Chapter 1

Linear Combinations and Span

In the introductory linear algebra chapter, you learned that vectors and matrices can be rotated, inverted, and added. In this chapter, we will explore linear combinations of vectors and the span of group of vectors. The **span** of a group of vectors is the set of vectors that can be made with linear combinations of the original group of vectors. We will offer mathematical and visual explanations later in the chapter. First, let's examine linear combinations.

1.1 Linear Combinations of Vectors

A linear combination is simply the addition of vectors with leading scalar multipliers. For example, $3\,[2,-1]+2\,[3,5]$ is a linear combination of the vectors $[2,-1]$ and $[3,5]$. Another way to say this is:

Linear Combination of Vectors

A linear combination of a list of n vectors, $v_1, v_2, \cdots, v_n$ takes the form:

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

where $a_1, a_2, \cdots, a_n \in \mathbb{R}$

Example: Find a linear combination of $[2, 1, -3]$ and $[1, -2, 4]$ that gives the vector $[17, -4, 2]$.

Solution: We are looking for a_1 and a_2 such that:

$$a_1\,[2, 1, -3] + a_2\,[1, -2, 4] = [17, -4, 2]$$

Looking at each dimension separately, we get the system of equations:

$$\begin{aligned}
2a_1 + 1a_2 &= 17\\
1a_1 - 2a_2 &= -4\\
-3a_1 + 4a_2 &= 2
\end{aligned}$$

If we can solve this system of equations, we will find a_1 and a_2. Let's multiply the first equation by 2 and add it to the second equation:

$$2\,[2a_1 + a_2] + [a_1 - 2a_2] = 2\,(17) + -4$$

$$4a_1 + 2a_2 + a_1 - 2a_2 = 34 - 4$$
$$5a_1 = 30$$
$$a_1 = 6$$

Now we can take a_1 and substitute it back into any equation in our system to find a_2. Let's use the third equation:

$$-3\,(6) + 4a_2 = 2$$
$$-18 + 4a_2 = 2$$
$$4a_2 = 20$$
$$a_2 = 5$$

Therefore, $6\,[2, 1, -3] + 5\,[1, -2, 4] = [17, -4, 2]$.

Exercise 1 Linear Combinations

Working Space

Find a linear combination of the first two vectors that yields the third vector.

1. $[1, 2]$, $[-3, 1]$, $[4, 5]$
2. $[9, 4]$, $[0, 1]$, $[-5, 3]$
3. $[7, -2]$, $[-8, 4]$, $[6, -2]$

Answer on Page ??

Sometimes, a set of vectors cannot be combined to make a specific vector. Take the pair of vectors we have looked at before: $[2, 1, -3]$ and $[1, -2, 4]$. Can we find a combination to make vector $[17, -4, 5]$? Let's try. We define a_1 and a_2 such that:

$$a_1 [2, 1, -3] + a_2 [1, -2, 4] = [17, -4, 5]$$

Which creates the system of equations:

$$2a_1 + a_2 = 17$$
$$a_1 - 2a_2 = -4$$
$$-3a_1 + 4a_2 = 5$$

We have two variables (a_1 and a_2) and three equations. Let's use the first two to find a_1 and a_2, then check our answers by substituting our solutions into the third equation. First, we'll multiply the second equation by -2 and add that to the first equation:

$$2a_1 + a_2 + (-2)(a_1 - 2a_2) = 17 + (-2)(-4)$$
$$2a_1 + a_2 - 2a_1 + 4a_2 = 17 + 8$$
$$5a_2 = 25$$
$$a_2 = 5$$

Substituting for a_2 back into the first equation and solving for a_1:

$$2a_1 + 5 = 17$$
$$2a_1 = 12$$
$$a_1 = 6$$

Now, let's check if $a_1 = 6$, $a_2 = 5$ is a solution to the third equation:

$$-3(6) + 4(5) = 5$$
$$-18 + 20 = 2 \neq 5$$

Therefore, there is no linear combination of the vectors $[2, 1, -3]$ and $[1, -2, 4]$ that yields $[17, -4, 5]$.

1.2 Span

The span of a list of vectors is all the vectors than can be made from a linear combination of those vectors. Above, we saw that $[17, -4, 2]$ is in the span of $[2, 1, -3]$ and $[1, -2, 4]$ while $[17, -4, 5]$ is not.

Span of a List of Vectors

The set of all linear combinations of a list of vectors, $v_1, v_2, \cdots, v_n$ is call the span of $v_1, v_2, \cdots, v_n$ and is denoted by $\text{span}(v_1, v_2, \cdots, v_n)$. Mathematically,

$$\text{span}(v_1, v_2, \cdots, v_n) = \{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n | a_1, a_2, \cdots, a_n \in \mathbb{R}\}$$

The span of an empty list of vectors is $\{0\}$.

1.3 Linear Independence

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

Appendix A

Answers to Exercises

Answer to Exercise ?? (on page ??)

1. We are looking for a_1 and a_2 such that:

$$a_1 [1, 2] + a_2 [-3, 1] = [4, 5]$$

Which creates the system of equations:

$$a_1 - 3a_2 = 4$$

$$2a_1 + a_2 = 5$$

We can multiply the first equation by -2 and add it to the second to solve for a_2:

$$-2 (a_1 - 3a_2) + 2a_1 + a_2 = -2 (4) + 5$$

$$6a_2 + a_2 = -8 + 5$$
$$7a_2 = -3$$
$$a_2 = -\frac{3}{7}$$

Substituting a_2 back into an equation and solving for a_1:

$$a_1 - 3\left(-\frac{3}{7}\right) = 4$$

$$a_1 + \frac{9}{7} = 4$$
$$a_1 = \frac{19}{7}$$

Therefore, $\frac{19}{7} [1, 2] - \frac{3}{7} [-3, 1] = [4, 5]$.

2. We are looking for a_1 and a_2 such that:

$$a_1 [9, 4] + a_2 [0, 1] = [-5, 3]$$

Which creates the system of equations:

$$9a_1 = -5$$

$$4a_1 + a_2 = 3$$

We can find a_1 from the first equation:

$$a_1 = -\frac{5}{9}$$

Substituting for a_1 back into the second equation and solving for a_2:

$$4\left(-\frac{5}{9}\right) + a_2 = 3$$

$$a_2 - \frac{20}{9} = 3$$

$$a_2 = \frac{47}{9}$$

Therefore, $-\frac{5}{9}[9,4] + \frac{47}{9}[0,1] = [-5,3]$.

3. We are looking for a_1 and a_2 such that:

$$a_1[7,-2] + a_2[-8,4] = [6,-2]$$

Which yields the system of equations:

$$7a_1 - 8a_2 = 6$$

$$-2a_1 + 4a_2 = -2$$

Doubling the second equation and adding it to the first:

$$7a_1 - 8a_2 + 2(-2a_1 + 4a_2) = 6 + 2(-2)$$

$$7a_1 - 8a_2 - 4a_1 + 8a_2 = 6 - 4$$

$$3a_1 = 2$$

$$a_1 = \frac{2}{3}$$

Substituting for a_1 back into the second equation and solving for a_2:

$$-2\left(\frac{2}{3}\right) + 4a_2 = -2$$

$$-\frac{4}{3} + 4a_2 = -2$$

$$4a_2 = -\frac{2}{3}$$

$$a_2 = -\frac{1}{6}$$

Therefore, $\frac{2}{3}[7,-2] - \frac{1}{6}[-8,4] = [6,-2]$

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