Chapter 1

Methods of Integration

1.1 u-substitution

Sometimes a function's antiderivative isn't obvious. Take this integral for example:

$$\int 4x\sqrt{1+2x^2}\,dx$$

We can solve this integral using *u-substitution*. Recall from implicit differentiation that if $u = f(x)$, then we can also say $du = f'(x)dx$. Let's set u so that it is equal to the statement under the square root sign:

$$u = 1 + 2x^2$$

Taking the derivative of both sides, we see that

$$du = (4x)dx$$

How does this help us evaluate the integral? First, let's rearrange the integrand a bit:

$$\int 4x\sqrt{1+2x^2}\,dx = \int \sqrt{1+2x^2}4x\,dx$$

We can substitute $u = 1 + 2x^2$ and $du = 4xdx$ to get:

$$= \int \sqrt{u}\,du$$

That is a much nicer integral! We can evaluate this integral using the Power Rule:

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2}$$

We can now substitute $u = 1 + 2x^2$ back into our solution to yield:

$$= \frac{2}{3}(1 + 2x^2)^{3/2}$$

Feel free to double-check this answer by taking the derivative using the Chain Rule. You should get the original integrand, $4x\sqrt{1 + 2x^2}$, back.

As you may have guessed, u-substitution is a method to help us "undo" the Chain Rule. Recall that the Chain Rule states:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

If we integrate both sides we see that:

$$f(g(x)) = \int f'(g(x))g'(x)\, dx$$

Which leads us to the formal definition of the u-substitution method:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\int f(g(x))g'(x)\, dx = \int f(u)\, du$

Let's apply u-substitution to a definite integral:

Example: Evaluate $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx$.

Solution: Recall that $\frac{d}{dx} \ln x = \frac{1}{x}$. Letting $\ln x = u$, it follows that $\frac{dx}{x} = du$. Rearranging the integral and substituting:

$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx = \int_e^{e^4} \frac{1}{\sqrt{\ln x}} \frac{dx}{x}$$
$$= \int_{x=e}^{x=e^4} \frac{1}{\sqrt{u}}\, du$$

Proceeding from here, there are two options: you can find the value of u at $x = e$ and $x = e^4$ and change the limits of the integral OR you can evaluate the integral, resubstitute back for x and then evaluate the result with the original limits. We will show both to demonstrate each method and show they have the same result.

Method 1: change the limits of integration When $x = e$, $u = \ln e = 1$. And when $x = e^4$, $u = \ln e^4 = 4$. Therefore, we can change the limits of the integral to:

$$\int_1^4 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_1^4 = 2\left[\sqrt{4} - \sqrt{1}\right] = 2(2-1) = 2$$

Method 2: keep the limits of integration and resubstitute for u:

$$\int_{x=e}^{x=e^4} \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_{x=e}^{x=e^4} = 2\sqrt{\ln x}\Big|_e^{e^4}$$
$$= 2\left[\sqrt{\ln e^4} - \sqrt{\ln e}\right] = 2(\sqrt{4} - \sqrt{1}) = 2(2-1) = 2$$

Which is the same result as method 1. When done correctly, either method will yield the correct result. Choose the method you prefer.

Exercise 1

Using the substitution $u = x^2 - 3$, rewrite $\int_{-1}^4 x(x^2-3)^5\,dx$ in terms of u.

Working Space

Answer on Page ??

Exercise 2

Evaluate $\int_1^\infty xe^{-x^2}\,dx$.

Working Space

Answer on Page ??

1.2 Partial Fractions

We can integrate rational functions by using partial fractions to decompose a complex rational function into simpler ones. Suppose we wanted to integrate $f(x) = \frac{4x+5}{x^2+x-2}$:

$$\int \frac{4x+5}{x^2+x-2}\,dx = \int \left(\frac{3}{x-1} + \frac{1}{x+2}\right)\,dx$$
$$= 3\ln|x-1| + \ln|x+2| + C$$

Example: Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$

Solution: We start by defining:

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

Multiplying both sides by $(x+1)^2(x+2)$:

$$x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Since there are only 2 roots to $(x+1)^2(x+2)$, we will equate the coefficients to find A, B, and C.

$$x^2+x+1 = A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)$$
$$x^2+x+1 = Ax^2+3Ax+2A+Bx+2B+Cx^2+2Cx+C$$
$$x^2+x+1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$

For this equation to be true, we know that:

$$A+C=1$$
$$3A+B+2C=1$$
$$2A+2B+C=1$$

Solving for each, you should find that:

$$A=-2$$
$$B=1$$
$$C=3$$

And therefore,

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$

Substituting this into our integral,

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx = \int \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}\right]\,dx$$

$$= -2\ln|x+1| + \frac{-1}{x+1} + 3\ln|x+2| + C = \ln\left|\frac{(x+2)^3}{(x+1)^2}\right| - \frac{1}{x+1} + C$$

Example: Evaluate $\int \frac{2x^2-x+4}{x^3+4x}\,dx$

Solution: We begin by factoring the denominator:

$$x^3 + 4x = x(x^2+4)$$

Which cannot be factored further. Therefore, we define:

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

Which implies that:

$$2 = A + B$$

$$C = -1$$

$$4A = 4$$

Therefore, $A = 1$, $B = 1$, and $C = -1$ and we can say that:

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \left[\frac{1}{x} + \frac{x-1}{x^2+4}\right]\,dx$$

$$= \int \left[\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right]\,dx$$

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

A useful identity that we used here is

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Exercise 3

Evaluate $\int_0^1 \frac{5x+8}{x^2+3x+2}\,dx$ without a calculator.

Working Space

Answer on Page ??

Exercise 4

Use the method of partial fractions to evaluate the following integrals:

1. $\int \frac{4x}{x^3+x^2+x+1}\,dx$
2. $\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx$
3. $\int \frac{x^3+2x}{x^4+4x^2+3}\,dx$

Working Space

Answer on Page ??

1.3 Integrating Powers of Trigonometric Functions

1.3.1 Odd powers of sine and cosine

Consider the integral:

$$\int \cos^5 x \, dx$$

To evaluate this integral, we may first try u-substitution, with $u = \cos x$. But then $du = -\sin x dx$, and there is no sine function in the integral. Instead, let's take advantage of the identity:

$$\cos^2 x = 1 - \sin^2 x$$

We begin by separating $\cos^5 x$ into three terms:

$$\int \cos x \cdot \cos^2 x \cdot \cos^2 x \, dx$$

Now we substitute in $\cos^2 x = 1 - \sin^2 x$:

$$\int \cos x \left(1 - \sin^2 x\right) \left(1 - \sin^2 x\right) \, dx$$

$$= \int \left(1 - sin^2 x\right)^2 \cos x \, dx$$

Now we have sine and cosine functions, which makes u-substitution possible. Let $u = \sin x$, then $du = \cos x dx$:

$$\int \left(1 - u^2\right)^2 \, du = \int \left(1 - 2u^2 + u^4\right) \, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C|_{u=\sin x} = \sin x - \frac{2}{3}\sin^2 x + \frac{1}{5}\sin^5 x + C$$

Exercise 5 Integrating odd powers of sine and cosine

Use the identity $\sin^2\theta + \cos^2\theta = 1$ and u-substitution to evaluate the following integrals:

1. $\int \sin^2 x \cos^3 x \, dx$
2. $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$
3. $\int \sin^3\theta \cos^4\theta \, d\theta$
4. $\int_0^{\pi/2} \sin^5 x \, dx$
5. $\int t \cos^5\left(t^2\right) dt$

Working Space

Answer on Page ??

1.3.2 Even powers of sine and cosine

Consider $\int \cos^4 x \, dx$. Let's try applying the identity $\cos^2 x = 1 - \sin^2 x$:

$$\int \cos^4 x \, dx = \int \cos^2 x \left(1 - \sin^2 x\right) \, dx$$

We can't use u-substitution for this. If we were to set $u = \sin x$, we see $du = \cos x dx$, and we would have an extra $\cos x$ left. Instead, let's use the double-angle identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$:

$$\int \cos^4 x \, dx = \int \left(\cos^2 x\right)^2 \, dx$$

$$= \frac{1}{4}\int (1 + \cos 2x)^2 \, dx = \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right) \, dx$$

We can apply the double-angle identity again, noting that $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$:

$$= \frac{1}{4}\int \left[1 + \cos 2x + \frac{1}{2}(1 + \cos 4x)\right] \, dx$$

$$= \frac{1}{4}\left[x + \frac{1}{2}\sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right] + C$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \frac{1}{2}\sin 2x + \frac{1}{8}\sin 4x\right] + C$$

In general, when the powers of sine and cosine are both even, apply the double-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Until the powers are reduced sufficiently to be able to evaluate the integral.

Exercise 6 Integrating even powers of sine and cosine

Use the double-angle identities to evaluate the following integrals:

1. $\int_0^{\pi/2} \cos^2\theta\, d\theta$
2. $\int_0^{\pi} \cos^4 2t\, dt$
3. $\int \sin^2 x \cos^2 x\, dx$
4. $\int_0^{2\pi} \sin^2 \frac{1}{3}\theta\, d\theta$
5. $\int_0^{\pi} \sin^2 t \cos^4 t\, dt$

Working Space

Answer on Page ??

1.3.3 Integrating powers of secant and tangent

There are a few identities that we won't prove here, but will be useful:

1. $\int \tan x\, dx = \ln|\sec x| + C$
2. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

Consider the integral $\int \tan^6 x \sec^4 x\, dx$. We know that $\frac{d}{dx}\tan x = \sec^2 x$, so we want to arrange things such that we can use the u-substitution $u = \tan x$ and $du = \sec^2 x dx$. Let's separate terms so we can easily see our intended du:

$$\int \tan^6 x \sec^2 x \sec^2 x \, dx$$

What to do with the extra $sec^2 x$? We already know that $\sin^2\theta + \cos^2\theta = 1$. If we divide both sides by $\cos^2\theta$, we get the identity $\tan^2\theta + 1 = \sec^2\theta$. So we can replace the extra $sec^2 x$ with $\tan^2 x + 1$:

$$\int \tan^6 x \left(\tan^2 x + 1\right) \sec^2 x \, dx =$$

$$\int \tan^8 x \sec^2 x \, dx + \int \tan^6 x \sec^2 x \, dx$$

We can now apply our u-substitution, as outlined above:

$$= \int u^8 \, du + \int u^6 \, du = \frac{1}{9}u^9 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C$$

This method works best when the secant is raised to an even power. Reserve a $\sec^2$ to use for the u-substitution and express the remaining secants in terms of the tangent.

We can use a similar method when the power of tangent is odd. We will evaluate $\int \tan^5 x \sec^3 x \, dx$. First, remember that $\frac{d}{dx}\sec x = \sec x \tan x$. We'll use this for a u-substitution, so let's pull out a secant factor and a tangent factor:

$$\int tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x \sec x \tan x \, dx$$

We re-write $tan^4 x$ in terms of $\sec x$, since we will use the u-substitution $u = \sec x$ and $du = \sec x \tan x dx$:

$$\int \tan^4 x \sec^2 x \sec x \tan x \, dx = \int \left(1 + \sec^2 x\right)^2 \sec^2 x \sec x \tan x \, dx$$

$$= \int \left(1 + u^2\right)^2 u^2 \, du = \int \left(1 + 2u^2 + u^4\right) u^2 \, du$$

$$= \int \left(u^2 + 2u^4 + u^6\right) \, du = \frac{1}{3}u^3 + \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{3}\sec^3 x + \frac{2}{5}\sec^5 x + \frac{1}{7}\sec^7 x + C$$

When the power of tangent is even and the power of secant is odd, you will need to try additional methods, including using integration by pats (explained below) or the identities for $\int \tan x \, dx$ and $\int \sec x \, dx$ given above.

Exercise 7 Integrating Powers of Secant and Tangent

Working Space

1. $\int \tan x \sec^3 x \, dx$
2. $\int \tan^4 x \sec^6 x \, dx$
3. $\int_0^{\pi/4} \sec^6 \theta \tan^6 \theta \, d\theta$

Answer on Page ??

1.4 Integration by Parts

Recall the Product Rule for derivatives:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

If we integrate both sides, we find that:

$$f(x) \cdot g(x) = \int \left[f(x) \cdot g'(x) + f'(x) \cdot g(x) \right] \, dx$$
$$f(x) \cdot g(x) = \int f(x)g'(x) \, dx + \int f'(x)g(x) \, dx$$

Rearranging,

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx$$

This identity allows us to perform **integration by parts** , a powerful method that allows us to evaluate integrals of complex functions.

Example: Evaluate $\int x \cos x \, dx$.

Solution: We may be tempted to try u-substitution, but that won't work because $\frac{d}{dx}\cos x$ is not proportional to x and $\frac{d}{dx}x$ is not proportional to $\cos x$. Let us define $f(x) = x$ and $g'(x) = \cos x$. This implies $f'(x) = 1$ and $g(x) = \sin x$. Then we can say that:

$$\int x \cos x \, dx = \int f(x) g'(x) \, dx$$

Using the identity $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$ and substituting for $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$, we see that:

$$\begin{aligned}
\int x \cos x \, dx &= [x \sin x] - \int 1 \cdot \sin x \, dx \\
&= x \sin x - \int \sin x \, dx \\
&= x \sin x - (-\cos x + C) = x \sin x + \cos x + C
\end{aligned}$$

(recall that C is the integration constant). You can check your results by taking the derivative: you should get the original integrand back. Let's check our result in this case:

$$\begin{aligned}
\frac{d}{dx}[x \sin x + \cos x + C] &= \frac{d}{dx}[x \sin x] + \frac{d}{dx}\cos x \\
&= x\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x) - \sin x \\
&= x \cos x + |SINx - \sin x = x \cos x
\end{aligned}$$

How did we choose that $f(x)$ should be x and $g(x)$ should be $\sin x$ in the example above? In general, you want to choose such that the resulting integral is simpler than the one we started with. This means you want to choose f such that f' is *less complex* or a *lower order* than f.

To illustrate this, let's re-evalute the example above, but this time let $f(x) = \cos x$ and $g'(x) = x$. Then we can say that $f'(x) = -\sin x$ and $g(x) = \frac{1}{2}x^2$. Substituting this into the integration by parts identity, we find that:

$$\int x \cos x \, dx = \frac{1}{2}x^2 \cos x - \int -\frac{1}{2}x^2 \sin x \, dx$$

Now the integral on the right side is more complex than the one we started with (on the left)! A good general rule for integration by parts is that *if* the two functions in the original integral are a polynomial and a sine or cosine function, set the polynomial to be $g(x)$ and the trigonometric function to be $f'(x)$. The polynomial will be differentiated and become *less* complex, while integrating the trigonometric function won't make it *more* complex.

Integration by parts is valid for definite integrals as well. Mathematically, this means:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\,dx$$

Which is the same as:

$$\int_a^b f(x)g'(x)\,dx = (f(b)g(b)) - (f(a)g(a)) - \int_a^b f'(x)g(x)\,dx$$

Let's see one more example that incorporates both u-substitution and integration by parts.

Example: Evaluate $\int \frac{\arcsin \ln x}{x}\,dx$

Solution: First, we notice that $\ln x$ and $\frac{1}{x}$ both appear in the integrand. Let us define $u = \ln x$. Then $du = \frac{dx}{x}$:

$$\int \arcsin \ln x \frac{dx}{x} = \int \arcsin u\,du$$

For integration by parts, if we let $\arcsin u = f(u)$ and $du = g'(u)$, it follows that $f'(u) = \frac{1}{\sqrt{1-u^2}}$ and $g(u) = u$. Then we can say that:

$$\int \arcsin u\,du = \arcsin u \cdot u - \int \frac{u}{\sqrt{1-u^2}}\,du$$

We can use u-substitution again to evaluate the second integral (we will use v, since we have already said that $u = \ln x$). Let $v = 1 - u^2$, which means that $\frac{dv}{2} = (-u)du$. Substituting:

$$= u \cdot \arcsin u + \int \frac{1}{2\sqrt{v}}\,dv = u \cdot \arcsin u + \sqrt{v}$$

Substituting back for v:

$$= u \cdot \arcsin u + \sqrt{1-u^2}$$

And substituting back for u:

$$= \ln x \cdot \arcsin \ln x + \sqrt{1 - \ln^2 x}$$

Exercise 8

Let f be a function such that $\int f(x) \sin x \, dx = -f(x) \cos x + \int 4x^3 \sin x \, dx$. Give a possible expression for $f(x)$.

Working Space

Answer on Page ??

Exercise 9

Evaluate the following integrals using integration by parts:

1. $\int_0^1 x \sin \frac{\pi}{2} x \, dx$
2. $\int e^{\theta} \cos \theta \, d\theta$
3. $\int (1-t)^2 \cos \beta t \, dt$

Working Space

Answer on Page ??

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

If $u = x^2 - 3$, then $du = 2xdx$ and $x(x^2-3)^5 dx = \frac{1}{2}u^5 du$. When $x = -1$, $u = -2$ and when $x = 4$, $u = 13$. Putting it all together, we find an equivalent integral is $\frac{1}{2}\int_{-2}^{13} u^5\, du$.

Answer to Exercise 2 (on page 3)

Letting $u = -x^2$, then $du = -2xdx$ and $xdx = \frac{-1}{2}du$. Substituting u and du into the integral, we have $\int_{x=1}^{x=\infty} \frac{-1}{2}e^u\, du$, which equals $\frac{-1}{2}e^u = \frac{-1}{2}e^{-x^2}|_1^\infty$. Evaluating the statement, we get $\frac{-1}{2}(e^{-\infty} - e^{-1}) = \frac{-1}{2}(0 - \frac{1}{e}) = \frac{1}{2e}$

Answer to Exercise 3 (on page 6)

We cannot use u-substitution because $\frac{d}{dx}(x^2 + 3x + 2) \neq n(5x + 8)$. We will use partial fractions to simplify the integrand. Settig up: $\frac{5x+8}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$. Rearranging, we find $5x + 8 = A(x + 2) + B(x + 1)$. Letting $x = -2$, we find that $B = 2$. And taking $x = -1$, we find $A = 3$. Therefore, $\int_0^1 \frac{5x+8}{x^2+3x+2}\, dx = \int_0^1 \frac{3}{x+1}\, dx + \int_0^1 \frac{2}{x+2}\, dx$. Evaluating the integrals, we get $3\ln(x+1)|_0^1 + 2\ln(x+2)|_0^1 = 3(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) = 3\ln 2 + 2\ln\frac{3}{2} = \ln 8 + \ln\frac{9}{4} = \ln\frac{8\cdot 9}{4} = \ln 18$.

Answer to Exercise 4 (on page 6)

1. Let $\frac{4x}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$. Rearranging, we see that $4x = A(x^2+1)+(Bx+C)(x+1)$. Which means that $4x = Ax^2 + A + Bx^2 + Bx + Cx + C$, which implies that $A + B = 0$ and $B + C = 4$ and $A + C = 0$. Solving this system of equations, we see that $A = -2$, $B = 2$, and $C = 2$. So we can say that $\int \frac{4x}{x^3+x^2+x+1}\, dx = \int \left[\frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\right] dx$. Which evaluates to $-2\ln|x+1| + \ln|x^2+1| + 2\arctan(x) + K$, where K is the constant of integration.

2. Since the order of x is greater in the numerator, first we divide and see that $\frac{x^3-4x+1}{x^2-3x+2} =$

$(x+3)+\frac{3x-5}{x^2-3x+2}$. Now let $\frac{3x-5}{x^2-3x+2}=\frac{A}{x-2}+\frac{B}{x-1}$, which means that $3x-5=A(x-1)+B(x-2)$. Solving, we find that $A=1$ and $B=2$. Therefore, $\int_{-1}^{0}\frac{x^3-4x+1}{x^2-3x+2}\,dx=\int_{-1}^{0}\left[x+3+\frac{1}{x-2}+\frac{2}{x-1}\right]dx$ which evaluates to $\frac{1}{2}x^2+3x+\ln|x-2|+\ln|x-1|\big|_{x=-1}^{x=0}=\frac{5}{2}-\ln(3)$.

3. Note that $\frac{x^3+2x}{x^4+4x^2+3}=\frac{x^3+2x}{(x^2+1)(x^2+3)}$. Then let $\frac{x^3+2x}{(x^2+1)(x^2+3)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}$. Then $x^3+2x=(A+C)x^3+(B+D)x^2+(3A+C)x+(3B+D)$ which implies that $A+C=1$, $B+D=0$, $3A+C=2$, and $3b+D=0$. Solving this system of equations, we see that $A=C=\frac{1}{2}$ and $B=D=0$, which means that $\frac{x^3+2x}{x^4+4x^2+3}=\frac{x}{2(x^2+1)}+\frac{x}{2(x^3+3)}$. And therefore, $int\frac{x^3+2x}{x^4+4x^2+3}\,dx=\int\left[\frac{x}{2(x^2+1)}+\frac{x}{2(x^3+3)}\right]dx=\frac{1}{4}\ln|x^2+1|+\frac{1}{4}\ln|x^2+3|+K$, where K is the constant of integration.

Answer to Exercise 5 (on page 8)

1. $\int\sin^2x\cos^3x\,dx=\int\sin^2x\cos x\left(1-\sin^2x\right)dx=\int\sin^2x\cos x\,dx-\int\sin^4x\cos x\,dx$. Applying the u-substitution $u=\sin x$ and $du=\cos x dx$: $\int\sin^2x\cos x\,dx-\int\sin^4x\cos x\,dx=\int u^2\,du-\int u^4\,du=\frac{1}{3}u^3-\frac{1}{5}u^5+C|_{u=\sin x}=\frac{1}{3}\sin^3x-\frac{1}{5}\sin^5x+C$

2. $\int_0^{\pi/2}\sin^7\theta\cos^5\theta\,d\theta=\int_0^{\pi/2}\sin^7\theta\left(\cos^2\theta\right)^2\cos\theta\,d\theta=\int_0^{\pi/2}\sin^7\theta\left(1-\sin^2\theta\right)^2\cos\theta\,d\theta=\int_0^{\pi/2}\sin^7\theta\left(1-2\sin^2\theta+\sin^4\theta\right)\cos\theta\,d\theta=\int_0^{\pi/2}\left(\sin^7\theta-2\sin^9\theta+\sin^{11}\theta\right)\cos\theta\,d\theta$. Substituting $u=\sin\theta$ and $du=\cos\theta d\theta$: $\int_0^{\pi/2}\left(\sin^7\theta-2\sin^9\theta+\sin^{11}\theta\right)\cos\theta\,d\theta=\int_{\theta=0}^{\theta=\pi/2}\left(u^7-2u^9+u^{11}\right)du=\left[\frac{1}{8}u^8-\frac{1}{5}u^{10}+\frac{1}{12}u^{12}\right]_{\theta=0}^{\theta=\pi/2}=\left[\frac{1}{8}\sin^8\theta-\frac{1}{5}\sin^{10}\theta+\frac{1}{12}\sin^{12}\theta\right]_{\theta=0}^{\theta=\pi/2}=\frac{1}{8}-\frac{1}{5}+\frac{1}{12}=\frac{1}{120}$

3. $\int\sin^3\theta\cos^4\theta\,d\theta=\int\sin\theta\left(1-\cos^2\theta\right)\cos^4\theta\,d\theta=\int\left(\cos^4\theta-\cos^6\theta\right)\sin\theta\,d\theta$. Substituting $u=\cos\theta$ and $du=-\sin\theta d\theta$: $\int\left(u^6-u^4\right)du=\frac{1}{7}u^7-\frac{1}{5}u^5+C=\frac{1}{7}\cos^7\theta-\frac{1}{5}\cos^5\theta+C$

4. $\int_0^{\pi/2}\sin^5x\,dx=\int_0^{\pi/2}\sin^4x\sin x\,dx=\int_0^{\pi/2}\left(1-\cos^2x\right)^2\sin x\,dx=\int_0^{\pi/2}\left(1-2\cos^2x+\cos^4x\right)\sin x\,dx$. Substituting $u=\cos x$ and $du=-\sin x dx$: $\int_{x=0}^{x=\pi/2}\left(-1+2u^2-u^4\right)du=\left[-u+\frac{2}{3}u^3-\frac{1}{5}u^5\right]_{x=0}^{x=\pi/2}=\left[-\cos x+\frac{2}{3}\cos^3x-\frac{1}{5}\cos^5x\right]_{x=0}^{x=\pi/2}=1-\frac{2}{3}+\frac{1}{5}=\frac{8}{15}$

5. First, we substitute for t^2: $u=t^2$ and $\frac{du}{2}=(t)dt$. $\int t\cos^5\left(t^2\right)dt=\int\frac{1}{2}\cos^5u\,du=\frac{1}{2}\int\cos^4u\cos u\,du=\frac{1}{2}\int\left(1-\sin^2u\right)^2\cos u\,du$. Substituting again, we set $v=\sin u$ and $dv=\cos u du$: $\frac{1}{2}\int\left(1-2v^2+v^4\right)dv=\frac{1}{2}\left[v-\frac{2}{3}v^3+\frac{1}{5}v^5\right]+C$. Back-substituting $v=\sin u$: $\frac{1}{2}\left(\sin u-\frac{2}{3}\sin^3u+\frac{1}{5}\sin^5u\right)+C$. And back-substituting $u=t^2$: $\frac{1}{2}\left(\sin t^2-\frac{2}{3}\sin^3t^2+\frac{1}{5}\sin^5t^2\right)+C=\frac{1}{2}\sin t^2-\frac{1}{3}\sin^3t^2+\frac{1}{10}\sin^5t^2+C$.

Answer to Exercise ?? (on page ??)

1. $\int_0^{\pi/2} \cos^2\theta \, d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right)+\frac{1}{2}\left(\sin 2\cdot\frac{\pi}{2}-\sin 0\right)\right] = \frac{1}{2}\left[\frac{\pi}{2}+\frac{1}{2}\sin\pi\right] = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

2. $\int_0^{\pi} \cos^4 2t\, dt = \int_0^{\pi}\left(\cos^2 2t\right)^2 dt = \int_0^{\pi}\left[\frac{1}{2}(1+\cos 4t)\right]^2 dt = \frac{1}{4}\int_0^{\pi}\left(1+2\cos 4t+\cos^2 4t\right) dt = \frac{1}{4}\int_0^{\pi}\left[1+2\cos 4t+\frac{1}{2}(1+\cos 8t)\right] dt = \frac{1}{4}\int_0^{\pi}\left[\frac{3}{2}+2\cos 4t+\frac{1}{2}\cos 8t\right] dt = \frac{1}{4}\left[\frac{3}{2}t+\frac{1}{2}\sin 4t+\frac{1}{16}\sin 8t\right]$ $\frac{1}{4}\left[\left(\frac{3\pi}{2}-0\right)+\frac{1}{2}(\sin 4\pi-\sin 0)+\frac{1}{16}(\sin 8\pi-\sin 0)\right] = \frac{3\pi}{8}$

3. $\int \sin^2 x\cos^2 x\, dx = \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\, dx = \frac{1}{4}\int\left(1-\cos^2 2x\right) dx = \frac{1}{4}\int\left[1-\frac{1}{2}(1+\cos 4x)\right]$ $\frac{1}{4}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right) dx = \frac{1}{8}\int 1-\cos 4x\, dx = \frac{1}{8}\left[x-\frac{1}{4}\sin 4x\right]+C = \frac{1}{32}\left[4x-\sin 4x\right]+C$

4. $\int_0^{2\pi}\sin^2\left(\frac{1}{3}\theta\right) d\theta = \int_0^{2\pi}\frac{1}{2}\left(1-\cos\left(\frac{2}{3}\theta\right)\right) d\theta = \frac{1}{2}\left[\theta-\frac{3}{2}\sin\left(\frac{2}{3}\theta\right)\right]_{\theta=0}^{\theta=2\pi} = \frac{1}{2}\left[(2\pi-0)-\frac{3}{2}\left(\sin\frac{4\pi}{3}-\sin 0\right)\right]$ $\frac{1}{2}\left(2\pi-\frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right)\right) = \pi+\frac{3\sqrt{3}}{8}$

5. $\int_0^{\pi}\sin^2 t\cos^4 t\, dt = \int_0^{\pi}\frac{1}{2}(1-\cos 2t)\left(\frac{1}{2}(1+\cos 2t)\right)^2 dt = \frac{1}{8}\int_0^{\pi}(1-\cos 2t)\left(1+2\cos 2t+\cos^2 2t\right) dt$ $\frac{1}{8}\int_0^{\pi}\left(1+2\cos 2t+\cos^2 2t-\cos 2t-2\cos^2 2t-\cos^3 2t\right) dt = \frac{1}{8}\int_0^{\pi}\left(1+\cos 2t-\cos^2 2t-\cos^3 2t\right) dt$ $\frac{1}{8}\int_0^{\pi}\left(1+\cos 2t-\frac{1}{2}(1+\cos 4t)\right) dt - \frac{1}{8}\int_0^{\pi}\cos 2t\cos^2 2t\, dt = \frac{1}{8}\int_0^{\pi}\left(\frac{1}{2}+\cos 2t-\frac{1}{2}\cos 4t\right) dt -$ $\frac{1}{16}\int_0^{\pi}\cos 2t\left(1-\sin^2 2t\right) dt$. Let $u=\sin 2t$, then $du = 2\cos 2t dt$ and $\cos 2t dt = \frac{du}{2}$.
   Then the previous integrals are equal to $\frac{1}{8}\left[\frac{t}{2}+\frac{1}{2}\sin 2t-\frac{1}{8}\sin 4t\right]_{t=0}^{t=\pi} - \frac{1}{16}\int_{t=0}^{t=\pi}\frac{1}{2}\left(1-u^2\right) du =$ $\frac{1}{8}\left[\left(\frac{\pi}{2}-0\right)+\frac{1}{2}(\sin 2\pi-\sin 0)-\frac{1}{8}(\sin 4\pi-\sin 0)\right]-\frac{1}{32}\left[u-\frac{1}{3}u^3\right]_{t=0}^{t=\pi} = \frac{1}{8}\left(\frac{\pi}{2}\right)-\frac{1}{32}\left[\sin 2t-\frac{1}{3}\sin^3 2t\right]$ $\frac{\pi}{16}-\frac{1}{32}\left[(\sin 2\pi-\sin 0)-\frac{1}{3}\left(\sin^3 2\pi-\sin^3 0\right)\right] = \frac{\pi}{16}-\frac{1}{32}(0) = \frac{\pi}{16}$

Answer to Exercise ?? (on page ??)

1. $\int \tan x\sec^3 x\, dx = \int \sec^2 x\tan x\sec x\, dx$. Let $u=\sec x$, then $du = \sec x\tan x dx$. Substituting, $\int \sec^2 x\tan x\sec x\, dx = \int u^2\, du = \frac{1}{3}u^3+C = \frac{1}{3}\sec^3 x+C$

2. $\int \tan^4 x\sec^6 x\, dx = \int \tan^4 x\sec^4 x\sec^2 x\, dx = \int \tan^4 x\left(1+\tan^2 x\right)^2\sec^2 x\, dx$. Let $u = \tan x$, then $du = \sec^2 x dx$. Substituting, $\int \tan^4 x\left(1+\tan^2 x\right)^2\sec^2 x\, dx = \int u^4\left(1+u^2\right)^2 du = \int u^4\left(1+2u^2+u^4\right) du = \int u^4+2u^6+u^8\, du = \frac{1}{5}u^5+\frac{2}{7}u^7+\frac{1}{9}u^9+C = \frac{1}{5}\tan^5 x+\frac{2}{7}\tan^7 x+\frac{1}{9}\tan^9 x+C$

3. $\int_0^{\pi/4}\sec^6\theta\tan^6\theta\, d\theta = \int_0^{\pi/4}\sec^4\theta\tan^6\theta\sec^2\theta\, d\theta = \int_0^{\pi/4}\left(1+\tan^2\theta\right)^2\tan^6\theta\sec^2\theta\, d\theta$. Let $u=\tan\theta$ and $du=\sec^2\theta d\theta$. Substituting, $\int_0^{\pi/4}\left(1+\tan^2\theta\right)^2\tan^6\theta\sec^2\theta\, d\theta = \int_{\theta=0}^{\theta=\pi/4}\left(1+u^2\right)^2 u^6\, du = \int_{\theta=0}^{\theta=\pi/4}\left(1+2u^2+u^4\right)u^6\, du = \int_{\theta=0}^{\theta=\pi/4}u^6+2u^8+u^{10}\, du = \left[\frac{1}{7}u^7+\frac{2}{9}u^9+\frac{1}{11}u^{11}\right]_{\theta=0}^{\theta=\pi/4} = \left[\frac{1}{7}\tan^7\theta+\frac{2}{9}\tan^9\theta+\frac{1}{11}\tan^{11}\theta\right]_{\theta=0}^{\theta=\pi/4} = \frac{1}{7}\left(1^7-0^7\right)+\frac{2}{9}\left(1^9-0^9\right)+\frac{1}{11}\left(1^{11}-0^{11}\right) = \frac{1}{7}+\frac{2}{9}+\frac{1}{11} = \frac{316}{693}$

Answer to Exercise ?? (on page ??)

This question takes the form of integration by parts. That is, $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$. If we let $g(x) = -\cos x$, then $g'(x) = \sin x$. The structure of the equation implies that $f'(x) = 4x^3$ and therefore that f could be $f(x) = x^4$.

Answer to Exercise ?? (on page ??)

1. $\frac{4}{\pi^2}$. Let $f = x$ and $g' = \sin\frac{\pi}{2}x dx$. Then $f' = dx$ and $g = -\frac{2}{\pi}\cos\frac{\pi}{2}x$. Which implies that $\int_0^1 x\sin\frac{\pi}{2}x\,dx = \left[\frac{-2}{\pi}\cos\frac{\pi}{2}x\right]_{x=0}^{x=1} - \int_0^1 \frac{-2}{\pi}\cos\frac{\pi}{2}x\,dx$. Evaluating $\left[\frac{-2x}{\pi}\cos\frac{\pi}{2}x\right]_{x=0}^{x=1} = \left(\frac{-2}{\pi}\cos\frac{\pi}{2}\right) - (0\cos 0) = 0 - 0 = 0$. Therefore, $\int_0^1 x\sin\frac{\pi}{2}x\,dx = \int_0^1 \frac{2}{\pi}\cos\frac{\pi}{2}x\,dx = \frac{2}{\pi}\left[\frac{2}{\pi}\sin\frac{\pi}{2}x\right]_0^1 = \frac{4}{\pi^2}\left[\sin\frac{\pi}{2} - \sin 0\right] = \frac{4}{\pi^2}$.

2. $\frac{e^\theta}{2}(\sin\theta + \cos\theta)$. Let $f = e^\theta$ and $g' = \cos\theta d\theta$. Then $f' = e^\theta d\theta$ and $g = \sin\theta$ and according to integration be parts $\int e^\theta\cos\theta\,d\theta = e^\theta\sin\theta - \int e^\theta\sin\theta\,d\theta$. We can also evaluate $\int e^\theta\sin\theta\,d\theta$ using integration by parts. Let $f = e^\theta$ and $g' = \sin\theta d\theta$. Then $f' = e^\theta d\theta$ and $g = -\cos\theta$ and according to integration by parts $\int e^\theta\cos\theta\,d\theta = e^\theta\sin\theta - \left[-e^\theta\cos\theta - \int -e^\theta\cos\theta\,d\theta\right] = e^\theta\sin\theta + e^\theta\cos\theta - \int e^\theta\cos\theta\,d\theta$. We can rearrange this to solve for $\int e^\theta\cos\theta\,d\theta$: $2\int e^\theta\cos\theta\,d\theta = e^\theta\sin\theta + e^\theta\cos\theta \to \int e^\theta\cos\theta\,d\theta = \frac{e^\theta}{2}(\sin\theta + \cos\theta)$.

3. $\frac{(1-t)^2}{\beta}\sin\beta t + \frac{2(t-1)}{\beta^2}\cos\beta t - \frac{2}{\beta^3}\sin\beta t$. Let $f = (1-t)^2$ and $g' = \cos\beta t dt$. Then $f' = -2(1-t)dt$ and $g = \frac{1}{\beta}\sin\beta t$. Then using integration by parts $\int(1-t)^2\cos\beta t\,dt = \frac{(1-t)^2}{\beta}\sin\beta t - \int\frac{(-2)(1-t)}{\beta}\sin\beta t\,dt = \frac{(1-t)^2}{\beta}\sin\beta t + \frac{2}{\beta}\int(1-t)\sin\beta t\,dt$. We use integration by parts again to evaluate $\int(1-t)\sin\beta t\,dt$. Let $f = 1 - t$ and $g' = \sin\beta t dt$. Then $f' = -dt$ and $g = -\frac{1}{\beta}\cos\beta t$. Then $\int(1-t)\sin\beta t\,dt = (1-t)\left(-\frac{1}{\beta}\right)\cos\beta t - \int\left(-\frac{1}{\beta}\right)\cos\beta t\ - dt = \frac{t-1}{\beta}\cos\beta t - \int\frac{\cos\beta t}{\beta}\,dt = \frac{t-1}{\beta}\cos\beta t - \frac{1}{\beta^2}\sin\beta t$. Substituting this back in for $\int(1-t)\sin\beta t\,dt$, we see that $\int(1-t)^2\cos\beta t\,dt = \frac{(1-t)^2}{\beta}\sin\beta t + \frac{2}{\beta}\left[\frac{t-1}{\beta}\cos\beta t - \frac{1}{\beta^2}\sin\beta t\right] = \frac{(1-t)^2}{\beta}\sin\beta t + \frac{2(t-1)}{\beta^2}\cos\beta t - \frac{2}{\beta^3}\sin\beta t$.

Index