

# Linear Algebra I HW2

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**Problem 0.0.1.** Let  $V$  be the set of real numbers. Regard  $V$  as a vector space over the field of rational numbers, with the usual operations. Prove that this vector space is not finite-dimensional.

**Proof.** If  $V$  is finite dimensional, then any real number  $r \in V$  can be represented by the linear combination of finite many rational numbers. However, rational numbers are closed under addition and multiplication, so if  $r$  is irrational, then it is impossible to represent  $r$  as the linear combination of finite many rational number. ■

**Problem 0.0.2.** Consider the differentiation transformation on  $V = \mathbb{R}[x]$ , which is defined by

$$D(f(x)) = f'(x) \quad \forall f(x) \in V.$$

Find the range and null space of this transformation.

**Proof.** Since for all  $f(x) = a_0 + a_1x + \cdots + a_nx^n$ , we know

$$F(x) = C + a_0x + \frac{a_1}{2}x^2 + \cdots + \frac{a_i}{i+1}x^{i+1} + \cdots + \frac{a_n}{n+1}x^{n+1}$$

has  $F'(x) = f(x)$ , where  $C$  can be any element in  $\mathbb{R}$ . Also, we have  $F(x) \in \mathbb{R}[x]$  and  $D(F(x)) = f(x)$ , so  $f(x) \in \text{Im}(D)$ , which means  $\text{Im}(D) = \mathbb{R}[x]$ .

Now if for some  $f(x) \in \mathbb{R}[x]$ , we have  $D(f(x)) = 0$ , then we know  $f(x)$  is a constant function, so

$$\ker D = \{f(x) \in \mathbb{R}[x] \mid f(x) = c \text{ for some } c \in \mathbb{R}\}.$$

**Problem 0.0.3.** Let  $V$  be the vector space of all  $n \times n$  matrices over the field  $F$ , and let  $B$  be a fixed  $n \times n$  matrix. If

$$T(A) = AB - BA,$$

verify that  $T$  is a linear transformation from  $V$  into  $V$ .

**Proof.** It is trivial that  $T$  is a map from  $V$  into  $V$ . Now we show that  $T$  is a linear map. Note that for all  $A, C \in M_{n \times n}(F)$ , we have

$$\begin{aligned} T(\alpha A + C) &= (\alpha A + C)B - B(\alpha A + C) \\ &= \alpha AB + CB - \alpha BA - BC \\ &= \alpha(AB - BA) + (CB - BC) \\ &= \alpha T(A) + T(C). \end{aligned}$$

Hence,  $T$  is a linear map. ■

**Problem 0.0.4.** Let  $V$  be the set of all complex numbers regarded as a vector space over the field of real numbers (usual operations). Find a function from  $V$  into  $V$  which is a linear transformation on the above vector space, but which is not a linear transformation on  $C^1$ , i.e., which is not complex linear.

**Proof.** We can consider  $T : V \rightarrow V$ , where

$$T(a + bi) = (a + b) + bi \quad \forall a + bi \in V \text{ with } a, b \in \mathbb{R}.$$

We first show that it is a linear transformation on  $V$ . For any  $\alpha \in \mathbb{R}$ , and  $a' + b'i \in V$  with  $a', b' \in \mathbb{R}$ ,

$$\begin{aligned} T(\alpha(a + bi) + (a' + b'i)) &= T((\alpha a + a') + (\alpha b + b')i) \\ &= (\alpha(a + b) + a' + b') + (\alpha b + b')i \\ &= (\alpha(a + b) + \alpha bi) + (a' + b' + b'i) \\ &= \alpha T(a + bi) + T(a' + b'i). \end{aligned}$$

Hence,  $T$  is linear on  $V$ . Now we show that  $T$  is not complex linear. For  $\alpha = p + qi \in \mathbb{C}$ , where  $p, q \in \mathbb{R}$ , we know for any  $a + bi \in V$  with  $a, b \in \mathbb{R}$ , we have

$$\begin{aligned} T((p + qi)(a + bi)) &= T((pa - qb) + (pb + qa)i) \\ &= (pa - qb + pb + qa) + (pb + qa)i, \end{aligned}$$

but we also have

$$\begin{aligned} (p + qi)T(a + bi) &= (p + qi)((a + b) + bi) \\ &= (p + qi)(a + b) + (p + qi)bi \\ &= pa + pb + qai + qbi + pbi - qb \\ &= (pa + pb - qb) + (qa + qb + pb)i \end{aligned}$$

Note that

$$T((p + qi)(a + bi)) - (p + qi)T(a + bi) = qa - qbi = q(a - bi).$$

If we pick  $a \neq b$  and  $q \neq 0$ , then  $T((p + qi)(a + bi)) \neq (p + qi)T(a + bi)$ . Thus,  $T$  is not complex linear. ■

**Problem 0.0.5.** Let  $V$  be a vector space and  $T$  a linear transformation from  $V$  into  $V$ . Prove that the following two statements about  $T$  are equivalent.

- (a) The intersection of the range of  $T$  and the null space of  $T$  is the zero subspace of  $V$ .
- (b) If  $T(T(\alpha)) = 0$ , then  $T(\alpha) = 0$ .

**Proof.**

$$\begin{aligned} &\ker T \cap \operatorname{Im} T = \{0\} \\ \Leftrightarrow &\text{If } w \in \ker T \text{ for some } w \in \operatorname{Im} T, \text{ then } w = 0. \\ \Leftrightarrow &\text{If } T(w) = 0 \text{ for some } w \in \operatorname{Im} T, \text{ then } w = 0. \\ \Leftrightarrow &\text{If } T(T(\alpha)) = 0 \text{ for some } \alpha \in V, \text{ then } T(\alpha) = 0. \end{aligned}$$