Introduction to Analysis I

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Abstract

The lecture note of 2025 Fall Introduction to Analysis I by professor 崔茂培.

Contents

Chapter 1

Basic Things

Lecture 1

1.1 Natural Numbers

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The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, \dots\}$. There exists an addition operation

$$1 + 1 = 2 \quad 1 + 1 + 1 = 3 \quad \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n.$$

1.2 Integers

The set of integers is $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. There is a zero element 0 such that $z + 0 = z$ for any $z \in \mathbb{Z}$. Also, for $n \in \mathbb{N}$, we have $n + (-n) = 0$ and $n - m = n + (-m)$ for all $n, m \in \mathbb{N}$.

$$\mathbb{Z} \xRightarrow[\text{introduce division}]{} \mathbb{Q} \xRightarrow[\text{Completeness axiom}]{} \mathbb{R}$$

1.3 Field

Next, we introduce the concept of field.

Definition 1.3.1 (Fields)**.** A field is a set F together with two binary operations, called addition($+$) and multiplication($*$), such that the following properties hold:

(a) $a + b = b + a$, $a \cdot b = b \cdot a$ for $a, b \in F$.

(b) $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in F$.

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$.

(d) There are distince element 0 and 1 such that $a + 0 = a$, $a \cdot 1 = a$ for $a \in F$.

(e) For each $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$. If $a \neq 0$, there is an element $\frac{1}{a}$ or a^{-1} in F such that $a \cdot \frac{1}{a} = 1$, or $a \cdot a^{-1} = 1$.

Remark 1.3.1. If $a \in F$, then $a + a \in F$. We denote $a + a$ by $2 \cdot a$. Similarly,

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = n \cdot a,$$

and

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

if $a \in F$ and $n \in \mathbb{N}$.

Remark 1.3.2. In a field, we have subtraction and division $a - b = a + (-b)$ for $a, b \in F$. If $b \neq 0$, then $\frac{a}{b} = a \cdot b^{-1}$ for $a, b \in F$.

In a field F, we have

$$\begin{aligned}(a+b)^2 &= (a+b)\cdot(a+b) \\ &= (a+b)\cdot a + (a+b)\cdot b \\ &= a \cdot a + b \cdot a + a \cdot b + b \cdot b \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2.\end{aligned}$$

Example 1.3.1.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

if $b \neq 0$ and $d \neq 0$.

Proof.

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= a \cdot b^{-1} + c \cdot d^{-1} \\ &= ab^{-1}dd^{-1} + cd^{-1}bb^{-1} \\ &= adb^{-1}d^{-1} + cbd^{-1}b^{-1} \\ &= \frac{ad+bc}{bd}.\end{aligned}$$

Notice that this is true since we have commutativity in multiplication and

$$d^{-1}b^{-1} = (bd)^{-1} = \frac{1}{bd}.$$

∎

Example 1.3.2. The set of rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$ is a field.

Example 1.3.3. The set of real numbers is also a field.

Example 1.3.4. $F_2 = \{0, 1\}$ is also a field since we can define addition and multiplication like $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$, and $0 \cdot 0 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$.

1.4 Order Relation

Next, we introduce the order relation. The real number system is ordered by the relation $<$, which has the following properties.

(f) For each pair of real numbers a and b, exactly one of the follwing is true: $a = b, a < b, b < a$.

(g) If $a < b$ and $b < c$, then $a < c$.

(h) If $a < b$, then $a + c < b + c$ for any c, and if $0 < c$, then $a \cdot c < b \cdot c$.

Definition 1.4.1. A field with an order relation satisfy (f) to (h) is called an ordered field.

Example 1.4.1. The set of rational numbers is an ordered field.

Example 1.4.2. F_2 is not an ordered field.

Proof. If $0 < 1$, then $1 = 0 + 1 < 1 + 1 = 0$, which is a contradiction. If $1 < 0$, then $0 = 1 + 1 < 0 + 1 = 1$, which is also a contradiction. ∎

Notation. In an ordered field, we use $a \leq b$ to denote either $a < b$ or $a = b$.

1.5 Absolute Value and Triangle Inequality

Next, we define the absolute value of a real number

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a \leq 0; \end{cases}$$

Theorem 1.5.1 (Triangle Inequality)**.**

$$|a + b| \leq |a| + |b|$$

for all $a, b \in \mathbb{R}$.

Corollary 1.5.1.

$$||a| - |b|| \leq |a - b| \quad \text{and} \quad ||a| - |b|| \leq |a + b|$$

Proof. We write

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Similarly we have

$$|b| \leq |b - a| + |a|.$$

So

$$-|b - a| \leq |a| - |b| \leq |a - b|.$$

Thus,

$$||a| - |b|| \leq |a - b|.$$

∎

1.6 Supremum and Infimum

Next, we introduce the notion of supremum of a subset of real numbers.

Definition 1.6.1. Let S be a subset of $\mathbb{R}$,

(1) we say b is an upper bound of S if $x \leq b$ for all $x \in S$.

(2) If B is an upper bound of S, and no number smaller than B is an upper bound of S, then B is called the supremum or the least upper bound of S. We write $B = \sup S$.

Corollary 1.6.1. If $B = \sup S$, then

(1) $x \in S$ implies $x \leq B$

(2) If $b < B$, then b is not an upper bound of S, i.e. there exists $x_1 \in S$ such that $b < x_1$.

Definition 1.6.2. Let S be a subset of $\mathbb{R}$,

(1) we say b is an lower bound of S if $x \geq b$ for all $x \in S$.

(2) If α is an lower bound of S, and no number bigger than α is an lower bound of S, then α is called the infimum or the greatest lower bound of S. We write $\alpha = \inf S$.

Corollary 1.6.2. If $\alpha = \inf S$, then

(1) $x \in S$ implies $x \geq \alpha$

(2) If $\alpha < a$, then a is not an lower bound of S, i.e. there exists $x_1 \in S$ such that $x_1 < a$.

Notation (Interval Notation).

$$\begin{aligned}(a,b) &= \{x \mid a < x < b\}\\ (a,b] &= \{x \mid a < x \leq b\}\\ [a,b) &= \{x \mid a \leq x < b\}\end{aligned}$$

Example 1.6.1. $S = \{x \mid x < 0\} = (-\infty, 0)$, then $\sup S = 0$ but $\inf S$ does not exists.

Example 1.6.2. $S_1 = \{-1, -2, -3, -4, \dots\} = \{-n \mid n \in \mathbb{N}\}$, then $\sup S = -1$, but $\inf S$ does not exist.

Definition 1.6.3 (Nonempty Sets). A nonempty set is that a set has at least one element. The empty set, written as $\varnothing$, is the set has no elements at all.

Example 1.6.3. $S = \left\{x \mid x \in \mathbb{Q}, x < \sqrt{2}\right\}$

In $\mathbb{Q}$, $\sup S$ does not exist. In $\mathbb{R}$, $\sup S = \sqrt{2}$.

Theorem 1.6.1 (Completeness axiom). If a nonempty set of real numbers (an ordered field) is bounded above, then it has a least upper bound or $\sup S$ exists.

Remark 1.6.1. This is an extra axiom that can't be derived from the properties of ordered field.

Remark 1.6.2. Up to "isomorphism", there is exactly one complete ordered field: the field of real numbers.

Remark 1.6.3. From now, we assume $\mathbb{R}$ satisfies the completeness axiom. Thus, any nonempty subset $S \subseteq \mathbb{R}$ that is bounded above, we have $\sup S$ exists.

We can prove the following property of $\sup S$.

Theorem 1.6.2. If $S \subseteq \mathbb{R}$ is bounded above, then $\sup S$ is the unique real number B such that

(i) $x \leq B$ for all $x \in S$

(ii) for every $\varepsilon > 0$, there exist an $x_0 \in S$ such that $B - \varepsilon < x_0$.

Proof. (i), (ii) follows from the definition. We prove the uniqueness. Suppose $B_1 = \sup S = B_2$. We want to show $B_1 = B_2$. Suppose $B_1 \neq B_2$. Then either $B_1 < B_2$ or $B_2 < B_1$. However, if either one is true, then the other one cannot be $\sup S$. ∎

Theorem 1.6.3 (Archimedean Property). If $p > 0$ and $\varepsilon > 0$, then there exists an $n \in \mathbb{N}$ such that $p < n\varepsilon$.

Proof. We prove this contradiction. Suppose it is not true. This implies $n\varepsilon \leq p$ for all $n \in \mathbb{N}$. Consider $S = \{n\varepsilon \mid n \in \mathbb{N}\}$, then p is an upper bound of S, so S is bounded above by p, so we know $B = \sup S$ exists. Hence, $n\varepsilon \leq B$ for all $n \in \mathbb{N}$, so we have $(n+1)\varepsilon \leq B$, which means

$$n\varepsilon \leq B - \varepsilon$$

for all $n \in \mathbb{N}$. This implies $B - \varepsilon$ is also an upper bound of S, which is a contradiction. ∎

1.7 Density of other number system

Theorem 1.7.1. Every nonempty subset of the integers that is bounded below has a least element.

Proof. We first introduce an axiom:

> **Theorem 1.7.2** (Well-Ordering principle). Every non-empty subset of the natural numbers has a least element.

> **Note 1.7.1.** Here, $\mathbb{N}$ can be $\{0, 1, 2, \dots\}$ or $\{1, 2, 3, \dots\}$, which is not that important.

Now we call this subset of integers as S, and suppose we have m as a lower bound of S, then define $S' = \{s - m \mid s \in S\}$, then we know S' is a nonempty subset of $\mathbb{N}$, then by well-ordering principle we know there is a least element in S' and thus there is also a least element in S. ∎

Corollary 1.7.1. Every nonempty subset of the integers that is bounded above has a greatest element.

Proof. Suppose M is an upper bound, then define a set $S' = \{M - s \mid s \in S\}$, then by well-ordering principle we know $M - a$ is the least element of S' for some $a \in S$, so we have $M - x \geq M - a$ for all $x \in S$, which means $a \geq x$ for all $x \in S$ and since $a \in S$, so a is the greatest element of S. ∎

Theorem 1.7.3. The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{p}{q}$ such that $a < \frac{p}{q} < b$.

Proof. Let $a, b \in \mathbb{R}$, $a < b$. By Archimedean Property, $\exists q \in \mathbb{N}$ such that $q(b - a) > 1$. Let $S = \{m \mid m \text{ is an integer with } m > qa\}$, since we know $S \neq \varnothing$ and S is bounded below. Hence, $p = \inf S$ exists and is an integer by the last theorem. So $qa < p$ and $p - 1 \leq qa$, which means $qa < p \leq qa + 1 < qb$, so we have $a < \frac{p}{q} < b$. ∎

Lecture 2

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Definition 1.7.1 (Floor Function). For any real number x, the floor function of x is denoted by $\lfloor x \rfloor$, and is defined by the formula $\lfloor n \rfloor$ if $n \leq x < n + 1$ where $n \in \mathbb{Z}$.

Corollary 1.7.2.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

Example 1.7.1. $\lfloor 3.7 \rfloor = 3$, $\lfloor -1.2 \rfloor = -2$.

Now by floor function, we can reprove Theorem 1.7.3.

Theorem 1.7.4 (Density of rational number in real number Again)**.** The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{q}{p}$ such that $a < \frac{q}{p} < b$.

Reprove Theorem 1.7.3. Since $a < b$, so we know $b - a > 0$. Now by Archimedean Property, we know there exists $q \in \mathbb{N}$ such that $q(b - a) > 1$. Let $p = \lfloor qa \rfloor + 1$, we have

$$\lfloor qa \rfloor \leq qa < \lfloor qa \rfloor + 1 = p.$$

From our construction, $qb > qa + 1$, so we have

$$p = \lfloor qa \rfloor + 1 \leq qa + 1 < qb,$$

hence we have

$$qa \leq p \leq qb.$$

■

Note 1.7.2. For some reason, p, q in Theorem 1.7.3 and Theorem 1.7.4 are reversed.

Definition 1.7.2 (irrational number)**.** x is called irrational if x is not rational.

Example 1.7.2. $\sqrt{2}$ is irrational.

Theorem 1.7.5. Let $r \in \mathbb{Q}$ and $x \in \mathbb{R} \backslash \mathbb{Q}$, then

1. $r + x$ is irrational.
2. If $r \neq 0$, then rx is irrational.

sketch of proof.

1. If $r + x = q \in \mathbb{Q}$, then $x = q - r \in \mathbb{Q}$, contradiction.
2. If $rx = q \in \mathbb{Q}$, then $x = \frac{q}{r} \in \mathbb{Q}$ since $r \neq 0$.

■

Theorem 1.7.6 (irrational number dense in real number)**.** The set of irrational number is dense in real number. That is, if $a, b \in \mathbb{R}$ and $a < b$, then there exists a irrational number t such that $a < t < b$.

Proof. By density of rational number, we can find $a < r_1 < r_2 < b$ where $r_1, r_2 \in \mathbb{Q}$, and then let $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$, then we know

$$a < r_1 < t < r_2 < b.$$

Note 1.7.3. We should use Theorem 1.7.5 and the fact that $\sqrt{2}$ is irrational.

■

Definition 1.7.3 (bounded set)**.** A set $S \subseteq \mathbb{R}$ is bounded if there are numbers a, b s.t. $a \leq x \leq b$ for all $x \in S$.

Corollary 1.7.3. A bounded non-empty set in $\mathbb{R}$ has a unique supremum and a unique infimum and $\inf S \leq \sup S$.

1.8 Extended real number system

The real number system, together with ∞ and $-\infty$, then we have the following properties:

(a) If $a \in \mathbb{R}$, then $a + \infty = \infty + a = \infty$ and $a - \infty = -\infty + a = -\infty$, and $\frac{a}{\infty} = \frac{a}{-\infty} = 0$.

(b) If $a > 0$, then $a \cdot \infty = \infty \cdot a = \infty$ and $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$

(c) If $a < 0$, then $a \cdot \infty = \infty \cdot a = -\infty$ and $a \cdot -\infty = -\infty \cdot a = \infty$ and $\infty + \infty = \infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$ and $-\infty - \infty = \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$ and $|-\infty| = |\infty| = \infty$

However, there are some indeterminate form:

Theorem 1.8.1. The following things are not defined:

$$\infty - \infty, \ 0 \cdot \infty, \ \frac{\infty}{\infty}, \text{ and } \frac{0}{0}.$$

1.9 Mathematical Induction

Theorem 1.9.1 (Peano's Postulate)**.** The natural numbers satisfy the following properties

(a) $\mathbb{N}$ is nonempty.

(b) For each natural number n, there exists a unique rational number n called the successor of n.

(c) There exists a natural number $\overline{n}$ that is not the sucessor of any natural number.

(d) Different natural numbers have different sucessors, that is, $n \neq m$ implies $n' \neq m'$.

(e) The only subset of $\mathbb{N}$ that contains $\overline{n}$ and also contains the sucessor of every one of its element is $\mathbb{N}$.

Theorem 1.9.2 (Principle of Mathematical Induction)**.** Let $p_1, p_2, \ldots, p_n$ be propositions, one for each positive integers, such that

(a) p_1 is true.

(b) for each positive integer n, p_n implies p_{n+1}.

then p_n is true for each $n \in \mathbb{N}$.

Proof. Let $M = \{n \mid n \in \mathbb{N} \text{ and } p_n \text{ is true}\}$, then from (a) we know $1 \in M$ and from (b) we know $n \in M$ implies $n + 1 \in M$. Hence, from (e) of Peano's Postulate, we know $M = \mathbb{N}$. ■

Chapter 2

Metric Space

2.1 Definition and examples

Definition 2.1.1. Suppose $x_n \in \mathbb{R}$ for $n \geq m$. We use the notation $(x_n)_{n=m}^{\infty}$ to denote the sequence of numbers

$$x_m, x_{m+1}, \ldots$$

We first recall the definition of a convergent sequence.

Definition 2.1.2 (Convergent Sequence)**.** We say that a sequence $(x_n)_{n=m}^{\infty}$ of real numbers converges to x if for every $\varepsilon > 0$, there exists an $N \geq m$ s.t. $|x_n - x| \leq \varepsilon$ for all $n \geq N$.

> **Notation.** We write $\lim_{n\to\infty} x_n = x$.

On $\mathbb{R}$, we can define the distance function between two points $x, y \in \mathbb{R}$ by $d(x, y) = |x - y|$. We'll discuss this more later.

Lemma 2.1.1. Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number, then $(x_n)_{n=m}^{\infty}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Proof. Assume $(x_n)_{n=m}^{\infty}$ converges to x. Let $\varepsilon > 0$ be arbitrary real number. By definition, there exists an $N \geq m$ such that $|x_n - x| \leq \varepsilon$ for all $n \geq N$. But $d(x_n, x) = |x_n - x|$ by the definition. Hence, $\forall \varepsilon > 0$, $\exists N \geq m$ such that $d(x_n, x) \leq \varepsilon$ fpr all $n \geq N$. This implies that $\forall \varepsilon > 0$, $\exists N \geq m$ such that $|d(x_n, x) - 0| \leq \varepsilon$ for all $n \geq N$. This implies $\lim_{n\to\infty} d(x_n, x) = 0$.

The proof of the other side is the same but writing the above proof from bottom to top again.

■

Definition 2.1.3 (Metric Space)**.** A metric space (X, d) is the space of X of objects(called points), together with a distance function or metric $d : X \times X \to [0, \infty)$ which associates to each x, y of points in X a nonnegative number $d(x, y) \geq 0$, the following. Furthermore, the metric must satisfy 4 axioms.

(a) For any $x \in X$, $d(x, x) = 0$.

(b) (Positivity) For any distinct $x, y \in X$, we have $d(x, y) > 0$.

(c) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.

(d) (Triangle inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Example 2.1.1. On $\mathbb{R}$, we can define $d(x, y) = |x - y|$.

Proof.

- $d(x, y) = |x - y| \geq 0$.

- $d(x, y) = 0$ iff $|x - y| = 0$ iff $x = y$.
- $|x - y| = |y - x|$, so $d(x, y) = d(y, x)$
- $|x - z| \leq |x - y| + |y - z|$ for all $x, y, z \in \mathbb{R}$.

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Example 2.1.2. Let (X, d) be a metric space and $Y \subseteq X$, then Y inherits a natural distance function

$$d|_{Y \times Y} : Y \times Y \to [0, \infty)$$

defined by $d|_{Y \times Y}(\alpha, \beta) = d(\alpha, \beta)$ for all $\alpha, \beta \in Y$.

Note 2.1.1. $(Y, d|_{Y \times Y})$ is called a metric subspace of (X, d). It is obvious that $d|_{Y \times Y}$ is a metric on Y.

Recall $\mathbb{R}^n$. Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

Definition 2.1.4 (l^2-metric)**.** The l^2-metric is defined by

$$d_2(x, y) = \left(\sum_{i=1}^{n} (x_n - y_n)^2 \right)^{\frac{1}{2}} \text{ (or we called } d_{l_2}(x, y)\text{).}$$

Definition 2.1.5 (l^1-metric(taxicab metric))**.** The l^1-metric is defined by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| (\text{or we called } d_{l_1}(x, y))$$

Definition 2.1.6 (l^∞-metric)**.** The l^∞-metric is defined by

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

Exercise 2.1.1. Verify they are all metrics.

Note 2.1.2. Actually we have to define inner product and norm first and then we can use the triangle inequality of norm to prove d_2 is a metric. (See lecture notes by professor)

Lecture 3

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Definition 2.1.7 (Cartesian Product)**.** Let A, B be sets. The cartesian product of A and B is defined by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Similarly, the cartesian product of $X_1, X_2, \dots, X_n$ is

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \dots, x_n) \mid x_i \in X_i \ \forall 1 \leq i \leq n\}.$$

Definition 2.1.8 (Functions)**.** Let $X_1, X_2, \dots, X_n$ be sets and let Y be another set. A fuction of n variables with codomains is a map $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ which assigns each n-tuple $(x_1, x_2, \dots, x_n)$ with $x_i \in X_i$ a unique element $f(x_1, x_2, \dots, x_n)$.

Definition. We talk about the definition of domain, codomain, and range:

Definition 2.1.9. The domain of f is $X_1 \times X_2 \times \cdots \times X_n$ and Y is the codomain of f.

Definition 2.1.10. The range of f is

$$\{f(x_1, x_2, \ldots, x_n) \in Y \mid x_i \in X_i \ \forall i\}.$$

In the definition of metric space, we write (X, d) to emphasize our set X and d is a distance function defined on $X \times X$, i.e.

$$d : X \times X \to [0, \infty) \subseteq \mathbb{R},$$

where

$$d : (x, y) \mapsto d(x, y)$$

for $x, y \in X$. Let (X, d) be a metric space and $Y \subseteq X$. Then $(Y, d|_{Y \times Y})$ is also a metric space with distance function defined by

$$d|_{Y\times Y} \to [0, \infty)$$

and

$$d|_{Y\times Y} : (\alpha, \beta) \mapsto d(\alpha, \beta) \text{ for } \alpha, \beta \in Y.$$

Example 2.1.3. Recall the Taxi-cab metric, it can be used in cryptography. For example, for two binary strings, we know

$$d_1((10010), (10101)) = 3 = \text{the number of mismatched bits.}$$

Example 2.1.4. Recall the l^∞-metric. Suppose two jobs where each consists of 3 tasks, and the time (in hours) to complete each task is represented by a vector

$$x = (2, 4, 6), \ \ y = (3, 7, 5),$$

so

$$d_\infty(x, y) = \max\{|2 - 3|, |4 - 7|, |6 - 5|\} = 3.$$

Definition 2.1.11 (Lipschitz equivalent metrics)**.** Let (X, d_1) and (X, d_2) be two metrics on X. We say d_1 and d_2 are Lipschitz equivalent if $\exists c_1, c_2 > 0$ s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y) \quad \forall x, y \in X$$

Remark 2.1.1. They will have same topology (defined later).

Proposition 2.1.1. For all $x, y \in \mathbb{R}^n$,

$$d_2(x, y) \leq d_1(x, y) \leq \sqrt{n} d_2(x, y) \tag{2.1}$$

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} d_\infty(x, y) \tag{2.2}$$

Remark 2.1.2.

$$\begin{aligned} d_\infty(x, y) &\geq \frac{1}{\sqrt{n}} d_2(x, y) \\ &\geq \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} d_1(x, y) = \frac{1}{n} d_1(x, y). \end{aligned}$$

Also,

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y).$$

Remark 2.1.3. d_1, d_2, d_∞ are all Lipschitz equivalent.

proof of Proposition 2.1.1 . Recall $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, then

$$d_1(x,y) = \sum_{i=1}^n |x_i - y_i|, \quad d_2(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

By Cauchy-Schurwatz inequality,

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^n |x_i - y_i| \\ &\le \left(\sum_{i=1}^n |x_i - y_i|\right)^{\frac{1}{2}} \left(\sum_{i=1}^n 1^2\right)^{\frac{1}{2}} = \sqrt{n} d_2(x,y). \end{aligned}$$

Now we show that $d_1(x,y) \ge d_2(x,y)$.

$$\begin{aligned} (d_1(x,y))^2 &= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\ &= \sum_{i=1}^n |x_i - y_i|^2 + 2 \sum_{1 \le i < j \le n} |x_i - y_i||x_j - y_j| \\ &\ge \sum_{i=1}^n |x_i - y_i|^2 = d_2(x,y)^2. \end{aligned}$$

Hence, we have $d_1(x,y) \ge d_2(x,y)$.

Now we show that $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$. Note that

$$d_2(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}, \quad d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|.$$

For each i, we know

$$|x_i - y_i| \le d_\infty(x,y),$$

so

$$d_2(x,y)^2 \le \sum_{i=1}^n d_\infty(x,y)^2 = n d_\infty(x,y)^2,$$

so $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$. ∎

Definition 2.1.12 (Discrete metric). Let X be any set, define the discrete metric:

$$d_{\text{disc}} : X \times X \to \{0,1\}$$

where

$$d_{\text{disc}}(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \ne y. \end{cases}$$

Why this is a metric? Because

- $d_{\text{disc}}(x,y) \ge 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$.
- $d_{\text{disc}}(x,y) = d_{\text{disc}}(y,x)$ by definition.
- $d_{\text{disc}}(x,z) \le d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z)$?

proof of triangle inequality in discrete metric. We first consider the case that $x = z$, then

$$d_{\text{disc}}(x, z) = 0,$$

so it is obviously that the triangle inequality is true.

Now if $x \neq z$, then either $y \neq z$ or $y \neq x$ must happen, so the triangle inequality must be true. ∎

Example 2.1.5. We can define

$$d(x, x) = 0, \quad d(x, y) = \text{minimal length of a path from } x \text{ to } y,$$

then this is also a metric.

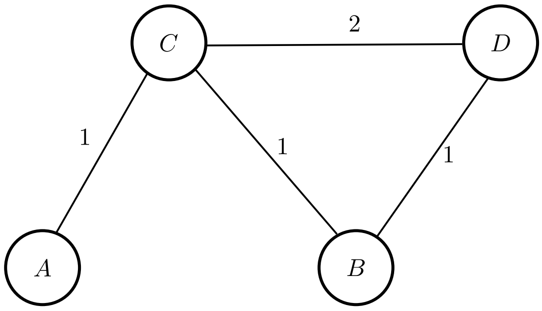

Figure 2.1: Graph metrics

Definition 2.1.13 (Convergence in metric space). Let m be an integer, (X, d) be a metric space, and let $\left(X^{(n)}\right)_{n=m}^{\infty}$ be a sequence of points in X. Let $x \in X$. We say that $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges to x with respect to d iff

$$\lim_{n\to\infty} d\left(X^{(n)}, x\right) = 0,$$

where $\lim_{n\to\infty} d\left(X^{(n)}, x\right) = 0$ iff for every $\varepsilon > 0$, $\exists N \geq m$ s.t. $d\left(X^{(n)}, x\right) \leq \varepsilon$ for all $n \geq N$.

Notation. We also write $\lim_{n\to\infty} X^{(n)} = x$ in (X, d).

Remark 2.1.4. Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges to x in (X, d), then $\left(X^{(n)}\right)_{n=m_1}^{\infty}$ also converges to x in (X, d) if $m_1 \geq m$.

Example 2.1.6. Let $\left(X^{(n)}\right)_{n=1}^{\infty}$ denote the sequence $X^{(n)} = \left(\frac{1}{n}, \frac{1}{n}\right)$ in $\mathbb{R}^2$, then what will this sequence converges to for different metric?

Proof.

- If the metric is d_1, then

$$d_1\left(X^{(n)}, (0,0)\right) = \left|\frac{1}{n} - 0\right| + \left|\frac{1}{n} - 0\right| = \frac{2}{n},$$

so

$$\lim_{n\to\infty} d_1\left(X^{(n)}, (0,0)\right) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

- If the metric is d_2, then

$$d_2\left(X^{(d)}, (0,0)\right) = \sqrt{\left(\frac{1}{n} - 0\right)^2 + \left(\frac{1}{n} - 0\right)^2} = \frac{\sqrt{2}}{n}.$$

  Hence, under l_2-metric $\left\{X^{(n)}\right\}$ also converges to 0.

- If the metric is d_∞, then

$$d_\infty\left(X^{(n)}, (0,0)\right) = \max\left\{\left|\frac{1}{n}\right|, \left|\frac{1}{n}\right|\right\} = \frac{1}{n},$$

  so it also converges to 0.

- If the metric is discrete metric, then however, it will not converges to $(0,0)$ since

$$\lim_{n\to\infty} d_{\text{disc}}\left(X^{(n)}, (0,0)\right) = \lim_{n\to\infty} d_{\text{disc}}\left(\left(\frac{1}{n}, \frac{1}{n}\right), (0,0)\right) = 1.$$

⊛

Definition. Let $f : X \to Y$ be a function with domain X and codomain Y. The range of $f = \{f(x) \mid x \in X\} \subseteq Y$.

Definition 2.1.14 (injective)**.** We say f is injective or one-to-one if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Definition 2.1.15 (surjective)**.** We say f is surjective or onto if for every $y \in Y$, $\exists x \in X$ s.t. $f(x) = y$.

Definition 2.1.16 (bijective)**.** We say f is bijective if f is injective and surjective.

Corollary 2.1.1. If f is bijective, then there exists $f^{-1} : Y \to X$ defined by $f^{-1}(y) = x$ if $f(x) = y$. We also have

$$\begin{aligned} f\left(f^{-1}(y)\right) &= y \ \forall y \in Y \\ f^{-1}\left(f(x)\right) &= x \ \forall x \in X. \end{aligned}$$

Example 2.1.7. $\lim_{n\to\infty} \frac{1}{n} = 0$ in $(\mathbb{R}, d)$, where d is the standard metric in $\mathbb{R}$, which is defined by

$$d(x, y) = |x - y|.$$

But in different metric, $\lim_{n\to\infty} \frac{1}{n}$ may not be 0.

Proof. Define $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} x, & \text{if } 0 < x < 1; \\ 1, & \text{if } x = 0; \\ 0, & \text{if } x = 1. \end{cases}$$

f is bijective on $[0,1]$ to $[0,1]$

Define another metric d^1 on $[0,1]$ by

$$d^1(x,y) = d(f(x), f(y)).$$

We want to show that d^1 is also a metric on $[0,1]$.

- $d^1(x,y) = d(f(x), f(y)) = |f(x) - f(y)| \geq 0$
- $d^1(x,y) = 0$ iff $f(x) = f(y)$ iff $x = y$ since f is injective.
- The triangle inequality is trivially true since we can just use the triangle inequality in d.

In fact, $\lim_{n\to\infty} \frac{1}{n} = 1$ in $\left([0,1], d^1\right)$ since

$$\lim_{n\to\infty} d^1\left(\frac{1}{n}, 1\right) = \lim_{n\to\infty} d\left(\frac{1}{n}, 0\right) = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0.$$

⊛

2.2 Some point set topology of metric space

Definition 2.2.1 (ball)**.** Let (X,d) be a metric space. let $x_0 \in X$ and $r > 0$. We define the ball $B_{(X,d)}(x_0, r)$ in X, centered at x_0 and with radius r in the metric d, to the set

$$B_{(X_0,d)}(X_0, Y) := \{x \in X \mid d(x_0, x) < r\}.$$

Sometimes, we write it as $B_X(x_0, r)$ or $B(x_0, r)$.

Example 2.2.1. In $\mathbb{R}^2$,

$$B_{(\mathbb{R}^2, d_2)}\left((0,0), 1\right) = \left\{(x,y) \mid d_2((x,y),(0,0)) = \sqrt{x^2+y^2} < 1\right\},$$

and

$$B_{(\mathbb{R}^2, d_1)}((0,0), 1) = \{(x,y) \mid d_1((x,y),(0,0)) = |x| + |y| < 1\},$$

and

$$B_{(\mathbb{R}^2, d_\infty)}((0,0), 1) = \{(x,y) \mid d_\infty((x,y),(0,0)) = \max\{|x|, |y|\} < 1\},$$

also we can consider the d_{disc} case but I am too lazy to write it down.

Notation. Let $E \subseteq X$, we will write

$$X \setminus E := \{x \in X \mid x \notin E\}.$$

Definition. Let (X,d) be a metric space and $E \subseteq X$. For a point $x_0 \in X$,

Definition 2.2.2 (interior point)**.** x_0 is an interior point of E if $\exists r > 0$ s.t. $B(x_0, r) \subseteq E$.

Definition 2.2.3 (exterior point)**.** x_0 is an exterior point of E if $\exists r > 0$ s.t. $B(x_0, r) \subseteq X \setminus E$.

Definition 2.2.4 (boundary point)**.** x_0 is a boundary point of E if it is neither an interior point nor an exterior point of E.

Proposition 2.2.1. x_0 is a boundary point of E iff for all $r > 0$, $B(x_0, r) \cap E \neq \varnothing$ and $B(x_0, r) \cap (X \setminus E) \neq \varnothing$.

Lecture 4

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Theorem 2.2.1. Let (X, d_1) and (X, d_2) be metrics on X, and suppose d_1 and d_2 are Lipschitz equivalent, then for any sequence $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$, then for any $x \in X$

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_1) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_2).$$

Proof. Since d_1, d_2 are Lipschitz equivalent, so there exists $c_1, c_2 > 0$ s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y).$$

($\Rightarrow$) Given $\frac{\varepsilon}{c_2} > 0$, since $\lim_{n\to\infty} x^{(n)} = x$ in (X, d_1), so there exists N s.t. $N \geq m$ and

$$d_1(x^{(n)}, x) \leq \frac{\varepsilon}{c_2} \text{ for } n \geq N.$$

This implies $d_2(x^{(n)}, x) \leq c_2 d_1(x^{(n)}, x) \leq \varepsilon$ for $n \geq N$, which means

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_2).$$

($\Leftarrow$) Similar.

■

Remark 2.2.1. On $\mathbb{R}^n$, the metrics d_1, d_2, d_∞ are Lipschitz equivalent, that is,

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_1) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_2) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_\infty)$$

Proposition 2.2.2. Let (X, d_{disc}) be a discrete metric space, and $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$. Then

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_{\text{disc}}) \Leftrightarrow \exists N \geq m \text{ s.t. } x^{(n)} = x \text{ for } n \geq N.$$

Proof. ($\Leftarrow$) Easy.

($\Rightarrow$) Given $\frac{1}{2} > 0$, there exists $N \geq m$ s.t. $d(x_n, x) < \frac{1}{2}$ for $n \geq N$, but $d(x_n, x) < \frac{1}{2}$ implies $d(x_n, x) = 0$, which means $x_n = x$ for all $n \geq N$.

■

Definition. We define the interior, exterior, and boundary point again.

Definition 2.2.5. The set of interior points is denoted by

$$\text{Int}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq E\}.$$

Definition 2.2.6. The set of exterior points is denoted by

$$\text{Ext}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq X \setminus E\}.$$

Definition 2.2.7. A point is a boundary points if it is neithe an interior point nor an exterior point, and we define

$$\partial E = \{x \in X \mid x \notin \text{Int}(E) \text{ and } x \notin \text{Ext}(E)\}.$$

Remark 2.2.2.

1. $$x_0 \notin \text{Int}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (X \setminus E) \neq \varnothing.$$

2. $$x_0 \notin \text{Ext}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (E) \neq \varnothing.$$

3. $\text{Int}(X \setminus E) = \text{Ext}(E)$.

4. $\partial E = \partial(X \setminus E)$ since

$$x_0 \in \partial E \Leftrightarrow x \notin \text{Int}(E) \text{ and } \text{Ext}(E) \Leftrightarrow x_0 \notin \text{Int}(E) \text{ and } x_0 \notin \text{Int}(X \setminus E).$$

Also,

$$x_0 \in \partial(X \setminus E) \Leftrightarrow x \notin \text{Int}(X \setminus E) \text{ and } \text{Ext}(X \setminus E) \Leftrightarrow x_0 \notin \text{Int}(X \setminus E) \text{ and } x_0 \notin \text{Int}(E).$$

Hence, acutually $\partial E = \partial(X \setminus E)$.

Proposition 2.2.3.

$$x_0 \in \partial E \Leftrightarrow \text{ For any } r > 0, B_X(x_0, r) \cap E \neq \varnothing \text{ and } B_X(x_0, r) \cap (X \setminus E) \neq \varnothing$$

Example 2.2.2. Let $(\mathbb{R}, d)$ be the usual metric on $\mathbb{R}$, where

$$d(x, y) = |x - y|.$$

Then, we know in this space,

$$\begin{aligned} B_{\mathbb{R}}(x_0, r) &= \{x \in \mathbb{R} \mid d(x, x_0) < r\} \\ &= \{x \in \mathbb{R} \mid |x - x_0| < r\} \\ &= \{x \in \mathbb{R} \mid -r + x_0 < x < r + x_0\}. \end{aligned}$$

Hence, suppose $E = [1, 2)$, then $\text{Int}(E) = (1, 2)$ since we know $B(x_0, r) = (x_0 - r, x_0 + r)$, so for all $x \in (1, 2)$, we know there is an open ball $B(x_0, r) \subseteq [1, 2)$ for some $r > 0$. Also, consider the endpoint $1, 2$, we can verify that these two points are not interior points. Besides, consider the points not in $[1, 2]$, it is trivial that they cannot be interior points.

Example 2.2.3. We consider (X, d_{disc}). Let $E \subseteq X$. If $x \in E$, we know

$$B\left(x, \frac{1}{2}\right) = \left\{y \mid d(y, x) < \frac{1}{2}\right\} = \{x\} \subseteq E.$$

Hence, $E \subseteq \text{Int}(E)$. Besides, for all $x \in \text{Int}(E)$, we know there exists $r > 0$ s.t. $B(x_0, r) \subseteq E$, also we know $x_0 \in B(x_0, r) \subseteq E$, so $x_0 \in E$, and thus $\text{Int}(E) \subseteq E$. Hence, $E = \text{Int}(E)$. Similarly, $\text{Int}(X \setminus E) = X \setminus E$. Suppose there is a $x \in X$ s.t. $x \in \partial E$, then $x \notin \text{Int}(E) = E$ and $x \notin \text{Ext}(E) = \text{Int}(X \setminus E) = X \setminus E$, so such x does not exist.

Definition 2.2.8 (Closure)**.** Let (X, d) be a metric space, and let $E \subseteq X$ and $x_0 \in X$. We say x_0 is a adherent point of E if for every $r > 0$, $B(x_0, r) \cap E \neq \varnothing$. The set of adeherent points is called the closure of E, and denoted by $\overline{E}$.

Proposition 2.2.4 (TFAE)**.**

(a) x_0 is an adherent point of E.

(b) x_0 is either an interior point or a boundary point of E.

(c) $\exists$ a sequence $\left\{X^{(n)}\right\}_{n=1}^{\infty}$ in E which converges to x_0 in (X,d).

proof from (a) to (b). Suppose $x_0 \in \overline{E}$, then $B(x_0,r) \cap E \neq \varnothing$ for all $r > 0$. If $\exists s > 0$ s.t. $B(x_0,s) \subseteq E$, then $x_0 \in \mathrm{Int}(E)$. If such s does not exists, then we know

$$B(x_0,r) \cap E \neq \varnothing \text{ and } B(x_0,r) \cap (X \setminus E) \neq \varnothing \text{ for all } r > 0,$$

so we can use Proposition 2.2.1 to conclude that x_0 must be a boundary point. ■

proof from (b) to (c). Since either $x_0 \in \mathrm{Int}(E)$ or $x_0 \in \partial E$. If $x_0 \in \mathrm{Int}(E)$, then $x_0 \in E$, then we can choose $X^{(n)} = x_0$ for all $n \geq 1$. If $x_0 \in \partial E$, then given $n \in \mathbb{N}$, $\exists x_n \in B\left(x_0, \frac{1}{n}\right) \cap E \neq \varnothing$. Hence, $x_n \in E$ and $d(x_n, x_0) < \frac{1}{n}$. Pick such x_n to form $\left\{X^{(n)}\right\}_{n=1}^{\infty}$, then we know this sequence converges to x_0. ■

proof from (c) to (a). Suppose $\left\{X^{(n)}\right\} \subseteq E$ s.t. $\lim_{n\to\infty} d\left(X^{(n)}, x_0\right) = 0$, then we want to show $x_0 \in \overline{E}$. Given any $r > 0$, choose $N \geq 1$ s.t.

$$d\left(X^{(n)}, x_0\right) < r \text{ when } n \geq N.$$

This implies for $n \geq N$, $X^{(n)} \in E$ and $X^{(n)} \in B(x_0,r)$, so we know $E \cap B(x_0,r) \neq \varnothing$ for all $r > 0$, which means $x_0 \in \overline{E}$. ■

Remark 2.2.3. The equation (a) and (b) implies $\overline{E} = \mathrm{Int}(E) \cup \partial E$.

An alternative proof. Since we know $X = \mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E$ by Theorem 2.2.2, and $\overline{E} \subseteq X$, so

$$\begin{aligned}\overline{E} = \overline{E} \cap X &= \overline{E} \cap (\mathrm{Int}(E) \cup \mathrm{ext}(E) \cup \partial E)\\ &= (\overline{E} \cap \mathrm{Int}(E)) \cup (\overline{E} \cap \mathrm{Ext}(E)) \cup (\overline{E} \cap \partial E).\end{aligned}$$

Also, notice that

$$\overline{E} \cap \mathrm{Int}(E) = \mathrm{Int}(E) \quad \overline{E} \cap \mathrm{Ext}(E) = \varnothing \quad \overline{E} \cap \partial E = \partial E,$$

so $\overline{E} = \mathrm{Int}(E) \cup \partial E$. ■

Corollary 2.2.1. $\overline{E} = \mathrm{Int}(E) \cup \partial E$.

Theorem 2.2.2. Let (X,d) be a metric space and $E \subseteq X$. Then,

$$X = \mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E$$

Remark 2.2.4. ∂E could be empty. (See previous example.)

Corollary 2.2.2. Let (X,d) be a metric space and $E \subseteq X$. Then

$$\overline{E} = \mathrm{Int}(E) \cup \partial E = X \setminus \mathrm{Ext}(E).$$

Lemma 2.2.1. $\overline{E} = E \cup \partial E$

Proof. We first show that $E \cup \partial E \subseteq \overline{E}$. For every point $x \in E$, we know $x \in B(x,r)$ for all $r > 0$, so $B(x,r) \cap E \neq \varnothing$. Also, by definition, we know $\partial E \subseteq \overline{E}$, so we're done.

Next, we show that $\overline{E} \subseteq E \cup \partial E$. For every $x \in \overline{E}$, if $x \in E$, then $x \in E \cup \partial E$. If not, since $x \in \overline{E}$, so $B(x,r) \cap E \neq \varnothing$ for all $r > 0$. Also, since $x \notin E$, and $x \in B(x,r)$, so $B(x,r) \cap (X \setminus E) \neq \varnothing$, otherwise $x \in B(x,r) \subseteq E$, which is a contradiction. Now we know for every $r > 0$, $B(x,r) \cap E \neq \varnothing$ and $B(x,r) \cap (X \setminus E) \neq \varnothing$, so $x \in \partial E$. ■

Lemma 2.2.2 (Discarded). If $x \in \mathrm{Int}(E)$, then $x \in E$. In other words, $\mathrm{Int}(E) \subseteq E$.

Proof. If $x \in \mathrm{Int}(E)$, then there exists $r > 0$ s.t. $B(x,r) \subseteq E$, and thus $x \in B(x,r) \subseteq E$, which means $x \in E$. ■

Note 2.2.1. I thought we need Lemma 2.2.2 to prove Theorem 2.2.3, but I found it needless. Nevertheless, I still want to keep it since I think it is useful in some elsewhere.

Definition 2.2.9. Let (X,d) be a metric space and $E \subseteq X$. We say E is closed if $\partial E \subseteq E$. We say E is open if it doesn't contain any boundary points i.e. $\partial E \cap E \ = \varnothing$.

Theorem 2.2.3. E is closed if and only if $\overline{E} = E$.

Proof.

$$E \text{ is closed } \Rightarrow \partial E \subseteq E \Rightarrow \overline{E} = E \cup \partial E = E.$$
$$E = \overline{E} = E \cup \partial E \Rightarrow \partial E \subseteq E \Rightarrow E \text{ is closed.}$$

■

Theorem 2.2.4. E is open. $\Leftrightarrow \mathrm{Int}(E) = E$.

proof of ($\Rightarrow$) **.** E is open means $\partial E \cap E = \varnothing$. Fix $x \in E$, since $x \notin \partial E$, so $\exists r > 0$ s.t. $B(x,r) \cap E = \varnothing$ or $B(x,r) \cap (X \setminus E) = \varnothing$. Since $x \in E$ and $x \in B(x,r)$, so $B(x,r) \cap (X \setminus E) = \varnothing$, which means $B(x,r) \subseteq E$, so $x \in \mathrm{Int}(E)$. Now we know $E \subseteq \mathrm{Int}(E)$. Also, we know $\mathrm{Int}(E) \subseteq E$ by Lemma 2.2.2. Hence, $\mathrm{Int}(E) = E$. ■

proof of ($\Leftarrow$) **.** If $\mathrm{Int}(E) = E$, then given any $x \in E = \mathrm{Int}(E)$, there exists $r > 0$ s.t. $B(x,r) \subseteq E$. Hence, $B(x,r) \cap (X \setminus E) = \varnothing$, so $x \notin \partial E$, and thus $E \cap \partial E = \varnothing$. ■

Theorem 2.2.5. If $E \subseteq X$, then E is open $\Leftrightarrow X \setminus E$ is closed.

proof of ($\Rightarrow$)**.** Since we can write $X = \mathrm{Int}(E) \cupdot \mathrm{Ext}(E) \cupdot \partial E$, and E is open, so

$$X \setminus E = (\mathrm{Int}(E) \cupdot \mathrm{Ext}(E) \cupdot \partial E) \setminus E = (\mathrm{Int}(E) \cupdot \mathrm{Ext}(E) \cupdot \partial E) \setminus \mathrm{Int}(E) = \mathrm{Ext}(E) \cupdot \partial E.$$

by Theorem 2.2.4. Now we want to show that $\partial(X \setminus E) \subseteq X \setminus E$, and we know

$$X \setminus E = \mathrm{Ext}(E) \cupdot \partial E = \mathrm{Ext}(E) \cupdot \partial(X \setminus E)$$

since $\partial E = \partial(X \setminus E)$. Hence, we have $\partial(X \setminus E) \subseteq X \setminus E$. ■

proof of $\Leftarrow$**.** Suppose $X \setminus E$ is closed, then $\partial(X \setminus E) \subseteq X \setminus E$, and since $\partial E = \partial(X \setminus E)$, so $\partial E \subseteq X \setminus E$, and thus $\partial E \cap E = \varnothing$, which means E is open. ■

Lecture 5

Definition 2.2.10. Let (X,d) be a metric space, $E \subseteq X$ and $x_0 \in E$. We say x_0 is an adherent point if for every $r > 0$, $B(x_0, r) \cap E \neq \varnothing$, and we denote $\overline{E}$ to the set of all adherent points. 16 Sep. 10:20

Remark 2.2.5. $E \subseteq \overline{E}$, since given any $x_0 \in E$ and $r > 0$, $x_0 \in B(x_0, r)$, so $B(x_0, r) \cap E \neq \varnothing$, and thus $E \subseteq \overline{E}$.

Remark 2.2.6. $\partial E \subseteq \overline{E}$. Given $x_0 \in \partial E$, we know for any $r > 0$, $B(x_0, r) \cap E \neq \varnothing$, so $x_0 \in \overline{E}$.

Proposition 2.2.5. $x_0 \in \overline{E}$ if and only if there exists $\left(X^{(n)}\right)_{n=1}^{\infty}$ s.t. $\lim_{n\to\infty} X^{(n)}$ exists and $\lim_{n\to\infty} X^{(n)} = x_0$.

proof of ($\Rightarrow$). Given $n \in \mathbb{N}$. Consider $B\left(x_0, \frac{1}{n}\right)$. We know $B\left(x_0, \frac{1}{n}\right) \cap E \neq \varnothing$. Choose $X^{(n)} \in B\left(x_0, \frac{1}{n}\right) \cap E$, then $d\left(x_0, X^{(n)}\right) < \frac{1}{n}$, which means $\lim_{n\to\infty} d\left(x_0, X^{(n)}\right) = 0$. Hence, there exists $\left(X^{(n)}\right) \in E$ s.t. $\lim_{n\to\infty} X^{(n)} = x_0$. ■

proof of $\Leftarrow$. There exists N s.t. $X^{(n)} \in B(x_0, r)$ when $n \geq N$. Given any $r > 0$, since $\lim_{n\to\infty} X^{(n)} = x_0$, so $\lim_{n\to\infty} d\left(X^{(n)}, x_0\right) = 0$. Hence, there exists N s.t. $d\left(X^{(n)}, x_0\right) < r$ when $n \geq N$. Hence, when $n \geq N$, we have $X^{(n)} \subseteq B(x_0, r)$, since we know $X^{(n)} \in E$ for all n, so we know $B(x_0, r) \cap E \neq \varnothing$, so $x_0 \in \overline{E}$. ■

Proposition 2.2.6. Let (X, d) be a metric space and $E \subseteq X$, then

$$X = \operatorname{Int}(E) \mathbin{\dot\cup} \operatorname{Ext}(E) \mathbin{\dot\cup} \partial E.$$

Corollary 2.2.3. Let (X, d) be a metric space and $E \subseteq X$. Then,

$$\overline{E} = \operatorname{Int}(E) \cup \partial E = X \setminus \operatorname{Ext}(E) = E \cup \partial E.$$

Proof. Since

$$\begin{aligned}
\overline{E} &= \overline{E} \cap X = \overline{E} \cap (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E) \\
&= \left(\overline{E} \cap \operatorname{Int}(E)\right) \cup \left(\overline{E} \cap \operatorname{Ext}(E)\right) \cup \left(\overline{E} \cap \partial E\right) = \operatorname{Int}(E) \mathbin{\dot\cup} \partial E.
\end{aligned}$$

Also,

$$X \setminus E = (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E) \setminus \operatorname{Ext}(E) = \operatorname{Int}(E) \cup \partial E = \overline{E}.$$

Besides, we know $\operatorname{Int}(E) \subseteq E \subseteq \overline{E}$, so

$$\overline{E} = \operatorname{Int}(E) \cup \partial E \subseteq \overline{E} \cup \partial E = (\operatorname{Int}(E) \cup \partial E) \cup \partial E = \operatorname{Int}(E) \cup \partial E \subseteq E \cup \partial E.$$

■

Definition 2.2.11. Let (X, d) be a metric space and $E \subseteq X$. We say E is open iff $\partial E \cap E \neq \varnothing$. We say E is closed iff $\partial E \subseteq E$.

Proposition 2.2.7.

$$E \text{ is open} \Leftrightarrow \operatorname{Int}(E) = E \Leftrightarrow X \setminus E \text{ is closed}.$$

proof of E is open $\Leftrightarrow \operatorname{Int}(E) = E$.

($\Rightarrow$) Since E is open, so $\partial E \cap E = \varnothing$. Hence,

$$\begin{aligned}
E &= E \cap X = E \cap (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E) \\
&= (E \cap \operatorname{Int}(E)) \cup (E \cap \operatorname{Ext}(E)) \cup (E \cap \partial E) = \operatorname{Int}(E) \cup (E \cap \partial E) = \operatorname{Int}(E)
\end{aligned}$$

since $E \cap \operatorname{Ext}(E) = \varnothing$ and we know $\partial E \cap E = \varnothing$.

■

proof of E is open $\Leftrightarrow X \setminus E$ is closed.

($\Rightarrow$) $X = \mathrm{Int}(E) \cup \mathrm{Ext}(E) \cup \partial E$, so

$$X \setminus E = (\mathrm{Int}(E) \cup \mathrm{Ext}(E) \cup \partial E) \setminus \mathrm{Int}(E) = \mathrm{Ext}(E) \cup \partial E = \mathrm{Int}(X \setminus E) \cup \partial(X \setminus E).$$

Hence, $\partial(X \setminus E) \subseteq X \setminus E$, which means $X \setminus E$ is closed.

($\Leftarrow$) $X \setminus E$ is closed, then $\partial(X \setminus E) \subseteq X \setminus E$, but $\partial E = \partial(X \setminus E)$, so $\partial E \subseteq X \setminus E$, and thus $\partial E \cap E = \varnothing$.

■

Remark 2.2.7. If $\partial E = \varnothing$, then E is open and closed.

Remark 2.2.8. If a set S is closed and open, then S is clopen.

Remark 2.2.9. Let (X, d) be a metric space, then $\varnothing$ is clopen, and we can deduce that X is also clopen since X is the complement of $\varnothing$ and we know S is open iff $X \setminus S$ is closed.

Remark 2.2.10. In $(\mathbb{R}, d)$, where d is the standard metric, then the only clopen set is $\mathbb{R}$ or $\varnothing$.

Remark 2.2.11. Let (X, d_{disc}) be the discrete metric space on X. Let E be any set, then E is open and closed. Given $x_0 \in E$, we know $B_{\mathrm{disc}}\left(x_0, \frac{1}{2}\right) \subseteq E$, so $x_0 \in \mathrm{Int}(E)$, which means $E = \mathrm{Int}(E)$, so E is open. Now since $X \setminus E$ is also open, so E is closed. Thus, E is clopen.

Proposition 2.2.8. The following hold:

(a) E is open iff $E = \mathrm{Int}(E)$.

(b) E is closed iff every convergent sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ in E, then the limit $\lim_{n\to\infty} X^{(n)} \in E$.

(c) Let $r > 0$, then

(i)
$$\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\} \text{ is closed.}$$

(ii)
$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\} \text{ is open.}$$

(d) Any singleton $\{x_0\}$ where $x_0 \in X$ is closed.

(e) E is open iff $X \setminus E$ is closed.

(f) (i) If $E_1, \ldots, E_n$ are open sets in X, then $E_1 \cap E_2 \cap \cdots \cap E_n$ is open.
(ii) If $F_1, \ldots, F_n$ are closed, then $F_1 \cup \cdots \cup F_n$ is closed.

(g) (i) If $\{E_\alpha\}_{\alpha \in I}$ is any collection of open sets in X, then $\bigcup_{\alpha \in I} E_\alpha$ is open.
(ii) If $\{F_\alpha\}_{\alpha \in I}$ is any collection of closed sets in X, then $\bigcap_{\alpha \in I} F_\alpha$ is closed.

(h) (i) If $E \subseteq X$, then $\mathrm{Int}(E)$ is the largest open set that contained in E i.e. $\mathrm{Int}(E)$ is open and if $V \subseteq E$ and V is open, then $V \subseteq \mathrm{Int}(E)$.
(ii) If $E \subseteq X$, then $\overline{E}$ is the smallest closed set containing E i.e. $\overline{E}$ is closed and if $E \subseteq K$ and K is closed, then $\overline{E} \subseteq K$.

proof of (b).

($\Rightarrow$) Since E is closed, so $\overline{E} = E$, and we know every convergent sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ converges to x_0 with $x_0 \in \overline{E}$. Thus, we have $x_0 \in E$.

($\Leftarrow$) Assume that every convergent sequence in E has its limit in E. We want to prove that E is closed, i.e. that $X \setminus E$ is open.

Take any point $y \in X \setminus E$. Suppose, for contradiction, that every ball around y meets E. That is, for each $k \in \mathbb{N}$ there exists a point

$$x^{(k)} \in E \cap B\big(y, \tfrac{1}{k}\big).$$

Then, by construction, we have $x^{(k)} \to y$.

By our assumption, the limit of any convergent sequence from E must lie in E. Hence $y \in E$, contradicting the fact that $y \in X \setminus E$.

Therefore, there must exist some radius $r > 0$ such that

$$B(y, r) \cap E = \varnothing,$$

which means $B(y, r) \subseteq X \setminus E$. Thus every point of $X \setminus E$ is an interior point, so $X \setminus E$ is open. Hence E is closed.

■

proof of (c). If $y \in B(x_0, r)$, then $d(x_0, y) < r$. Let $\varepsilon = r - d(x_0, y) > 0$, then we claim that $B(y, \varepsilon) \subseteq B(x_0, r)$. Given $z \in B(y, \varepsilon)$, then $d(z, y) < \varepsilon$, then use triangle inequality we know $z \in B(x_0, r)$. Now we show that $\overline{B}(x_0, r)$ is closed. To show that $\overline{B}(x_0, r)$ is closed, it sufficies to show that $X \setminus \overline{B}(x_0, r)$ is open. Note that

$$X \setminus \overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) > r\}.$$

Let $y \in X \setminus \overline{B}(x_0, r)$, then define $\varepsilon = d(x_0, y) - r > 0$, then we can similarly prove that $B(y, \varepsilon) \subseteq X \setminus \overline{B}(x_0, r)$. Hence, $X \setminus \overline{B}(x_0, r) = \operatorname{Int}(X \setminus \overline{B}(x_0, r))$, and thus it is open. ■

proof of (d). It sufficies to show that $X \setminus \{x_0\}$ is open. Given $y \in X \setminus \{x_0\}$, so we can show that

$$B\left(y, \frac{d(y, x_0)}{2}\right) \subseteq X \setminus \{x_0\}.$$

Hence, $y \in \operatorname{Int}(X \setminus \{x_0\})$, and thus $X \setminus \{x_0\}$ is open. ■

proof of (f). Given $x_0 \in E_1 \cap E_2 \cap \cdots \cap E_n$, then $x_0 \in E_i$ for all $1 \le i \le n$. Thus, there exists $r_i > 0$ s.t.

$$B(x_0, r_i) \subseteq E_i \quad \text{for each } 1 \le i \le n.$$

Let $r = \min\{r_1, \ldots, r_n\} > 0$, then we know $B(x_0, r) \subseteq B(x_0, r_i) \subseteq E_i$ for all $1 \le i \le n$. Hence, $E_1 \cap \cdots \cap E_n$ is open. Now if $F_1, \ldots, F_n$ are closed, then $X \setminus F_1, \ldots, X \setminus F_n$ are open. Since we know $\bigcap_{i=1}^{n}(X \setminus F_i)$ is open, so $X \setminus (\bigcup_{i=1}^{n} F_i)$ is open, so $\bigcup_{i=1}^{n} F_i$ is closed. ■

proof of (g). Suppose $x_0 \in \bigcup_{\alpha \in I} E_\alpha$, then there exists $\mathcal{B} \in I$ s.t. $x_0 \in E_{\mathcal{B}}$. Now since $E_{\mathcal{B}}$ is open, so there exists $r_{x_0} > 0$ s.t.

$$B(x_0, r_{x_0}) \subseteq E_{\mathcal{B}} \subseteq \bigcup_{i \in \alpha} E_\alpha.$$

Also, $\bigcup_{\alpha \in I} E_\alpha$ is open. Hence,

$$\left(X \setminus \left(\bigcap_{\alpha \in I} F_i\right)\right) = \bigcup_{\alpha \in I} (X \setminus F_i)$$

is open, so we have $\bigcap_{\alpha \in I} F_\alpha$ is closed.

Remark 2.2.12.

(1) $\bigcap_{\alpha\in I} E_\alpha$ may NOT be open. For example,

$$\bigcap_{i=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)=\{0\},$$

which is closed.

(2) $\bigcup_{\alpha\in I} F_\alpha$ may NOT be closed. For example,

$$\bigcup_{i=1}^{\infty}\left[-1+\frac{1}{n},1-\frac{1}{n}\right]=(-1,1),$$

which is open.

■

proof of (h). Let $x \in \mathrm{Int}(E)$, then there exists $r_x > 0$ s.t. $B(x, r_x) \subseteq \mathrm{Int}(E)$ since $\mathrm{Int}(E)$ is open. Now if we have $V \subseteq E$ and V is open, then $y \in V$ implies there exists $s > 0$ s.t. $B(y, s) \subseteq V$, and thus $B(y, s) \subseteq E$ since $V \subseteq E$. Hence, we know $y \in \mathrm{Int(E)}$, and thus $V \subseteq \mathrm{Int}(E)$. ■

To show $\overline{E}$ is closed, it suffices to show that $X \setminus \overline{E}$ is open. Note that

$$\overline{E}=X\setminus \mathrm{Ext}(E)=X\setminus \underbrace{\mathrm{Int}(X\setminus E)}_{\text{open}},$$

so $\overline{E}$ is closed. Now if $E \subseteq K$ and K is closed, then $x \in \overline{E}$ means $B(x, r) \cap E \neq \varnothing$ for all $r > 0$. Hence, $B(x, r) \cap K \neq \varnothing$ since $E \subseteq K$, so $x \in \overline{K} = K$. Thus, $\overline{E} \subseteq K$.

Note 2.2.2. remember to write why $\mathrm{Int}(E)$ is open.

proof of $\mathrm{Int}(E)$ is open. Since for all $x \in \mathrm{Int}(E)$, $\exists r_x > 0$ s.t. $B(x, r_x) \subseteq E$, so

$$\mathrm{Int}(E)=\bigcup_{x\in \mathrm{Int}(E)} B(x,r_x),$$

and by (g) in Proposition 2.2.8, we know $\mathrm{Int}(E)$ is open. ■

Appendix