Combinatorics I (Math 7701) Fall 2025
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Exercise Sheet 6

Due date: 15:30, Dec 9th, to be submitted on COOL.

Working with your partner, you should try to solve all of the exercises below. You should then submit solutions to four of the problems, with each of you writing two, clearly indicating the author of each solution. Note that each problem is worth 10 points, and starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting on COOL, please send your solutions by e-mail.

Exercise 1 Recall that the Euler totient function $\phi(n)$ counts the numbers in $[n]$ that are relatively prime to n. Prove that $n = \sum_{d|n} \phi(d)$, where the sum is over all $d \in \mathbb{N}$ that divide n.

Exercise 2 Recall that a chain C in a poset $(P, \leq)$ is a subset $C \subseteq P$ where every pair of elements $x, y \in C$ is comparable. Consider the infinite poset $(2^{\mathbb{N}}, \subseteq)$, where the elements are (possibly infinite) subsets of $\mathbb{N}$, ordered by inclusion.

(a) Show that this poset has a countably infinite[1] chain.

(b)* Is there an uncountable[2] chain in the poset?

Exercise 3 Let $n \geq 1$ be odd. For a family $\mathcal{F} \subseteq \binom{[n]}{k}$ of subsets of size k of a ground set $[n]$, we define its *shadow* $\partial\mathcal{F} = \{G \in \binom{[n]}{k-1} : \exists F \in \mathcal{F} \text{ such that } G \subset F\}$ to be the $(k-1)$-sets that are contained in some set of $\mathcal{F}$.

(a) Show that for every $\mathcal{F} \subseteq \binom{[n]}{\frac{n+1}{2}}$, we have $|\partial\mathcal{F}| \geq |\mathcal{F}|$.

(b) Prove that if $\mathcal{A} \subseteq 2^{[n]}$ is an antichain of size $\binom{n}{\frac{n+1}{2}}$ with $\mathcal{A} \notin \{\binom{[n]}{\frac{n-1}{2}}, \binom{[n]}{\frac{n+1}{2}}\}$, then there must be some non-empty $\mathcal{F} \subsetneq \binom{[n]}{\frac{n+1}{2}}$ with $|\partial\mathcal{F}| = |\mathcal{F}|$.

(c) Deduce that the only maximum antichains in $2^{[n]}$ are $\binom{[n]}{\frac{n-1}{2}}$ and $\binom{[n]}{\frac{n+1}{2}}$.

[1] A set S is countably infinite if there is a bijection from S to $\mathbb{N}$. In other words, you can list the elements $S = \{s_1, s_2, s_3, s_4, \ldots\}$, with every element of S appearing within some finite number of terms.

[2] A set is uncountable if it is infinite, but not countably so. That is, there is no injection from S into $\mathbb{N}$.

Exercise 4 Let $\mathcal{B}_n = \left(2^{[n]}, \subseteq\right)$ be the Boolean poset of all subsets of $[n]$, ordered by inclusion. A symmetric chain in $\mathcal{B}_n$ is a chain $S_k \subseteq S_{k+1} \subseteq \ldots \subseteq S_{n-k}$ such that $|S_i| = i$ for all i. Prove that for all $n \geq 1$, $\mathcal{B}_n$ can be partitioned into symmetric chains.

Exercise 5 A family $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of subsets of $[n]$ is said to be *separating* if for any two elements $1 \leq i < j \leq n$, there is some set $F \in \mathcal{F}$ such that $|F \cap \{i, j\}| = 1$; that is, the set F differentiates between i and j.

(a) Prove that the smallest separating family has size $\lceil \log_2 n \rceil$.

The family is said to be *strongly separating* if even more is true: for every $1 \leq i < j \leq n$, there are sets $F, G \in \mathcal{F}$ such that $F \cap \{i, j\} = \{i\}$ and $G \cap \{i, j\} = \{j\}$.

(b) Prove that the smallest strongly separating family has size m, where m is the smallest natural number satisfying $\binom{m}{\lceil m/2 \rceil} \geq n$.[3]

Exercise 6 Consider the poset $(P, \leq)$ whose Hasse diagram is given in Figure 1.

(a) For every pair $i, j \in [6]$, determine value of the Möbius function $\mu(i, j)$.

(b) Prove that for any finite poset $(P, \leq)$ and all $x, y \in P$, $\mu(x, y)$ is always an integer.

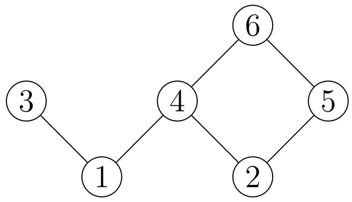

Figure 1: The Hasse diagram of a poset.

[3]When n is large, we have $m = \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1)$.