Chapter 1

Methods of Integration

1.1 u-substitution

Sometimes a function's antiderivative isn't obvious. Take this integral for example:

$$\int 4x\sqrt{1+2x^2}\,dx$$

We can solve this integral using *u-substitution*. Recall from implicit differentiation that if $u = f(x)$, then we can also say $du = f'(x)dx$. Let's set u so that it is equal to the statement under the square root sign:

$$u = 1 + 2x^2$$

Taking the derivative of both sides, we see that

$$du = (4x)dx$$

How does this help us evaluate the integral? First, let's rearrange the integrand a bit:

$$\int 4x\sqrt{1+2x^2}\,dx = \int \sqrt{1+2x^2}4x\,dx$$

We can substitute $u = 1 + 2x^2$ and $du = 4xdx$ to get:

$$= \int \sqrt{u}\,du$$

That is a much nicer integral! We can evaluate this integral using the Power Rule:

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2}$$

We can now substitute $u = 1 + 2x^2$ back into our solution to yield:

$$= \frac{2}{3}(1+2x^2)^{3/2}$$

Feel free to double-check this answer by taking the derivative using the Chain Rule. You should get the original integrand, $4x\sqrt{1+2x^2}$, back.

As you may have guessed, u-substitution is a method to help us ”undo” the Chain Rule. Recall that the Chain Rule states:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

If we integrate both sides we see that:

$$f(g(x)) = \int f'(g(x))g'(x)\,dx$$

Which leads us to the formal definition of the u-substitution method:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\int f(g(x))g'(x)\,dx = \int f(u)\,du$

1.2 Partial Fractions

We can integrate rational functions by using partial fraction to decompose a complex rational function into simpler ones. Recall that if we want to add terms with different denominators, we cross-multiply to create a common denominator:

$$\frac{3}{x-1} + \frac{1}{x+2} = \frac{3(x+2)}{(x-1)(x+2)} + \frac{1(x-1)}{(x+2)(x-1)} = \frac{3(x+2)+(x-1)}{(x+2)(x-1)} = \frac{4x+5}{x^2+x-2}$$

The reverse of this process is called **partial fractions** . Suppose we wanted to integrate $f(x) = \frac{4x+5}{x^2+x-2}$:

$$\begin{aligned}\int \frac{4x+5}{x^2+x-2}\,dx &= \int \left(\frac{3}{x-1} + \frac{1}{x+2}\right)\,dx \\ &= 3\ln|x-1| + \ln|x+2| + C\end{aligned}$$

Let $g(x)$ be a rational function such that

$$g(x) = \frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials. If $g(x)$ is proper (that is, the degree of P is less than the degree of Q) then we can express $g(x)$ as the sum of simpler rational fractions. If $g(x)$ is improper (that is, the degree of P is greater than or equal to the degree of Q), then we must first perform long division to obtain a remainder, $R(x)$, where the degree of R is less than the degree of Q:

$$g(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

1.2.1 Improper fractions

What is $\int \frac{x^3+x}{x-1}\,dx$. Using long division, we see that:

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

(see figure 1.1 for an explanation). Then we can also say that:

$$\int \frac{x^3+x}{x-1}\,dx = \int \left[x^2 + x + 2 + \frac{2}{x-1}\right]\,dx$$

And therefore:

$$\int \frac{x^3+x}{x-1}\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + C$$

$$\begin{array}{r@{\,}l}
 & \quad x^2 + x + 2 \\ \cline{2-2}
x - 1 & \big) \; x^3 + 0x^2 + x \\
 & -(x^3 - x^2) \\ \cline{2-2}
 & \qquad\quad x^2 + x \\
 & \qquad -(x^2 - x) \\ \cline{2-2}
 & \qquad\qquad\quad 2x \\
 & \qquad\qquad -(2x - 2) \\ \cline{2-2}
 & \qquad\qquad\qquad\quad 2
\end{array}$$

Figure 1.1: $(x^3+x) \div (x-1)$ with the long division method

1.2.2 Proper fractions

When the order of the numerator is less than or equal to the denominator, there are three further possibilities.

No repeated linear factors

In the first case, the denominator, $Q(x)$ is composed of distinct linear factors. In this case, we can say that $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$, where no factor is repeated (including constant multiples). Then, there exists $A, B, C, \cdots$ such that:

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \cdots$$

Let's see an example of this by decomposing $\frac{4x^2-7x-12}{x(x+2)(x-3)}$. We start by defining A, B, and C, such that:

$$\frac{4x^2-7x-12}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$

Multiplying both sides by $x(x+2)(x-3)$ we get:

$$4x^2 - 7x - 12 = A(x+2)(x-3) + B(x)(x-3) + C(x)(x+2)$$

We have 3 unknowns and only one equation! Don't worry: remember this equation is true for all x, so we can choose a convenient value of x to isolate each unknown in turn. Starting, let $x = 0$. Then:

$$4(0)^2 - 7(0) - 12 = A(0+2)(0-3) + B(0)(x-3) + C(0)(x+2)$$

$$-12 = A(2)(-3) + 0 + 0$$

Notice that the B and C disappear, and we can solve for A:

$$A = \frac{-12}{-6} = 2$$

We can solve for B by setting $x = -2$ and for C by setting $x = 3$ (notice, we've used all three zeroes of the denominator polynomial):

$$4(-2)^2 - 7(-2) - 12 = A(-2+2)(-2-3) + B(-2)(-2-3) + C(-2)(-2+2)$$

$$4(4) + 14 - 12 = 0 + B(-2)(-5) + 0$$

$$16 + 2 = 10B$$

$$B = \frac{9}{5}$$

and

$$4(3)^2 - 7(3) - 12 = A(3+2)(3-3) + B(3)(3-3) + C(3)(3+2)$$

$$4(9) - 21 - 12 = 0 + 0 + C(3)(5)$$

$$36 - 33 = 15C$$

$$C = \frac{1}{5}$$

And we can decompose our original fraction:

$$\frac{4x^2-7x-12}{x(x+2)(x-3)} = \frac{2}{x} + \frac{9}{5(x+2)} + \frac{1}{5(x-3)}$$

You can check your answer by cross-multiplying and adding. You should get the same rational function back.

Repeated linear factors

The second case is if $Q(x)$ has repeated factors (such as $x^2+8x+16=(x+4)^2$). Suppose the first linear factor, (a_1x+b_1) is repeated r times (that is, $Q(x)$ contains the factor $(a_1x+b_1)^r$). Then instead of $\frac{A}{a_1x+b_1}$ we should write:

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}$$

Let's look at a concrete example to see how this works: **Example**: Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$

Solution: We start by defining:

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}$$

Multiplying both sides by $(x+1)^2(x+2)$:

$$x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2$$

Since there are only 2 roots to $(x+1)^2(x+2)$, we will use another method called "equating the coefficients" to find A, B, and C. We start by expanding the right side of the equation:

$$x^2+x+1=A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)$$

Distributing and combining, we find that:

$$x^2+x+1=Ax^2+3Ax+2A+Bx+2B+Cx^2+2Cx+C$$
$$x^2+x+1=(A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

For this equation to be true, we know that:

$$A+C=1$$
$$3A+B+2C=1$$
$$2A+2B+C=1$$

(That is, the coefficient for x^2 on the left, 1, must be equal to the coefficient for x^2 on the right, $(A+C)$, and so on.) We now have a system of 3 equations and 3 unknowns. When you solve for each, you should find that:

$$A=-2$$
$$B=1$$
$$C=3$$

And therefore,

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}$$

Substituting this into our integral,

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}\,dx=\int\left[\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}\right]\,dx$$

$$=-2\ln|x+1|+\frac{-1}{x+1}+3\ln|x+2|+C=\ln\left|\frac{(x+2)^3}{(x+1)^2}\right|-\frac{1}{x+1}+C$$

Irreducible quadratic factors

Sometimes we cannot express a polynomial as the product of two linear statements (that is, terms in the form $ax+b$). Take x^2+1, which cannot be expressed as the product of real, linear terms. What do you do if something like x^2+1 is in the denominator? Then when we write an expression for $\frac{P(x)}{Q(x)}$ we include a term in the form:

$$\frac{Ax+B}{ax^2+bx+c}$$

For example, we can write

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

Example: Evaluate $\int \frac{2x^2-x+4}{x^3+4x}\,dx$

Solution: We begin by factoring the denominator:

$$x^3+4x = x(x^2+4)$$

Which cannot be factored further. Therefore, we define:

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2-x+4 = A(x^2+4)+(Bx+C)x$$

$$2x^2-x+4 = Ax^2+4A+Bx^2+Cx$$

Which implies that:

$$2 = A+B$$

$$C = -1$$

$$4A = 4$$

Therefore, $A=1$, $B=1$, and $C=-1$ and we can say that:

$$\begin{aligned}\int \frac{2x^2-x+4}{x^3+4x}\,dx &= \int \left[\frac{1}{x} + \frac{x-1}{x^2+4}\right] dx \\ &= \int \left[\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right] dx \\ &= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C\end{aligned}$$

A useful identity that we used here is

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

1.3 Integration by Parts

1.4 Practice

Exercise 1

Using the substitution $u = x^2 - 3$, rewrite $\int_{-1}^{4} x(x^2-3)^5 dx$ in terms of u.

Working Space

Answer on Page 9

Exercise 2

Evaluate $\int_0^1 \frac{5x+8}{x^2+3x+2}\,dx$ without a calculator.

Working Space

Answer on Page 9

Exercise 3

Let f be a function such that $\int f(x)\sin x\,dx = -f(x)\cos x + \int 4x^3 \sin x\,dx$. Give a possible expression for $f(x)$.

Working Space

Answer on Page 9

Exercise 4

Evaluate $\int_1^\infty xe^{-x^2}\,dx$.

Working Space

Answer on Page 9

Exercise 5

Use the method of partial fractions to evaluate the following integrals:

1. $\int \frac{4x}{x^3+x^2+x+1}\,dx$
2. $\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2}\,dx$
3. $\mathrm{int}\frac{x^3+2x}{x^4+4x^2+3}\,dx$

Working Space

Answer on Page ??

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

If $u = x^2 - 3$, then $du = 2xdx$ and $x(x^2-3)^5dx = \frac{1}{2}u^5du$. When $x = -1$, $u = -2$ and when $x = 4$, $u = 13$. Putting it all together, we find an equivalent integral is $\frac{1}{2}\int_{-2}^{13} u^5\, du$.

Answer to Exercise 2 (on page 6)

We cannot use u-substitution because $\frac{d}{dx}(x^2 + 3x + 2) \neq n(5x + 8)$. We will use partial fractions to simplify the integrand. Settig up: $\frac{5x+8}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$. Rearranging, we find $5x + 8 = A(x + 2) + B(x + 1)$. Letting $x = -2$, we find that $B = 2$. And taking $x = -1$, we find $A = 3$. Therefore, $\int_0^1 \frac{5x+8}{x^2+3x+2}\, dx = \int_0^1 \frac{3}{x+1}\, dx + \int_0^1 \frac{2}{x+2}\, dx$. Evaluating the integrals, we get $3\ln(x+1)|_0^1 + 2\ln(x+2)|_0^1 = 3(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) = 3\ln 2 + 2\ln\frac{3}{2} = \ln 8 + \ln\frac{9}{4} = \ln\frac{8\cdot 9}{4} = \ln 18$.

Answer to Exercise 3 (on page 6)

This question takes the form of integration by parts. That is, $\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx$. If we let $g(x) = -\cos x$, then $g'(x) = \sin x$. The structure of the equation implies that $f'(x) = 4x^3$ and therefore that f could be $f(x) = x^4$.

Answer to Exercise 4 (on page 7)

Letting $u = -x^2$, then $du = -2xdx$ and $xdx = \frac{-1}{2}du$. Substituting u and du into the integral, we have $\int_{x=1}^{x=\infty} \frac{-1}{2}e^u\, du$, which equals $\frac{-1}{2}e^u = \frac{-1}{2}e^{-x^2}|_1^\infty$. Evaluating the statement, we get $\frac{-1}{2}(e^{-\infty} - e^{-1}) = \frac{-1}{2}(0 - \frac{1}{e}) = \frac{1}{2e}$

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