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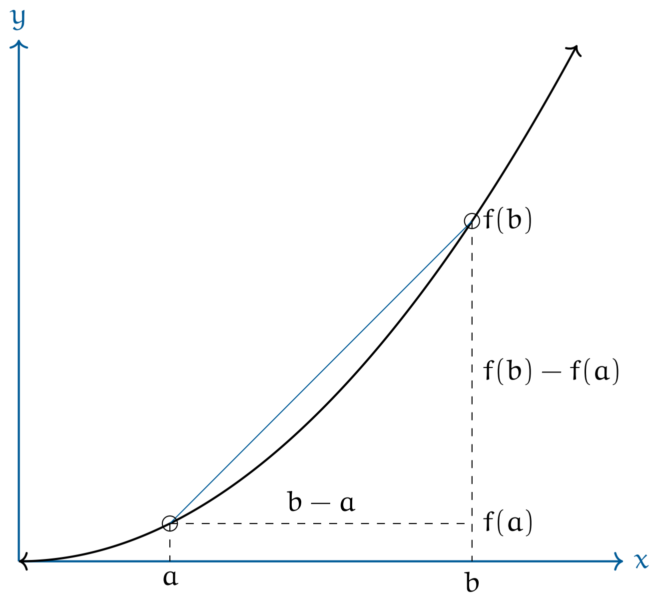
Chapter 1

Differentiation

We have done some differentiation, but you haven't been given the real definition because it is based on limits.

The idea is that we can find the slope between two points on the graph a and b like this:

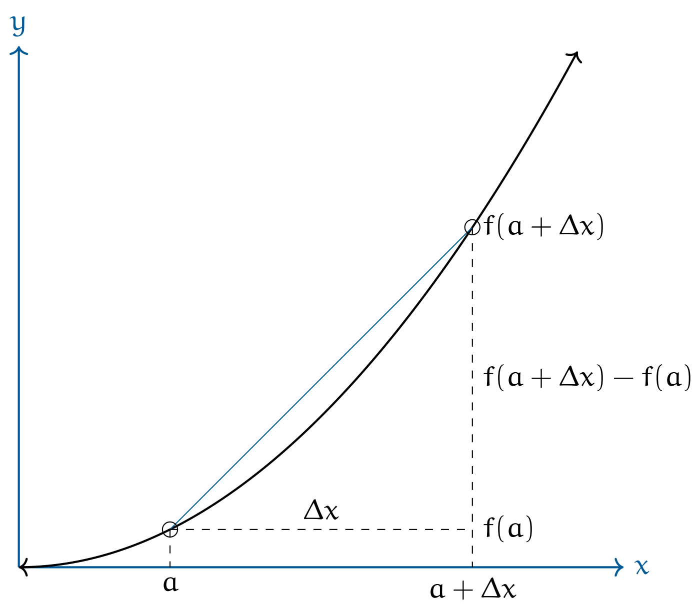
$$m = \frac{f(b) - f(a)}{b - a}$$



If we want to find the slope at a we take the limit of this as the b goes to a:

$$f'(a) = \lim_{b \to b} \frac{f(b) - f(a)}{b - a}$$

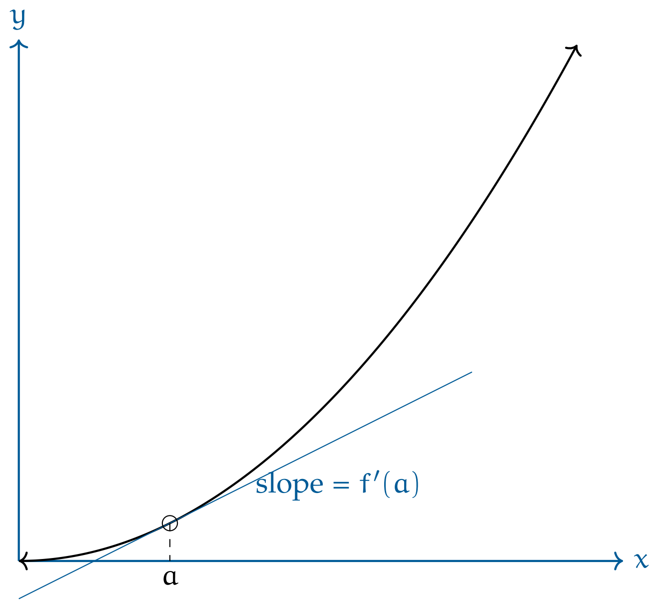
This idea is usually expressed using Δx as the difference between b and a:



Then the formula becomes:

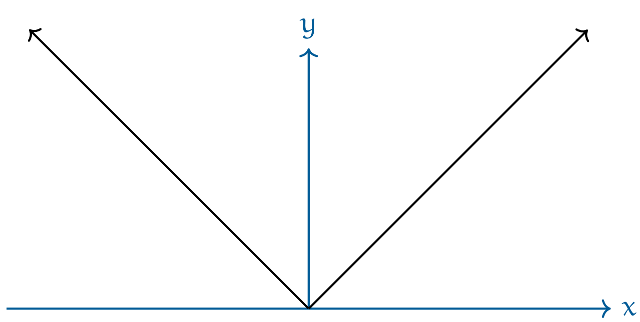
$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Now, at any point a we can compute the slope of the line tangent to the function at a:



1.1 Differentiability

Warning: Not every function is differentiable everywhere. For example, if $f(x) = |x|$, you get a corner at zero.



To the left of zero, the slope is -1. To the right of zero, the slope is 1. At zero? The derivative is not defined.

If a function has a derivative everywhere, it is said to be *differentiable*. Generally, you can think of differentiable functions as smooth – their graphs have no corners.

1.2 Using the definition of derivative

Let's say that you want to know the slope of $f(x) = -3x^2$ at $x = 2$. Using the definition of the derivative, that would be:

$$f'(2) = \lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-3(2+\Delta x)^2 - \left(-3(2)^2\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-12 - 12\Delta x + -3(\Delta x)^2 + 12}{\Delta x} = -12$$

Chapter 2

Derivatives

In calculus, the derivative of a function represents the rate at which the function is changing at a particular point. It is a fundamental concept that has vast applications in various fields, including physics.

2.1 Definition

The derivative of a function $f(x)$ at a point x is defined as the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.1}$$

provided this limit exists. In words, the derivative of f at x is the limit of the rate of change of f at x as the change in x approaches zero.

2.2 Applications in Physics

In physics, derivatives play a vital role in describing how quantities change with respect to one another.

2.2.1 Velocity and Acceleration

In kinematics, the derivative of the position function with respect to time gives the velocity function, and further taking the derivative of the velocity function gives the acceleration function. For example, if $s(t)$ represents the position of an object at time t, then the velocity $v(t)$ and acceleration $a(t)$ are given by:

$$v(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} \tag{2.2}$$

2.2.2 Force and Momentum

In mechanics, the derivative of the momentum of an object with respect to time gives the net force acting on the object, as stated by Newton's second law of motion:

$$F = \frac{dp}{dt} \tag{2.3}$$

where F is the force, p is the momentum, and t is the time.

Chapter 3

Rules for Finding Derivatives

Derivatives play a key role in calculus, providing us with a means of calculating rates of change and the slopes of curves. Here, we present some common rules used to calculate derivatives.

3.1 Constant Rule

The derivative of a constant is zero. If c is a constant and x is a variable, then:

$$\frac{d}{dx}c = 0 \tag{3.1}$$

3.2 Power Rule

For any real number n, the derivative of x^n is:

$$\frac{d}{dx}x^n = nx^{n-1} \tag{3.2}$$

3.3 Product Rule

The derivative of the product of two functions is:

$$\frac{d}{dx}(fg) = f'g + fg' \tag{3.3}$$

where f' and g' denote the derivatives of f and g, respectively.

3.4 Quotient Rule

The derivative of the quotient of two functions is:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \tag{3.4}$$

3.5 Chain Rule

The derivative of a composition of functions is:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \tag{3.5}$$

3.6 Conclusion

These rules form the basis for calculating derivatives in calculus. Many more complex rules and techniques are built upon these fundamental rules.

Chapter 4

Optimization

Optimization is a branch of mathematics that involves finding the best solution from all feasible solutions. In the field of operations research, optimization plays a crucial role. Whether it is minimizing costs, maximizing profits, or reducing the time taken to perform a task, optimization techniques are employed to make decisions effectively and efficiently.

4.1 Optimization Problems

An optimization problem consists of maximizing or minimizing a real function by systematically choosing the values of real or integer variables from within an allowed set. This function is known as the objective function.

A standard form of an optimization problem is:

$$\underset{x}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad g_i(x) \leq 0, ; i = 1, \ldots, m \quad h_j(x) = 0, ; j = 1, \ldots, p$$

where

- $f(x)$ is the objective function,
- $g_i(x) \leq 0$ are the inequality constraints,
- $h_j(x) = 0$ are the equality constraints.

4.2 Types of Optimization Problems

There are different types of optimization problems, including but not limited to:

- **Linear Programming:** The objective function and the constraints are all linear.
- **Integer Programming:** The solution space is restricted to integer values.
- **Nonlinear Programming:** The objective function and/or the constraints are nonlinear.
- **Stochastic Programming:** The objective function and/or constraints involve random variables.

These problems are solved using different techniques and algorithms, many of which are a subject of active research.

4.3 Applications

Optimization techniques have a wide variety of applications in many fields such as economics, engineering, transportation, and scheduling problems.

Appendix A

Answers to Exercises

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