Introduction to Mathematical Analysis
Homework 9 Due November 21 (Friday), 2025
Please submit your homework online in PDF format.

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1. (15 pts) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Let $S_n = \sum_{k=0}^{n} a_k$ be the partial sums of $\sum a_n$. Denote the radius of convergence of $\sum_{n=0}^{\infty} S_n x^n$ by r.

   (1) Show that $r \le R$.

   (2) Show that $\min\{1, R\} \le r$. *Hint: The power series $\sum_{n=0}^{\infty} S_n x^n$ can be seen as the Cauchy product between $\sum_{n=0}^{\infty} a_n x^n$ and a specific power series that you need to choose,).*

2. (30 pts) For each real t, define

$$f_t(x) = \begin{cases} \dfrac{xe^{xt}}{e^x - 1}, & x \in \mathbb{R},\ x \neq 0, \\ 1, & x = 0. \end{cases}$$

   (a) Show that there exists $\delta > 0$ such that f_t admits a power series expansion in x for all $|x| < \delta$.

   Hint. Write

$$f_t(x) = e^{xt} g(x)$$

   Where

$$g(x) = \begin{cases} \dfrac{x}{e^x - 1}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

   Both e^{xt} and $g(x)$ are analytic near 0. Also $g(x) = \frac{1}{h(x)}$ where $h(x) = \dfrac{e^x - 1}{x}$ for $x \neq 0$ and we can express it as an power series in x. Then may use the fact that if h is analytic on $\mathbb{R}$ and $h(0) \neq 0$, then $1/h$ is analytic on a smaller interval $(-\delta, \delta)$.

   (b) Define $P_0(t), P_1(t), P_2(t), \ldots$ by the equation

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!}, \qquad x \in (-\delta, \delta),$$

   and use the identity

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}$$

   to prove that

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0)\, t^{n-k}.$$

   (Hint: $f_t(x) = e^{tx} f_0(x)$ and $f_0(x) = g(x)$.) This shows that each function P_n is a polynomial. These are the *Bernoulli polynomials*. The numbers $B_n := P_n(0)$ $(n = 0, 1, 2, \ldots)$ are called the *Bernoulli numbers*. Derive the following further properties:

   (c) $B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$, if $n = 2, 3, \ldots$

   (d) $P_n'(t) = n\, P_{n-1}(t)$, if $n = 1, 2, \ldots$

   (e) $P_n(t+1) - P_n(t) = n\, t^{n-1}$, if $n = 1, 2, \ldots$

   (f) $P_n(1 - t) = (-1)^n P_n(t)$

   (g) $B_{2n+1} = 0$, if $n = 1, 2, \ldots$

   (h)

$$1^n + 2^n + \cdots + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}, \qquad (n = 2, 3, \ldots).$$

3. (15 pts) **Exercise 4.2.7.** Show that for every integer $n \geq 3$, we have

$$0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{n!}.$$

(Hint: first show that $(n+k)! > 2^k n!$ for all $k = 1, 2, 3, \ldots$.) Conclude that $n!e$ is not an integer for every $n \geq 3$. Deduce from this that e is irrational. *(Hint: prove by contradiction.)*

4. (10 pts) **Exercise 4.5.6** Prove that the natural logarithm function $\ln x$ is real analytic on $(0, +\infty)$. Hint: For any $a > 0$, consider the change of variable $y = x - a$.

5. (10 pts)

   Exercise 4.5.7 Let $f : (0, \infty) \to \mathbb{R}$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that $f(x) = Ce^x$ for some positive constant C; justify your reasoning. *(Hint: there are basically three different proofs available. One proof uses the logarithm function, another proof uses the function e^{-x}, and a third proof uses power series. Of course, you only need to supply one proof.)*

6. (10 pts) **Exercise 4.5.8** Let $m > 0$ be an integer. Prove

$$\lim_{x \to +\infty} \frac{e^x}{x^m} = +\infty.$$

   without using the L'Hopital's rule.

   (Hint: $e^x \geq \sum_{k=0}^{m+1} \frac{x^k}{k!}$ for $x > 0$.)

7. (10 pts) **Exercise 4.5.9** Let $P(x)$ be a polynomial, and let $c > 0$. Show that there exists a real number $N > 0$ such that $e^x > |P(x)|$ for all $x > N$; thus an exponentially growing function, no matter how small the growth rate c, will eventually overtake any given polynomial $P(x)$, no matter how large. *(Hint: use Exercise 4.5.8.)*

You can do the following problems to practice. You don't have to submit the following problems.

1. **Exercise 4.5.4** Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by setting $f(x) := \exp(-1/x)$ when $x > 0$, and $f(x) := 0$ when $x \leq 0$. Prove that f is infinitely differentiable, and $f^{(k)}(0) = 0$ for every integer $k \geq 0$, but that f is *not* real analytic at 0.

2. In class, we proved that the function $f(x) = a^x$ is continuous on $\mathbb{Q}$ for $a > 1$. Let $n \in \mathbb{N}$. Prove that f is uniformly continuous on the rational interval

$$[-n, n] \cap \mathbb{Q}.$$

   Remark. If this is true, then $f(x) = a^x$ admits a unique continuous extension to all real numbers $x \in [-n, n]$.

3. Define the sequence

$$\forall n \geq 1, \qquad S_n = \sum_{k=1}^{n} \ln k.$$

   (1) Show that for every $k \geq 2$, we have

$$\int_{k-1}^{k} \ln t \, dt \;\leq\; \ln k \;\leq\; \int_{k}^{k+1} \ln t \, dt.$$

   Deduce that

$$S_n = n \ln n - n + o(n).$$

(2) By considering the sequence $(A_n)_{n\geq 1}$, defined by

$$\forall n \geq 1, \qquad A_n = S_n - n\ln n + n,$$

show that $A_n - A_{n-1} \sim \dfrac{1}{2n}$ and deduce that

$$A_n \sim \frac{1}{2}\ln n.$$

(3) Let

$$D_n := S_n - n\ln n + n - \frac{1}{2}\ln n \qquad \text{for } n \geq 1.$$

Show that

$$D_n - D_{n-1} \sim -\frac{1}{12n^2}.$$

(4) Show that D_n converges to some D_∞ when $n \to \infty$. Deduce that there exists some constant $C > 0$ such that

$$n! \sim C\left(\frac{n}{e}\right)^n \sqrt{n}.$$

(5) Using the expression of $I_{2n} = \int_0^{\pi/2} \sin^{2n} x\, dx = \frac{\pi}{2} \cdot \frac{(2n)!}{2^{2n}(n!)^2} = \sqrt{\frac{\pi}{4n}}\,(1+o(1))$ (proved in the following), show that

$$C = \sqrt{2\pi}.$$

(6) Show that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + o\left(\frac{1}{n}\right)\right).$$

4. Let $\mathcal{P}$ be the set of all the primes. In this exercise, we will prove that $\displaystyle\sum_{p\in\mathcal{P}} \frac{1}{p}$ is divergent.

(1) Show that for $s > 1$, we have

$$-\sum_{p\in\mathcal{P}} \log\left(1 - \frac{1}{p^s}\right) = \log\zeta(s).$$

(2) Deduce that there exists $M > 0$ such that for any $s > 1$, we have

$$\left|\sum_{p\in\mathcal{P}} \frac{1}{p^s} - \log\zeta(s)\right| < M.$$

(3) Show that as $s \to 1^+$, we have $\zeta(s) \to +\infty$.

(4) Conclude that

$$\sum_{p\in\mathcal{P}} \frac{1}{p}$$

is divergent.

Theorem (Wallis Integrals — Factorial Version)**.** For each integer $n \geq 0$, define

$$I_n := \int_0^{\pi/2} \sin^n x\, dx.$$

Then:

(a)

$$I_0 = \frac{\pi}{2}, \qquad I_1 = 1.$$

(b) For all $n \geq 2$,

$$nI_n = (n-1)I_{n-2}.$$

(c) For each $m \in \mathbb{N}$,

$$I_{2m-1} = \frac{2^{2m-1}(m-1)!\,m!}{(2m)!}, \qquad I_{2m} = \frac{\pi}{2} \cdot \frac{(2m)!}{2^{2m}(m!)^2}.$$

(d) For all $n \geq 1$,

$$I_n I_{n-1} = \frac{\pi}{2n}.$$

(e) As $n \to \infty$,

$$I_n = \sqrt{\frac{\pi}{2n}}\,(1 + o(1)).$$

(f) In particular,

$$I_{2n} = \frac{\pi}{2} \cdot \frac{(2n)!}{2^{2n}(n!)^2}.$$

Proof. **(a) Direct computation.** We directly compute:

$$I_0 = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}, \qquad I_1 = \int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = 1.$$

(b) Recurrence formula.
For $n \geq 2$,

$$I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x\,dx.$$

Using $u = \sin^{n-1} x$ and $dv = \sin x\,dx$ gives

$$nI_n = (n-1)I_{n-2}.$$

(c) Explicit formulas.
Iterating the recurrence gives:
Odd case: $n = 2m - 1$,

$$I_{2m-1} = \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdots \frac{2}{3} \cdot I_1 = \frac{2^{2m-1}(m-1)!\,m!}{(2m)!}.$$

Even case: $n = 2m$,

$$I_{2m} == \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{1}{2} \cdot I_0 = \frac{\pi}{2} \cdot \frac{(2m)!}{2^{2m}(m!)^2}.$$

(d) Product formula.
For $n = 2m$,

$$I_{2m} I_{2m-1} = \left(\frac{\pi}{2} \cdot \frac{(2m)!}{2^{2m}(m!)^2}\right)\left(\frac{2^{2m-1}(m-1)!\,m!}{(2m)!}\right) = \frac{\pi}{2(2m)} = \frac{\pi}{2n}.$$

For $n = 2m + 1$, a similar calculation yields

$$I_{2m+1} I_{2m} = \frac{\pi}{2(2m+1)} = \frac{\pi}{2n}.$$

Thus,

$$I_n I_{n-1} = \frac{\pi}{2n}.$$

(e) Asymptotics.

The product identity gives

$$I_{n+1}I_n = \frac{\pi}{2(n+1)}, \qquad I_nI_{n-1} = \frac{\pi}{2n}.$$

Since $I_{n+1} \le I_n \le I_{n-1}$,

$$\frac{\pi}{2(n+1)} \le I_n^2 \le \frac{\pi}{2n}.$$

Multiplying by $\frac{2n}{\pi}$:

$$\frac{n}{n+1} \le \frac{2n}{\pi} I_n^2 \le 1.$$

Taking square roots:

$$\sqrt{\frac{n}{n+1}} \le \sqrt{\frac{2n}{\pi}}\, I_n \le 1.$$

Thus,

$$I_n \sim \sqrt{\frac{\pi}{2n}}.$$

(f) Even case formula.

Directly from part (c),

$$I_{2n} = \frac{\pi}{2} \cdot \frac{(2n)!}{2^{\,2n}(n!)^2}.$$

$\square$