Linear Algebra I HW1

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September 7, 2025

Problem 0.0.1. Let W be a set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$\begin{aligned} 2x_1 - x_2 \quad &+ \tfrac{4}{3}x_3 - x_4 && = 0 \\ x_1 \quad &+ \tfrac{2}{3}x_3 && -x_5 = 0 \\ 9x_1 - 3x_2 \quad &+ 6x_3 - 3x_4 && -3x_5 = 0 \end{aligned}$$

Find a finite set of vectors which spans W.

Proof. We can first write the system of equations into the matrix form and then use Gaussian elimination.

$$\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 9 & -3 & 6 & -3 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & \frac{2}{3} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, we can need to solve

$$\begin{cases} x_1 - \dfrac{1}{2}x_2 + \dfrac{2}{3}x_3 - \dfrac{1}{2}x_4 = 0 \\ \dfrac{1}{2}x_2 + \dfrac{1}{2}x_4 - x_5 = 0. \end{cases}$$

So we know $(x_{1,2}, x_3, x_4, x_5) = \left(t - \frac{2}{3}a, b, a, 2t - b, t\right)$ for some $a, b, t \in \mathbb{R}^5$, and thus we know the set

$$S = \left\{(1,0,0,2,1), \left(-\frac{2}{3}, 0, 1, 0, 0\right), (0,1,0,-1,0)\right\}$$

spans W. ■

Problem 0.0.2. Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$, or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. (The last type of subspace is, intuitively, a straight line through the origin.)

Proof. We first give a claim:

Claim. Suppose V is a vector space, and if W is a subspace of V, then $\dim W \leq \dim V$.

Proof. Since $W \subseteq V$, so suppose $k = \dim V$ and B_1 is a basis of V, then $W \subseteq V = \operatorname{span} B_1$, which means if there is a basis of W, say B_2, then $|B_2| \leq |B_1|$, which means $\dim W \leq \dim V$. ⊛

Now also we know $\dim \mathbb{R}^2 = 2$ since $\{(0,1), (1,0)\}$ is a linearly independent set and spans $\mathbb{R}^2$. Thus, if there is a subspace of $\mathbb{R}^2$, say W, then $\dim W = 0, 1, 2$. If $\dim W = 0$, then W is the zero subspace. If $\dim W = 1$, then W consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. If $\dim W = 2$, then we can give a claim first:

Claim. Suppose $W \subseteq V$ and they are both vector spaces, then if $\dim V = \dim W$, then $V = W$.

Proof. Suppose by contradiction, there exists $v \in V \setminus W$, and suppose B is a basis of W, then we know $B \cup \{v\}$ is linearly independent in V. However, $|B \cup \{v\}| > \dim V$, which is a contradiction. ⊛

By this claim, we know $W = \mathbb{R}^2$ if $\dim W = 2$. ■

Alternative proof (not sure about if the above one is legal). Suppose W is a subspace of $\mathbb{R}^2$. If $W = \{(0,0)\}$, then it is the zero subspace. If not, there is some $v \neq (0,0)$ s.t. $v \in W$. However, since W is a vector space, so $\operatorname{span}\{v\} \subseteq W$. If the equal sign holds, then W consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. If the equal signs does not holds, then there exists $v, w \in W$ s.t. w is not a scalar multiple of v.

Claim. $\operatorname{span}\{v, w\} = \mathbb{R}^2$.

Proof. It is trivial that $\operatorname{span}\{v, w\} \in \mathbb{R}^2$. Now we show that $\mathbb{R}^2 \subseteq \operatorname{span}\{v, w\}$. First assume $v = (v_1, v_2)$ and $w = (w_1, w_2)$. For all $r = (c_1, c_2) \in \mathbb{R}^2$, we know the systems of equations

$$\begin{cases} v_1 x + w_1 y = c_1 \\ v_2 x + w_2 y = c_2 \end{cases}$$

has a unique solution by Cramer's rule learnt in high school. Hence, we know $r = xv + yw \in \operatorname{span}\{v, w\}$, and we're done. ⊛

Besides, it is trivial that a subspace cannot be bigger than the original vector space, that is, there does not exists $v \in W$ s.t. $v \notin \mathbb{R}^2$. Hence, W cannot be bigger, and thus we have concluded all the cases of W. ■

Problem 0.0.3. Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

Proof. Suppose each W_i is not contained in the other, then there exists u, v s.t.

$$u \in W_2 \setminus W_1 \quad v \in W_1 \setminus W_2.$$

Thus, we know $u + v \in W_1 \cup W_2$ since $W_1 \cup W_2$ is a vector space. Also, we know $u + v \in W_1$ or $u + v \in W_2$. Now if $u + v \in W_1$, then $u = u + v + (-v) \in W_1$, which is a contradiction, and if $u + v \in W_2$, we have $v = u + v + (-u) \in W_2$, which is also a contradiction. Hence, we know either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ should happen. ■

Problem 0.0.4. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

(a) Prove that V_e and V_o are subspaces of V.

(b) Prove that $V_e + V_o = V$.

(c) Prove that $V_e \cap V_o = \{0\}$.

Proof.

(a) First note that $V_e \subseteq V$ and $V_o \subseteq V$, then

- For all $f, g \in V_e$ and $\alpha \in F$, we define $h(x) = \alpha f(x) + g(x)$, and we know $h \in V_e$ since

  $$h(-x) = \alpha f(-x) + g(-x) = \alpha f(x) + g(x) = h(x).$$

  Hence, V_e is a subspace of V.
- For all $f, g \in V_o$ and $\alpha \in F$, we define $h(x) = \alpha f(x) + g(x)$, and we know $h \in V_o$ since

  $$h(-x) = \alpha f(-x) + g(-x) = -\alpha f(x) - g(x) = -h(x).$$

  Hence, V_o is a subspace of V.

(b) We first show that $V \subseteq V_e + V_o$. Since for all $f \in V$, we know

$$f(x) = \left(\frac{f(x) + f(-x)}{2}\right) + \left(\frac{f(x) - f(-x)}{2}\right),$$

where

$$f_e(x) = \left(\frac{f(x) + f(-x)}{2}\right) \in V_e \quad f_o(x) = \left(\frac{f(x) - f(-x)}{2}\right) \in V_o$$

since

$$f_e(-x) = \left(\frac{f(-x)+f(x)}{2}\right) = f_e(x)$$
$$f_o(-x) = \left(\frac{f(-x)-f(x)}{2}\right) = -f_o(x).$$

Now we show that $V_e + V_o \subseteq V$. This is trivial since for all $f \in V_e + V_o$ we have $f = g + h$ for some $g \in V_e \subseteq V$ and $h \in V_o \subseteq V$, so $f = g + h \in V$ since V is a vector space. Hence, $V_e + V_o \subseteq V$.
Now we have $V \subseteq V_e + V_o$ and $V_e + V_o \subseteq V$, so we can conclude that $V = V_e + V_o$.

(c)

■

Problem 0.0.5. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are unique vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

Proof. Since $W_1 + W_2 = V$, so we know for each vector $\alpha \in V$, it can be representend as $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$. If there are two differet (α_1, α_2)-pairs to represent α, say $\alpha = \alpha_1 + \alpha_2 = \alpha_1' + \alpha_2'$ where $\alpha_1, \alpha_1' \in W_1$ and $\alpha_2, \alpha_2' \in W_2$, then we know $\alpha_1 - \alpha_1' = \alpha_2' - \alpha_2$. However, since $\alpha_1 - \alpha_1' \in W_1$ and $\alpha_2' - \alpha_2 \in W_2$, so $\alpha_1 - \alpha_1' = \alpha_2' - \alpha_2 \in W_1 \cap W_2 = \{0\}$, which means $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$ and thus the (α_1, α_2)-pair is unique. ■