Contents

Chapter 1

Differentiating Polynomials

If you had a function that gave you the height of an object, it would be handy to be able to figure out a function that gave you the velocity at which it was rising or falling. The process of converting the position function into a velocity function is known as *differentiation* or *finding the derivative*.

There are a bunch of rules for finding a derivative, but differentiating polynomials only requires three:

- The derivative of a sum is equal to the sum of the derivatives.
- The derivative of a constant is zero.
- The derivative of a nonconstant monomial at^b (a and b are constant numbers, t is time) is abt^{b-1}

So, for example, if I tell you that the height in meters of quadcopter at second t is given by $2t^3 - 5t^2 + 9t + 200$. You could tell me that its vertical velocity is $6t^2 - 10t + 9$

Exercise 1 Differentation of polynomials

Differentiate the following polynomials.

Working Space

Answer on Page 23

Notice that the degree of the derivative is one less than the degree of the original polynomial. (Unless, of course, the degree of the original is already zero.)

Now, if you know that a position is given by a polynomial, you can differentate it to find the object's velocity at any time.

The same trick works for acceleration: Let's say you know a function that gives an object's velocity. To find its acceleration at any time, you take the derivative of the velocity function.

Exercise 2 Differentation of polynomials in Python

Working Space

Write a function that returns the derivative of a polynomial in `poly.py`. It should look like this:

```
def derivative_of_polynomial(pn):
  ...Your code here...
```

When you test it in `test.py`, it should look like this:

```
# 3x**3 + 2x + 5
p1 = [5.0, 2.0, 0.0, 3.0]
d1 = poly.derivative_of_polynomial(p1)
# d1 should be 9x**2 + 2
print("Derivative of", poly.polynomial_to_string(p1),"is", poly.polynomial_to_string(d1))

# Check constant polynomials
p2 = [-9.0]
d2 = poly.derivative_of_polynomial(p2)
# d2 should be 0.0
print("Derivative of", poly.polynomial_to_string(p2),"is", poly.polynomial_to_string(d2))
```

Answer on Page 23

Chapter 2

Python Classes

The built-in types, like strings have functions associated with them. So, for example, if you needed a string converted to uppercase, you would call it's `upper()` function: -

```
my_string = "houston, we have a problem!"
louder_string = my_string.upper()
```

This would set `louder_string` to "HOUSTON, WE HAVE A PROBLEM!" When a function is associated with a datatype like this, it called a *method*. A datatype with methods is known as a *class*. The data of that type is known as *instance* of that class. For example, in the example, we would say "`my_string` is an instance of the class `str`. `str` has a method called `upper`"

The function `type` will tell you the type of any data:

```
  print(type(my_string))
```

This will output

```
<class 'str'>
```

A class can also define operators. `+`, for example, is redefined by `str` to concatenate strings together:

```
long_string = "I saw " + "15 people"
```

2.1 Making a Polynomial class

You have created a bunch of useful python functions for dealing with polynomials. Notice how each one has the word "polynomial" in the function name like `derivative_of_polynomial`. Wouldn't it be more elegant if you had a Polynomial class with a `derivative` method? Then you could use your polynomial like this:

```
a = Polynomial([9.0, 0.0, 2.3])
b = Polynomial([-2.0, 4.5, 0.0, 2.1])

print(a, "plus", b , "is", a+b)
print(a, "times", b , "is", a*b)
print(a, "times", 3 , "is", a*3)
print(a, "minus", b , "is", a-b)

c = b.derivative()

print("Derivative of", b ,"is", c)
```

And it would output:

```
2.30x^2 + 9.00 plus 2.10x^3 + 4.50x + -2.00 is 2.10x^3 + 2.30x^2 + 4.50x + 7.00
2.30x^2 + 9.00 times 2.10x^3 + 4.50x + -2.00 is 4.83x^5 + 29.25x^3 + -4.60x^2 + 40.50x + -18.00
2.30x^2 + 9.00 times 3 is 6.90x^2 + 27.00
2.30x^2 + 9.00 minus 2.10x^3 + 4.50x + -2.00 is -2.10x^3 + 2.30x^2 + -4.50x + 11.00
Derivative of 2.10x^3 + 4.50x + -2.00 is 6.30x^2 + 4.50
```

Create a file for your class definition called `Polynomial.py`. Enter the following:

```
class Polynomial:
    def __init__(self, coeffs):
        self.coefficients = coeffs.copy()

    def __repr__(self):
```

```
        # Make a list of the monomial strings
        monomial_strings = []

        # For standard form we start at the largest degree
        degree = len(self.coefficients) - 1

        # Go through the list backwards
        while degree >= 0:
            coefficient = self.coefficients[degree]

            if coefficient != 0.0:
                # Describe the monomial
                if degree == 0:
                    monomial_string = "{:.2f}".format(coefficient)
                elif degree == 1:
                    monomial_string = "{:.2f}x".format(coefficient)
                else:
                    monomial_string = "{:.2f}x^{}".format(coefficient, degree)

                # Add it to the list
                monomial_strings.append(monomial_string)

            # Move to the previous term
            degree = degree - 1

        # Deal with the zero polynomial
        if len(monomial_strings) == 0:
            monomial_strings.append("0.0")

        # Separate the terms with a plus sign
        return " + ".join(monomial_strings)

    def __call__(self, x):
        sum = 0.0
        for degree, coefficient in enumerate(self.coefficients):
            sum = sum + coefficient * x ** degree
        return sum

    def __add__(self, b):
        result_length = max(len(self.coefficients), len(b.coefficients))
        result = []
        for i in range(result_length):
            if i < len(self.coefficients):
                coefficient_a = self.coefficients[i]
            else:
                coefficient_a = 0.0
```

```
            if i < len(b.coefficients):
                coefficient_b = b.coefficients[i]
            else:
                coefficient_b = 0.0
            result.append(coefficient_a + coefficient_b)

        return Polynomial(result)

    def __mul__(self, other):

        # Not a polynomial?
        if not isinstance(other, Polynomial):
            # Try to make it a constant polynomial
            other = Polynomial([other])

        # What is the degree of the resulting polynomial?
        result_degree = (len(self.coefficients) - 1) + (len(other.coefficients) - 1)

        # Make a list of zeros to hold the coefficents
        result = [0.0] * (result_degree + 1)

        # Iterate over the indices and values of a
        for a_degree, a_coefficient in enumerate(self.coefficients):

            # Iterate over the indices and values of b
            for b_degree, b_coefficient in enumerate(other.coefficients):

                # Calculate the resulting monomial
                coefficient = a_coefficient * b_coefficient
                degree = a_degree + b_degree

                # Add it to the right bucket
                result[degree] = result[degree] + coefficient

        return Polynomial(result)

    __rmul__ = __mul__

    def __sub__(self, other):
        return self + other * -1.0

    def derivative(self):

        # What is the degree of the resulting polynomial?
        original_degree = len(self.coefficients) - 1
```

```
        if original_degree > 0:
            degree_of_derivative = original_degree - 1
        else:
            degree_of_derivative = 0

        # We can ignore the constant term (skip the first coefficient)
        current_degree = 1
        result = []

        # Differentiate each monomial
        while current_degree < len(self.coefficients):
            coefficient = self.coefficients[current_degree]
            result.append(coefficient * current_degree)
            current_degree = current_degree + 1

        # No terms? Make it the zero polynomial
        if len(result) == 0:
            result.append(0.0)

        return Polynomial(result)
```

Create a second file called `test_polynomial.py` to test it:

```
from Polynomial import Polynomial

a = Polynomial([9.0, 0.0, 2.3])
b = Polynomial([-2.0, 4.5, 0.0, 2.1])

print(a, "plus", b , "is", a+b)
print(a, "times", b , "is", a*b)
print(a, "times", 3 , "is", a*3)
print(a, "minus", b , "is", a-b)

c = b.derivative()

print("Derivative of", b ,"is", c)

slope = c(3)
print("Value of the derivative at 3 is", slope)
```

Run the test code:

```
python3 test_polynomial.py
```

Chapter 3

Common Polynomial Products

In math and physics, you will run into certain kinds of polynomials over and over again. In this chapter, I am going to cover some patterns that you will want to start to recognize.

3.1 Difference of squares

Watch **Polynomial special products: difference of squares** from Khan Academy at https://youtu.be/uNweU6I4Icw.

If you are asked what is $(3x-7)(3x+7)$, you would use the distributive property to expand that to $(3x)(3x) + (3x)(7) + (-7)(3x) + (-7)(7)$. Two of the terms cancel each other, so this is $(3x)^2 - (7)^2$. This would simplify to $9x^2 - 49$

You will see this pattern a lot. Anytime you see $(a + b)(a - b)$, you should immediately recognize it equals $a^2 - b^2$. (Note that the order doesn't matter: $(a-b)(a+b)$ also a^2-b^2.)

Working the other way is important too: anytime you see a^2-b^2, that you should recognize that you can change that into the product $(a+b)(a-b)$. Making something into a product like this is known as *factoring*. You probably have done prime factorization of numbers like $42 = 2 \times 3 \times 7$. In the next couple of chapters you will learn to factorize polynomials.

Exercise 3 Difference of Squares

Working Space

Simply the following products

1. $(2x - 3)(2x + 3)$
2. $(7 + 5x^3)(7 - 5x^3)$
3. $(x - a)(x + a)$
4. $(3 - \pi)(3 + \pi)$
5. $(-4x^3 + 10)(-4x^3 - 10)$
6. $(x + \sqrt{7})(x - \sqrt{7})$ Factor the following polynomials:
7. $x^2 - 9$
8. $49 - 16x^6$
9. $\pi^2 - 25x^8$
10. $x^2 - 5$

Answer on Page 24

We are often interested in the roots of a polynomial, that is we want to know "For what values of x does the polynomial evaluate to zer?" For example, when you deal with falling bodies, the first question you might ask would be "How many seconds before the hammer hits the ground?" Once you have factored a polynomial into binomials, you can easily find the roots.

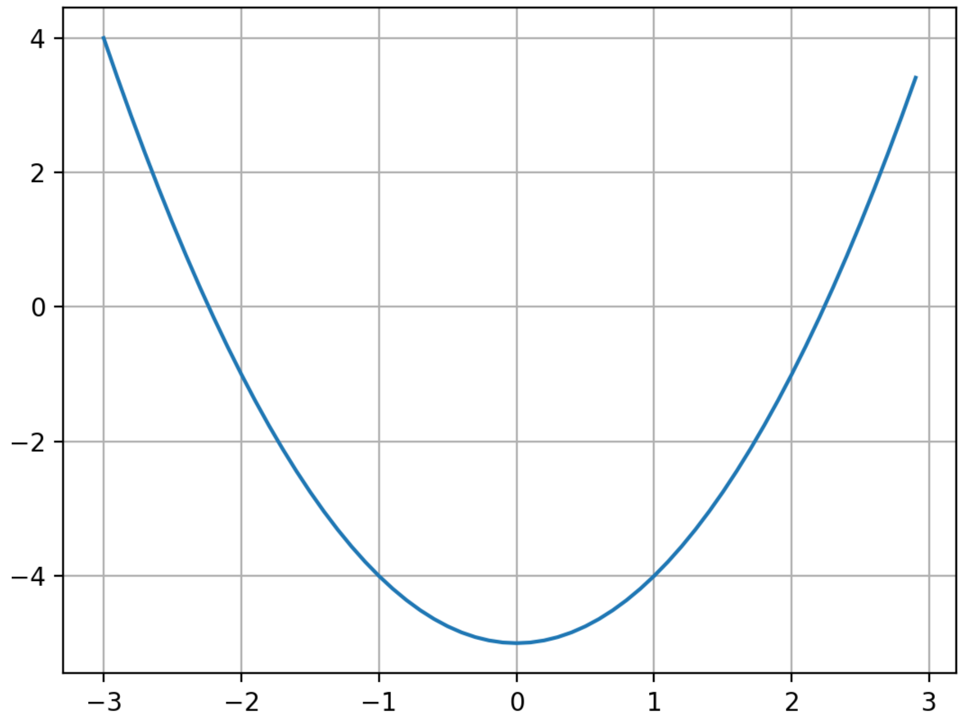
For example, what are the roots of $x^2 - 5$? You just factored it into $(x + \sqrt{5})(x - \sqrt{5})$ This product is zero if and only if one of the factors is zero. The first factor is only zero when x is $-\sqrt{5}$. The second factor is zero only when x is $\sqrt{5}$. Those are the only two roots of this polynomial.

Let's check that result. $\sqrt{5}$ is a little more than 2.2. Using your Python code, you can graph the polynomial:

```
import poly.py
import matplotlib.pyplot as plt
```

```
# x**2 - 5
pn = [-5.0, 0.0, 1.0]

# These lists will hold our x and y values
x_list = []
y_list = []

# Start at x=-3
current_x =-3.0

# End at x=3.0
while current_x < 3.0:
    current_y = poly.evaluate_polynomial(pn, current_x)

    # Add x and y to respective lists
    x_list.append(current_x)
    y_list.append(current_y)

    # Move x forward
    current_x += 0.1

# Plot the curve
plt.plot(x_list, y_list)
plt.grid(True)
plt.show()
```

You should get a plot like this:



It does, indeed, seem to cross the x-axis near -2.2 and 2.2.

3.2 Powers of binomials

You can raise whole polynomials to exponents. For example,

$$(3x^3+5)^2 = (3x^3+5)(3x^3+5)$$
$$= 9x^6 + 15x^3 + 15x^3 + 25 = 9x^6 + 30x^3 + 25$$

A polynomial with two terms is called a *binomial*. $5x^9 - 2x^4$, for example, is a binomial. In this section, we are going to develop some handy techniques for raising a binomial to some power.

Looking at the previous example, you can see that for any monomials a and b, $(a+b)^2 = a^2 + 2ab + b^2$. So, for example, $(7x^3+\pi)^2 = 49x^6 + 14\pi x^3 + \pi^2$

Exercise 4 Squaring binomials

Simply the following

1. $(x+1)^2$
2. $(3x^5+5)^2$
3. $(x^3-1)^2$
4. $(x-\sqrt{7})^2$

Working Space

Answer on Page 24

What about $(x+2)^3$? You can do it as two separate multiplications:

$$\begin{aligned}(x+2)^3 &= (x+2)(x+2)(x+2) \\ &= (x+2)(x^2+4x+4) = x^3+4x^2+4x+2x^2+8x+8 \\ &= x^3+6x^2+12x+8\end{aligned}$$

And, in general, we can say that for any monomials a and b, $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

What about higher powers? $(a+b)^4$, for example? You could use the distributive property four times, but it starts to get pretty tedious.

Here is a trick. This is known as *Pascal's triangle*

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\ 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\ \ldots \end{array}$$

Each entry is the sum of the two above it.

The coefficients of each term are given by the entries in Pascal's triangle:

$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

Exercise 5 Using Pascal's Triangle

Working Space

1. What is $(x + \pi)^5$?

Answer on Page 25

CHAPTER 4

Factoring Polynomials

We factor a polynomial into two or more polynomials of lower degree. For example, let's say that you wanted to factor $5x^3 - 45x$. You would note that you can factor out $5x$ from every term. Thus,

$$5x^3 - 45x = (5x)(x^2 - 9)$$

And then, you might notice that the second factor looks like the difference of squares, so

$$5x^3 - 45x = (5x)(x + 3)(x - 3)$$

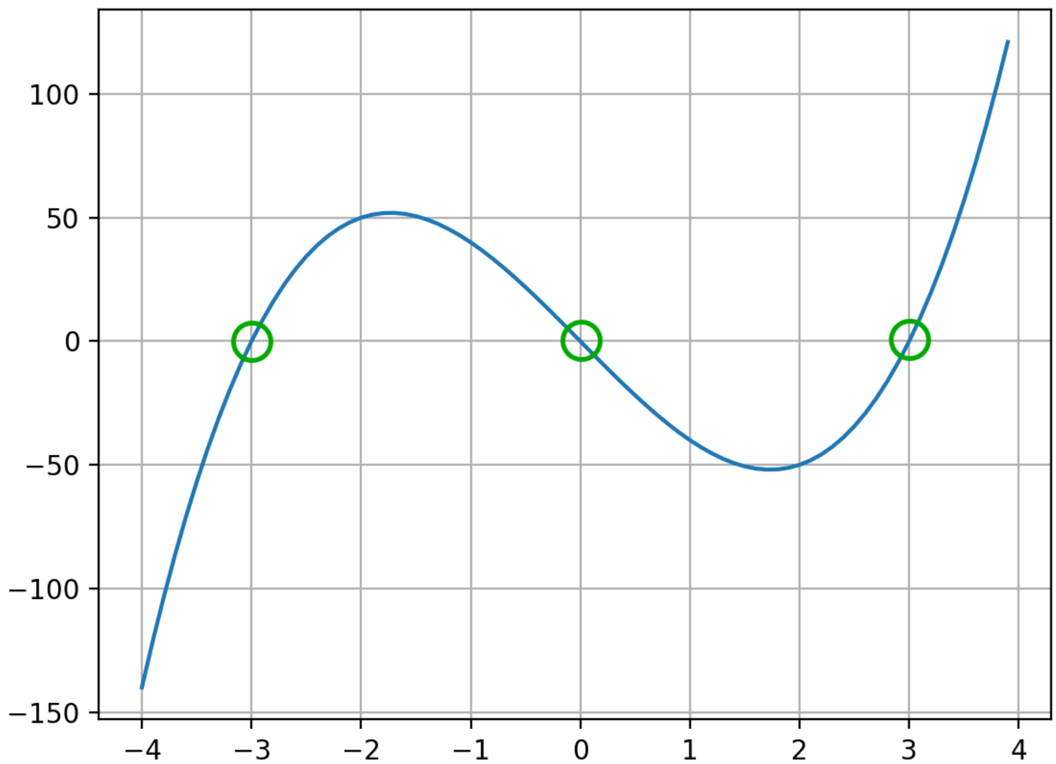
That is as far as we can factorize this polynomial.

Why do we care? The factors make it easy to find the roots of the polynomial. This polynomial evaluates to zero if and only if at least one of the factors is zero. Here we see that

- The factor $(5x)$ is zero when x is zero.
- The factor $(x + 3)$ is zero when x is -3.
- The factor$(x - 3)$ is zero when x is 3.

So looking at the factorization, you can see that $5x^3 - 45x$ is zero when x is 0, -3, or 3.

This is a graph of that polynomial with its roots circled:



4.1 How to factor polynomials

The first step when you are trying to factor a polynomial is to find the greatest common divisor for all the terms, and pull that out. In this case, the greatest common divisor will also be a monomial: its degree is the least of the degrees of the terms, its coefficient will be the greatest common divisor of the coefficients of the terms.

For example, what can you pull out of this polynomial?

$$12x^{1}00 + 30x^{3}1 + 42x^{1}7$$

The greatest common divisor of the coefficients (12, 30, and 42) is 6. The least of the degrees of terms (100, 31, and 17) is 17. So you can pull out $6x^{1}7$:

$$12x^{1}00 + 30x^{3}1 + 42x^{1}7 = (6x^{1}7)(2x^{8}3 + 5x^{1}4 + 7)$$

Exercise 6 Factoring out the GCD monomial

Working Space

Answer on Page 25

So, now you have the product of a monomial and a polynomial. If you are lucky, the polynomial part looks familiar, like the difference of squares or a row from Pascal's triangle.

Often you are trying factor a quadratic like $x^2 + 5x + 6$ in a pair of binomials. In this case, the result would be $(x + 3)(x + 2)$. Let's check that:

$$(x + 3)(x + 2) = (x)(x) + (3)(x) + (2)(x) + (3)(2) = x^2 + 5x + 6$$

Notice that 3 and 2 multiply to 6 and add to 5. If I were trying to factor $x^2 + 5x + 6$, I would ask myself"What are two numbers that when multiplied equal 6 and when added equal 5?" And I would might guess wrong a couple of times. For example, I might say to myself "Well, 6 times 1 is 6. Maybe those work. But 6 and 1 add 7. So those don't work."

Solving these sorts of problems are like solving a Sudoku puzzle: you try things and realize they are wrong, so you backtrack and try something else.

The numbers are sometimes negative. For example, $x^2 + 3x - 10$ factors into $(x + 5)(x - 2)$.

Exercise 7 Factoring quadratics

Working Space

Answer on Page 25

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

Answer to Exercise 2 (on page 5)

```
def derivative_of_polynomial(pn):

    # What is the degree of the resulting polynomial?
    original_degree = len(pn) - 1
    if original_degree > 0:
        degree_of_derivative = original_degree - 1
    else:
        degree_of_derivative = 0

    # We can ignore the constant term (skip the first coefficient)
    current_degree = 1
```

```
    result = []

    # Differentiate each monomial
    while current_degree < len(pn):
        coefficient = pn[current_degree]
        result.append(coefficient * current_degree)
        current_degree = current_degree + 1

    # No terms? Make it the zero polynomial
    if len(result) == 0:
        result.append(0.0)

    return result
```

Answer to Exercise 3 (on page 14)

$(2x - 3)(2x + 3) = 4x^2 - 9$

$(7 + 5x^3)(7 - 5x^3) = 49 - 25x^6$

$(x - a)(x + a) = x^2 - a^2$

$(3 - \pi)(3 + \pi) = 9 - \pi^2$

$(-4x^3 + 10)(-4x^3 - 10) = 16x^6 - 100$

$(x + \sqrt{7})(x - \sqrt{7}) = x^2 - 7$

$x^2 - 9 = (x + 3)(x - 3)$

$49 - 16x^6 = (7 + 4x^3)(7 + 4^3)$

$\pi^2 - 25x^8 = (\pi + 5x^4)(\pi - 5x^4)$

$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$

Answer to Exercise 4 (on page 17)

$(x + 1)^2 = x^2 + 2x + 1$

$(3x^5 + 5)^2 = 9x^10 + 30x^5 + 25$

$(x^3 - 1)^2 = x^6 - 2x^3 + 1$

$(x - \sqrt{7})^2 = x^2 - 2x\sqrt{7} + 7$

Answer to Exercise 5 (on page 18)

$(x + \pi)^5 = x^5 + 5\pi x^4 + 10\pi^2 x^3 + 10\pi^3 + x^2 + 5\pi^2 x + \pi^5$

Answer to Exercise 6 (on page 21)

Answer to Exercise 7 (on page 21)

Index