Introduction to Probability

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February 24, 2026

Abstract

Lecture note of Introduction to Probability.

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Chapter 1

Combinatorial Analysis

Lecture 1

24 Feb.

Definition 1.0.1 (Gerolamo Cardano (1501-1576) Basic Probability Model)**.**

- Sample space: set of all possible outcomes.
- Event: $E \subseteq S$, the set of outcomes we are interested in.
- Probability: $\mathbb{P}(E) \in [0, 1]$.

Remark 1.0.1. In (finite) uniform model,

$$\mathbb{P}(E) = \frac{|E|}{|S|}.$$

Example 1.0.1. Rolling a (fair) die, then what is the probability of getting a six?

Proof. $S = \{1, 2, 3, 4, 5, 6\}, E = \{6\}$, then $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{1}{6}$. ⊛

Example 1.0.2. If we roll a fair die, then what is the probability of rolling a prime?

Proof. $S = \{1, 2, 3, 4, 5, 6\}, E = \{2, 3, 5\}$, then $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{3}{6} = \frac{1}{2}$. ⊛

Example 1.0.3. Standard deck of 52 cards. Draw a random card, then what is the probability of getting an ace?

Answer. $\frac{1}{13}$. ⊛

Example 1.0.4. Roll two fair dice, what is the probability of the sum being 7?

Proof.

$$S = \{(1, 1), (1, 2), \dots, (1, 6), (2, 1), (2, 2), \dots, (6, 6)\},$$

and

$$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\},$$

so $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$. ⊛

1.1 Counting Rule

Theorem 1.1.1 (Counting Rule)**.** If a set S is a disjoint union

$$S = S_1 \uplus S_2 \uplus \ldots \uplus S_n,$$

i.e. $S_i \cap S_j = \varnothing$ for all $i \neq j$, then

$$|S| = \sum_{i=1}^{n} |S_i|.$$

Example 1.1.1. Roll two fair dice. What is the probability of having at least one odd number?

Proof. $E = \{\text{at least one odd roll}\}$, then $E = E_1 \cup E_2$ where $E_1 = \{\text{first die is odd}\}$ and $E_2 = \{\text{second die is odd}\}$. However, $E_1 \cap E_2 \neq \varnothing$. Thus, instead, we define $E_1' = \{\text{first die is odd}\}$ and $E_2' = \{\text{first die is even and second die is odd}\}$, then we know

$$E = E_1' \uplus E_2',$$

so we have $|E| = |E_1'| + |E_2'|$, and $S = \{(x, y) : x, y \in [6]\}$, and we know

$$E_1' = \{(x, y) \mid x_1 \in \{1, 3, 5\}, y \in [6]\}$$

and

$$E_2' = \{(x, y) \mid x \in \{2, 4, 6\}, y \in \{1, 3, 5\}\},$$

which gives $|E_1'| = 18$ and $|E_2'| = 9$, and thus

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{18 + 9}{36} = \frac{3}{4}.$$

⊛

Theorem 1.1.2 (Product Rule)**.** If a set S is the Cartesian product of sets $S_1, S_2, \ldots, S_n$, i.e.

$$S = S_1 \times S_2 \times \cdots \times S_n = \{(a_1, a_2, \ldots, a_n) : \forall i \in [n], a_i \in S_i\},$$

then $|S| = |S_1| \times |S_2| \times \ldots |S_n| = \prod_{i=1}^{n} |S_i|$.

Remark 1.1.1. Informally, if a big chain can be broken into a sequence of smaller choice, then the total number of options is the product of the number of options for each small choice.

Example 1.1.2. Roll two fair dice. What is the probability that the sum is odd?

Proof. $S = \{(x, y) \mid x, y \in [6]\}$. Also, we know

$$E = \{\text{sum is odd}\} = \{\text{exactly one die is odd and the other is even}\} = E_1 \uplus E_2,$$

where $E_1 = \{\text{first is odd and second is even}\}$ and $E_2 = \{\text{first is even and second is odd}\}$. Note that

$$E_1 = \{1, 3, 5\} \times \{2, 4, 6\} \text{ and } E_2 = \{2, 4, 6\} \times \{1, 3, 5\},$$

so $|E_1| = |E_2| = 9$. By the sum rule, we know $|E| = |E_1| + |E_2| = 9 + 9 = 18$, and so

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{18}{36} = \frac{1}{2}.$$

⊛

Theorem 1.1.3 (Advanced Product Rule). If we are making a series of n choices, and for the i-th choice, we always have k_i options available, then the total number of options is

$$k_1 \times k_2 \times \cdots \times k_n = \prod_{i=1}^{n} k_i.$$

Example 1.1.3. Roll two fair dice. What is the probability that the sum is odd?

Proof. We know $S = \{(x, y) : x, y \in [6]\}$, and $E = \{(x, y) \in S : 2 \nmid x + y\}$.

- First question: Which roll is odd?
- Second question: What is the first roll? How many options?
- Third question: What is the second roll? How many options?

For the first question, we have 2 options, the first and the second. For the second question, we know there are 3 options since we need the first die to be even or odd, and similarly we know the second roll also has 3 options. Hence, $|E| = 2 \times 3 \times 3 = 18$, and thus $\mathbb{P}(E) = \frac{1}{2}$ since $|S| = 36$. ⊛

1.1.1 Permutations

Claim 1.1.1. There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

ways to order n distinct elements.

Proof. Use the advanced product rule. For the first option, we have n choices, and the second has $n - 1$ options, and so on. ⊛

1.1.2 Combinations

Question. How many subsets of size r are there of an n -element set?

Definition 1.1.1. The binomial coefficient $\binom{n}{r}$, "n choose r" counts the number of r-element subsets of an n-element set.

Claim 1.1.2. $\forall 0 \leq r \leq n$, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. We count the number of ways of ordering all n items in two different ways. (Double counting)

- First method: Direct permutation, which has $n!$ ways.
- Second method:
  - Step 1: Choose which elements will be in the front r, which has $\binom{n}{r}$ elements.
  - Step 2: Order these r elements, which has $r!$ methods.
  - Step 3: Order the remaining $n - r$ elements, which has $(n - r)!$ methods.

  Thus, by advanced product rule, we know

$$n! = \binom{n}{r} r!(n-r)! \Rightarrow \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

⊛

Observation. For all $0 \le r \le n$,

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof.

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

■

Observation. Choosing a subset of r elements is equivalent to choose the $n-r$ elements that don't go in the subset. In fact, it means $\binom{n}{r} = \binom{n}{n-r}$.

Proposition 1.1.1 (Pascal's identity). $\forall 1 \le r \le n$,

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

Proof.

$$\binom{n}{r} + \binom{n}{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-(r+1))!} = \frac{(n+1)!}{r!(n+1-r)!}.$$

■

Appendix