Contents

Chapter 1

Limits

The asymptotic behavior we see in rational functions suggests that we need to expand our vocabulary of function characteristics. We examined vertical asymptotes and end behavior through graphs and tables and discussed them in English. The language of limits enables us to discuss these attributes with greater efficiency.

Let us revisit an example from the previous chapter. This function has a hole at $x = 1$, a vertical asymptote at $x = 3$, and a horizontal asymptote of $y = 1$.

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$

First, consider the vertical asymptote. We see that the graph goes down as it hugs the left side of the vertical asymptote, and goes up as it hugs the right side. We can describe these behaviors as the left- and right-hand limits, respectively. We say that the left-hand limit of f at $x = 3$ is negative infinity. Another way of communicating this is to say that as x approaches 3 from the left, the function approaches negative infinity. Symbolically, we summarize this as $\lim_{x \to 3^-} f(x) = -\infty$.

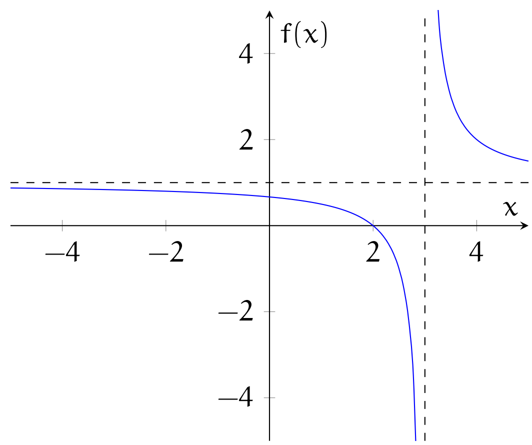

Figure 1.1: Graph of $f(x) = \frac{x^2-3x+2}{x^2-4x+3}$ with asymptotes

Similarly, the right-hand limit of f at $x = 3$ is positive infinity. In other words, as x approaches 3 from the right, the function approaches positive infinity. Symbolically, we write $\lim_{x\to 3^+} f(x) = \infty$.

The limit of a function at a particular x-value is the y-value that the function approaches as it approaches the given x-value. In the previous example, we could only specify the left- and right-hand limits, because they were different. In cases where the left- and right-hand limits are equal, we can say that the function has a limit there. The hole in our function f is one such value. We see that as we approach the hole from both the left and right, the function takes on values near $\frac{1}{2}$. This is more apparent numerically:

x	0.9	0.99	0.999	1	1.001	1.01	1.1
f(x)	0.5238	0.5025	0.5003	undefined	0.4998	0.4975	0.4737

The left-hand and right-hand limits of f at 1 are both $\frac{1}{2}$. Since they are equal, we can also say that the limit of f at 1 is $\frac{1}{2}$. This allows us to efficiently discuss the behavior of f at 1, even though the function is not defined there since substituting 1 into the function gives division by zero.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = \frac{1}{2}$$

We can also talk about limits at x-values where nothing weird is happening, that is, no hole or vertical asymptote. For example, as x approaches 4 from the left and right, y approaches 2.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
f(x)	2.1111	2.0101	2.0010	2	1.9990	1.9901	1.9091

In this case, since nothing weird is happening, the limit is equal to the function value. This is an example of continuity, which we will discuss in more detail in the next chapter. By contrast, at the vertical asymptote $x = 1$, since the left- and right-hand limits are not equal, we say the function does not have a limit, or the limit does not exist.

Finally, let us consider the horizontal asymptote of f. The graph hugs the line $y = 1$ as x goes far to the left and far to the right. We say that as x approaches negative infinity, f approaches 1, and likewise, that as x approaches positive infinity, f approaches 1. We write these symbolically as $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 1$.

Exercise 1 Limits Practice 1

Determine the left- and right-hand limits of the function as x approaches the given values. At x-values where the limit exists, determine it.

1. $p(x) = \frac{x+3}{x^2+9x+18}, x = -6, -5, -3, \infty$

Working Space

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We have seen two weird behaviors of rational functions at certain x-values: holes and vertical asymptotes. Now we will examine another type of weird behavior: jumps. This is a characteristic of some piecewise defined functions. In piecewise defined functions, the domain is divided into two or more pieces, and a different expression is used to give the y-value depending on which piece contains the x-value. One common piecewise defined function is the floor function, sometimes denoted $\lfloor x \rfloor$. The standard floor function rounds any real number down to the nearest integer. So, for a price quoted in dollars and cents, the floor would be just the number of dollars.

When x is exactly 1, the function value is 1: the number of dollars in a price of \$1.00. When x is any number greater than 1 but less than 2, the function value is still 1. Also, $\lfloor 1.01 \rfloor$, $\lfloor 1.5 \rfloor$, and $\lfloor 1.99999 \rfloor$ are all 1. As we continue to look to the right, once x equals

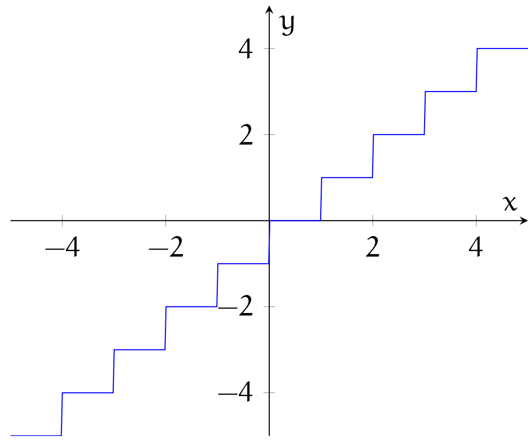

Figure 1.2: Graph of $y = \lfloor x \rfloor$

exactly 2, h jumps up to the value 2. So, $\lim_{x \to 2^-} \lfloor x \rfloor = 1$, while $\lim_{x \to 2^+} \lfloor x \rfloor = 2$.

Besides rational and piecewise defined functions, there are other functions with interesting limits. Consider the standard exponential function, $y = e^x$.

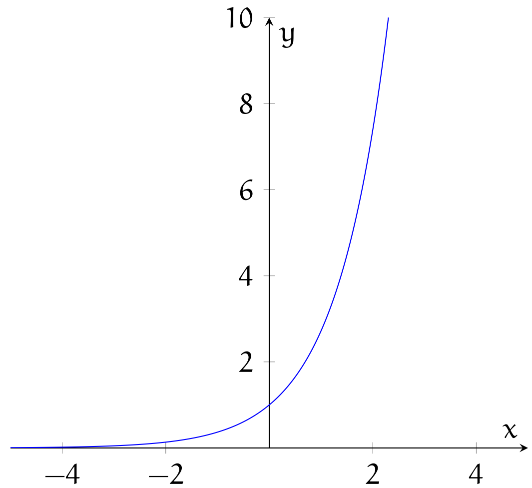

Figure 1.3: Graph of $y = e^x$

As x increases, y increases without bound; that is, $\lim_{x \to \infty} e^x = \infty$. However, looking far to the left, we see that y hugs the x-axis. This is because raising e to a large negative exponent is the same as 1 divided by e raised to a large positive exponent; that is, 1 divided by a very large number, which yields a very small positive number. In limit notation, $\lim_{x \to -\infty} e^x = 0$. This example illustrates that horizontal asymptotes need not model end behavior in both directions. Note that this reasoning holds for $y = b^x$ for any $b > 1$, so all such functions have the same horizontal asymptote, $y = 0$.

We know that the natural logarithm function, $y = \ln x$, is the inverse of $y = e^x$. Since inverse functions swap the role of x and y, it stands to reason that a horizontal asymptote in one function corresponds with a vertical asymptote in the other function, and that is indeed the case.

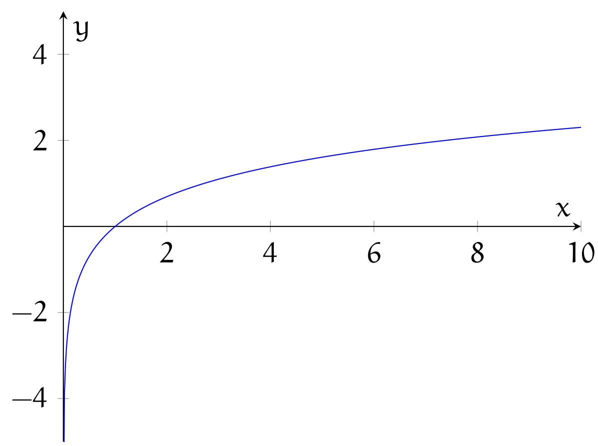

Figure 1.4: Graph of $y = \ln x$

An untransformed logarithm function is defined only for positive inputs. That is because it is not possible to find an exponent of a positive number which will yield a negative or zero result. What type of exponent on a positive number yields a number near zero? That would be a large-magnitude negative number. So, on the logarithm graph, large negative y-values correspond with x-values only slightly greater than zero. So, $\ln x$ (and $\log_2 x$, and indeed $\log_b x$ for any $b > 1$) approaches negative infinity as x approaches 0 from the right. There is no left-hand limit at 0, however. In limit notation, $\lim_{x \to 0^+} \ln x = -\infty$.

Exercise 2 Limits Practice 2

State the asymptotes of the following transformed exponential and logarithmic functions. Give the limit statement which describes the behavior of the function along the asymptote.

1. $y = 3^x + 1, y = \log_2(x-4), y = 2^{1-x}, y = \log_{10}(-2x)$

Working Space

Answer on Page 19

We conclude this chapter by considering two functions which each have two horizontal asymptotes. These two seemingly obscure functions are quite important in data science.

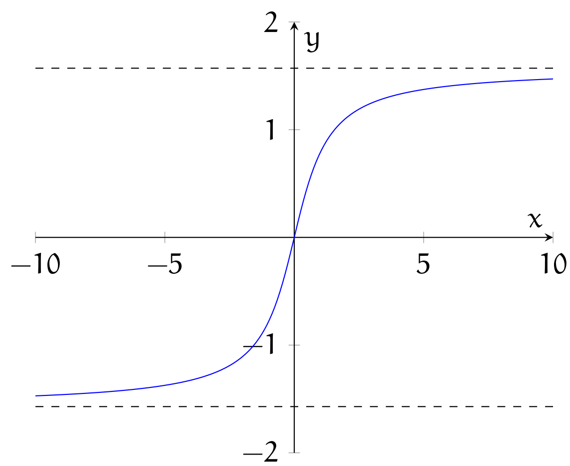

Figure 1.5: Graph of $y = \arctan x$

We know that the arctangent, or inverse tangent, function is the inverse of the piece of the tangent function which passes through the origin. The vertical asymptotes bounding this piece become horizontal asymptotes when the function is inverted.

Here are the equation and graph of the logistic function:

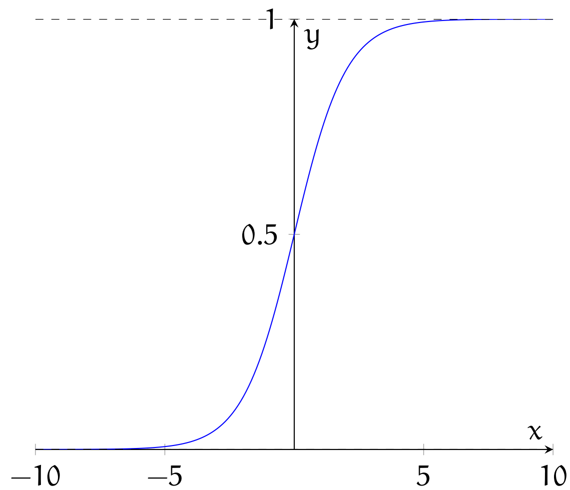

Figure 1.6: Graph of the logistic function, $y = \frac{1}{1+e^{-x}}$

For large magnitude negative values of x, the exponential term in the denominator becomes a very large positive value. The fraction thus becomes a positive number very close to zero. For large magnitude positive values of x, that exponential term becomes a very small positive number. Adding it to 1 yields a denominator just barely greater than 1. Dividing 1 by this number thus yields a function value just barely less than 1. So, the logistic function yields values between 0 and 1, though never equaling either of these values exactly. It is precisely this characteristic which makes the logistic function so useful.

Exercise 3 Limits Practice 3

Using limit notation, state the limits as x approaches negative and positive infinity for the inverse tangent and logistic functions.

Working Space

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Chapter 2

Rational Functions

We have discussed addition, subtraction, and multiplication of polynomials. What about division?

A quotient of polynomials is called a rational expression. When the polynomials are factored and the stars align, we can simplify the rational expression to a single polynomial, just like we might reduce a fraction to lowest terms.

Example

$$\begin{aligned} \frac{(x+1)(x+5)}{x+5} &= (x+1) * \frac{x+5}{x+5} \\ &= x+1 \end{aligned} \tag{2.1}$$

What if the polynomials are not factored? Factor them first.

Example

$$\frac{x^2+6x+5}{x+5} = \frac{(x+1)(x+5)}{x+5}$$

and simplify as in the previous example.

Now, let us consider a rational expression which can be simplified to a single polynomial - but in the denominator.

Example

$$\begin{aligned} \frac{x+5}{x^2+6x+5} &= \frac{x+5}{(x+1)(x+5)} \\ &= \frac{1}{x+1} * \frac{x+5}{x+5} \\ &= \frac{1}{x+1} \end{aligned} \tag{2.2}$$

Consider this expression as a function: $f(x) = \frac{1}{x+1}$. As you might have guessed, this is called a rational function. We did not bother looking at the result of the previous example as a function, because we already know that function type: it is a line with slope 1 and y-intercept 1. But this rational function is another animal entirely. Let us examine our first rational function with a familiar concept: the y-intercept.

y-intercept: $f(0) = \frac{1}{0+1} = \frac{1}{1} = 1$. The graph contains the point $(0, 1)$.

Does f have an x-intercept? That would be an x-value where $f(x) = 0$. But a fraction equals 0 only when its numerator equals 0; since the numerator of this expression is always 1, f has no x-intercept.

Knowing the y-intercept, and that there is no x-intercept, is a comforting start. But things get weird when we consider a concept that has previously seemed quite simple: domain. Recall that the domain of a function is the set of all values which can be used as inputs. In this case, the domain includes all real numbers, with one exception. The number -1 is not a valid input because $f(-1) = \frac{1}{-1+1} = \frac{1}{0}$, which is undefined. So, we say that the domain is all real numbers except -1. This means the graph contains a point corresponding to every x-value except -1.

There is no point at $x = -1$, but there is a point at every other x-value, such as, say, -1.1, or -0.99999. So what is happening near $x = -1$?

x	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9
f(x)	-10	-100	-1000	1000	100	10

The function is going haywire: as we choose x-values closer and closer to -1, the resulting function values are larger and larger in magnitude. Also, they are negative on one side, but positive on the other. So how does a graph go from y-values of -10, to -100, to -1000, all in a space of less than 0.1 on the x-axis? And then all of a sudden to big positive numbers on the other side of $x = -1$? All without ever crossing the x-axis (since there is no x-intercept)? Let us look at the graph.

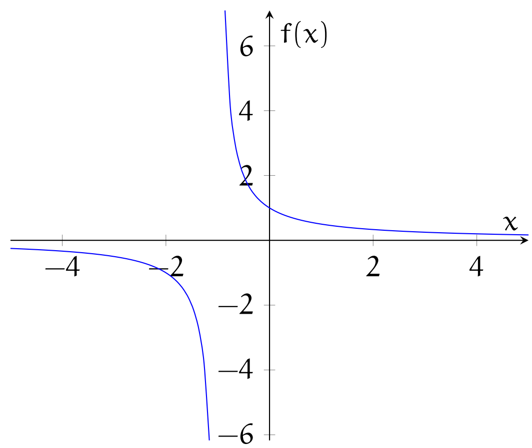

Figure 2.1: Graph of $f(x) = \frac{1}{x+1}$

We can see the y-intercept we found above. We can also see that the graph has no x-intercept, as expected. The phenomenon occurring at $x = -1$ is called a vertical asymptote. One other interesting feature of this graph is how it hugs the x-axis toward the left and right edges of the window. This makes the line $y = 0$ (the x-axis) a horizontal asymptote for this function. We can see why this is happening numerically by considering what happens for x-values far from 0. In this function, the result is a fraction with a numerator of 1 and a denominator that is large in size: a fraction that is close to 0.

x	-1000	-100	-10	10	100	1000
f(x)	-0.001	-0.01	-0.1	0.1	0.01	0.001

Let us examine another rational function. Begin by factoring to see if the function can be simplified.

$$g(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$

Consider the domain of g before continuing. Which values of x are valid inputs? Since substituting $x = 1$ or $x = 3$ would result in division by 0, these are not valid inputs. The domain of g is all real numbers except 1 and 3.

Now, for any x-value except 1, $\frac{x-1}{x-1} = 1$. This means that, for all x-values but 1, we can cancel those factors, leaving $g(x) = \frac{x-2}{x-3}$. (We will talk more about what is happening at $x = 1$ in a moment.)

This function has both x- and y-intercepts: y-intercept: $g(0) = \frac{0-2}{0-3} = \frac{2}{3}$. The graph contains the point $(0, \frac{2}{3})$. x-intercept: $g(x) = 0$ where the numerator equals 0 and the denominator does not equal 0. Since $x - 2 = 0$ when $x = 2$, the x-intercept is 2 and the graph contains the point $(2, 0)$.

The graph of g has a vertical asymptote at any x-value where substitution would result in dividing a nonzero number by zero. Thus, g has a vertical asymptote at $x = 3$.

Does g have a horizontal asymptote? Let us see what happens when we substitute x-values far from 0.

x	-1000	-100	-10	10	100	1000
g(x)	0.999	0.990	0.923	1.143	1.010	1.001

As we move further away from the y-axis, the y-values become closer to 1. The horizontal asymptote describes the end behavior of the function, or what the graph looks like far from the y-axis. In this case, if we ignore the portion close to the y-axis, the graph begins to look like the line $y = 1$, making this the horizontal asymptote of g.

So, what is happening at $x = 1$? The value is not in the domain of the function, but there is no vertical asymptote there. That is because substituting any other value for x, even values very close to 1, into $\frac{(x-1)(x-2)}{(x-1)(x-3)}$ gives the exact same number as substituting into $\frac{x-2}{x-3}$. So, there is a hole in the graph at $x = 1$, but nothing strange is happening on either side of 1. (Depending on the graphing software, the hole may not be visible.)

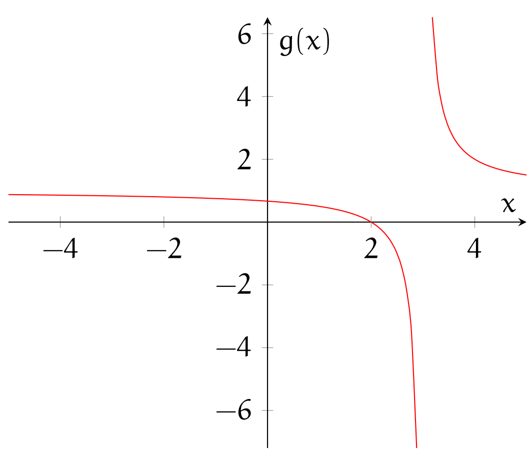

Figure 2.2: Graph of $g(x) = \frac{x^2-3x+2}{x^2-4x+3}$

Exercise 4 Rational Functions Practice 1

Determine the x- and y-intercepts and horizontal and vertical asymptotes of the rational function:

1. $\frac{2x+5}{x+4}$

Working Space

Answer on Page 20

In those examples, common factors cancel, leaving one polynomial. Of course, there is no guarantee that any two polynomials will have common factors, or even be factorable at all. Now, we consider an example which cannot be simplified. We will focus on just the asymptotes here.

$$h(x) = \frac{x^2}{x-1}$$

We see that the x-value 1 gives division of a non-zero number by zero, giving a vertical asymptote at $x = 1$. How about a horizontal asymptote? We examine values of h for values of x far from 0.

x	-1000	-100	-10	10	100	1000
h(x)	-999	-99	-9	11	101	1001

Rather than seeing function values leveling off as in the previous examples, we see function values that grow in size along with x. The function h has no horizontal asymptote. Let us examine the graph:

This function exhibits a different type of end behavior: that of a line with slope 1. To see that, cover up the portion of the graph near the y-axis and focus on the left and right. The rather dull and time-consuming technique of polynomial long division can be used to rewrite the function as a quotient and a remainder. Feel free to watch the Khan Academy video on the topic, but let us instead use our knowledge of factoring techniques and a clever little trick.

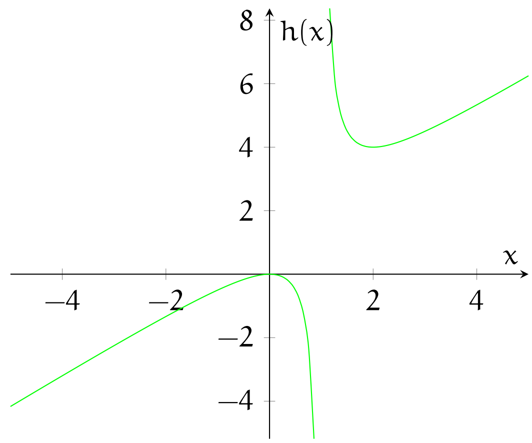

Figure 2.3: Graph of $h(x) = \frac{x^2}{x-1}$

$$
\begin{aligned}
h(x) &= \frac{x^2}{x-1} \\
&= \frac{x^2-1+1}{x-1} \\
&= \frac{x^2-1}{x-1} + \frac{1}{x-1} \\
&= \frac{(x-1)(x+1)}{x-1} + \frac{1}{x-1} \\
&= x+1+\frac{1}{x-1}
\end{aligned} \tag{2.3}
$$

We obtain a quotient of $x+1$ and a remainder of 1. It is the quotient which determines the end behavior of the graph. Why? Substituting x-values far from zero makes the remainder term very small, since it becomes a fraction with a large denominator but a numerator of only 1. So for x-values far from zero, the y-value is x plus 1 plus a very small number, so small that we can justifiably ignore it. This means that far from the y-axis, the function acts like the quotient: the line $y = x+1$. We call this line an oblique asymptote. See below how the graph of $h(x)$ hugs that line.

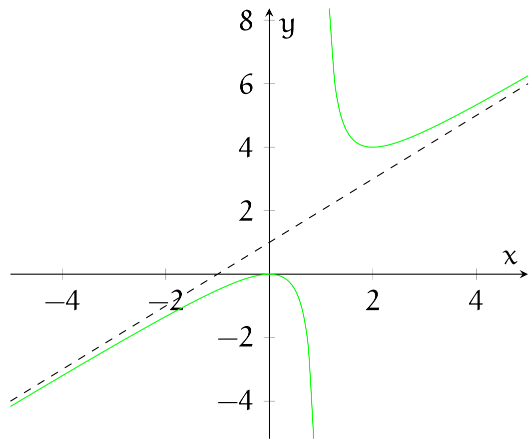

Figure 2.4: Graph of $h(x) = \frac{x^2}{x-1}$ and its oblique asymptote $y = x + 1$

Exercise 5 Rational Functions Practice 2

Factor and simplify the rational function, then determine any holes and vertical and oblique asymptotes of the rational function.

1. $\frac{x^3+2x^2}{x^2+x}$

Working Space

Answer on Page 20

We have seen lines act as end behaviors. Are there other possibilities? Sure! Here is an example with parabolic end behavior.

$$k(x) = \frac{x^3}{x-2}$$

We use our add-subtract trick to reveal the quotient, which describes the end behavior.

$$\begin{aligned} h(x) &= \frac{x^3}{x-2} \\ &= \frac{x^3-8+8}{x-2} \\ &= \frac{x^3-8}{x-2} + \frac{8}{x-2} \\ &= \frac{(x-2)(x^2+2x+4)}{x-2} + \frac{8}{x-2} \\ &= x^2+2x+4+\frac{8}{x-2} \end{aligned} \tag{2.4}$$

The quotient, $x2+2x+4$, should describe the end behavior. We confirm by graphing both k and the quotient - the parabolic asymptote.

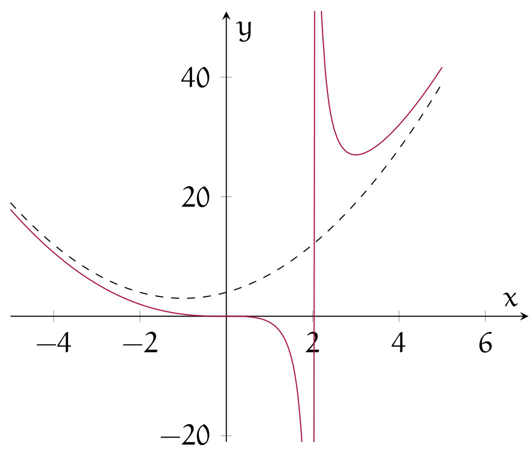

Figure 2.5: Graph of $k(x) = \frac{x^3}{x-2}$ and its parabolic asymptote $y = x^2 + 2x + 4$

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

$$\lim_{x \to -6^-} p(x) = -\infty, \lim_{x \to -6^+} p(x) = \infty$$

$$\lim_{x \to -5^-} p(x) = \lim_{x \to -5^+} p(x) = \lim_{x \to -5} p(x) = 1$$

$$\lim_{x \to -3^-} p(x) = \lim_{x \to -3^+} p(x) = \lim_{x \to -3} p(x) = \frac{1}{3}$$

$\lim_{x \to \infty} p(x) = 0$ called simply a limit, although it is a left-hand limit

Answer to Exercise 2 (on page 8)

$$\lim_{x \to -\infty} 3^x + 1 = 1; \lim_{x \to 4^+} \log_2(x - 4) = -\infty; \lim_{x \to \infty} 2^{1-x} = 0; \lim_{x \to 0^-} \log_{10}(-2x) = -\infty$$

Answer to Exercise 3 (on page 9)

$\lim_{x\to-\infty}\tan^{-1}x=-\frac{\pi}{2}, \lim_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}; \lim_{x\to-\infty}\frac{1}{1+e^{-x}}=0, \lim_{x\to\infty}\frac{1}{1+e^{-x}}=1$

Answer to Exercise 4 (on page 15)

x-intercept: $(-5/2, 0)$; y-intercept: $(0, 5/4)$; horizontal asymptote: $y=2$; vertical asymptote: $x=-4$

Answer to Exercise 5 (on page 17)

Factored form: $\frac{x^2(x+2)}{x(x+1)}$; hole: $(0,0)$; vertical asymptote: $x=-1$; oblique asymptote: $y=x+1$

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