

Introduction to Analysis I HW9

B13902024 張沂魁

November 15, 2025

Problem 0.0.1 (15pts). Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R . Let $S_n = \sum_{k=0}^n a_k$ be the partial sums of $\sum a_n$. Denote the radius of convergence of $\sum_{n=0}^{\infty} S_n x^n$ by r .

- (1) Show that $r \leq R$.
- (2) Show that $\min\{1, R\} \leq r$. *Hint: The power series $\sum_{n=0}^{\infty} S_n x^n$ can be seen as the Cauchy product between $\sum_{n=0}^{\infty} a_n x^n$ and a specific power series that you need to choose.*

Problem 0.0.2 (30pts). For each real t , define

$$f_t(x) = \begin{cases} \frac{x e^{xt}}{e^x - 1}, & x \in \mathbb{R}, x \neq 0, \\ 1, & x = 0. \end{cases}$$

- (a) Show that there exists $\delta > 0$ such that f_t admits a power series expansion in x for all $|x| < \delta$.

Hint. Write

$$f_t(x) = e^{xt} g(x)$$

Where

$$g(x) = \begin{cases} \frac{x}{e^x - 1}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Both e^{xt} and $g(x)$ are analytic near 0. Also $g(x) = \frac{1}{h(x)}$ where $h(x) = \frac{e^x - 1}{x}$ for $x \neq 0$ and we can express it as an power series in x . Then may use the fact that if h is analytic on \mathbb{R} and $h(0) \neq 0$, then $1/h$ is analytic on a smaller interval $(-\delta, \delta)$.

- (b) Define $P_0(t), P_1(t), P_2(t), \dots$ by the equation

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!}, \quad x \in (-\delta, \delta),$$

and use the identity

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}$$

to prove that

$$P_n(t) = \sum_{k=0}^n \binom{n}{k} P_k(0) t^{n-k}.$$

(Hint: $f_t(x) = e^{tx} f_0(x)$ and $f_0(x) = g(x)$.) This shows that each function P_n is a polynomial. These are the *Bernoulli polynomials*. The numbers $B_n := P_n(0)$ ($n = 0, 1, 2, \dots$) are called the *Bernoulli numbers*. Derive the following further properties:

- (c) $B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \text{ if } n = 2, 3, \dots$
- (d) $P'_n(t) = n P_{n-1}(t), \quad \text{if } n = 1, 2, \dots$
- (e) $P_n(t+1) - P_n(t) = n t^{n-1}, \quad \text{if } n = 1, 2, \dots$
- (f) $P_n(1-t) = (-1)^n P_n(t)$
- (g) $B_{2n+1} = 0, \quad \text{if } n = 1, 2, \dots$
- (h)

$$1^n + 2^n + \dots + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}, \quad (n = 2, 3, \dots).$$

Problem 0.0.3 (15pts **Exercise 4.2.7**). Show that for every integer $n \geq 3$, we have

$$0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{n!}.$$

(Hint: first show that $(n+k)! > 2^k n!$ for all $k = 1, 2, 3, \dots$) Conclude that $n!e$ is not an integer for every $n \geq 3$. Deduce from this that e is irrational. (Hint: prove by contradiction.)

Problem 0.0.4 (10pts **Exercise 4.5.6**). Prove that the natural logarithm function $\ln x$ is real analytic on $(0, +\infty)$. Hint: For any $a > 0$, consider the change of variable $y = x - a$.

Problem 0.0.5 (10pts **Exercise 4.5.7**). Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that $f(x) = Ce^x$ for some positive constant C ; justify your reasoning. (Hint: there are basically three different proofs available. One proof uses the logarithm function, another proof uses the function e^{-x} , and a third proof uses power series. Of course, you only need to supply one proof.)

Problem 0.0.6 (10pts **Exercise 4.5.8**). Let $m > 0$ be an integer. Prove

$$\lim_{x \rightarrow +\infty} \frac{e^x}{x^m} = +\infty.$$

without using the L'Hopital's rule.

(Hint: $e^x \geq \sum_{k=0}^{m+1} \frac{x^k}{k!}$ for $x > 0$.)

Problem 0.0.7 (10pts **Exercise 4.5.9**). Let $P(x)$ be a polynomial, and let $c > 0$. Show that there exists a real number $N > 0$ such that $e^x > |P(x)|$ for all $x > N$; thus an exponentially growing function, no matter how small the growth rate c , will eventually overtake any given polynomial $P(x)$, no matter how large. (Hint: use Exercise 4.5.8.)