

# Introduction to Analysis I HW2

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**Problem 0.0.1 (11pts).** If  $(X, d)$  is a metric space, define

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that  $d'$  is also a metric on  $X$ .

Note that  $0 \leq d'(x, y) < 1$  for all  $x, y \in X$ .

**Proof.** In the first three properties we are going to check, they are all true since we can directly these properties on  $d$  to conclude that these properties are also true on  $d'$ .

- We know  $d'(x, x) = \frac{d(x, x)}{1 + d(x, x)} = 0$  for every  $x \in X$ .
- For every distinct  $x, y \in X$ , we have

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} > 0.$$

- For any  $x, y \in X$ , we have  $d'(x, y) = d'(y, x)$ , which is trivial.
- For any  $x, y, z \in X$ , suppose

$$a = d(x, z) \quad b = d(x, y) \quad c = d(y, z),$$

we want to show that

$$\frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c},$$

where we know  $a, b, c \geq 0$  and  $a \leq b + c$ . By directly computing, we know it is equivalent to

$$\begin{aligned} a(1 + b)(1 + c) &\leq (1 + a)(1 + c)b + (1 + a)(1 + b)c \\ \Leftrightarrow a(1 + b + c + bc) &\leq (1 + a + c + ac)b + (1 + a + b + ab)c \\ \Leftrightarrow a &\leq b(1 + c) + c(1 + b + ab) = b + c + 2bc + abc. \end{aligned}$$

Hence, we know this inequality holds because we know  $a, b, c \geq 0$ .

■

**Problem 0.0.2 (12 pts) exercise 1.2.4.** Let  $(X, d)$  be a metric space,  $x_0$  be a point in  $X$ , and  $r > 0$ . Let  $B$  be the open ball

$$B := B(x_0, r) = \{x \in X : d(x, x_0) < r\},$$

and let  $C$  be the closed ball

$$C := \{x \in X : d(x, x_0) \leq r\}.$$

- Show that  $\overline{B} \subseteq C$ .
- Give an example of a metric space  $(X, d)$ , a point  $x_0$ , and a radius  $r > 0$  such that  $\overline{B} \neq C$ .

**Proof.**

- For all  $b \in \overline{B}$ , we know for all  $r' > 0$ , we have  $B(b, r') \cap B(x_0, r) \neq \emptyset$ . Now if  $d(b, x_0) > r$ , say  $\varepsilon = d(b, x_0) - r > 0$ . Suppose  $z \in B(b, \varepsilon)$ , we have

$$\begin{aligned} d(z, x_0) &\geq d(b, x_0) - d(z, b) \\ &> d(b, x_0) - \varepsilon = r \end{aligned}$$

by triangle inequality. However, this means  $z \notin B(x_0, r)$ . Hence,  $B(b, \varepsilon) \cap B(x_0, r) = \emptyset$ , which is a contradiction. By this, we know  $d(b, x_0) \leq r$  for all  $b \in \overline{B}$ , so  $\overline{B} \subseteq C$ .

(b) Suppose the metric space is  $(\mathbb{R}, d_{\text{disc}})$ , where  $d_{\text{disc}}$  is the discrete metric defined by

$$d_{\text{disc}} = \begin{cases} 1, & \text{if } x \neq y; \\ 0, & \text{if } x = y, \end{cases}$$

and suppose  $x_0 = 0$  and  $r = 1$ . Thus, we know  $\overline{B} = B \cup \partial B$ , but notice that

$$B = \{x \in X \mid d(x, 0) < 1\} = \{0\},$$

and  $\partial B = \emptyset$  since for all  $x \neq 0$ , we know

$$B\left(x, \frac{1}{2}\right) = \{x\} \subseteq X \setminus B(0, 1),$$

so we know  $\text{Ext}(B) = \mathbb{R} \setminus \{0\}$ . Also, we know  $\text{Int}(B) = \{0\}$  since  $B(0, 1) \subseteq B$  and  $\text{Ext}(B) \cap \text{Int}(B) = \emptyset$ , so  $\partial B = \emptyset$ . Now we know  $\overline{B} = B \cup \partial B = \{0\}$ , but

$$C = \{x \in X \mid d(x, 0) \leq 1\} = \mathbb{R},$$

so  $\overline{B} \neq C$ .

■

**Problem 0.0.3 (21pts).** Two metrics  $d_1$  and  $d_2$  on a set  $X$  are said to be *Lipschitz equivalent* if there exist constants  $C_1 > 0$  and  $C_2 > 0$  such that

$$C_1 d_2(x, y) \leq d_1(x, y) \leq C_2 d_2(x, y) \quad \text{for all } x, y \in X.$$

Let  $E \subset X$ .

- (a) Prove that  $E$  is open in  $(X, d_1)$  if and only if  $E$  is open in  $(X, d_2)$ .
- (b) Prove that  $E$  is closed in  $(X, d_1)$  if and only if  $E$  is closed in  $(X, d_2)$ .
- (c) Two metrics  $d_1$  and  $d_2$  on a set  $X$  are said to be *topologically equivalent* if they induce the same topology on  $X$ . That is, a set  $U \subset X$  is open in  $(X, d_1)$  if and only if it is open in  $(X, d_2)$ . Give examples of topologically equivalent metrics that are not Lipschitz equivalent.

**Proof.** In the following text, if we write  $\text{Int}_1, \text{Int}_2, B_1, B_2$ , then the number of the subscript means it is under which metric. For example,  $\text{Int}_1(E)$  means the interior points of  $E$  in  $(X, d_1)$ , and the others are similarly defined.

- (a)  $(\Rightarrow)$  If  $E$  is open in  $(X, d_1)$ , then we know  $E = \text{Int}_1(E)$ . Thus,  $\forall x_0 \in E, \exists r > 0$  s.t.

$$B_1(x_0, r) = \{x \in X \mid d_1(x, x_0) < r\} \subseteq E.$$

However, it means for all  $x_0 \in E$ , we know

$$B_2\left(x_0, \frac{r}{C_2}\right) = \left\{x \in X \mid d_2(x, x_0) < \frac{r}{C_2}\right\} \subseteq B_1(x_0, r) \subseteq E$$

because for all  $x \in B_2\left(x_0, \frac{r}{C_2}\right)$ , we have  $d_2(x, x_0) < \frac{r}{C_2}$ , so it must have  $d_1(x, x_0) < r$  since

$$d_1(x, x_0) \leq C_2 d_2(x, x_0) < r.$$

Hence, we have  $E \subseteq \text{Int}_2(E)$ .

Also, for every  $x \in \text{Int}_2(E)$ , we know there exists  $r > 0$  s.t.  $B_2(x, r) \subseteq E$ , and also  $x \in B_2(x, r)$ , so  $x \in E$ , which means  $\text{Int}_2(E) \subseteq E$ .

Hence, we have  $\text{Int}_2(E) = E$ , which means  $E$  is open in  $(X, d_2)$ .

( $\Leftarrow$ ) Since we know

$$\frac{1}{C_2}d_1(x, y) \leq d_2(x, y) \leq \frac{1}{C_1}d_1(x, y) \quad \forall x, y \in X,$$

so we can just use the same method in the ( $\Rightarrow$ )'s proof to prove ( $\Leftarrow$ ) direction.

(b)

$$\begin{aligned} E \text{ is closed in } (X, d_1) &\Leftrightarrow X \setminus E \text{ is open in } (X, d_1) \\ &\Leftrightarrow X \setminus E \text{ is open in } (X, d_2) \quad (\text{by (a)}) \\ &\Leftrightarrow E \text{ is closed in } (X, d_2). \end{aligned}$$

(c) For  $X = \mathbb{R}$ ,  $d_1 = |x - y|$ , and  $d_2 = \frac{d_1}{1+d_1}$ , we claim that  $d_1$  and  $d_2$  are not Lipschitz equivalent and are topologically equivalent.

**Note 0.0.1.** In the course, we have shown that  $d_1$  is a metric, and in [Problem 0.0.1](#) we have shown that  $d_2$  is a metric.

**Claim 0.0.1.**  $d_1$  and  $d_2$  are not Lipschitz equivalent.

**Proof.** Note that  $d_1(x, y)$  can be arbitrarily large in  $\mathbb{R}$  and  $d_2(x, y) < 1$  for any  $x, y \in \mathbb{R}$ , so there does not exist a constant  $c$  s.t.  $d_1(x, y) < cd_2(x, y)$ , which means  $d_1$  and  $d_2$  are not Lipschitz equivalent.  $\otimes$

Now we show that a set  $U \subseteq \mathbb{R}$  is open in  $(\mathbb{R}, d_1)$  if and only if  $U$  is open in  $(\mathbb{R}, d_2)$ .

First notice that

$$d_2(x, y) = \frac{d_1(x, y)}{1 + d_1(x, y)} \Leftrightarrow d_1(x, y) = \frac{d_2(x, y)}{1 - d_2(x, y)}.$$

( $\Rightarrow$ ) If  $U$  is open in  $(\mathbb{R}, d_1)$ , then for all  $u \in U$ , there exists  $r > 0$  s.t.

$$B_1(u, r) = \{x \in X \mid d_1(x, u) < r\} \subseteq X.$$

Also, we know

$$d_1(x, u) < r \Leftrightarrow \frac{d_2(x, u)}{1 - d_2(x, u)} < r \Leftrightarrow d_2(x, u) < \frac{r}{1 + r}.$$

Thus, we know in  $(\mathbb{R}, d_2)$ , for all  $u \in U$ , there exists  $\frac{r}{1+r} > 0$  s.t.

$$B_2\left(u, \frac{r}{1+r}\right) = \left\{x \in X \mid d_2(x, u) < \frac{r}{1+r}\right\} \subseteq X,$$

which means  $\text{Int}_2(U) = U$  and thus  $U$  is open in  $(\mathbb{R}, d_2)$ .

( $\Leftarrow$ ) If  $U$  is open in  $(\mathbb{R}, d_2)$ , then for all  $u \in U$ , there exists  $r > 0$  s.t.

$$B_2(u, r) = \{x \in X \mid d_2(x, u) < r\} \subseteq X.$$

Besides, we can let  $r < 1$ . (If  $r \geq 1 > r_2$ , then  $B_2(u, r_2) \subseteq B(u, r) \subseteq X$ , and then we can let  $r = r_2$ .) Also, we know

$$d_2(x, u) < r \Leftrightarrow \frac{d_1(x, u)}{1 + d_1(x, u)} < r \Leftrightarrow d_1(x, u) < \frac{r}{1 - r}.$$

Notice that since  $0 < r < 1$ , so  $\frac{r}{1-r} > 0$ . Thus, we know in  $(\mathbb{R}, d_2)$ , for all  $u \in U$ , there exists  $\frac{r}{1-r} > 0$  s.t.

$$B_1\left(u, \frac{r}{1-r}\right) = \left\{x \in X \mid d_1(x, u) < \frac{r}{1-r}\right\} \subseteq X,$$

which means  $\text{Int}_1(U) = U$  and thus  $U$  is open in  $(\mathbb{R}, d_1)$ . ■

**Problem 0.0.4 (15 pts).** Let  $\mathcal{M}_n = M_n(\mathbb{R})$  denote the set of all  $n \times n$  real matrices. Define a function on  $\mathcal{M}_n \times \mathcal{M}_n$  by

$$\rho(A, B) = \text{rank}(A - B).$$

Then  $\rho$  is a metric on  $\mathcal{M}_n$  and it is topologically equivalent to the discrete metric on  $\mathcal{M}_n$ .

**Proof.** We first show that  $\rho$  is a metric on  $\mathcal{M}_n$ .

- For all  $A \in \mathcal{M}_n$ , we know  $\rho(A, A) = \text{rank}(A - A) = \text{rank } 0 = 0$ .
- For any distinct  $A, B \in \mathcal{M}_n$ , we know there is a row of  $A - B$  not equal to 0-vector, so  $\text{rank}(A - B) > 0$ .
- For  $A, B \in \mathcal{M}_n$ , we know  $\text{rank}(A - B) = \text{rank}(B - A)$ , so  $\rho(A, B) = \rho(B, A)$ .
- For  $A, B, C \in \mathcal{M}_n$ , we want to show  $\text{rank}(A - C) \leq \text{rank}(A - B) + \text{rank}(B - C)$ . Suppose  $A - B = X, B - C = Y$ , then we want to show  $\text{rank}(X + Y) \leq \text{rank } X + \text{rank } Y$ , which is equivalent to show

$$\dim \text{Im}(X + Y) \leq \dim(\text{Im } X) + \dim(\text{Im } Y).$$

Notice that

$$\text{Im}(X + Y) = \{w \mid (X + Y)v = w \text{ for some } v\} \subseteq \{a + b \mid a \in \text{Im } X, b \in \text{Im } Y\} = \text{Im } X + \text{Im } Y.$$

Hence, we have  $\dim \text{Im}(X + Y) \leq \dim(\text{Im } X + \text{Im } Y)$ . Also, we know

$$\dim(\text{Im } X + \text{Im } Y) = \dim \text{Im } X + \dim \text{Im } Y - \dim \text{Im } X \cap \text{Im } Y \leq \dim \text{Im } X + \dim \text{Im } Y.$$

Hence, we know  $\dim \text{Im}(X + Y) \leq \dim \text{Im } X + \dim \text{Im } Y$ .

Now we prove that  $\rho$  is topologically equivalent to the discrete metric on  $\mathcal{M}_n$ , called  $d_{\text{disc}}$ . Now we show that for any set  $U \subseteq \mathcal{M}_n$ ,  $U$  is open in  $(\mathcal{M}_n, \rho)$  and  $(\mathcal{M}, d_{\text{disc}})$ . For any  $U \subseteq \mathcal{M}_n$ , and for all  $u \in U$ , we know  $B_\rho(u, \frac{1}{2}) = \{u\} \subseteq U$ , so  $U = \text{Int}_\rho(U)$ , which means  $U$  is open in  $(\mathcal{M}_n, \rho)$ . Similarly, for all  $u \in U$ ,  $B_{\text{disc}}(u, \frac{1}{2}) = \{u\} \subseteq U$ , so we can similarly conclude that  $U$  is open in  $(\mathcal{M}_n, d_{\text{disc}})$ . Hence, we can say that  $U \subseteq \mathcal{M}_n$  is open in  $(\mathcal{M}, \rho)$  if and only if  $U$  is open in  $(\mathcal{M}_n, d_{\text{disc}})$ , so these two metrics are topologically equivalent. ■

**Problem 0.0.5 (20 pts).** Let  $E$  be a subset of a metric space  $(X, d)$ . Prove the following:

- (a) The boundary of  $E$  is a closed set.
- (b)  $\partial E = \overline{E} \cap \overline{X \setminus E}$
- (c) If  $E$  is clopen (closed and open), what is  $\partial E$ ?
- (d) Give an example of  $S \subset \mathbb{R}$  such that  $\partial(\partial S) \neq \emptyset$ , and infer that “the boundary of the boundary  $\partial \circ \partial$  is not always zero.”

**Proof.**

- (a) We want to show that  $\partial(\partial E) \subseteq \partial E$ . For all  $x \in \partial(\partial E)$ , if  $x \in \partial E$ , then we're done. Now

consider the second case:  $x \in \text{Int}(E) \cup \text{Ext}(E)$ . Note that for all  $r > 0$ , we have

$$B(x, r) \cap \partial E \neq \emptyset \quad B(x, r) \cap (X \setminus \partial E) \neq \emptyset.$$

Case 1:  $x \in \text{Int}(E)$ .

We know there exists  $r' > 0$  s.t.  $B(x, r') \subseteq E$ . If there exists  $c \in B(x, r') \cap \partial E$ , then we know  $c \in B(x, r') \subseteq E$ , so  $c \in E$ . Also, we know

$$B(c, r'') \cap E \neq \emptyset \quad B(c, r'') \cap (X \setminus E) \neq \emptyset \quad \forall r'' > 0.$$

Now suppose  $\varepsilon = d(c, x) < r'$ . If we pick some  $r'' < r' - \varepsilon$ , then for all  $p \in B(c, r'')$ , we have  $d(p, c) < r''$ , and by triangle inequality we have

$$d(p, x) \leq d(p, c) + d(c, x) < r'' + \varepsilon < r' - \varepsilon + \varepsilon = r',$$

which means  $p \in B(x, r')$ . Hence,  $B(c, r'') \subseteq B(x, r') \subseteq E$ , which means  $B(c, r'') \cap (X \setminus E) = \emptyset$ , and this is a contradiction, so we know there does not exist  $x \in \partial(\partial E)$  s.t.  $x \in \text{Int}(E)$ .

Case 2:  $x \in \text{Ext}(E)$ .

We know there exists  $r' > 0$  s.t.  $B(x, r') \subseteq X \setminus E$ . If there exists  $c \in B(x, r') \cap \partial E$ , then we know  $c \in B(x, r') \subseteq X \setminus E$ , so  $c \in X \setminus E$ . Also, we know

$$B(c, r'') \cap E \neq \emptyset \quad B(c, r'') \cap (X \setminus E) \neq \emptyset \quad \forall r'' > 0.$$

Now suppose  $\varepsilon = d(c, x) < r'$ . If we pick some  $r'' < r' - \varepsilon$ , then for all  $p \in B(c, r'')$ , we have  $d(p, c) < r''$ , and by triangle inequality we have

$$d(p, x) \leq d(p, c) + d(c, x) < r'' + \varepsilon < r' - \varepsilon + \varepsilon = r',$$

which means  $p \in B(x, r')$ . Hence,  $B(c, r'') \subseteq B(x, r') \subseteq X \setminus E$ , which means  $B(c, r'') \cap E = \emptyset$ , and this is a contradiction, so we know there does not exist  $x \in \partial(\partial E)$  s.t.  $x \in \text{Ext}(E)$ .

(b)

$$\begin{aligned} \text{a point } x \in \partial E &\Leftrightarrow \begin{cases} B(x, r) \cap E \neq \emptyset \\ B(x, r) \cap (X \setminus E) \neq \emptyset \end{cases} \\ &\Leftrightarrow x \in \overline{E} \text{ and } x \in \overline{X \setminus E}. \\ &\Leftrightarrow x \in \overline{E} \cap x \in \overline{X \setminus E}. \end{aligned}$$

(c)

(d)

■

**Problem 0.0.6 (21 pts).** In a metric space  $(X, d)$ , if subsets satisfy  $A \subseteq S \subseteq \overline{A}$ , where  $\overline{A}$  is the closure of  $A$ , then  $A$  is said to be *dense* in  $S$ . For example, the set  $\mathbb{Q}$  of rational numbers is dense in  $\mathbb{R}$ .

(a) If  $A$  is dense in  $S$  and  $S$  is dense in  $T$ , prove that  $A$  is dense in  $T$ .

(b) If  $A$  is dense in  $S$  and if  $B$  is open in  $S$ , prove that

$$B \subseteq \overline{A \cap B}.$$

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(c) If each of  $A$  and  $B$  is dense in  $S$  and if  $B$  is open in  $S$ , prove that

$A \cap B$  is dense in  $S$ .