Linear Algebra I HW5

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Secion 1.4

Problem 0.0.1. Suppose R and R' are 2×3 row-reduced echelon matrices and that the systems $RX = 0$ and $R'X = 0$ have exactly the same solutions. Prove that $R = R'$.

Proof. If $RX = 0$ and $R'X = 0$ have exactly the same solutions, then $\ker R = \ker R'$, and by rank and nullity theorem we know $\operatorname{rank} R = \operatorname{rank} R'$.

- Case 1: $\operatorname{rank} R = \operatorname{rank} R' = 0$, the only 2×3 matrices with rank 0 is the zero matrix, so if $\operatorname{rank} R = \operatorname{rank} R' = 0$, then $R = R' = 0$.
- Case 2: $\operatorname{rank} R = \operatorname{rank} R' = 1$, then since R and R' are in row-reduced echelon form, so suppose
$$R = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \quad R' = \begin{pmatrix} 1 & a' & b' \\ 0 & 0 & 0 \end{pmatrix},$$
and then we know
$$\begin{cases} x_1 + ax_2 + bx_3 = 0 \\ x_1 + a'x_2 + b'x_3 = 0 \end{cases}$$
have same solutions (x_1, x_2, x_3). Since $x_1 + ax_2 + bx_3 = 0$ and $x_1 + a'x_2 + b'x_3 = 0$ are both planes in $\mathbb{R}^3$, so we must have these two planes coincide, and thus
$$(1, a, b) \parallel (1, a', b'),$$
which means $a = a'$ and $b = b'$, so $R = R'$.
- Case 3: $\operatorname{rank} R = \operatorname{rank} R' = 2$, then since there are two types of row-reduced echelon form 2×3 matrices with rank 2, which are
$$T_1 = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix} \quad T_2 = \begin{pmatrix} 1 & c & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
but we should note that it is impossible that R is in T_1 form and R' is in T_2 form or the converse occurs, otherwise WLOG suppose R is in T_1 form and R' is in T_2 form, then $RX = 0$ has some solutions (x_1, x_2, x_3) with $x_3 \neq 0$ but the solutions of $R'X = 0$ must be $(x_1, x_2, 0)$, so their solutions are not the same.

  Now if R, R' are both in T_1 form, so suppose
$$R = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix} \quad R' = \begin{pmatrix} 1 & 0 & a' \\ 0 & 1 & b' \end{pmatrix},$$
we know the solutions of $RX = 0$ are $(-ax_3, -bx_3, x_3)$ and the solutions of $R'X = 0$ are $(-a'x_3, -b'x_3, x_3)$, so we must have $a = a'$ and $b = b'$, and thus $R = R'$.

  Now if R and R' are both in T_2 form, then suppose
$$R = \begin{pmatrix} 1 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R' = \begin{pmatrix} 1 & c' & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
we know the solutions of $RX = 0$ are $(-cx_2, x_2, 0)$ and the solutions of $R'X = 0$ are $(-c'x_2, x_2, 0)$, so we must have $c = c'$, and thus $R = R'$.

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Section 3.2

Problem 0.0.2. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$. Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

Proof. Since $\operatorname{rank} T^2 = \operatorname{rank} T$, so by rank and nullity theorem we know $\dim\ker T^2 = \dim\ker T$, and since $\ker T \subseteq \ker T^2$, so $\ker T = \ker T^2$. Now suppose $s \in \ker T \cap \operatorname{Im} T$, then we know $T(s) = 0$ and $s = T(v)$ for some $v \in V$, so $T(s) = T(T(v)) = T^2(v)$, and since $T(s) = 0$, so $v \in \ker T^2 = \ker T$, so $T(v) = 0$, and thus $s = T(v) = 0$. ■

Problem 0.0.3. Let p, m, and n be positive integers and F a field. Let V be the space of $m \times n$ matrices over F and let W be the space of $p \times n$ matrices over F. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

Proof.

($\Rightarrow$) If T is invertible, then T is bijective, which means $\dim V = \dim W$, and since $\dim V = m \times n$ and $\dim W = p \times n$, so $m = p$. Now since T is bijective, so $\ker T = \{0\}$, so suppose $A = (a_{ij})_{m\times n}$, then we know

$$B\begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} = 0 \text{ has only trivial solution } \forall 1 \leq i \leq n,$$

which means B is injective and thus invertible since $B \in M_{m\times m}(F)$.

($\Leftarrow$) Now if $p = m$ and B is invertible, then B^{-1} exists, so we can define $T^{-1}W \to V$ as $T^{-1}(X) = B^{-1}X$, then we have

$$TT^{-1}(X) = T\left(B^{-1}X\right) = BB^{-1}X = X$$

and

$$T^{-1}T(A) = T^{-1}(BA) = B^{-1}BA = A,$$

so T^{-1} is the inverse function of T, which means T is invertible.

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Section 3.5

Problem 0.0.4. If A and B are $n \times n$ matrices over the field F, show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$. Now show that similar matrices have the same trace.

Proof. Suppose $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$, then

$$\operatorname{Tr}(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki},$$

and we know

$$\operatorname{Tr}(BA) = \sum_{j=1}^{n}\sum_{s=1}^{n} b_{js}a_{sj} = \sum_{s=1}^{n}\sum_{j=1}^{n} b_{js}a_{sj} = \sum_{s=1}^{n}\sum_{j=1}^{n} a_{sj}b_{js} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \operatorname{Tr}(AB).$$

Now suppose A and B are similar, then $A = P^{-1}BP$ for some matrix P, then we know

$$\operatorname{Tr}(A) = \operatorname{Tr}\left(P^{-1}(BP)\right) = \operatorname{Tr}\left((BP)P^{-1}\right) = \operatorname{Tr}\left(B\left(PP^{-1}\right)\right) = \operatorname{Tr}(B).$$

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Problem 0.0.5. Let V be the vector space of all polynomial functions p from $\mathbb{R}$ into $\mathbb{R}$ which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)\,dx, \qquad f_2(p) = \int_0^2 p(x)\,dx, \qquad f_3(p) = \int_0^{-1} p(x)\,dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is the dual.

Proof. Suppose $\{p_1, p_2, p_3\}$ is a basis of V, and its dual basis is $\{f_1, f_2, f_3\}$, then suppose

$$\begin{aligned} p_1(x) &= a_1 x^2 + b_1 x + c_1 \\ p_2(x) &= a_2 x^2 + b_2 x + c_2 \\ p_3(x) &= a_3 x^2 + b_3 x + c_3, \end{aligned}$$

we want to solve

$$\begin{cases} \frac{1}{3}a_1 + \frac{1}{2}b_1 + c_1 = 1 \\ \frac{8}{3}a_1 + 2b_1 + 2c_1 = 0 \\ -\frac{1}{3}a_1 + \frac{1}{2}b_1 - c_1 = 0 \end{cases} \qquad \begin{cases} \frac{1}{3}a_2 + \frac{1}{2}b_2 + c_2 = 0 \\ \frac{8}{3}a_2 + 2b_2 + 2c_2 = 1 \\ -\frac{1}{3}a_2 + \frac{1}{2}b_2 - c_2 = 0 \end{cases} \qquad \begin{cases} \frac{1}{3}a_3 + \frac{1}{2}b_3 + c_3 = 0 \\ \frac{8}{3}a_3 + 2b_3 + 2c_3 = 0 \\ -\frac{1}{3}a_3 + \frac{1}{2}b_3 - c_3 = 1 \end{cases}$$

since we know $f_i(p_j) = \delta_{ij}$ by the definition of dual basis. By solving these system of equations, we know

$$\begin{aligned} p_1(x) &= -\frac{3}{2}x^2 + x + 1 \\ p_2(x) &= \frac{1}{2}x^2 - \frac{1}{6} \\ p_3(x) &= -\frac{1}{2}x^2 + x - \frac{1}{2}. \end{aligned}$$

Also, we can check that $\{p_1, p_2, p_3\}$ is linearly independent since each of them cannot be represented as the linear combination of the other 2 elements, so $\{p_1, p_2, p_3\}$ is linearly independent and thus a basis of V. ■