HW Template
作業模板

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Problem 0.0.1. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) Show that for every fixed direction $v \in \mathbf{R}^2$, the limit

$$\lim_{t\to 0} \frac{f(tv) - f(0)}{t}$$

exists.

(b) Show that f is *not* differentiable at $(0,0)$ in the sense of Definition 6.2.2.

(c) Explain precisely which part of the definition of differentiability fails.

Problem 0.0.2. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ and suppose that for some linear map $L : \mathbf{R}^n \to \mathbf{R}^m$ one has

$$f(x_0 + h) = f(x_0) + L(h) + R(h),$$

where the remainder satisfies

$$\|R(h)\| \leq C\|h\|^{1+\alpha}$$

for some constants $C > 0$ and $\alpha > 0$.

(a) Prove that f is differentiable at x_0 with derivative L.

(b) Show that if $\alpha = 0$, the conclusion may fail by constructing a counterexample.

(a). It is suffices to show that

$$\lim_{h\to 0} \frac{\|f(x_0+h) - f(x_0) - L(h)\|}{\|h\|} = 0.$$

Notice that

$$f(x_0+h) - f(x_0) - L(h) = R(h)$$

and

$$\|R(h)\| \leq C\|h\|^{1+\alpha},$$

so

$$0 \leq \lim_{h\to 0} \frac{\|f(x_0+h) - f(x_0) - L(h)\|}{\|h\|} \leq \lim_{h\to 0} \frac{C\|h\|^{1+\alpha}}{\|h\|} = \lim_{h\to 0} C\|h\|^{\alpha} = 0,$$

and hence

$$\lim_{h\to 0} \frac{\|f(x_0+h) - f(x_0) - L(h)\|}{\|h\|} = 0$$

by squeeze theorem. ∎

(b). Suppose that $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = |x|$, $x_0 = 0$, $L(h) = 0$, $R(h) = |x|$, $C = 1$ and $\alpha = 0$. Notice that

$$f(x_0+h) = |h| = 0 + 0 + |h| = f(x_0) + L(h) + R(h)$$

and

$$\|R(h)\| = h \leq 1 \cdot \|h\|^{1+0} = h,$$

so it satisfies the condition. However, f is not differentiable at 0 because

$$\lim_{x\to 0^+} \frac{|h|}{h} \neq \lim_{x\to 0^-} \frac{|h|}{h}.$$

∎

Problem 0.0.3. Let $D = \{(x, y) \in \mathbb{R}^2 : \ x \geq 0 \text{ and } 0 \leq y \leq x^3\}$ and $x_0 = (0, 0)$. Define $f : D \to \mathbb{R}$ by $f(x, y) = 2x + \sqrt{y}$.

We say that f is *differentiable at x_0 on D with derivative L* if $L : \mathbb{R}^2 \to \mathbb{R}$ is linear and

$$\lim_{\substack{h \to 0 \\ x_0 + h \in D}} \frac{|f(x_0 + h) - f(x_0) - L(h)|}{\|h\|} = 0.$$

(a) Explain briefly why f could *never* be differentiable at $(0, 0)$ if the domain were the entire first quadrant $\mathbb{R}^2_{\geq 0} = \{(x, y) : x \geq 0, \ y \geq 0\}$.

(b) Prove that f *is* differentiable at $(0, 0)$ on the restricted domain D.

(c) Describe the set of all valid derivatives $L(x, y)$ for f at $(0, 0)$ on D.

(a). Note that

$$\lim_{\substack{h \to 0 \\ x_0 + h \in \mathbb{R}^2_{\geq 0}}} \frac{|f((0,0) + h) - f(0,0) - L(h)|}{\|h\|} = \lim_{\substack{h \to 0 \\ x_0 + h \in \mathbb{R}^2_{\geq 0}}} \frac{|f(h) - L(h)|}{\|h\|}.$$

Now let $h = (a, b)$ for $a, b \in \mathbb{R}$, then

$$\lim_{\substack{h \to 0 \\ x_0 + h \in \mathbb{R}^2_{\geq 0}}} \frac{|f(h) - L(h)|}{\|h\|} = \lim_{\substack{(a,b) \to (0,0) \\ (a,b) \in \mathbb{R}^2_{\geq 0}}} \frac{\left|2a + \sqrt{b} - aL(1,0) - bL(0,1)\right|}{\sqrt{a^2 + b^2}},$$

but if we approach (a, b) to $(0, 0)$ along $a = 0$ and $b \geq 0$, then

$$\frac{\left|2a + \sqrt{b} - aL(1,0) - bL(0,1)\right|}{\sqrt{a^2 + b^2}} = \frac{\sqrt{b} - bL(0,1)}{b} = \frac{1}{\sqrt{b}} - L(0,1),$$

but if $(a, b) \to (0, 0)$, then this limit diverges, which shows

$$\lim_{\substack{h \to 0 \\ x_0 + h \in \mathbb{R}^2_{\geq 0}}} \frac{|f((0,0) + h) - f(0,0) - L(h)|}{\|h\|} \text{ does not exist,}$$

i.e. f could never be differentiable at $(0, 0)$ no matter what L we pick. ■

(b). If we pick $L : \mathbb{R}^2 \to \mathbb{R}$ s.t. $L(1, 0) = 2$ and $L(0, 1) = 0$ and suppose $h = (a, b)$, then

$$\lim_{\substack{h \to 0 \\ x_0 + h \in D}} \frac{|f((0,0) + h) - f(0,0) - L(h)|}{\|h\|} = \lim_{\substack{(a,b) \to (0,0) \\ (a,b) \in D}} \frac{\sqrt{b}}{\sqrt{a^2 + b^2}},$$

and note that for $(a, b) \in D$, we know $0 \leq b \leq a^3$, so $b^{\frac{1}{3}} \leq a$, and thus

$$0 \leq \frac{\sqrt{b}}{\sqrt{a^2 + b^2}} \leq \sqrt{\frac{b}{b^{\frac{2}{3}} + b^2}} = \sqrt{\frac{b^{\frac{1}{3}}}{1 + b^{\frac{4}{3}}}},$$

where we know

$$\sqrt{\frac{b^{\frac{1}{3}}}{1 + b^{\frac{4}{3}}}} \to 0 \text{ as } (a, b) \to (0, 0),$$

so by squeeze theorem, we know

$$\lim_{\substack{(a,b) \to (0,0) \\ (a,b) \in D}} \frac{\sqrt{b}}{\sqrt{a^2 + b^2}} = 0,$$

which shows f is differentiable at $(0, 0)$ on the restricted domain D. ■

(c). If $L(x,y)$ is a valid derivative for f at $(0,0)$ on D, then

$$\lim_{\substack{(a,b)\to(0,0)\\(a,b)\in D}} \frac{\left|2a+\sqrt{b}-aL(1,0)-bL(0,1)\right|}{\sqrt{a^2+b^2}} = 0.$$

Without lose of generality, suppose

$$2a+\sqrt{b}-aL(1,0)-bL(0,1) > 0,$$

while the < 0 case can be solved similarly. Now note that

$$\frac{a(2-L(1,0)+\sqrt{b}-bL(0,1))}{\sqrt{a^2+b^2}} = \frac{a(2-L(1,0))}{\sqrt{a^2+b^2}} + \frac{\sqrt{b}}{\sqrt{a^2+b^2}} - \frac{bL(0,1)}{\sqrt{a^2+b^2}}.$$

In (b), we have shown that

$$\lim_{\substack{(a,b)\to(0,0)\\(a,b)\in D}} \frac{\sqrt{b}}{\sqrt{a^2+b^2}} = 0.$$

Also, we know

$$\lim_{\substack{(a,b)\to(0,0)\\(a,b)\in D}} \frac{bL(0,1)}{\sqrt{a^2+b^2}} = \lim_{\substack{(a,b)\to(0,0)\\(a,b)\in D}} \sqrt{b}L(0,1)\frac{\sqrt{b}}{\sqrt{a^2+b^2}} = 0.$$

However,

$$\sqrt{\frac{1}{1+a^4}} = \frac{a}{a^2+a^6} \le \frac{a}{\sqrt{a^2+b^2}} \le \frac{a}{\sqrt{a^2}} = 1$$

when $(a,b)\in D$ and we know

$$\lim_{\substack{(a,b)\to(0,0)\\(a,b)\in D}} \sqrt{\frac{1}{1+a^4}} = 1,$$

so by squeeze theorem we have

$$\lim_{\substack{(a,b)\to(0,0)\\(a,b)\in D}} \frac{a}{\sqrt{a^2+b^2}} = 1.$$

Thus, if $L(1,0)\neq 2$, then

$$\lim_{\substack{(a,b)\to(0,0)\\(a,b)\in D}} \frac{a(2-L(1,0)+\sqrt{b}-bL(0,1))}{\sqrt{a^2+b^2}} \neq 0,$$

and thus $L(x,y)$ is a valid derivative for f at $(0,0)$ on D if and only if $L(1,0)=2$, i.e. $L(x,y) = 2x+by$ for any constant b.

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Problem 0.0.4. Let $E\subset \mathbf{R}^n$, let $f: E\to\mathbf{R}^m$, let x_0 be an interior point of E, and let $1\le j\le n$.

Show that $\frac{\partial f}{\partial x_j}(x_0)$ exists if and only if $D_{e_j}f(x_0)$ and $D_{-e_j}f(x_0)$ exist and are negatives of each other. In this case,

$$\frac{\partial f}{\partial x_j}(x_0) = D_{e_j}f(x_0).$$

Problem 0.0.5. **Exercise 6.3.3.** Let $f:\mathbf{R}^2\to\mathbf{R}$ be defined by

$$f(x,y) = \frac{x^3}{x^2+y^2} \quad \text{for } (x,y)\neq(0,0), \qquad f(0,0)=0.$$

Show that f is not differentiable at $(0,0)$, even though it is directionally differentiable in every direction at $(0,0)$. Explain why this does not contradict Theorem 6.3.8.

Proof. For fixed $v=(a,b)\in\mathbb{R}^2\setminus\{0\}$,

$$\lim_{t\to 0}\frac{f(tv)-f((0,0))}{t}=\lim_{t\to 0}\frac{\frac{a^3t^3}{a^2t^2+b^2t^2}}{t}=\frac{a^3}{a^2+b^2}=D_vf((0,0)),$$

the limit exists, and hence f is directionally differentiable in every direction at $(0,0)$. However,

$$D_{(1,0)}f((0,0))=1, D_{(0,1)}f((0,0))=0, \text{and } D_{(1,1)}f((0,0))=\frac{1}{2},$$

so

$$D_{(1,0)}f((0,0))+D_{(0,1)}f((0,0))\neq D_{(1,1)}f((0,0)),$$

so there doesn't exist linear map L such that $D_vf(x_0)=L(v)$ at every direction v, and hence f is not differentiable.

The theorem 6.3.8 requires that all partial derivative $\frac{\partial f}{\partial x_i}$ are continuous at x_0, however we will show that this hypothesis is not true in this problem, and hence does not contradict the theorem.

Notice that

$$\frac{\partial f}{\partial x}=\frac{3x^2(x^2+y^2)-x^3(2x)}{(x^2+y^2)^2}=\frac{x^4+3x^2y^2}{(x^2+y^2)^2}.$$

First, we approach the limit along $y=0$,

$$\lim_{t\to 0}\frac{\partial f}{\partial x}(t,0)=1.$$

Then, we approach the limit along $x=0$,

$$\lim_{t\to 0}\frac{\partial f}{\partial x}(0,t)=0.$$

So the limit depends on the path, and hence $\frac{\partial f}{\partial x}$ is NOT continuous at $(0,0)$.

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Problem 0.0.6. Let $E\subset\mathbb{R}^n$, let $f:E\to\mathbb{R}^m$, and let x_0 be an interior point of E.

Assume that there exists a linear map $A:\mathbb{R}^n\to\mathbb{R}^m$ such that for every unit vector $v\in S^{n-1}$,

$$\lim_{t\to 0^+}\frac{f(x_0+tv)-f(x_0)}{t}=A(v).$$

Suppose moreover that the above convergence is *uniform in v on the unit sphere S^{n-1}*, meaning that

$$\lim_{t\to 0^+}\sup_{v\in S^{n-1}}\left\|\frac{f(x_0+tv)-f(x_0)}{t}-A(v)\right\|=0.$$

Equivalently, for every $\varepsilon>0$ there exists $\delta>0$ such that for all $0<t<\delta$ and all $v\in S^{n-1}$,

$$\left\|\frac{f(x_0+tv)-f(x_0)}{t}-A(v)\right\|<\varepsilon.$$

Prove that f is differentiable at x_0 and that $f'(x_0)=A$.

Hint. For $h\neq 0$, write

$$h=\|h\|\,v \quad \text{with } v=\frac{h}{\|h\|}\in S^{n-1}.$$

Use linearity of A to rewrite

$$A(h)=\|h\|A(v),$$

and compare

$$\frac{f(x_0+h)-f(x_0)-A(h)}{\|h\|}$$

with

$$\frac{f(x_0+\|h\|v)-f(x_0)}{\|h\|}-A(v).$$

Then apply the assumed uniform convergence.

Proof. We want to show

$$\lim_{\substack{h\to 0\\ x_0+h\in E}}\frac{\|f(x_0+h)-f(x_0)-A(h)\|}{\|h\|}=0,$$

or equivalently, for all $\varepsilon>0$, there exists $\delta>0$ s.t. $0<\|h\|<\delta$ and $x_0+h\in E$ implies

$$\frac{\|f(x_0+h)-f(x_0)-A(h)\|}{\|h\|}<\varepsilon.$$

Now fix any $\varepsilon>0$, we know there exists $\delta'>0$ s.t. for any $0<\|h\|<\delta'$ and $v\in S^{n-1}$ we have

$$\left\|\frac{f(x_0+\|h\|v)-f(x_0)}{\|h\|}-A(v)\right\|<\varepsilon$$

due to the uniform convergence. Note that $x_0\in\operatorname{Int}(E)$, so there exists $r>0$ s.t. $B(x_0,r)\subseteq E$. Now we know for any $h=\|h\|v$ and $v\in S^{n-1}$ s.t. $0<\|h\|<\min\{\delta',r\}$,

$$\begin{aligned}\varepsilon>\left\|\frac{f(x_0+\|h\|v)-f(x_0)}{\|h\|}-A(v)\right\|&=\left\|\frac{f(x_0+h)-f(x_0)}{\|h\|}-A(v)\right\|\\&=\left\|\frac{f(x_0+h)-f(x_0)-\|h\|A(v)}{\|h\|}\right\|\\&=\left\|\frac{f(x_0+h)-f(x_0)-A(\|h\|v)}{\|h\|}\right\|\\&=\left\|\frac{f(x_0+h)-f(x_0)-A(h)}{\|h\|}\right\|,\end{aligned}$$

so we can pick $\delta=\min\{\delta',r\}$, and we're done. ■