Contents

Chapter 1

Practice with Polynomials

At this point, you know all the pieces necessary to solve problems involving polynomials. In this chapter, you are going to practice using all of these ideas together.

Watch Khan Academy's **Polynomial identities introduction here**: https://youtu.be/EvNKKyhLSpQ Also watch the follow up here: https://youtu.be/-6qiO49Q180

FIXME: Lots of practice problems here

Chapter 2

Graphing Polynomials

In using polynomials to solve real-world problems, it is often handy to know what the graph of the polynomial looks like. You have many of the tools you need to start to sketch out the the graphs:

- To find where the graph crosses the y-axis, you can evaluate the polynomial at $x = 0$.
- To find where the graph crosses the x-axis, you can find the roots of the polynomial.
- To find the level spots on the graph (often the top of a hump or the bottom of a dip), you can take the derivative of the polynomial (which is a polynomial), and find the roots of that.

FIXME: Diagram of those things

For example, if you wanted to graph the polynomial $f(x) = -x^3 - x^2 + 6x$, you might plug in a few values that are easy to compute:

- $f(-2) = -8$

- $f(-1) = -6$
- $f(0) = 0$
- $f(1) = 4$
- $f(2) = 0$

So, right away we know two roots: $x = 0$ and $x = 2$. Are there others? We won't know until we factor the polynomial:

$$\begin{aligned} -x^3 - x^2 + 6x &= (-1x)(x^2 + x - 6) \\ &= (-1x)(x+3)(x-2) \end{aligned}$$

So, yes, there is a third root: $x = -3$

What about the level spots? $f'(x) = -3x^2 - 2x + 6$. Where is that zero?

$$\begin{aligned} -3x^2 - 2x + 6 &= 0 \\ x^2 + \frac{2}{3}x - 2 &= 0 \end{aligned}$$

We have a formula for quadratics like this:

$$\begin{aligned} x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} &= -\frac{\frac{2}{3}}{2} \pm \frac{\sqrt{\left(\frac{2}{3}\right)^2 - 4(-2)}}{2} \\ &= -\frac{1}{3} \pm \frac{\sqrt{\frac{4}{9} + 8}}{2} \\ &= -\frac{1}{3} \pm \frac{\sqrt{\frac{85}{9}}}{2} \\ &= -\frac{1}{3} \pm \frac{\sqrt{85}}{6} \\ &\approx 1.20 \text{ and } -1.87 \end{aligned}$$

Now you might plug those numbers in:

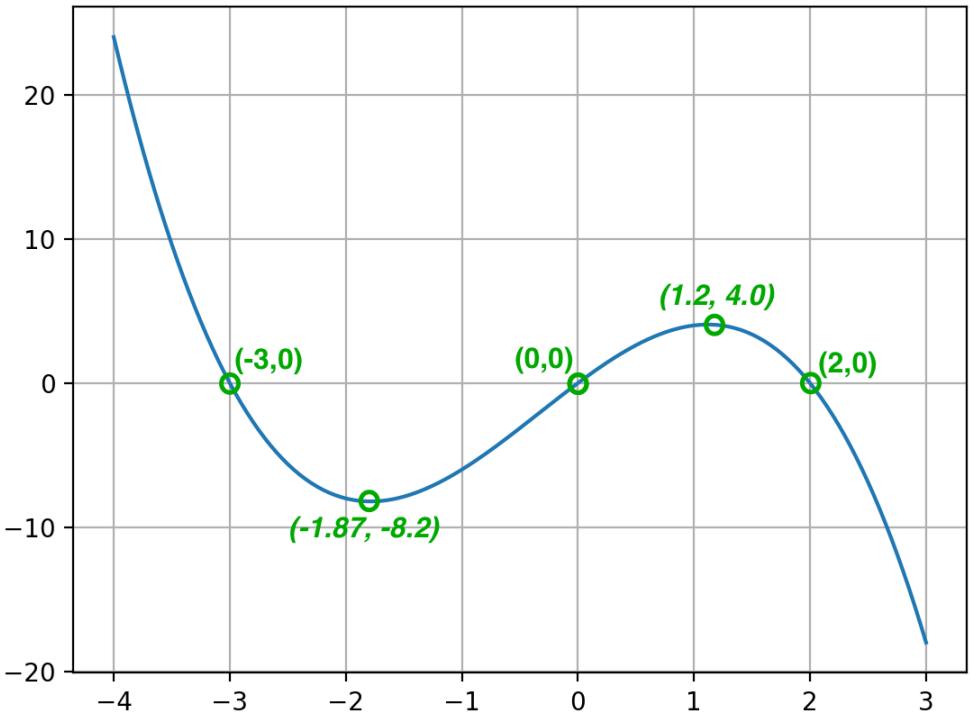
- $f(1.2) \approx 4.0$
- $f(-1.87) \approx -8.2$

2.1 Leading term in graphing

There is one more trick you need before you can draw a good graph of a polynomial. As you go father and farther to the left and right, where does the function go? That is, does the graph go up on both ends (like a smile)? Or does it go down on both ends (like a frown)? Or does the negative end go down (frowny) while the positive end go up (smiley)? Or does the negative go up (smiley) and the positive end go down (frowny)?

Assuming the polynomial is not constant, there are only those four possibilties. It is determined entirely by the leading term of the polynomial. If the degree of the leading term is even, both ends go in the same direction (both are smiley or both are frowny). If the coefficient of the leading term is positive, the positive end is smiley.

The graph we are working on has a leading term of $-1x^3$. The degree is odd, thus the ends go in different directions. The coefficient is negative, so the positive end points down. Now you can draw the graph, which should look something like this:



Chapter 3

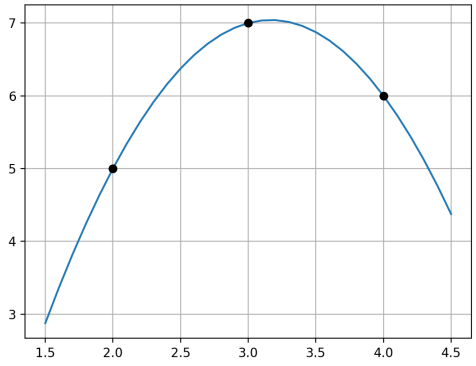
Interpolating with Polynomials

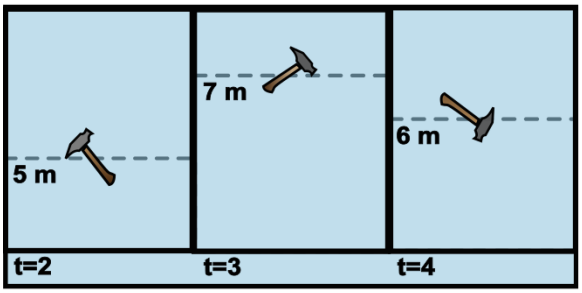
Let's say someone on a distant planet records video of a hammer being throw up into the air. They send you three random frames of the hammer in flight. Each frame has a timestamp and you can clearly see how high the hammer is in each one. Can you create a 2nd degree polynomial that explains the entire flight of the hammer?

That is, you have three points $(t_0, h_0), (t_1, h_1), (t_2, h_2)$. Can you find a, b, c such that the graph of $at^2 + bt + c = t$ passes through all three points?

The answer is yes. In fact, given any n points, there is exactly one $n-1$ degree polynomial that passes through all the points.

There are a lot of variables floating around. Let's make it concrete: The photos are taken at $t = 2$ seconds, $t = 3$ seconds, and $t = 4$ seconds. In those photos, the height of the hammer is $5m$, $7m$, and $6m$. So, we want our polynomial to pass through these points: (2, 5), (3, 7), (4,6).





How can you find that polynomial? Let's do it in small steps. Can you create a 2nd degree polynomial that is not zero at $t = 2$, but is zero at $t = 3$ and $t = 4$? Yes, you can: $(x-3)(x-4)$ has exactly two roots at $t = 3$ and $t = 4$. The value of this polynomial at $t = 2$ is $(2-3)(2-4) = 2$. We really want it to be 5m, so we can divide the whole polynomial by 2 and multiply it by 5.

Now we have the polynomial:

$$f_0(x) = \frac{5}{(2-3)(2-4)}(x-3)(x-4) = \frac{5}{2}x^2 - \frac{35}{2}x + 30$$

This is a second degree polynomial that is 5 at $t = 2$ and 0 at $t = 3$ and $t = 4$.

Now we create a polynomial that is 7 at $t = 3$ and 0 at $t = 2$ and $t = 4$:

$$f_1(x) = \frac{7}{(3-2)(3-4)}(x-2)(x-4) = -7x^2 + 42x - 56$$

Finally, we create a polynomial that is 6 at $t = 4$ and zero at $t = 2$ and $t = 3$:

$$f_2(x) = \frac{6}{(4-2)(4-3)}(x-2)(x-3) = 3x^2 - 15x + 18$$

Adding these three polynomials together gives you a new polynomial that touches all three points:

$$f(x) = \frac{5}{2}x^2 - \frac{35}{2}x + 30 - 7x^2 + 42x - 56 + 3x^2 - 15x + 18 = -\frac{3}{2}x^2 + \frac{19}{2}x - 8$$

You can test this with your `Polynomial` class. Create a file called `test_interpolation.py`. Add this code:

```
from Polynomial import Polynomial
import matplotlib.pyplot as plt
```

```
in_x = [2,3,4]
in_y = [5,7,6]

pn = Polynomial([-8, 19/2, -3/2])
print(pn)

# These lists will hold our x and y values
x_list = []
y_list = []

# Starting x
current_x = 1.5

while current_x <= 4.5:
    # Evaluate pn at current_x
    current_y = pn(current_x)

    # Add x and y to respective lists
    x_list.append(current_x)
    y_list.append(current_y)

    # Move x forward
    current_x += 0.05

# Plot the curve
plt.plot(x_list, y_list)

# Plot black circles on the given points
plt.plot(in_x, in_y, "ko")
plt.grid(True)
plt.show()
```

You should get a nice plot that shows the graph of the polynomial passing through those three points.

In general, then, if you give me any three points $(t_0, h_0), (t_1, h_1), (t_2, h_2)$, here is a second degree polynomial that pass through all three points:

$$\frac{h_0}{(t_0 - t_1)(t_0 - t_2)}(x - t_1)(x - t_2) + \frac{h_1}{(t_1 - t_0)(t_1 - t_2)}(x - t_0)(x - t_2) + \frac{h_2}{(t_2 - t_0)(t_2 - t_1)}(x - t_0)(x - t_1)$$

What if you are given 9 points $((t_0, h_0), (t_1, h_1), \ldots, (t_8, h_8))$ and want to find a 8th degree polynomial that passes through all of them? Just what you would expect:

$$\frac{h_0}{(t_0 - t_1)(t_0 - t_2)\ldots(t_0 - t_8)}(x - t_1)(x - t_2)\ldots(x - t_8) + \ldots + \frac{h_8}{(t_8 - t_0)\ldots(t_8 - t_7)}(x - t_0)\ldots(x - t_7)$$

FIXME: Do I need to define summation and prod here?

The general solution is, given n points, the $n-1$ degree polynomial that goes through them is

$$y = \sum_{i=0}^{n} \left(\prod_{\substack{0 \le j \le n \\ j \neq i}} \frac{x - t_j}{t_i - t_j} \right) h_i$$

That would be tedious for a person to compute, but computers love this stuff. Let's create a method that creates instances of Polynomial using interpolation.

3.1 Interpolating polynomials in python

Your method will take two lists of numbers, one contains x-values and the other contains y-values. So comment out the line that creates the polynomial in `test_interpolation.py` and create it from two lists:

```
in_x = [2,3,4]
in_y = [5,7,6]
# pn = Polynomial([-8, 19/2, -3/2])
pn = Polynomial.from_points(in_x, in_y)
print(pn)
```

Add the following method to your Polynomial class in `Polynomial.py`

```
    @classmethod
    def from_points(cls, x_values, y_values):
        coef_count = len(x_values)

        # Sums start with a zero polynomial
        sum_pn = Polynomial([0.0] * coef_count)
        for i in range(coef_count):

            # Products start with the constant 1 polynomial
            product_pn = Polynomial([1.0])
            for j in range(coef_count):

                # Must skip j=i
                if j != i:
                    # (1x - x_values[j]) has a root at x_values[j]
                    factor_pn = Polynomial([-1 * x_values[j], 1])
                    product_pn = product_pn * factor_pn
```

```
            # Scale so product_pn(x_values[i]) = y_values[i]
            scale_factor  = y_values[i] / product_pn(x_values[i])
            scaled_pn = scale_factor * product_pn

            # Add it to the sum
            sum_pn = sum_pn + scaled_pn

        return sum_pn
```

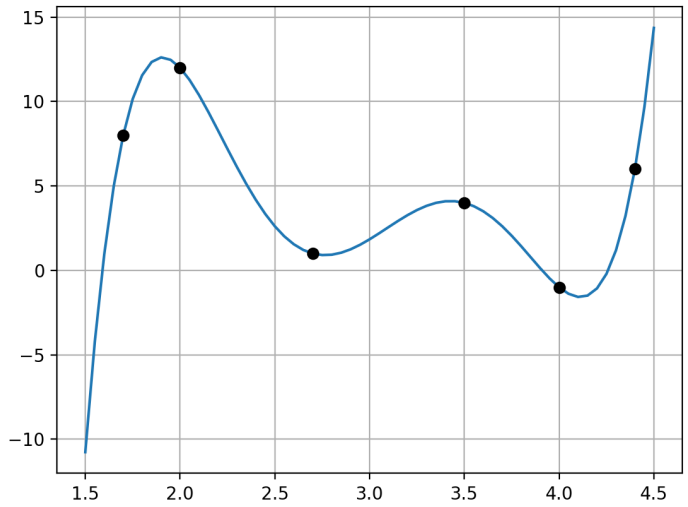
It should work exactly the same as before. You should get the same polynomial printed out as before. You shoud get the same plot of the curve passing through the three points.

How about five points? Change `in_x` and `in_y` at the start of `test_interpolation.py`:

```
in_x = [1.7, 2, 2.7, 3.5, 4, 4.4]
in_y = [8, 12, 1, 4, -1, 6]
```

You should get a polynomial that passes through all five points:

$$11.21x^5 - 171.05x^4 + 1019.44x^3 - 2957.53x^2 + 4161.78x - 2258.75$$



It should look like this:

Appendix A

Answers to Exercises

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