Introduction to Analysis I HW7

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Problem 0.0.1 (15pts). Assume that (S, d) is a metric space, and let $f_n, f : S \to \mathbb{R}$ be real-valued functions. Suppose that $f_n \to f$ uniformly on S, and there exists a constant $M > 0$ such that

$$|f_n(x)| \leq M \quad \text{for all } x \in S \text{ and all } n.$$

Let $g : \overline{B(0; M)} \to \mathbb{R}$ be continuous, where

$$B(0; M) = \{y \in \mathbb{R} : |y| < M\}.$$

Define

$$h_n(x) = g(f_n(x)), \qquad h(x) = g(f(x)), \quad x \in S.$$

Prove that $h_n \to h$ uniformly on S.

Problem 0.0.2 (15pts). Let $f_n(x) = x^n$. The sequence $\{f_n\}$ converges pointwise but not uniformly on $[0, 1]$. Let g be continuous on $[0, 1]$ with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on $[0, 1]$.

Problem 0.0.3 (15pts). Assume that $g_{n+1}(x) \leq g_n(x)$ for each x in T and each $n = 1, 2, \ldots$, and suppose that $g_n \to 0$ uniformly on T. Prove that

$$\sum (-1)^{n+1} g_n(x)$$

converges uniformly on T.

Problem 0.0.4 (15pts).

$$f_n(x) = \frac{x}{1 + nx^2}, \quad x \in \mathbb{R}, \ n = 1, 2, \ldots$$

Find the limit function f of the sequence $\{f_n\}$ and the limit function g of the sequence $\{f_n'\}$.

(a) Prove that $f'(x)$ exists for every x but that $f'(0) \neq g(0)$. For what values of x is $f'(x) = g(x)$?

(b) In what subintervals of $\mathbb{R}$ does $f_n \to f$ uniformly?

(c) In what subintervals of $\mathbb{R}$ does $f_n' \to g$ uniformly?

Problem 0.0.5 (15pts). Prove that

$$\sum x^n(1 - x)$$

converges pointwise but not uniformly on $[0, 1]$, whereas

$$\sum (-1)^n x^n (1 - x)$$

converges uniformly on $[0, 1]$. This illustrates that uniform convergence of $\sum f_n(x)$ along with pointwise convergence of $\sum |f_n(x)|$ does not necessarily imply uniform convergence of $\sum |f_n(x)|$.

Problem 0.0.6 (15pts). Let

$$f_n(x) = \frac{1}{n} e^{-n^2 x^2}, \quad x \in \mathbb{R}, \ n = 1, 2, \ldots$$

Prove that $f_n \to 0$ uniformly on $\mathbb{R}$, that $f_n' \to 0$ pointwise on $\mathbb{R}$, but that the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

Problem 0.0.7 (10pts). Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on $[0, 1]$

and assume that $f_n \to f$ uniformly on $[0, 1]$. Prove or disprove

$$\lim_{n \to \infty} \int_0^{1-1/n} f_n(x)\, dx = \int_0^1 f(x)\, dx.$$