# Introduction to Analysis I HW9

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**Problem 0.0.1** (15pts). Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R$. Let $S_n = \sum_{k=0}^{n} a_k$ be the partial sums of $\sum a_n$. Denote the radius of convergence of $\sum_{n=0}^{\infty} S_n x^n$ by $r$.

(1) Show that $r \leq R$.

(2) Show that $\min\{1, R\} \leq r$. *Hint: The power series $\sum_{n=0}^{\infty} S_n x^n$ can be seen as the Cauchy product between $\sum_{n=0}^{\infty} a_n x^n$ and a specific power series that you need to choose,).*

**Problem 0.0.2** (30pts). For each real $t$, define

$$f_t(x) = \begin{cases} \dfrac{xe^{xt}}{e^x - 1}, & x \in \mathbb{R},\ x \neq 0, \\ 1, & x = 0. \end{cases}$$

(a) Show that there exists $\delta > 0$ such that $f_t$ admits a power series expansion in $x$ for all $|x| < \delta$.

*Hint.* Write

$$f_t(x) = e^{xt} g(x)$$

Where

$$g(x) = \begin{cases} \dfrac{x}{e^x - 1}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Both $e^{xt}$ and $g(x)$ are analytic near 0. Also $g(x) = \frac{1}{h(x)}$ where $h(x) = \dfrac{e^x - 1}{x}$ for $x \neq 0$ and we can express it as an power series in $x$. Then may use the fact that if $h$ is analytic on $\mathbb{R}$ and $h(0) \neq 0$, then $1/h$ is analytic on a smaller interval $(-\delta, \delta)$.

(b) Define $P_0(t), P_1(t), P_2(t), \ldots$ by the equation

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!}, \qquad x \in (-\delta, \delta),$$

and use the identity

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}$$

to prove that

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0)\, t^{n-k}.$$

(Hint: $f_t(x) = e^{tx} f_0(x)$ and $f_0(x) = g(x)$.) This shows that each function $P_n$ is a polynomial. These are the *Bernoulli polynomials*. The numbers $B_n := P_n(0)$ $(n = 0, 1, 2, \ldots)$ are called the *Bernoulli numbers*. Derive the following further properties:

(c) $B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0,$ if $n = 2, 3, \ldots$

(d) $P_n'(t) = n\, P_{n-1}(t), \quad$ if $n = 1, 2, \ldots$

(e) $P_n(t+1) - P_n(t) = n\, t^{n-1}, \quad$ if $n = 1, 2, \ldots$

(f) $P_n(1-t) = (-1)^n P_n(t)$

(g) $B_{2n+1} = 0, \quad$ if $n = 1, 2, \ldots$

(h)

$$1^n + 2^n + \cdots + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}, \qquad (n = 2, 3, \ldots).$$

**Problem 0.0.3** (15pts **Exercise 4.2.7**). Show that for every integer $n \geq 3$, we have

$$0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{n!}.$$

*(Hint: first show that $(n+k)! > 2^k n!$ for all $k = 1, 2, 3, \ldots$.)* Conclude that $n!e$ is not an integer for every $n \geq 3$. Deduce from this that $e$ is irrational. *(Hint: prove by contradiction.)*

**Proof.** For $n \geq 3$, we know

$$(n+k)! = (n+k)(n+k-1)\ldots(n+1)n! > 2^k$$

for all $k = 1, 2, 3, \ldots$ since $n + i > 2$ for all $1 \leq i \leq k$. Hence,

$$\sum_{k=1}^{\infty} \frac{1}{(n+k)!} < \sum_{k=1}^{\infty} \frac{1}{2^k n!} = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{n!} \left( \frac{\frac{1}{2}}{1 - \frac{1}{2}} \right) = \frac{1}{n!},$$

and since $\frac{1}{(n+k)!} > 0$ for all $k = 1, 2, 3, \ldots$, so

$$0 < \sum_{k=1}^{\infty} \frac{1}{(n+k)!} < \frac{1}{n!}.$$

Thus, we know

$$\begin{aligned} n!e = n! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots \right) &= n! \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!} + \sum_{k=1}^{\infty} \frac{1}{(n+k)!} \right) \\ &< n! \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!} + \frac{1}{n!} \right) = n! \sum_{m=0}^{n} \frac{1}{m!} + 1. \end{aligned}$$

Also, we know

$$n!e = n! \sum_{m=0}^{\infty} \frac{1}{m!} > n! \sum_{m=0}^{n} \frac{1}{m!},$$

so

$$n! \sum_{m=0}^{n} \frac{1}{m!} < n!e < n! \sum_{m=0}^{n} \frac{1}{m!} + 1,$$

which means $n!e$ is not an integer. Now if $e$ is rational, then $e = \frac{q}{p}$ for some $q \in \mathbb{Z}$ and $p \in \mathbb{N}$, so $n!e = \frac{n!q}{p}$, and if we pick $n = \max\{3, p\}$, then we know $\frac{n!q}{p}$ is an integer since $p \mid n!$, but $\frac{n!q}{p} = n!e$ is not an integer, so it is a contradiction, and thus $e$ is irrational. ∎

**Problem 0.0.4** (10pts **Exercise 4.5.6**). Prove that the natural logarithm function $\ln x$ is real analytic on $(0, +\infty)$. Hint: For any $a > 0$, consider the change of variable $y = x - a$.

**Problem 0.0.5** (10pts **Exercise 4.5.7**). Let $f : (0, \infty) \to \mathbb{R}$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that $f(x) = Ce^x$ for some positive constant $C$; justify your reasoning. *(Hint: there are basically three different proofs available. One proof uses the logarithm function, another proof uses the function $e^{-x}$, and a third proof uses power series. Of course, you only need to supply one proof.)*

**Problem 0.0.6** (10pts **Exercise 4.5.8**). Let $m > 0$ be an integer. Prove

$$\lim_{x \to +\infty} \frac{e^x}{x^m} = +\infty.$$

without using the L'Hopital's rule.
*(Hint: $e^x \geq \sum_{k=0}^{m+1} \frac{x^k}{k!}$ for $x > 0$.)*

**Proof.** Note that for $x > 0$,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} > \sum_{k=0}^{m+1} \frac{x^k}{k!}.$$

Thus,

$$\frac{e^x}{x^m} \geq \left( \sum_{k=0}^{m} \frac{1}{x^{m-k} k!} \right) + \frac{x}{(m+1)!} > \frac{x}{(m+1)!}.$$

Since $\lim_{x\to+\infty} \frac{x}{(m+1)!} = +\infty$, so $\lim_{m\to+\infty} \frac{e^x}{x^m} = +\infty$.

∎

**Problem 0.0.7** (10pts **Exercise 4.5.9**). Let $P(x)$ be a polynomial, and let $c > 0$. Show that there exists a real number $N > 0$ such that $e^{cx} > |P(x)|$ for all $x > N$; thus an exponentially growing function, no matter how small the growth rate $c$, will eventually overtake any given polynomial $P(x)$, no matter how large. *(Hint: use Exercise 4.5.8.)*

**Proof.** If $P(x)$ is a constant polynomial, then there exists $N > 0$ s.t. $x > N$ implies $e^{cx} > |P(x)|$ since $e^{cx}$ is strictly increasing and has no upper bound. Now suppose $P(x) = a_m x^m + \cdots + a_1 x + a_0$ for some $m \geq 1$, then by Problem 6 we know for all $m \in \mathbb{N}$ we have

$$\lim_{cx\to\infty} \frac{e^{cx}}{c^m x^m} = \infty,$$

so for all $i = 1, 2, \ldots, m$, there exists $N_i > 0$ s.t. $cx \geq N_i$ implies

$$\frac{e^{cx}}{c^i x^i} > (m+1) \left| \frac{a_i}{c^i} \right| \Leftrightarrow e^{cx} > (m+1)|a_i||x^i|.$$

Also, we know there exists $N_0 > 0$ s.t. $x > N_0$ implies $e^{cx} > (m+1)|a_0|$ since $e^{cx}$ is strictly increasing and has no upper bound. Hence, we know $x > \max\left\{N_0, \left\lceil \frac{N_1}{c} \right\rceil, \left\lceil \frac{N_2}{c} \right\rceil, \ldots, \left\lceil \frac{N_m}{c} \right\rceil\right\}$ implies

$$(m+1) \cdot e^{cx} > \sum_{i=0}^{m} (m+1)|a_i||x^i| = (m+1)\left|P(x)\right|,$$

which means

$$e^{cx} > |P(x)|,$$

so we're done. ∎