February 26, 2026

Introduction to Mathematical Analysis II
Homework 1 Due March 6 (Friday), 2026
Please submit your group homework online in PDF format. You are expected to work collaboratively within your group. Please do not obtain complete solutions directly from AI tools. The purpose of this assignment is to develop your own mathematical reasoning and problem-solving skills.

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1. (20 pts)

   Define $f : \mathbf{R}^2 \to \mathbf{R}$ by
   $$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

   (a) Show that for every fixed direction $v \in \mathbf{R}^2$, the limit
   $$\lim_{t \to 0} \frac{f(tv) - f(0)}{t}$$
   exists.

   (b) Show that f is *not* differentiable at $(0,0)$ in the sense of Definition 6.2.2.

   (c) Explain precisely which part of the definition of differentiability fails.

2. (15 pts) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ and suppose that for some linear map $L : \mathbf{R}^n \to \mathbf{R}^m$ one has
   $$f(x_0 + h) = f(x_0) + L(h) + R(h),$$
   where the remainder satisfies
   $$\|R(h)\| \leq C\|h\|^{1+\alpha}$$
   for some constants $C > 0$ and $\alpha > 0$.

   (a) Prove that f is differentiable at x_0 with derivative L.

   (b) Show that if $\alpha = 0$, the conclusion may fail by constructing a counterexample.

3. (15 pts) Let $E = \{(x,y) \in \mathbf{R}^2 : y \geq 0\}$, and define
   $$f(x,y) = 0 \quad \text{for all } (x,y) \in E.$$

   (a) Show that f is differentiable at $(0,0)$ in the sense of Definition 6.2.2.

   (b) Show that every linear map $L : \mathbf{R}^2 \to \mathbf{R}$ satisfying $L(x,0) = 0$ for all x also satisfies the differentiability condition.

   (c) Conclude that the derivative at $(0,0)$ is not uniquely determined.

4. (15 pts) **Exercise 6.3.2.** Let $E \subset \mathbf{R}^n$, let $f : E \to \mathbf{R}^m$, let x_0 be an interior point of E, and let $1 \leq j \leq n$.

   Show that $\frac{\partial f}{\partial x_j}(x_0)$ exists if and only if $D_{e_j} f(x_0)$ and $D_{-e_j} f(x_0)$ exist and are negatives of each other. In this case,
   $$\frac{\partial f}{\partial x_j}(x_0) = D_{e_j} f(x_0).$$

5. (15 pts) **Exercise 6.3.3.** Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0), \qquad f(0, 0) = 0.$$

Show that f is not differentiable at $(0, 0)$, even though it is directionally differentiable in every direction at $(0, 0)$. Explain why this does not contradict Theorem 6.3.8.

6. (20 pts) Let $E \subset \mathbf{R}^n$, let $f : E \to \mathbf{R}^m$, and let x_0 be an interior point of E.

Assume that there exists a linear map $A : \mathbf{R}^n \to \mathbf{R}^m$ such that for every unit vector $v \in S^{n-1}$,

$$\lim_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t} = A(v).$$

Suppose moreover that the above convergence is *uniform in v on the unit sphere S^{n-1}*, meaning that

$$\lim_{t \to 0^+} \sup_{v \in S^{n-1}} \left\| \frac{f(x_0 + tv) - f(x_0)}{t} - A(v) \right\| = 0.$$

Equivalently, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $0 < t < \delta$ and all $v \in S^{n-1}$,

$$\left\| \frac{f(x_0 + tv) - f(x_0)}{t} - A(v) \right\| < \varepsilon.$$

Prove that f is differentiable at x_0 and that $f'(x_0) = A$.

Hint. For $h \neq 0$, write

$$h = \|h\| \, v \quad \text{with } v = \frac{h}{\|h\|} \in S^{n-1}.$$

Use linearity of A to rewrite

$$A(h) = \|h\| A(v),$$

and compare

$$\frac{f(x_0 + h) - f(x_0) - A(h)}{\|h\|}$$

with

$$\frac{f(x_0 + \|h\| v) - f(x_0)}{\|h\|} - A(v).$$

Then apply the assumed uniform convergence.

You can do the following problems to practice. You don't have to submit the following problems.

1. (Exercise 6.3.4.) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be differentiable and suppose that $f'(x) = 0$ for all $x \in \mathbf{R}^n$.

   (a) Show that f is constant on $\mathbf{R}^n$.
   (b) For a greater challenge, replace $\mathbf{R}^n$ by an open connected subset $\Omega \subset \mathbf{R}^n$ and prove the same result.

2. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x_1, x_2) := |x_1|.$$

   (a) Show that for every $v = (v_1, v_2) \in \mathbf{R}^2$, the one-sided directional derivative

$$D_v f(0, 0) = \lim_{t \to 0^+} \frac{f(tv) - f(0, 0)}{t}$$

   exists, and compute it explicitly.
   (b) Show that the map $v \mapsto D_v f(0, 0)$ is *not* linear.
   (c) Conclude that f is not differentiable at $(0, 0)$.

(*Comment:* This illustrates precisely why Lemma 6.3.5 is only one-way: differentiability forces linear dependence on v, but existence of $D_v f(x_0)$ for all v does not.)

3. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

(a) Show that the partial derivatives $\dfrac{\partial f}{\partial x}(0, 0)$ and $\dfrac{\partial f}{\partial y}(0, 0)$ both exist, and compute their values.

(b) Show that f is *not* differentiable at $(0, 0)$. (*Hint:* Test the differentiability definition along a suitable curve such as $(x, y) = (t, t)$ or (t, t^2) and compare the size of $f(x, y) - f(0, 0) - L(x, y)$ for any candidate linear map L.)

(c) Prove that at least one of the partial derivative functions $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial f}{\partial y}$ fails to be continuous at $(0, 0)$. (*Hint:* Compute $\dfrac{\partial f}{\partial x}(0, y)$ for $y \neq 0$ by restricting to the line $y = \text{const}$, and examine the limit as $y \to 0$.)

This problem shows that the existence of partial derivatives at a point is not enough for differentiability; the continuity hypothesis in Theorem 6.3.8 is genuinely needed.