

Introduction to Analysis I

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Abstract

The lecture note of 2025 Fall Introduction to Analysis I by professor 崔茂培.

Contents

1	Basic Things	2
1.1	Natural Numbers	2
1.2	Integers	2
1.3	Field	2
1.4	Order Relation	3
1.5	Absolute Value and Triangle Inequality	4
1.6	Supremum and Infimum	4
1.7	Density of other number system	6
1.8	Extended real number system	8
1.9	Mathematical Induction	8
2	Metric Space	9
2.1	Definition and examples	9

Chapter 1

Basic Things

Lecture 1

1.1 Natural Numbers

2 Sep. 09:10

The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, \dots\}$. There exists an addition operation

$$1 + 1 = 2 \quad 1 + 1 + 1 = 3 \quad \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n.$$

1.2 Integers

The set of integers is $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. There is a zero element 0 such that $z + 0 = z$ for any $z \in \mathbb{Z}$. Also, for $n \in \mathbb{N}$, we have $n + (-n) = 0$ and $n - m = n + (-m)$ for all $n, m \in \mathbb{N}$.

$$\mathbb{Z} \xrightarrow[\text{introduce division}]{} \mathbb{Q} \xrightarrow[\text{Completeness axiom}]{} \mathbb{R}$$

1.3 Field

Next, we introduce the concept of field.

Definition 1.3.1 (Fields). A field is a set F together with two binary operations, called addition(+) and multiplication(*), such that the following properties hold:

- (a) $a + b = b + a$, $a \cdot b = b \cdot a$ for $a, b \in F$.
- (b) $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in F$.
- (c) $a \cdot (b + c) = a \cdot b + a \cdot c$.
- (d) There are distinct element 0 and 1 such that $a + 0 = a$, $a \cdot 1 = a$ for $a \in F$.
- (e) For each $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$. If $a \neq 0$, there is an element $\frac{1}{a}$ or a^{-1} in F such that $a \cdot \frac{1}{a} = 1$, or $a \cdot a^{-1} = 1$.

Remark. If $a \in F$, then $a + a \in F$. We denote $a + a$ by $2 \cdot a$. Similarly,

$$\underbrace{a + a + \dots + a}_{n \text{ times}} = n \cdot a,$$

and

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

if $a \in F$ and $n \in \mathbb{N}$.

Remark. In a field, we have subtraction and division $a - b = a + (-b)$ for $a, b \in F$. If $b \neq 0$, then $\frac{a}{b} = a \cdot b^{-1}$ for $a, b \in F$.

In a field F , we have

$$\begin{aligned} (a+b)^2 &= (a+b) \cdot (a+b) \\ &= (a+b) \cdot a + (a+b) \cdot b \\ &= a \cdot a + b \cdot a + a \cdot b + b \cdot b \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2. \end{aligned}$$

Example.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

if $b \neq 0$ and $d \neq 0$.

Proof.

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= a \cdot b^{-1} + c \cdot d^{-1} \\ &= ab^{-1}dd^{-1} + cd^{-1}bb^{-1} \\ &= adb^{-1}d^{-1} + cbd^{-1}b^{-1} \\ &= \frac{ad + bc}{bd}. \end{aligned}$$

Notice that this is true since we have commutativity in multiplication and

$$d^{-1}b^{-1} = (bd)^{-1} = \frac{1}{bd}.$$

■

Example. The set of rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$ is a field.

Example. The set of real numbers is also a field.

Example. $F_2 = \{0, 1\}$ is also a field since we can define addition and multiplication like $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$, and $0 \cdot 0 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$.

1.4 Order Relation

Next, we introduce the order relation. The real number system is ordered by the relation $<$, which has the following properties.

- (f) For each pair of real numbers a and b , exactly one of the following is true: $a = b, a < b, b < a$.
- (g) If $a < b$ and $b < c$, then $a < c$.
- (h) If $a < b$, then $a + c < b + c$ for any c , and if $0 < c$, then $a \cdot c < b \cdot c$.

Definition 1.4.1. A field with an order relation satisfy (f) to (h) is called an ordered field.

Example. The set of rational numbers is an ordered field.

Example. F_2 is not an ordered field.

Proof. If $0 < 1$, then $1 = 0 + 1 < 1 + 1 = 0$, which is a contradiction. If $1 < 0$, then $0 = 1 + 1 < 0 + 1 = 1$, which is also a contradiction. ■

Notation. In an ordered field, we use $a \leq b$ to denote either $a < b$ or $a = b$.

1.5 Absolute Value and Triangle Inequality

Next, we define the absolute value of a real number

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a \leq 0; \end{cases}$$

Theorem 1.5.1 (Triangle Inequality).

$$|a + b| \leq |a| + |b|$$

for all $a, b \in \mathbb{R}$.

Corollary 1.5.1.

$$||a| - |b|| \leq |a - b| \quad \text{and} \quad ||a| - |b|| \leq |a + b|$$

Proof. We write

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Similarly we have

$$|b| \leq |b - a| + |a|.$$

So

$$-|b - a| \leq |a| - |b| \leq |a - b|.$$

Thus,

$$||a| - |b|| \leq |a - b|. \quad \blacksquare$$

1.6 Supremum and Infimum

Next, we introduce the notion of supremum of a subset of real numbers.

Definition 1.6.1. Let S be a subset of \mathbb{R} ,

- (1) we say b is an upper bound of S if $x \leq b$ for all $x \in S$.
- (2) If B is an upper bound of S , and no number smaller than B is an upper bound of S , then B is called the supremum or the least upper bound of S . We write $B = \sup S$.

Corollary 1.6.1. If $B = \sup S$, then

- (1) $x \in S$ implies $x \leq B$

(2) If $b < B$, then b is not an upper bound of S , i.e. there exists $x_1 \in S$ such that $b < x_1$.

Definition 1.6.2. Let S be a subset of \mathbb{R} ,

- (1) we say b is a lower bound of S if $x \geq b$ for all $x \in S$.
- (2) If α is a lower bound of S , and no number bigger than α is a lower bound of S , then α is called the infimum or the greatest lower bound of S . We write $\alpha = \inf S$.

Corollary 1.6.2. If $\alpha = \inf S$, then

- (1) $x \in S$ implies $x \geq \alpha$
- (2) If $\alpha < a$, then a is not a lower bound of S , i.e. there exists $x_1 \in S$ such that $x_1 < a$.

Notation (Interval Notation).

$$(a, b) = \{x \mid a < x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

Example. $S = \{x \mid x < 0\} = (-\infty, 0)$, then $\sup S = 0$ but $\inf S$ does not exist.

Example. $S_1 = \{-1, -2, -3, -4, \dots\} = \{-n \mid n \in \mathbb{N}\}$, then $\sup S = -1$, but $\inf S$ does not exist.

Definition 1.6.3 (Nonempty Sets). A nonempty set is that a set has at least one element. The empty set, written as \emptyset , is the set has no elements at all.

Example. $S = \{x \mid x \in \mathbb{Q}, x < \sqrt{2}\}$

In \mathbb{Q} , $\sup S$ does not exist. In \mathbb{R} , $\sup S = \sqrt{2}$.

Theorem 1.6.1 (Completeness axiom). If a nonempty set of real numbers (an ordered field) is bounded above, then it has a least upper bound or $\sup S$ exists.

Remark. This is an extra axiom that can't be derived from the properties of ordered field.

Remark. Up to "isomorphism", there is exactly one complete ordered field: the field of real numbers.

Remark. From now, we assume \mathbb{R} satisfies the completeness axiom. Thus, any nonempty subset $S \subseteq \mathbb{R}$ that is bounded above, we have $\sup S$ exists.

We can prove the following property of $\sup S$.

Theorem 1.6.2. If $S \subseteq \mathbb{R}$ is bounded above, then $\sup S$ is the unique real number B such that

- (i) $x \leq B$ for all $x \in S$
- (ii) for every $\varepsilon > 0$, there exist an $x_0 \in S$ such that $B - \varepsilon < x_0$.

Proof. (i), (ii) follows from the definition. We prove the uniqueness. Suppose $B_1 = \sup S = B_2$. We want to show $B_1 = B_2$. Suppose $B_1 \neq B_2$. Then either $B_1 < B_2$ or $B_2 < B_1$. However, if either one is true, then the other one cannot be $\sup S$. ■

Theorem 1.6.3 (Archimedean Property). If $p > 0$ and $\varepsilon > 0$, then there exists an $n \in \mathbb{N}$ such that $p < n\varepsilon$.

Proof. We prove this contradiction. Suppose it is not true. This implies $n\varepsilon \leq p$ for all $n \in \mathbb{N}$. Consider $S = \{n\varepsilon \mid n \in \mathbb{N}\}$, then p is an upper bound of S , so S is bounded above by p , so we know $B = \sup S$ exists. Hence, $n\varepsilon \leq B$ for all $n \in \mathbb{N}$, so we have $(n+1)\varepsilon \leq B$, which means

$$n\varepsilon \leq B - \varepsilon$$

for all $n \in \mathbb{N}$. This implies $B - \varepsilon$ is also an upper bound of S , which is a contradiction. ■

1.7 Density of other number system

Theorem 1.7.1. Every nonempty subset of the integers that is bounded below has a least element.

Proof. We first introduce an axiom:

Theorem 1.7.2 (Well-Ordering principle). Every non-empty subset of the natural numbers has a least element.

Note. Here, \mathbb{N} can be $\{0, 1, 2, \dots\}$ or $\{1, 2, 3, \dots\}$, which is not that important.

Now we call this subset of integers as S , and suppose we have m as a lower bound of S , then define $S' = \{s - m \mid s \in S\}$, then we know S' is a nonempty subset of \mathbb{N} , then by well-ordering principle we know there is a least element in S' and thus there is also a least element in S . ■

Corollary 1.7.1. Every nonempty subset of the integers that is bounded above has a greatest element.

Proof. Suppose M is an upper bound, then define a set $S' = \{M - s \mid s \in S\}$, then by well-ordering principle we know $M - a$ is the least element of S' for some $a \in S$, so we have $M - x \geq M - a$ for all $x \in S$, which means $a \geq x$ for all $x \in S$ and since $a \in S$, so a is the greatest element of S . ■

Theorem 1.7.3. The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{p}{q}$ such that $a < \frac{p}{q} < b$.

Proof. Let $a, b \in \mathbb{R}$, $a < b$. By [Archimedean Property](#), $\exists q \in \mathbb{N}$ such that $q(b - a) > 1$. Let $S = \{m \mid m \text{ is an integer with } m > qa\}$, since we know $S \neq \emptyset$ and S is bounded below. Hence, $p = \inf S$ exists and is an integer by the last theorem. So $qa < p$ and $p - 1 \leq qa$, which means $qa < p \leq qa + 1 < qb$, so we have $a < \frac{p}{q} < b$. ■

Lecture 2

Definition 1.7.1 (Floor Function). For any real number x , the floor function of x is denoted by $\lfloor x \rfloor$, and is defined by the formula $\lfloor n \rfloor$ if $n \leq x < n + 1$ where $n \in \mathbb{Z}$.

4 Sep. 10:20

Corollary 1.7.2.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

Example. $\lfloor 3.7 \rfloor = 3$, $\lfloor -1.2 \rfloor = -2$.

Now by floor function, we can reprove [Theorem 1.7.3](#).

Theorem 1.7.4 (Density of rational number in real number Again). The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{q}{p}$ such that $a < \frac{q}{p} < b$.

Reprove Theorem 1.7.3. Since $a < b$, so we know $b - a > 0$. Now by [Archimedean Property](#), we know there exists $q \in \mathbb{N}$ such that $q(b - a) > 1$. Let $p = \lfloor qa \rfloor + 1$, we have

$$\lfloor qa \rfloor \leq qa < \lfloor qa \rfloor + 1 = p.$$

From our construction, $qb > qa + 1$, so we have

$$p = \lfloor qa \rfloor + 1 \leq qa + 1 < qb,$$

hence we have

$$qa \leq p \leq qb.$$

■

Note. For some reason, p, q in [Theorem 1.7.3](#) and [Theorem 1.7.4](#) are reversed.

Definition 1.7.2 (irrational number). x is called irrational if x is not rational.

Example. $\sqrt{2}$ is irrational.

Theorem 1.7.5. Let $r \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, then

1. $r + x$ is irrational.
2. If $r \neq 0$, then rx is irrational.

sketch of proof.

1. If $r + x = q \in \mathbb{Q}$, then $x = q - r \in \mathbb{Q}$, contradiction.
2. If $rx = q \in \mathbb{Q}$, then $x = \frac{q}{r} \in \mathbb{Q}$ since $r \neq 0$.

■

Theorem 1.7.6 (irrational number dense in real number). The set of irrational number is dense in real number. That is, if $a, b \in \mathbb{R}$ and $a < b$, then there exists a irrational number t such that $a < t < b$.

Proof. By [density of rational number](#), we can find $a < r_1 < r_2 < b$ where $r_1, r_2 \in \mathbb{Q}$, and then let $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$, then we know

$$a < r_1 < t < r_2 < b.$$

Note. We should use [Theorem 1.7.5](#) and the fact that $\sqrt{2}$ is irrational.

■

Definition 1.7.3 (bounded set). A set $S \subseteq \mathbb{R}$ is bounded if there are numbers a, b s.t. $a \leq x \leq b$ for all $x \in S$.

Corollary 1.7.3. A bounded non-empty set in \mathbb{R} has a unique supremum and a unique infimum and $\inf S \leq \sup S$.

1.8 Extended real number system

The real number system, together with ∞ and $-\infty$, then we have the following properties:

- (a) If $a \in \mathbb{R}$, then $a + \infty = \infty + a = \infty$ and $a - \infty = -\infty + a = -\infty$, and $\frac{a}{\infty} = \frac{a}{-\infty} = 0$.
- (b) If $a > 0$, then $a \cdot \infty = \infty \cdot a = \infty$ and $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$
- (c) If $a < 0$, then $a \cdot \infty = \infty \cdot a = -\infty$ and $a \cdot (-\infty) = -\infty \cdot a = \infty$ and $\infty + \infty = \infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$ and $-\infty - \infty = \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$ and $|\infty| = |-\infty| = \infty$

However, there are some indeterminate form:

Theorem 1.8.1. The following things are not defined:

$$\infty - \infty, 0 \cdot \infty, \frac{\infty}{\infty}, \text{ and } \frac{0}{0}.$$

1.9 Mathematical Induction

Theorem 1.9.1 (Peano's Postulate). The natural numbers satisfy the following properties

- (a) \mathbb{N} is nonempty.
- (b) For each natural number n , there exists a unique rational number n called the successor of n .
- (c) There exists a natural number \bar{n} that is not the sucessor of any natural number.
- (d) Different natural numbers have different sucessors, that is, $n \neq m$ implies $n' \neq m'$.
- (e) The only subset of \mathbb{N} that contains \bar{n} and also contains the sucessor of every one of its element is \mathbb{N} .

Theorem 1.9.2 (Principle of Mathematical Induction). Let p_1, p_2, \dots, p_n be propositions, one for each positive integers, such that

- (a) p_1 is true.
- (b) for each positive integer n , p_n implies p_{n+1} .

then p_n is true for each $n \in \mathbb{N}$.

Proof. Let $M = \{n \mid n \in \mathbb{N} \text{ and } p_n \text{ is true}\}$, then from (a) we know $1 \in M$ and from (b) we know $n \in M$ implies $n + 1 \in M$. Hence, from (e) of [Peano's Postulate](#), we know $M = \mathbb{N}$. ■

Chapter 2

Metric Space

2.1 Definition and examples

Definition 2.1.1. Suppose $x_n \in \mathbb{R}$ for $n \geq m$. We use the notation $(x_n)_{n=m}^{\infty}$ to denote the sequence of numbers

$$x_m, x_{m+1}, \dots$$

We first recall the definition of a convergent sequence.

Definition 2.1.2 (Convergent Sequence). We say that a sequence $(x_n)_{n=m}^{\infty}$ of real numbers converges to x if for every $\varepsilon > 0$, there exists an $N \geq m$ s.t. $|x_n - x| \leq \varepsilon$ for all $n \geq N$.

Notation. We write $\lim_{n \rightarrow \infty} x_n = x$.

On \mathbb{R} , we can define the distance function between two points $x, y \in \mathbb{R}$ by $d(x, y) = |x - y|$. We'll discuss this more later.

Lemma 2.1.1. Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number, then $(x_n)_{n=m}^{\infty}$ converges to x if and only if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.

Proof. Assume $(x_n)_{n=m}^{\infty}$ converges to x . Let $\varepsilon > 0$ be arbitrary real number. By definition, there exists an $N \geq m$ such that $|x_n - x| \leq \varepsilon$ for all $n \geq N$. But $d(x_n, x) = |x_n - x|$ by the definition. Hence, $\forall \varepsilon > 0, \exists N \geq m$ such that $d(x_n, x) \leq \varepsilon$ for all $n \geq N$. This implies that $\forall \varepsilon > 0, \exists N \geq m$ such that $|d(x_n, x) - 0| \leq \varepsilon$ for all $n \geq N$. This implies $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.

The proof of the other side is the same but writing the above proof from bottom to top again.

Definition 2.1.3 (Metric Space). A metric space (X, d) is the space of X of objects (called points), together with a distance function or metric $d : X \times X \rightarrow [0, \infty)$ which associates to each x, y of points in X a nonnegative number $d(x, y) \geq 0$, the following. Furthermore, the metric must satisfy 4 axioms.

- (a) For any $x \in X$, $d(x, x) = 0$.
- (b) (Positivity) For any distinct $x, y \in X$, we have $d(x, y) > 0$.
- (c) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.
- (d) (Triangle inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Example. On \mathbb{R} , we can define $d(x, y) = |x - y|$.

Proof. • $d(x, y) = |x - y| \geq 0$.

- $d(x, y) = 0$ iff $|x - y| = 0$ iff $x = y$.
- $|x - y| = |y - x|$, so $d(x, y) = d(y, x)$
- $|x - z| \leq |x - y| + |y - z|$ for all $x, y, z \in \mathbb{R}$.

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Example. Let (X, d) be a metric space and $Y \subseteq X$, then Y inherits a natural distance function

$$d|_{Y \times Y} : Y \times Y \rightarrow [0, \infty)$$

defined by $d|_{Y \times Y}(\alpha, \beta) = d(\alpha, \beta)$ for all $\alpha, \beta \in Y$.

Note. $(Y, d|_{Y \times Y})$ is called a metric subspace of (X, d) . It is obvious that $d|_{Y \times Y}$ is a metric on Y .

Recall \mathbb{R}^n . Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

Definition 2.1.4 (l^2 -metric). The l^2 -metric is defined by

$$d_2(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}} \quad (\text{or we called } d_{l_2}(x, y)).$$

Definition 2.1.5 (l^1 -metric(taxicab metric)). The l^1 -metric is defined by

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i| \quad (\text{or we called } d_{l_1}(x, y))$$

Definition 2.1.6 (l^∞ -metric). The l^∞ -metric is defined by

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

Exercise. Verify they are all metrics.

Note. Actually we have to define inner product and norm first and then we can use the triangle inequality of norm to prove d_2 is a metric. (See lecture notes by professor)

Appendix