

Math 2213 Introduction to Analysis
Homework 3 Due September 25 (Thursday), 2025
Please submit your homework online in PDF format.

- (1) (10 pts) Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d) , and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set

$$S = \{x^{(n)} : n \geq m\}.$$

Is the converse true?

- (2) (20 pts) The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

- (a) Given any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X , we introduce the *formal limit*

$$\text{LIM}_{n \rightarrow \infty} x_n.$$

We say that two formal limits $\text{LIM}_{n \rightarrow \infty} x_n$ and $\text{LIM}_{n \rightarrow \infty} y_n$ are equal if

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Show that this equality relation obeys the reflexive, symmetry, and transitive axioms, i.e. that it is an equivalence relation.

- (b) Let \overline{X} be the space of all formal limits of Cauchy sequences in X , modulo the above equivalence relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \rightarrow [0, \infty)$ by

$$d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) := \lim_{n \rightarrow \infty} d(x_n, y_n).$$

Show that this function is well-defined (the limit exists and does not depend on the choice of representatives) and that it satisfies the axioms of a metric. Thus $(\overline{X}, d_{\overline{X}})$ is a metric space.

- (c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.
(d) We identify an element $x \in X$ with the corresponding constant Cauchy sequence (x, x, x, \dots) , i.e. with the formal limit $\text{LIM}_{n \rightarrow \infty} x$. Show that this is legitimate: for $x, y \in X$,

$$x = y \iff \text{LIM}_{n \rightarrow \infty} x = \text{LIM}_{n \rightarrow \infty} y.$$

With this identification, show that

$$d(x, y) = d_{\overline{X}}(x, y),$$

and thus (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

- (e) Show that the closure of X in \overline{X} is \overline{X} itself. (This explains the choice of notation.)

- (f) Finally, show that the formal limit agrees with the actual limit: if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X that converges in X , then

$$\lim_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} x_n \quad \text{in } \overline{X}.$$

- (3) (20 pts) In the following, all the sets are subsets of a metric space (X, d) .

- (a) If $\overline{A} \cap \overline{B} = \emptyset$, then

$$\partial(A \cup B) = \partial A \cup \partial B.$$

- (b) For a finite family $\{A_i\}_{i=1}^n \subseteq X$, show that

$$\text{int}\left(\bigcap_{i=1}^n A_i\right) = \bigcap_{i=1}^n \text{int}(A_i).$$

- (c) For an arbitrary (possibly infinite) family $\{A_\alpha\}_{\alpha \in F} \subseteq X$, prove that

$$\text{int}\left(\bigcap_{\alpha \in F} A_\alpha\right) \subseteq \bigcap_{\alpha \in F} \text{int}(A_\alpha).$$

- (d) Give an example where the inclusion in part (c) is strict (i.e., equality fails).

- (e) For any family $\{A_\alpha\}_{\alpha \in F} \subseteq M$, prove that

$$\bigcup_{\alpha \in F} \text{int}(A_\alpha) \subseteq \text{int}\left(\bigcup_{\alpha \in F} A_\alpha\right).$$

- (f) Give an example of a finite collection F in which equality does not hold in part (e).

- (4) (10 pts) Let (X, d) be a metric space and $Y \subset X$ be an open subset. For any subset $A \subset Y$, show that A is open in Y if and only if it is open in X .

- (5) (20 pts) On the space $(0, 1]$, we may consider the topology induced by the metric space (\mathbb{R}, d) defined by $d(x, y) = |x - y|$. Alternatively, we may also define a distance d' on $(0, 1]$, given by

$$d'(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|, \quad \forall x, y \in (0, 1].$$

- (a) Show that d' is a metric on $(0, 1]$
 (b) Let $x \in (0, 1]$ and $\varepsilon > 0$. Let $B = B_d(x, \varepsilon) = \{y \mid |y - x| < \varepsilon\} \cap (0, 1]$ be the open ball centered at x of radius ε for the metric d in $(0, 1]$. Show that for any $y \in B$, we may find $\varepsilon' > 0$ such that

$$B_{d'}(y, \varepsilon') \subseteq B = B_d(x, \varepsilon).$$

- (c) Show that an open ball in $((0, 1], d')$ is also an open ball in $((0, 1], d)$.
 (d) Conclude that the metric spaces $((0, 1], d)$ and $((0, 1], d')$ are topologically equivalent, that is, a set A is open in one space if and only if it is also open in the other one.

(e) Is $((0, 1], d')$ a complete metric space? How about $((0, 1], d)$?

(6) (20 pts)

(a) We say that a family of closed balls

$$(\overline{B}(x_n, r_n))_{n \geq 1}$$

is a *decreasing sequence of closed balls* if the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Give an example of a decreasing sequence of closed balls in a complete metric space with empty intersection.

(b) We say that a family of closed balls

$$(\overline{B}(x_n, r_n))_{n \geq 1}$$

is a *decreasing sequence of closed balls with radii tending to zero* if

$$r_n \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Show that a metric space (M, d) is complete if and only if every decreasing sequence of closed balls with radii going to zero has a nonempty intersection.