

# Combinatorics I

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### **Abstract**

The lecture note of Combinatorics I by Shagnik Das, where the NTU cool site is <https://cool.ntu.edu.tw/courses/55532/>.

# Contents

<b>1</b>	<b>Chatting</b>	<b>2</b>
1.1	Prime Numbers . . . . .	2
<b>2</b>	<b>Elementary Counting Principles</b>	<b>4</b>
2.1	Sum Rule . . . . .	4
2.2	Product Rule . . . . .	6
2.3	Double-Counting argument . . . . .	7

# Chapter 1

## Chatting

### Lecture 1

#### 1.1 Prime Numbers

2 Sep. 15:30

**Theorem 1.1.1** (Euclid  $\approx$  300 BCE). There are infinitely many primes.

**proof.** (Saidak, 2006).

- Every natural number has at least one prime factor.
- No prime divides  $n$  and  $n + 1$ , for any  $n \in \mathbb{N}$ .

Consider a sequence of pronic number

$$p_1 = 2, p_{n+1} = p_n(p_n + 1).$$

Then the number of prime factors of  $p_n$  is strictly increasing in  $n$ :  $p_{n+1}$  has all the factors of  $p_n$  together with the (distinct) ones of  $p_n + 1$ .

**Example.**  $p_1 = 2, p_2 = 6, p_3 = 42, p_4 = 1806$ , where the prime factors of them are  $\{2\}, \{2, 3\}, \{2, 3, 7\}, \{2, 3, 7, 43\}$ .

■

##### 1.1.1 How many prime numbers are there?

**Definition 1.1.1.** We define

$$\pi(n) = |\{p : 1 \leq p \leq n : p \text{ is prime}\}|.$$

**Note.** By Saidak's proof, we know  $\pi(p_n) \geq n$ . In fact,  $\pi(p_n) \geq \log_2 n$ .

**Theorem 1.1.2** (Legendre,  $\approx$  1800 LE ).

$$\pi(n) \approx \frac{n}{\ln n} \Leftrightarrow \lim_{n \rightarrow \infty} \frac{\pi(n) \ln n}{n} = 1$$

**Note.** Proven by Hadamard and independently de la Vallée Poussin(1896).

**Theorem 1.1.3** (Better Approximation).

Dirichlet:  $\pi(x) \approx Li(x) = \int_2^x \frac{1}{\ln t} dt$ .

Known:  $\pi(n) = Li(n) + O\left(ne^{-a\sqrt{\ln n}}\right)$

Believed:  $\pi(n) = Li(n) + O(\sqrt{n} \ln n)$

## Chapter 2

# Elementary Counting Principles

Fundamental problem: Given a set  $S$ , and we want to determine  $|S|$ .

### 2.1 Sum Rule

**Theorem 2.1.1 (Sum Rule).** If  $S = \cup_{i=1}^k S_i$ , then  $|S| = \sum_{i=1}^k |S_i|$ .

**Note.**  $\cup$  means disjoint union.

**Example.** A drawer contains 8 pairs of yellow socks, 5 pairs of blue socks, and 3 pairs of red socks. How many socks are there in total.

**Informal proof.**  $2 \times (8 + 5 + 3) = 32$ . ■

**Proof.** Let  $S$  be the set of socks in the drawer, then  $S = \cup_{p \in P} S_p$ , where  $P$  is the set of pairs of socks, and  $S_p$  is the set of two socks in the pair where  $p \in P$ . By the sum rule,

$$|S| = \sum_{p \in P} |S_p| = \sum_{p \in P} 2 = 2|P| = 32.$$

$P = P_{\text{yellow}} \cup P_{\text{blue}} \cup P_{\text{red}}$ . By the sum rule,

$$|P| = |P_{\text{yellow}}| + |P_{\text{blue}}| + |P_{\text{red}}| = 8 + 5 + 3 = 16.$$
 ■

**Note.** Sum rule is the basis for case analysis arguments. It needs two requirements:

- Cover each case.
- Cover each case exactly once.

**Example.** Counting subset of a general set.

**Notation.** If  $X$  is a set, and  $k \in \mathbb{N} \cup \{0\}$ , then

$$\binom{X}{k} = \{T : T \subseteq X, |T| = k\}.$$

We define the binomial coefficient as

$$\binom{|X|}{k} = \left| \binom{X}{k} \right|.$$

i.e. Given  $n \geq k \geq 0$ ,  $\binom{n}{k}$  is the number of  $k$ -element subsets of a set of size  $n$ . ■

**Proposition 2.1.1** (Pascal's relation). If  $n \geq k \geq 1$ , then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

**Proof. Proof.** Let  $X$  be an  $n$ -element set (e.g.  $X = [n] = \{1, 2, \dots, n\}$ ), and let  $S = \binom{X}{k} = \{T \subseteq X : |T| = k\}$ . Then, by definition,  $\binom{n}{k} = |S|$ . For each  $k$ -element subset, we can ask: "Do you contain  $n$ ?" Let

$$S_0 = \{T : T \subseteq X, n \notin T, |T| = k\},$$

and

$$S_1 = \{T : T \subseteq X, n \in T, |T| = k\}.$$

Then,  $S = S_0 \cup S_1$ . By the sum rule,  $|S| = |S_0| + |S_1|$ . Observe that

$$\begin{aligned} S_0 &= \{T \subseteq [n], n \notin T, |T| = k\} \\ &= \{T \subseteq [n-1], |T| = k\}, \end{aligned}$$

so by definition,

$$|S_0| = \left| \binom{[n-1]}{k} \right| = \binom{n-1}{k}.$$

$$S_1 = \{T \subseteq [n], n \in T, |T| = k\}.$$

Let

$$S'_1 = \{T' \subseteq [n-1], |T'| = k-1\},$$

then we know a bijection from  $S_1$  to  $S'_1$ :

$$T \in S_1 \longleftrightarrow T \setminus \{n\} \in S'_1.$$

**Theorem 2.1.2** (bijection rule). Given two sets  $S$  and  $S'$ , if there is a bijection  $f : S \rightarrow S'$ , then  $|S| = |S'|$ .

By this rule, we know

$$|S_1| = |S'_1| = \left| \binom{[n-1]}{k-1} \right| = \binom{n-1}{k-1}.$$

Hence,

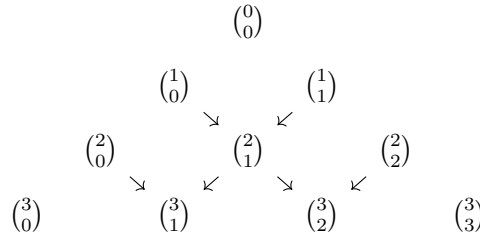
$$\binom{n}{k} = |S| = |S_0| + |S_1| = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

■

### 2.1.1 Pascal's Triangle

We can use Pascal's relation to compute  $\binom{n}{k}$ .

**Note.** Boundary case:  $\binom{n}{0} = 1$ ,  $\binom{n}{n} = 1$ . Also,  $\binom{n}{k} = 0$  for  $k = -1, n+1$ .



## 2.2 Product Rule

**Theorem 2.2.1.** If  $S = S_1 \times S_2 \times \cdots \times S_k = \{(x_1, x_2, \dots, x_k), x_i \in S_i\}$ , then  $|S| = \prod_{i=1}^k |S_i|$ .

**Proof.** Induction on  $k$ :

Base case:  $k = 1$ , trivial.

Induction step: separate into cases based on choice of  $x_{k+1} \in S_{k+1}$ . Let

$$S(x) = \{(x_1, \dots, x_{k+1}) \in S, x_{k+1} = x \in S_{k+1}\},$$

then

$$S = \bigcup_{x \in S_{k+1}} S(x) \rightarrow |S| = \sum_{x \in S_{k+1}} |S(x)|.$$

But  $S(x) = S_1 \times S_2 \times \cdots \times \{x\}$ , which is in bijection with  $S_1 \times S_2 \times \cdots \times S_k$ . By induction rule,

$$|S(x)| = |S_1 \times S_2 \times \cdots \times S_k| \quad \forall x$$

Hence,

$$\begin{aligned} |S| &= \sum_{x \in S_{k+1}} |S(x)| = \sum_{x \in S_{k+1}} |S_1 \times S_2 \times \cdots \times S_k| \\ &= |S_1 \times S_2 \times \cdots \times S_k| \times |S_{k+1}| = |S_1| \times |S_2| \times \cdots \times |S_{k+1}|. \end{aligned}$$

■

**Example.** Consider binary strings of length  $n$ .

**Proof.**

$$S = \{0, 1\}^n \Rightarrow |S| = |\{0, 1\}^n| = |\{0, 1\}|^n = 2^n.$$

■

**Definition 2.2.1 (Power Set).** Given a finite set  $X$ , let  $2^X$  denote the set of all subsets of  $X$  (also denoted  $\mathcal{P}(X)$ ), which is called the power set.

**Corollary 2.2.1.**  $|2^X| = 2^{|X|}$ .

**Proof.** Without loss of generality,  $X = [n]$ . We build a bijection between  $2^{[n]}$  and the set of binary strings of length  $n$ . Suppose for every  $T \in 2^{[n]}$ , we have  $\chi_T = (x_1, x_2, \dots, x_n)$ , where

$$x_i = \begin{cases} 1, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases}$$

Then,

$$|2^{[n]}| = |\{0, 1\}^n| = 2^n.$$

■



## 2.3 Double-Counting argument

If we count a set in two different ways, the answer should be equal.

**Example.** Count  $2^{[n]}$ .

**Proof.**

1. Product rule  $\rightarrow 2^n$ .
2. Use the sum rule, split the subsets by size.

$$2^{[n]} = \binom{[n]}{0} \cup \binom{[n]}{1} \cup \dots \cup \binom{[n]}{n}$$

Hence, we have the following proposition:

**Proposition 2.3.1.** For all  $n \geq 0$ ,

$$2^n = \sum_{k=0}^n \binom{n}{k}.$$

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## Lecture 2

5 Sep. 13:10

# Appendix