Chapter 1

# Double Integrals Over Non-Rectangular Regions

Now that we've seen how to evaluate double integrals over rectangular regions, let's consider non-rectangular regions. Suppose we are interested in the integral of a function, $f(x, y)$, over a region, $D$, exists such that it can be bounded by inside a rectangular region, $R$ (see figure 1.1). We can then define a new function:

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

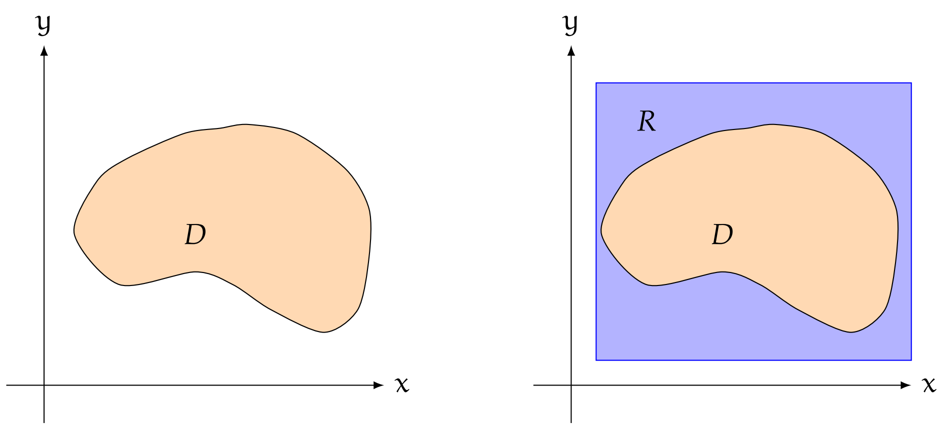



Figure 1.1: We can find a rectangular region, $R$, that completely encloses $D$

Then, we can see that:

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA$$

Which makes sense intuitively, since integrating over $F$ outside of $D$ doesn't contribute anything to the integral, and the integral of $F$ inside $D$ is equal to the integral of $f$ inside $D$. In general, there are two types of regions for $D$. A region is **type I** if it lies between two continuous functions of $x$ and can be defined thusly:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

Some type I regions are shown in figure 1.2. To evaluate $\iint_D f(x, y)\, dA$, we begin by choosing a rectangle $R = [a, b] \times [c, d]$ such that $D$ is completely contained in $R$. We again

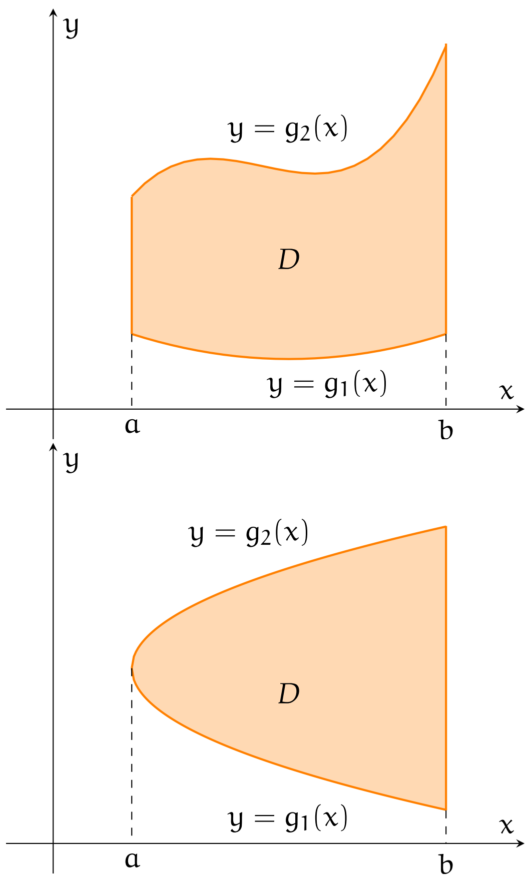



Figure 1.2: Two examples of type I domains

define $F(x, y)$ such that $F(x, y) = f(x, y)$ on $D$ and $F = 0$ outside of $D$. Then, by Fubini's theorem:

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Since $F(x, y) = 0$ when $y \le g_1(x)$ or $y \ge g_2(x)$, we know that:

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

Substituting this into the iterated integral above, we see that for a type I region $D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Another way to visualize the double integral over a type I region is shown in figure 1.3. For any value of $x \in [a, b]$, we know that $g_1(x) \le y \le g_2(x)$. The inner integral represents moving along one blue line from $y = g_1(x)$ to $y = g_2(x)$ and integrating with respect to $y$. Then, for the outer integral, we integrate with respect to $x$, which is represented by moving the line from $x = a$ to $x = b$.

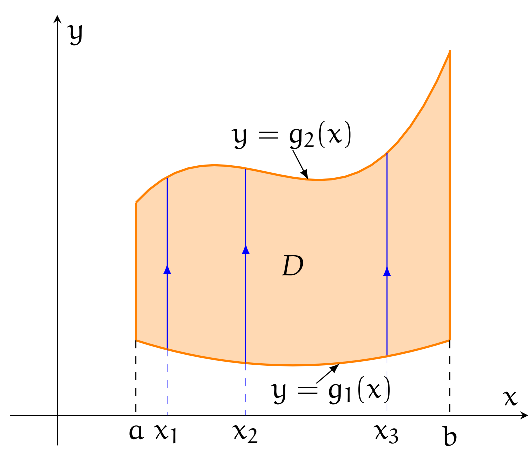



Figure 1.3: On type I domains, for a given value of $x$, $g_1(x) \le y \le g_2(x)$

A **type II** region is a region such that we can define the limits of $x$ in terms of $y$ (see figure 1.4). That is, a type II region can be defined as:

$$D = \{(x, y) \mid c \le y \le d, h_1(y0 \le x \le h_2(y)\}$$

And in a similar manner to above, we can show that:

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

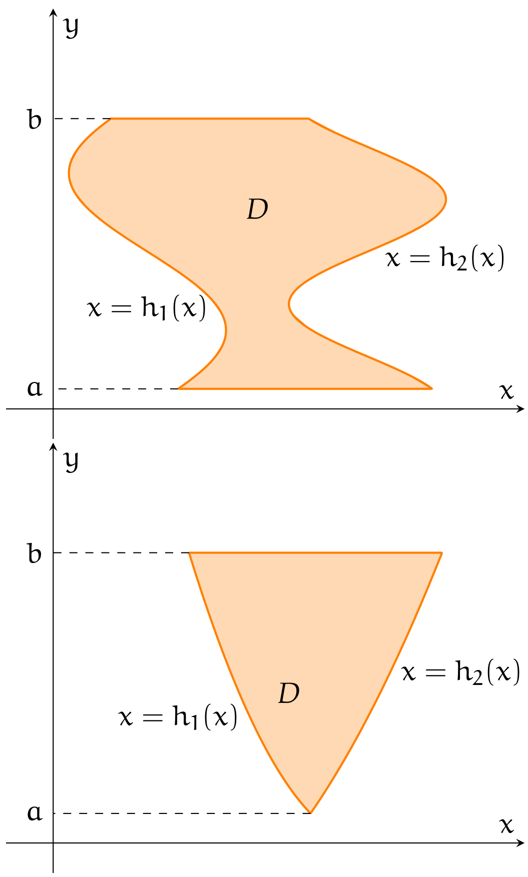



Figure 1.4: Two examples of type II domains

You can annotate type II regions with horizontal lines to show that, for a given $y$ values, all $x$ values in the region are contained in $h_1(y) \leq x \leq h_2(y)$ (see figure 1.5).

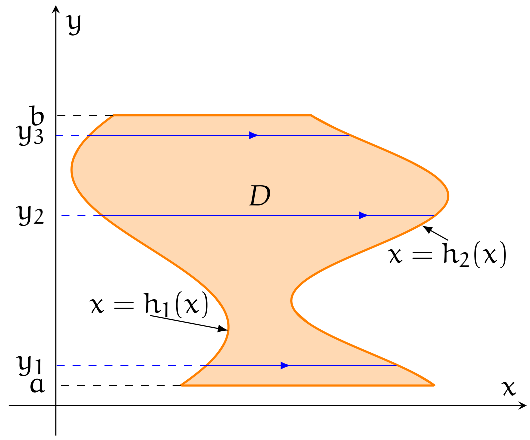



Figure 1.5: On type II domains, for a given value of $y$, $h_1(y) \leq x \leq h_2(y)$

## 1.1 Determining Region Type

Many regions can be described as either type I or type II. Consider the region between the curves $y = \frac{3}{2}(x-1)$ and $y = \frac{1}{2}(x-1)^2$ (see figure 1.6).[fix me classifying domains examples and explanations]

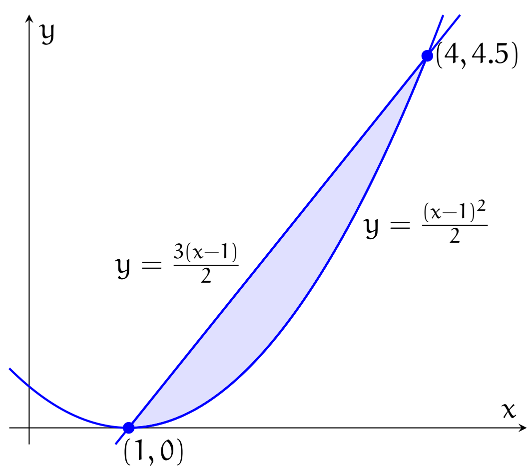



Figure 1.6: The region that lies between $y = \frac{(x-1)^2}{2}$ and $y = \frac{3(x-1)}{2}$ can be classified as type I or type II

**Example**: Evaluate $\iint_D (2x + y)\, dA$, where $D$ is the region bounded by the parabolas $y = 3x^2$ and $y = 2 + x^2$. Region $D$ is shown in figure 1.7.

**Solution**:This is a type I region, since for a given $x$, $y \in [3x^2, 2 + x^2]$. We can define region

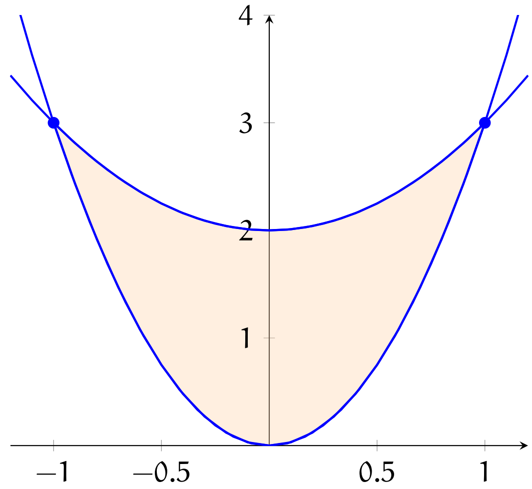



Figure 1.7: Region $D$ is bounded above by $y = 2 + x^2$ and below by $y = 3x^2$

$D$ as $D = \{(x, y) \mid -1 \le x \le 1, 3x^2 \le y \le 2 + x^2\}$. Therefore,

$$\iint_D (2x + y)\, dA = \int_{-1}^{1} \int_{3x^2}^{2+x^2} (2x + y)\ dy\, dx$$

$$= \int_{-1}^{1} \left[ \int_{3x^2}^{2+x^2} 2x\ dy + \int_{3x^2}^{2+x^2} y\ dy \right]\ dx$$

$$= \int_{-1}^{1} \left[ 2xy\Big|_{y=3x^2}^{y=2+x^2} + \frac{1}{2} y^2\Big|_{y=3x^2}^{y=2+x^2} \right]\ dx$$

$$= \int_{-1}^{1} \left[ 2x\left(2 + x^2 - 3x^2\right) + \frac{1}{2}\left((2 + x^2)^2 - (3x^2)^2\right) \right]\ dx$$

$$= \int_{-1}^{1} \left[ 2 + 4x + 2x^2 - 4x^3 - 4x^4 \right]\ dx$$

$$= \left[ 2x + 2x^2 + \frac{2}{3}x^3 - x^4 - \frac{4}{5}x^5 \right]_{x=-1}^{x=1}$$

$$= \left( 2 + 2 + \frac{2}{3} - 1 - \frac{4}{5} \right) - \left( -2 + 2 - \frac{2}{3} - 1 + \frac{4}{5} \right)$$

$$= 4 + \frac{4}{3} - \frac{8}{5} = \frac{56}{15}$$

### *Exercise 1* Double Integrals over Non-Rectangular Regions

*Working Space*

Evaluate the double integral.

1. $\iint_D e^{-y^2}\, dA$, $D = \{(x, y) \mid 0 \leq y \leq 3, 0 \leq x \leq 2y\}$.
2. $\iint_D x \sin y \, dA$, $D$ is bounded by $y = 0$, $y = x^2$, $x = 2$.
3. $\iint_D (2y - x)\ dA$, $D$ is bounded by the circle with center at the origin and radius 3.

*Answer on Page 13*

## 1.2 Double Integrals in Other Coordinate Systems

Consider a region composed of a semi-circular ring (see figure **??**). Describing the region in Cartesian coordinates is complicated: you would have to split it into 3 regions (see figure ...). However, in polar coordinates, we can describe the whole region in one statement:

$$D = \{(r, \theta) \mid 1 \leq r \leq 4,\ 0 \leq \theta \leq \pi\}$$

There are many instances where a region is simpler to describe in polar coordinates, so

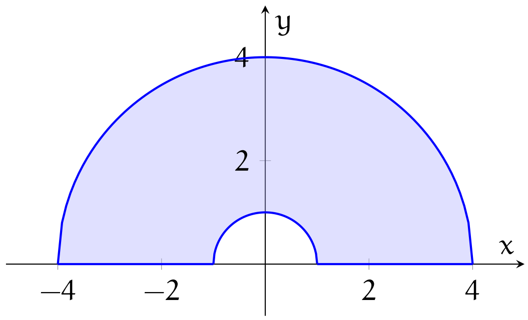



Figure 1.8: A semi-circular ring

how do we take double integrals in polar coordinates? Suppose we want to integrate some function, $f(x, y)$, over a polar rectangle described by $D = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$ (see figure 1.9). Similar to Cartesian coordinates, we can divide this region in to many smaller polar rectangles, with each subrectangle defined by $D_{ij} = \{(r, \theta) \mid r_{i-1} \le r \le r_i,\ \theta_{i-1} \le \theta \le \theta_i\}$. And the center of each subrectangle has polar coordinates $(r_i^*, \theta_j^*)$, where:

$$r_i^* - \frac{1}{2}\left(r_{i-1} + r_i\right)$$

$$\theta_j^* = \frac{1}{2}\left(\theta_{j-1} + \theta_j\right)$$

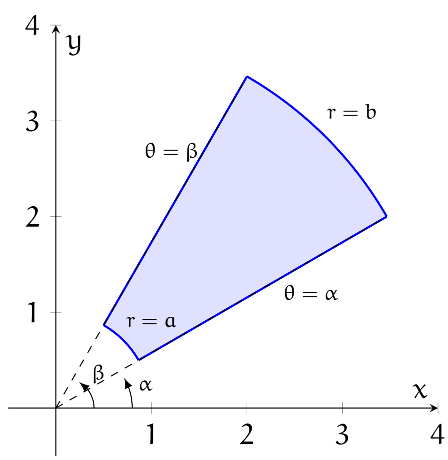



Figure 1.9: A polar rectangle described by $D = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$

Each subrectangle is a larger radius sector minus a smaller radius sector, each with the same central angle, $\Delta\theta = \theta_j - \theta_{j-1}$. Then the total area of each subrectangle is given by:

$$\Delta A_i = \frac{1}{2}\left(r_i\right)^2 \delta\theta - \frac{1}{2}\left(r_{i-1}\right)^2 \Delta\theta = \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta$$

Substituting $\left(r_i^2 - r_{i-1}^2\right) = \left(r_i + r_{i-1}\right)\left(r_i - r_{i-1}\right)$, we see that:

$$\Delta A_i = \frac{1}{2}\left(r_i + r_{i-1}\right)\left(r_i - r_{i-1}\right)\Delta\theta$$

Recall that we have defined $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$. Additionally, $\Delta r = r_i - r_{i-1}$. Substituting this, we find a simplified expression for the area of each subrectangle:

$$\Delta A_i = r_i^* \Delta r \Delta \theta$$

And therefore the Riemann sum of $f(x, y)$ over the region is:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) \Delta A_i$$

(Recall that to convert from Cartesian to polar coordinates, we use $x = r\cos\theta$ and $y = r\sin\theta$). Substituting for $\delta A_i$:

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) r_i^* \Delta r \Delta \theta$$

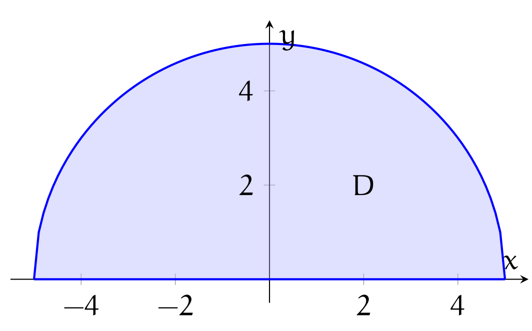
Taking the limit as $n \to \infty$, the Riemann sum becomes the double integral:

$$\int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

And therefore, if $f$ is continuous on the polar rectangle $a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, then:

$$\iint_D f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

**Example**: Evaluate $\iint_D x^2 y \, dA$, where $D$ is the semi-circle shown below.




**Solution**: Since the region is a semi-circle with radius 5, we can describe $D$ as $D = \{(r, \theta) \mid 0 \leq r \leq 5, \; 0 \leq \theta \leq \pi\}$. Therefore,

$$\iint_D x^2 y \, dA = \int_0^{\pi} \int_0^5 (r\cos\theta)^2 (r\sin\theta) \, r \, d\theta \, dr$$

$$= \int_0^{\pi} \int_0^{5} r^4 \cos^2\theta \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \cos^2\theta \sin\theta \left[\frac{1}{5} r^5\right]_{r=0}^{r=5} d\theta$$

$$= \int_0^{\pi} \cos^2\theta \sin\theta \frac{5^5}{5} \, d\theta = 625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

Using $u$-substitution, let $u = \cos\theta$. Then $-du = \sin\theta d\theta$ and therefore:

$$625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = 625 \int_{\theta=0}^{\theta=\pi} -u^2 \, du$$

$$= -625 \frac{1}{3} u^3 |_{\theta=0}^{\theta=\pi} = -625 \frac{1}{3} \left(\cos^3\theta\right) |_{\theta=0}^{\theta=\pi}$$

$$= -\frac{625}{3} \left[(-1)^3 - (1)^3\right] = -\frac{625}{3} (-2) = \frac{1250}{3}$$

## *Exercise 2* Changing to Polar Coordinates

Evaluate the following iterated integrals by converting to polar coordinates:

1. $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$
2. $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy$
3. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

*Working Space*

*Answer on Page 13*

## *Exercise 3* Using Polar Coordinates in Multiple Integration

*Working Space*

Find the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the $xy$-plane.

*Answer on Page 15*

## *Exercise 4* The volume of a pool

A circular swimming pool has a 40-ft diameter. The depth of the pool is constant along the north-south axis and increases from 3 feet at the west end to 10 feet at the east end. What is the total volume of water in the pool?

*Working Space*

*Answer on Page 16*

*This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.*

Appendix A

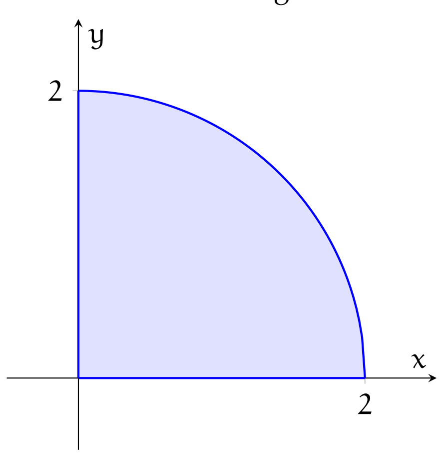
# Answers to Exercises

## Answer to Exercise 1 (on page 7)

1. $\iint_D e^{-y^2}\, dA = \int_0^3 \int_0^{2y} e^{-y^2}\, dx\, dy = \int_0^3 \left[ e^{-y^2} x|_{x=0}^{x=2y} \right] dy = \int_0^3 2ye^{-y^2}\, dy = -e^{-y^2}|_{y=0}^{y=3} = 1 - e^{-9} \approx 0.9999$
2. $\iint_D x \sin y\, dA = \int_0^2 \int_0^{x^2} x \sin y\, dy\, dx = \int_0^2 x \int_0^{x^2} \sin y\, dy\, dx = \int_0^2 x\, [-\cos y]_{y=0}^{y=x^2}$
$= \int_0^2 x \left(\cos 0 - \cos x^2\right) dx = \int_0^2 \left(x - x\cos x^2\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{2}\sin x^2\right]_{x=0}^{x=2} = \frac{1}{2}(2)^2 - \frac{1}{2}\left(\sin 2^2 - \sin 0\right)$
$= 2 - \frac{1}{2}(\sin 4 - 0) = 2 - \frac{\sin 4}{2} \approx 2.378$
3. We can describe the region as $D = \{(x,y) \mid -3 \le x \le -3, -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}\}$. Therefore, $\iint_D (2y - x)\, dA = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (2x - y)\, dy\, dx = \int_{-3}^{3} \left[2xy - \frac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$
$= \int_{-3}^{3} \left[2x\left(\sqrt{9-x^2} + \sqrt{9-x^2}\right) - \frac{1}{2}\left(9 - x^2 - (9-x^2)\right)\right] dx = \int_{-3}^{3} 4x\sqrt{9-x^2}\, dx$. Let $u = 9 - x^2$, then $du = -2x$ and $4x = -2du$. Substituting, $\int_{-3}^{3} 4x\sqrt{9-x^2}\, dx = \int_{x=-3}^{x=3} -2\sqrt{u}\, du = -2 \cdot \frac{2}{3}u^{3/2}|_{x=-3}^{x=3} = -\frac{4}{3}\left[(9-x^2)\right]_{x=-3}^{x=3} = 0$

## Answer to Exercise 2 (on page 10)

1. Let's visualize the region in the $xy$-plane:




The region is a quarter-circle that can be described with $D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le$

$\theta \le \pi/2\}$. Then we can re-write the integral in polar coordinates:

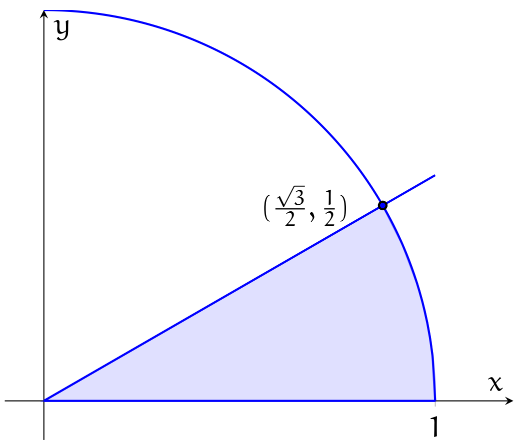
$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^2 re^{-r^2}\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=2} d\theta = \int_0^{\pi/2} \left(-\frac{1}{2}\right)\left[e^{-4}-1\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - e^{-4}\, d\theta = \frac{1}{2}\left(1-\frac{1}{e^4}\right)\int_0^{\pi/2} 1\, d\theta$$

$$= \frac{1}{2}\left(1-\frac{1}{e^4}\right)\theta\Big|_{\theta=0}^{\theta=\pi/2} = \frac{\pi}{4}\left(1-\frac{1}{e^4}\right)$$

2. The region is bounded by the $x$-axis, the line $y = x/\sqrt{3}$, and the circle $x^2+y^2=1$:




We see that the region defined in polar coordinates is $D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi/6\}$ . And therefore:

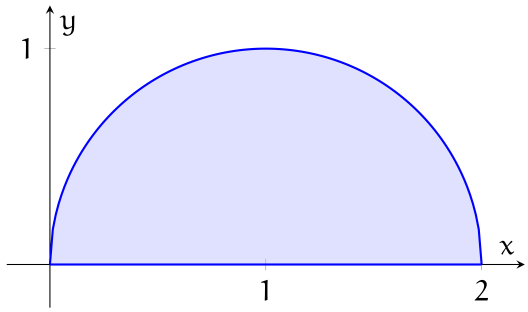
$$\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy = \int_0^{\pi/6} \int_0^1 r\,(r\cos\theta)\,(r\sin\theta)^2\, dr\, d\theta$$

$$= \int_0^{\pi/6} \left[\cos\theta\sin^2\theta\right] d\theta \cdot \int_0^1 r^4\, dr$$

$$= \left(\frac{1}{3}\sin^3\theta\Big|_{\theta=0}^{\theta=\pi/6}\right)\cdot\left(\frac{1}{5}r^5\Big|_{r=0}^{r=1}\right)$$

$$= \frac{1}{15}\cdot\left(\frac{1}{2}\right)^3 = \frac{1}{120}$$

3. Visualizing the region:




We see that the region is the top half of a circle of radius 1 centered at (1, 0). In polar coordinates, this region is $D = \{(r, \theta) \mid 0 \le r \le 2\cos\theta, 0 \le \theta \le \pi/2\}$. And therefore:

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r\sqrt{r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{3}\left[r^3\right]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2} \cos\theta\left(1-\sin^2\theta\right)\,d\theta$$

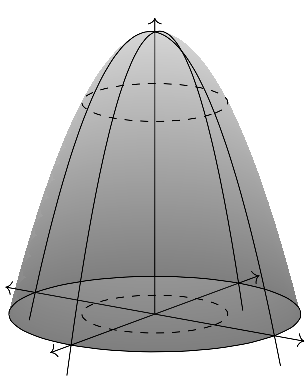
$$= \frac{8}{3}\left[\int_0^{\pi/2} \cos\theta\,d\theta - \int_0^{\pi/2} \cos\theta\sin^2\theta\,d\theta\right]$$

$$= \frac{8}{3}\left[(\sin\theta)_{\theta=0}^{\theta=\pi/2} - \left(\frac{1}{3}\sin^3\theta\right)_{\theta=0}^{\theta=\pi/2}\right]$$

$$= \frac{8}{3}\left[(1-0) - \frac{1}{3}\left(1^3-0^3\right)\right] = \frac{8}{3}\cdot\frac{2}{3} = \frac{16}{9}$$

## Answer to Exercise 3 (on page 11)

We are finding the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the $xy$-plane.



We can use polar coordinates to simplify the double integral. In polar coordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so $x^2 + y^2 = r^2$. The volume under the surface and above the $xy$-plane is given by

$$V = \int\int (4 - r^2) r \, dr \, d\theta, \tag{1.1}$$

where $r$ ranges from 0 to 2 (since $4 - r^2 \geq 0$ if $0 \leq r \leq 2$) and $\theta$ ranges from 0 to $2\pi$.

Hence,

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta \\
&= \int_0^{2\pi} (8 - 4) \, d\theta \\
&= \int_0^{2\pi} 4 \, d\theta \\
&= [4\theta]_0^{2\pi} \\
&= 8\pi.
\end{aligned}$$

So the volume of the solid is $8\pi$ cubic units.

## Answer to Exercise 4 (on page 11)

Let's describe the footprint of the pool as a 20-foot radius circle centered at the origin (that is, a region $D = \{(r, \theta) \mid 0 \leq r \leq 20,\ 0 \leq \theta \leq 2\pi\}$). Further, let's take north-south as parallel to the y-axis and east-west as parallel to the x-axis. Then the depth of water is then given by $z = f(x, y) = \frac{7}{40}x + \frac{13}{2}$ over the footprint of the pool. And the total volume of water is given by:

$$\begin{aligned}
&\int_0^{2\pi} \int_0^{20} r \left(\frac{7}{40} r\cos\theta + \frac{13}{2}\right) dr \, d\theta \\
&= \int_0^{2\pi} \int_0^{20} \left[\frac{7}{40} r^2 \cos\theta + \frac{13}{2} r\right] dr \, d\theta \\
&= \int_0^{2\pi} \left[\frac{7\cos\theta}{40} \int_0^{20} r^2 \, dr + \frac{13}{2} \int_0^{20} r \, dr\right] d\theta \\
&= \int_0^{2\pi} \left[\frac{7\cos\theta}{40} \left(\frac{1}{3} r^3\right)\Bigg|_{r=0}^{r=20} + \frac{13}{2} \left(\frac{1}{2} r^2\right)\Bigg|_{r=0}^{r=20}\right] d\theta
\end{aligned}$$

$$= \int_0^{2\pi} \left[ \frac{1400}{3} \cos\theta + 1300 \right] \, d\theta = \left[ \frac{1400}{3} \sin\theta + 1300\theta \right]_{\theta=0}^{\theta=2\pi}$$

$$= 2600\pi \text{ cubic feet}$$