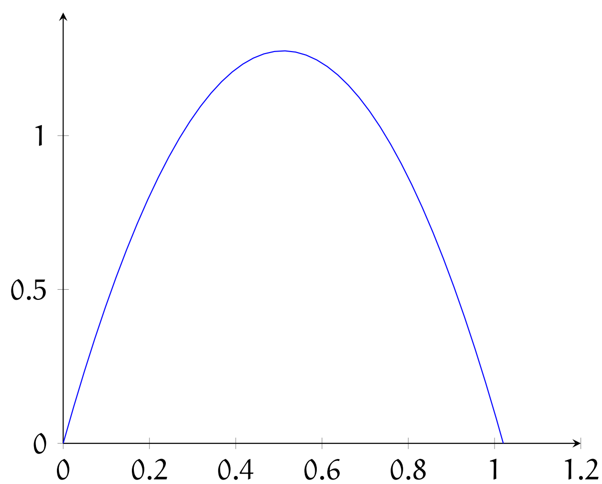
CHAPTER 1

First and Second Derivatives and the Shape of a Function

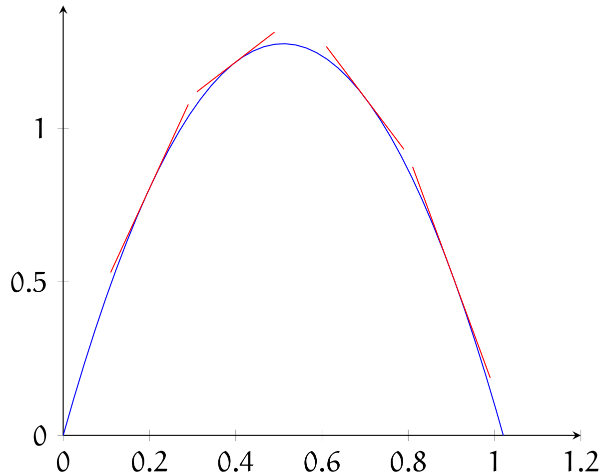
1.1 Using first derivatives to describe a function

1.1.1 Increasing and Decreasing Intervals

Let's re-examine our graph showing the height of a hammer tossed in the air:



As you can see, the hammer reaches its peak around $t \approx 0.5s$. Let's add tangent lines just before and after the peak of the hammer's path so we can more easily examine how the slope of the graph changes:



As you can see, the slope changes from positive to negative as t increases. That implies that $f'(t)$ also changes from positive to negative. In fact, at the highest point of the hammer's flight, the slope (and therefore $f'(t)$) is exactly zero! In general,

1. If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

Example 1: Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing.

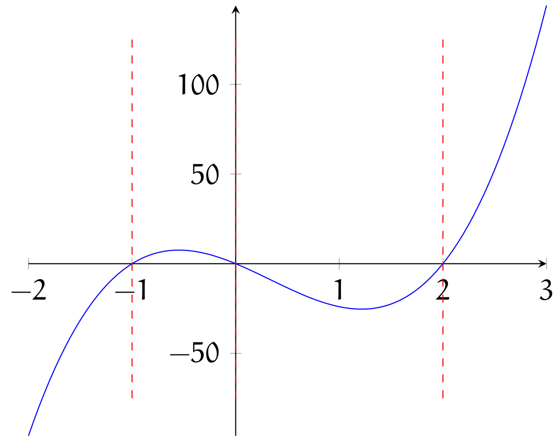
Solution: We want to find the intervals where $f'(x) > 0$. First, we take the derivative to find $f'(x)$:

$$f'(x) = 12x^3 - 12x^2 - 24x$$

It will be easier to analyze the value of $f'(x)$ if we factor it so:

$$f'(x) = 12x(x-2)(x+1)$$

To determine where $f'(x) > 0$, we start by finding where $f'(x) = 0$ (in this case, this is true when $x = -1, 0, 2$). These values of x are called *critical values*, and we will use them to divide $f'(x)$ into intervals. On each of these intervals, $f'(x)$ must be always positive or always negative. This is shown graphically below:



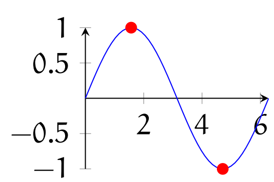
As you can see above, $f'(x) > 0$ on two intervals: $x \in (-1, 0)$ and $x \in (2, \infty)$. These are open intervals because $f'(x) = 0$ at $x = -1$, $x = 0$, and $x = 2$. But what if we had a more complex function, or didn't have the resources to graph it? We can use a table to help us analyze the value of $f'(x)$ (and therefore the behavior of $f(x)$). For each interval around the critical values, we can determine if $f'(x)$ is positive or negative by noting the value of the factors of $f'(x)$, which are $12x$, $x - 2$, and $x + 1$ in this case. For example, for $x < -1$, $12x < 0$, $(x - 2) < 0$, and $(x + 1) < 0$. Three negatives multiplied together is also negative. Therefore, for $x < -1$, $f'(x)$ is negative and $f(x)$ is decreasing. We can analyze all of the intervals similarly and log the results in a table:

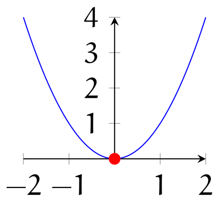
x	$12x$	$x - 2$	$x + 1$	$f'(x)$	$f(x)$
$x < -1$	negative	negative	negative	negative	decreasing
$-1 < x < 0$	negative	negative	positive	positive	increasing
$0 < x < 2$	positive	negative	positive	negative	decreasing
$2 < x$	positive	positive	positive	positive	increasing

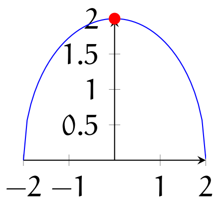
Notice the table method yields the same result as examining the graph: $f(x)$ is increasing for $x \in (0, -1)$ and $x \in (2, \infty)$, which can also be written as $x \in (-1, 0) \cup (2, \infty)$.

1.1.2 Local Extrema

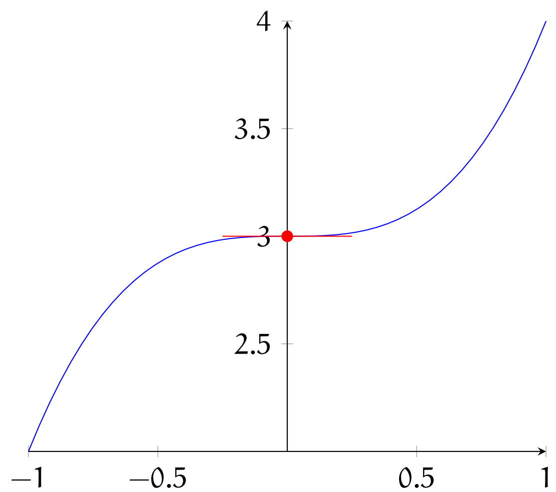
Examine the graphs of x^2, $\sin x$, and $y = \sqrt{4 - x^2}$ below. Each has a dot at a local extreme (either a local minimum or local maximum). Sketch what you think the tangent line to the graph would be at each local extreme. Use this to estimate the value of the derivative at that point.



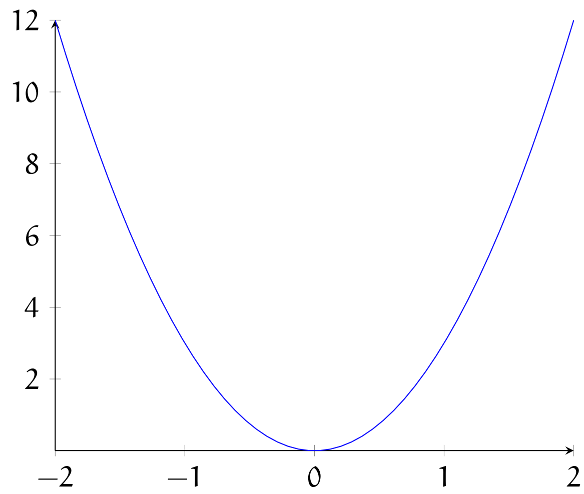




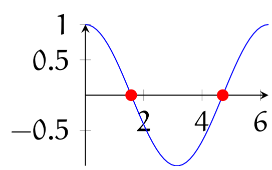
You should notice that all of the tangent lines are horizontal. Since the tangent lines at these local extrema have a slope of 0, that tells us $f'(x) = 0$ at these points too. In fact, for *all* local minima and maxima, the value of the derivative is zero at that point. However, the converse statement is not necessarily true: just because the derivative is zero at some $x = c$ it does not mean there is a local extrema at $f(c)$. Consider $f(x) = x^3 + 3$:



At $x = 0$, $f'(x) = 0$, but there is not a local extreme. For a local extreme to exist, the graph of $f(x)$ must change from increasing to decreasing, or vice versa. Look at the graph of $x^3 + 3$ above: the function is increasing for $x < 0$ and $x > 0$. Another way of saying this is to note that the graph of $f'(x)$ touches but does not cross the x-axis in this case:



If $f(x)$ changes from increasing to decreasing, then $f'(x)$ is changing from positive to negative (i.e. crossing the x-axis). Look at the derivative of $f(x) = \sin x$, $f'(x) = \cos x$, on the graph below. The x-values where local extrema exist on $f(x)$ are marked in red:



As you can see, local extrema are indicated when $f'(x)$ crosses the x-axis. If $f'(x)$ is negative to the left of $x = c$ and positive to the right, then $f(x)$ has a local minimum at $x = c$. On the other hand, if $f'(x)$ is positive to the left of $x = c$ and negative to the right, then $f(x)$ has a local maximum at $x = c$. Any value of $x = c$ where $f'(c) = 0$ is called a **critical number**. Values where $f(c)$ does not exist is also a critical number.

1.1.3 Practice: Interval of Increasing and Decreasing, Local Extrema

Exercise 1

Let f be the function given by $f(x) = 300x - x^3$. On which of the following intervals is f increasing?

Working Space

Answer on Page ??

Exercise 2

Find the intervals on which $f(x) = x^3 - 3x^2 - 9x + 4$ is increasing or decreasing. Then, find all local minimum and/or maximum values of $f(x)$.

Working Space

Answer on Page ??

1.1.4 Global Extrema

Now that we've learned how to identify local minima and maxima, let's expand the discussion to include global extrema. A global extreme is an absolute minimum or maximum value of a function over a particular interval or the entire domain of the function. Let's examine the graph of $f(x) = x^4 - 5x^3 + 6x^2$ over the domain $x \in [-1, 4]$.

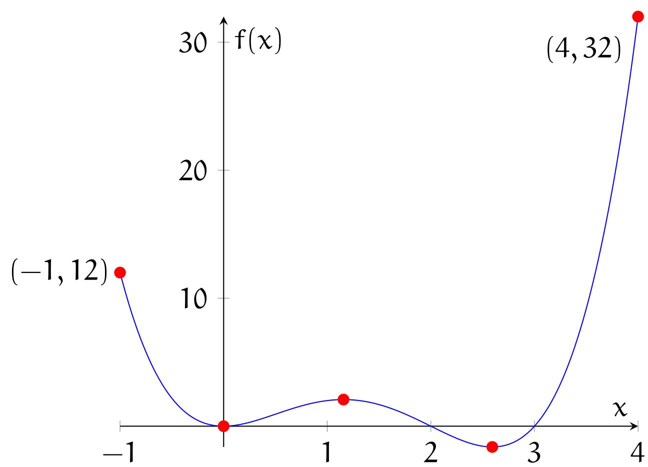


Figure 1.1: Graph of $f(x) = x^4 - 5x^3 + 6x^2$

As you can see, $f(x)$ has two local minima and one local maximum. Additionally, the endpoints are labeled. To determine the *global* extrema, we need to examine the any local extrema (identified here graphically, but you can also identify them mathematically using that you learned in the "Local Extrema" subsection) AND the endpoints of the domain (or the function's behavior at $\pm\infty$).

In the case of $f(x) = x^4 - 5x^3 + 6x^2, \text{ for} x \in [-1, 4]$, the global maximum value is 32 at $x = 4$ and the global minimum is -1.623 at $x = 2.593$.

If a function is continuous on an interval, then there must exist a global maximum and global minimum on that interval. These global extrema may also be local extrema (as is the case for $f(2.593)$ in the example above) or not (as is the case for $f(4)$). Applying the Closed Interval Method is a straightforward way to identify global (absolute) extrema. To find the global extrema of a continuous function, f, on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from steps 1 and 2 is the absolute maximum; the smallest of the values is the absolute minimum.

Let's use the Closed Interval Method to determine the global extrema for the function $g(x) = x - 3\sin x$ on the interval $x \in [0, 2\pi]$.

To find the value of g at any critical numbers, we must first identify the critical numbers. Recall that critical numbers are values where the first derivative of the function is 0 or does not exist. To find critical numbers, we set g' equal to 0:

$$g'(x) = 1 - 3\cos x = 0$$

$$3\cos x = 1$$

$$\cos x = \frac{1}{3}$$

$$x = 1.23, 5.052$$

Now, we substitute these critical numbers back into $g(x)$:

$$g(1.23) \approx -1.60$$

$$g(5.052) = 7.881$$

Now we need to check the endpoints:

$$g(0) = 0 - 3 * 0 = 0$$

$$g(2\pi) = 2\pi - 3 * 0 = 2\pi \approx 6.28$$

The results are presented in the table below:

x	$g(x)$
0	0
1.23	-1.60
5.052	7.881
6.28	6.28

Therefore, for $g(x) = x - 3\sin x$ on the interval $x \in [0, 2\pi]$, the global maximum is $g(5.052) = 7.881$ and the global minimum is $g(1.23) = -1.60$.

1.1.5 Practice: Global Extrema

Exercise 3

Let f be the function defined by $f(x) = \frac{\ln x}{x}$. What is the absolute maximum value of f?

Working Space

Answer on Page ??

Exercise 4

Find the global minimum and maximum values on the stated interval.

1. $f(x) = 12 + 4x - x^2$, $[0, 5]$
2. $f(t) = \frac{\sqrt{t}}{1+t^2}$, $[0, 2]$
3. $f(t) = 2\cos t + \sin 2t$, $[0, \frac{\pi}{2}]$
4. $f(x) = \ln x^2 + x + 1$, $[-1, 1]$

Working Space

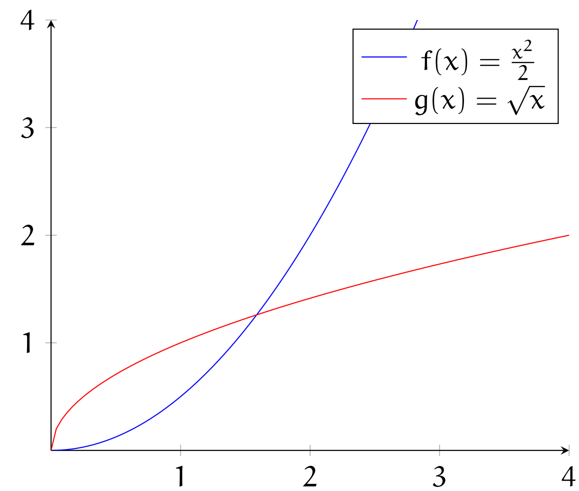
Answer on Page ??

1.2 Using second derivatives to describe a function

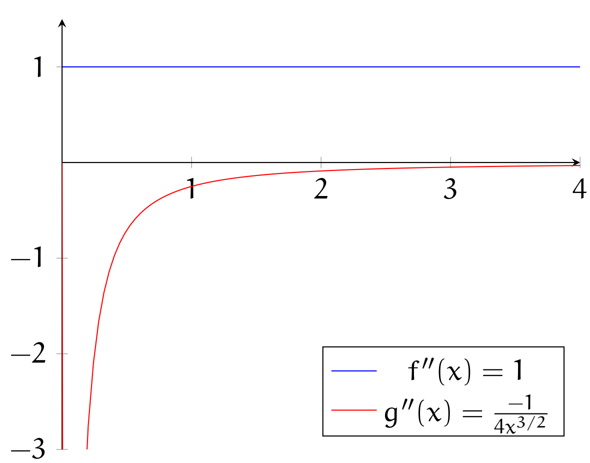
1.2.1 Concavity

Let's examine two increasing functions, $f(x) = \frac{x^2}{2}$ and $g(x) = \sqrt{x}$:

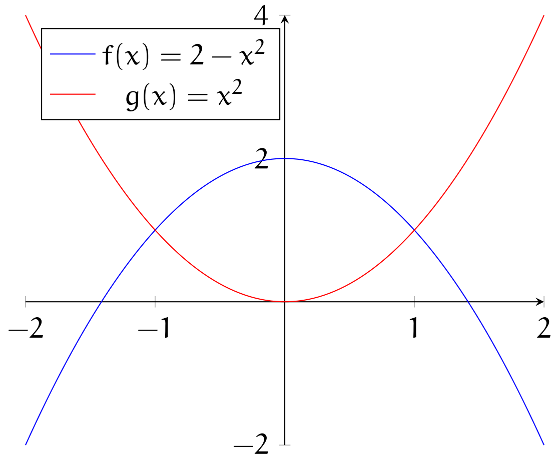
Even though both of these functions are increasing, they have different shapes. $f(x)$ looks like bowl. On the other hand, $g(x)$ looks like an upside-down bowl. These shapes are called *concave up* (in the case of $f(x)$) and *concave down* (in the case of $g(x)$). Both functions are increasing on the interval $x \in [0, 4]$, and therefore both $f'(x)$ and $g'(x)$ are positive on the stated interval. Let's look at their second derivatives, $f''(x)$ and $g''(x)$:



4
3
2
1
1
2
3
4
$f(x) = \frac{x^2}{2}$
$g(x) = \sqrt{x}$



1
1
2
3
4
−1
−2
−3
$f''(x) = 1$
$g''(x) = \frac{-1}{4x^{3/2}}$



As you can see, $f''(x) > 0$ and $g''(x) < 0$. The second derivative tells us if a function is concave up or concave down. In general:

1. If $f''(x) > 0$ for all x in a given interval, then the graph of f is concave up on the interval.
2. If $f''(x) < 0$ for all x in a given interval, then the graph of f is concave down on the interval.

Additionally, the second derivative can help us determine if there is a local minimum or maximum at critical numbers. Look at the graphs of $f(x) = 2 - x^2$ and $g(x) = x^2$, which both have first derivatives equal to 0 at $x = 0$:

When the graph is concave up, there is a local minimum where the first derivative equals 0. When the graph is concave down, there is a local maximum where the first derivative equals 0. This is summarized with the Second Derivative Test:

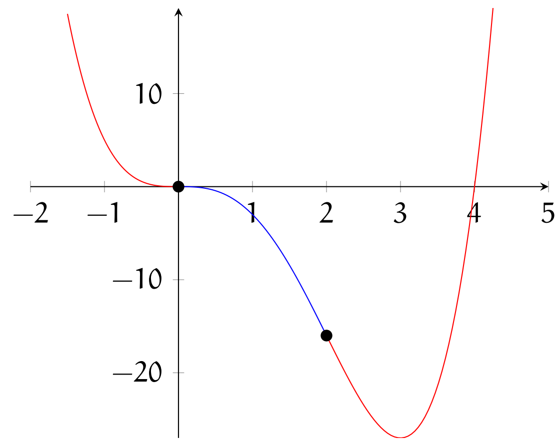
Suppose f'' is continuous near c. Then,

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

1.2.2 Inflection Points

If f is concave up when $f'' > 0$ and concave down when $f'' < 0$, what about when $f'' = 0$? This is the value at which f changes from concave up to concave down (or vice versa), which is called an *inflection point*. Similar to local extrema with f', if there is an inflection point at $x = c$, then $f''(c) = 0$, but the converse is not necessarily true. To check if $x = c$ is an inflection point, then f'' should change signs on either side of $x = c$ (either from positive to negative to from negative to positive).

Look at the graph of $f(x) = x^4 - 4x^3$. The concave up areas are shown in red, and the concave down in blue:



Let's examine f'' to confirm the inflection points are at $(0, 0)$ and $(2, -16)$. First, we note that $f''(x) = 12x^2 - 24x$. Factoring, we see that $f''(x) = 12x(x - 2)$, which has zeroes at $x = 0$ and $x = 2$. For $x < 0$, $f'' > 0$, and for $0 < x < 2$, $f'' < 0$; therefore, there is an inflection point in f at $(0, 0)$.

Exercise 5

Prove that the other inflection point for $f(x) = x^4 - 4x^3$ is $(2, -16)$.

Working Space

Answer on Page ??

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

First, we will find f' and set it equal to zero:

$$f'(x) = 300 - 3x^2 = 0$$

$$300 = 3x^2 \to x = \pm\sqrt{100} = \pm 10$$

(Note: $f'(x) = 3(10 - x)(10 + x)$, which implies roots at $x = \pm 10$. Now we will evaluate the value of $f'(x)$ for $x < -10$, $-10 < x < 10$, and $x > 10$.

Value of x	(10-x)	(10+x)	$f'(x)$	$f(x)$ behavior
$x < -10$	positive	negative	negative	decreasing
$-10 < x < 10$	positive	positive	positive	increasing
$x > 10$	negative	positive	negative	decreasing

Therefore, the fucntion is increasing on the interval $x \in [-10, 10]$ because $f'(x) > 0$ for $x \in [-10, 10]$.

Answer to Exercise 2 (on page 6)

Given $f(x) = x^3 - 3x^2 - 9x + 4$, it follows that $f'(x) = 3x^2 - 6x - 9$. Factoring, we find that $f'(x) = 9(x - 3)(x + 1)$ and $f'(x) = 0$ when $x = 3$ and $x = -1$. We construct our table to help us analyze the value of $f'(x)$ and behavior of $f(x)$ on the whole domain of the function:

Value of x	$(x - 3)$	$(x + 1)$	$f'(x)$	$f(x)$ behavior
$x < -1$	negative	negative	positive	increasing
$-1 < x < 3$	negative	positive	negative	decreasing
$x > 3$	positive	positive	positive	increasing

So, $f(x)$ is increasing for $x \in (-\infty, -1) \cup (3, \infty)$ and decreasing for $x \in (-1, 3)$. Since $f'(-1) = 0$ and changes from positive to negative, $f(x)$ has a local maximum at $x = -1$. And since $f'(3) = 0$ and changes from negative to positive, $f(x)$ has a local minimum at $x = 3$.

Answer to Exercise 3 (on page 8)

First, we identify any critical numbers:

$$f'(x) = \frac{x * (\frac{1}{x}) - \ln x * 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Recall that critical numbers are values where $f'(x) = 0$ or does not exist. We might identify $x = 0$ as a critical number, but the presence of $\ln x$ limits the domain of the function to $x \in (0, \infty)$, excluding $x = 0$. For all $x \in (0, \infty)$, $f'(x)$ exists. So, we look for values where $f'(x) = 0$.

$$\frac{1 - \ln x}{x^2} = 0$$
$$1 - \ln x = 0$$
$$1 = \ln x$$
$$x = e$$

Finding the value of $f(x)$ at $x = e$:

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

Because the domain of $f(x)$ is on an *open interval*, instead of checking the endpoints directly, we'll take the limits as x approaches 0 and ∞.

$$\lim_{x \to 0} \frac{\ln x}{x} = -\infty < \frac{1}{e}$$
$$\lim_{x \to \infty} \frac{\ln x}{x} = 0 < \frac{1}{e}$$

Therefore, the absolute maximum values of $f(x) = \frac{\ln x}{x}$ is $\frac{1}{e}$ at $x = e$.

Answer to Exercise 4 (on page 8)

1. $f'(x) = 4 - 2x$ and to find the critical numbers, we set $f'(x) = 0$:

   $$4 - 2x = 0$$
   $$x = 2$$

   We evaluate $f(x)$ at $x = 0, 2, 5$:

   $$f(0) = 12 + 4(0) - 0^2 = 12$$
   $$f(2) = 12 + 4(2) - 2^2 = 12 + 8 - 4 = 16$$
   $$f(5) = 12 + 4(5) - 5^2 = 12 + 20 - 25 = 7$$

   Therefore, the global maximum is $f(2) = 16$ and the global minimum is $f(5) = 7$.

2.

Answer to Exercise ?? (on page ??)

Noting that $f''(2) = 0$, we examine the value of f'' around $x = 2$. For $0 < x < 2$, $f'' < 0$, which indicates f is concave down in the domain $x \in (0, 2)$. For $x > 2$, $f'' > 0$, which indicates f is concave up. Therefore, there is an inflection point at $x = 2$ for f. Recalling that $f(x) = x^4 - 4x^3$, we find the coordinate of the inflection point by substituting $x = 2$:

$$f(2) = 2^4 - 4 * 2^3 = 16 - 4 * 8 = 16 - 32 = -16$$

Therefore, $f(x)$ has an inflection point at $(2, -16)$.