

Linear Algebra I HW8

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Problem 0.0.1. Let T be the linear operator on \mathbb{R}^4 which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}.$$

Under what conditions on a, b , and c is T diagonalizable?

Proof. Suppose

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix},$$

then $\det(xI - A) = x^4$, so if A is diagonalizable, then we must have $\dim \ker(A - 0I) = \dim \ker A = 4$, which means $\text{rank } A = 0$ by rank and nullity theorem, so $a = b = c = 0$. ■

Problem 0.0.2. Let A and B be $n \times n$ matrices over the field F . Prove that if $(I - AB)$ is invertible, then $I - BA$ is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

Proof. Since $I - AB$ is not invertible, so $\det(I - AB) \neq 0$, and thus we know 1 is not an eigenvalue of AB , so there does not exist $v \neq 0$ s.t. $ABv = v$. Now suppose by contradiction, $\det(I - BA) = 0$, then we know 1 is an eigenvalue of BA , so there exists $w \neq 0$ s.t. $BAw = w$, so $AB(Aw) = Aw$. Now note that $Aw \neq 0$, otherwise $w = BA w = B0 = 0$, which is impossible. Hence, if we suppose $v = Aw$, then $ABv = v$ and $v \neq 0$, which is a contradiction, so $\det(I - BA) \neq 0$, and thus $I - BA$ is invertible.

Now suppose $X = (I - AB)^{-1}$, so we have $(I - AB)X = X(I - AB) = I$, which gives

$$\begin{aligned} X - XAB &= I \text{ and } X - ABX = I, \\ \Rightarrow XAB &= ABX = X - I. \end{aligned}$$

Thus, we know

$$\begin{aligned} (I - BA)(I + BXA) &= I + BXA - BA - BABXA \\ &= I + BXA - BA - B(X - I)A \\ &= I + BXA - BA - (BXA - BA) = I, \end{aligned}$$

so $(I - BA)^{-1} = I + BXA = I + B(I - AB)^{-1}A$, and we're done. ■

Problem 0.0.3 (Exercise 9). Use the result of Exercise 8 to prove that, if A and B are $n \times n$ matrices over the field F , then AB and BA have precisely the same characteristic values in F .

Remark 0.0.1 (Exercise 8). Let A and B be $n \times n$ matrices over the field F . Prove that if $(I - AB)$ is invertible, then $I - BA$ is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

Proof. If x is an eigenvalue of AB , then

- Case 1: $x \neq 0$, then there exists $v \neq 0$ s.t. $ABv = xv$, and thus $BA(Bv) = xBv$. Now we claim that $Bv \neq 0$. If not, then $xv = ABv = A0 = 0$, and since $x \neq 0$, so $v = 0$, which is a contradiction. Now since $Bv \neq 0$, so Bv is an eigenvector for x of BA , so x is an eigenvalue of BA .
- Case 2: $x = 0$, then $ABv = 0$ for some $v \neq 0$ and we have two subcases:

- Subcase 1: A is invertible, then we know $Bv = 0$ for $v \neq 0$, and since A is surjective, so there exists p s.t. $Ap = v \neq 0$, and thus $BAp = Bv = 0$. Note that $p \neq 0$ otherwise $v = Ap = 0$ and it is a contradiction, so we know 0 is an eigenvalue of BA .
- Subcase 2: A is not invertible, so there exists $w \neq 0$ s.t. $Aw = 0$, and thus $BAw = B0 = 0$, which means 0 is an eigenvalue of BA .

Thus, we have shown that all eigenvalues of AB are eigenvalues of BA . Similarly, we can use same arguments to show all eigenvalues of BA are eigenvalues of AB , and thus AB and BA have precisely the same characteristic values in F . ■

Problem 0.0.4 (Exercise 12). Use the result of Exercise 11 to prove the following: If A is a 2×2 matrix with complex entries, then A is similar over \mathbb{C} to a matrix of one of the two types

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}.$$

Remark 0.0.2 (Exercise 11). Let N be a 2×2 complex matrix such that $N^2 = 0$. Prove that either $N = 0$ or N is similar over \mathbb{C} to

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Proof. Since \mathbb{C} is algebraically closed, so we can always get 2 eigenvalues of A . If two eigenvalues of A are distinct, then we know A is diagonalizable and thus it is similar to a matrix of type $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. If A 's two eigenvalues are same, say they are both λ , then if A is diagonalizable, then A is also similar to a matrix of the type $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Now we consider the case that A is not diagonalizable. In this case, we know $\dim E(\lambda) = 1$, say $v \neq 0$ and $v \in E(\lambda)$, then we can extend $\{v\}$ to $B = \{w, v\}$, which is a basis of \mathbb{C}^2 . Then, we know

$$A \sim \begin{pmatrix} x & 0 \\ y & \lambda \end{pmatrix},$$

for some $x, y \in \mathbb{C}$. Then, we know $A(w) = xw + yv$, so if we pick $w' = \frac{1}{y}w$, we know

$$A(w') = A\left(\frac{1}{y}w\right) = \frac{1}{y}(xw + yv) = xw' + v,$$

and note that $B' = \{w', v\}$ is still a basis of \mathbb{C}^2 , so we know

$$A \sim \begin{pmatrix} x & 0 \\ 1 & \lambda \end{pmatrix}.$$

Note that

$$\lambda + \lambda = \text{Tr}(A) = \text{Tr} \begin{pmatrix} x & 0 \\ 1 & \lambda \end{pmatrix} = x + \lambda,$$

so we know $x = \lambda$, and we're done. ■

Problem 0.0.5. Let V be the space of $n \times n$ matrices over F . Let A be a fixed $n \times n$ matrix over F . Let T be the linear operator “left multiplication by A ” on V . Is it true that A and T have the same characteristic values?

Proof. The answer is true. If λ is an eigenvalue of A , then $\exists v \neq 0$ s.t. $Av = \lambda v$, so if we construct

a matrix $M \in V$ by all the n the columns of M are v , then we know

$$AM = A(v, v, \dots, v) = (Av, Av, \dots, Av) = (\lambda v, \lambda v, \dots, \lambda v) = \lambda(v, v, \dots, v) = \lambda M,$$

and $M \neq 0$ is trivial since $v \neq 0$. Hence, λ is an eigenvalue of T .

Now if λ is an eigenvalue of T , then there exists $M \neq 0$ s.t. $AM = \lambda M$. Suppose the i -th column of M is not zero column, say this column is called c_i , then we know $Ac_i = \lambda c_i$ since

$$AM = A(\dots, c_i, \dots) = (\dots, Ac_i, \dots) = \lambda M = \lambda(\dots, c_i, \dots) = (\dots, \lambda c_i, \dots).$$

Hence, λ is an eigenvalue of A .

By above arguments, we know A and T have same eigenvalues. ■