Chapter 1

Sequences in Calculus

We have introduced sequences in a previous chapter. Now, we will examine them in more detail in a calculus context. You already know about arithmetic and geometric sequences, but not all sequences can be classified as arithmetic or geometric. Take the famous Fibonacci sequence, {1, 1, 2, 3, 5, 8, ...}, which can be explicitly defined as $a_n = a_{n-1} + a_{n-2}$, with $a_1 = a_2 = 1$. There is no common difference or common ratio, so the Fibonacci sequence is not arithmetic or geometric. Another example is $a_n = \sin \frac{n\pi}{6}$, which will cycle through a set of values.

Sequences have many real-world applications, including compound interest and modeling population growth. In later chapters, you will learn that the sum of all the values in a sequence is a series and how to use series to describe functions. In order to be able to do all that, first we need to talk in-depth about sequences.

Some sequences are defined explicitly, like $a_n = \sin \frac{n\pi}{6}$, while others are defined recursively, like $a_n = a_{n-1} + a_{n-2}$.

Example: Write the first 5 terms for the explicitly defined sequence $a_n = \frac{n}{n+1}$.

Solution: We can construct a table to keep track of our work:

n	work	a_n
1	$\frac{1}{1+1}$	$\frac{1}{2}$
2	$\frac{2}{2+1}$	$\frac{2}{3}$
3	$\frac{3}{3+1}$	$\frac{3}{4}$
4	$\frac{4}{4+1}$	$\frac{4}{5}$
5	$\frac{5}{5+1}$	$\frac{5}{6}$

So the first five terms are $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$.

Exercise 1

Write the first 5 terms for each sequence.

1. $a_n = \frac{2^n}{2n+1}$
2. $a_n = \cos \frac{n\pi}{2}$
3. $a_1 = 1$, $a_{n+1} = 5a_n - 3$
4. $a_1 = 6$, $a_{n+1} = \frac{a_n}{n+1}$

Working Space

Answer on Page 9

1.1 Convergence and Divergence

You can visualize a sequence on an xy-plane or a number line. Figures 1.1 and 1.2 show visualizations of the sequence $a_n = \frac{n}{n+1}$. To visualize this on the xy-plane, we take points such that $x = n$ and $y = a_n$, where n is a positive integer. What do you notice about this sequence? As n increases, a_n gets closer and closer to 1.

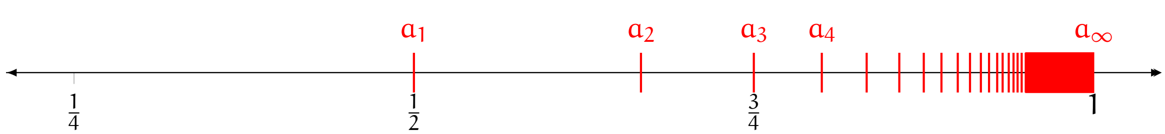

Figure 1.1: $a_n = \frac{n}{n+1}$ on a number line

Because a_n approaches a specific number as $n \to \infty$, we call the series $a_n = \frac{n}{n+1}$ *convergent*. We prove a sequence is convergent by taking the limit as n approaches ∞. If the limit exists and approaches a specific number, the sequence is convergent. If the limit does not exist or approaches $\pm\infty$, the sequence is divergent.

We can see graphically that $\lim_{n\to\infty} \frac{n}{n+1} = 1$, so that sequence is convergent. What about $b_n = \frac{n}{\sqrt{10+n}}$? Is b_n convergent or divergent?

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{n/n}{\sqrt{\frac{10}{n^2} + \frac{n}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

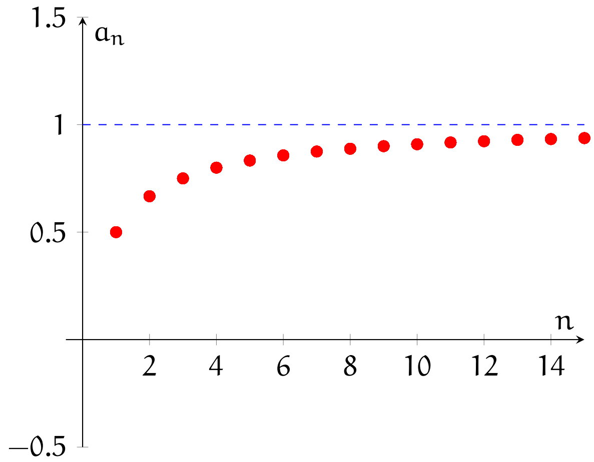

Figure 1.2: $a_n = \frac{n}{n+1}$ on an xy-plane

Therefore, the sequence $b_n = \frac{n}{\sqrt{10+n}}$ is divergent.

Here is another example of a divergent sequence: $c_n = \sin \frac{n\pi}{2}$. The graph is shown in figure 1.3. As you can see, the value of c_n oscillates between 1, 0, and -1 without approaching a specific number. This means that c_n does not approach a particular number as $n \to \infty$ and the sequence is divergent.

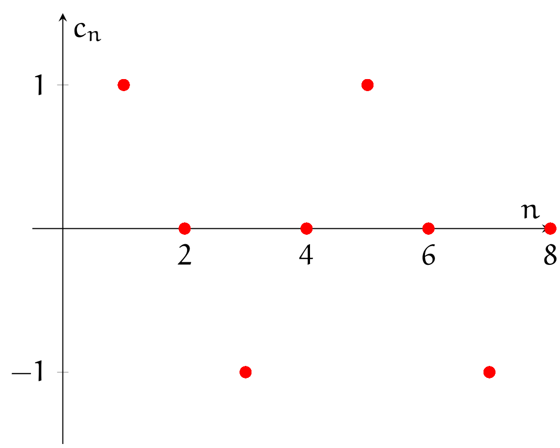

Figure 1.3: $c_n = \sin \frac{n\pi}{2}$ on an xy-plane

Exercise 2

Classify each sequence as convergent or divergent. If the sequence is convergent, find the limit as $n \to \infty$.

1. $a_n = \frac{3+5n^2}{n+n^2}$
2. $a_n = \frac{n^4}{n^3-2n}$
3. $a_n = 2 + (0.86)^n$
4. $a_n = \cos \frac{n\pi}{n+1}$
5. $a_n = \sin n$

Working Space

Answer on Page 9

1.2 Evaluating limits of sequences

Recall that a sequence can be considered a function where the domain is restricted to positive integers. If there is some $f(x)$ such that $a_n = f(n)$ when n is an integer, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$ (see figure 1.4). This means that all the rules that apply to the limits of functions also apply to the limits of sequences, including the Squeeze Theorem and l'Hospital's rule.

What is $\lim_{n\to\infty} \frac{\ln n}{n}$? First, we will try to compute the limit directly:

$$\lim_{n\to\infty} \frac{\ln n}{n} =$$

$$\frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n} = \frac{\infty}{\infty}$$

This is undefined, but fits the criteria for Ll'Hospital's rule:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{d}{dn} \ln n}{\frac{d}{dn} n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

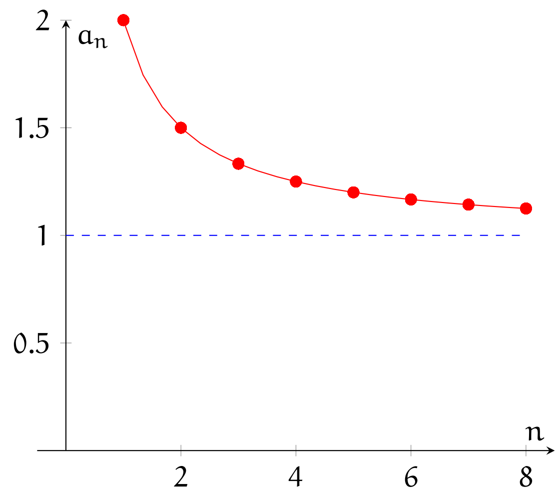

Figure 1.4: The limit of the function is the same as the limit of the sequence

Here's an example that requires the Squeeze Theorem: is the sequence $a_n = \frac{n!}{n^n}$ convergent or divergent? First trying to take the limit directly, we see that:

$$\lim_{n\to\infty} \frac{n!}{n^n} = \frac{\infty}{\infty}$$

which is undefined. Because the factorial cannot be described as a continuous function, we can't use l'Hospital's rule. We can examine this sequence graphically (see figure 1.5) and mathematically. We examine it mathematically by writing out a few terms to get an idea of what happens to a_n as n gets large:

$$a_1 = \frac{1!}{1^1} = 1$$

$$a_2 = \frac{2!}{2^2} = \frac{1\cdot 2}{2\cdot 2}$$

$$a_3 = \frac{3!}{3^3} = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}$$

$$\cdots$$

$$a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot n\cdots n}$$

From examining the graph in figure 1.5, we can guess that $\lim_{n\to\infty} a_n = 0$. Let's prove that mathematically. We can rewrite our expression for a_n as n gets large:

$$a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot n\cdots n} = \frac{1}{n}\left(\frac{2\cdot 3\cdots n}{n\cdot n\cdots n}\right)$$

The expression inside the parentheses is less than 1, therefore $0 < a_n < \frac{1}{n}$. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, by Squeeze Theorem we know that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. Therefore, the sequence $a_n = \frac{n!}{n^n}$ is convergent.

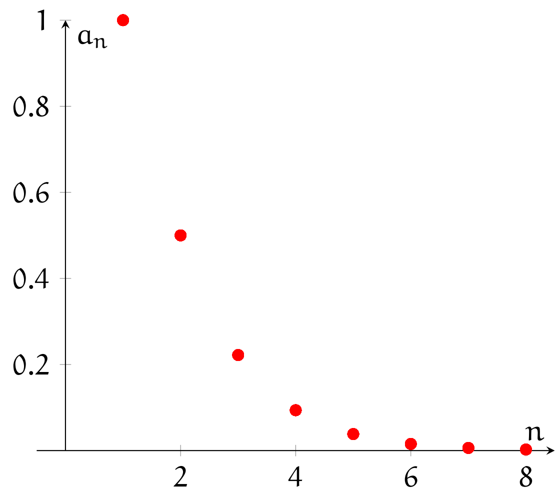

Figure 1.5: $a_n = \frac{n!}{n^n}$

[[FIX ME intro]] If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. For example, what is $\lim_{n\to\infty} \sin \frac{\pi}{n}$? Well, we know that $\lim_{n\to\infty} \frac{\pi}{n} = 0$ and that the sine function is continuous at 0. Therefore, $\lim_{n\to\infty} \sin \frac{\pi}{n} = \sin \lim_{n\to\infty} \frac{\pi}{n} = \sin 0 = 0$.

1.3 Monotonic and Bounded sequences

Just like functions, sequences can be increasing or decreasing. A sequence is increasing if $a_n < a_{n+1}$ for $n \geq 1$. Similarly, a sequence is decreasing if $a_n > a_{n+1}$ for $n \geq 1$. If a sequence is strictly increasing or decreasing, it is called *monotonic*.

The sequence $a_n = \frac{1}{n+6}$ is decreasing. We prove this formally by comparing a_n to a_{n+1}:

$$\frac{1}{n+6} > \frac{1}{(n+1)+6} = \frac{1}{n+7}$$

Is the sequence $a_n = \frac{n}{n^2+1}$ increasing or decreasing? First, we find an expression for a_{n+1}:

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$$

Since the degree of n is greater in the denominator, we have a guess that the sequence is decreasing. To prove this, we check if $a_n > a_{n+1}$ is true:

$$\frac{n}{n^2+1} > \frac{n+1}{n^2+2n+2}$$

We can cross-multiply, because $n > 0$ and the denominators are positive:

$$(n)(n^2+2n+2) > (n+1)(n^2+1)$$

$$n^3 + 2n^2 + 2n > n^3 + n^2 + n + 1$$

Subtracting $(n^3 + n^2 + n)$ from both sides we see that:

$$n^2 + n > 1$$

Which is true for all $n \geq 1$. Therefore, $a_n > a_{n+1}$ for all $n \geq 1$ and the sequence is decreasing.

A sequence is *bounded above* if there is some number M such that $a_n \leq M$ for all $n \geq 1$. And a sequence is *bounded below* if there is some other number m such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above and below, then it is a *bounded sequence*.

If a sequence is both *monotonic* and *bounded*, then it must be convergent.

Example: is the sequence given by $a_n = 4$ and $a_{n+1} = \frac{1}{2}(a_n + 7)$ bounded above, below, both, or neither?

We start by calculating the first several terms:

Term	Work	Value
a_1	$a_1 = 4$	4
a_2	$= \frac{1}{2}(4 + 7)$	5.5
a_3	$= \frac{1}{2}(5.5 + 7)$	6.25
a_4	$= \frac{1}{2}(6.25 + 7)$	6.625
a_5	$= \frac{1}{2}(6.625 + 7)$	6.8125
a_6	$= \frac{1}{2}(6.8125 + 7)$	6.90625
a_7	$= \frac{1}{2}(6.90625 + 7)$	6.953125
a_8	$= \frac{1}{2}(6.953125 + 7)$	6.9765625

The sequence is increasing, so it is bounded below by the initial term, $a_1 = 4$, and we can state that $a_n \geq 4$. Examining the computed terms, we see that $a_n \to 7$ as n grows larger. We can guess that this sequence is bounded above, with $a_n \leq 7$. We can prove this by induction. Suppose that there is some k such that $a_k < 7$ (which is true for a_1, etc.). Then

$$a_k < 7$$

$$a_k + 7 < 14$$
$$\frac{1}{2}(a_k + 7) < \frac{1}{2}(14)$$
$$a_{k+1} < 7$$

Therefore, $a_n < 7$ for all n and the sequence is bounded above. Because the sequence is monotonic and bounded, we know the sequence is convergent and, therefore, that the limit of a_n as $n \to \infty$ exists.

1.4 Applications of Sequences

1.4.1 Compound Interest

When you put money in a bank account or invest it, your money may accumulate *compound interest*. Compound interest is interest paid on the principal (the amount you originally invested) and previously accumulated interest. For example, suppose you put \$100 in a savings account that earns 3.25% annually. After one year, the bank will pay $(\$100) \cdot (0.0325) = \$3.25$ into your savings account. Now you have \$103.25. The *next* year, your bank will pay you 3.25% interest on the entire amount in your bank account (the \$100 principal plus the \$3.25 in earned interest). This comes to $\$103.25 \cdot 0.0325 = 3.36$ and your new account total will be \$106.61. This continues year over year and can build up quite a bit of money. How can we model compound interest as a sequence? Let's write out how we calculate the first few years of interest accumulation:

$$a_0 = 100 \text{ (this represents our principal investment)}$$

$$a_1 = a_0 + a_0(0.0325) \text{ (the total value of the bank account will be the previous year's value plus the interest)}$$

We can combine terms to see that:

$$a_1 = a_2 1.10325$$

Similarly,

$$a_2 = a_1(1.0325) = [a_0(1.0325)](1.0325) = a_0(1.0325)^2$$
$$a_3 = a_2(1.0325) = [a_0(1.0325)^2](1.0325) = a_0(1.0325)^3$$

Do you see the pattern? For an initial investment of \$100 at an annual rate of 3.25% compounded annually, the value of the bank account in the n^{th} year is given by:

$$a_n = 100(1.0325)^n$$

The general form of this formula with an initial investment of P and annual interest rate r is:

$$a_n = P(1 + r)^n$$

Exercise 3

You invest $1500 at 5%, compounded annually. Write an explicit formula that describes the value of your investment every year. What will your investment be worth after 10 years? Is the sequence convergent or divergent? Explain.

Working Space

Answer on Page ??

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

1. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$
2. 0, -1, 0, 1, 0
3. 1, 2, 7, 32, 157
4. 6, 3, 1, $\frac{1}{4}$, $\frac{1}{20}$

Answer to Exercise 2 (on page 4)

1. convergent, 5
2. divergent
3. convergent, 2
4. convergent, -1
5. divergent

Answer to Exercise ?? (on page ??)

Out principal is $P = 1500$ and the interest rate is $r = 0.06$. After n years, your investment will be worth $a_n = 1500(1.06)^n$. For $n = 10$, your investment will be valued at $a_{10} = \$1500(1.06)^{10} = \$2686.27$ (that's over \$1000 in interest!). To determine if the sequence is convergent or divergent, we examine the limit as $n \to \infty$:

$$\lim_{n\to\infty} 1500(1.06)^n = 1500 \cdot \lim_{n\to\infty} (1.06)^n = 1500 \cdot \infty = \infty$$

The sequence is divergent.