

## Sequences in Calculus

We have introduced sequences in a previous chapter. Now, we will examine them in more detail in a calculus context. You already know about arithmetic and geometric sequences, but not all sequences can be classified as arithmetic or geometric. Take the famous Fibonacci sequence,  $\{1, 1, 2, 3, 5, 8, \dots\}$ , which can be explicitly defined as  $a_n = a_{n-1} + a_{n-2}$ , with  $a_1 = a_2 = 1$ . There is no common difference or common ratio, so the Fibonacci sequence is not arithmetic or geometric. Another example is  $a_n = \sin \frac{n\pi}{6}$ , which will cycle through a set of values.

Sequences have many real-world applications, including compound interest and modeling population growth. In later chapters, you will learn that the sum of all the values in a sequence is a series and how to use series to describe functions. In order to be able to do all that, first we need to talk in-depth about sequences.

Some sequences are defined explicitly, like  $a_n = \sin \frac{n\pi}{6}$ , while others are defined recursively, like  $a_n = a_{n-1} + a_{n-2}$ .

Example: Write the first 5 terms for the explicitly defined sequence  $a_n = \frac{n}{n+1}$ .

Solution: We can construct a table to keep track of our work:

n	work	$a_n$
1	$\frac{1}{1+1}$	$\frac{1}{2}$
2	$\frac{2}{2+1}$	$\frac{2}{3}$
3	$\frac{3}{3+1}$	$\frac{3}{4}$
4	$\frac{4}{4+1}$	$\frac{4}{5}$
5	$\frac{5}{5+1}$	$\frac{5}{6}$

So the first five terms are  $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$ .

**Exercise 1**

Write the first 5 terms for each sequence.

*Working Space*

1.  $a_n = \frac{2^n}{2n+1}$

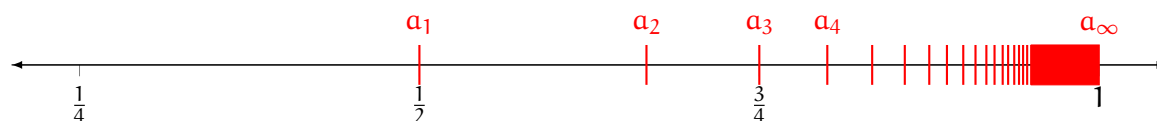
2.  $a_n = \cos \frac{n\pi}{2}$

3.  $a_1 = 1, a_{n+1} = 5a_n - 3$

4.  $a_1 = 6, a_{n+1} = \frac{a_n}{n+1}$

*Answer on Page ??***1.1 Convergence and Divergence**

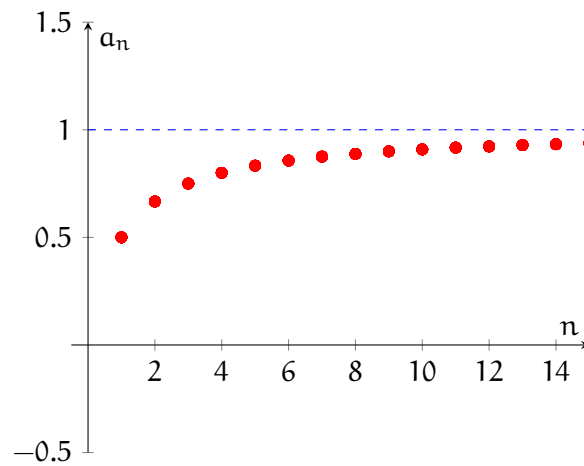
You can visualize a sequence on an  $xy$ -plane or a number line. Figures ?? and ?? show visualizations of the sequence  $a_n = \frac{n}{n+1}$ . To visualize this on the  $xy$ -plane, we take points such that  $x = n$  and  $y = a_n$ , where  $n$  is a positive integer. What do you notice about this sequence? As  $n$  increases,  $a_n$  gets closer and closer to 1.

Figure 1.1:  $a_n = \frac{n}{n+1}$  on a number line

Because  $a_n$  approaches a specific number as  $n \rightarrow \infty$ , we call the series  $a_n = \frac{n}{n+1}$  *convergent*. We prove a sequence is convergent by taking the limit as  $n$  approaches  $\infty$ . If the limit exists and approaches a specific number, the sequence is convergent. If the limit does not exist or approaches  $\pm\infty$ , the sequence is divergent.

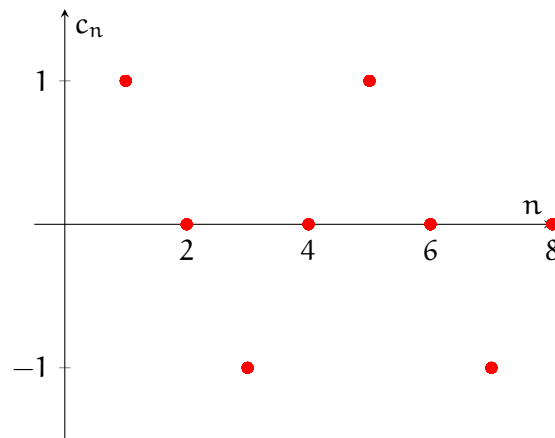
We can see graphically that  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ , so that sequence is convergent. What about  $b_n = \frac{n}{\sqrt{10+n}}$ ? Is  $b_n$  convergent or divergent?

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{\sqrt{10+n}} &= \lim_{n \rightarrow \infty} \frac{n/n}{\sqrt{\frac{10}{n^2} + \frac{n}{n^2}}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty \end{aligned}$$

Figure 1.2:  $a_n = \frac{n}{n+1}$  on an  $xy$ -plane

Therefore, the sequence  $b_n = \frac{n}{\sqrt{10+n}}$  is divergent.

Here is another example of a divergent sequence:  $c_n = \sin \frac{n\pi}{2}$ . The graph is shown in figure ???. As you can see, the value of  $c_n$  oscillates between 1, 0, and -1 without approaching a specific number. This means that  $c_n$  does not approach a particular number as  $n \rightarrow \infty$  and the sequence is divergent.

Figure 1.3:  $c_n = \sin \frac{n\pi}{2}$  on an  $xy$ -plane

**Exercise 2**

Classify each sequence as convergent or divergent. If the sequence is convergent, find the limit as  $n \rightarrow \infty$ .

*Working Space*

1.  $a_n = \frac{3+5n^2}{n+n^2}$

2.  $a_n = \frac{n^4}{n^3-2n}$

3.  $a_n = 2 + (0.86)^n$

4.  $a_n = \cos \frac{n\pi}{n+1}$

5.  $a_n = \sin n$

*Answer on Page ??*

**1.2 Evaluating limits of sequences**

Recall that a sequence can be considered a function where the domain is restricted to positive integers. If there is some  $f(x)$  such that  $a_n = f(n)$  when  $n$  is an integer, then  $\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x)$  (see figure ??). This means that all the rules that apply to the limits of functions also apply to the limits of sequences, including the Squeeze Theorem and l'Hospital's rule.

What is  $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$ ? First, we will try to compute the limit directly:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln n}{n} &= \\ \frac{\lim_{n \rightarrow \infty} \ln n}{\lim_{n \rightarrow \infty} n} &= \frac{\infty}{\infty} \end{aligned}$$

This is undefined, but fits the criteria for l'Hospital's rule:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln n}{n} &= \lim_{n \rightarrow \infty} \frac{\frac{d}{dn} \ln n}{\frac{d}{dn} n} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = 0 \end{aligned}$$

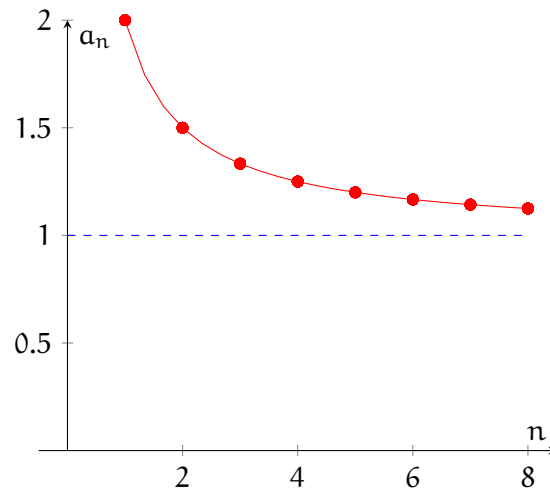


Figure 1.4: The limit of the function is the same as the limit of the sequence

Here's an example that requires the Squeeze Theorem: is the sequence  $a_n = \frac{n!}{n^n}$  convergent or divergent? First trying to take the limit directly, we see that:

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = \frac{\infty}{\infty}$$

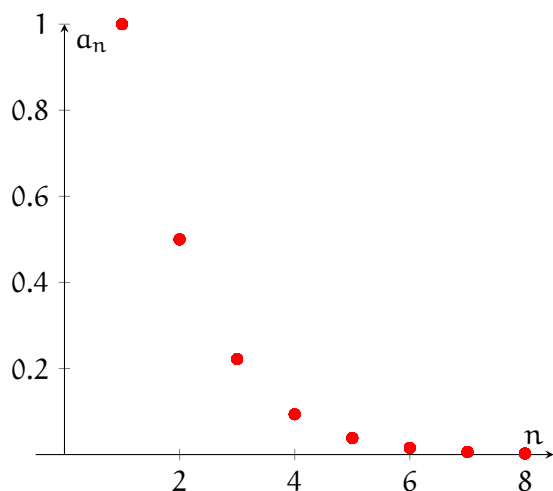
which is undefined. Because the factorial cannot be described as a continuous function, we can't use l'Hospital's rule. We can examine this sequence graphically (see figure ??) and mathematically. We examine it mathematically by writing out a few terms to get an idea of what happens to  $a_n$  as  $n$  gets large:

$$\begin{aligned} a_1 &= \frac{1!}{1^1} = 1 \\ a_2 &= \frac{2!}{2^2} = \frac{1 \cdot 2}{2 \cdot 2} \\ a_3 &= \frac{3!}{3^3} = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3} \\ &\quad \dots \\ a_n &= \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} \end{aligned}$$

From examining the graph in figure ??, we can guess that  $\lim_{n \rightarrow \infty} a_n = 0$ . Let's prove that mathematically. We can rewrite our expression for  $a_n$  as  $n$  gets large:

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \left( \frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n} \right)$$

The expression inside the parentheses is less than 1, therefore  $0 < a_n < \frac{1}{n}$ . Since  $\lim_{n \rightarrow \infty} 0 = 0$  and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , by Squeeze Theorem we know that  $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$ . Therefore, the sequence  $a_n = \frac{n!}{n^n}$  is convergent.

Figure 1.5:  $a_n = \frac{n!}{n^n}$ 

[[FIX ME intro]] If  $\lim_{n \rightarrow \infty} a_n = L$  and the function  $f$  is continuous at  $L$ , then  $\lim_{n \rightarrow \infty} f(a_n) = f(L)$ . For example, what is  $\lim_{n \rightarrow \infty} \sin \frac{\pi}{n}$ ? Well, we know that  $\lim_{n \rightarrow \infty} \frac{\pi}{n} = 0$  and that the sine function is continuous at 0. Therefore,  $\lim_{n \rightarrow \infty} \sin \frac{\pi}{n} = \sin \lim_{n \rightarrow \infty} \frac{\pi}{n} = \sin 0 = 0$ .

### 1.3 Monotonic and Bounded sequences

Just like functions, sequences can be increasing or decreasing. A sequence is increasing if  $a_n < a_{n+1}$  for  $n \geq 1$ . Similarly, a sequence is decreasing if  $a_n > a_{n+1}$  for  $n \geq 1$ . If a sequence is strictly increasing or decreasing, it is called *monotonic*.

The sequence  $a_n = \frac{1}{n+6}$  is decreasing. We prove this formally by comparing  $a_n$  to  $a_{n+1}$ :

$$\frac{1}{n+6} > \frac{1}{(n+1)+6} = \frac{1}{n+7}$$

Is the sequence  $a_n = \frac{n}{n^2+1}$  increasing or decreasing? First, we find an expression for  $a_{n+1}$ :

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$$

Since the degree of  $n$  is greater in the denominator, we have a guess that the sequence is decreasing. To prove this, we check if  $a_n > a_{n+1}$  is true:

$$\frac{n}{n^2+1} > \frac{n+1}{n^2+2n+2}$$

We can cross-multiply, because  $n > 0$  and the denominators are positive:

$$(n)(n^2+2n+2) > (n+1)(n^2+1)$$

$$n^3 + 2n^2 + 2n > n^3 + n^2 + n + 1$$

Subtracting  $(n^3 + n^2 + n)$  from both sides we see that:

$$n^2 + n > 1$$

Which is true for all  $n \geq 1$ . Therefore,  $a_n > a_{n+1}$  for all  $n \geq 1$  and the sequence is decreasing.

A sequence is *bounded above* if there is some number  $M$  such that  $a_n \leq M$  for all  $n \geq 1$ . And a sequence is *bounded below* if there is some other number  $m$  such that  $a_n \geq m$  for all  $n \geq 1$ . If a sequence is bounded above and below, then it is a *bounded sequence*.

If a sequence is both *monotonic* and *bounded*, then it must be convergent.

Example: is the sequence given by  $a_n = 4$  and  $a_{n+1} = \frac{1}{2}(a_n + 7)$  bounded above, below, both, or neither?

We start by calculating the first several terms:

Term	Work	Value
$a_1$	$a_1 = 4$	4
$a_2$	$= \frac{1}{2}(4 + 7)$	5.5
$a_3$	$= \frac{1}{2}(5.5 + 7)$	6.25
$a_4$	$= \frac{1}{2}(6.25 + 7)$	6.625
$a_5$	$= \frac{1}{2}(6.625 + 7)$	6.8125
$a_6$	$= \frac{1}{2}(6.8125 + 7)$	6.90625
$a_7$	$= \frac{1}{2}(6.90625 + 7)$	6.953125
$a_8$	$= \frac{1}{2}(6.953125 + 7)$	6.9765625

The sequence is increasing, so it is bounded below by the initial term,  $a_1 = 4$ , and we can state that  $a_n \geq 4$ . Examining the computed terms, we see that  $a_n \rightarrow 7$  as  $n$  grows larger. We can guess that this sequence is bounded above, with  $a_n \leq 7$ . We can prove this by induction. Suppose that there is some  $k$  such that  $a_k < 7$  (which is true for  $a_1$ , etc.). Then

$$\begin{aligned} a_k &< 7 \\ a_k + 7 &< 14 \\ \frac{1}{2}(a_k + 7) &< \frac{1}{2}(14) \\ a_{k+1} &< 7 \end{aligned}$$

Therefore,  $a_n < 7$  for all  $n$  and the sequence is bounded above. Because the sequence is monotonic and bounded, we know the sequence is convergent and, therefore, that the limit of  $a_n$  as  $n \rightarrow \infty$  exists.

## 1.4 Applications of Sequences

### 1.4.1 Compound Interest

You previously learned about compound interest and modeled the accumulation of compound interest by  $P_n = P_0(1 + r)^n$ , where  $P_0$  is the principal investment,  $r$  is the yearly interest rate, and  $n$  is the number of elapsed years. This sequence describes the value of an investment accumulating interest, but most people add to their savings on a regular schedule. We can write a sequence to model the value of a savings account that the owner makes regular deposits into.

Suppose you open a savings account with an initial deposit of \$3,000 and you plan to deposit an additional \$1,200 at the end of every year. If your savings account has an annual interest rate of 3.25%, how long will it take you to save \$10,000?

We can write a recursive definition for the sequence. At the end of each year, the account will gain the interest on the entirety of the previous year's balance plus \$1200:

$$P_n = P_{n-1}(1 + 0.0325) + \$1200$$

With an initial investment  $P_0 = \$3000$ . We can write out the first few terms to find how many years it will take to save \$10,000:

Year	Savings
0	\$3,000
1	\$4,297.50
2	\$5,637.17
3	\$7,020.38
4	\$8,448.54
5	\$9,923.12
6	\$11,445.62

When you put money in a bank account or invest it, your money may accumulate *compound interest*. Compound interest is interest paid on the principal (the amount you originally invested) and previously accumulated interest. For example, suppose you put \$100 in a savings account that earns 3.25% annually. After one year, the bank will pay  $(\$100) \cdot (0.0325) = \$3.25$  into your savings account. Now you have \$103.25. The *next* year, your bank will pay you 3.25% interest on the entire amount in your bank account (the \$100 principal plus the \$3.25 in earned interest). This comes to  $\$103.25 \cdot 0.0325 = \$3.36$  and your new account total will be \$106.61. This continues year over year and can build up quite a bit of money. How can we model compound interest as a sequence? Let's write out how we calculate the first few years of interest accumulation:

$$a_0 = 100 \text{ (this represents our principal investment)}$$

$$a_1 = a_0 + a_0(0.0325) \text{ (the total value of the bank account will be the previous year's value plus the interest)}$$



We can combine terms to see that:

$$a_1 = a_0 1.0325$$

Similarly,

$$a_2 = a_1(1.0325) = [a_0(1.0325)](1.0325) = a_0(1.0325)^2$$

$$a_3 = a_2(1.0325) = [a_0(1.0325)^2](1.0325) = a_0(1.0325)^3$$

Do you see the pattern? For an initial investment of \$100 at an annual rate of 3.25% compounded annually, the value of the bank account at the end of the  $n^{\text{th}}$  year is given by:

$$a_n = 100(1.0325)^n$$

The general form of this formula with an initial investment of  $P$  and annual interest rate  $r$  is:

$$a_n = P(1 + r)^n$$

which you may recall from the chapter on compound interest.

### Exercise 3

You invest \$1500 at 5%, compounded annually. Write an explicit formula that describes the value of your investment every year. What will your investment be worth after 10 years? Is the sequence convergent or divergent? Explain.

Working Space

Answer on Page ??

### Exercise 4

Working Space

Answer on Page ??

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*This is a draft chapter from the Kontinua Project. Please see our website (<https://kontinua.org/>) for more details.*

# Answers to Exercises

## Answer to Exercise ?? (on page ??)

1.  $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$
2. 0, -1, 0, 1, 0
3. 1, 2, 7, 32, 157
4. 6, 3, 1,  $\frac{1}{4}, \frac{1}{20}$

## Answer to Exercise ?? (on page ??)

1. convergent, 5
2. divergent
3. convergent, 2
4. convergent, -1
5. divergent

## Answer to Exercise ?? (on page ??)

Out principal is  $P = 1500$  and the interest rate is  $r = 0.06$ . After  $n$  years, your investment will be worth  $a_n = 1500(1.06)^n$ . For  $n = 10$ , your investment will be valued at  $a_{10} = \$1500(1.06)^{10} = \$2686.27$  (that's over \$1000 in interest!). To determine if the sequence is convergent or divergent, we examine the limit as  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} 1500(1.06)^n = 1500 \cdot \lim_{n \rightarrow \infty} (1.06)^n = 1500 \cdot \infty = \infty$$

The sequence is divergent.

