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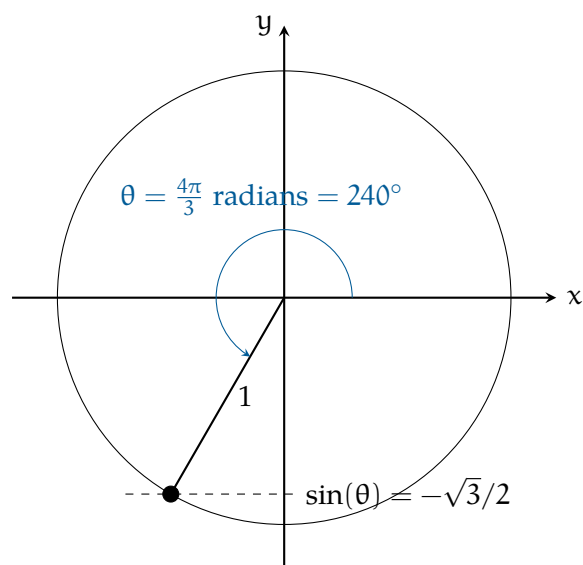
## CHAPTER 1

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# Trigonometric Functions

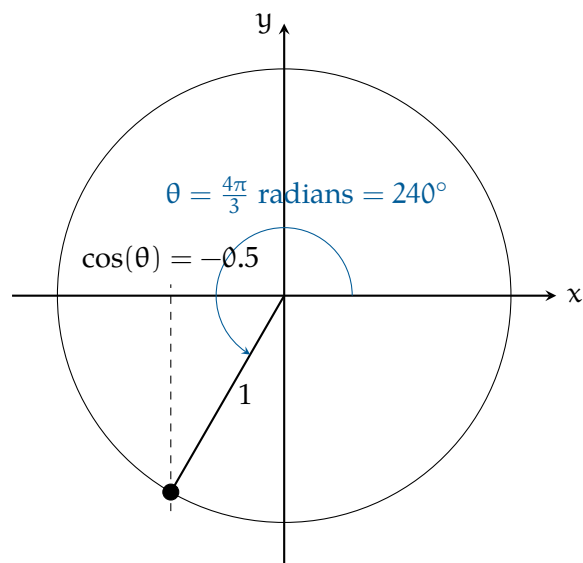
As mentioned earlier, in a right triangle where one angle is  $\theta$ , the sine of  $\theta$  is the length of the side opposite  $\theta$  divided by the length of the hypotenuse.

The sine function is defined for any real number. We treat that real number  $\theta$  as an angle, we draw a ray from the origin out to the unit circle. The y value of that point is the sine. So, for example, the  $\sin(\frac{4\pi}{3})$  is  $-\sqrt{3}/2$



(Note that in this section, we will be using radians instead of degrees unless otherwise noted. While degrees are more familiar to most people, engineers and mathematicians nearly always use radians when solving problems. Your calculator should have a radians mode and a degrees mode. You want to be in radians mode.)

Similarly, we define cosine using the unit circle: to find the cosine of  $\theta$ , we draw a ray from the origin at the angle  $\theta$ . The x component of the point where the ray intersects the unit circle is the cosine of  $\theta$ .



From this description, it is easy to see why  $\sin(\theta)^2 + \cos(\theta)^2 = 1$ . They are the legs of a right triangle with a hypotenuse of length 1.

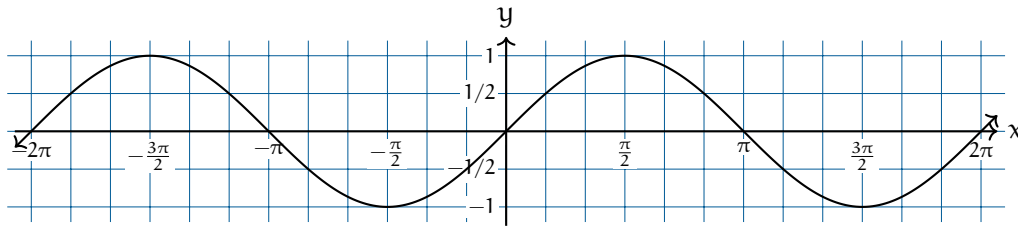
It should also be easy to see why  $\sin(\theta) = \sin(\theta + 2\pi)$ : Each time you go around the circle,

you come back to where you started.

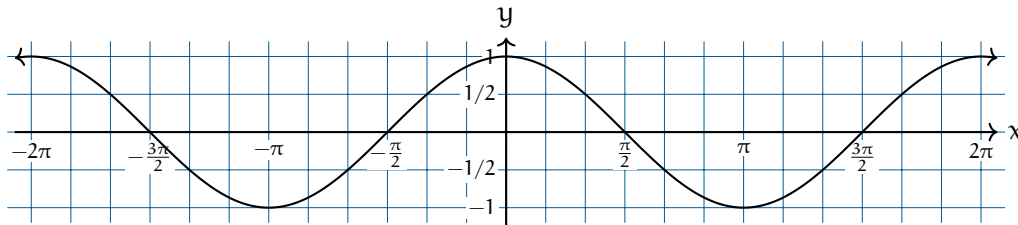
Can you see why  $\cos(\theta) = \sin(\theta + \pi/2)$ ? Turn the picture sideways.

## 1.1 Graphs of sine and cosine

Here is a graph of  $y = \sin(x)$ :



It looks like waves, right? It goes forever to the left and right. Remembering that  $\cos(\theta) = \sin(\theta + \pi/2)$ , we can guess what the graph of  $y = \cos(x)$  looks like:



## 1.2 Plot cosine in Python

Create a file called `cos.py`:

```
import numpy as np
import matplotlib.pyplot as plt

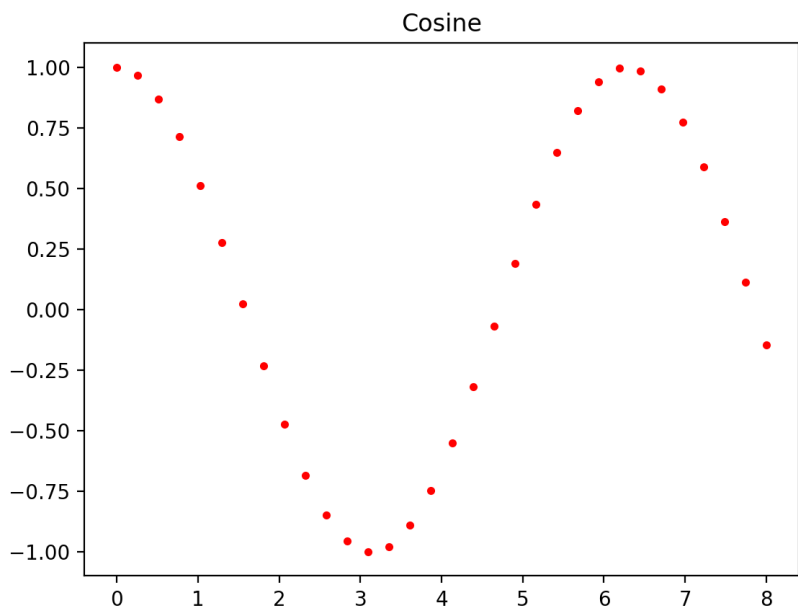
until = 8.0

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
```

```
fig, ax = plt.subplots()
ax.plot(thetas, cosines, 'r.', label="Cosine")
ax.set_title("Cosine")
plt.show()
```

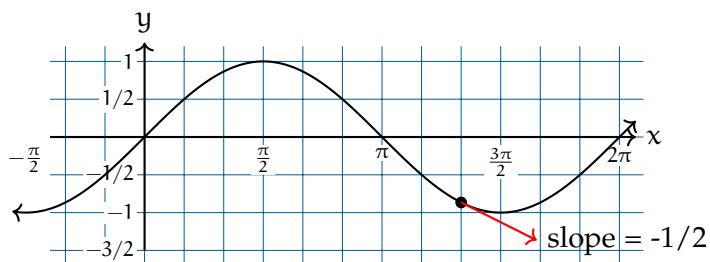
This will plot 32 points on the cosine wave between 0 and 8. When you run it, you should see something like this:



### 1.3 Derivatives of sine and cos

Here is a wonderful property of sine and cosine functions: At any point  $\theta$ , the slope of the sine graph at  $\theta$  equals  $\cos(\theta)$ .

For example, we know that  $\sin(4\pi/3) = -(1/2)\sqrt{3}$  and  $\cos(4\pi/3) = -1/2$ . If we drew a line tangent to the sine curve at this point, it would have a slope of  $-1/2$ :



We say “The derivative of the sine function is the cosine function.”

Can you guess the derivative of the cosine function? For any  $\theta$ , the slope of the graph of the  $\cos(\theta)$  is  $-\sin(\theta)$ .

## 1.4 A weight on a spring

Let’s say you fill a rollerskate with heavy rocks and attach it to the wall with a stiff spring. If you push the skate toward the wall and release it, it will roll back and forth. Engineers would say “The skate will oscillate.”

Intuitively, you can probably guess:

- If the spring is stronger, the skate will oscillate more times per minute.
- If the rocks are lighter, the skate will oscillate more times per minute.

The force that the spring exerts on the skate is proportional to how far its length is from its relaxed length. When you buy a spring, the manufacturer advertises its “spring rate”, which is in pounds per inch or newtons per meter. If a spring has a rate of 5 newtons per meter, which means that if stretch or compress it 10 cm, it will push back with a force of 0.5 newtons. If you stretch or compress it 20 cm, it will push back with a force of 1 newton.

Let’s write a simulation of the skate-on-a-spring. Duplicate `cos.py`, and name the new copy `spring.py`. Add code to implement the simulation:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Constants
mass = 100 # kg
spring_constant = -1 # newtons per meter displacement
time_step = 0.01 # s

# Initial state
displacement = 1.0 # height above equilibrium in meters
velocity = 0.0
time = 0.0 # seconds

# Lists to gather data
```

```
displacements = []
times = []

# Run it for a little while
while time <= until:
    # Record data
    displacements.append(displacement)
    times.append(time)

    # Calculate the next state
    time += time_step
    displacement += time_step * velocity
    force = spring_constant * displacement
    acceleration = force / mass
    velocity += acceleration

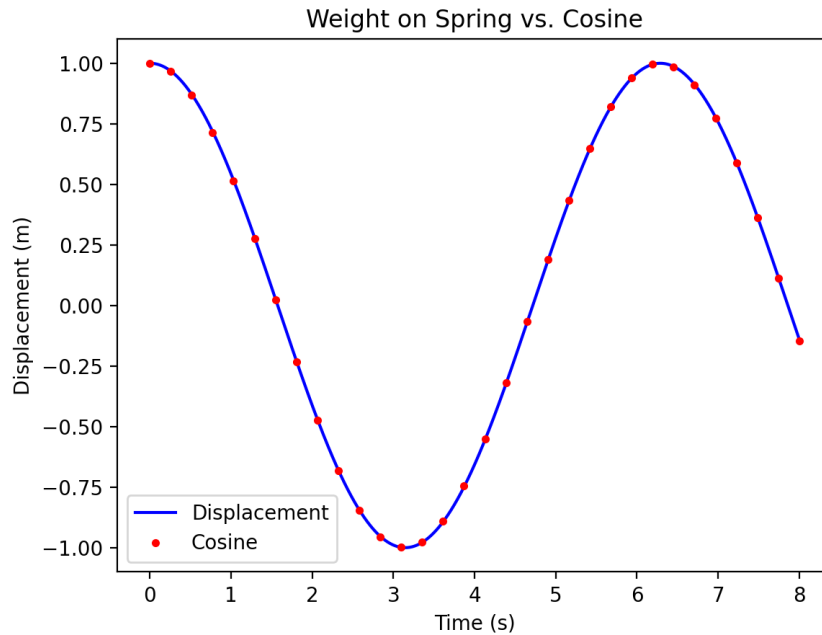
# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(times, displacements, 'b', label="Displacement")
ax.plot(thetas, cosines, 'r.', label="Cosine")

ax.set_title("Weight on Spring vs. Cosine")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Displacement (m)")
ax.legend()
plt.show()
```

When you run it, you should get a plot of your spring and the cosine graph on the same plot.





The position of the skate is following a cosine curve. Why?

Because a sine or cosine waves happen whenever the acceleration of an object is proportional to -1 times its displacement. Or in symbols:

$$a \propto -p$$

where  $a$  is acceleration and  $p$  is the displacement from equilibrium.

Remember that if you take the derivative of the displacement, you get the velocity. And if you take the derivative of that, you get acceleration. So, the weight on the spring must follow a function  $f$  such that

$$f(t) \propto -f''(t)$$

Remember that the derivative of the  $\sin(\theta)$  is  $\cos(\theta)$ .

And the derivative of the  $\cos(\theta)$  is  $-\sin(\theta)$

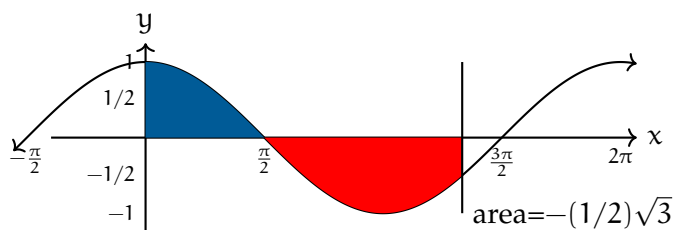
Thus these sorts of waves have an almost-magical power: their acceleration is proportional to -1 times their displacement.

Thus sine waves of various magnitudes and frequencies are ubiquitous in nature and

technology.

## 1.5 Integral of sine and cosine

If we take the area between the graph and the  $x$  axis of the cosine function (and if the function is below the  $x$  axis, it counts as negative area), from 0 to  $4\pi/3$ , we find that it is equal to  $-(1/2)\sqrt{3}$



We say “The integral of the cosine function is the sine function.”

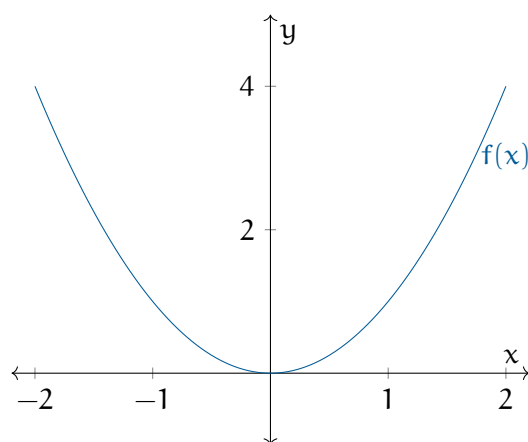


## CHAPTER 2

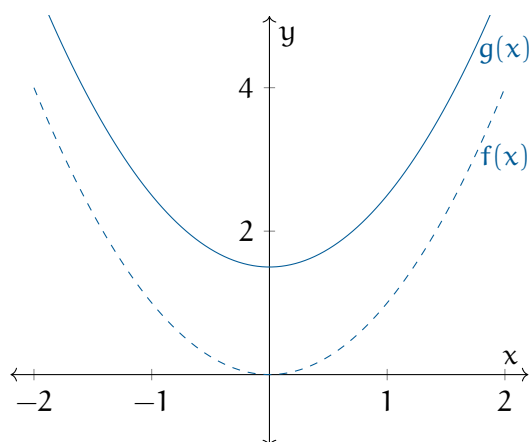
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# Transforming Functions

Let's say I gave you the graph of a function  $f$ , like this:



And then I tell you that the function  $g(x) = f(x) + 1.5$ . Can you guess what the graph of  $g$  would look like? It is the same graph, just translated up 1.5:



There are four kinds of transformations that we do all the time:

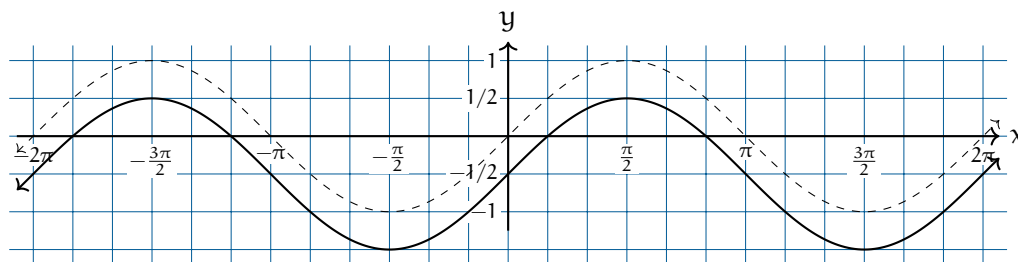
- Translation up and down in the direction of  $y$  axis (the one you just saw)
- Translation left and right in the direction of the  $x$  axis
- Scaling up and down along the  $y$  axis
- Scaling up and down along the  $x$  axis

Now I will demonstrate each of the four using the graph of  $\sin(x)$ .

## 2.1 Translation up and down

When you add a positive constant to a function, you translate the whole graph up that much. A negative constant translates it down.

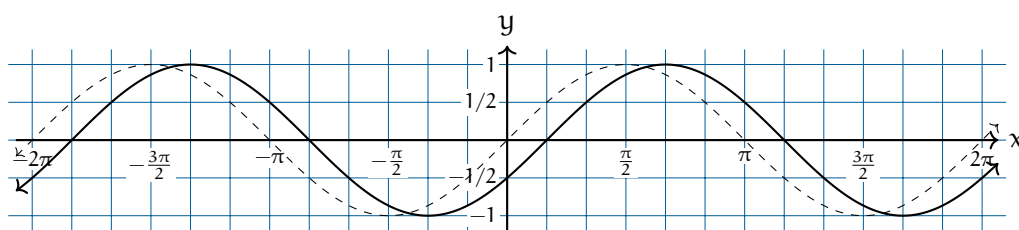
Here is the graph of  $\sin(x) - 0.5$ :



## 2.2 Translation left and right

When you add a positive number to  $x$  before running it through  $f$ , you translate the graph to the left that much. Adding a negative number translates the graph to the right.

Here is the graph of  $\sin(x - \pi/6)$ :



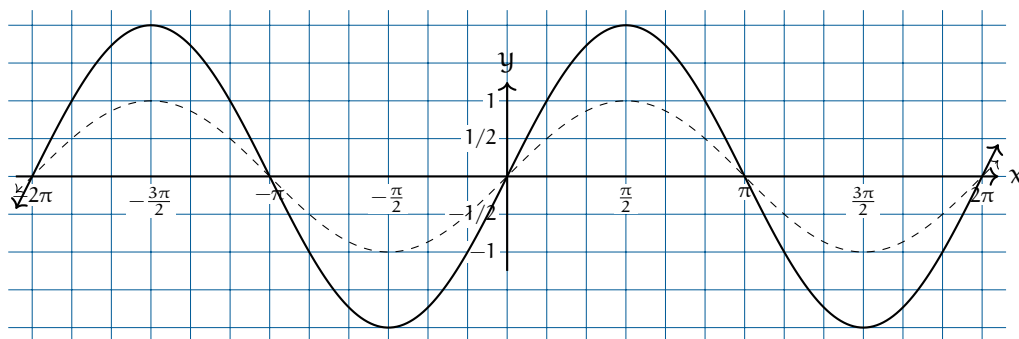
Notice the sign:

- Add to  $x$  before processing with the function translates the graph to the *left*.
- Subtract from  $x$  before processing with the function translates the graph to the *right*.

## 2.3 Scaling up and down in the $y$ direction

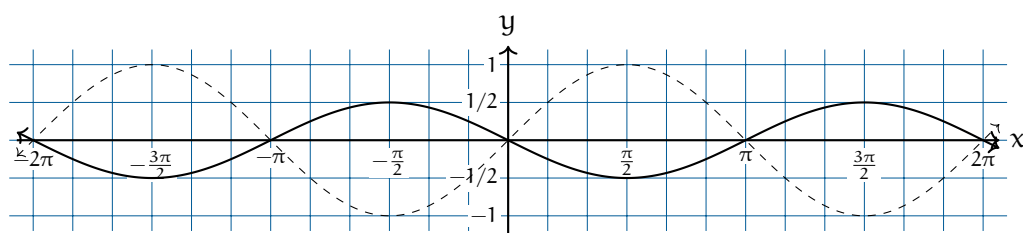
To scale the function up and down, you multiply the result of the function by a constant. If the constant is larger than 1, it stretches the function up and down.

Here is  $y = 2 \sin(x)$ :



With a wave like this, we speak of its *Amplitude*, which you can think of as its height. The baseline that this wave oscillates around is zero. The maximum distance that it gets from that baseline is its amplitude. Thus, the amplitude here has been increased from 1 to 2.

If you multiply by a negative number, the function gets flipped. Here is  $y = -0.5 \sin(x)$

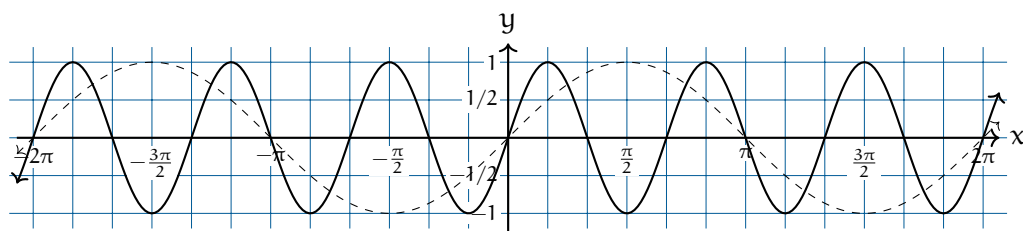


Amplitude is never negative. Thus, the amplitude of this wave is 0.5.

## 2.4 Scaling up and down in the $x$ direction

If you multiply  $x$  by a number larger than 1 before running it through the function, the graph gets compressed toward zero.

Here is  $y = \sin(3x)$ :

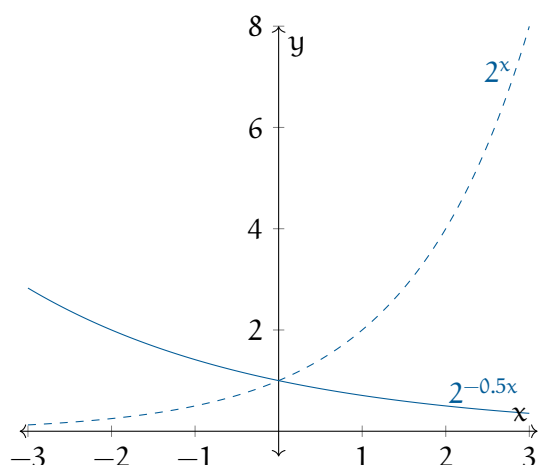


The distance between two peaks of a wave is known as its *wavelength*. The original wave had a wavelength of  $2\pi$ . The compressed wave has a wavelength of  $2\pi/3$ .

If you multiply  $x$  by a number smaller than 1, it will stretch the function out, away from the  $y$  axis.

If you multiply  $x$  by a negative number, it will flip the function around the  $y$  axis.

Here is  $y = 2^{(-0.5x)}$ . Notice that it has flipped around the  $y$  axis and is stretched out along the  $x$  axis.

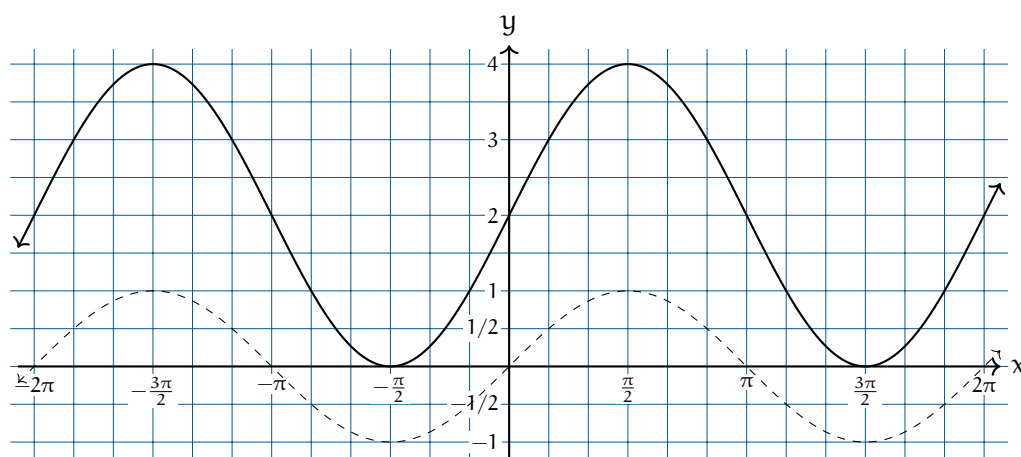


<b>Reflection over x-axis</b>	$(x, y) \rightarrow (x, -y)$
<b>Reflection over y-axis</b>	$(x, y) \rightarrow (-x, y)$
<b>Translation</b>	$(x, y) \rightarrow (x + a, y)$
<b>Dilation</b>	$(x, y) \rightarrow (kx, ky)$
<b>Rotation 90° counterclockwise</b>	$(x, y) \rightarrow (-y, x)$
<b>Rotation 180°</b>	$(x, y) \rightarrow (-x, -y)$

## 2.5 Order is important!

We can combine these transformations. This allows us, for example, to translate a function up 2 and then scale along the y axis by 3.

Here is  $y = 2.0(\sin(x) + 1)$ :



A function is often a series of steps. Here are the steps in  $f(x) = 2(\sin(x) + 1)$ :

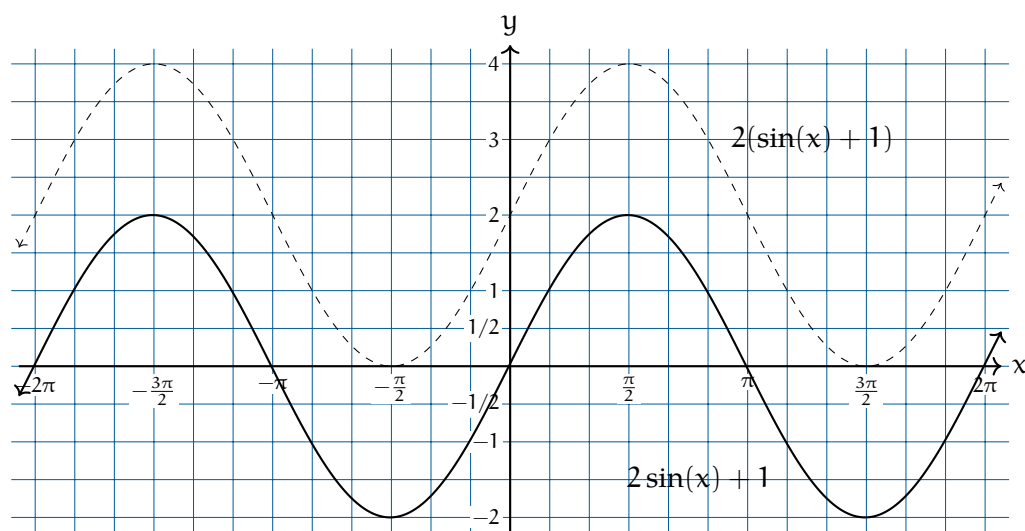
1. Take the sine of  $x$
2. Add 1 to that
3. Multiply that by 2

What if we change the order? Here are the steps in  $g(x) = 2\sin(x) + 1$ :

1. Take the sine of  $x$

2. Multiply that by 2

3. Add 1 to that

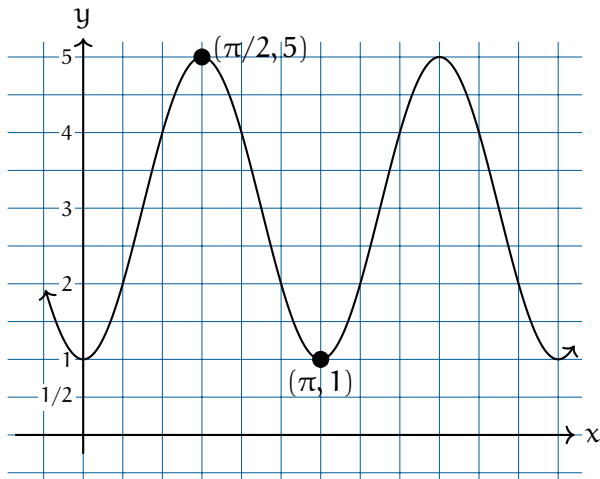


The moral: You can do multiple transformations of your function, but the order in which you do them is important.



**Exercise 1**      **Transforms**

Find a function that creates a sine wave such that the top of the first crest is at the point  $(\frac{\pi}{2}, 5)$  and the bottom of the trough that follows is at  $(\pi, 1)$ .

*Working Space**Answer on Page 19*



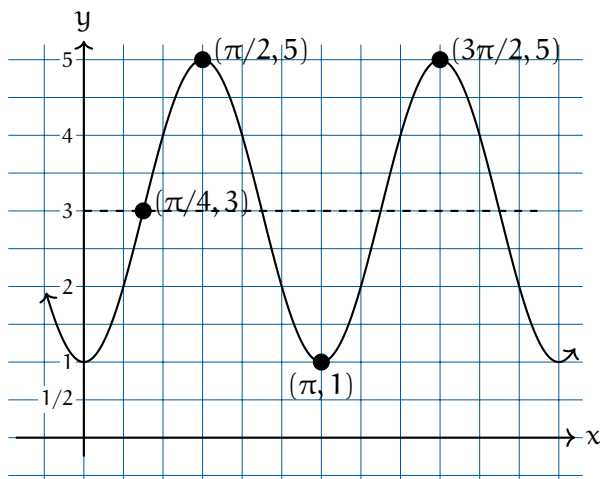


## APPENDIX A

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# Answers to Exercises

### Answer to Exercise 1 (on page 17)



This wave has an amplitude of 2. Its baseline has been translated up to 3.

This wave has wavelength of  $\pi$ . A sine wave usually has a wavelength of  $2\pi$ , so we need to compress the  $x$  axis by a factor of 2.

The wave first crosses its baseline at  $\pi/4$ . The sine wave starts by crossing its baseline, so we need to translate the curve right by  $\pi/4$ .

$$f(x) = 2 \sin\left(2x - \frac{\pi}{4}\right) + 3$$