Introduction to Analysis I HW3

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Problem 0.0.1 (16pts).

(a) Let

$$X := \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

be the space of absolutely convergent sequences. Define the ℓ^1 and ℓ^∞ metrics on this space by

$$d_{\ell^1}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) := \sum_{n=0}^{\infty} |a_n - b_n|,$$
$$d_{\ell^\infty}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) := \sup_{n\in\mathbb{N}} |a_n - b_n|.$$

Show that these are both metrics on X, but show that there exist sequences

$$x^{(1)}, x^{(2)}, \ldots$$

of elements of X (i.e. sequences of sequences) which are convergent with respect to the d_{ℓ^∞} metric but not with respect to the d_{ℓ^1} metric. Conversely, show that any sequence which converges in the d_{ℓ^1} metric automatically converges in the d_{ℓ^∞} metric.

(b) Let (X, d_{ℓ^1}) be the metric space from part (a). For each natural number n, let $e^{(n)} = (e_j^{(n)})_{j=0}^{\infty}$ be the sequence in X such that

$$e_j^{(n)} := \begin{cases} 1, & \text{if } n = j, \\ 0, & \text{if } n \neq j. \end{cases}$$

Show that the set

$$\{e^{(n)} : n \in \mathbb{N}\}$$

is a closed and bounded subset of X, but is not compact.

(This is despite the fact that (X, d_{ℓ^1}) is even a complete metric space—a fact which we will not prove here. The problem is not that X is incomplete, but rather that it is "infinite-dimensional," in a sense that we will not discuss here.)

(a). We first show that d_{ℓ^1} is a metric:

- For any $(a_n)_{n=0}^{\infty} \in X$, we have

$$d_{\ell^1}\left((a_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} |a_n - a_n| = 0.$$

- For any distinct $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty} \in X$, we have

$$d_{\ell^1}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} |a_n - b_n| > 0.$$

- For any $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty} \in X$, we have

$$d_{\ell^1}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) = d_{\ell^1}\left((b_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\right).$$

- For any $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty} \in X$, we have

$$\begin{aligned} d_{\ell^1}\left((a_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}\right) &= \sum_{n=0}^{\infty} |a_n - c_n| \le \sum_{n=0}^{\infty} |a_n - b_n| + |b_n - c_n| \\ &= d_{\ell^1}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) + d_{\ell^1}\left((b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}\right). \end{aligned}$$

We then show that d_{l^∞} is also a metric:

- For any $(a_n)_{n=0}^\infty \in X$, we have
$$d_{\ell^\infty}\left((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty\right) = \sup_{n\in\mathbb{N}} |a_n - a_n| = 0.$$
- For any distinct $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty \in X$, we have
$$d_{\ell^\infty}\left((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty\right) = \sup_{n\in\mathbb{N}} |a_n - b_n| > 0.$$
- For any $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty \in X$, we have
$$d_{\ell^\infty}\left((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty\right) = d_{\ell^\infty}\left((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty\right).$$
- For any $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$, we have
$$\begin{aligned} d_{\ell^\infty}\left((a_n)_{n=0}^\infty, (c_n)_{n=0}^\infty\right) &= \sup_{n\in\mathbb{N}} |a_n - c_n| \le \sup_{n\in\mathbb{N}} |a_n - b_n| + |b_n - c_n| \\ &\le \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n| \\ &= d_{\ell^\infty}\left((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty\right) + d_{\ell^\infty}\left((b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty\right). \end{aligned}$$

Now we show that there exists a sequence of X, say $\left(x^{(n)}\right)_{n=1}^\infty$ s.t. $\left(x^{(n)}\right)_{n=1}^\infty$ converges with respect to d_{ℓ^∞} but not to d_{ℓ^1}. Now we let $\left(x^{(n)}\right)_{n=1}^\infty$ to be

$$x_n^{(k)} = \begin{cases} \dfrac{1}{k}, & \text{if } 0 \le n \le k-1; \\ 0, & \text{if } n \ge k. \end{cases}$$

We first show that $(x_n)_{n=1}^\infty$ converges with respect to d_{ℓ^∞}. Note that

$$d_{\ell^\infty}\left(x^{(p)}, (0)\right) = \left|\frac{1}{p} - 0\right| = \frac{1}{p}$$

where (0) is the sequence with all entries 0. Hence, for every $\varepsilon > 0$, then there exists $N > 0$ s.t. $\frac{1}{N} < \varepsilon$, and thus for all $p \ge N$, we have

$$d_{\ell^\infty}\left(x^{(p)}, 0\right) = \frac{1}{p} \le \frac{1}{N} < \varepsilon.$$

Now we show that $\left(x^{(n)}\right)_{n=1}^\infty$ does not converge with respect to d_{ℓ^1}. Suppose for contradiction, $\left(x^{(n)}\right)_{n=1}^\infty$ converges with respect to d_{ℓ^1}, then $\left(x^{(n)}\right)_{n=1}^\infty$ is a Cauchy sequence in (X, d_{ℓ^1}) since every convergent sequence is a Cauchy sequence. Now if $\left(x^{(n)}\right)_{n=1}^\infty$ is a Cauchy sequence, then for all $\varepsilon > 0$, there exists $N > 0$ s.t. $p, q \ge N$ implies $d_{\ell^1}\left(x^{(p)}, x^{(q)}\right) < \varepsilon$. Now if we pick some $\varepsilon < 1$, and the corresponding N is N_ε, and let $q = N_\varepsilon$, then we know for all $p > 2N_\varepsilon > N_\varepsilon$, we must have

$$\begin{aligned} 1 > \varepsilon > d_{\ell^1}&\left(x^{(p)}, x^{(N_\varepsilon)}\right) \\ &= \sum_{n=0}^{\infty} \left|x_n^{(p)} - x_n^{(N_\varepsilon)}\right| = \sum_{n=0}^{p} \left|x_n^{(p)} - x_n^{(N_\varepsilon)}\right| \\ &= \sum_{n=0}^{N_\varepsilon} \left|\frac{1}{p} - \frac{1}{N_\varepsilon}\right| + \sum_{n=N_\varepsilon+1}^{p} \left|\frac{1}{p} - 0\right| \\ &= N_\varepsilon\left(\frac{1}{p} - \frac{1}{N_\varepsilon}\right) + \frac{p - N_\varepsilon}{p} = 2 - \frac{2N_\varepsilon}{p} > 1, \end{aligned}$$

which is a contradiction. Hence, $(x_n)_{n=1}^{\infty}$ cannot be Cauchy with respect to d_{ℓ^1}, and thus it does not converge with respect to d_{ℓ^1}.

Now we show that any sequence converges in the d_{ℓ^1} metric automatically converges in the d_{ℓ^∞} metirc. If $(x_n)_{n=1}^{\infty}$ converges to y, then for all $\varepsilon > 0$, there exists $N > 0$ s.t. $k \geq N$ implies

$$\sum_{n=0}^{\infty} \left| x_n^{(k)} - y_n \right| < \varepsilon,$$

and thus for all $k \geq N$, we have $\sup_{n \in \mathbb{N}} \left| x_n^{(k)} - y_n \right| < \varepsilon$. Hence, $\left(x^{(n)}\right)_{n=1}^{\infty}$ also converges to y in the d_{ℓ^∞} metric. ■

(b). We first show that $\left\{e^{(n)}\right\}_{n=1}^{\infty}$ is closed. Suppose $\left\{e^{(n_j)}\right\}_{j=1}^{\infty} \subseteq \left\{e^{(n)}\right\}_{n=1}^{\infty}$ converges to some $y \in X$, then for all $\varepsilon > 0$, there exists $N > 0$ s.t. $k \geq N$ implies $\sum_{n=0}^{\infty} \left| e_n^{(n_k)} - y_n \right| < \varepsilon$. Then we do case analysis:

- Case 1: $\{n_k\}_{k=1}^{\infty}$ has no constant tail, that is, there does not exists $N' > 0$ s.t. $k \geq N'$ implies $n_k = n_{N'}$. If we pick some $k' > k \geq N$ with $n_k \neq n_{k'}$ (we can do this since the sequence has no constant tail), then we will have

  $$d_{\ell^1}\left(e^{n_k}, y\right) = \sum_{n=0}^{\infty} \left| e_n^{(n_k)} - y_n \right| = |1 - y_{n_k}| + \sum_{n \neq n_k} |y_n| < \varepsilon.$$

  Hence, we must have $y_{n_k} = 1$ and $y_n = 0$ for all $n \neq n_k$, otherwise the above equation cannot holds for all $\varepsilon > 0$. However, if we write down the same equation but replace n_k with $n_{k'}$, that is,

  $$d_{\ell^1}\left(e^{n_{k'}}, y\right) = \sum_{n=0}^{\infty} \left| e_n^{(n_{k'})} - y_n \right| = |1 - y_{n_{k'}}| + \sum_{n \neq n_{k'}} |y_n| < \varepsilon,$$

  then we have $y_{n_{k'}} = 1$ and $y_n = 0$ for all $n \neq n_{k'}$, but this means $y_{n_k} = 1$ and $y_{n_k} = 0$ since $n_k \neq n_{k'}$, so this is a contradiction, and so it is impossible that $\{n_k\}_{k=1}^{\infty}$ has no constant tail if $\left\{e^{(n_j)}\right\}_{j=1}^{\infty}$ converges.

- Case 2: $\{n_k\}_{k=1}^{\infty}$ has constant tail i.e. there exists $N' > 0$ s.t. $k \geq N'$ implies $n_k = n_{N'}$. If so, we will show that $\left\{e^{(n_k)}\right\}_{k=1}^{\infty}$ converges to $y \in X$ s.t.

  $$y_n = \begin{cases} 1, & \text{if } n = n_{N'}; \\ 0, & \text{if } n \neq n_{N'}. \end{cases}$$

  Here we know for all $\varepsilon > 0$, if $k \geq N'$, then

  $$d_{\ell^1}\left(e^{(n_k)}, y\right) = \sum_{n=0}^{\infty} \left| e_n^{(n_k)} - y_n \right| = |1 - y_{n_k}| + \sum_{n \neq n_k} |y_n| = 0 + 0 = 0 < \varepsilon$$

  since for all $k \geq N'$ we have $n_k = n_{N'}$. Now since the limit of a sequence is unique, so $\left\{e^{(n_k)}\right\}_{k=1}^{\infty}$ converges to this y and does not converge to any other $y' \in X$. Note that $y \in \left\{e^{(n)}\right\}_{n=1}^{\infty}$, so $\left\{e^{(n_k)}\right\}_{k=1}^{\infty}$ converges in $\left\{e^{(n)}\right\}_{n=1}^{\infty}$.

Since we have discussed all cases, so we know if $\left\{e^{(n_k)}\right\}_{k=1}^{\infty}$ converges, then it must converges in $\left\{e^{(n)}\right\}_{n=1}^{\infty}$, which means $\left\{e^{(n)}\right\}_{n=1}^{\infty}$ is closed.

Now we show that $\left\{e^{(n)}\right\}_{n=1}^{\infty}$ is bounded. Note that

$$e^{(n)} \in B_{\left(X, d_{\ell^1}\right)}((0), 1.1) \quad \forall n \in \mathbb{N}$$

since

$$d_{\ell^1}\left(e^{(n)}, (0)\right) = 1 < 1.1 \quad \forall n \in \mathbb{N}.$$

Hence, we have $\left\{e^{(n)}\right\}_{n=1}^{\infty} \subseteq B_{\left(X,d_{\ell^1}\right)}\left((0),1.1\right)$, and thus $\left\{e^{(n)}\right\}_{n=1}^{\infty}$ is bounded.

Now we show that $\left\{e^{(n)}\right\}_{n=1}^{\infty}$ is not compact. Consider $\left\{e^{(n)}\right\}_{n=1}^{\infty}$ itself, which is a subsequence of $\left\{e^{(n)}\right\}_{n=1}^{\infty}$. Since it corresponds to the Case 1 above, so it does not converges in (X, d_{ℓ^1}), and thus there is a subsequence of $\left\{e^{(n)}\right\}_{n=1}^{\infty}$ that does not converge, and thus $\left\{e^{(n)}\right\}_{n=1}^{\infty}$ is not compact. ■

Problem 0.0.2 (24pts). A metric space (X, d) is called *totally bounded* if for every $\varepsilon > 0$, there exists a natural number n and a finite number of balls

$$B(x^{(1)}, \varepsilon),\ B(x^{(2)}, \varepsilon),\ \ldots,\ B(x^{(n)}, \varepsilon)$$

which cover X (i.e. $X = \bigcup_{i=1}^{n} B(x^{(i)}, \varepsilon)$).

(a) Show that every totally bounded space is bounded.

(b) Show the following stronger version of Proposition 1.5.5: if (X, d) is compact, then it is complete and totally bounded. *Hint:* if X is not totally bounded, then there is some $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls. Then use Exercise 8.5.20 (on page 182 of Analysis I)to find an infinite sequence of balls $B(x^{(n)}, \varepsilon/2)$ which are disjoint from each other. Use this to construct a sequence which has no convergent subsequence.

(c) Conversely, show that if X is complete and totally bounded, then X is compact. *Hint:* if $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X, use the total boundedness hypothesis to recursively construct a sequence of subsequences $(x^{(n;j)})_{n=1}^{\infty}$ of $(x^{(n)})_{n=1}^{\infty}$ for each positive integer j, such that for each j the elements of the sequence $(x^{(n;j)})_{n=1}^{\infty}$ are contained in a single ball of radius $1/j$. Also ensure that each sequence $(x^{(n;j+1)})_{n=1}^{\infty}$ is a subsequence of the previous one $(x^{(n;j)})_{n=1}^{\infty}$. Then show that the "diagonal" sequence $(x^{(n;n)})_{n=1}^{\infty}$ is a Cauchy sequence, and then use the completeness hypothesis.

Problem 0.0.3 (16pts).

(a) A metric space (X, d) is compact if and only if every sequence in X has at least one limit point in X.

(b) Let (X, d) have the property that every open cover of X has a finite subcover. Show that X is compact.

Hint: If X is not compact, then by part (a) there is a sequence $(x^{(n)})_{n=1}^{\infty}$ with no limit points. Then for every $x \in X$ there exists a ball $B(x, \varepsilon)$ containing x which contains at most finitely many elements of this sequence. Now use the hypothesis.

(a).

($\Rightarrow$) Suppose (X, d) is compact, then for all sequence $\{a_n\}_{n=1}^{\infty} \subseteq X$, we know there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converges to some $L \in X$. Now we claim that L is a limit point of $\{a_n\}_{n=1}^{\infty}$. For all $\varepsilon > 0$, we know there exists $N_\varepsilon > 0$ s.t. $k \geq N_\varepsilon$ implies $d\left(a_{n_k}, L\right) < \varepsilon$. Hence, given any $\varepsilon > 0$ and $N > 0$, we know there exists $k \geq \max\{N_\varepsilon, N\}$ s.t. $d\left(a_{n_k}, L\right) < \varepsilon$. Note that $n_k \geq k$ since $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ and thus $1 \leq n_1 \leq n_2 < \ldots$. By this, we know $n_k \geq k \geq N$, and $d\left(a_{n_k}, L\right) < \varepsilon$, which means L is a limit point of $\{a_n\}_{n=1}^{\infty}$. Thus, every sequence in X has at least one limit point in X.

($\Leftarrow$) If every sequence in X has at least one limit point in X, then consider a sequence $\{a_n\}_{n=1}^{\infty}$, and suppose L is a limit point of $\{a_n\}_{n=1}^{\infty}$. Then for all $\varepsilon > 0$ and $N_\varepsilon > 0$, we know there exists $n_\varepsilon > N_\varepsilon$ s.t. $d\left(a_{n_\varepsilon}, L\right) < \varepsilon$. Now we construct a subsequence $\left\{a_{n_p}\right\}_{p=1}^{\infty}$ s.t. $d\left(a_{n_p}, L\right) < \frac{1}{p}$. First, we pick $\varepsilon = \frac{1}{1}$ and $N_1 = 1$, then there is a $n_1 > N_1$ s.t. $d\left(a_{n_1}, L\right) < \frac{1}{1}$. Then this is the a_{n_1} we want. Next, we pick $\varepsilon = \frac{1}{2}$ and $N_2 = n_1 + 1$, then there is a $n_2 > N_2 > n_1$ s.t. $d\left(a_{n_2}, L\right) < \frac{1}{2}$. By repeating this step, we can construct $\left\{a_{n_p}\right\}_{p=1}^{\infty}$. Note that $1 \leq n_1 < n_2 < \ldots$, so $\left\{a_{n_p}\right\}_{p=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ and $d\left(a_{n_p}, L\right) < \frac{1}{p}$ for all

$p \geq 1$. Now we claim that $\{a_{n_p}\}_{p=1}^{\infty}$ converges to L. For all $\varepsilon > 0$, we can pick some $N > 0$ s.t. $\frac{1}{N} < \varepsilon$, then for all $k \geq N$, we have

$$d(a_{n_k}, L) < \frac{1}{k} < \frac{1}{N} < \varepsilon.$$

Hence, we know $\{a_{n_p}\}_{p=1}^{\infty}$ converges to L and thus (X, d) is compact.

■

(b). If (X, d) is not compact, then $\exists \left(x^{(n)}\right)_{n=1}^{\infty}$ which has no limit point by (a). Thus, for all $L \in X$ and for all $\varepsilon > 0$, we know there exists some $N > 0$ s.t. $n \geq N$ implies $d\left(x^{(n)}, L\right) \geq \varepsilon$. Now for all $x \in X$, we can pick some $\varepsilon_x > 0$ so that $X \subseteq \bigcup_{x \in X} B(x, \varepsilon_x)$, and by the hypothesis given in the problem, we know $X \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon_{x_i})$ for some x_i's in X. Now since for every $1 \leq j \leq n$, there exists $N_j > 0$ s.t. $n \geq N_j$ implies $d\left(x^{(n)}, x_j\right) \geq \varepsilon_{x_j}$, so $B(x_j, \varepsilon_{x_j})$ contains at most $N_j - 1$ points of $\left(x^{(n)}\right)_{n=1}^{\infty}$. Hence, $\bigcup_{i=1}^{n} B(x_i, \varepsilon_{x_i})$ contains finitely many points of $\left(x^{(n)}\right)_{n=1}^{\infty}$. However,

$$\left(x^{(n)}\right)_{n=1}^{\infty} \subseteq X \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon_{x_i}),$$

so this is a contradiction. Hence, (X, d) is compact. ■

Problem 0.0.4 (10pts). Let (X, d) be a compact metric space. Suppose that $(K_\alpha)_{\alpha \in I}$ is a collection of closed sets in X with the property that any finite subcollection of these sets necessarily has non-empty intersection, thus

$$\bigcap_{\alpha \in F} K_\alpha \neq \varnothing \quad \text{for all finite } F \subseteq I.$$

(This property is known as the *finite intersection property*.)

Show that the entire collection has non-empty intersection, thus

$$\bigcap_{\alpha \in I} K_\alpha \neq \varnothing.$$

Show by counterexample that this statement fails if X is not compact.

Problem 0.0.5 (24pts).

(a) Let (X, d) be a metric space, and let $(E, d|_{E \times E})$ be a subspace of (X, d). Let $\iota_{E \to X} : E \to X$ be the inclusion map, defined by setting

$$\iota_{E \to X}(x) := x \quad \text{for all } x \in E.$$

Show that $\iota_{E \to X}$ is continuous.

(b) Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Let E be a subset of X (which we give the induced metric $d_X|_{E \times E}$), and let $f|_E : E \to Y$ be the restriction of f to E, thus

$$f|_E(x) := f(x) \quad \text{when } x \in E.$$

If $x_0 \in E$ and f is continuous at x_0, show that $f|_E$ is also continuous at x_0. (Is the converse of this statement true? Explain.)

Conclude that if f is continuous, then $f|_E$ is continuous. Thus restriction of the domain of a function does not destroy continuity.

Hint: use part (a).

(c) Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Suppose that the image $f(X)$ of X is contained in some subset $E \subseteq Y$ of Y. Let $g : X \to E$ be the function which is the same as f but with the codomain restricted from Y to E, thus $g(x) = f(x)$ for all $x \in X$.

Note on codomain: The *codomain* of a function is the declared target set of the function, in contrast to the *image* (or range), which is the set of values the function actually takes. So while f is originally defined with codomain Y, its values all lie in the smaller set $E \subseteq Y$. Therefore, one can equivalently regard f as a function $g : X \to E$. The metric on E is the one *induced from* Y, i.e. $d_Y|_{E\times E}$.

Show that for any $x_0 \in X$, f is continuous at x_0 if and only if g is continuous at x_0. Conclude that f is continuous if and only if g is continuous.

(Thus the notion of continuity is not affected if one restricts the codomain of the function.)

Problem 0.0.6 (20pts). Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \mapsto Y$ is a function from X to Y.

(a) Prove that f is continuous on X if, and only if,

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset A of X.

(b) Prove that f is continuous on X if, and only if, f is continuous on every compact subset of X.

Hint: If $x_n \to p$ in X, the set $\{p, x_1, x_2, \dots\}$ is compact.