Introduction to Analysis I HW8

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Problem 0.0.1 (25pts). Give examples of a formal power series

$$\sum_{n=0}^{\infty} c_n x^n$$

centered at 0 with radius of convergence 1, which

(a) diverges at both $x = 1$ and $x = -1$;

(b) diverges at $x = 1$ but converges at $x = -1$;

(c) converges at $x = 1$ but diverges at $x = -1$;

(d) converges at both $x = 1$ and $x = -1$;

(e) converges pointwise on $(-1, 1)$, but does not converge uniformly on $(-1, 1)$.

(a). Suppose $(c_n)_{n=0}^{\infty} = (1)_{n=0}^{\infty}$, then the radius of convergence $R = \frac{1}{\limsup_{n\to\infty} 1} = 1$. Also, when $x = 1$, the power series is $\sum_{n=0}^{\infty} 1$, which diverges, while $x = -1$, $\sum_{n=0}^{\infty}(-1)^n$ also diverges since it oscillates. ■

(b). Suppose $(c_n)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n=1}^{\infty}$, then if the radius of convergence is R, then

$$R = \frac{1}{\limsup_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}}}.$$

Note that

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} n^{-\frac{1}{n}} = 1$$

since suppose $y_n = n^{-\frac{1}{n}}$, then $\ln y_n = -\frac{1}{n}\ln n$ for $y > 0$, and thus

$$\lim_{n\to\infty} \ln y_n = \lim_{n\to\infty} -\frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0,$$

which means $\lim_{n\to\infty} y_n = e^0 = 1$. Note that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-series test and $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges since $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ decreasing so we can use alternating series test. Hence, the power series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

diverges at $x = 1$ but converges at $x = -1$. ■

(c). Suppose $(c_n) = \left((-1)^n\frac{1}{n}\right)_{n=1}^{\infty}$, then the radius converges is

$$R = \frac{1}{\limsup_{n\to\infty} \left|(-1)^n\frac{1}{n}\right|^{\frac{1}{n}}} = \frac{1}{\limsup_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}}} = 1$$

by (b). Now if $x = 1$, then the series becomes

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n},$$

which has been argued to be convergent in (b). If $x = -1$, then the series becomes

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}(-1)^n = \sum_{n=1}^{\infty}\frac{1}{n},$$

which has been argued to be convergent in (b). Hence, this power series converges at $x = 1$ but diverges at $x = -1$. ■

(d). Suppose $(c_n)_{n=1}^{\infty} = \left(\frac{1}{n^2}\right)_{n=1}^{\infty}$, then the radius of convergence is

$$R = \frac{1}{\limsup_{n\to\infty}\left(\frac{1}{n^2}\right)^{\frac{1}{n}}} = 1$$

since $\lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1$. Now if $x = 1$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by p-series test, while if $x = -1$, then the series becomes

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2},$$

which converges by alternating series test. Hence, this power series converges at both $x = 1$ and $x = -1$. ■

(e). Consider the power series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

then we know the radius of convergence of this power series is 1. Now we show that it converges pointwise on $(-1, 1)$ but not uniformly on $(-1, 1)$. First, suppose

$$S_N(x) = \sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1-x},$$

then for all $\varepsilon > 0$ and fixed $x \in (-1, 1)$, there exists $N > 0$ s.t. $x^{N+1} < (1-x)\varepsilon$, so for all $n \geq N$ we have

$$\left| \frac{1 - x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \frac{x^{n+1}}{1-x} \leq \frac{x^{N+1}}{1-x} < \varepsilon,$$

so it converges pointwise on $(-1, 1)$. Now we show that it does not converge uniformly on $(-1, 1)$. If so, then for some fixed $N \in \mathbb{N}$ we have

$$\frac{x^{N+1}}{1-x} < \varepsilon, \quad \forall x \in (-1, 1) \text{ and } \varepsilon > 0,$$

but

$$\lim_{x\to 1^-} \frac{x^{N+1}}{1-x} = \infty,$$

so this is impossible, and thus $\sum_{n=0}^{\infty} x^n$ does not converge uniformly on $(-1, 1)$. ■

Problem 0.0.2 (25pts **Exercise 4.2.7.**). Let $m \geq 0$ be a positive integer, and let $0 < r$ be real numbers. Prove the identity

$$\frac{r}{r-x} = \sum_{n=0}^{\infty} x^n r^{-n}$$

for all $x \in (-r, r)$.

Using Proposition 4.2.6, conclude the identity

$$\frac{r}{(r-x)^{m+1}} = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}$$

for all integers $m \geq 0$ and all $x \in (-r, r)$. Also explain why the series on the right-hand side is absolutely convergent.

Proof. Fix $x \in (-r, r)$. Suppose

$$S_N(x) = \sum_{n=0}^{N} x^n r^{-n} = \sum_{n=0}^{N} \left(\frac{x}{r}\right)^n = \frac{1 - \left(\frac{x}{r}\right)^{N+1}}{1 - \frac{x}{r}},$$

then we know

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \frac{1 - \left(\frac{x}{r}\right)^{N+1}}{1 - \frac{x}{r}} = \frac{1}{1 - \frac{x}{r}} = \frac{r}{r - x}$$

since $\left|\frac{x}{r}\right| < 1$. Hence,

$$\frac{r}{r - x} = \sum_{n=0}^{\infty} x^n r^{-n}, \quad \forall x \in (-r, r).$$

Now we show

$$\frac{r}{(r - x)^{m+1}} = \sum_{n=m}^{\infty} \frac{n!}{m!(n - m)!} x^{n-m} r^{-n}$$

for all integers $m \geq 0$ and all $x \in (-r, r)$ by induction.

- Base case: for $m = 0$, we have proved that it is true.
- Now suppose for $m = k - 1$ this is true, then

$$\frac{r}{(r - x)^k} = \sum_{n=k-1}^{\infty} \frac{n!}{(k - 1)!(n - k + 1)!} x^{n-k+1} r^{-n},$$

then by Proposition 4.2.6 we know it is differentiable for all $x \in (-r, r)$ and thus

$$k \cdot r(r - x)^{-k-1} = \sum_{n=k}^{\infty} \frac{n!}{(k - 1)!(n - k + 1)!} \cdot (n - k + 1) x^{n-k} r^{-n},$$

which gives

$$\frac{r}{(r - x)^{k+1}} = \sum_{n=k}^{\infty} \frac{n!}{k!(n - k)!} x^{n-k} r^{-n}.$$

Hence, this statement is true for $m = k$ and all $x \in (-r, r)$, and thus the induction is finished. Now we show

$$\sum_{n=m}^{\infty} \frac{n!}{m!(n - m)!} x^{n-m} r^{-n}$$

converges absolutely for all $x \in (-r, r)$. We can first rewrite the series by letting $k = n - m$, so

$$\sum_{n=m}^{\infty} \frac{n!}{m!(n - m)!} x^{n-m} r^{-n} = \sum_{k=0}^{\infty} \frac{(k + m)!}{m!k!} x^k r^{-k-m}.$$

Hence, we know

$$\lim_{k\to\infty} \left| \frac{\frac{(k+m+1)!}{m!(k+1)!} x^{k+1} r^{-k-m-1}}{\frac{(k+m)!}{m!k!} x^k r^{-k-m}} \right| = \lim_{k\to\infty} \left| \frac{(k + m + 1)x}{(k + 1)r} \right| = \left| \frac{x}{r} \right| < 1,$$

and thus by ratio test we know this power series converges absolutely.

■

Problem 0.0.3 (25pts). Let E be a subset of $\mathbb{R}$, let a be an interior point of E, and let $f : E \to \mathbb{R}$

be a function which is real analytic at a and has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

at a which converges on the interval $(a-r,\, a+r)$. Let $(b-s,\, b+s)$ be any subinterval of $(a-r,\, a+r)$ for some $s > 0$.

(a) Prove that $|a-b| \leq r-s$, so in particular $|a-b| < r$.

(b) Show that for every $0 < \varepsilon < r$, there exists a $C > 0$ such that $|c_n| \leq C(r-\varepsilon)^{-n}$ for all integers $n \geq 0$. *(Hint: what do we know about the radius of convergence of the series* $\sum_{n=0}^{\infty} c_n(x-a)^n$*?)*

(c) Show that the numbers $d_0, d_1, \ldots$, given by the formula

$$d_m := \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}(b-a)^{n-m} c_n \qquad \text{for all integers } m \geq 0,$$

are well-defined, in the sense that the above series is absolutely convergent. *(Hint: use (b) and the comparison test, Corollary 7.3.2, followed by Exercise 4.2.7.)*

(d) Show that for every $0 < \varepsilon < s$ there exists a $C > 0$ such that

$$|d_m| \leq C(s-\varepsilon)^{-m}$$

for all integers $m \geq 0$. *(Hint: use the comparison test, and Exercise 4.2.7.)*

(e) Show that the power series $\sum_{m=0}^{\infty} d_m (x-b)^m$ is absolutely convergent for $x \in (b-s,\, b+s)$ and converges to $f(x)$. (You may need Fubini's theorem for infinite series, Theorem 8.2.2 of *Analysis I*, as well as Exercise 4.2.5. One may also need to use a variant of the d_m in which the c_n are replaced by $|c_n|$.)

Note. You can use Exercise 4.2.5. Let a, b be real numbers, and let $n \geq 0$ be an integer. Prove the identity

$$(x-a)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m$$

for any real number x.

(f) Conclude that f is real analytic at b, and thus analytic at every point in $(a-r,\, a+r)$.

Problem 0.0.4 (25pts).

(a) If each $a_n \geq 0$ and if $\sum a_n$ diverges, show that $\sum a_n x^n \to +\infty$ as $x \to 1^-$. (Assume $\sum a_n x^n$ converges for $|x| < 1$.)

(b) If each $a_n \geq 0$ and if $\lim_{x \to 1^-} \sum a_n x^n$ exists and equals A, prove that $\sum a_n$ converges and has sum A.