Introduction to Analysis I HW10

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Problem 0.0.1 (15pts **Exercise 4.7.8**). Let $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ be the tangent function $\tan(x) := \sin(x)/\cos(x)$. Show that tan is differentiable and monotone increasing, with

$$\frac{d}{dx}\tan(x) = 1 + \tan(x)^2,$$

and that $\lim_{x\to\pi/2}\tan(x) = +\infty$ and $\lim_{x\to-\pi/2}\tan(x) = -\infty$. Conclude that tan is in fact a bijection from $(-\pi/2, \pi/2) \to \mathbb{R}$, and thus has an inverse function

$$\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$$

(this function is called the *arctangent function*). Show that $\tan^{-1}$ is differentiable and

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}.$$

Problem 0.0.2 (15pts **Exercise 4.7.9**). Recall the arctangent function $\tan^{-1}$ from Exercise 4.7.8. By modifying the proof of Theorem 4.5.6(e), establish the identity

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

for all $x \in (-1, 1)$. Using Abel's theorem (Theorem 4.3.1) to extend this identity to the case $x = 1$, conclude in particular the identity

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

(Note that the series converges by the alternating series test, Proposition 7.2.11.) Conclude in particular that $4 - \frac{4}{3} < \pi < 4$. (One can of course compute $\pi = 3.1415926\ldots$ to much higher accuracy, though if one wishes to do so it is advisable to use a different formula than the one above, which converges very slowly.)

Problem 0.0.3 (30pts **Exercise 4.7.10**). Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) := \sum_{n=1}^{\infty} 4^{-n}\cos(32^n \pi x).$$

(a) Show that this series is uniformly convergent, and that f is continuous.

(b) Show that for every integer j and every integer $m \geq 1$, we have

$$\left| f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right)\right| \geq 4^{-m}.$$

Hint: use the identity

$$\sum_{n=1}^{\infty} a_n = \left(\sum_{n=1}^{m-1} a_n\right) + a_m + \sum_{n=m+1}^{\infty} a_n$$

for certain sequences a_n. Also, use the fact that the cosine function is periodic with period 2π, as well as the geometric series formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for any $|r| < 1$. Finally, you will need the inequality $|\cos(x) - \cos(y)| \leq |x - y|$ for any real numbers x and y; this can be proven by using the mean value theorem.

(c) Using (b), show that for every real number x_0, the function f is not differentiable at x_0. (Hint: for every x_0 and every $m \geq 1$, there exists an integer j such that $j \leq 32^m x_0 \leq j + 1$, thanks to Exercise 5.4.3.)

(d) Explain briefly why the result in (c) does not contradict Corollary 3.7.3.

Problem 0.0.4 (20pts). (a) Prove that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

or all integers n and all real θ. This is the classical *DeMoivre's theorem*.

(b) By equating imaginary parts in DeMoivre's formula, prove that

$$\sin n\theta = \sin^n\theta \left\{ \binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - \cdots \right\}.$$

(c) If $0 < \theta < \pi/2$, prove that

$$\sin(2m+1)\theta = \sin^{2m+1}\theta \; P_m(\cot^2\theta)$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - \cdots.$$

Use this to show that P_m has zeros at the m distinct points

$$x_k = \cot^2\left(\frac{\pi k}{2m+1}\right), \qquad k = 1, 2, \ldots, m.$$

(d) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} \cot^2\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

Problem 0.0.5 (20pts). This exercise outlines a simple proof of the formula $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$. Start with the inequality

$$\sin x < x < \tan x, \qquad 0 < x < \frac{\pi}{2},$$

take reciprocals, and square each member to obtain

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x.$$

Now put $x = \dfrac{k\pi}{2m+1}$, where k and m are integers with $1 \le k \le m$, and sum on k to obtain

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) < \frac{(2m+1)^2}{\pi^2}\sum_{k=1}^{m}\frac{1}{k^2} < m + \sum_{k=1}^{m}\cot^2\left(\frac{k\pi}{2m+1}\right).$$

Use the formula in problem 4(d) to deduce the inequality

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < \sum_{k=1}^{m}\frac{1}{k^2} < \frac{2m(m+1)\pi^2}{3(2m+1)^2}.$$

Now let $m \to \infty$ to obtain $\zeta(2) = \pi^2/6$.