Chapter 1

Sequences in Calculus

We have introduced sequences in a previous chapter. Now, we will examine them in more detail in a calculus context. You already know about arithmetic and geometric sequences, but not all sequences can be classified as arithmetic or geometric. Take the famous Fibonacci sequence, {1, 1, 2, 3, 5, 8, ...}, which can be explicitly defined as $a_n = a_{n-1} + a_{n-2}$, with $a_1 = a_2 = 1$. There is no common difference or common ratio, so the Fibonacci sequence is not arithmetic or geometric. Another example is $a_n = \sin \frac{n\pi}{6}$, which will cycle through a set of values. In later chapters, you will learn that the sum of all the values in a sequence is a series and how to use series to describe functions. In order to be able to do all that, first we need to talk in-depth about sequences.

Some sequences are defined explicitly, like $a_n = \sin \frac{n\pi}{6}$, while others are defined recursively, like $a_n = a_{n-1} + a_{n-2}$.

Example: Write the first 5 terms for the explicitly defined sequence $a_n = \frac{n}{n+1}$.

Solution: We can construct a table to keep track of our work:

n	work	a_n
1	$\frac{1}{1+1}$	$\frac{1}{2}$
2	$\frac{2}{2+1}$	$\frac{2}{3}$
3	$\frac{3}{3+1}$	$\frac{3}{4}$
4	$\frac{4}{4+1}$	$\frac{4}{5}$
5	$\frac{5}{5+1}$	$\frac{5}{6}$

So the first five terms are $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$.

Exercise 1

Write the first 5 terms for each sequence.

1. $a_n = \frac{2^n}{2n+1}$
2. $a_n = \cos \frac{n\pi}{2}$
3. $a_1 = 1$, $a_{n+1} = 5a_n - 3$
4. $a_1 = 6$, $a_{n+1} = \frac{a_n}{n+1}$

Working Space

Answer on Page 7

1.1 Convergence and Divergence

You can visualize a sequence on an xy-plane or a number line. Figures 1.1 and 1.2 show visualizations of the sequence $a_n = \frac{n}{n+1}$. To visualize this on the xy-plane, we take points such that $x = n$ and $y = a_n$, where n is a positive integer. What do you notice about this sequence? As n increases, a_n gets closer and closer to 1.

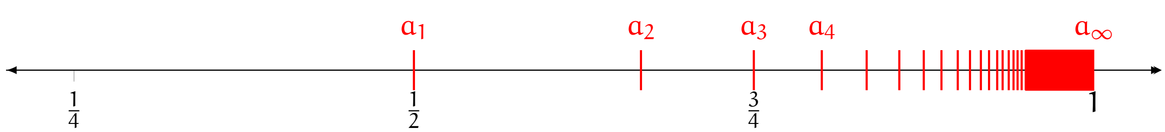

Figure 1.1: $a_n = \frac{n}{n+1}$ on a number line

Because a_n approaches a specific number as $n \to \infty$, we call the series $a_n = \frac{n}{n+1}$ *convergent*. We prove a sequence is convergent by taking the limit as n approaches ∞. If the limit exists and approaches a specific number, the sequence is convergent. If the limit does not exist or approaches $\pm\infty$, the sequence is divergent.

We can see graphically that $\lim_{n\to\infty} \frac{n}{n+1} = 1$, so that sequence is convergent. What about $b_n = \frac{n}{\sqrt{10+n}}$? Is b_n convergent or divergent?

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{n/n}{\sqrt{\frac{10}{n^2} + \frac{n}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

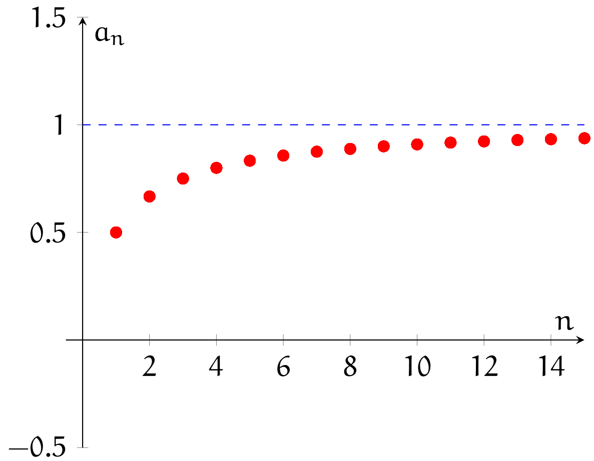

Figure 1.2: $a_n = \frac{n}{n+1}$ on an xy-plane

Therefore, the sequence $b_n = \frac{n}{\sqrt{10+n}}$ is divergent.

Here is another example of a divergent sequence: $c_n = \sin \frac{n\pi}{2}$. The graph is shown in figure 1.3. As you can see, the value of c_n oscillates between 1, 0, and -1 without approaching a specific number. This means that c_n does not approach a particular number as $n \to \infty$ and the sequence is divergent.

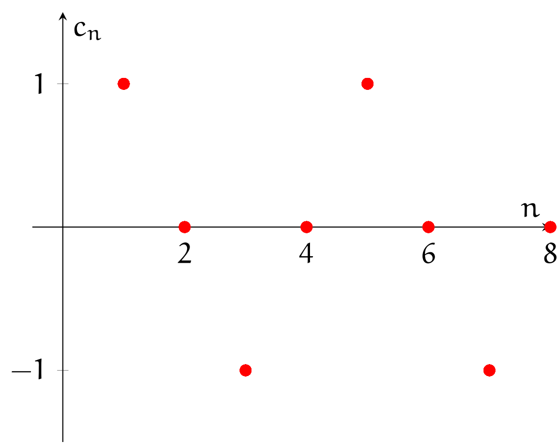

Figure 1.3: $c_n = \sin \frac{n\pi}{2}$ on an xy-plane

Exercise 2

Classify each sequence as convergent or divergent. If the sequence is convergent, find the limit as $n \to \infty$.

1. $a_n = \frac{3+5n^2}{n+n^2}$
2. $a_n = \frac{n^4}{n^3-2n}$
3. $a_n = 2 + (0.86)^n$
4. $a_n = \cos\frac{n\pi}{n+1}$
5. $a_n = \sin n$

Working Space

Answer on Page 7

1.2 Evaluating limits of sequences

Recall that a sequence can be considered a function where the domain is restricted to positive integers. If there is some $f(x)$ such that $a_n = f(n)$ when n is an integer, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$ (see figure 1.4). This means that all the rules that apply to the limits of functions also apply to the limits of sequences, including the Squeeze Theorem and l'Hospital's rule.

What is $\lim_{n\to\infty} \frac{\ln n}{n}$? First, we will try to compute the limit directly:

$$\lim_{n\to\infty} \frac{\ln n}{n} =$$

$$\frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n} = \frac{\infty}{\infty}$$

This is undefined, but fits the criteria for Ll'Hospital's rule:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln n}{\frac{d}{dn} n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

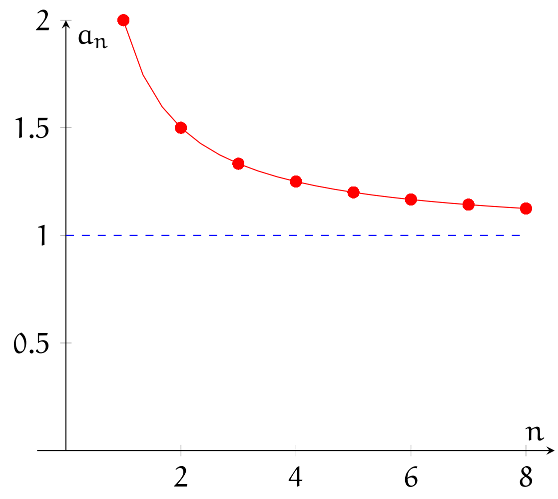

Figure 1.4: The limit of the function is the same as the limit of the sequence

Here's an example that requires the Squeeze Theorem: is the sequence $a_n = \frac{n!}{n^n}$ convergent or divergent? First trying to take the limit directly, we see that:

$$\lim_{n\to\infty} \frac{n!}{n^n} = \frac{\infty}{\infty}$$

which is undefined. Because the factorial cannot be described as a continuous function, we can't use l'Hospital's rule. We can examine this sequence graphically (see figure 1.5) and mathematically. We examine it mathematically by writing out a few terms to get an idea of what happens to a_n as n gets large:

$$a_1 = \frac{1!}{1^1} = 1$$

$$a_2 = \frac{2!}{2^2} = \frac{1 \cdot 2}{2 \cdot 2}$$

$$a_3 = \frac{3!}{3^3} = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$

$$\cdots$$

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdots n}$$

From examining the graph in figure 1.5, we can guess that $\lim_{n\to\infty} a_n = 0$. Let's prove that mathematically. We can rewrite our expression for a_n as n gets large:

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n}\right)$$

The expression inside the parentheses is less than 1, therefore $0 < a_n < \frac{1}{n}$. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, by Squeeze Theorem we know that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. Therefore, the sequence $a_n = \frac{n!}{n^n}$ is convergent.

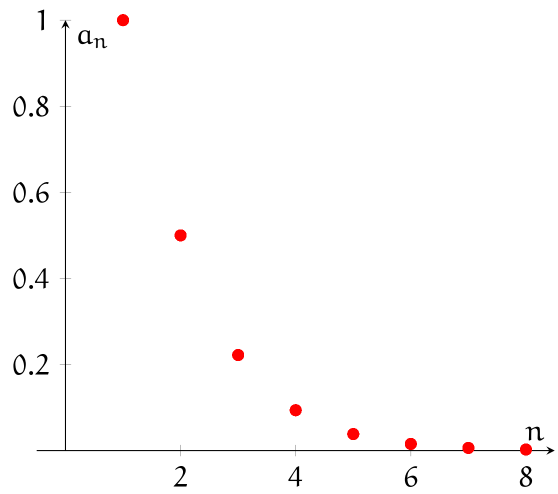

Figure 1.5: $a_n = \frac{n!}{n^n}$

[[FIX ME intro]] If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. For example, what is $\lim_{n\to\infty} \sin \frac{\pi}{n}$? Well, we know that $\lim_{n\to\infty} \frac{\pi}{n} = 0$ and that the sine function is continuous at 0. Therefore, $\lim_{n\to\infty} \sin \frac{\pi}{n} = \sin \lim_{n\to\infty} \frac{\pi}{n} = \sin 0 = 0$.

1.3 Monotonic and Bounded sequences

Just like functions, sequences can be increasing or decreasing. A sequence is increasing if $a_n < a_{n+1}$ for $n \geq 1$. Similarly, a sequence is decreasing if $a_n > a_{n+1}$ for $n \geq 1$. If a sequence is strictly increasing or decreasing, it is called *monotonic*.

The sequence $a_n = \frac{1}{n+6}$ is decreasing. We prove this formally by comparing a_n to a_{n+1}:

$$\frac{1}{n+6} > \frac{1}{(n+1)+6} = \frac{1}{n+7}$$

Is the sequence $a_n = \frac{n}{n^2+1}$ increasing or decreasing? First, we find an expression for a_{n+1}:

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$$

Since the degreen of n is greater in the denominator, we have a guess that the sequence is decreasing. To prove this, we check if $a_n > a_{n+1}$ is true:

$$\frac{n}{n^2+1} > \frac{n+1}{n^2+2n+2}$$

We can cross-multiply, because $n > 0$ and the denominators are positive:

$$(n)(n^2+2n+2) > (n+1)(n^2+1)$$

$$n^3 + 2n^2 + 2n > n^3 + n^2 + n + 1$$

Subtracting $(n^3 + n^2 + n)$ from both sides we see that:

$$n^2 + n > 1$$

Which is true for all $n \geq 1$. Therefore, $a_n > a_{n+1}$ for all $n \geq 1$ and the sequence is decreasing.

A sequence is *bounded above* if there is some number M such that $a_n \leq M$ for all $n \geq 1$. And a sequence is *bounded below* if there is some other number m such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above and below, then it is a *bounded sequence*.

If a sequence is both *monotonic* and *bounded*, then it must be convergent.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

1. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$
2. 0, -1, 0, 1, 0
3. 1, 2, 7, 32, 157
4. 6, 3, 1, $\frac{1}{4}$, $\frac{1}{20}$

Answer to Exercise 2 (on page 4)

1. convergent, 5
2. divergent
3. convergent, 2
4. convergent, -1
5. divergent