Introduction to Analysis I HW10

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Problem 0.0.1 (15pts **Exercise 4.7.8**). Let $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ be the tangent function $\tan(x) := \sin(x)/\cos(x)$. Show that tan is differentiable and monotone increasing, with

$$\frac{d}{dx}\tan(x) = 1 + \tan(x)^2,$$

and that $\lim_{x\to\pi/2}\tan(x) = +\infty$ and $\lim_{x\to-\pi/2}\tan(x) = -\infty$. Conclude that tan is in fact a bijection from $(-\pi/2, \pi/2) \to \mathbb{R}$, and thus has an inverse function

$$\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$$

(this function is called the *arctangent function*). Show that $\tan^{-1}$ is differentiable and

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}.$$

Proof. Since $\sin(x)$ and $\cos(x)$ are both differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\cos(x) \neq 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\tan(x)$ is differentiable and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(x)}{\cos(x)} = \frac{(\sin(x))'\cos(x) - \sin(x)\,(\cos(x))'}{\cos^2(x)} \\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = 1 + \frac{\sin^2(x)}{\cos^2(x)} = 1 + \tan^2(x).\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = 1 + \tan^2(x) \geq 1 > 0$$

for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and thus $\tan(x)$ is monotone increasing. Note that

$$\lim_{x\to\frac{\pi}{2}}\tan(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^-}\tan(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\sin(x)}{\cos(x)},$$

and $\sin(x) \to 1$ and $\cos(x) \to 0^+$ as $x \to \left(\frac{\pi}{2}\right)^-$, so

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\sin(x)}{\cos(x)} = +\infty.$$

Similarly, since $\sin(x) \to -1$ and $\cos(x) \to 0^+$ as $x \to \left(-\frac{\pi}{2}\right)^+$, so

$$\lim_{x\to-\frac{\pi}{2}}\tan(x) = \lim_{x\to\left(-\frac{\pi}{2}\right)^+}\tan(x) = \lim_{x\to\left(-\frac{\pi}{2}\right)^+}\frac{\sin(x)}{\cos(x)} = -\infty.$$

Hence, we have shown that

$$\lim_{x\to\frac{\pi}{2}}\tan(x) = +\infty, \qquad \lim_{x\to\left(-\frac{\pi}{2}\right)}\tan(x) = -\infty.$$

By this, we know $\tan(x)$ is a bijection from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, and thus the inverse function $\tan^{-1}(x)$ is well-defined. Also, since

$$\tan\left(\tan^{-1}(x)\right) = x,$$

so by differentiating on both sides, we have

$$\begin{aligned}1 = (x)' &= \left(\tan\left(\tan^{-1}(x)\right)\right)' = \tan'\left(\tan^{-1}(x)\right)\cdot\left(\tan^{-1}(x)\right)' \\ &= \left(1 + \left(\tan\left(\tan^{-1}(x)\right)\right)^2\right)\cdot\left(\tan^{-1}(x)\right)' = (1+x^2)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x)\right).\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2}.$$

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Problem 0.0.2 (15pts **Exercise 4.7.9**). Recall the arctangent function $\tan^{-1}$ from Exercise 4.7.8. By modifying the proof of Theorem 4.5.6(e), establish the identity

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for all $x \in (-1, 1)$. Using Abel's theorem (Theorem 4.3.1) to extend this identity to the case $x = 1$, conclude in particular the identity

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

(Note that the series converges by the alternating series test, Proposition 7.2.11.) Conclude in particular that $4 - \frac{4}{3} < \pi < 4$. (One can of course compute $\pi = 3.1415926\ldots$ to much higher accuracy, though if one wishes to do so it is advisable to use a different formula than the one above, which converges very slowly.)

Problem 0.0.3 (30pts **Exercise 4.7.10**). Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) := \sum_{n=1}^{\infty} 4^{-n}\cos(32^n \pi x).$$

(a) Show that this series is uniformly convergent, and that f is continuous.

(b) Show that for every integer j and every integer $m \geq 1$, we have

$$\left| f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right)\right| \geq 4^{-m}.$$

Hint: use the identity

$$\sum_{n=1}^{\infty} a_n = \left(\sum_{n=1}^{m-1} a_n\right) + a_m + \sum_{n=m+1}^{\infty} a_n$$

for certain sequences a_n. Also, use the fact that the cosine function is periodic with period 2π, as well as the geometric series formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for any $|r| < 1$. Finally, you will need the inequality $|\cos(x) - \cos(y)| \leq |x - y|$ for any real numbers x and y; this can be proven by using the mean value theorem.

(c) Using (b), show that for every real number x_0, the function f is not differentiable at x_0. (Hint: for every x_0 and every $m \geq 1$, there exists an integer j such that $j \leq 32^m x_0 \leq j+1$, thanks to Exercise 5.4.3.)

(d) Explain briefly why the result in (c) does not contradict Corollary 3.7.3.

Problem 0.0.4 (20pts). (a) Prove that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

or all integers n and all real θ. This is the classical *DeMoivre's theorem*.

(b) By equating imaginary parts in DeMoivre's formula, prove that

$$\sin n\theta = \sin^n \theta \left\{ \binom{n}{1} \cot^{n-1} \theta - \binom{n}{3} \cot^{n-3} \theta + \binom{n}{5} \cot^{n-5} \theta - \cdots \right\}.$$

(c) If $0 < \theta < \pi/2$, prove that

$$\sin(2m+1)\theta = \sin^{2m+1}\theta \, P_m(\cot^2 \theta)$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1} x^m - \binom{2m+1}{3} x^{m-1} + \binom{2m+1}{5} x^{m-2} - \cdots .$$

Use this to show that P_m has zeros at the m distinct points

$$x_k = \cot^2\left(\frac{\pi k}{2m+1}\right), \qquad k = 1, 2, \ldots, m.$$

(d) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} \cot^2\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

Problem 0.0.5 (20pts). This exercise outlines a simple proof of the formula $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$. Start with the inequality

$$\sin x < x < \tan x, \qquad 0 < x < \frac{\pi}{2},$$

take reciprocals, and square each member to obtain

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x.$$

Now put $x = \frac{k\pi}{2m+1}$, where k and m are integers with $1 \le k \le m$, and sum on k to obtain

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right).$$

Use the formula in problem 4(d) to deduce the inequality

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{2m(m+1)\pi^2}{3(2m+1)^2}.$$

Now let $m \to \infty$ to obtain $\zeta(2) = \pi^2/6$.

Claim 0.0.1. $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$.

Proof. If $0 < x < \frac{\pi}{2}$, then

$$\sin x = \int_0^x \cos t \, \mathrm{d}t < \int_0^x 1 \, \mathrm{d}t = x$$

and

$$\tan(x) = \int_0^x \sec^2(t) \, \mathrm{d}t = \int_0^x \frac{1}{\cos^2(t)} \, \mathrm{d}t > \int_0^x 1 \, \mathrm{d}t = x,$$

so we have

$$\sin x < x < \tan x.$$

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Take reciprocals, and square each member, we have

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x \text{ for all } 0 < x < \frac{\pi}{2}.$$

Now put $x = \frac{k\pi}{2m+1}$, where k and m are integers with $1 \leq k \leq m$, then we know for all such x we have $0 < x < \frac{\pi}{2}$, so we can sum up all k to obtain

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right),$$

and by 4(d) we know

$$\sum_{k=1}^{m} \cot^2\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)}{3},$$

so apply this formula we have

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{2m(m+1)\pi^2}{3(2m+1)^2},$$

and since

$$\lim_{m\to\infty} \frac{m(2m-1)\pi^2}{3(2m+1)^2} = \lim_{m\to\infty} \frac{(4m-1)\pi^2}{12(2m+1)} = \lim_{m\to\infty} \frac{4\pi^2}{24} = \frac{\pi^2}{6}$$

and

$$\lim_{m\to\infty} \frac{2m(m+1)\pi^2}{3(2m+1)^2} = \lim_{m\to\infty} \frac{(4m+2)\pi^2}{12(2m+1)} = \lim_{m\to\infty} \frac{4\pi^2}{24} = \frac{\pi^2}{6}$$

by L'Hôpital's rule, so by Squeeze theorem we know

$$\zeta(2) = \lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

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