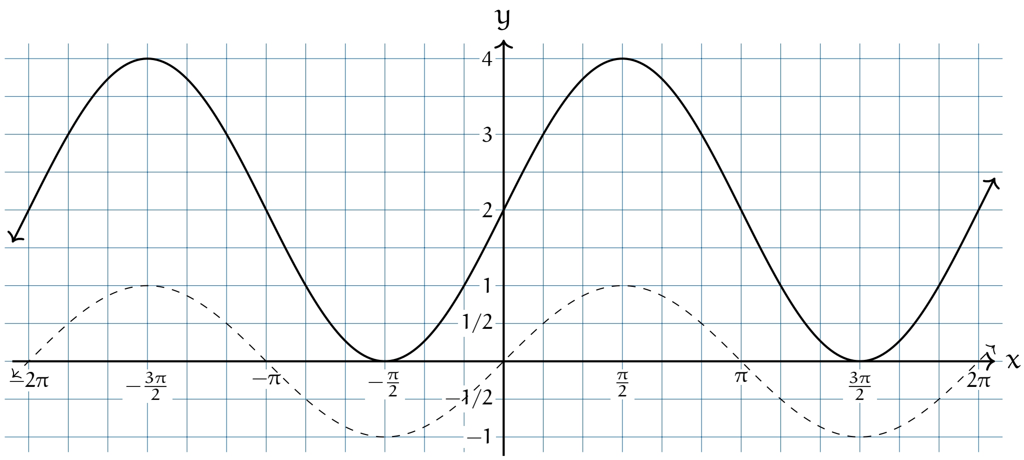


Reflection over *x*-axis	$(x, y) \to (x, -y)$
Reflection over *y*-axis	$(x, y) \to (-x, y)$
Translation	$(x, y) \to (x + a, y + b)$
Dilation	$(x, y) \to (kx, ky)$
Rotation 90° counterclockwise	$(x, y) \to (-y, x)$
Rotation 180°	$(x, y) \to (-x, -y)$

3.5 Order is important!

We can combine these transformations. This allows us, for example, to translate a function up 2 and then scale along the y axis by 3.

Here is $y = 2.0(\sin(x) + 1)$:

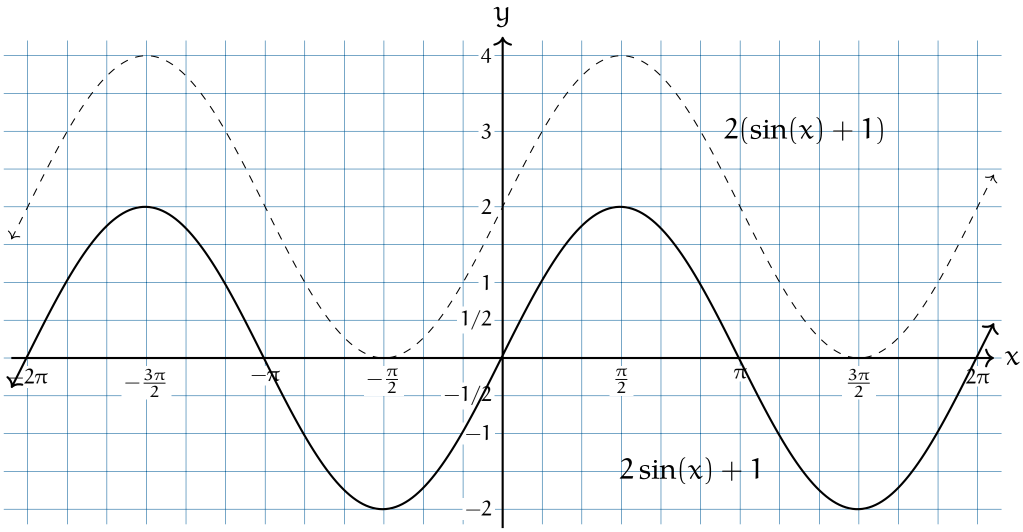


A function is often a series of steps. Here are the steps in $f(x) = 2(\sin(x) + 1)$:

1. Take the sine of x
2. Add 1 to that
3. Multiply that by 2

What if we change the order? Here are the steps in $g(x) = 2\sin(x) + 1$:

1. Take the sine of x
2. Multiply that by 2
3. Add 1 to that

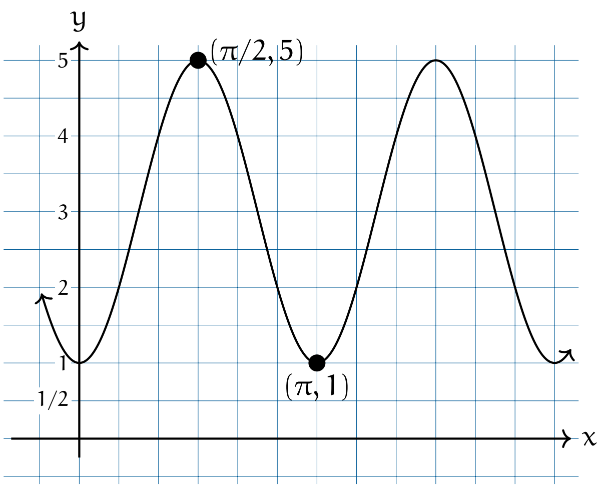


The moral: You can do multiple transformations of your function, but the order in which you do them is important.

Exercise 5 Transforms

Working Space

Find a function that creates a sine wave such that the top of the first crest is at the point $(\frac{\pi}{2}, 5)$ and the bottom of the trough that follows is at $(\pi, 1)$.



Answer on Page 23

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

We start by writing out the limit:

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos x + h - \cos x}{h}$$

Applying the sum formula for $\cos(x+h)$, we get:

$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

Rearranging to group the $\cos x$ and applying the Difference Rule:

$$= \lim_{h\to 0}\frac{\cos x\cos h - \cos x}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h}$$

Applying the Constant Multiple Rule:

$$= \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h}$$

Recalling that $\lim_{h\to 0}\frac{\sin h}{h} = 1$,

$$= \cos x\lim_{h\to a}\frac{\cos h - 1}{h} - \sin x\cdot 1$$

Recalling that $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$:

$$= \cos x\cdot 0 - \sin x = -\sin x$$

Therefore, $\frac{d}{dx}\cos x = -\sin x$

Answer to Exercise 2 (on page 7)

1. $\frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$
2. $\sec t[\sec^2 t + \tan^2 t]$
3. $\frac{4 - \tan\theta + \theta\sec^2\theta}{(4-\tan\theta)^2}$
4. $2\sec t\tan t + \csc t\cot t$
5. $\frac{2}{(1+\cos\theta)^2}$
6. $\cos^2 x - \sin^2 x$

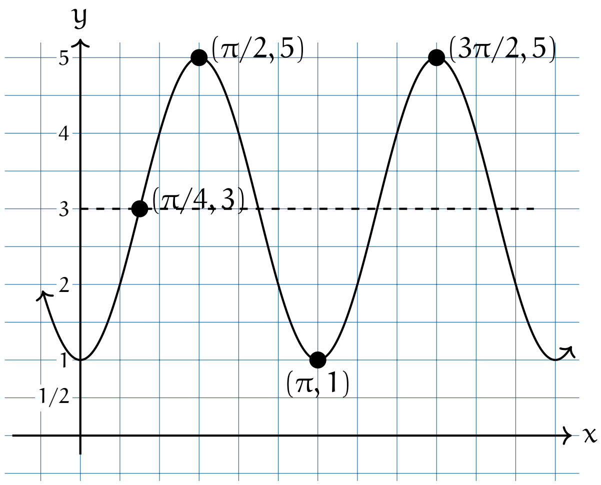
Answer to Exercise 3 (on page 10)

1. $\sec x + C$

Answer to Exercise 4 (on page 13)

1. By the chain rule, $f'(x) = 2\arctan x \times \frac{d}{dx}\arctan x = 2\arctan x\frac{1}{1+x^2}$
2. By the Product rule, $f'(x) = x\frac{d}{dx}\text{arcsec}\,(x^3) + \text{arcsec}\,(x^3)$. Further, by the chain rule, $\frac{d}{dx}\text{arcsec}\,(x^3) = \frac{1}{(x^3)\sqrt{(x^3)^2-1}} \times \frac{d}{dx}(x^3) = \frac{3x^2}{x^3\sqrt{x^6-1}}$. Therefore, $f'(x) = \frac{3}{\sqrt{x^6-1}} + \text{arcsec}\,(x^3)$
3. By the chain rule, $f'(x) = \frac{1/x}{\sqrt{1-(1/x)^2}} \times -\frac{1}{x^2} = -\frac{1}{x^3\sqrt{1-\frac{1}{x^2}}}$

Answer to Exercise 5 (on page 20)



This wave has an amplitude of 2. Its baseline has been translated up to 3.

This wave has wavelength of π. A sine wave usually has a wavelength of 2π, so we need to compress the x axis by a factor of 2.

The wave first crosses its baseline at $pi/4$. The sine wave starts by crossing its baseline, so we need to translate the curve right by $\pi/4$.

$$f(x) = 2\sin(2x - \frac{\pi}{4}) + 3$$