Chapter 1

Methods of Integration

1.1 u-substitution

Sometimes a function's antiderivative isn't obvious. Take this integral for example:

$$\int 4x\sqrt{1+2x^2}\,dx$$

We can solve this integral using *u-substitution*. Recall from implicit differentiation that if $u = f(x)$, then we can also way $du = f'(x)dx$. Let's set u so that it is equal to the statement under the square root sign:

$$u = 1 + 2x^2$$

Taking the derivative of both sides, we see that

$$du = (4x)dx$$

How does this help us evaluate the integral? First, let's rearrange the integrand a bit:

$$\int 4x\sqrt{1+2x^2}\,dx = \int \sqrt{1+2x^2}4x\,dx$$

We can substitute $u = 1 + 2x^2$ and $du = 4xdx$ to get:

$$= \int \sqrt{u}\,du$$

That is a much nicer integral! We can evaluate this integral using the Power Rule:

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2}$$

We can now substitute $u = 1 + 2x^2$ back into our solution to yield:

$$= \frac{2}{3}(1+2x^2)^{3/2}$$

Feel free to double-check this answer by taking the derivative using the Chain Rule. You should get the original integrand, $4x\sqrt{1+2x^2}$ back.

As you may have guessed, u-substitution is a method to help us "undo" the Chain Rule. Recall that the Chain Rule states:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

If we integrate both sides we see that:

$$f(g(x)) = \int f'(g(x))g'(x)\, dx$$

Which leads us to the formal definition of the u-substitution method:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\int f(g(x))g'(x)\, dx = \int f(u)\, du$

1.2 Partial Fractions

1.3 Integration by Parts

1.4 Practice

Exercise 1

Using the substitution $u = x^2 - 3$, rewrite $\int_{-1}^{4} x(x^2-3)^5 dx$ in terms of u.

Working Space

Answer on Page 3

Exercise 2

Evaluate $\int_0^1 \frac{5x+8}{x^2+3x+2}\, dx$ without a calculator.

Working Space

Answer on Page 3

Exercise 3

Let f be a function such that $\int f(x)\sin x\,dx = -f(x)\cos x + \int 4x^3 \sin x\,dx$. Give a possible expression for $f(x)$.

Working Space

Answer on Page 3

Exercise 4

Evaluate $\int_1^\infty x e^{-x^2}\,dx$.

Working Space

Answer on Page 3

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 1)

If $u = x^2 - 3$, then $du = 2x dx$ and $x(x^2-3)^5 dx = \frac{1}{2}u^5 du$. When $x = -1$, $u = -2$ and when $x = 4$, $u = 13$. Putting it all together, we find an equivalent integral is $\frac{1}{2}\int_{-2}^{13} u^5\, du$.

Answer to Exercise 2 (on page 1)

We cannot use u-substitution because $\frac{d}{dx}(x^2 + 3x + 2) \neq n(5x + 8)$. We will use partial fractions to simplify the integrand. Settig up: $\frac{5x+8}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$. Rearranging, we find $5x + 8 = A(x + 2) + B(x + 1)$. Letting $x = -2$, we find that $B = 2$. And taking $x = -1$, we find $A = 3$. Therefore, $\int_0^1 \frac{5x+8}{x^2+3x+2}\, dx = \int_0^1 \frac{3}{x+1}\, dx + \int_0^1 \frac{2}{x+2}\, dx$. Evaluating the integrals, we get $3\ln(x+1)|_0^1 + 2\ln(x+2)|_0^1 = 3(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) = 3\ln 2 + 2\ln\frac{3}{2} = \ln 8 + \ln\frac{9}{4} = \ln\frac{8\cdot 9}{4} = \ln 18$.

Answer to Exercise 3 (on page 2)

This question takes the form of integration by parts. That is, $\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx$. If we let $g(x) = -\cos x$, then $g'(x) = \sin x$. The structure of the equation implies that $f'(x) = 4x^3$ and therefore that f could be $f(x) = x^4$.

Answer to Exercise 4 (on page 2)

Letting $u = -x^2$, then $du = -2x dx$ and $x dx = \frac{-1}{2} du$. Substituting u and du into the integral, we have $\int_{x=1}^{x=\infty} \frac{-1}{2} e^u\, du$, which equals $\frac{-1}{2}e^u = \frac{-1}{2}e^{-x^2}|_1^\infty$. Evaluating the statement, we get $\frac{-1}{2}(e^{-\infty} - e^{-1}) = \frac{-1}{2}(0 - \frac{1}{e}) = \frac{1}{2e}$