Combinatorics I (Math 7701) Fall 2025
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Exercise Sheet 2

Due date: 23:59, Oct 9th, to be submitted on COOL.

Working with your partner, you should try to solve all of the exercises below. You should then submit solutions to four of the problems, with each of you writing two, clearly indicating the author of each solution. Note that each problem is worth 10 points, and starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting on COOL, please send your solutions by e-mail.

Exercise 1 A professor is getting a room ready for an exam. The room has d desks in one long row, and there will be s students taking the exam. Before the students enter the exam room, the professor wants to place the examination papers on the desks in advance. How many ways can this be done if:

(a) the examination papers are identical?

(b) the examination papers already have the names of the students printed on them?

(c) the examination papers are identical, but the professor wants to have at least two empty desks between each pair of students?

Solution (黃子恆): Given at the last few pages of this file.

Exercise 2 Prove that for every $n \geq 1$, the Stirling numbers of the first kind are unimodal in k; that is, there is some $m(n)$ such that

$$s_{n,0} < s_{n,1} < \ldots < s_{n,m(n)-1} \leq s_{n,m(n)} > s_{n,m(n)+1} > \ldots > s_{n,n}.$$

Moreover, either $m(n) = m(n-1)$ or $m(n) = m(n-1) + 1$.

Exercise 3 Show that the average number of cycles in a permutation of length n is exactly H_n, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number.
Solution (張沂魁): Note that we want to calculate

$$\frac{\sum_{k=1}^{n} k \cdot s_{n,k}}{n!}$$

since $s_{n,k}$ represents the number of permutation of n with exactly k cycles and $n!$ is the total number of permutations of n. Now since we know

$$x(x+1)\dots(x+(n-1)) = x^{\overline{n}} = \sum_{k=0}^{n} s_{n,k}x^k,$$

so if we differentiate both side, we have

$$\sum_{i=0}^{n-1} \frac{x(x+1)\dots(x+n-1)}{x+i} = \sum_{k=1}^{n} k s_{n,k} x^{k-1},$$

and if we plug $x = 1$ into both sides, we have

$$\sum_{i=0}^{n-1} \frac{n!}{1+i} = \sum_{k=1}^{n} k \cdot s_{n,k},$$

so

$$\frac{\sum_{k=1}^{n} k \cdot s_{n,k}}{n!} = \frac{\sum_{i=0}^{n-1} \frac{n!}{1+i}}{n!} = \sum_{i=0}^{n-1} \frac{1}{1+i} = \sum_{i=1}^{n} \frac{1}{i} = H_n.$$

Exercise 4* Show that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Solution (張沂魁): We first give the generating function of (a_n), where a_n is the number of partition of n into odd parts. Note that it is

$$A(x) = \prod_{j\geq 1} \frac{1}{1-x^{2j-1}} = \prod_{j\geq 1} \left(1 + x^{2j-1} + \left(x^{2j-1}\right)^2 + \dots\right)$$

since for each j, $\frac{1}{1-x^{2j-1}}$ represents the number of parts of size $2j-1$, and note that $2j-1$ must be odd, so the coefficient of x^n in $A(x)$ is just a_n. More precisely, we can contribute 1 to the coefficient of x^n in $A(x)$ if and only if

$$\left(x^{2j_1-1}\right)^{i_1} \cdot \left(x^{2j_2-1}\right)^{i_2} \cdots = x^n,$$

which corresponds to a partition into odd parts: use i_p many $2j_p - 1$s to get a partition of n for all p, where $j_1 < j_2 < \dots$.

Now we talk about the generating function of (b_n), where b_n is the number of partitions of n into distinct parts. Note that it is

$$B(x) = \prod_{k\geq 1} \left(1 + x^k\right)$$

since we can contribute 1 to the coefficient of x^n in $B(x)$ if and only if

$$x^{k_1} x^{k_2} \cdots = x^n$$

for some $k_1 < k_2 < \ldots$, which corresponds to a parition of n into distinct parts. Now we show that in fact $A(x) = B(x)$. Since for all $j \in \mathbb{N}$, we have

$$\frac{1}{1 - x^{2j-1}} = \left(1 + x^{2j-1}\right)\left(1 + \left(x^{2j-1}\right)^2\right)\left(1 + \left(x^{2j-1}\right)^{2^2}\right)\cdots = \prod_{p\geq 0}\left(1 + \left(x^{2j-1}\right)^{2^p}\right),$$

so in fact

$$A(x) = \prod_{j\geq 1}\prod_{p\geq 0}\left(1 + \left(x^{2j-1}\right)^{2^p}\right) = \prod_{j\geq 1}\prod_{p\geq 0}\left(1 + x^{(2j-1)2^p}\right),$$

but if we define $f : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N}$ by $f(j, p) = (2j - 1)2^p$, we can show that f is bijective:

- Injective: If $f(j_1, p_1) = f(j_2, p_2)$, then $(2j_1 - 1)2^{p_1} = (2j_2 - 1)2^{p_2}$, but if $p_1 \neq p_2$, then one side have more 2 as its factor than the other side, which is impossible, so $p_1 = p_2$. However, if $p_1 = p_2$, then $2j_1 - 1 = 2j_2 - 1$, which means $j_1 = j_2$, so f is injective.
- Surjective: For $n \in \mathbb{N}$, if n is odd, then $n = n \cdot 2^0 = f\left(\frac{n+1}{2}, 0\right)$. If $2 \mid n$, then $n = \left(\frac{n}{2}\right) \cdot 2^1$, and if $\frac{n}{2}$ is odd, then $f\left(\frac{\frac{n}{2}+1}{2}, 1\right) = n$, and if $\frac{n}{2}$ is even, we can repeat this step. Since n has finitely many 2 as its factor, so this algorithm stops, and we can conclude that f is surjective.

Now we know f is bijective, so in fact

$$A(x) = \prod_{j\geq 1}\prod_{p\geq 0}\left(1 + x^{(2j-1)2^p}\right) = \prod_{k\geq 1}\left(1 + x^k\right) = B(x),$$

so comparing the coefficient of x^n in $A(x)$ and $B(x)$, we know they are same, and thus the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Exercise 5 A group[1] of five pirates, called Alice, Bob, Charles, Diana and Erik, have a treasure of 100 identical gold coins that they need to divide between themselves.

(a) How many ways can they divide the coins?

Unfortunately, Alice, Bob, Charles, Diana and Erik do not care about your answer to (a).[2] They will divide the coins according to the traditional rules of piracy. Alice will first suggest a division of the coins - for instance, she might suggest that they each get twenty coins.

Once she has made a suggestion, the pirates (including her) will vote — either they accept the division, or they do not. If a strict majority (strictly more than half) accept the division, then that is how they divide the coins, and the matter is settled.

[1] In the non-mathematical sense of the word.

[2] After all, they are pirates, not combinators.

However, if a majority reject the proposal, or the vote is split evenly, then they kill Alice, and the next pirate (Bob — they proceed alphabetically) makes a proposal instead. They repeat the same process until a division is agreed upon by the remaining pirates.

The pirates all have the following priorities, which they use when deciding how to vote:

1. **Staying alive:** above all, the pirates want to survive — they prefer an outcome where they are alive with 0 coins to one where they die.
2. **Greed:** provided the pirates can stay alive, they want to get as many coins as possible — they prefer an outcome where they are alive with $n+1$ coins to one where they are alive with n coins.
3. **Violence:** all other things being equal, the pirates would like to kill as many other pirates as possible – they prefer an outcome where they are alive with n coins and $k+1$ pirates die to one where they are alive with n coins and k pirates die.

(b) Given these rules, what division should Alice propose?

Exercise 6 In this exercise, you will determine the general solution to recurrence relations where repeated roots are allowed.

(a) Suppose we have the polynomial $\prod_{i=1}^{q}(z-\lambda_i)^{k_i}$, let $k=\sum_{i=1}^{q}k_i$, and let $R(z)$ be a polynomial of degree at most $k-1$. Prove that there are unique polynomials $R_i(z)$, $1\le i\le q$, such that $\deg(R_i)\le k_i-1$ and

$$\frac{R(z)}{\prod_{i=1}^{q}(z-\lambda_i)^{k_i}}=\sum_{i=1}^{q}\frac{R_i(z)}{(z-\lambda_i)^{k_i}}.$$

(b) Let $R_i(z)$ be a polynomial of degree at most k_i-1. Show that there are unique constants $A_{i,j}$, $1\le j\le k_i$, such that

$$\frac{R_i(z)}{(z-\lambda_i)^{k_i}}=\sum_{j=1}^{k_i}\frac{A_{i,j}}{(z-\lambda_i)^{j}}.$$

(c) Suppose the sequence $(b_n)_{n\ge 0}$ has the generating function $b(x)=(1-\lambda x)^{-j}$. Show that the sequence is given by $b_n=\binom{j+n-1}{n}\lambda^n$.

(d) Let $(a_n)_{n\ge 0}$ be a sequence determined by a recurrence relation with characteristic polynomial $p(z)=\prod_{i=1}^{q}(z-\lambda_i)^{k_i}$; that is, the characteristic polynomial has distinct roots λ_i, each appearing with multiplicity k_i. Show that the solution must take the form[3]

$$a_n=\sum_{i=1}^{q}\left(\sum_{j=1}^{k_i}A_{i,j}\binom{n-1+j}{n}\right)\lambda_i^n,$$

where the coefficients $A_{i,j}$ can be determined from the initial conditions.

[3] The solution is often presented in the more convenient (but equivalent) form $\sum_{i=1}^{q}\left(\sum_{j=0}^{k_i-1}\tilde{A}_{i,j}n^j\right)\lambda_i^n$.

Solution (黄子恆): Given at the last few pages of this file.

Hints

The following QR codes contain hints for some of the homework exercises. You should be able to decode them using any QR code scanner, including this one: `https://zxing.org/w/decode.jspx`.

Exercise 6 Also available at `https://i.ibb.co/5hDxjwxD/s2e6.png`.

1.(a) Choose s desks from d desks to place the examination paper in advance $\Rightarrow \binom{d}{s}$ ways.

(b) After choosing s desks from d desks, since the papers are distinct, we need to consider its arrangement $\Rightarrow \binom{d}{s} \times s! = \frac{d!}{(d-s)!}$ ways.

(c) Number the students from lefts to right, with the one on the far left is number 1 and the one on the far right is number s. Then suppose there has l_i desks between number i student and number $i+1$ student. Besides, say there has l_0 desks on the left of number 1 student and l_s desks on the right of number s student. Then from the problem's assumption, $l_i \geq 2$ for $i=1,2,3,\cdots,s-1$, $l_j \geq 0$ for $j=0$ or s, and $l_0+l_1+l_2+\cdots+l_s = d-s$.

Now, define $t_i = \begin{cases} l_i & \text{if } i=0 \text{ or } s \\ l_i-2 & \text{if } i\neq 0 \text{ or } s \end{cases}$, so $t_i \geq 0 \ \forall i$ and

$t_0+t_1+\cdots t_s = d-s-2(s-1) = d-3s+2$.

Consider the stars and bars diagram. There has $d-3s+2$ stars and s bars. So there has $\binom{d-3s+2+s}{s} = \binom{d-2s+2}{s}$ ways.

6. (a) Let $M_j(z) = [\prod_{i=1}^{g} (z-\lambda_i)^{k_i}] / (z-\lambda_j)^{k_j}$.

<u>Claim.</u> $\exists !\ F_i(z)$ s.t. $R(z) \equiv F_i(z) \pmod{(z-\lambda_i)^{k_i}}$ and $F_i(z)$ has the factor $M_i(z)$ and $\deg F_i(z) \le \deg R(z)$.

<u>pf.</u> Note that $F_i(z) \equiv \begin{cases} R(z) & \pmod{(z-\lambda_i)^{k_i}} \\ 0 & \pmod{M_i(z)} \end{cases}$.

Since $(z-\lambda_i)^{k_i}$ and $M_i(z)$ are relatively prime, by Euclidean algorithm, there exists polynomial $a_i(z)$ s.t. $a_i(z)M_i(z) \equiv 1 \pmod{(z-\lambda_i)^{k_i}}$. Consider $R(z)a_i(z)$ divided by $(z-\lambda_i)^{k_i}$, say $R(z)a_i(z) = p_i(z)(x-\lambda_i)^{k_i} + q_i(z)$ with $\deg q_i(z) \le k_i - 1$. Let $F_i(z) = q_i(z)M_i(z)$.

Since $\deg q_i(z)M_i(z) \le (k_i - 1) + (k - k_i) = k-1 = \deg R(z)$ and $q_i(z)M_i(z) \equiv R(z)a_i(z)M_i(z) \equiv R(z) \pmod{(z-\lambda_i)^{k_i}}$, we have proven the existence. Also, note that $q_i(z)$ is unique and $M_i(z)$ is given, $F_i(z)$ is also unique.

Now, say $F_i(z) = s_i(z)(z-\lambda_i)^{k_i} + t_i(z)$ with $\deg t_i(z) \le k_i - 1$.

Follow the notation above, we have $R(z) \equiv F_i(z) \equiv t_i(z)$ $\forall i$.

By Chinese Remainder theorem, $\exists$ a polynomial $d(z)$ such that

$$R(z) = d(z) \cdot \prod_{i=1}^{g} (z-\lambda_i)^{k_i} + \sum_{i=1}^{g} [t_i(z)\, a_i(z) M_i(z)]$$

where $a_i(z)M_i(z) \equiv 1 \pmod{(z-\lambda_i)^{k_i}}$. Here, d should be unique since $\deg R(z) < k$ and $\deg(\prod_{i=1}^{g} (z-\lambda_i)^{k_i}) = k$.

$\forall i$, say $t_i(z)a_i(z) = g_i(z)(z-\lambda_i)^{k_i} + h_i(z)$ with $\deg h_i(z) \le k_i - 1$.

$\Rightarrow \sum_{i=1}^{g} [t_i(z)\, a_i(z) M_i(z)] = \sum_{i=1}^{g} (h_i(z) M_i(z)) + (\sum_{i=1}^{g} g_i(z)) \cdot \prod_{i=1}^{g} (z-\lambda_i)^{k_i}$.

Note that $\deg(h_i(z)M_i(z)) \le \deg R(z)$. Let $k(z) = -\sum_{i=1}^{g} g_i(z)$.

$\Rightarrow R(z) = \sum_{i=1}^{g} (h_i(z) M_i(z))$

$\Rightarrow R(z) / \prod_{i=1}^{g} (z-\lambda_i)^{k_i} = \sum_{i=1}^{g} (h_i(z) / (z-\lambda_i)^{k_i})$.

Since $h_i(z)$ is unique for all i, let $R_i(z) = h_i(z)$, we are done!

(b) Let $\mathbb{R}[z]_{\le k_i-1}$ be the vector spaces of all polynomials of z of degree at most k_i-1. $\dim(\mathbb{R}[z]_{k_i-1}) = k_i$.
Note that $R_i(z) \in \mathbb{R}[z]_{\le k_i-1}$ and $\{(z-\lambda_i)^s \mid 0 \le s \le k_i-1, s \in \mathbb{Z}\}$ is a basis for $\mathbb{R}[z]_{\le k_i-1}$. So $\exists!$ constant $a_{i,1}, a_{i,2}, \cdots, a_{i,k_i}$ s.t. $R_i(z) = \sum_{t=0}^{k_i-1}\left(a_{i,t+1}(z-\lambda_i)^t\right)$.
$\Rightarrow R_i(z)/(z-\lambda_i)^{k_i} = \sum_{t=0}^{k_i-1}\left(a_{i,t+1}/(z-\lambda_i)^{k_i-t}\right)$.
Let $A_{i,j} = a_{i,k_i-j}$, then we are done!

(c) Claim. If $b(x) = (1-\lambda x)^{-j}$, then $b_n = \binom{j+n-1}{n}\lambda^n$

pf. We'll prove it by induction.
As $j=1$, $b(x) = 1+\lambda x+(\lambda x)^2+(\lambda x)^3+\cdots = \sum_{i=0}^{\infty}(\lambda x)^i$.
$\Rightarrow b_n = \lambda^n = \binom{1+n-1}{n}\lambda^n$. The claim holds.
Assume the statement holds up to $j=k$. Then for $j=k+1$, we first set $a(x) = (1-\lambda x)b(x) = (1-\lambda x)^{-k}$. By the induction hypothesis, $a_n = \binom{k+n-1}{n}\lambda^n$.
Also, $(1-\lambda x)b(x) = \sum_{i=0}^{\infty} b_i x^i - \sum_{i=1}^{\infty}\lambda b_{i-1}x^i = \sum_{i=0}^{\infty} a_i x^i$.
We have a recurrence relation $a_n = b_n - \lambda b_{n-1}$ and $a_0 = b_0 = 1$.
By $b_n = \lambda b_{n-1} + \binom{k+n-1}{n}\lambda^n$ and $\lambda^i b_{n-i} = \lambda^{i+1}b_{n-i-1} + \binom{k+i-1}{i}\lambda^n$

$$
\begin{aligned}
\Rightarrow \quad b_n &= \lambda b_{n-1} + \binom{k+n-1}{n}\lambda^n \\
\lambda b_{n-1} &= \lambda^2 b_{n-2} + \binom{k+n-2}{n-1}\lambda^n \\
&\vdots \\
\lambda^{n-1}b_1 &= \lambda^n b_0 + \binom{k}{1}\lambda^n \\
+)\quad \lambda^n b_0 &= \lambda^n = \binom{k}{0}\lambda^n \\
\hline
b_n &= \lambda^n \times \left(\binom{k+n-1}{n} + \binom{k+n-2}{n-1} + \cdots + \binom{k+1}{2} + \binom{k}{1} + \binom{k}{0}\right) \\
&= \lambda^n \times \left(\binom{k+n-1}{n} + \binom{k+n-2}{n-1} + \cdots + \binom{k+1}{2} + \binom{k+1}{1}\right) \\
&= \cdots = \binom{k+n}{n}\lambda^n = \binom{(k+1)+n-1}{n}\lambda^n.
\end{aligned}
$$

By induction, the claim holds for all integers $j \ge 1$.

(d) Since the characteristic polynomial is $p(z) = \prod_{i=1}^{q} (z-\lambda_i)^{k_i}$, the recurrence relation shall be $a_n = \sum_{i=1}^{k} \alpha_{k-i} a_{n-i}$ where $1-\alpha_{k-1}x - \alpha_{k-2}x^2 - \cdots - \alpha_0 x^k = \prod_{i=1}^{q}(1-\lambda_i x)^{k_i}$.

From the recurrence relation, we have

$$A(x) := \sum_{i=0}^{\infty} a_i x^i$$

$$-\alpha_{k-1} \cdot x \cdot A(x) = \sum_{i=1}^{\infty} -\alpha_{k-1} a_{i-1} x^i$$

$$-\alpha_{k-2} \cdot x^2 \cdot A(x) = \sum_{i=2}^{\infty} -\alpha_{k-2} a_{i-2} x^i$$

$$\vdots$$

$$+) \quad -\alpha_0 \cdot x^k \cdot A(x) = \sum_{i=k}^{\infty} -\alpha_0 a_{i-k} x^i$$

$$\prod_{i=1}^{q} (1-\lambda_i x)^{k_i} \cdot A(x) = \sum_{i=k}^{\infty} \left(a_i - \sum_{j=1}^{k} \alpha_{k-j} a_{i-j}\right) x^i + r(x) = r(x)$$

where $r(x)$ is a polynomial of degree at most $k-1$.

Hence, $A(x) = \sum_{i=0}^{\infty} a_i x^i = r(x) / \prod_{i=1}^{q} (1-\lambda_i x)^{k_i}$

$$= \left[(-1)^k \cdot \left(\prod_{i=1}^{q} \lambda_i^{-k_i}\right) \cdot r(x)\right] / \prod_{i=1}^{q} \left(x - \frac{1}{\lambda_i}\right)^{k_i}.$$

Let $R(x) = (-1)^k \cdot \left(\prod_{i=1}^{q} \lambda_i^{-k_i}\right) \cdot r(x)$ and $\lambda_i' = \frac{1}{\lambda_i}$. Then apply the conclusions in (a) and (b), we have

$$A(x) = R(x) / \prod_{i=1}^{q} (x-\lambda_i')^{k_i} = \sum_{i=1}^{q} \left(R_i(x) / (x-\lambda_i')^{k_i}\right) = \sum_{i=1}^{q} \sum_{j=1}^{k_i} \frac{B_{i,j}}{(x-\lambda_i')^j}$$

where $R_i(x)$'s are polynomials of degree at most $k_i - 1$ and $B_{i,j}$'s are constants.

$$\Rightarrow A(x) = \sum_{i=1}^{q} \sum_{j=1}^{k_i} \left(B_{i,j}(-\lambda_i)^j / (1-\lambda_i x)^j\right) = \sum_{i=1}^{q} \sum_{j=1}^{k_i} \left(A_{i,j} / (1-\lambda_i x)^j\right)$$

where $A_{i,j} = B_{i,j}(-\lambda_i)^j$ is also constant.

Use the conclusion in (c), we have

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{i=1}^{q} \sum_{j=1}^{k_i} \left(A_{i,j} \times \sum_{n=0}^{\infty} \left[\binom{j+n-1}{n} \lambda_i^n x^n\right]\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=1}^{q} \sum_{j=1}^{k_i} A_{i,j} \binom{j+n-1}{n} \lambda_i^n x^n\right)$$

Hence, $a_n = \sum_{i=1}^{q} \left(\sum_{j=1}^{k_i} A_{i,j} \binom{j+n-1}{n}\right) \lambda_i^n$.