# Combinatorics I

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**Abstract**

The lecture note of Combinatorics I by Shagnik Das, where the NTU cool site is https://cool.ntu.edu.tw/courses/55532/.

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# Chapter 1

# Chatting

## Lecture 1

## 1.1 Prime Numbers

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**Theorem 1.1.1** (Euclid $\approx$ 300 BCE)**.** There are infinitely many primes.

**proof. (Saidak, 2006).**

- Every natural number has at least one prime factor.
- No prime divides $n$ and $n+1$, for any $n \in \mathbb{N}$.

Consider a sequence of pronic number

$$p_1 = 2,\ p_{n+1} = p_n(p_n + 1).$$

Then the number of prime factors of $p_n$ is strictly increasing in $n$: $p_{n+1}$ has all the factors of $p_n$ together with the (disstinct) ones of $p_n + 1$.

**Example.** $p_1 = 2, p_2 = 6, p_3 = 42, p_4 = 1806$, where the prime factors of them are $\{2\}$, $\{2,3\}, \{2,3,7\}, \{2,3,7,43\}$.

■

### 1.1.1 How many prime numbers are there?

**Definition 1.1.1.** We define

$$\pi(n) = |\{p : 1 \leq p \leq n : p \text{ is prime}\}|.$$

**Note.** By Saidak's proof, we know $\pi(p_n) \geq n$. In fact, $\pi(p_n) \geq \log_2 n$.

**Theorem 1.1.2** (Legendre, $\approx$ 1800 LE )**.**

$$\pi(n) \approx \frac{n}{\ln n} \Leftrightarrow \lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1$$

**Note.** Proven by Hadamard and independently de la Vallée Poussin(1896).

**Theorem 1.1.3** (Better Approximation).
Dirichlet: $\pi(x) \approx Li(x) = \int_2^x \frac{1}{\ln t} dt$ .
Known: $\pi(n) = Li(n) + O\left(ne^{-a\sqrt{\ln n}}\right)$
Believed: $\pi(n) = Li(n) + O\left(\sqrt{n} \ln n\right)$

# Chapter 2

# Elementary Counting Principles

Fundemental problem: Given a set $S$, and we want to determine $|S|$.

## 2.1 Sum Rule

**Theorem 2.1.1** (Sum Rule). If $S = \dot{\bigcup}_{i=1}^{k} S_i$, then $|S| = \sum_{i=1}^{k} |S_i|$.

**Note.** $\dot{\bigcup}$ means disjoint union.

**Example.** A drawer contains 8 pairs of yellow socks, 5 pairs of blue socks, and 3 pairs of red socks. How many socks are there in total.

**Informal proof.** $2 \times (8 + 5 + 3) = 32$. ■

**Proof.** Let $S$ be the set of socks in the drawer, then $S = \dot{\bigcup}_{p \in P} S_p$, where $P$ is the set of pairs of socks, and $S_p$ is the set of two socks in the pair where $p \in P$. By the sum rule,

$$|S| = \sum_{p \in P} |S_p| = \sum_{p \in P} 2 = 2|P| = 32.$$

$P = P_{\text{yellow}} \dot{\cup} P_{\text{blue}} \dot{\cup} P_{\text{red}}$. By the sum rule,

$$|P| = |P_{\text{yellow}}| + |P_{\text{blue}}| + |P_{\text{red}}| = 8 + 5 + 3 = 16.$$

■

**Note.** Sum rule is the basis for case analysis arguments. It needs two requirements:

- Cover each case.
- Cover each case exactly once.

**Example.** Counting subset of a general set.

**Notation.** If $X$ is a set, and $k \in \mathbb{N} \cup \{0\}$, then

$$\binom{X}{k} = \{T : \ T \subseteq X, \ |T| = k\}.$$

We define the binomial coefficient as

$$\binom{|X|}{k} = \left|\binom{X}{k}\right|.$$

i.e. Given $n \geq k \geq 0$, $\binom{n}{k}$ is the number of $k$-element subsets of a set of size $n$. ■

**Proposition 2.1.1** (Pascal's relation)**.** If $n \geq k \geq 1$, then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

**Proof. Proof.** Let $X$ be an $n$-element set (e.g. $X = [n] = \{1, 2, \ldots, n\}$), and let $S = \binom{X}{k} = \{T \subseteq X : |T| = k\}$. Then, by definition, $\binom{n}{k} = |S|$. For each $k$-element subset, we can ask: "Do you contain $n$?" Let

$$S_0 = \{T : T \subseteq X, n \notin T, |T| = k\},$$

and

$$S_1 = \{T : T \subseteq X, n \in T, |T| = k\}.$$

Then, $S = S_0 \uplus S_1$. By the sum rule, $|S| = |S_0| + |S_1|$. Observe that

$$\begin{aligned} S_0 &= \{T \subseteq [n], n \notin T, |T| = k\} \\ &= \{T \subseteq [n-1], |T| = k\}, \end{aligned}$$

so by definition,

$$|S_0| = \binom{|[n-1]|}{k} = \binom{n-1}{k}.$$

$$S_1 = \{T \subseteq [n], n \in T, |T| = k\}.$$

Let

$$S_1' = \{T' \subseteq [n-1], |T'| = k-1\},$$

then we know a bijection from $S_1$ to $S_1'$:

$$T \in S_1 \longleftrightarrow T \setminus \{n\} \in S_1'.$$

**Theorem 2.1.2** (bijection rule)**.** Given two sets $S$ and $S'$, if there is a bijection $f : S \to S'$, then $|S| = |S'|$.

By this rule, we know

$$|S_1| = |S_1'| = \binom{|[n-1]|}{k-1} = \binom{n-1}{k-1}.$$

Hence,

$$\binom{n}{k} = |S| = |S_0| + |S_1| = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

■

### 2.1.1 Pascal's Triangle

We can use Pascal's relation to compute $\binom{n}{k}$.

**Note.** Boundary case: $\binom{n}{0} = 1$, $\binom{n}{n} = 1$. Also, $\binom{n}{k} = 0$ for $k = -1, n+1$.

$$
\begin{tikzcd}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} \arrow[dr] & & \binom{1}{1} \arrow[dl] & & \\
 & \binom{2}{0} \arrow[dr] & & \binom{2}{1} \arrow[dl] \arrow[dr] & & \binom{2}{2} \arrow[dl] & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{tikzcd}
$$

## 2.2 Product Rule

**Theorem 2.2.1.** If $S = S_1 \times S_2 \times \cdots \times S_k = \{(x_1, x_2.., x_k), x_i \in S_i\}$, then $|S| = \prod_{i=1}^{k} |S_i|$.

**Proof.** Induction on $k$:
Base case: $k = 1$, trivial.
Induction step: seperate into cases bases on choice of $x_{k+1} \in S_{k+1}$. Let

$$S(x) = \{(x_1, \dots, x_{k+1}) \in S, x_{k+1} = x \in S_{k+1}\},$$

then

$$S = \bigcupdot_{x \in S_{k+1}} S(x) \to |S| = \sum_{x \in S_{k+1}} |S(x)|.$$

But $S(x) = S_1 \times S_2 \times \cdots \times \{x\}$, which is in bijection with $S_1 \times S_2 \times \cdots \times S_k$. By induction rule,

$$|S(x)| = |S_1 \times S_2 \times \cdots \times S_k| \quad \forall x$$

Hence,

$$\begin{aligned} |S| = \sum_{x \in S_{k+1}} |S(x)| &= \sum_{x \in S_{k+1}} |S_1 \times S_2 \times \dots S_k| \\ &= |S_1 \times S_2 \times \cdots \times S_k| \times |S_{k+1}| = |S_1| \times |S_2| \times \cdots \times |S_{k+1}|. \end{aligned}$$

∎

**Example.** Consider binary strings of length $n$.

**Proof.**

$$S = \{0,1\}^n \Rightarrow |S| = |\{0,1\}^n| = |\{0,1\}|^n = 2^n.$$

∎

**Definition 2.2.1** (Power Set). Given a finite set $X$, let $2^X$ denote the set of all subsets of $X$ (also denoted $\mathcal{P}(x)$), which is called the power set.

**Corollary 2.2.1.** $\left|2^X\right| = 2^{|X|}$.

**Proof.** Without lose of generality, $X = [n]$. We build a bijection between $2^{[n]}$ and the set of binary string of length $n$. Suppose for every $T \in 2^{[n]}$, we have $\chi_T = (x_1, x_2, \dots, x_n)$, where

$$x_i = \begin{cases} 1, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases}$$

Then,

$$\left|2^{[n]}\right| = |\{0,1\}^n| = 2^n.$$

∎

## 2.3 Double-Counting argument

If we count a set in two different ways, the answer should be equal.

**Example.** Count $2^{[n]}$.

**Proof.**

1. Product rule $\to 2^n$.
2. Use the sum rule, split the subsets by size.

$$2^{[n]} = \binom{[n]}{0} \cup \binom{[n]}{1} \cup \ldots \cup \binom{[n]}{n}$$

Hence, we have the following proposition:

**Proposition 2.3.1.** For all $n \geq 0$,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

⊛

## 2.4 Permutations

## Lecture 2

**As previously seen.** Instead of choosing the subsets all at once, we could pick one element at a time, then we can try to use product rule. 5 Sep. 13:10

**Example.** Consider

$$\binom{[10]}{3}.$$

**Proof.** At the choice of the first element, we have 10 choices, the second one has 9 choisce, while the third one has 8 choice, but we didn't consider the order of each picked elements. ⊛

**Definition 2.4.1.** Given a set $X$ and $k \in \mathbb{N} \cup \{0\}$, a $k$-permutation of $X$ is

- an ordered choice of $k$ distinct elements from $X$.
- a $k$-tuple $(x_1, x_2, \ldots, x_k)$ with $x_i \in X$ and $x_i \neq x_j$ for each $i \neq j$.
- an injection $f : [k] \to X$.

where these 3 statements are equivalent.

**Notation.** $X^{\underline{k}} = \{k\text{-permutation of } X\} \subseteq X^k$ where $X^k = X \times X \times \cdots \times X$ allows repitition of the elements but $X^{\underline{k}}$ don't allow repitition.

**Note.** If $|X| = n$, then

$$n^{\underline{k}} = \left|X^{\underline{k}}\right|.$$

**Definition 2.4.2.**

- a $n$-permutation is a $n$-permutation of $[n]$.
- a $X$-permutation is a $|X|$-permutation of $X$.

**Theorem 2.4.1** (Generalized Product Rule). Suppose we are enumerating $S$, and can uniquely determine an element $s \in S$ through a series of $k$ questions, if $i$-th problem always has $n_i$ possible outcomes, independently to the permutation, then

$$|S| = n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i$$

**Proof.** Can make a bijection from $S$ to

$$[n_1] \times [n_2] \times \cdots \times [n_k].$$

Map each element in $S$ to the index of its answer in the series of answer.

Our moral is when counting we don't care about what the options are but only how many options. ∎

**Proposition 2.4.1.**

$$\begin{aligned} n^{\underline{k}} &= n(n-1)\dots(n-(k-1)) \\ &= \prod_{i=0}^{k-1}(n-i) = \frac{n!}{(n-k)!}. \end{aligned}$$

**Proof.** Use the generalized product rule.

Question $i$: What is the $i$-th element in the $k$-permutation of $[n]$?

We can choose anything except what we're alreafy chosen, so there are $i-1$ forbidden choices and thus there are $n-(i-1)$ possible choices. ∎

**Proposition 2.4.2.** For all $0 \leq k \leq n$,

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k^{\underline{k}}} = \frac{\left(\frac{n!}{(n-k)!}\right)}{k!} = \frac{n!}{k!(n-k)!}.$$

**Proof.** Double-count $[n]^{\underline{k}}$ i.e. $k$-permutation of $[n]$.

- Direct counting $\left|[n]^{\underline{k}}\right| = n^{\underline{k}}$.
- First choose the $k$ elements to appear in the $k$-permutation, $\binom{n}{k}$ options, then choose the order in which they appear, $k^{\underline{k}}$ options.

  Then, by the generalized product rule, the number of $k$-permutation of $[n]$ is $\binom{n}{k} \cdot k^{\underline{k}}$.

Hence,

$$n^{\underline{k}} = \left|[n]^{\underline{k}}\right| = \binom{n}{k} \cdot k^{\underline{k}}.$$

∎

**Corollary 2.4.1.** We can then use this result to reprove Pascal's Property again.

**Proof.** ∎

**Exercise.** 6 players at the tennis club want to have three matches involving all the players? How many ways can we arrange the games.

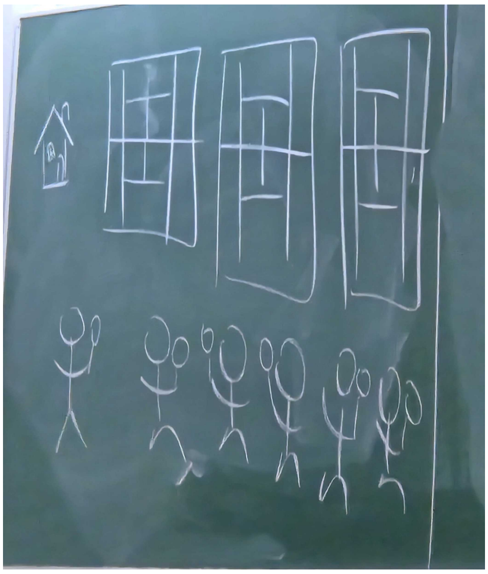

Figure 2.1: Tennis Games

**Proof.** We only care about who plays against whom, not about which court or who versus first, e.t.c.

The arrangement of games is a set of three disjoint pairs of players.

$$\{\{1,2\},\{3,4\},\{5,6\}\} \neq \{\{1,3\},\{2,4\},\{5,6\}\}.$$

Double-count the arrangements of games where counts do matter.

- Choose a pair of players for Court A: $\binom{6}{2}$
- Choose a pair of players for Court B: $\binom{4}{2}$
- Choose a pair of players for Court C: $\binom{2}{2}$

Generalized product rule tells

$$\text{number of choices} = \binom{6}{2}\binom{4}{2}\binom{2}{2} = 90.$$

Second count: First gets a set of 3 pairs, say there are $x$ possibilities , and assign the three pairs to 3 courts, so there are 3! , so $x \cdot 3! = 90$, and thus $x = \frac{90}{3!} = 15$. ■

## Lecture 3

Actually we have an alternitive prove: 9 Sep. 15:30

**proof by direct computation.**

- Q1: Who's the opponent for the 1-st player? There are 5 choices.
- Q2: Who plays the next lowest numbered player? There are 3 choices.

The left 2 players are the opponents to each other. Hence, there are $3 \times 5 = 15$possible pairings. ■

More generally, if we have $n = 2k$ players to pair up, then the first proof gives there are

$$\frac{\binom{n}{2}\binom{n-2}{2}\dots\binom{2}{2}}{\left(\frac{n}{2}\right)!}$$

possible pairings, while the second proof gives that there are

$$(n-1)\cdot(n-3)\cdot(n-5)\dots \coloneqq (n-1)!! \neq ((n-1)!)!.$$

By this, we know these two numbers must be equal, or more rigorously, we can write

$$\frac{\binom{n}{2}\binom{n-2}{2}\dots\binom{2}{2}}{\left(\frac{n}{2}\right)!} = 2^n \cdot \frac{\frac{n(n-1)}{2}\,\frac{(n-2)(n-3)}{2}\dots}{n(n-2)(n-4)\dots 2} = (n-1)\cdot(n-3)\cdot\dots$$

**Example.** How mant shortest routes on the grid are there from $(0,0)$ to $(n,m)$?

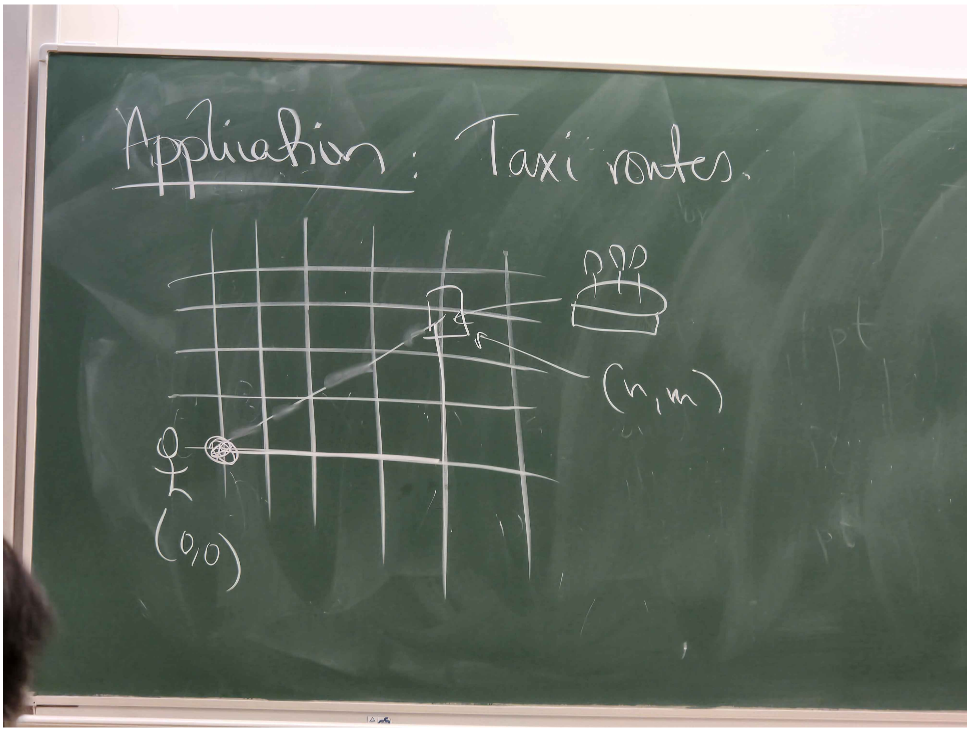


Figure 2.2: Taxi routes

**Proof.** Shortest route is of length $n+m$,$m$ up-steps and $n$ right-steps. We can think of a shortest route to be a binary string of length $n+m$ with $n$ 1s and $m$ 0s, so we want to count how many such binary strings are there. Choose $n$ of them to be 1s, while the other are 0s. Hence, there are $\binom{n+m}{n}$ possibilities. ⊛

## 2.5 Binomial Theorem

**Theorem 2.5.1** (Binomial Theorem). For any $n \in \mathbb{N} \cup \{0\}$, and $x, y \in \mathbb{R}$, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

**Example.** $(x+y)^0 = 1 = \sum_{k=0}^{0} x^k y^{0-k}$.

**Example.** $(x+y)^1 = x+y$, while

$$\sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x.$$

**proof of binomial theorem.**

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)\dots(x+y)}_{n \text{ factors}}$$

From each factor, we pick a term $x$ or $y$, multuply chosen factors together. If we choose $k$ $x$'s, then we must choose $n-k$ $y$'s, so the monomial is $x^k y^{n-k}$, where the coefficient of $x^k y^{n-k}$ is the number of ways of choosing $k$ $x$'s. Also, the possible monomials are $x^k y^{n-k}$ for $k = 0, 1, 2, \dots, n$. Hence, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

■

We can use this formula to derive identities for the binomial coefficients, by plugging in values for $x$ and $y$.

**Example.** $x = 1, y = 1$.

**Proof.**

$$2^n = (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

⊛

**Example.** $y = -1, x = 1$.

**Proof.**

$$(x+y)^n = (-1+1)^n = 0^n = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \sum_{2 \mid k} \binom{n}{k} - \sum_{2 \nmid k} \binom{n}{k}$$

⊛

**Corollary 2.5.1.**

$$\sum_{2 \mid k} \binom{n}{k} = \sum_{2 \nmid k} \binom{n}{k}$$

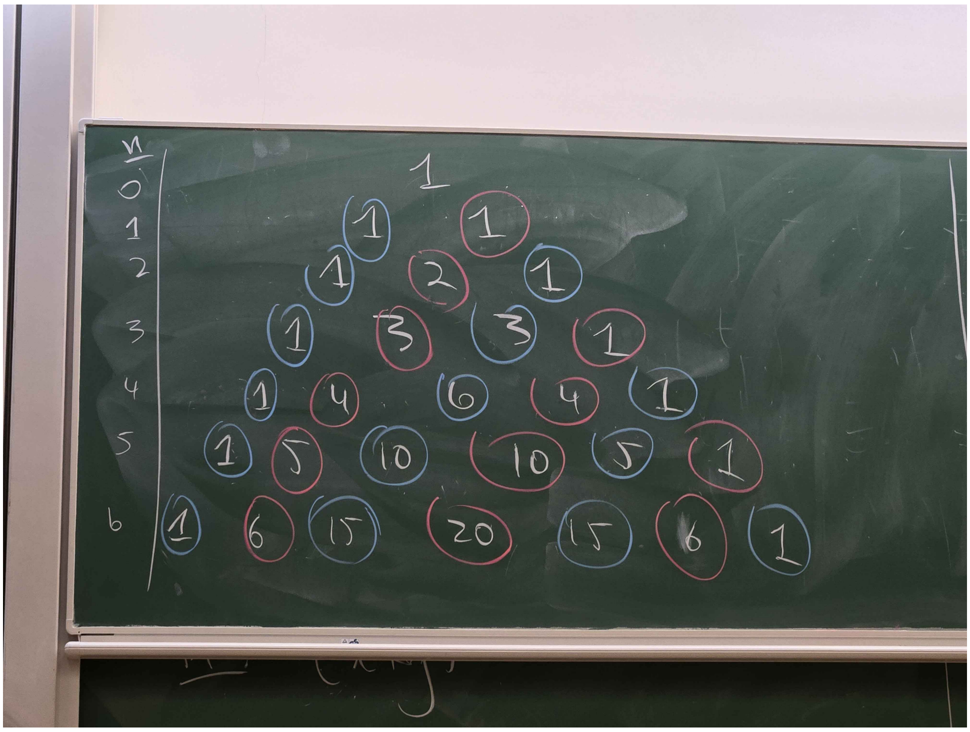


Figure 2.3: The sum of even terms is equal to the sum of odd terms.

**Theorem 2.5.2.** $\forall n \geq k$, we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

**Proof.**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-(n-k))!} = \binom{n}{n-k}.$$

**Remark.** Choosing a subset of $k$ elements from $n$ is equivalent to choose $n-k$ elements to discard, and we can build a bijection between these two methods.

■

For $n$ even.
Consider the bijection

$$S \mapsto S\triangle\{n\} = \begin{cases} S - \{n\}, & \text{if } n \in S; \\ S \cup \{n\}, & \text{if } n \notin S. \end{cases}$$

Hence,

$$|S\triangle\{n\}| \subseteq \{|S|-1, |S|+1\},$$

so if $|S|$ is odd, then $S\triangle\{n\}$ is even, and vice versa. We know this is a bijection (self-inverse), so we have odd-sized sets to even-sized set. Hence, $\sum_{2|k}\binom{n}{k} = \sum_{2\nmid k}\binom{n}{k}$.

**Example.** $x = 2, y = 1$.

**Proof.**

$$(2+1)^n = 3^n = \sum_{k=0}^{n}\binom{n}{k}2^k$$

Counting partitions $[n] = A \cup B \cup C$, each element has a choice of 3 sets to go into. Hence, the product rule says there are $3^n$ partitions, while RHS uses sum rule bases on $k = |A \cup B|$. ⊛

## 2.6 Divisor Function

**Definition 2.6.1** (Divisor Functions). Given a natural number $n \in \mathbb{N}$, let $d(n)$ count the number of divisors of $n$.

**Example.**

$$\begin{aligned} d(1) &= 1 = |\{1\}| \\ d(2) &= 2 = |\{1,2\}| \\ d(3) &= 2 = |\{1,3\}| \\ d(4) &= 3 = |\{1,2,4\}| \\ d(5) &= 2 = |\{1,5\}|. \end{aligned}$$

**Corollary 2.6.1.** $d(n) = 2$ if and only if $n$ is a prime.

Now we want to compute the average value of $d(n)$.

**Definition 2.6.2.**

$$\bar{d}(n) = \frac{\sum_{i=1}^{n} d(i)}{n}.$$

We can use double-counting. First, notice that

$$d(i) = \sum_{\substack{j \in [i] \\ j \mid i}} 1.$$

Hence,

$$\sum_{i=1}^{n} d(i) = \sum_{i=1}^{n} \sum_{\substack{j \in [i] \\ j \mid i}} 1.$$

We can exchange the order of summation:

$$n\bar{d}(n) = \sum_{i=1}^{n} d(i) = \sum_{i=1}^{n} \sum_{j : j \mid i} 1 = \sum_{j=1}^{n} \sum_{\substack{j \in [n] \\ j \mid i}} 1.$$

For fixed $j$, we know

$$\sum_{\substack{i \in [n] \\ j \mid i}} 1 = \left\lfloor \frac{n}{j} \right\rfloor.$$

Hence, we have

$$n\bar{d}(n) = \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor,$$

which is equivalent to

$$\bar{d}(n) = \frac{1}{n} \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor.$$

Observe that

$$\frac{n}{j} - 1 \leq \left\lfloor \frac{n}{j} \right\rfloor \leq \frac{n}{j},$$

so

$$H_n - 1 = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{n}{j} - 1\right) \leq \overline{d}(n) \leq \frac{1}{n}\sum_{j=1}^{n}\frac{n}{j} = \sum_{j=1}^{n}\frac{1}{j} = H_n \approx \ln n.$$

Hence,

$$H_n - 1 \leq \overline{d}(n) \leq H_n,$$

which gives $\overline{d}(n) \sim \ln n$.

# Chapter 3

# Partitions

How many ways can we divide $n$ items into $k$ groups? Need to specify details to get well-posed questions.

1. Items distinguishable or not?
2. Groups distinguishable or not?
3. Can we have empty groups? Can we have group with more than one item?

**Example.** Professors has 49 students, to distribute 3000% between the students.

**Proof.** Indistinguishable items: percentage points.
Distinguishable groups: students $k = 49$. No restriction on sizes of groups. Formally, we are enumerating

$$S = \left\{ (x_1, x_2, \ldots, x_{49}) \mid x_i \geq 0, x_i \in \mathbb{Z}, \sum_{i=1}^{49} x_i = 3000 \right\}$$

⊛

# Appendix