Chapter 1

Methods of Integration

1.1 u-substitution

Sometimes a function's antiderivative isn't obvious. Take this integral for example:

$$\int 4x\sqrt{1+2x^2}\,dx$$

We can solve this integral using *u-substitution*. Recall from implicit differentiation that if $u = f(x)$, then we can also say $du = f'(x)dx$. Let's set u so that it is equal to the statement under the square root sign:

$$u = 1 + 2x^2$$

Taking the derivative of both sides, we see that

$$du = (4x)dx$$

How does this help us evaluate the integral? First, let's rearrange the integrand a bit:

$$\int 4x\sqrt{1+2x^2}\,dx = \int \sqrt{1+2x^2}4x\,dx$$

We can substitute $u = 1 + 2x^2$ and $du = 4xdx$ to get:

$$= \int \sqrt{u}\,du$$

That is a much nicer integral! We can evaluate this integral using the Power Rule:

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2}$$

We can now substitute $u = 1 + 2x^2$ back into our solution to yield:

$$= \frac{2}{3}(1 + 2x^2)^{3/2}$$

Feel free to double-check this answer by taking the derivative using the Chain Rule. You should get the original integrand, $4x\sqrt{1 + 2x^2}$, back.

As you may have guessed, u-substitution is a method to help us "undo" the Chain Rule. Recall that the Chain Rule states:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

If we integrate both sides we see that:

$$f(g(x)) = \int f'(g(x))g'(x)\, dx$$

Which leads us to the formal definition of the u-substitution method:

If u = $g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\int f(g(x))g'(x)\, dx = \int f(u)\, du$

Let's apply u-substitution to a definite integral:

Example: Evaluate $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx$.

Solution: Recall that $\frac{d}{dx} \ln x = \frac{1}{x}$. Letting $\ln x = u$, it follows that $\frac{dx}{x} = du$. Rearranging the integral and substituting:

$$\begin{aligned}\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx &= \int_e^{e^4} \frac{1}{\sqrt{\ln x}} \frac{dx}{x} \\ &= \int_{x=e}^{x=e^4} \frac{1}{\sqrt{u}}\, du\end{aligned}$$

Proceeding from here, there are two options: you can find the value of u at $x = e$ and $x = e^4$ and change the limits of the integral OR you can evaluate the integral, resubstitute back for x and then evaluate the result with the original limits. We will show both to demonstrate each method and show they have the same result.

Method 1: change the limits of integration When $x = e$, $u = \ln e = 1$. And when $x = e^4$, $u = \ln e^4 = 4$. Therefore, we can change the limits of the integral to:

$$\int_1^4 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_1^4 = 2\left[\sqrt{4} - \sqrt{1}\right] = 2(2-1) = 2$$

Method 2: keep the limits of integration and resubstitute for u:

$$\int_{x=e}^{x=e^4} \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_{x=e}^{x=e^4} = 2\sqrt{\ln x}\Big|_e^{e^4}$$
$$= 2\left[\sqrt{\ln e^4} - \sqrt{\ln e}\right] = 2(\sqrt{4} - \sqrt{1}) = 2(2-1) = 2$$

Which is the same result as method 1. When done correctly, either method will yield the correct result. Choose the method you prefer.

Exercise 1

Using the substitution $u = x^2 - 3$, rewrite $\int_{-1}^{4} x(x^2-3)^5\,dx$ in terms of u.

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Exercise 2

Evaluate $\int_1^\infty xe^{-x^2}\,dx$.

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1.2 Partial Fractions

We can integrate rational functions by using partial fractions to decompose a complex rational function into simpler ones. Suppose we wanted to integrate $f(x) = \frac{4x+5}{x^2+x-2}$:

$$\int \frac{4x+5}{x^2+x-2}\,dx = \int \left(\frac{3}{x-1} + \frac{1}{x+2}\right) dx$$
$$= 3\ln|x-1| + \ln|x+2| + C$$

Example: Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$

Solution: We start by defining:

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

Multiplying both sides by $(x+1)^2(x+2)$:

$$x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Since there are only 2 roots to $(x+1)^2(x+2)$, we will equate the coefficients to find A, B, and C.

$$x^2+x+1 = A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)$$
$$x^2+x+1 = Ax^2+3Ax+2A+Bx+2B+Cx^2+2Cx+C$$
$$x^2+x+1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$

For this equation to be true, we know that:

$$A+C = 1$$
$$3A+B+2C = 1$$
$$2A+2B+C = 1$$

Solving for each, you should find that:

$$A = -2$$
$$B = 1$$
$$C = 3$$

And therefore,

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$

Substituting this into our integral,

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx = \int \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}\right] dx$$

$$= -2\ln|x+1| + \frac{-1}{x+1} + 3\ln|x+2| + C = \ln\left|\frac{(x+2)^3}{(x+1)^2}\right| - \frac{1}{x+1} + C$$

Example: Evaluate $\int \frac{2x^2-x+4}{x^3+4x}\,dx$

Solution: We begin by factoring the denominator:

$$x^3 + 4x = x(x^2+4)$$

Which cannot be factored further. Therefore, we define:

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

Which implies that:

$$2 = A + B$$

$$C = -1$$

$$4A = 4$$

Therefore, $A = 1$, $B = 1$, and $C = -1$ and we can say that:

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \left[\frac{1}{x} + \frac{x-1}{x^2+4}\right]\,dx$$

$$= \int \left[\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right]\,dx$$

$$= \ln|x| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

A useful identity that we used here is

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Exercise 3

Evaluate $\int_0^1 \frac{5x+8}{x^2+3x+2}\,dx$ without a calculator.

Working Space

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Exercise 4

Use the method of partial fractions to evaluate the following integrals:

1. $\int \frac{4x}{x^3+x^2+x+1}\,dx$
2. $\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx$
3. $\int \frac{x^3+2x}{x^4+4x^2+3}\,dx$

Working Space

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1.3 Integration by Parts

Recall the Product Rule for derivatives:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

If we integrate both sides, we find that:

$$f(x) \cdot g(x) = \int \left[f(x) \cdot g'(x) + f'(x) \cdot g(x)\right] \, dx$$
$$f(x) \cdot g(x) = \int f(x)g'(x) \, dx + \int f'(x)g(x) \, dx$$

Rearranging,

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx$$

This identity allows us to perform **integration by parts** , a powerful method that allows us to evaluate integrals of complex functions.

Example: Evaluate $\int x \cos x \, dx$.

Solution: We may be tempted to try u-substitution, but that won't work because $\frac{d}{dx} \cos x$ is not proportional to x and $\frac{d}{dx} x$ is not proportional to $\cos x$. Let us define $f(x) = x$ and $g'(x) = \cos x$. This implies $f'(x) = 1$ and $g(x) = \sin x$. Then we can say that:

$$\int x \cos x \, dx = \int f(x)g'(x) \, dx$$

Using the identity $\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx$ and substituting for $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$, we see that:

$$\int x \cos x \, dx = [x \sin x] - \int 1 \cdot \sin x \, dx$$
$$= x \sin x - \int \sin x \, dx$$
$$= x \sin x - (-\cos x + C) = x \sin x + \cos x + C$$

(recall that C is the integration constant). You can check your results by taking the derivative: you should get the original integrand back. Let's check our result in this case:

$$\begin{aligned}\frac{d}{dx}\left[x\sin x+\cos x+C\right] &= \frac{d}{dx}\left[x\sin x\right]+\frac{d}{dx}\cos x\\ &= x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)-\sin x\\ &= x\cos x+|SINx-\sin x = x\cos x\end{aligned}$$

How did we choose that $f(x)$ should be x and $g(x)$ should be $\sin x$ in the example above? In general, you want to choose such that the resulting integral is simpler than the one we started with. This means you want to choose f such that f' is *less complex* or a *lower order* than f.

To illustrate this, let's re-evalute the example above, but this time let $f(x) = \cos x$ and $g'(x) = x$. Then we can say that $f'(x) = -\sin x$ and $g(x) = \frac{1}{2}x^2$. Substituting this into the integration by parts identity, we find that:

$$\int x\cos x\,dx = \frac{1}{2}x^2\cos x - \int -\frac{1}{2}x^2\sin x\,dx$$

Now the integral on the right side is more complex than the one we started with (on the left)! A good general rule for integration by parts is that *if* the two functions in the original integral are a polynomial and a sine or cosine function, set the polynomial to be $g(x)$ and the trigonometric function to be $f'(x)$. The polynomial will be differentiated and become *less* complex, while integrating the trigonometric function won't make it *more* complex.

Integration by parts is valid for definite integrals as well. Mathematically, this means:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\,dx$$

Which is the same as:

$$\int_a^b f(x)g'(x)\,dx = (f(b)g(b)) - (f(a)g(a)) - \int_a^b f'(x)g(x)\,dx$$

Let's see one more example that incorporates both u-substitution and integration by parts.

Example: Evaluate $\int \frac{\arcsin \ln x}{x}\,dx$

Solution: First, we notice that $\ln x$ and $\frac{1}{x}$ both appear in the integrand. Let us define $u = \ln x$. Then $du = \frac{dx}{x}$:

$$\int \arcsin \ln x \frac{dx}{x} = \int \arcsin u \, du$$

For integration by parts, if we let $\arcsin u = f(u)$ and $du = g'(u)$, it follows that $f'(u) = \frac{1}{\sqrt{1-u^2}}$ and $g(u) = u$. Then we can say that:

$$\int \arcsin u \, du = \arcsin u \cdot u - \int \frac{u}{\sqrt{1-u^2}} \, du$$

We can use u-substitution again to evaluate the second integral (we will use v, since we have already said that $u = \ln x$). Let $v = 1 - u^2$, which means that $\frac{dv}{2} = (-u)du$. Substituting:

$$= u \cdot \arcsin u + \int \frac{1}{2\sqrt{v}} \, dv = u \cdot \arcsin u + \sqrt{v}$$

Substituting back for v:

$$= u \cdot \arcsin u + \sqrt{1 - u^2}$$

And substituting back for u:

$$= \ln x \cdot \arcsin \ln x + \sqrt{1 - \ln^2 x}$$

Exercise 5

Let f be a function such that $\int f(x) \sin x \, dx = -f(x) \cos x + \int 4x^3 \sin x \, dx$. Give a possible expression for $f(x)$.

Working Space

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Exercise 6

Evaluate the following integrals using integration by parts:

1. $\int_0^1 x \sin \frac{\pi}{2} x \, dx$
2. $\int e^\theta \cos \theta \, d\theta$
3. $\int (1-t)^2 \cos \beta t \, dt$

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Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

If $u = x^2 - 3$, then $du = 2x dx$ and $x(x^2-3)^5 dx = \frac{1}{2}u^5 du$. When $x = -1$, $u = -2$ and when $x = 4$, $u = 13$. Putting it all together, we find an equivalent integral is $\frac{1}{2}\int_{-2}^{13} u^5\, du$.

Answer to Exercise 2 (on page 3)

Letting $u = -x^2$, then $du = -2x dx$ and $x dx = \frac{-1}{2} du$. Substituting u and du into the integral, we have $\int_{x=1}^{x=\infty} \frac{-1}{2} e^u\, du$, which equals $\frac{-1}{2} e^u = \frac{-1}{2} e^{-x^2}|_1^\infty$. Evaluating the statement, we get $\frac{-1}{2}(e^{-\infty} - e^{-1}) = \frac{-1}{2}(0 - \frac{1}{e}) = \frac{1}{2e}$

Answer to Exercise 3 (on page 6)

We cannot use u-substitution because $\frac{d}{dx}(x^2+3x+2) \neq n(5x+8)$. We will use partial fractions to simplify the integrand. Settig up: $\frac{5x+8}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$. Rearranging, we find $5x + 8 = A(x+2) + B(x+1)$. Letting $x = -2$, we find that $B = 2$. And taking $x = -1$, we find $A = 3$. Therefore, $\int_0^1 \frac{5x+8}{x^2+3x+2}\, dx = \int_0^1 \frac{3}{x+1}\, dx + \int_0^1 \frac{2}{x+2}\, dx$. Evaluating the integrals, we get $3\ln(x+1)|_0^1 + 2\ln(x+2)|_0^1 = 3(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) = 3\ln 2 + 2\ln\frac{3}{2} = \ln 8 + \ln\frac{9}{4} = \ln\frac{8\cdot 9}{4} = \ln 18$.

Answer to Exercise 4 (on page 6)

1. Let $\frac{4x}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$. Rearranging, we see that $4x = A(x^2+1) + (Bx+C)(x+1)$. Which means that $4x = Ax^2 + A + Bx^2 + Bx + Cx + C$, which implies that $A + B = 0$ and $B + C = 4$ and $A + C = 0$. Solving this system of equations, we see that $A = -2$, $B = 2$, and $C = 2$. So we can say that $\int \frac{4x}{x^3+x^2+x+1}\, dx = \int \left[\frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\right] dx$. Which evaluates to $-2\ln|x+1| + \ln|x^2+1| + 2\arctan(x) + K$, where K is the constant of integration.

2. Since the order of x is greater in the numerator, first we divide and see that $\frac{x^3-4x+1}{x^2-3x+2} =$

$(x+3) + \frac{3x-5}{x^2-3x+2}$. Now let $\frac{3x-5}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1}$, which means that $3x - 5 = A(x-1) + B(x-2)$. Solving, we find that $A = 1$ and $B = 2$. Therefore, $\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx = \int_{-1}^{0} \left[x+3+\frac{1}{x-2}+\frac{2}{x-1}\right] dx$ which evaluates to $\frac{1}{2}x^2 + 3x + \ln|x-2| + \ln|x-1|\big|_{x=-1}^{x=0} = \frac{5}{2} - \ln(3)$.

3. Note that $\frac{x^3+2x}{x^4+4x^2+3} = \frac{x^3+2x}{(x^2+1)(x^2+3)}$. Then let $\frac{x^3+2x}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$. Then $x^3+2x = (A+C)x^3 + (B+D)x^2 + (3A+C)x + (3B+D)$ which implies that $A + C = 1$, $B + D = 0$, $3A + C = 2$, and $3b + D = 0$. Solving this system of equations, we see that $A = C = \frac{1}{2}$ and $B = D = 0$, which means that $\frac{x^3+2x}{x^4+4x^2+3} = \frac{x}{2(x^2+1)} + \frac{x}{2(x^3+3)}$. And therefore, $int\frac{x^3+2x}{x^4+4x^2+3}\,dx = \int \left[\frac{x}{2(x^2+1)} + \frac{x}{2(x^3+3)}\right] dx = \frac{1}{4}\ln|x^2+1| + \frac{1}{4}\ln|x^2+3| + K$, where K is the constant of integration.

Answer to Exercise 5 (on page 9)

This question takes the form of integration by parts. That is, $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$. If we let $g(x) = -\cos x$, then $g'(x) = \sin x$. The structure of the equation implies that $f'(x) = 4x^3$ and therefore that f could be $f(x) = x^4$.

Answer to Exercise 6 (on page 10)

1. $\frac{4}{\pi^2}$. Let $f = x$ and $g' = \sin\frac{\pi}{2}x dx$. Then $f' = dx$ and $g = -\frac{2}{\pi}\cos\frac{\pi}{2}x$. Which implies that $\int_0^1 x\sin\frac{\pi}{2}x\,dx = \left[\frac{-2}{\pi}\cos\frac{\pi}{2}x\right]_{x=0}^{x=1} - \int_0^1 \frac{-2}{\pi}\cos\frac{\pi}{2}x\,dx$. Evaluating $\left[\frac{-2x}{\pi}\cos\frac{\pi}{2}x\right]_{x=0}^{x=1} = \left(\frac{-2}{\pi}\cos\frac{\pi}{2}\right) - (0\cos 0) = 0 - 0 = 0$. Therefore, $\int_0^1 x\sin\frac{\pi}{2}x\,dx = \int_0^1 \frac{2}{\pi}\cos\frac{\pi}{2}x\,dx = \frac{2}{\pi}\left[\frac{2}{\pi}\sin\frac{\pi}{2}x\right]_0^1 = \frac{4}{\pi^2}\left[\sin\frac{\pi}{2} - \sin 0\right] = \frac{4}{\pi^2}$.

2. $\frac{e^\theta}{2}(\sin\theta + \cos\theta)$. Let $f = e^\theta$ and $g' = \cos\theta d\theta$. Then $f' = e^\theta d\theta$ and $g = \sin\theta$ and according to integration be parts $\int e^\theta\cos\theta\,d\theta = e^\theta\sin\theta - \int e^\theta\sin\theta\,d\theta$. We can also evaluate $\int e^\theta\sin\theta\,d\theta$ using integration by parts. Let $f = e^\theta$ and $g' = \sin\theta d\theta$. Then $f' = e^\theta d\theta$ and $g = -\cos\theta$ and according to integration by parts $\int e^\theta\cos\theta\,d\theta = e^\theta\sin\theta - \left[-e^\theta\cos\theta - \int -e^\theta\cos\theta\,d\theta\right] = e^\theta\sin\theta + e^\theta\cos\theta - \int e^\theta\cos\theta\,d\theta$. We can rearrange this to solve for $\int e^\theta\cos\theta\,d\theta$: $2\int e^\theta\cos\theta\,d\theta = e^\theta\sin\theta + e^\theta\cos\theta \to \int e^\theta\cos\theta\,d\theta = \frac{e^\theta}{2}(\sin\theta + \cos\theta)$.

3. $\frac{(1-t)^2}{\beta}\sin\beta t + \frac{2(t-1)}{\beta^2}\cos\beta t - \frac{2}{\beta^3}\sin\beta t$. Let $f = (1-t)^2$ and $g' = \cos\beta t dt$. Then $f' = -2(1-t)dt$ and $g = \frac{1}{\beta}\sin\beta t$. Then using integration by parts $\int(1-t)^2\cos\beta t\,dt = \frac{(1-t)^2}{\beta}\sin\beta t - \int\frac{(-2)(1-t)}{\beta}\sin\beta t\,dt = \frac{(1-t)^2}{\beta}\sin\beta t + \frac{2}{\beta}\int(1-t)\sin\beta t\,dt$. We use integration by parts again to evaluate $\int(1-t)\sin\beta t\,dt$. Let $f = 1-t$ and $g' = \sin\beta t dt$. Then $f' = -dt$ and $g = -\frac{1}{\beta}\cos\beta t$. Then $\int(1-t)\sin\beta t\,dt = (1-t)\left(-\frac{1}{\beta}\right)\cos\beta t -$

$\int \left(-\frac{1}{\beta}\right) \cos \beta t \, - \, dt = \frac{t-1}{\beta} \cos \beta t - \int \frac{\cos \beta t}{\beta} \, dt = \frac{t-1}{\beta} \cos \beta t - \frac{1}{\beta^2} \sin \beta t$. Substituting this back in for $\int (1-t) \sin \beta t \, dt$, we see that $\int (1-t)^2 \cos \beta t \, dt = \frac{(1-t)^2}{\beta} \sin \beta t + \frac{2}{\beta}\left[\frac{t-1}{\beta} \cos \beta t - \frac{1}{\beta^2} \sin \beta t\right] = \frac{(1-t)^2}{\beta} \sin \beta t + \frac{2(t-1)}{\beta^2} \cos \beta t - \frac{2}{\beta^3} \sin \beta t$.

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