Linear Algebra I

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Abstract

The lecture note of Linear Algebra I by professor 余正道.

Contents

Chapter 1

Vector Space

Lecture 1

1.1 Introduction to vector and vector space

3 Sep. 10:20

In high school, our vectors are in $\mathbb{R}^2$ and $\mathbb{R}^3$, and we have define the addition and scalar multiplication of vectors.

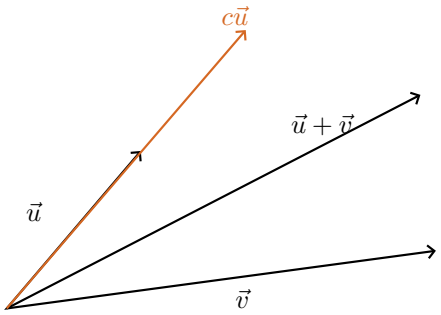

Figure 1.1: Vectors in $\mathbb{R}^2$

Example 1.1.1. $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n \mid a_i \in \mathbb{R})\}$

With this type of space, we can define addition and multiplication as

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = \{a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\}$$
$$\alpha \cdot (a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$$

Also, if we define a space:

Example 1.1.2. $V = \{\text{function } f : (a, b) \to \mathbb{R}\}$, where (a, b) is an open interval.

then this can also be a vector space after defining addtion and multiplication.

Note 1.1.1. In a vector space, we have to make sure the existence of 0-element, which means $0(x) = 0$.

Now we give a more abstract example:

Example 1.1.3. Suppose S is any set, then define $V = \{\text{all functions from } S \text{ to } \mathbb{R}\}$

If we define $(f + g)(s) = f(s) + g(s)$ and $(\alpha \cdot f)(s) = \alpha \cdot f(s)$, and $0(s) = 0$, then this is also a vector space.

Put some linear conditions

Example 1.1.4. In $\mathbb{R}^n$, fix $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, if we define

$$W = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\},$$

then this is also a vector space.

However, if we have

$$W' = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = 1\},$$

then this is not a vector space because it is not close.

Example 1.1.5. In $V = \{(a, b) \to \mathbb{R}\}$ or $W_1 = \{\text{polynomial defined on } (a, b)\}$, these are both vector space.

Remark 1.1.1. In the later course, we will learn that W_1 is a subspace of V.

Example 1.1.6. If we furtherly defined $W_1^{(k)} = \{\text{polynomial degree } \leq k\}$, then this is also a vector space.

Remark 1.1.2. $W_1^{(k)}$ is actually isomorphic to $\mathbb{R}^{k+1}$ since

$$a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \leftrightarrow (a_0, a_1, a_2, \ldots, a_n).$$

Example 1.1.7. $W_2 = \{\text{continuous function on } (a, b)\}$ and $W_3 = \{\text{differentiable functions}\}$ are also both vector spaces.

Example 1.1.8. $W_4 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0\right\}$ and $W_5 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -f\right\}$ are both vector spaces.

Proof.

$$W_4 = \{a_0 + a_1x\}$$
$$W_5 = \{a_1 \cos x + a_2 \sin x\}$$

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1.2 Formal definition of vector spaces

1.2.1 Vector Spaces Over $\mathbb{R}$

Definition 1.2.1. Suppose V is a non-empty set equipped with

- addition: $V \times V \to V$, that is, given $u, v \in V$, defining $u + v \in V$
- scalare multiplication: $\mathbb{R} \times V \to V$, that is, given $\alpha \to \mathbb{R}$ and $v \in V$, we need to have $\alpha v \in V$

Also, we need some good properties or conditions

- For addition,
  - $u + v = v + u$
  - $(u + v) + w = u + (v + w)$
- There exists $0 \in V$ such that $u + 0 = u = 0 + u$

- Given $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0 = (-v) + v$
- For scalar multiplication,
  - $1 \cdot v = v$ for all $v \in V$
  - $(\alpha\beta)v = \alpha \cdot (\beta v)$ for all $\alpha, \beta \in \mathbb{R}$ and $v \in V$.
- For addition and multiplication,
  - $\alpha(u + v) = \alpha u + \alpha v$
  - $(\alpha + \beta)u = \alpha u + \beta u$

Lecture 2

1.3 Vector Space over general field

5 Sep. 10:20

Now we introduce the concept of field.

Definition 1.3.1 (Field). A set F with $+$ and $\cdot$ is called a **field** if

- $\alpha + \beta = \beta + \alpha$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- There exists $0 \in F$ such that $\alpha + 0 = 0 + \alpha = \alpha$.
- For $\alpha \in F$, there exists $-\alpha$ such that $\alpha + (-\alpha) = 0$.
- $\alpha\beta = \beta\alpha$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
- $\exists 1$ such that $1 \neq 0$ and $1 \cdot \alpha = \alpha$.
- For $\alpha \neq 0$, $\exists \alpha^{-1} \in F$ such that $\alpha\alpha^{-1} = 1$.
- $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Example 1.3.1. $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are all fields but $\mathbb{Z}$ is not.

Example 1.3.2. $\{0, 1\}$ is also a field.

Now we know the concept of filed, so we can make a vector space over a field.

Theorem 1.3.1 (Cancellation law). Suppose $v_1, v_2, w \in V$, a vector space, then if $v_1 + w = v_2 + w$, then $v_1 = v_2$.

Proof.

$$v_1 = v_1 + (w + (-w)) = (v_1 + w) + (-w) = (v_2 + w) + (-w) = v_2 + (w + (-w)) = v_2.$$

■

Theorem 1.3.2. The zero vector 0 is unique.

Proof. Suppose we have $0, 0'$ both zero vector, then for some $0 = 0 + 0' = 0'$. ■

Theorem 1.3.3. For any $v \in V$, $0 \cdot u = 0$.

Proof. $0 \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u$, so $0 = 0 \cdot u$ by cancellation law. ■

Theorem 1.3.4. $(-1) \cdot u = -u$.

Theorem 1.3.5. Given any $u \in V$ is unique, $-u$ is unique.

1.4 Subspaces

Definition 1.4.1 (subspace). Let V be a vector space. A non-empty subset $W \subseteq V$ is called a subspace of V if W is itself a vector space under $+$ and $\cdot$ on V.

Example 1.4.1. $M_n(F) = \{n \times n \text{ matrix with entries in } F\}$ is a vector space, and

$$U_n(F) = \left\{ \begin{pmatrix} a_{11} & & & \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \right\}$$

is a subspace of $M_n(F)$.

Proposition 1.4.1. Suppose V is a vector space, and $W \subseteq V$ is non-empty, then

$$W \text{ is a subspace } \Leftrightarrow \text{ For } u, v \in W, \alpha \in F, \text{ we have } u + v \in W \text{ and } \alpha \cdot u \in W.$$

proof of $\Rightarrow$. Clear. ■

proof of $\Leftarrow$. First, we would want to check $0 \in W$, and we can pick any $u \in W$, and pick $\alpha = -1$, so we know $-u \in W$, and thus $0 = u + (-u) \in W$. ■

Corollary 1.4.1. If we want to check W is a subspace, we just need to check for $u, v \in W$, $\alpha \in F$, $u + \alpha v \in W$ or not.

1.5 Linear Combination

Definition 1.5.1 (Linear combinaiton). Given $v_1, v_2, \ldots, v_n \in V$, a linear combination of them is a vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Proposition 1.5.1. Given $v_1, v_2, \ldots, v_n \in V$,

1. $W = \{\text{all linear combinations of } v_, \ldots, v_n\}$ is a subspace.
2. This subspace is the smallest subspace containing $v_1, \ldots, v_n$. That is, if $W' \subseteq V$ is a subspace containing $v_1, \ldots, v_n$, then $W \subseteq W'$.

Notation. $\text{span}\{v_1, v_2, \ldots, v_n\} = \{\text{all linear combinations of } v_1, v_2, \ldots, v_n\}$

1.6 Linearly independent

Definition. Now we talk about the linear dependence and linear independence.

Definition 1.6.1 (Linearly dependent). $v_1, v_2, \ldots, v_n$ are linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zeros.

Definition 1.6.2 (Linearly independent). $v_1, v_2, \ldots, v_n$ are called linearly independent if they are not linearly dependent.

Corollary 1.6.1. Say $\alpha_i \neq 0$, then $v_i \in \operatorname{span}\{\hat{v_1}, \hat{v_2}, \ldots, \hat{v_k}\}$ suppose the corresponding α_i of $\hat{v_1}, \ldots, \hat{v_k}$ are not zeros.

Corollary 1.6.2. Linearly independent means if $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Corollary 1.6.3. Linearly independent meeans if $\sum \alpha_i v_i = \sum \beta_i v_i$, then $\alpha_i = \beta_i$ for all i.

Example 1.6.1.

- $v \in V$ is linearly independent iff $v \neq 0$.
- $v, w \in V$ are linearly independent iff v is not a scaler of w and w is not a scalar of v.

Lemma 1.6.1. $v_1, \ldots, v_n$ are linearly independent iff $v_i \notin \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$.

1.7 Basis

Definition. We now talking about basis

Definition 1.7.1 (Basis). $B = \{v_1, v_2, \ldots, v_n\}$ is called a basis of V if B spans V and B is linearly independent.

Definition 1.7.2 (Dimension). In this case, n is called the dimension of V, and denoted by $\dim V$.

Notation. $\operatorname{span}\{v_1, v_2, \ldots, v_n\} = \langle v_1, v_2, \ldots, v_n \rangle$

Notation. $\operatorname{span}(S) = \langle S \rangle$

Theorem 1.7.1. For any $v \in V$, it has a unique expression $v = \sum_{i=1}^{n} \alpha_i v_i$.

Lecture 3

10 Sep 10:20

As previously seen. A basis of a vector space V is a set $\{v_1, v_2, \ldots, v_n\}$ that is linearly independent and simultaneously spans V. That is, suppose we have $\sum a_i v_i = 0$ for some scalars a_i, then $a_i = 0$ for all i. Also, we call the number n, the dimension of V.

Example 1.7.1. Suppose we have $V = F^n = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \in F\}$, then we have a **standard basis**, which is

$$\begin{aligned} e_1 &= (1, 0, \ldots, 0) \\ e_2 &= (0, 1, \ldots, 0) \\ &\vdots \\ e_n &= (0, 0, \ldots, 1) \end{aligned}$$

since $\{e_i\}_{i=1}^n$ is linearly independent and for every $\vec{a} = (a_1, \ldots, a_n)$, we know

$$\vec{a} = \sum_{i=1}^{n} a_i e_i.$$

Example 1.7.2. Suppose

$$V = M_{n\times n}(F) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \ddots & & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \cdots & & \alpha_{nn} \end{pmatrix} \right\},$$

then we know

$$\{e_{ij}\}_{1\le i,j\le n} = \begin{pmatrix} 0 & 0 & & \\ 0 & & & \\ & & 1 & \\ 0 & & & 0 \\ 0 & & & & 0 \end{pmatrix},$$

where the 1 is in the i-th row and j-th column.

Theorem 1.7.2. Suppose V is a vector space, and $V = \langle v_1, v_2, \ldots, v_n\rangle$ and $\{w_1, w_2, \ldots, w_m\}$ is linearly independent, then $m \le n$. Furtheremore, one can make

$$\langle w_1, w_2, \ldots, w_m, v_{m+1}, \ldots, v_n\rangle = V$$

after rearrangement of $v_1, \ldots, v_n$.

Proof. We can do induction on m. It is trivial that $m = 0$ is true. Suppose the statement holds for a fixed m with $m \le n$. Let $w_1, w_2, \ldots, w_{m+1}$ be linearly independent. In particular, $w_1, w_2, \ldots, w_m$ is linearly independent.

Claim 1.7.1. $m + 1 \le n$.

Proof. Otherwise, if $m + 1 > n$, then since $m \le n$, so $m = n$. Hence, by induction hypothesis, we know $\langle w_1, w_2, \ldots, w_m\rangle = V$. However, by Lemma 1.7.1 and the note following it, we know

$$\{w_1, w_2, \ldots, w_m\} \cup \{w_{m+1}\}$$

can not be linearly indepndent since $w_{m+1} \in V = \langle w_1, \ldots, w_m\rangle$. ⊛

Now we know $m + 1 \le n$. By induction hypothesis, we know

$$\langle w_1, w_2, \ldots, w_m, v_{m+1}, \ldots, v_n\rangle = V$$

Claim 1.7.2. One of $v_{m+1}, \ldots, v_n$ can be replaced by w_{m+1}.

Proof. Since

$$w_{m+1} = \sum_{i=1}^{m} \alpha_i w_i + \sum_{j=m+1}^{n} \beta_j v_j.$$

Trivially, one of $\beta_j \neq 0$, say $\beta_{m+1} \neq 0$. Check

$$\langle w_1, \ldots, w_m, w_{m+1}, v_{m+2}, \ldots, v_n \rangle = V.$$

⊛

■

Corollary 1.7.1. If $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ are bases of V, then $n = m$.

Remark 1.7.1. Corollary 1.7.1 tells us $\dim V$ is well-defined, which means the size of the bases of a vector space is unique.

Corollary 1.7.2. Suppose $\dim V = n$, then if $\langle v_1, v_2, \ldots, v_m \rangle = V$, then $m \geq n$. If $\{w_1, w_2, \ldots, w_m\}$ is linearly independent, then $m \leq n$. Also, any $\{v_i\}_{i=1}^{m}$ with $m > n$ is linearly dependent.

Lemma 1.7.1. Suppose $v_1, v_2, \ldots, v_n$ is linearly independent. If $w \notin \langle v_1, v_2, \ldots, v_n \rangle$, then

$$\{v_1, v_2, \ldots, v_n, w\}$$

is linearly independent.

Proof. Suppose $\sum_{i=1}^{n} \alpha_i v_i + \alpha_{i+1} w = 0$, then if $\alpha_{i+1} = 0$, we know $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ since $\{v_i\}_{i=1}^{n}$ is linearly independent. If $\alpha_{i+1} \neq 0$, then $w = \frac{1}{\alpha_{i+1}} \sum_{i=1}^{n} \alpha_i v_i \in \langle v_1, v_2, \ldots, v_n \rangle$, which is a contradiction. ■

Note 1.7.1. The reverse of Lemma 1.7.1 is still correct and is trivial. That is, if $w \notin \{v_1, \ldots, v_n\}$ and $\{v_1, v_2, \ldots, v_n, w\}$ is linearly independent, then $\{v_1, \ldots, v_n\}$ is linearly independent.

Corollary 1.7.3. If $W \subseteq V$ is a subspace of V, then $\dim W \leq \dim V$.

Proof. If $\dim V = n$, and $\{w_i\}_{i=1}^{m}$ is a basis of W, then this basis is linearly independent in V, which means $m \leq n$ by Theorem 1.7.2. ■

Corollary 1.7.4. If $v_1, v_2, \ldots, v_m$ is linearly independent, then $\{v_1, v_2, \ldots, v_m\}$ forms a basis after adding some $v_{m+1}, \ldots, v_n$ to it.

Theorem 1.7.3 (Dual version). If $\langle v_1, v_2, \ldots, v_n \rangle = V$, then $\{v_1, v_2, \ldots, v_m\}$ forms a basis after rearrangement, where $m \leq n$.

Remark 1.7.2. Most of the time, we consider finite-dimensional vector spaces.

Remark 1.7.3 (Examples of ∞-dim vector space).

- $V = \{\text{all polynomials over } F\} = F[x] = \{a_0 + a_1 x + \cdots + a_n x^n \text{ for some } n \text{ where } a_i \in F\}$.

- $$W = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\}.$$

  Notice that
  $$W' = \{\text{convergent sequence}\} \subseteq W.$$
  and
  $$W'' = l^2 = \left\{(a_i) \mid \sum_{i=0}^{\infty} a_i^2 \text{ finite}\right\} \subseteq W'$$

Remark 1.7.4. We define $\dim\{0\} = 0$, which is the only vector space with dimension 0, and we define $\langle\varnothing\rangle = \{0\}$, which means $\varnothing$ is the basis of $\{0\}$.

Note 1.7.2. We call a subspace $W \subsetneq V$ is proper.

1.8 More on subspaces

Theorem 1.8.1. If W_1 and W_2 are subspace of V, then $W_1 \cap W_2$ is a subspace.

Theorem 1.8.2. If W_1, W_2 are subspaces of V, then $W_1 + W_2$ is still a subspace of V.

Remark 1.8.1. If W_1, W_2 are subspaces of V, then $W_1 \cup W_2$ may not be a subspace. (See HW1).

Remark 1.8.2. In fact, $W_1 \cap W_2$ is the largest subspaces contained in W_1 and W_2.

Remark 1.8.3. In fact, $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2.

Corollary 1.8.1. Suppose S is the index set, and for all $i \in S$, W_i is a subspace of V, then
$$\bigcap_{i \in S} W_i = \{v \in V \mid v \in W_i \ \forall i\}$$
is also a subspace of V.

Corollary 1.8.2. Suppose S is the index set, and for all $i \in S$, W_i is a subspace of V, then
$$\sum_{i \in S} W_i = \{w_{i_1} + w_{i_2} + \cdots + w_{i_n} \text{ for some } i_j \in S\}$$
is also a subspace of V.

Proposition 1.8.1 (Dimension theorem). Suppose $W_1, W_2 \subseteq V$ are subspaces of V, then
$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Lecture 4

In calculus, $f : \mathbb{R} \to \mathbb{R}$ is called continuous if $f\left(\lim_{x \to a} x\right) = \lim_{x \to a} f(x)$. 12 Sep 10:20

Definition 1.8.1 (Linear transformation). Suppose V, W are vector spaces over F. A function

$$\begin{aligned} T : V &\to W \\ v &\mapsto T(v) \end{aligned}$$

is called a linear transformation or a linear map if

$$T(u+v) = T(u) + T(v) \quad T(\alpha v) = \alpha T(v),$$

or equivalently,

$$T(\alpha u + v) = \alpha T(u) + T(v).$$

Corollary 1.8.3. Suppose T is a linear transformation, then

$$T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i T(u_i).$$

Example 1.8.1. Suppose $V = \{\text{functions from } (-1,1) \text{ to } \mathbb{R}\}$, and define $T_a(f) = f(a)$, then T_a is a linear transformation.

Example 1.8.2. Consider the space of column vectors,

$$F^n = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \mid \alpha_i \in F \right\},$$

and define $A = (a_{ij}) \in M_{n\times n}(F)$ by

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

then if we have $T_A : F^n \to F^m$ where

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

then T_A is a linear map.

Note 1.8.1.

$$\begin{pmatrix} & \vdots & \\ \alpha_{i1} & \cdots & \alpha_{in} \\ & \vdots & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vdots \\ \sum_{j=1}^{n} a_{ij} x_j \\ \vdots \end{pmatrix}$$

Example 1.8.3. Consider row of vector space,

$$F^m = \{(\alpha_1, \ldots, \alpha_m) \mid \alpha_i \in F\},$$

and $A \in M_{m\times n}(F)$, then if $T_A : F^m \to F^n$ where

$$T_A : u = (u_1, \dots, u_m) \mapsto (u_1, \dots, u_m) \cdot A$$

is a linear map.

Obeserve that a linear map $T : V \to W$ is determined by $T(v_i)$, where $\{v_1, \dots, v_n\}$ is a basis of V.

Proposition 1.8.2. Suppose $\{v_1, v_2, \dots, v_n\}$ is a basis of V, then pick any $w_1, \dots, w_n \in W$. Then there is a unique linear map $T : V \to W$ satisfying $T(v_i) = w_i$.

Proof. Since any $v \in V$ has a unique representation $v = \sum_{i=1}^n \alpha_i v_i$. Hence, for a linear map $T : V \to W$, and for any $v \in V$, we know

$$T(v) = T\left(\sum_{i=1} \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i T(v_i) = \sum_{i=1}^n \alpha_i w_i.$$

Hence, if such map exists, then it must be unique. Now we have to show the existence of this map. Now if we define a map

$$T\left(\sum_{i=1}^n \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i w_i,$$

then we can check this is a linear map. ■

Example 1.8.4. Suppose F^n is the span of column vectors, and $A \in M_{m\times n}(F)$, and define $T_A(v) = Av$, then we can check $T_A(e_i) = c_i$, where c_i is the i-th column of A. This is the linear map that sends e_i to $c_i \in F^m$. If we pick $c_1, c_2, \dots, c_n \in F^m$, then there is a unique map sending e_i to c_i. In fact, this map is

$$T_A : v \mapsto Av$$

, where the i-th column of A is c_i.

Definition. Given $T : V \to W$, where T is linear.

Definition 1.8.2 (Kernel). The kernel/nullspace of T is defined as

$$\ker(T) = \{v \in V \mid T(v) = 0\} \subseteq V.$$

Definition 1.8.3 (Image). The image/range of T is defined as

$$\mathrm{Im}(T) = \{T(v) \mid v \in V\} \subseteq W.$$

Remark 1.8.4. Kernel and Image are subspaces.

Lecture 5

17 Sep. 10:20

As previously seen. Given such a linear map $T : V \to W$, we define

$$\ker T = T^{-1}(0) \quad \text{kernel/null space of } T$$
$$\mathrm{Im}\, T = T(V) \quad \text{image/range of } T,$$

and $\ker T$ is a subspaces of V, and $\mathrm{Im}\, T$ is a subspace of W.

Definition. Now we define the nullity and rank of a linear map.

Definition 1.8.4 (nullity). The nullity of T is the number

$$\nu(T) = \dim \ker T.$$

Definition 1.8.5 (rank). The rank of T is the number $\operatorname{rank} T = \dim \operatorname{Im} T$.

Example 1.8.5. Suppose $T : F^n \to F^m$, where F^n is the column space of dimension n, then $T = T_A$ for a matrix $A \in M_{m\times n}(F)$ and $T_A(v) = Av$.

Proof. Suppose $A = (c_1, c_2, \dots, c_n)$, where c_i is the i-th column vector of A. Consider the standard basis $\{e_1, e_2, \dots, e_n\}$ of F^n, where e_i is the column vector with i-th position 1 and the other entries are all 0's. Then, $T_A(e_i) = c_i \in F^m$. Explicitly,

$$T_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1c_1 + \cdots + x_nc_n$$

since we know

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_i x_i e_i.$$

and $T_A(e_i) = c_i$. In this case,

$$\begin{aligned} \ker T_A &= \text{all linear relations among } c_1, \dots, c_n \subseteq F^n \\ \operatorname{Im} T_A &= \operatorname{span}\{c_1, \dots, c_n\} \subseteq F^m. \end{aligned}$$

If we want to solve $\ker T_A$, then we need to solve

$$0 = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Hence, we have to solve

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0. \end{cases}$$

Given $A = (c_1, \dots, c_n)_{m\times n}$, then the column rank is $\dim\langle c_1, \dots, c_m\rangle$. If we rewrite $A = (r_1, \dots, r_m)^t$, where r_i is the i-th row of A, then the row rank is $\dim\langle r_1, r_2, \dots, r_m\rangle$. Since we can define $S_A : F^m \to F^n$, where

$$v = (x_1, \dots, x_m) \mapsto vA.$$

Remark 1.8.5. In fact, column rank is equal to row rank in a matrix, and we will prove it later.

⊛

Theorem 1.8.3 (rank and nullity theorem). Suppose $T : V \to W$ is a linear map, then

$$\nu(T) + \operatorname{rank} T = \dim V.$$

Proof. Since $\ker T \subseteq V$, so take a basis $\{v_1, \dots, v_\nu\}$ of $\ker T$, and $\operatorname{Im} T \subseteq W$, so take a basis $\{w_1, \dots, w_r\}$ of $\operatorname{Im} T$. Take u_j s.t. $T(u_j) = w_j$.

Claim 1.8.1. $S = \{v_1, \dots, v_\nu, u_1, \dots, u_r\}$ forms a basis of V.

Proof. We first show that S is linearly independent. Suppose $\sum \alpha_i v_i + \sum \beta_j u_j = 0$. Apply T on it, we get

$$0 = \sum \alpha_i T(v_i) + \sum \beta_j T(u_j) = \sum \alpha_i T(v_i) + \sum \beta_j w_j = \sum \beta_j w_j.$$

However, $\{w_j\}$ is linearly independent, so $\beta_j = 0$ for all j. Now we know $\sum \alpha_i v_i = 0$, which means $\alpha_i = 0$ for all i, so S is linearly independent. Now we want to show $\langle S \rangle = V$. Given $v \in V$, we know $T(v) \in \operatorname{Im} T$, and thus we can represent it as $T(v) = \sum \beta_j w_j$. We want to show

$$v = \sum \alpha_i v_i + \sum \beta_j u_j.$$

Thus, we want to show $v - \sum \beta_j u_j \in \ker T$, but note that

$$T\left(v - \sum \beta_j u_j\right) = T(v) - \sum \beta_j w_j = \sum \beta_j w_j - \sum \beta_j w_j = 0,$$

so we're done, and thus we have

$$v - \sum \beta_j u_j = \sum \alpha_i v_i$$

for some α_i's, and we're done. ⊛

Hence, $\dim V = |S| = \nu T + \operatorname{rank} T$. ■

Remark 1.8.6. If $\dim V > \dim W$, then $\nu(T) > 0$. Since, $\operatorname{rank} T \leq \dim W$, so if $\dim V > \dim W$, then we have $\nu(T) = \dim V - \operatorname{rank} T \geq \dim V - \dim W > 0$.

As previously seen. A map $f : X \to Y$ is called one-to-one or 1-1 or injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. f is called onto, surjective if $f(X) = Y$. f is called bijective if it is both 1-1 and onto. In this case, there is the inverse map $f^{-1} : Y \to X$ with $y \mapsto x$ if $f(x) = y$.

Proposition 1.8.3. Let $T : V \to W$ be linear, then T is injective iff $\ker T = \{0\}$.

Proof.

(⇒) If $v \in \ker T$, then since $T(0) = 0$, so $v = 0$.

(⇐) If $T(v_1) = T(v_2)$, then $T(v_1 - v_2) = 0$, which means $v_1 - v_2 \in \ker T = \{0\}$, so $v_1 = v_2$, which means T is linear.

■

Proposition 1.8.4. If $T : V \to W$ is a linear map, and if b is a basis of V, then T is injective if and only if $T(b)$ is linearly independent.

Proof.

(⇒) Suppose $v_1, v_2, \dots, v_n$ is a basis of V and we want to show $T(v_1), \dots, T(v_n)$ is linearly inde-

pendent. Suppose $\sum \alpha_i T(v_i) = 0$, then $T\left(\sum \alpha_i v_i\right) = 0$, so $\sum \alpha_i v_i = 0$, and thus $\alpha_i = 0$ for all i.

($\Leftarrow$) T sends one particular basis $v_1, \dots, v_n$ to a linearly independent set. We want to show $\ker T = \{0\}$. Suppose $v \in \ker T$, then if $v = \sum \alpha_i v_i$, we have

$$0 = T\left(\sum \alpha_i v_i\right) = \sum \alpha_i T(v_i),$$

but since $\{T(v_i)\}$ is linearly independent, so $\alpha_i = 0$ for all i, which means $v = 0$.

■

Proposition 1.8.5. If $T : V \to W$ is a linear map, then TFAE

(a) T is surjective

(b) T sends any basis to a generating set.

(c) T sends one basis to a generating set.

Theorem 1.8.4 (isomorphism)**.** Suppose $T : V \to W$ is linear and bijective, then there is the inverse map $T^{-1} : W \to V$, and T^{-1} is also linear. In this case, $T : V \to W$ is called an isomorphism.

Definition 1.8.6. If T is both injective and surjective, then T is an isomorphism.

Remark 1.8.7. If there is an isomorphism from V to W, we say V is isomorphic to W, or V and W are isomorphic.

Example 1.8.6 (Coordinates)**.** If $\dim V = n$, then V is isomorphic to F^n, we write $V \simeq F^n$.

Proof. In fact, given an order basis $B = \{v_1, \dots, v_n\}$ of V, then we know $v = \sum_{i=1}^n \alpha_i v_i$, where

$$v = \sum_{i=1}^{n} \alpha_i v_i \mapsto [v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix},$$

and this is a bijection. Note that this map is well-defined since any v has unique coordinate under B. Hence, we have $v_i \mapsto [v_i]_B = e_i$. ⊛

Hence, if $T : V \to W$, and we know $V \simeq F^n$ and $W \simeq F^m$, and we know there is a matrix sends F^n to F^m, called $[T]_{B'}^{B}$, and we can use it to represent the transformation from V to W, which is T.

Exercise 1.8.1. $T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$.

Proof. Suppose $T(v_3) = w_1 + w_2$, we want to show $v_3 = v_1 + v_2$. Hence, we need to check

$$w_1 + w_2 = T\left(T^{-1}(w_1) + T^{-1}(w_2)\right) = T\left(T^{-1}(w_1)\right) + T\left(T^{-1}(w_2)\right) = w_1 + w_2,$$

which is true. ■

Lecture 6

As previously seen. T is called an isomorphism if T is both injective and surjective. 19 Sep. 10:20

Proposition 1.8.6. Suppose $\dim V = \dim W = n$, then TFAE

(i) T is an isomorphism.

(ii) T is injective.

(iii) T is surjective.

(iv) T sends any basis of V to a basis of W.

(v) T sends one basis to a basis.

Example 1.8.7. Suppose $A \in M_{m\times n}(F)$, say $A = (c_1, c_2, \dots, c_n)$, then T_A is injective if and only if $\{c_1, \dots, c_n\}$ is linearly independent. (which means $n \leq m$).

Proof. Since $T_A(e_i) = c_i$ and $\{e_i\}_{i=1}^n$ forms a basis. ⊛

Example 1.8.8. Following the last example, T_A is surjective if and only if $\{c_1, c_2, \dots, c_n\}$ spans W. (which means $n \geq m$).

1.9 Space of linear maps

Consider

$$\{f : V \to W\},$$

and then we can define addition and multiplication by

$$(f+g)(v) = f(v) + g(v) \quad (\alpha \cdot f)(v) = \alpha f(v).$$

Hence, we know it is a vector space. Now if we collect all linear maps, say

$$\mathcal{L}(V, W) = \{\text{linear } T : V \to W\}.$$

Observe that $\mathcal{L}(V, W)$ is a vector space since we can similarly define the addition and multiplication.

Now if we have U, V, W, three vector spaces, and $f : U \to V$ is a linear map, then if we define a map

$$\begin{aligned} R_f : \mathcal{L}(V, W) &\to \mathcal{L}(U, W) \\ T &\mapsto T \circ f, \end{aligned}$$

then this map is linear. Similarly,

$$\begin{aligned} L_f : \mathcal{L}(W, U) &\to \mathcal{L}(W, V) \\ T &\mapsto f \circ T, \end{aligned}$$

then this is also a linear map.

Note 1.9.1. We just need to check something like

$$R_f(T+S) = R_f(T) + R_f(S) \quad R_f(\alpha T) = \alpha R_f(T).$$

Now if we consider

$$\begin{aligned} \mathcal{L}(V, W) \times \mathcal{L}(U, V) &\to \mathcal{L}(U, W) \\ (T, S) &\mapsto T \circ S, \end{aligned}$$

then this is also a linear map.

Example 1.9.1. $\mathcal{L}(F^n, F^m) = M_{m\times n}(F)$.

Proof. Check that

$$T_A + T_B = T_{A+B}.$$

Note 1.9.2. More precisely, they are isomorphic, that is, $\mathcal{L}(F^n, F^m) \cong M_{m\times n}(F)$.

⊛

Example 1.9.2. Consider

$$\mathcal{L}(F^n, F^m) \times \mathcal{L}(F^p, F^n) \to \mathcal{L}(F^p, F^m),$$

we know this is a linear map, and by Example 1.9.1, we know

$$M_{m\times n}(F) \times M_{n\times p}(F) \to M_{m\times p}(F)$$

is a linear map.

Proof. Check

$$(T_A \circ T_B)(v) = T_{AB}(v) \Leftrightarrow A(Bv) = (AB)(v).$$

⊛

Definition 1.9.1. We call

$$V \cong F^n$$

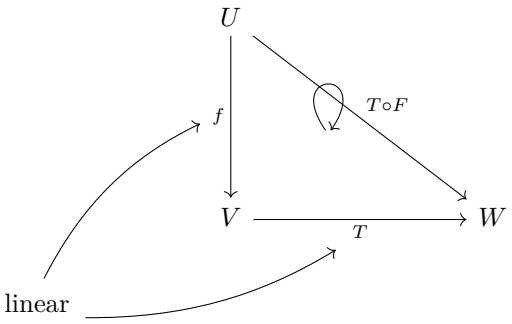
a basic isomorphisms if $\dim V = n$.

Corollary 1.9.1. $\mathcal{L}(F^n, F^m) \cong M_{m\times n}(F)$.

Remark 1.9.1. If you change F^n to V and F^m to W, then this is also correct since $F^n \cong V$ and $F^m \cong W$. (We suppose $\dim V = n$ and $\dim W = m$.)

Lecture 7

24 Sep. 10:20

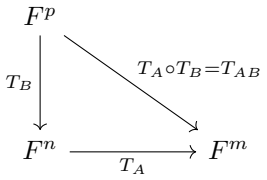


There is a special case,

$$\mathcal{L}(V, V) \coloneqq \mathcal{L}(V) = \{\text{linear } T : V \to V\},$$

which is the space of linear operators on V.

Now consider linear $T_A : F^n \to F^m, T_B : F^p \to F^m$, then we can define a map $T_{AB} = T_A \circ T_B$, and it will be a linear map.



Also, note that T_A, T_B corresponds to two matrices A, B, respectively, and it turns out that T_{AB} corresponds to the matrix AB. (Check)

Hence, $\mathcal{L}(F^n) = M_n(F)$.

A matrix P is called invertible if T_P is bijective. In this case,

$$F^n \underset{T_Q}{\overset{T_p}{\rightleftarrows}} F^m$$

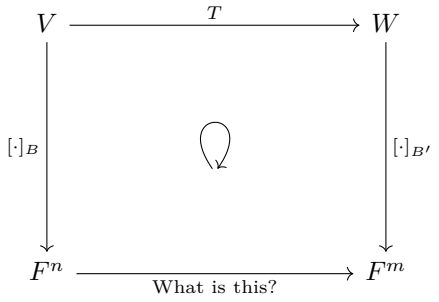
Hence, there exists $Q \in M_n(F)$ s.t. $QP = PQ = I_n$ since we know $T_P \circ T_Q = T_Q \circ T_P = I$.

Thus, we have

$$P = (c_1, c_2, \dots, c_n) \text{ invertible} \Leftrightarrow \{c_1, \dots, c_n\} \text{ is a basis.}$$

by Proposition 1.8.6.

1.10 Map/matrix correspondence



Take an ordered basis $B = \{v_1, v_2, \dots, v_n\}$ and $B' = \{w_1, \dots, w_m\}$, and says

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i \mapsto \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}.$$

Now consider the matrix

$$A = (\alpha_{ij}) = ([T(v_1)]_{B'}, [T(v_2)]_{B'}, \dots),$$

and then we called A the martix of T relative to B and B'. (matrix representative of T), and we denote this by $[T]^B_{B'}$.

Theorem 1.10.1.

$$[T(v)]_{B'} = [T]^B_{B'} [v]_B.$$

Theorem 1.10.2. We have $[\cdot]^B_{B'} : \mathcal{L}(V, W) \to M_{m \times n}(F)$, ans this matrix representative $[\cdot]^B_{B'}$ is an isomorphism, which means

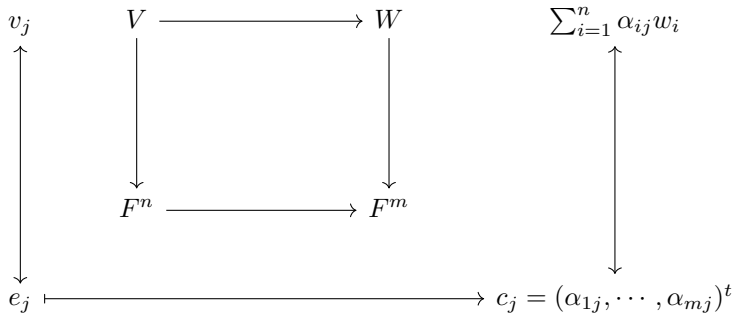
- $[T + S]^B_{B'} = [T]^B_{B'} + [S]^B_{B'}$.
- It is bijective.

Corollary 1.10.1. if $\dim V = n$ and $\dim W = m$, then

$$\dim(\mathcal{L}(V,W)) = \dim V \cdot \dim W.$$

Theorem 1.10.3.

$$[T]^B_{B'}[S]^{B''}_B = [T \circ S]^{B''}_{B'}.$$



Special case:

$$\mathcal{L}(V) \to M_n(F).$$

Take an ordered basis $B = \{v_1, \ldots, v_n\}$. If $T \in \mathcal{L}(V)$, then we can define $[T]_B = [T]^B_B$.

Corollary 1.10.2. Given $T : V \to W$. There are $B = \{v_1, \ldots, v_n\}$ and $B' = \{w_1, \ldots, w_m\}$ where B is a basis of V and B' is a basis of W and

$$[T]^B_{B'} = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

where $p = \operatorname{rank}(T)$.

Proof. We can let $B = \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$, where $\{v_{r+1}, \ldots, v_n\}$ is a basis of $\ker T$ and $T(v_1), \ldots, T(v_r)$ is a basis of $\operatorname{Im}(T)$, (Recall the proof in Theorem 1.8.3), then we can let $B' = \{T(v_1), \ldots, T(v_r), \ldots\}$. ■

Example 1.10.1. Suppose $V = \{\text{polynomials with degree} \le k\}$ and W is the space of polynomials with degree $\le k+1$, then if $T : V \to W$ and $p(x) \mapsto \int_0^x p(t)\,\mathrm{d}t$, then we know an ordered basis $B = \{1, x, x^2, \ldots, x^k\}$ and $B' = \{1, x, x^2, \ldots, x^{k+1}\}$, and then

$$[T]^B_{B'} = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ 0 & \frac{1}{2} & & \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & \frac{1}{k+1} \end{pmatrix}.$$

Example 1.10.2. Suppose V is the space of polynomials of degree $\le k$, and $B = \{1, x, x^j, \ldots, x^k\}$, and $B' = \{1, y, y^2, \ldots, y^k\}$ with $y = x - 1$. Then, if T is the identity transformation, note that

$$x^j = (y+1)^j = 1 + j \cdot y + \binom{j}{2} y^2 + \cdots + \binom{j}{j} y^j.$$

Hence, we have

$$[T]_{B'}^{B} = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \\ 0 & \binom{1}{1} & \binom{2}{1} & \\ 0 & 0 & \binom{2}{2} & \\ \vdots & \vdots & & \ddots \\ 0 & 0 & & \end{pmatrix}$$

Question. Given V, and B, B' are ordered basis, then what is the relation between $[v]_B$ and $[v]_{B'}$?

Answer. Change of bases. ⊛

Corollary 1.10.3.

$$[id]_{B'}^{B}[v]_B = [v]_{B'}.$$

Corollary 1.10.4.

$$[id]_{B'}^{B}[id]_{B}^{B'} = [id]_{B'}^{B'}.$$

Corollary 1.10.5. Given any $A \in M_{m\times n}(F)$. There are invertible matrices $P \in M_m(F)$ and $Q \in M_n(F)$ s.t.

$$PAQ = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

where p is the row rank of A.

Proof. Suppose $A = [T]_B^{B'}$, and by Corollary 1.10.2, we know there exists b, b' s.t. $[T]_b^{b'}$ is the matrix we want, then we can let $Q = [id]_{b'}^{B'}$ and $P = [id]_b^B$, and we're done. ■

Lecture 8

26 Sep. 10:20

Lemma 1.10.1. Consider

$$V' \xrightarrow{f} V \xrightarrow{T} W \xrightarrow{g} W'$$

- Suppose g is injective, then $\ker(g \circ T) = \ker T$.
- Suppose f is surjective, then $\operatorname{Im}(T \circ f) = \operatorname{Im} T$.

Definition 1.10.1 (Matrix Equivalence)**.** Let $A, B \in M_{m\times n}(\mathbb{F})$. We say that A and B are *equivalent* if there exist invertible matrices $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$ such that

$$B = PAQ.$$

Remark 1.10.1. Matrix equivalence means that one can obtain B from A by a sequence of invertible row and column operations.

Equivalently, if A represents a linear map $T : \mathbb{F}^n \to \mathbb{F}^m$, then B represents the same linear map with respect to different bases of the domain and codomain.

Theorem 1.10.4 (Row Rank Equals Column Rank)**.** Let $A \in M_{m\times n}(\mathbb{F})$ be any matrix over a field $\mathbb{F}$. Then

$$\text{row rank}(A) = \text{column rank}(A).$$

Proof. We prove this using invertible row and column operations.

Step 1: Reduce A to canonical form.

It is a standard fact that any matrix $A \in M_{m\times n}(\mathbb{F})$ can be transformed into a block matrix of the form

$$C = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times n},$$

by multiplying on the left and right by invertible matrices $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$:

$$C = PAQ.$$

Here $r = \text{rank}(A)$ and I_r is the $r \times r$ identity matrix. This uses Gaussian elimination (invertible row operations) and invertible column operations.

Step 2: Row and column ranks of C.

- The first r rows of C are linearly independent, and the remaining $m - r$ rows are zero. So

$$\text{row rank}(C) = r.$$

- The first r columns of C are linearly independent, and the remaining $n - r$ columns are zero. So

$$\text{column rank}(C) = r.$$

Step 3: Equivalence preserves row and column ranks.

We have $C = PAQ$.

1. *Left multiplication by P (row operations):* Multiplying A on the left by invertible P corresponds to invertible row operations. Row operations do not change the linear independence of the rows. Hence

$$\text{row rank}(PA) = \text{row rank}(A).$$

2. *Right multiplication by Q (column operations):* Each row of AQ is obtained by multiplying the corresponding row of A by Q:

$$\text{row}_i(AQ) = \text{row}_i(A) \cdot Q.$$

Since Q is invertible, this is an invertible linear transformation on $\mathbb{F}^n$, which preserves linear independence of the rows. Therefore

$$\text{row rank}(AQ) = \text{row rank}(A).$$

Note 1.10.1.

$$\sum_{i\in I} \alpha_i \text{row}_i(A) \cdot Q = 0 \Leftrightarrow \sum_{i\in I} \alpha_i \text{row}_i(A) = 0$$

since Q is invertible.

Combining the above, for $C = PAQ$ we get

$$\text{row rank}(C) = \text{row rank}(A) = r,$$

and similarly

$$\text{column rank}(C) = \text{column rank}(A) = r.$$

Step 4: Conclusion.

From Step 2 and Step 3, we have

$$\text{row rank}(A) = \text{row rank}(C) = r = \text{column rank}(C) = \text{column rank}(A).$$

Hence, the row rank of A equals the column rank of A. ■

Theorem 1.10.5. Two matrices A and B of same sizes are equivalent if and only if $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Proof. Suppose A, B equivalent, then $A = PBQ$ for some invertible P, Q. By Lemma 1.10.1, we know $\operatorname{Im}(BQ) = \operatorname{Im} B$, which gives $\operatorname{rank}(BQ) = \operatorname{rank} B$. Also, since $\ker(P(BQ)) = \ker(BQ)$, so $\operatorname{rank}(P(BQ)) = \operatorname{rank}(BQ)$ by rank and nullity theorem. Hence, we have $\operatorname{rank} A = \operatorname{rank}(PBQ) = \operatorname{rank}(BQ) = \operatorname{rank} B$.

Now if $\operatorname{rank} A = \operatorname{rank} B$, then we know

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = P'BQ',$$

so $A = P^{-1}P'BQ'Q^{-1}$, which means A, B are equivalent. ■

Theorem 1.10.6. Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces over a field $\mathbb{F}$. Let $B = \{v_1, \ldots, v_n\}$ be a basis for V and $C = \{w_1, \ldots, w_m\}$ be a basis for W. Let

$$A = [T]_{B,C} \in M_{m\times n}(\mathbb{F})$$

be the matrix of T with respect to the bases B and C. Then

$$\operatorname{rank}(A) = \dim(\operatorname{Im}(T)).$$

Proof. **Step 1: Express the image of T in terms of the basis.**

The matrix A is given by

$$A = \big[[T(v_1)]_C \; [T(v_2)]_C \; \ldots \; [T(v_n)]_C\big],$$

where $[T(v_j)]_C$ denotes the coordinate vector of $T(v_j)$ with respect to C.

Since B is a basis for V, any vector $v \in V$ can be written as

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

for some scalars $c_1, \ldots, c_n \in \mathbb{F}$. By linearity of T,

$$T(v) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

Thus, every vector in $\operatorname{Im}(T)$ is a linear combination of

$$\{T(v_1), T(v_2), \ldots, T(v_n)\},$$

and hence

$$\operatorname{Im}(T) = \operatorname{span}\{T(v_1), T(v_2), \ldots, T(v_n)\}.$$

Step 2: Relate $\operatorname{Im}(T)$ to the column space of A.

The column space of A, denoted $\operatorname{Col}(A)$, is

$$\operatorname{Col}(A) = \operatorname{span}\{[T(v_1)]_C, [T(v_2)]_C, \ldots, [T(v_n)]_C\}.$$

The coordinate mapping $[\cdot]_C : W \to \mathbb{F}^m$ is a linear isomorphism. In particular, it preserves linear independence and spanning sets. Therefore, the map

$$T(v_j) \longmapsto [T(v_j)]_C$$

establishes a linear isomorphism between $\operatorname{Im}(T)$ and $\operatorname{Col}(A)$:

$$\operatorname{Im}(T) \cong \operatorname{Col}(A).$$

Step 3: Compare dimensions.

Since isomorphic vector spaces have the same dimension,

$$\dim(\mathrm{Im}(T)) = \dim(\mathrm{Col}(A)).$$

By definition, the rank of A is the dimension of its column space:

$$\mathrm{rank}(A) = \dim(\mathrm{Col}(A)).$$

Combining these equalities, we obtain

$$\mathrm{rank}(A) = \dim(\mathrm{Im}(T)),$$

as desired.

This shows that the rank of a matrix representing a linear transformation is independent of the choice of bases B and C, since $\dim(\mathrm{Im}(T))$ depends only on T itself. ■

Lecture 9

Consider the system 1 Oct. 10:20

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_m. \end{cases}$$

We want to solve X s.t. $AX = Y$, where $A = (a_{ij})_{m\times n}$ and $Y = (y_i)_{i=1}^m$. Suppose $P \in M_{m\times m}(F)$ invertible, then if $B = PA$, we have $BX = Z$, which means doing row operations on the system. In this case, we call two systems are equivalent. We also call A, B are row equivalent.

Now we talk about the types of elementary row operations:

(i) Replace i-th row with $c \cdot r_i$ for some $c \neq 0$.

(ii) Replace r_i with $r_i + cr_j$ for some $j \neq i$.

(iii) Interchange r_i and r_j for some $i \neq j$.

One can use (i) and (ii) in finite steps, making A into row reduced form(REF) of A, which means

- first entry of a non-zero row is 1, we called it leading 1
- entries below and above leading 1 are 0.

If allowing (iii), we can make A into RREF(row reduced echelon form), which means REF and all zero rows are at the bottom.

Note that $AX = Y$ gives $PAX = PY$, so we can write $P(A \mid Y) = (PA \mid PY)$. Hence, we can do row operations on $(X \mid Y)$ so that the X part becomes REF or RREF to solve the system. The system will be like

$$\begin{aligned} x_{k_1} + \cdots + 0 + \cdots &= z_1 \\ x_{k_2} + \cdots + 0 &= z_2 \\ \vdots & \end{aligned}$$

Suppose for the first n rows, there are r non-zero rows. If there is some $z_i \neq 0$ for $i > r$, the system has no solution. If not, there is at least one solution, and there are $n - r$ free variables.

Note 1.10.2. If $n - r = 0$, then the system has unique solution, and if $n - r > 0$, then it has infinitely many solutions.

In the homogeneous case (i.e. $y_1 = y_2 = \cdots = y_m = 0$), we find $\nu(A) = n - r$. In this case, if $n > m$, then $n - r > m - r \geq 0$, so there are non-zero solutions to $AX = 0$.

Some consequences:

- If $A \in M_n(F)$, then TFAE
  - The system $AX = 0$ has only trivial solution (injective).
  - For any Y, $AX = Y$ has a (unique) solution (surjective).
  - A is invertible.

If P, Q are invertible, then $(PQ)^{-1} = Q^{-1}P^{-1}$. Also, by above mentioned things, we know every invertible matrix is a product of many elementary matrix, that is, $A = (E_1)^{-1}(E_2)^{-1}\dots(E_m)^{-1}$ since we know

$$(E_m \dots E_2 E_1)A = I_m.$$

Note 1.10.3. If A is invertible, then $AX = 0$ has only trivial solution, then its RREF is I, and thus A can be recovered to I by some row operations.

As previously seen. If $\{v_1, \dots, v_n\}$ is linearly independent and $\{w_1, \dots, w_m\}$ spans V, then $n \le m$.

Suppose $x_1v_1 + \cdots + x_nv_n = 0$, where

$$v_i = a_{1i}w_1 + a_{2i}w_2 + \cdots + a_{mi}w_m,$$

then we have

$$a_{i1}x_1 + \cdots + a_{im}x_n = 0$$

for all $1 \le i \le m$. If $n > m$, then there exists a non-zero solution to this system, which contradicts to the fact that $x_1 = x_2 = \cdots = x_n = 0$.

Corollary 1.10.6. For $A \in M_{m\times n}(F)$, we know there exists invertible P, Q s.t.

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Corollary 1.10.7. row rank is equal to col rank.

Question. How to show A invertible?

Answer. Check RREF of A is I_n or not. ⊛

Question. How to find A^{-1}?

Answer. Calculate $(A \mid I_n)$. ⊛

Chapter 2

Dual space

Consider a vector space V, and V is over a field F, then we call

$$V^* = \mathcal{L}(V, F).$$

Definition 2.0.1. Suppose V is a vector space over F (with basis $\{1\}$), then

- A linear functional f is a linear map $f : V \to F$.
- $V^* = \mathcal{L}(V, F)$ is called the dual space of V.

Example 2.0.1. Suppose $V = F^n$, then $V^* = M_{1\times n}(F)$.

Note that Suppose $f \in V^*$ corresponds to $(a_1, a_2, \ldots, a_n)$, then $f(e_i) = a_i$.

Example 2.0.2. Suppose $V = M_{n\times n}(F)$, then the tract map

$$\mathrm{tr} : M_{n\times n}(F) \to F \quad (a_{ij}) \mapsto \sum_{i=1}^{n} a_{ii}$$

is in V^*.

Example 2.0.3. We can define $E^*_{pq} \in V^*$ by

$$E^*_{pq}((a_{ij})) = a_{pq},$$

then $\left\{E^*_{ij}\right\}$ is a basis of V^*.

Example 2.0.4. Suppose

$$V = \{\text{continuous function } f : [p, q] \to \mathbb{R}\},$$

then we can define ev_s, the evaluation at s, by

$$\mathrm{ev}_s(f) = f(s),$$

and we can define $I : V \to \mathbb{R}$ with

$$I(f) = \int_p^q f(x)\,\mathrm{d}x,$$

then ev_s and I are both elements of V^*.

Lecture 10

3 Oct. 10:20

Definition 2.0.2. $A, B \in M_n(F)$ are called similar or $A \sim B$ iff $B = P^{-1}AP$.

Notation. We call $\mathcal{L}(V, F)$

$$V^* \quad \text{or} \quad V^\vee \quad \text{or} \quad V^t.$$

Theorem 2.0.1.

$$\dim V = \dim V^*.$$

Matrix relation proof. Since $V^* \simeq M_{1\times n}(F)$, where $n = \dim V$, so

$$\dim V^* = \dim M_{1\times n}(F) = n = \dim V.$$

■

Proof. Suppose $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of V, and define $B^* = \{v_1^*, v_2^*, \ldots, v_n^*\}$ as

$$v_i^*(v_j) = \delta_{ij}.$$

Note that $v_i^* \in \mathcal{L}(V, F)$ for all i. Note that for all $v = \sum_{i=1}^n \alpha_i v_i$, we have

$$v_i^*(v) = \alpha_i.$$

Check B^* is linearly independent: Suppose $f = \sum \alpha_i v_i^* = 0$, then we know $f(v_j) = \alpha_j = 0$ for all j. Also, note that B^* spans V^*. ■

Remark 2.0.1.

$$[v]_B = \begin{pmatrix} v_1^*(v) \\ \vdots \\ v_n^*(v) \end{pmatrix}$$

Example 2.0.5. Suppose $V = F^2$ and $B = \{e_1, e_2\}$, then V^* is identified with

$$\mathcal{L}\left(F^2, F\right) = M_{1\times 2}(F),$$

where $B^* = \{e_1^*, e_2^*\}$ with

$$e_1^* = (1, 0) \quad e_2^* = (0, 1).$$

Now if we know $T : V \to W$ is a linear map, then we can define $T^* : W^* \to V^*$ by

$$T^* : f \mapsto f \circ T,$$

and we called it the transpose of T. We will show that if $[T]_C^B = M$, then $[T^*]_{B^*}^{C^*} = N = M^t$, which means if $M = (m_{ij})_{m\times n}$ and $N = (n_{ij})_{n\times m}$, then $n_{ij} = m_{ji}$ for all i, j with $1 \le i \le n$ and $1 \le j \le m$.

Proof. Suppose $T^*\left(w_j^*\right) = \sum_{p=1}^n n_{pj} v_p^*$, then since

$$w_j^*(T(v_j)) = w_j^*\left(\sum_{q=1}^m m_{qi} w_q\right) = m_{ji},$$

so $n_{ij} = m_{ji}$. (See Remark 2.0.1) Note that the below one is the evaluation of the above equation at v_j. ■

Lecture 11

8 Oct. 10:20

Definition 2.0.3 (Annihilator). Let $S \subseteq V$ be a subset, then the annihilator $S^0 \subseteq V^*$ is the subset defined by

$$\{f \in V^* \mid f(x) = 0 \quad \forall x \in S\}.$$

Proposition 2.0.1. For all $S \subseteq V$, S^0 is a subspace of V^*.

Proof. For all $f, g \in S^0$, we know

$$(cf + g)(x) = cf(x) + g(x) = 0 \quad \forall x \in S,$$

so $cf + g \in S^0$. ■

Example 2.0.6. $\{0\}^0 = V^*$ and $V^0 = \{0\}$.

Proposition 2.0.2. If $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

Proof. If $f \in S_2^0$, then $f(x) = 0$ for all $x \in S_2$, so $f(x) = 0$ for all $x \in S_1$, and thus $f \in S_1^0$, which means $S_2^0 \subseteq S_1^0$. ■

Proposition 2.0.3. If $W = \langle S \rangle$, then $W^0 = S^0$.

Proof. Since $S \subseteq W$, so we know $W^0 \subseteq S^0$ by Proposition 2.0.2. Also, for all $f \in S^0$, we know for all $x \in \langle S \rangle$, $x = \sum \alpha_i x_i$ where x_i's are elements of S, so

$$f(x) = f\left(\sum \alpha_i x_i\right) = \sum \alpha_i f(x_i) = 0,$$

which means $S^0 \subseteq W^0$. ■

Example 2.0.7. Suppose $W_1 \subseteq W_2 \subseteq V$, then $W_1^0 \supseteq W_2^0 \supseteq V^0$.

Proposition 2.0.4. Suppose V is finite dimensional and $W \subseteq V$, then $\dim W + \dim W^0 = \dim V = \dim V^*$.

Proof. Let $\dim W = m$ and $\dim V = n$, and take $B = \{w_1, \ldots, w_m\}$ a basis of W and $C = \{w_1, \ldots, w_m, v_{m+1}, \ldots, v_n\}$ as a basis of V. If we take dual of C, suppose

$$C^* = \left\{w_1^*, w_2^*, \ldots, w_m^*, v_{m+1}^*, \ldots, v_n^*\right\},$$

and now we claim $\left\{v_{m+1}^*, \ldots, v_n^*\right\}$ is a basis of W^0. For all $f \in V^*$, we know $f = \sum_{i=1}^m \alpha_i w_i^* + \sum_{j=m+1}^n \beta_j v_j^*$. Now if $f \in W^0$, then we know $f(w) = 0$ for all $w \in W$, so $f(w_i) = 0$ for all w_i's, and thus

$$f(w_i) = \sum_{i=1}^m \alpha_i w_i^*(w_i) + \sum_{j=m+1}^n \beta_j v_j^*(w_i) = \alpha_i = 0,$$

so we know $f = \sum_{j=m+1}^n \beta_j v_j^*$, which means $f \in \langle v_{m+1}^*, \ldots, v_n^* \rangle$. Thus, $W^0 \subseteq \langle v_{m+1}^*, \ldots, v_n^* \rangle$ Also, $v_i^*(w) = 0$ for all $w \in W$, so we know $\langle v_{m+1}^*, \ldots, v_n^* \rangle \subseteq W^0$, and we're done. ■

Corollary 2.0.1. If $\dim V, \dim W < \infty$ and $T : V \to W$ is linear, and we define $T^* : W^* \to V^*$ as T's transpose, then $\operatorname{rank} T = \operatorname{rank} T^*$.

Proof. First we show that $\ker T^* = (\operatorname{Im} T)^0$. Suppose $f \in \ker T^*$, then

$$0 = T^*(f) = fT,$$

so $fT(v) = 0$ for all $v \in V$, so $f(w) = 0$ for all $w \in \operatorname{Im} T$, so $f \in (\operatorname{Im} T)^0$. Conversely, we can similarly show that $(\operatorname{Im} T)^0 \subseteq \ker T^*$, and we're done. Note that

$$\dim W^* - \operatorname{rank} T^* = \nu(T^*) = \dim\left(\operatorname{Im}(T)^0\right) = \dim W - \dim(\operatorname{Im} T) = \dim W - \operatorname{rank} T,$$

and since $\dim W = \dim W^*$, so we know $\operatorname{rank} T = \operatorname{rank} T^*$. ■

Corollary 2.0.2. Suppose A is a matrix, then its row rank and column rank are same.

Proof. By regarding A as a linear map T's corresponding matrix, then T^*'s corresponding matrix is A^t, and since we have shown that $\operatorname{rank} T = \operatorname{rank} T^*$, so A's row rank is equal to A^t's row rank, which is A's column rank. ■

2.1 Dual of Dual space/Evaluation

We first define that $V^{**} = (V^*)^*$, and we can define a linear map

$$\mathrm{ev} : V \to V^{**}, \quad x \mapsto \widetilde{x},$$

where $\widetilde{x}$ is the functional

$$\widetilde{x} : V^* \to F \quad f \mapsto f(x).$$

Theorem 2.1.1. ev is an isomorphism between V and V^{**}.

Proof. We can check $\widetilde{x}$, ev are linear easily. DIY

Lemma 2.1.1. If $v \in V$ is not zero, then there exists $f \in V^*$ s.t. $f(v) \neq 0$.

Proof. Take $B = \{v_1 = v, v_2, \ldots, v_n\}$ as a basis of V and take dual B^*, then $v_1^*(v) = 1$. ■

Claim 2.1.1. $\mathrm{ev} : V \to V^{**}$ is injective.

Proof. Suppose $v \in \ker \mathrm{ev}$, then $\widetilde{v} = 0$, which means $f(v) = 0$ for all $f \in V^*$, so $v = 0$ by Lemma 2.1.1, and thus ev is injective. ■

Since $\dim V = \dim V^* = \dim (V^*)^* = \dim V^{**}$, so injectivity implies bijectivity. ■

Corollary 2.1.1. If $T : V \to W$ is a linear map with inverse $S : W \to V$, then $T^* : W^* \to V^*$'s inverse is $S^* : V^* \to W^*$, where S^* is the transpose of S.

Corollary 2.1.2 (Matrix ver)**.** Suppose $A \in M_n(F)$ is invertible, then A^t is invertible, and

$$\left(A^t\right)^{-1} = \left(A^{-1}\right)^t.$$

Chapter 3

Eigenvalue and Eigenvector

Lecture 12

15 Oct. 10:20

Question. If V is a vector space and $\dim V < \infty$, if $T : V \to V$ is a linear map, then is there a basis of V,

$$B = \{v_1, v_2, \dots, v_n\}$$

s.t. $T(v_i) = \lambda_i v_i$ for some $\lambda_i \in F$ i.e.

$$[T]_B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Note that this question is equivalent to find some linearly independent $\{v_i\}_{i=1}^n$ s.t.

$$A \underbrace{\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}}_{P} = \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{pmatrix} = \underbrace{\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}}_{P} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

which means is there invertible P s.t. $P^{-1}AP$?

Question. Why we want to diagonalize a matrix?

Answer. If we have $A = PBP^{-1}$, then $A^k = PB^kP^{-1}$, and if B is diagonal, say

$$B = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix},$$

then

$$B^k = \begin{pmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{pmatrix},$$

and it is easy to compute. ⊛

One of the applications of diagonalization is about recurrence relation. If we have a sequence $\{a_i\}_{i=0}^{\infty}$, where

$$a_{k+2} = \alpha a_{k+1} + \beta a_k,$$

then suppose $v_k = (a_k, a_{k+1})^t$, then

$$v_k = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} a_{k-1} \\ a_k \end{pmatrix} = Av_{k-1},$$

so we have $v_k = A^k v_0$, and thus if we know diagonalization, then we can compute A^k quickly.

Now we talk about how to find λ, v s.t. $T(v) = \lambda v$. If $v = 0$, then it is trivial, so we suppose $v \neq 0$, and thus it is equivalent to find λ, v s.t.

$$(T - \lambda I)(v) = 0.$$

Definition 3.0.1 (Singular). A matrix or linear operator is singular if it is not invertible.

Thus, we want to find λ s.t. $T - \lambda I$ is singular since if $T - \lambda I$ is invertible, then $v = 0$.

Definition 3.0.2 (Adjoint of a matrix). If $A \in M_n(F)$, then we define the adjoint of A to be $\operatorname{adj}(A) \in M_n(F)$ where

$$(\operatorname{adj}(A))_{ij} = (-1)^{i+j} \det\left(A\left(j \mid i\right)\right),$$

where $A\left(j \mid i\right)$ is A deleting its j-th row and i-th column.

Note 3.0.1. If we look at $M_2(F)$ and $M_3(F)$, we can find that

$$A \cdot \operatorname{adj}(A) = \det(A) I.$$

In fact, this is true for square matrices of all sizes.

Remark 3.0.1. A is invertible iff $\det(A) \neq 0$.

Proof. We will later show the proof. ■

We first introduce some good properties:

(1) Multilinear.

(2) Alternating.

(3) $\det(I_n) = 1$.

Definition 3.0.3 (Multilinear). Consider a function D of n row vectors in F^n as its input, and the output is $D(v_1, v_2, \ldots, v_n) \in F$, then D is called multilinear or n-linear if

$$\begin{aligned} D(u + \alpha w, v_2, \ldots, v_n) &= D(u, v_2, \ldots, v_n) + \alpha D(w, v_2, \ldots, v_n) \\ &\vdots \\ D(v_1, v_2, \ldots, u + \alpha w) &= D(v_1, v_2, \ldots, u) + \alpha D(v_1, v_2, \ldots, w). \end{aligned}$$

Example 3.0.1. If we suppose $A \in M_n(F)$, and r_i is the i-th row of A, where $r_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, then If we define $D(A) = a_{ak_1} a_{2k_2} \ldots a_{nk_n}$, then in fact D is multilinear if we regard D as a function which takes n row vectors as its input.

Lemma 3.0.1. If D_1, D_2 are n-linear, then $D_1 + \alpha D_2$ is also n-linear. If D is n-linear, then D is determined by $D(v_1, \ldots, v_n)$ with $v_i \in \{e_i\}_{i=1}^n$.

Note 3.0.2. D is a function determined by n^n values since each v_i has n choices.

Definition 3.0.4 (Alternating). Suppose D is n-linear, then D is alternating if

$$D(v_1, \ldots, v_n) = 0$$

if $v_i = v_j$ for some $i \neq j$.

Lemma 3.0.2. If D is alternating, then

(1)
$$D(\dots, \overbrace{v_i + \alpha v_j}^{i\text{-th position}}, \dots) = D(\dots, \overbrace{v_i}^{i\text{-th position}}, \dots).$$

(2) If $\{v_1, v_2, \dots, v_n\}$ is linearly dependent, then $D(v_1, v_2, \dots, v_n) = 0$.

(3)
$$D(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -D(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

proof of (2). WLOG, say $v_i = \sum_{j \neq i} \alpha_j v_j$, then

$$D(v_1, \dots, v_n) = D\left(v_1, \dots, \sum_{j \neq i} \alpha_j v_j, \dots, v_n\right) = \sum_{j \neq i} \alpha_j D(v_1, \dots, \overbrace{v_j}^{i\text{-th position}}, \dots, v_n) = 0$$

since D is alternating.

■

proof of (3). Since

$$\begin{aligned} 0 &= D(\dots, v_i + v_j, \dots, v_i + v_j, \dots) \\ &= D(\dots, v_i, \dots, v_i, \dots) + D(\dots, v_i, \dots, v_j, \dots) + D(\dots, v_j, \dots, v_i, \dots) + D(\dots, v_j, \dots, v_j, \dots) \\ &= D(\dots, v_i, \dots, v_j, \dots) + D(\dots, v_j, \dots, v_i, \dots), \end{aligned}$$

so this is true. ■

Proposition 3.0.1. If D is n-linear and alternating, then it is determined by

$$D\left(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}\right),$$

where $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is any permutation on $[n]$.

Remark 3.0.2. In this case, there is at most one n-linear alternating D satisfying $D(e_1, \dots, e_n) = 1$.

Proof. Since D is alternating, so swaping e_i and e_j just turn the original value to negative. Thus, if $D(e_1, \dots, e_n) = 1$, then we know

$$D\left(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}\right)$$

is uniquely defined for all permutation σ. Now since D is determined by $D\left(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}\right)$, so D is uniquely defined. ■

Another approach/inductive construction

Theorem 3.0.1. There exists a function

$$\det_n : M_n(F) \to F,$$

s.t. $\det_n$ is n-linear(on rows) and alternating(on rows) and $\det(I_n) = 1$.

We can just define

$$\begin{cases} \det_1(a) = a \\ \det_n(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det_{n-1}(A(i \mid j)) \end{cases},$$

where $A(i \mid j)$ is A deleting i-th row and j-th column.

Note 3.0.3. The definition given above is the expansion along j-th column.

Note 3.0.4. Since we know there is at most one n-linear, alternating D satisfying $D(e_1, e_2, \ldots, e_n) = 1$, and we have constructed such D, and thus we can define this D to be the determinant function.

Lecture 13

Actually determinant can be defined on ring (we defined it on field before). 17 Oct. 10:20

Theorem 3.0.2. There is the determinant function

$$\det : M_n(R) \to R.$$

Now we talk more about expansion. We do expansion along a column. Suppose we have

$$\delta : M_{n-1}(R) \to R,$$

which is $(n-1)$-linear and alternating and $\delta(I_{n-1}) = 1$, then if we define $D_j = D : M_n(R) \to R$, which is the expansion along the j-th column, and it has

$$D(A = (a_{ij})) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \delta\left(A(i \mid j)\right).$$

Note 3.0.5. $C_{ij} = (-1)^{i+j}\delta\left(A(i \mid j)\right)$ is called the (i,j)-cofactor.

Theorem 3.0.3. D is n-linear and alternating, and $D(I_n) = 1$.

Proof. ■ DIY

Note 3.0.6. In the proof of alternating, we may need to use Lemma 3.0.2.

Note 3.0.7. We still regard D as a function taking n row vectors as its input.

As previously seen. If $D : M_n(R) \to R$ is n-linear, alternating, then

$$D\left((a_{ij})\right) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)} D \begin{pmatrix} e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n)} \end{pmatrix}$$

Proof. Suppose $A = (a_{ij})_{n\times n}$'s rows are $r_1, r_2, \dots, r_n$, then we know $r_i = \sum_{j_i=1}^{n} a_{ij_i} e_{j_i}$, so we know

$$\begin{aligned} D(A) &= \sum_{j_1=1}^{n} a_{1j_1} D(e_{j_1}, r_2, \dots, r_n) = \sum_{j_1=1}^{n} a_{1j_1} \left(\sum_{j_2=1}^{n} a_{2j_2} D(e_{j_1}, e_{j_2}, r_3, \dots, r_n) \right) \\ &= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} a_{1j_1} a_{2j_2} D(e_{j_1}, e_{j_2}, r_3, \dots, r_n) \\ &= \cdots = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} a_{2j_2} \dots a_{nj_n} D(e_{j_1}, e_{j_2}, \dots, e_{j_n}) \\ &= \sum_{\sigma\in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right) D(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}) \end{aligned}$$

since if $j_p = j_q$ for some $p \neq q$, then since D is alternating, so we know that term will be 0, and thus we just need to consider the terms with $j_p \neq j_q$ for any $p \neq q$. ■

Now we put things together:

Theorem 3.0.4.

(i) There is a function $\det : M_n(R) \to R$ satisfying n-linear, alternating, and $\det(I_n) = 1$.

(ii) If $D : M_n(R) \to R$ is n-linear, alternating, then $D(A) = D(I) \cdot \det(A)$.

(iii) For a permutation σ, if $\sigma = t_1 t_2 \dots t_n = t_1' t_2' \dots t_m'$, where t_i, t_i''s are transpositions, then $(-1)^n = (-1)^m$.

Remark 3.0.3. (ii) needs the fact that

$$D\left(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}\right) = (-1)^m D(e_1, e_2, \dots, e_n)$$

if σ is the composition of m traspositions.

Remark 3.0.4. (i) and (ii) hold for any R.

Now we introduce two formulas:

(1)
$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det\left(A(i \mid j)\right).$$

(2)
$$\operatorname{sgn} : \{\text{permutation}\} \to \{\pm 1\}, \quad \sigma \mapsto (-1)^m$$

if $\sigma = t_1 t_2 \dots t_m$ if t_i's are transpositions.

Thus, we know

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

by the proof above and Remark 3.0.3.

Lecture 14

29 Oct. 10:20

As previously seen. There is a unique function

$$\det : M_n(R) \to R$$

satisfying n-linear in rows, alternating, and $\det(I_n) = 1$. Also, if $D : M_n(R) \to R$ satisfies n-linear and alternating, then $D(A) = D(I) \cdot \det(A)$. Besides, det can be constructed inductively:

$$\det(A) = \sum_{i=1}^{n} a_{ij} c_{ij}$$

where $c_{ij} = (-1)^{i+j} \det(A(i \mid j))$ is the (i,j)-cofactor.

If $\sigma \in S_n$, and let $\sigma(I) = (e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)})$ (permuting the rows), then $\det(\sigma(I)) = (-1)^m$ if $\sigma = \tau_1 \tau_2 \ldots \tau_m$ where τ_i is a transposition since det is alternating, so exchange two rows in the function input change the sign of the output.

Corollary 3.0.1. For $\sigma \in S_n$, if $\sigma = \tau_1 \tau_2 \ldots \tau_p = \tau_1' \tau_2' \ldots \tau_q'$, then p and q are both even or both odds.

Definition 3.0.5. $\sigma \in S_n$ is called an even(resp. odd) permutation if $\sigma = \tau_1 \tau_2 \ldots \tau_m$ for m even(resp. odd). Thus, we can define

$$\operatorname{sgn} : S_n \to \{\pm 1\}, \quad \sigma \mapsto \det(\sigma(I)).$$

Hence, we can give a second method to construct det:

$$\det\left((a_{ij})_{n\times n}\right) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Example 3.0.2. If we want to calculate

$$\det \begin{pmatrix} 0 & 0 & & a_n \\ a_1 & 0 & & 0 \\ & & \ddots & \\ 0 & \cdots & a_{n-1} & 0 \end{pmatrix},$$

then we have two ways:

(1) expand along the last column.

(2) Suppose $A = (a_{ij})_{n\times n}$, where $a_{ii} = a_i$ for all i and $a_{ij} = 0$ for all $i \neq j$, then $\det A = a_1 a_2 \ldots a_n$, and the matrix given in the problem is from exchanging first row and second row of A, then exchange second row and third row, and keep going until exchanging the $n-1$-th row and n-th row, so the answer is $(-1)^{n-1} a_1 a_2 \ldots a_n$ since it takes $n-1$ times exchangement. (exchange rows in the inoput of an alternating function will change the sign of output.)

Example 3.0.3. Companion form of $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$:

$$A_f = \begin{pmatrix} 0 & 0 & \cdots & -a_n \\ 1 & 0 & \cdots & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & -a_1 \end{pmatrix}.$$

We can calculate $\det(xI - A_f) = f(x)$.

Theorem 3.0.5. Suppose $A, B \in M_n(R)$, where R is a ring with identity, then

$$\det(AB) = \det(A) \det(B).$$

Thus, we have $\det\left(P^{-1}\right) = \det(P)^{-1}$.

Proof. Let $D(A) = \det(AB)$, then we can check that D satisfies n-linear and alternating. If this were true, then $D(A) = D(I)\det(A)$, and $D(I) = \det(IB) = \det(B)$, so $D(A) = \det(A)\det(B)$ and thus we have

$$\det(AB) = \det(A)\det(B).$$

Note 3.0.8. Note that

$$D\begin{pmatrix} u_1 \\ \vdots \\ v+\alpha w \\ \vdots \\ u_n \end{pmatrix} = \det\left(\begin{pmatrix} u_1 \\ \vdots \\ v+\alpha w \\ \vdots \\ u_n \end{pmatrix} B\right) = \det\left(\begin{pmatrix} u_1 B \\ \vdots \\ vB+\alpha wB \\ \vdots \\ u_n B \end{pmatrix}\right) = D\begin{pmatrix} u_1 \\ \vdots \\ v \\ \vdots \\ u_n \end{pmatrix} + \alpha D\begin{pmatrix} u_1 \\ \vdots \\ w \\ \vdots \\ u_n \end{pmatrix},$$

and alternating can be proved similarly.

■

Theorem 3.0.6. If $A \sim B$, then $\det A = \det B$.

Theorem 3.0.7. $\det A^t = \det A$.

Proof. Note that

$$a_{1\sigma(1)} \dots a_{n\sigma(n)} = a_{\sigma^{-1}(1),1} \dots a_{\sigma^{-1}(n),n},$$

and $\operatorname{sgn}(\sigma) = \operatorname{sgn}\left(\sigma^{-1}\right)$.Hence, if we suppose $B = A^t$, then

$$\begin{aligned} \det(B) &= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod b_{i,\sigma(i)} \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod a_{\sigma(i),i} \\ &= \sum_{\tau:\tau=\sigma^{-1}} \operatorname{sgn}(\tau) \prod a_{i,\tau(i)} = \det(A). \end{aligned}$$

■

Exercise 3.0.1. Show that

$$\det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A)\det(D).$$

Theorem 3.0.8. Let $A \in M_n(R)$, then we can define the (classical) adjoint

$$\operatorname{adj}(A) = \widetilde{A} = (\widetilde{a_{ij}}),$$

where

$$\widetilde{a_{ij}} = (j,i)\text{-cofactor } c_{j,i} = (-1)^{i+j} \det\left(A(j \mid i)\right),$$

then $A\widetilde{A} = \widetilde{A}A = \det(A)I$. This means if A is invertible, then $A^{-1} = \frac{1}{\det(A)}\widetilde{A}$.

Proof. Note that the (i,i)-entry of $A\widetilde{A}$ is

$$\sum_{k=1}^{n} a_{ik}\widetilde{a_{ki}} = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det(A(i \mid k)) = \det(A),$$

while the (i,j)-entry for $i \neq j$ is

$$\sum_{k=1}^{n} a_{ik}\widetilde{a_{kj}} = \sum_{k=1}^{n}(-1)^{j+k}a_{ik}\det\left(A(j \mid k)\right)$$
$$= \det\begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \\ & & & \end{pmatrix} (j\text{-th row}) = 0$$

since det is alternating. Thus, $A\widetilde{A} = \det(A)I$. Similarly, we can show $\widetilde{A}A = \det(A)I$. ■

Theorem 3.0.9. Suppose $A \in M_n(F)$ is invertible, then consider the system

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

then $x_i = \frac{1}{\det(A)}\det(C_i)$, where C_i is the matrix A but replace the i-th column with $(c_1, c_2, \dots, c_n)^t$.

Proof. In fact,

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det(A)}\widetilde{A}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

and by comparing the entries, we know

$$\det(A)x_i = \sum_{j=1}^{n}(-1)^{i+j}c_j\det(A(j \mid i)) = \det(C_i).$$

■

Exercise 3.0.2. If $v_1, v_2, \dots, v_n \in \mathbb{R}^n$, then

$$\det(v_1, v_2, \dots, v_n) = \pm\text{volumn}.$$

Definition 3.0.6. For finite dimensional vector space V, suppose $T \in \mathcal{L}(V)$, then one can define $\det(T)$ by choosing an ordered basis B of V, and define

$$\det(T) := \det\left([T]_B\right).$$

Remark 3.0.5. This $\det(T)$ does not depend on the choice of B since

$$[T]_B \sim [T]_{B'}$$

for any two basis B, B' of V. This is because

$$[T]_{B'} = [id]_{B'}^{B}[T]_B[id]_B^{B'}.$$

Lecture 15

Definition 3.0.7. Let $T \in \mathcal{L}(V)$ (or a matrix $A \in M_n(F)$). A scalar $\lambda \in F$ is called an eigenvalue of T if $\exists v \neq 0$ s.t. $Tv = \lambda v$. Equivalently, $T - \lambda I$ is singular, or $\det(T - \lambda I) = 0$ or $\nu(T - \lambda I) > 0$. In

31 Oct. 10:20

this case, $E(\lambda) = \ker(T - \lambda I)$ is called the eigenspace and any vector in $E(\lambda)$ is called an eigenvector (for λ).

Remark 3.0.6. If A is not invertible, then $\det(A) = 0$ since there is a row of A is the linear combination of other rows, and det is n-linear and alternating.

Remark 3.0.7. Eigenvalues are also called characteristic values, proper value, spectral value.

If $A \in M_n(F)$ is the matrix representation of T, then

$$\det(T - \lambda I) = \det(A - \lambda I) = (-1)^n \det(\lambda I - A).$$

Definition 3.0.8. The polynomial $f(x) = \det(xI - A)$ is called the characteristic polynomial of T.

Remark 3.0.8. $f(x)$ does not depend on the choice of matrix representation since if we choose another $B = P^{-1}AP$, then

$$\begin{aligned}\det(xI - B) &= \det\left(xI - P^{-1}AP\right) = \det\left(P^{-1}(xI)P - P^{-1}AP\right) \\ &= \det\left(P^{-1}(xI - A)P\right) = \det\left(P^{-1}\right)\det(xI - A) = \det(P) = \det(xI - A).\end{aligned}$$

Remark 3.0.9. One can verify that for two similar matrices A, B, we have $\mathrm{Tr}(A) = \mathrm{Tr}(B)$.

Remark 3.0.10. Note that

$$f(x) = x^n - \mathrm{Tr}(T)x^{n-1} + \cdots + (-1)^n \det(T).$$

This is because x^n and x^{n-1} terms come from $(x - a_{11})(x - a_{22})\dots(x - a_{nn})$, and by Vieta's theorem, we know the coefficient of x^{n-1} is $\mathrm{Tr}(T)$. Also, $f(0) = \det(-A) = (-1)^n \det(A)$ is trivial.

Remark 3.0.11. For the coefficient of x^{n-1}, suppose $B = xI - A$, then we know

$$\det B = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \dots b_{n\sigma(n)},$$

so if some term contributes x^{n-1}, then at least $n - 1$ of $\sigma(i)$ is equal to i, which means all n of $\sigma(i)$'s are i, and thus the only term contributes x^{n-1} is $(x - a_{11})(x - a_{22})\dots(x - a_{nn})$.

Theorem 3.0.10. λ is an eigenvalue of T iff λ is a root of $f(x)$.

3.1 Diagonalization

Definition 3.1.1. $T \in \mathcal{L}(V)$ is called diagonalizable if $\exists$ matrix representation of T, which is a diagonal matrix. A matrix A is called diagonalizable if A is similar to a diagonal matrix.

If

$$[T]_B = \begin{pmatrix} \lambda_1 I_1 & & \\ & \ddots & \\ & & \lambda_r I_{m_r} \end{pmatrix}$$

and $\lambda_i \neq \lambda_j$ for any $i \neq j$ with

$$B = \bigcup_{i=1}^{r} \{v_{i1}, v_{i2}, \dots, v_{im_i}\},$$

then $f(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\dots(x-\lambda_r)^{m_r}$ splits (by plugging $[T]_B$ into $\det(xI-A)$), and we have $\dim(E(\lambda_i)) = \dim\ker(T-\lambda_i I) = m_i$, which can been seen by observing the rank of matrix $[T]_B - \lambda_i I$. Also, we can observe that $V = E(\lambda_1)+E(\lambda_2)+\cdots+E(\lambda_r)$, so $\dim V = \sum_{i=1}^r \dim E(\lambda_i)$ since

$$E(\lambda_i)\cap E(\lambda_j) = \{0\}$$

for any $i \neq j$.

Definition 3.1.2. Suppose λ is an eigenvalue of T and characteristic polynomial $f(x) = (x-\lambda)^m g(x)$ with $g(\lambda) \neq 0$. The algebraic multiplicity of λ, a-mult$(\lambda) = m$, and the geometric multiplicity g-mult$(\lambda) = \dim(E_\lambda) = \nu(T-\lambda I) \geq 1$.

Proposition 3.1.1. a-mult$(\lambda) \geq$ g-mult(λ).

Proof. Let $\{v_1,\dots,v_e\}$ be a basis of $E(\lambda)$, and extend it to a basis of V, say $B = \{v_1,\dots,v_e,\dots,v_n\}$. Hence,

$$A = [T]_B = \begin{pmatrix} \lambda I_e & B \\ 0 & D \end{pmatrix},$$

which gives

$$f(x) = \det(xI-A) = (x-\lambda)^e \det(xI-D),$$

note that $\det(xI-D)$ may have λ as a root, so the algebraic multiplicity of $\lambda \geq$ the geometric multiplicity of λ.

Note 3.1.1. If A is not diagonalizable, then we know $\det(xI-D)$ may have λ as its root.

■

Definition 3.1.3. Let $W_1, W_2, \dots, W_r$ be subspaces of V. We say W_i's are linearly independent if $w_1+w_2+\cdots+w_r = 0$ for $w_i \in W_i$, then $w_i = 0$ for all i.

Proposition 3.1.2. Let $W = W_1+W_2+\cdots+W_r$, then TFAE:

(i) W_i are linearly independent.

(ii) Any $w \in W$ has a unique expression

$$w = \sum_{i=1}^r w_i, \quad \forall w_i \in W_i.$$

(iii)

$$W_i \cap (W_1+W_2+\cdots+W_{i-1}+W_{i+1}+\cdots+W_r) = \{0\}.$$

(iv) $\dim W = \sum_{i=1}^r \dim W_i$.

(i) to (ii),(iii),(iv). ■ DIY

(ii) to (i). If $\sum w_i = 0$, then since $\sum 0 = 0$ and $0 \in W_i$ for all i, and 0 has unique expression, so $w_i = 0$ for all i. ■

(iii) to (i). If $\sum w_i = 0$ for $w_i \in W_i$, then

$$-w_i = w_1+w_2+\cdots+w_{i-1}+w_{i+1}+\cdots+w_r \in W_i \cap (W_1+W_2+\cdots+W_{i-1}+W_{i+1}+\cdots+W_r) = \{0\}$$

for all i, so $w_i = 0$ for all i. ■

(iv) to (i). If $\{v_{ij}\}_{j=1}^{m_i}$ is a basis of W_i, then $\{v_{ij}\}_{i,j}$ generates W. Also, we know $\dim W = \sum_{i=1}^{r} \dim W_i$, so $\{v_{ij}\}_{i,j}$ is a basis of W. Now if $\sum_{i=1}^{r} w_i = 0$, so we have $\sum_{i,j} \alpha_{ij} v_{ij} = 0$, and thus $\alpha_{ij} = 0$ for all i, j. Hence, $w_i = 0$ for all i. ■

Proposition 3.1.3. If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct eigenvalues of T, then $\{E(\lambda_i)\}_{i=1}^{r}$ are linearly independent.

Proof. Suppose $v_1 + v_2 + \cdots + v_r = 0$ for $v_i \in E(\lambda_i)$, then by applying T, we know $\lambda_1 v_1 + \cdots + \lambda_r v_r = 0$, so we have

$$(\lambda_2 - \lambda_1)v_2 + \cdots + (\lambda_r - \lambda_1)v_r = 0.$$

Hence, by this thought, suppose $v_1 + \cdots + v_m = 0$ for $v_i \in E(\lambda_i)$ and it is a shortest equality of a non-trivial relation. Then, we can always obtain a shorter non-trivial relation by above method, so it is a contradiction. ■

Corollary 3.1.1. If $\{v_{ij}\}_{j=1}^{m_i}$ is a basis of $E(\lambda_i)$, then $B = \bigcup_{i=1}^{r} \{v_{ij}\}_{j=1}^{m_i}$ is linearly independent.

Proof. Suppose $\sum_{i=1}^{r} \sum_{j=1}^{m_i} \alpha_{ij} v_{ij} = 0$, then since $\sum_{j=1}^{m_i} \alpha_{ij} v_{ij} \in W_i$, so since $\{E(\lambda_i)\}_{i=1}^{r}$ are linearly independent, so we know $\sum_{j=1}^{m_i} \alpha_{ij} v_{ij} \in W_i = 0$ for all i, and since $\{v_{ij}\}_{j=1}^{m_i}$ is a basis of $E(\lambda_i)$ for all i, so they are linearly independent, and thus we know $\alpha_{ij} = 0$ for all i, j, which shows B is linearly independent. ■

Corollary 3.1.2. Suppose $T \in \mathcal{L}(V)$ and has a characteristic polynomial

$$f(x) = \prod_{i=1}^{r} (x - \lambda_i)^{m_i}$$

with $\lambda_i \neq \lambda_j$ for any $i \neq j$, then TFAE:

(i) T is diagonalizable.

(ii) $\dim E(\lambda_i) = m_i$ for all i.

(iii) $V = \sum_{i=1}^{r} E(\lambda_i)$ (or any $v \in V$ is a linear combination of eigenvectors.)

(iv) $\dim V = \sum_{i=1}^{r} \dim E(\lambda_i)$.

Corollary 3.1.3. If the characteristic polynomial of a linear operator has degree n and has n distinct roots, then T is diagonalizable.

Proof. By (ii) of Corollary 3.1.2. ■

Corollary 3.1.4. If $T^2 = T$, then T is diagonalizable.

Lecture 16

Suppose V is a finite dimensional vector space, then fix $T \in \mathcal{L}(V)$, we have 5 Nov. 10:20

$$a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n \in \mathcal{L}(V),$$

which means $f(T) \in \mathcal{L}(V)$ where $f(x) = \sum_{k=0}^{n} a_k x^k \in F[x]$. We call V is an $F[x]$-module. (= "vector space over a ring") What makes the classification (structure theorem) simple. The answer is something like $F[x], \mathbb{Z}, \ldots$, the principal ideal domains(PID). Note that $F[x], \mathbb{Z}$ are Euclidean domain, which means that there is the degree map

$$\deg : F[x] \to \mathbb{Z}_{\geq 0} \text{ or } \deg : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$$

s.t. for any $a, b \in F[x]$ and $b \neq 0$, there exists unique $a = qb + r$ where $\deg r < \deg b$.

3.2 Minimal polynomial

Fix $T \in \mathcal{L}(V)$. For $g(x) = b_n x^n + \cdots + b_0 \in F[x]$, let $g(T) = b_n T^n + \cdots + b_0 \in \mathcal{L}(V)$. Note that

$$\begin{aligned} g(x) = g_1(x) \cdot g_2(x) &\Rightarrow g(T) = g_1(T) \cdot g_2(T). \\ g(x) = g_1(x) + g_(x) &\Rightarrow g(T) = g_1(T) + g_2(T). \\ \text{If } T(v) = \lambda v &\Rightarrow g(T)(v) = g(\lambda)(v). \end{aligned}$$

Definition 3.2.1. Suppose $T : V \to V$ is a linear operator, then we define

$$\begin{aligned} \mathrm{Ann}_T(V) &= \{\text{annihilator of } T\} \\ &= \{g(x) \in F[x] \mid g(T) = 0\} \\ &= \left\{\text{linear relations of } T^0, T^1, T^2, \cdots \in \mathcal{L}(V)\right\}. \end{aligned}$$

Note 3.2.1. There exists a non-trivial relation among $T^0, T^1, \ldots, T^{n_2}$ since $\dim \mathcal{L}(V) = n^2$.

Proposition 3.2.1. Let $m(x) = m_T(x)$ be a monic polynomial (leading coefficient is 1) in $\mathrm{Ann}_T(V)$ with minimal degree. Then,

$$\mathrm{Ann}_T(V) = F[x] \cdot m(x).$$

Proof. For any $g(x) \in \mathrm{Ann}_T(V)$, we have

$$g(x) = q(x) \cdot m(x) + r(x)$$

with $\deg r < \deg m$. Then,

$$0 = g(T) = q(T) \cdot m(T) + r(T) = r(T).$$

Since m is the "minimal degree" monic polynomial, so $r(x) = 0$. ■

Definition 3.2.2. This $m_T(x)$ is called the minimal polynomial of T.

Example 3.2.1. Suppose

$$A = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

then $m_A(x) = x^n$ since we can found that $A^n = 0$, so $m_A(x) \mid x^n$, so $m_A(x) = x^p$ for some $p \leq n$, and we can find that n is the minimal p s.t. $A^p = 0$.

Example 3.2.2. Suppose

$$B = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & -a_2 \\ & & \ddots & 0 & \vdots \\ & & & 1 & -a_{n-1} \end{pmatrix},$$

then we know $m_B(x) = f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

Remark 3.2.1. Check that $B(e_i) = e_{i+1}$ for $1 \le i \le n-1$ and $B(e_n) = \sum_{i=0}^{n-1} -a_i e_{i+1}$, and thus $f(B) = 0$ since it sends the standard basis to 0. Then, we can check that $\deg m_B(x) \ge n$, and we're done.

Remark 3.2.2. $f(B)(e_i) = f(B)B^{i-1}(e_1) = B^{i-1}f(B)(e_1) = 0$.

Example 3.2.3. Suppose

$$C = \begin{pmatrix} \lambda_1 I_{m_1} & & \\ & \ddots & \\ & & \lambda_r I_{m_r} \end{pmatrix},$$

then $m(x) = m_C(x) = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_r)$. This is beacause $Ce_k = \lambda e_k$ for some $\lambda = \lambda_1, \dots \lambda_r$, and thus

$$(C-\lambda_1)(C-\lambda_2)\dots(C-\lambda_r)(e_i) = 0$$

for all i, and thus we know

$$m_C(x) \mid (x-\lambda_1)\dots(x-\lambda_r).$$

Also, we can check that if $q(C) = 0$, then $(x-\lambda_i) \mid q(x)$ for all i by observing the matrix of $q(C)$.

Observe that if T is diagonalizable with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$, then

$$m_T(x) = \prod_{i=1}^{r}(x-\lambda_i),$$

and $\mathrm{ch}_T(x) \in \mathrm{Ann}_T(V)$(Cayley-Hamilton Theorem).

Proposition 3.2.2. If λ is some element of F, then

$$\mathrm{ch}_T(\lambda) = 0 \Leftrightarrow m_T(\lambda) = 0.$$

Proof.

($\Rightarrow$) Since there exists $v \neq 0$ s.t. $T(v) = \lambda v$, then

$$0 = m_T(T)(v) = m_T(\lambda)(v),$$

and since $v \neq 0$, so $m_T(\lambda) = 0$.

($\Leftarrow$) Write $m_T(x) = (x-\lambda)p(x)$, then $\exists v$ s.t. $p(T)(v) \neq 0$, so

$$0 = m_T(T)(v) = (T-\lambda)p(T)(v) = (T-\lambda)w,$$

and since $w \neq 0$, so $E(\lambda) \neq \{0\}$, so $(x-\lambda) \mid \mathrm{ch}_T(x)$.

■

3.3 Invariant subspaces

Definition 3.3.1. Suppose $T \in \mathcal{L}(V)$, then a subspace W is called T-invariant if $T(W) \subseteq W$. In this case, W is also $g(T)$-invariant for $g(x) \in F[x]$. Besides, we know T induces an operator $T_W = T|_W \in \mathcal{L}(W)$.

Example 3.3.1. If $ST = TS$, then $\ker(S)$ and $\mathrm{Im}(S)$ are T-invariant. In particular, $E(\lambda) = \ker(T-\lambda)$ is T-invariant.

Example 3.3.2. If W_1, W_2 are T-invariant, then $W_1 + W_2$ and $W_1 \cap W_2$ are T-invariant.

Proposition 3.3.1. Let W be T-invariant and $S = T_W \in \mathcal{L}(W)$, then we have $\mathrm{ch}_S(x) \mid \mathrm{ch}_T(x)$ and $m_S(x) \mid m_T(x)$.

Proof. Let $B = \{w_1, \dots, w_m\}$ be a basis of W, and extend it to a basis of V, say

$$\widetilde{B} = \{w_1, \dots, w_m, w_{m+1}, \dots, w_n\},$$

and suppose $A = [S]_B$ and $\widetilde{A} = [T]_{\widetilde{B}}$, then we know

$$\widetilde{A} = \begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \Rightarrow xI - \widetilde{A} = \begin{pmatrix} xI - A & C \\ 0 & xI - D \end{pmatrix},$$

so we know $\det\left(xI - \widetilde{A}\right) = \det(xI - A)\det(D - xI)$, which gives $\mathrm{ch}_{\widetilde{A}}(x) = \mathrm{ch}_A(x)\mathrm{ch}_D(x)$, so we proved the first part.

Now since $m_T(S) = m_T(T_W)$, and $m_T(T_W)(w) = m_T(T)(w) = 0$ for all $w \in W$, so $m_T(T_W) = 0$ and thus $m_T(S) = 0$, so $m_S(x) \mid m_T(x)$. ■

Definition 3.3.2. Let W be T-invariant, then

$$\begin{aligned} \mathrm{Ann}_T(V/W) &= \{f(x) \in F[x] \mid f(T)(v) \in W \quad \forall v\} \\ &= \{f(x) \in F[x] \mid f(T)(V) \subseteq W\}. \end{aligned}$$

In particular, we know $m_T(x) \in \mathrm{Ann}_T(V/W)$.

Lemma 3.3.1. Let $p(x) \in \mathrm{Ann}_T(V/W)$ be the monic polynomial of smallest degree, then

$$\mathrm{Ann}_T(V/W) = F[x] \cdot p(x).$$

Proof. Take $g \in \mathrm{Ann}_T(V/W)$, then $g = qp + r$, and

$$g(T)(v) = q(T)p(T)(v) + r(T)(v) \in W \quad \forall v \in V.$$

since $p(T)(v) \in W$ and W is $q(T)$-invariant, then $r(T)(v) \in W$, so $r(x) = 0$. ■

Theorem 3.3.1. T is diagonalizable if and only if $m_T(x) = \prod_{i=1}^{r}(x - \lambda_i)$ with $\lambda_i \neq \lambda_j$ for all $i \neq j$.

Proof.

($\Rightarrow$) We have shown in previous example.

($\Leftarrow$) Suppose $m_T(x) = \prod(x - \lambda_i)$ for $\lambda_i \neq \lambda_j$ for all $i \neq j$, and suppose

$$W = E(\lambda_1) + E(\lambda_2) + \cdots + E(\lambda_r),$$

then we know $W \subseteq V$ and W is T-invariant. Now if $W \neq V$, then let

$$\mathrm{Ann}_T(V/W) = F[x] \cdot p(x),$$

and WLOG we can suppose $p(x) = (x - \lambda_1)q(x)$ since $m_T(x) \in \mathrm{Ann}_T(V/W)$, and we can check that $p(x)$ cannot be a constant polynomial, otherwise $V = W$, which is a contradiction. Thus, there exists $v \in V$ s.t. $q(T)(v) \notin W$. Set

$$g(x) = \frac{m_T(x)}{x - \lambda_1} = (x - \lambda_2)\dots(x - \lambda_r),$$

then $g(x) = (x - \lambda_1)h(x) + g(\lambda_1)$. Note that $g(\lambda_1) \neq 0$, so if we pick $u = q(T)(v) \notin W$, then

$$g(T)(u) = h(T)(T - \lambda_1)(u) + g(\lambda_1)(u)$$

and $h(T)(T - \lambda_1)(u) = h(T)p(v) \in W$, and $g(\lambda_1)(u) \notin W$, and $g(T)(u) \in E(\lambda_1) \subseteq W$ since

$$(T - \lambda_1)g(T)(u) = (T - \lambda_1)(T - \lambda_2)\dots(T - \lambda_r)(u) = 0,$$

so we know this is a contradiction. Hence, $W = V$, so T is diagonalizable.

■

Lecture 17

7 Nov. 10:20

As previously seen. T is diagonalizable if and only if

$$\begin{cases} \mathrm{ch}_T(x) = \prod_{i=1}^{r}(x - \lambda_i)^{m_i}, \\ \dim E(\lambda_i) = m_i \quad \forall i \end{cases},$$

and we've learned that $m_T(x) = \prod_{i=1}^{r}(x - \lambda_i)$ for $\lambda_i \neq \lambda_j$.

3.4 Triangulanization and Cayley-Hamilton theorem

Definition 3.4.1. We call $T \in \mathcal{L}(V)$ triangulanizable if $\exists B = \{v_1, \dots\}$ s.t.

$$[T]_B = \begin{pmatrix} a_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{pmatrix},$$

i.e. $[T]_B$ is upper triangular. In particular, $T(v_k) \in \langle v_1, \dots, v_k \rangle$.

Corollary 3.4.1. If T is triangulanizable, then there exists a chain of T-invariant subspace $0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_n = V$ where $\dim W_k = k$ for all k.

Corollary 3.4.2. If T is triangulanizable, and

$$[T]_B = \begin{pmatrix} a_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{pmatrix},$$

then $\mathrm{ch}_T(x) = \prod_{i=1}^{n}(x - a_i)$ splits (e.g. This always holds for $F = \mathbb{C}$). (And by Cayley-Hamilton we know $m_T(x)$ splits completely).

Lemma 3.4.1. Suppose $m_T(x)$ splits. If W is a T-invariant proper subspace of V, then $\exists u \notin W$ (i.e. u and W are linearly independent), and $\lambda \in F$ s.t. $(T - \lambda)(u) \in W$.

Proof. Since we have

$$\mathrm{Ann}_T(V/W) = \{g(x) \in F[x] \mid g(T)(V) \subseteq W\} = F[x] \cdot p(x),$$

and we know $m_T(x) \in \mathrm{Ann}_T(V/W)$, so $p(x) = (x - \lambda)q(x)$ for some $\lambda \in F$, where $x - \lambda \mid m_T(x)$

since $W \neq V$ and $p(x) \mid m_T(x)$. Hence, there exists $v \notin W$ s.t. $u = q(T)(v) \notin W$. Thus, we know

$$(T-\lambda)(u) = (T-\lambda)q(T)(v) = p(T)(v) \in W.$$

■

Theorem 3.4.1. Suppose $m_T(x)$ splits, then T is triangulanizable.

Proof. Use induction (for finding a T-invariant chain). Suppose we have

$$0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_k \neq V,$$

where $\dim W_i = i$ for all i. Then, $\exists v_{k+1} \notin W_k$ and $\lambda \in F$ s.t.

$$(T-\lambda)(v_{k+1}) = \sum_{i=1}^{k} a_{i,k+1} v_i$$

by Lemma 3.4.1. Hence, $T(v_{k+1}) \in \langle v_1, v_2, \ldots, v_{k+1} \rangle$ and thus $\langle v_1, \ldots, v_{k+1} \rangle$ is T-invariant, so we can let $W_{k+1} = \langle v_1, \ldots, v_{k+1} \rangle$. ■

Theorem 3.4.2 (Cayley-Hamilton theorem). Let $f(x) = \mathrm{ch}_T(x)$ be the characteristic polynomial of T, then $f(T) = 0$.

Proof. We consider a matrix $A = (a_{ij})$, which is a matrix representation of T. We work over the commutative ring $F[A] = \left\{\sum_{i=0}^{m} a_i A^i\right\}$. Since $Ae_k = \sum_{i=1}^{n} a_{ik} e_k$, so if we let

$$B = (B_{ij}) = \begin{pmatrix} A - a_{11} & -a_{21} & & \\ \vdots & & & \\ -a_{1k} & \cdots & A - a_{kk} & \cdots \\ & & & \end{pmatrix},$$

we have $B_{k1}e_1 + \cdots + B_{kn}e_n = 0$. If we let $\mathrm{adj}(B) = (C_{ij})$, then

$$\begin{aligned} \det B &= C_{11}B_{11} + C_{12}B_{21} + \cdots + C_{1n}B_{n1} \\ 0 &= C_{11}B_{12} + C_{12}B_{22} + \cdots + C_{1n}B_{n2} \\ &\vdots \end{aligned}$$

We can check that $\det(e_k) = 0$ for all k, and $\det(B) = f(A)$, so we're done. ■

Alternative. First, recall that for any matrix $B \in M_n(\mathbb{C})$, one has

$$B \,\mathrm{adj}(B) = \det(B) I_n.$$

Take $B = A - xI$, we get

$$(A - xI)\,\mathrm{adj}(A - xI) = \det(A - xI) I_n = p_A(x) I_n.$$

Observation

Let

$$p_A(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Note that

$$A - xI = \begin{pmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x \end{pmatrix}.$$

Any minor of $A - xI$ is a polynomial of degree $\leq n-1$. Then we can write

$$\operatorname{adj}(A - xI) = B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1}.$$

For example,

$$\operatorname{adj} \begin{pmatrix} x^2 - 3x & 2 + 2x & x \\ 3 + x^2 & x & 2x \\ 4x & 3x^2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -3 & 2 & 1 \\ 0 & 1 & 2 \\ 4 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix} x^2.$$

Hence,

$$(A - xI)(B_0 + B_1 x + \cdots + B_{n-1} x^{n-1}) = a_n I x^n + a_{n-1} I x^{n-1} + \cdots + a_0 I.$$

By comparing coefficients, we get:

$$\begin{cases} a_n I = -B_{n-1}, \\ a_{n-1} I = AB_{n-1} - B_{n-2}, \\ a_{n-2} I = AB_{n-2} - B_{n-3}, \\ \quad \vdots \\ a_0 I = AB_0. \end{cases}$$

Multiplying each equation successively by appropriate powers of A (First equation multiplies A^n, the second one multiplies A^{n-1}, and so on), we obtain

$$a_n A^n + a_{n-1} A^{n-1} + \cdots + a_0 I = 0.$$

Thus,

$$p_A(A) = 0.$$

■

Chapter 4

Decompositions of spaces

Lecture 18

12 Nov. 10:20

4.1 Direct Sums

As previously seen. Let $W_1, \dots, W_r$ be subspaces of V. They are called linearly independent if $\sum w_i = 0$ with $w_i \in W_i$ for all i iff $w_i = 0$ for all i.

Let $W = W_1 + \cdots + W_r$, then TFAE:

(i) W_i's are linearly independent.

(ii) Any $w \in W$ has a unique expression $w = \sum_{i=1}^{r} w_i$ where $w_i \in W_i$.

(iii)
$$W_i \cap [W_1 + W_2 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_r] = \{0\}.$$

(iv) $\dim W = \sum_{i=1}^{r} \dim W_i$.

(v) If $\{v_{ij}\}_{j=1}^{m_i}$ is a basis of W_i, then $\{v_{ij}\}_{i,j}$ is a basis of W.

In this case, we write
$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_r,$$
and call it the direct sum.

Example 4.1.1. Let $T \in \mathcal{L}(V)$ with eigenvalues $\lambda_1, \dots, \lambda_r$ with $\lambda_i \neq \lambda_j$. Then,
$$W = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus E(\lambda_r).$$

4.2 Projections and idempotent decompositions

Definition 4.2.1. An operator $P \in \mathcal{L}(V)$ is called a projection if $P^2 = P$.

Remark 4.2.1. Note that if P is a projection, then suppose $W_1 = \operatorname{Im} P$ and $W_2 = \ker P$, then $V = W_1 \oplus W_2$. Suppose $v = v_1 + v_2$ with $v_i \in W_i$, then $Pv = Pv_1 + Pv_2 = v_1$. (Since $v_1 \in \operatorname{Im}(P)$, we have $v_1 = Pu$ for some u, so $Pv_1 = P^2u = Pu = v_1$.) Moreover, W_i's are P-invariant with $P|_{W_1} = \mathrm{id}$, $P|_{W_2} = 0$. $1 - P$ is a projection since $(1-P)^2 = 1 - 2P + P^2 = 1 - 2P + P = 1 - P$. In this case, we say P is a projection/idempotent onto W_1 and along W_2.

Remark 4.2.2. Since $V = \operatorname{Im} P \oplus \ker P$, so for all $v \in V$,
$$v = Ev + (v - Ev)$$

is the unique decomposition.

Theorem 4.2.1 (Idempotent decomposition). Suppose $P_i \in \mathcal{L}(V)$ satisfying $1 = \sum_{i=1}^r P_i$ and $P_iP_j = 0$ for all $i \neq j$. Let $V_i = \operatorname{Im}(P_i)$, then $V = \bigoplus_{i=1}^r V_i$, and P_1 is the projection onto V_1 along $V_2 \oplus \cdots \oplus V_r$.

Proof. We first show that P_i is a projection for all i. WLOG, suppose $i = 1$, then

$$P_1^2 = P_1 (1 - P_2 - P_3 - \cdots - P_r) = P_1.$$

Now since $1 = \sum_{i=1}^r P_i$, so for all $v \in V$ we have

$$v = \sum_{i=1}^r P_i v,$$

which means $V = V_1 + V_2 + \cdots + V_r$. Now if

$$v = \sum_{i=1}^r v_i, \quad \forall v_i \in V_i,$$

then note that if $x \in \operatorname{Im} P_i$, then $x \in \ker P_j$ for $i \neq j$ since $x = P_i w$ for some w and thus $P_j x = P_j P_i w = 0$. Hence,

$$P_i v = P_i \sum_{j=1}^r v_j = P_i v_i + \sum_{j \neq i} P_i v_j = P_i v_i = v_i$$

since $v_i \in \operatorname{Im} P_i$. Hence, $v_i = P_i v$ for all v_i and thus $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$. ■

Theorem 4.2.2. Let $V_i = \operatorname{Im} P_i$ for all i. Suppose $V = \bigoplus_{i=1}^r V_i$. Let P_i be the projection onto V_i along $V_1 + \cdots + V_{i-1} + V_{i+1} + \cdots + V_r$, then $\sum_i P_i = 1$ and $P_iP_j = 0$ for all $i \neq j$.

Proof. Explicitly, for any $v \in V$, write its unique expression

$$v = \sum_{i=1}^r v_i, \quad v_i \in V_i.$$

Then, $P_i v = P_i v_i = v_i$. Hence, we have $v = \sum v_i = \sum P_i v$ and

$$P_iP_j(v_1 + \cdots + v_r) = P_i \left(\sum_{l=1}^r P_j v_l \right) = P_iP_j(v_j) = P_i(v_j) = 0.$$

■

4.3 T-invariant decomposition

Proposition 4.3.1. Suppose $V = \bigoplus V_i$ and $T_i \in \mathcal{L}(V_i)$. Define a map

$$T : V \to V, \quad \sum v_i \mapsto \sum T_i(v_i),$$

then

(i) $T \in \mathcal{L}(V)$

(ii) V_i is T-invariant

(iii) Suppose $1 = \sum P_i$ is the corresponding idempotenet decomposition. Then $TP_i = P_iT$ $(= T_i$ in some sense).

Proof. Check

$$T(v + \alpha w) = Tv + \alpha T(w).$$

Now if $v \in V_i$, then the unique expression of v in V is $v = \sum_{j=1}^{r} v_j$ with $v_i = v$ and $v_j = 0$ for all $j \neq i$. So $T(v) = T_i(v_i) \in V_i$. Hence, V_i is T-invariant.

Suppose $v = \sum_i v_i$ where $v_i \in V_i$ for all i, then

$$TP_iv = TP_i(v_i) = Tv_i = T_iv_i,$$

and

$$P_iTv = P_i\left(\sum_i T_iv_i\right) = P_iT_iv_i = T_iv_i$$

since $T_iv_i \in V_i$. Hence, $TP_i = P_iT$. ■

Proposition 4.3.2. Suppose $V_i = \operatorname{Im} P_i$. Let $V = \bigoplus_{i=1}^{r} V_i$, corresponding to $1 = \sum_{i=1}^{r} P_i$. Let $T \in \mathcal{L}(V)$. Suppose $TP_i = P_iT$, then

(i) V_i is T-invariant.

(ii) Let $T_i = T|_{V_i}$, then $T = \bigoplus T_i$.

Proof. For $u \in V_i$, we have $u = P_iu$ and $Tu = TP_iu = P_i(Tu)$, so $Tu \in V_i = \operatorname{Im} P_i$. For any $v \in V$, we know

$$Tv = \sum T_iv_i$$

if $v = \sum v_i$ where $v_i \in V_i$ since

$$Tv = T\left(\sum P_iv\right) = \sum T(P_iv) = \sum Tv_i = \sum T_iv_i.$$

In this case, we write $T = \bigoplus T_i$ and if $B_i = \{v_{ij}\}_{j=1}^{m_i}$ is an ordered basis of V_i, and $B = \{v_{ij}\}_{i,j}$ is an ordered basis of V. ■

Example 4.3.1. Let $T \in \mathcal{L}(V)$, and let $f(x) = \operatorname{ch}_T(x)$ be its characteristic polynomial. Suppose $f(x) = g(x) \cdot h(x)$ with $g(x)$ and $h(x)$ coprime, then

$$1 = p(x)g(x) + q(x)h(x)$$

for some p, q. Thus,

$$1 = p(T)g(T) + g(T)h(T),$$

and let $P = p(T)g(T)$ and $Q = g(T)h(T)$, then $PT = TP$ and $QT = TQ$. Also, $PQ = 0$. Note that $PQ = 0$ since

$$PQ = p(T)q(T)f(T) = 0$$

by Cayley-Hamilton theorem. Thus, we know this gives an idempotenet decomposition.

Remark 4.3.1. $\operatorname{Im} P = \ker Q$, and $\operatorname{Im} Q = \ker P$. If we let $W_1 = \operatorname{Im} P$ and $W_2 = \operatorname{Im} Q$, we will see the characteristic polynomial of $T|_{W_1} = h(x)$ and $T|_{W_2} = g(x)$.

Lecture 19

Given $T \in \mathcal{L}(V)$ or $A \in M_n(F)$, we want to see the structures of T transparently, e.g. to compute A^k. 14 Nov. 10:30

(i) (constant) recursive sequence. Suppose $S_0, S_1, \ldots, S_{n-1}$ is given, and

$$S_{k+n} = \alpha_0S_k + \alpha_1S_{k+1} + \cdots + \alpha_{n-1}S_{k+n-1}.$$

Let $v_k = \begin{pmatrix} S_k \\ \vdots \\ S_{k+n-1} \end{pmatrix}$, then

$$v_{k+1} = \begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \end{pmatrix} v_k.$$

(ii) (linear homogeneous) ODE. (with constant coefficient)

$$y^{(n)} = \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 y' + \alpha_0 y.$$

Let $f(x) = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$, then

$$f'(x) = e^{Ax} \cdot C_{n\times 1},$$

where

$$e^{Ax} = 1 + Ax + \cdots + \frac{1}{k!} A^k x^k + \ldots$$

In fact, each entry of $A^k = O\left(d^k\right)$.

Now we can study $v_{k+1} = Av_k$, $f'(x) = Af(x)$ for any $A \in M_n(\mathbb{R})$.

Now can we make T or A diagonal? Note that the entries in the diagonal must be eigenvalues, which are the roots of $\mathrm{ch}_A(x)$. So it needs to split, say

$$\mathrm{ch}_T(x) = \sum_{i=1}^{r} (x - \lambda_i)^{m_i}, \quad \lambda_i \neq \lambda_j \text{ for } i \neq j.$$

Geometrically, see if $\dim E(\lambda_i) = m_i$. (In general, g-mult $\leq$ a-mult)

Algebraically, see if

$$\prod_{i=1}^{r} (T - \lambda_i) = 0.$$

What we can do?

Decompose V into smaller/simpler pieces. Hence, we can use idempotenet deomposition:

$$P_i P_j = \begin{cases} P_i, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}, \quad 1 = P_1 + \cdots + P_n, \quad V = W_1 \oplus \cdots \oplus W_n.$$

Chapter 5

Jordan Form

Lecture 20

19 Nov. 10:20

5.1 Congruence (Chinese Remainder Theorem)

Suppose n is a positive integer, then we called $\mathbb{Z}/(n)$ to be its remainder classes.

$$\mathbb{Z}/(105) \simeq \mathbb{Z}/(3) \times \mathbb{Z}/(5) \times \mathbb{Z}/(7)$$

is the Chinese remainder theorem for positive integers.

Now we extend it to polynomial rings. Let $T \in \mathcal{L}(V)$. Suppose $f(x) \in F[x]$ is monic s.t. $f(T) = 0$. Suppose $f(x) = g_1(x) \dots g_r(x)$ s.t. g_i, g_j are coprime for all $i \neq j$, i.e. $\exists p(x), q(x)$ s.t.

$$p(x)g_i(x) + q(x)g_j(x) = 1.$$

Let

$$h_i = g_1 \dots g_{i-1} g_{i+1} \dots g_r = \frac{f}{g_i},$$

then

Proposition 5.1.1.

(i) $\operatorname{Im}(h_i(T)) = \ker(g_i(T))$.

(ii) Let V_i be the subspace $\ker(g_i(T))$, then $V = \bigoplus V_i$ is a T-invariant decomposition.

proof of (i). Since h_i's are pairwises coprime, so there exists $\xi_i(x) \in F[x]$ s.t.

$$\sum_i \xi_i(x) h_i(x) = 1.$$

Hence, $1 = \sum_i P_i$ where $P_i = (\xi_i h_i)(T)$. We can check this is an idempotent decomposition:

$$P_i P_j = \left(\xi_i \frac{f}{g_i} \xi_j \frac{f}{g_j}\right)(T) = \left(\xi_i \xi_j \frac{f}{g_i g_j} f\right)(T) = 0$$

since $f(T) = 0$. So letting $W_i = P_i(V)$, then $V = \bigoplus W_i$ and it is a T-invariant decomposition. Now note that $W_i = \operatorname{Im}(h_i \xi_i)(T) \subseteq \operatorname{Im}(h_i(T)) \subseteq \ker(g_i(T))$. Note that the last $\subseteq$ holds since $g_i(T)h_i(T)(v) = f(v) = 0$ for all $v \in V$. Now we need to check $\ker g_i(T) \subseteq \operatorname{Im} \xi_i h_i(T)$. Suppose $v \in \ker g_i(T)$. We have

$$v = \sum_j (\xi_j h_j)(T)(v).$$

For $j = i$, $(\xi_i h_i)(T)(v) \in W_i$, which is what we want. For $j \neq i$,

$$(\xi_j h_j)(T)(v) = \left(\xi_j \frac{f}{g_i g_j} g_i\right)(T)(v) = 0$$

since $v \in \ker(g_i(T))$. Hence,

$$v = (\xi_i h_i)(T)(v) \in W_i = \operatorname{Im} \xi_i h_i(T).$$

Thus,

$$\ker g_i(T) \subseteq \operatorname{Im} \xi_i h_i(T) \subseteq \operatorname{Im} h_i(T).$$

■

proof of (ii). It follows from (i) since we can show $\operatorname{Im} h_i(T) = \ker(g_i(T))$. Note that

$$\operatorname{Im} P_i \subseteq \operatorname{Im} h_i(T) = \ker g_i(T) \subseteq \operatorname{Im} \xi_i h_i(T) = \operatorname{Im} P_i,$$

so $\operatorname{Im} P_i = \ker(g_i(T))$ and we have shown that $V = \bigoplus \operatorname{Im} P_i$, so we're done. ■

Canonical cases

We first take $f(x)$ to be the minimal polynomial of T, and suppose

$$f(x) = p_1(x)^{m_1} \dots p_r(x)^{m_r}$$

is a prime decomposition, i.e. every p_i monic and irreducible and $p_i \neq p_j$ for each $i \neq j$. Then, we have $V = \bigoplus V_i$ where $V_i = \ker(p_i(T)^{m_i})$. This is the primary decomposition theorem.

Similarly, take $f(x) = \operatorname{ch}_T(x)$, and write $f(x) = p_1(x)^{m_1} \dots p_r(x)^{m_r}$, which is a prime decomposition of f, suppose $g_i(x) = p_i(x)^{m_i}$, then

Proposition 5.1.2. Let $V_i = \ker(g_i(T))$, then $\dim V_i = \deg g_i(x)$. In fact, letting $T_i = T|_{V_i}$, then

$$\operatorname{ch}_{T_i}(x) = g_i(x).$$

Remark 5.1.1. Today, we only consider

$$f(x) = \prod_{i=1}^{r} (x - \lambda_i)^{m_i},$$

where $p_i(x) = x - \lambda_i$ and $g_i(x) = (x - \lambda_i)^{m_i}$.

Proof. We know $g_i(T_i) = 0$, so the minimal polynomial $m_i(x)$ of $T_i = (x - \lambda_i)^{\alpha_i}$. Let $\operatorname{ch}_i(x) = \operatorname{ch}_{T_i}(x)$, then since $\operatorname{ch}_i(x)$ and $m_i(x)$ have the same roots, so $\operatorname{ch}_i(x) = (x - \lambda_i)^{b_i}$. Note that $b_i = m_i$ since $T = \bigoplus T_i$, which means

$$\prod_{i=1}^{r} (x - \lambda_i)^{m_i} = \operatorname{ch}_T(x) = \prod_{i=1}^{r} \operatorname{ch}_i(x) = \prod_{i=1}^{r} (x - \lambda_i)^{b_i}.$$

Hence, $\operatorname{ch}_i(x) = (x - \lambda_i)^{m_i} = g_i(x)$. Hence, $\dim V_i = \deg \operatorname{ch}_i(x) = \deg g_i(x)$. ■

Nilpotent operators

We obtain: Suppose characteristic polynomial of $T \in \mathcal{L}(V)$ with

$$f(x) = \prod_{i=1}^{r} (x - \lambda_i)^{m_i},$$

then $V = \bigoplus V_i$ and $T = \bigoplus T_i$ with $\operatorname{ch}_{T_i}(x) = (x - \lambda_i)^{m_i}$.

Definition 5.1.1. $T \in \mathcal{L}(V)$ is called nilpotent (or order m) if $T^m = 0$ and $T^{m-1} \neq 0$ for some $m \geq 1$.

Now let

$$N = J_d(0) = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} \in M_d(F) = \text{ companion matrix of } x^d,$$

then

Theorem 5.1.1. Suppose T is nilpotent. There is a unique sequence

$$d_1 \geq d_2 \geq \cdots \geq d_r \quad (> 0)$$

s.t.

$$[T]_B = \begin{pmatrix} J_{d_1}(0) & & 0 \\ & \ddots & \\ 0 & & J_{d_r}(0) \end{pmatrix} = J_{d_1}(0) \oplus \cdots \oplus J_{d_r}(0).$$

Observation: If $A \sim J_{d_1}(0) \oplus \ldots$ and write $\nu\left(A^k\right) = \delta_1 + \cdots + \delta_k$, then

$$\#J_k(0) = \delta_k - \delta_{k-1}.$$

Proof: Suppose

$$A \sim \underbrace{\cdots}_{a:\text{size}>k} \underbrace{J_k(0) \oplus \cdots}_{b:\text{size}=k} \underbrace{\cdots}_{c:\text{size}<k},$$

and suppose the rank of A is $m = \delta_1 + \cdots + \delta_{k-1}$, then

$$\nu\left(A^{k-1}\right) = (k-1)(a+b) + m, \quad \nu\left(A^k\right) = k(a+b) + m, \quad \delta_k = a + b, \quad \nu\left(A^{k+1}\right) = (k+1)a + kb + m,$$

and $\delta_{k+1} = a$.

Lecture 21

Remark 5.1.2. From now on, the note is not the lecture note since I didn't go to the lecture. 26 Nov. 10:20

5.2 Cyclic Subspaces and Annihilators

Definition 5.2.1. If α is a any vector in V, the T-cyclic subspace generated by α is the subspace $Z(\alpha; T)$ of all vectors of the form $g(T)\alpha$, $g \in F[x]$. If $Z(\alpha; T) = V$, then α is called a cyclic vector for T.

Remark 5.2.1. Another way to describe $Z(\alpha; T)$ is that

$$Z(\alpha; T) = \text{span}\left\{T^k\alpha\right\}_{k\geq 0},$$

and thus α is a cyclic vector for T if and only if these vectors span V. However, the general operator T has no cyclic vector.

Definition 5.2.2. If α is any vector in V, the T-annihilator of α is the ideal $M(\alpha; T)$ in $F[x]$ consisting of all polynomials g over F such that $g(T)\alpha = 0$. The unique monic polynomial p_α which generates this ideal will also be called the T-annihilator of α.

Remark 5.2.2. $\deg p_\alpha > 0$ unless α is the zero vector.

Theorem 5.2.1. Let α be any non-zero vector in V and let p_α be the T-annihilator of α, then

(i) The degree of p_α is equal to the dimension of the cyclic subspace $Z(\alpha;T)$.

(ii) If the degree of p_α is k, then the vectors $\alpha, T\alpha, \dots, T^{k-1}\alpha$ form a basis for $Z(\alpha;T)$.

(iii) If U is the linear operator on $Z(\alpha;T)$ induced by T, then the minimal polynomial for U is p_α.

Proof. Let $g \in F[x]$, then $g = p_\alpha q + r$, where either $r = 0$ or $\deg(r) < \deg p_\alpha = k$. The polynomial $p_\alpha q \in \mathrm{Ann}_T(\alpha)$, so

$$g(T)\alpha = r(T)\alpha.$$

Since $r = 0$ or $\deg(r) < k$, the vector $r(T)\alpha$ is a linear combination of the vectors $\alpha, T\alpha, \dots T^{k-1}\alpha$, and thus

$$g(T)\alpha = r(T)\alpha \in \mathrm{span}\left\{\alpha, T\alpha, \dots, T^{k-1}\alpha\right\}.$$

Since g can be arbitrary polynomial of $F[x]$, so

$$Z(\alpha;T) \subseteq \mathrm{span}\left\{\alpha, T\alpha, \dots, T^{k-1}\alpha\right\},$$

and

$$\mathrm{span}\left\{\alpha, T\alpha, \dots, T^{k-1}\alpha\right\} \subseteq Z(\alpha;T)$$

is trivial, so

$$Z(\alpha;T) = \mathrm{span}\left\{\alpha, T\alpha, \dots, T^{k-1}\alpha\right\}.$$

Note that the set $\left\{\alpha, T\alpha, \dots, T^{k-1}\alpha\right\}$ is linearly independent, otherwise if

$$\sum_{i=0}^{k-1} \beta_i T^i \alpha = 0,$$

where β_i are not all zeros, then

$$d(x) = \sum_{i=0}^{k-1} \beta_i x^i \in \mathrm{Ann}_T(\alpha),$$

but $\deg d \leq k-1 < k = \deg p_\alpha$, so this is impossible. Hence,

$$\left\{\alpha, T\alpha, \dots, T^{k-1}\alpha\right\}$$

is a basis of $Z(\alpha;T)$, and thus we showed (i) and (ii). Now we show (c). Note that

$$p_\alpha(U)\left(g(T)\alpha\right) = p_\alpha(T)g(T)\alpha = g(T)p_\alpha(T)\alpha = g(T)0 = 0.$$

Hence, $p_\alpha(U) = 0$. Now if $h \in F[x]$ with $\deg h < k$ and $h(U) = 0$, then we know $h(U)\alpha = h(T)\alpha = 0$, so $h \in \mathrm{Ann}_T(\alpha)$ and thus $p_\alpha \mid h$, so this is impossible since $\deg p_\alpha > \deg h$. Hence, $\deg m_U \geq k$ and thus $m_U = p_\alpha$. ∎

Corollary 5.2.1. If $V = Z(\alpha;T)$ and $T \in \mathcal{L}(V)$, then $\deg m_T(x) = \dim V$. Also, since $\dim V = \deg \mathrm{ch}_T(x)$, so we have $m_T(x) = \mathrm{ch}_T(x)$ since $m_T(x) \mid \mathrm{ch}_T(x)$ and they are both monic.

Now if $U \in \mathcal{L}(W)$ where $\dim W = k$ and $W = Z(\alpha;U)$, then we know

$$\alpha, U\alpha, \dots, U^{k-1}\alpha$$

form a basis for W, and the annihilator p_α of α is $m_U(x) = \mathrm{ch}_U(x)$ by the above corollary. If we let $\alpha_i = U^{i-1}\alpha$ for $i = 1, 2, \dots, k$, then suppose $\mathcal{B} = \{\alpha_1, \dots, \alpha_k\}$, we know the action of U on $\mathcal{B}$ is

$$\begin{aligned} U\alpha_i &= \alpha_{i+1} \text{ for } i = 1, \dots, k-1 \\ U\alpha_k &= -c_0\alpha_1 - c_1\alpha_2 - \cdots - c_{k-1}\alpha_k \end{aligned}$$

where $p_\alpha = c_0 + c_1 x + \cdots + c_{k-1}x^{k-1} + x^k$. The expression for $U\alpha_k$ is true since $p_\alpha(U)\alpha = 0$. Hence,

$$[U]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{k-1} \end{pmatrix},$$

which is called the companion matrix of the monic polynomial p_α.

Theorem 5.2.2. If $U \in \mathcal{L}(W)$ where $\dim W < \infty$, then U has a cyclic vector if and only if there is some ordered basis for W in which U is represented by the companion matrix of $m_U(x)$.

Proof. We have shown that if U has a cyclic vector, then there is some ordered basis for W in which U is represented by the companion matrix of $m_U(x)$.

Now if $[U]_B$ is the companion matrix of $m_U(x)$, then we know

$$B = \{\alpha, U\alpha, \ldots, U^{k-1}\alpha\}$$

where $k = \dim W$. ■

Corollary 5.2.2. If A is the companion matrix of a monic polynomial p, then p is both $m_A(x)$ and $\mathrm{ch}_A(x)$.

Proof. If we regard A as the matrix representation of $T \in \mathcal{L}(V)$ with respect to the standard basis, then we know $V = Z(\alpha; T)$ for some α, and thus $m_T(x) = \mathrm{ch}_T(x)$. Also, we can check this matrix's characteristic polynomial is p by direct computing. ■

5.3 Cyclic Decompositions and the Rational Form

The primary purpose of this section is to prove that if T is any linear operator on a finite-dimensional vector space V, then there exist vectors $\alpha_1, \ldots, \alpha_r$ in V s.t.

$$V = Z(\alpha_1; T) \oplus Z(\alpha_2; T) \oplus \cdots \oplus Z(\alpha_r; T).$$

This will show for any operator $T : V \to V$,

$$T = T_1 \oplus T_2 \oplus \cdots \oplus T_r,$$

where T_i's all have cyclic vectors.

The cyclic theorem is closely related to the following question. If W is a T-invariant subspace, then is there another subspace W' s.t. $V = W \oplus W'$? Usually, the answer is yes, and there are many such W', each of them is called **complementary** to W.

Question. When a T-invariant subspace has a complementary subspace which is also invariant under T?

Now if we find such T-invariant W', then for $\beta \in V$, we know $\beta = \gamma + \gamma'$ where $\gamma \in W$ and $\gamma' \in W'$. If $f \in F[x]$, then

$$f(T)\beta = f(T)\gamma + f(T)\gamma'.$$

Since W and W' are both invariant under T, so we can find that $f(T)\beta \in W$ if and only if $f(T)\gamma' = 0$. Thus, if $f(T)\beta \in W$, then $f(T)\beta = f(T)\gamma$.

Definition 5.3.1. Let $T \in \mathcal{L}(V)$ and let W be a subspace of V. We say that W is T-admissible if

(i) W is invariant under T.

(ii) if $f(T)\beta$ is in W, there exists a vector γ in W s.t. $f(T)\beta = f(T)\gamma$.

Corollary 5.3.1. If W is invariant and has a complementary invariant subspace, then W is admissible.

Let us indicate how the admissibility property is involved in the attempt to obtain a decomposition

$$V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T).$$

Our basic method for arriving such a decomposition will be to inductively select the vectors $\alpha_1, \ldots, \alpha_r$. Suppose that by some process or another we have selected $\alpha_1, \ldots, \alpha_j$, and the subspace

$$W_j = Z(\alpha_1; T) + \cdots + Z(\alpha_j; T)$$

is proper. We would like to find a non-zero vector α_{j+1} such that

$$W_j \cap Z(\alpha_{j+1}; T) = \{0\}$$

because the subspace $W_{j+1} = W_j \oplus Z(\alpha_{j+1}; T)$ would then come at least one dimension nearer to exhausting V. But, why should any such α_{j+1} exist? If $\alpha_1, \alpha_2, \ldots, \alpha_j$ have been chosen so that W_j is a T-admissible subspace, then it is rather easy to see that we can find a suitable α_{j+1}. We choose some $\beta \notin W$. Consider the T-conductor $S(\beta; W)$, where

$$S(\beta; W) = \{g \in F[x] : g(T)\beta \in W\},$$

then $S(\beta; W)$ is an ideal and suppose $S(\beta; W) = (s(\beta; W))$. Let $f = s(\beta; W)$, then $f(T)\beta \in W$. Now since W is T-admissible, there is a $\gamma \in W$ s.t. $f(T)\beta = f(T)\gamma$. Let $\alpha = \beta - \gamma$ and let g be any polynomial. Since $\beta - \alpha \in W$, so $g(T)\beta \in W$ if and only if $g(T)\alpha \in W$. In other words, $S(\alpha; W) = S(\beta; W)$. Thus, f is also the T-conductor of α into W. But $f(T)\alpha = 0$, so $g(T)\alpha \in W$ iff $f \mid g$ iff $g(T)\alpha = p(T)f(T)\alpha = 0$. Thus,

$$W \cap Z(\alpha; T) = \{0\}.$$

Theorem 5.3.1 (Cyclic Decomposition Theorem). Let $T \in \mathcal{L}(V)$ where $\dim V < \infty$ and let W_0 be a proper T-admissible subspace of V. There exists non-zero vectors $\alpha_1, \alpha_2, \ldots, \alpha_r \in V$ with respective T-annihilators $p_1, p_2, \ldots, p_r$ s.t.

(i) $V = W_0 \oplus Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T)$;

(ii) p_k divides p_{k-1} for $k = 2, 3, \ldots, r$.

Furthermore, the integer r and the annihilators $p_1, p_2, \ldots, p_r$ are uniquely determined by (i), (ii), and the fact that no $\alpha_k = 0$.

Proof. The proof is rather long; hence, we shall divide it into four steps. For the first reading it may seem easier to take $W_0 = \{0\}$, although it does not produce any substantial simplication, Throughout the proof, we shall abbreviate $f(T)\beta$ to $f\beta$.

- Step 1: There exist non-zero vectors $\beta_1, \beta_2, \ldots, \beta_r \in V$ s.t.

  (a) $V = W_0 + Z(\beta_1; T) + \cdots + Z(\beta_r; T)$

  (b) If $1 \leq k \leq r$ and
  $$W_k = W_0 + Z(\beta_1; T) + \cdots + Z(\beta_k; T),$$
  then the conductor $p_k = s(\beta_k; W_{k-1})$ has maximum degree among all T-conductors into the subspace W_{k-1}, i.e. for every k we have
  $$\deg p_k = \max_{\alpha \in V} \deg s(\alpha; W_{k-1}).$$

  This step only depends upon the fact that W_0 is an invariant subspace. If W is a proper T-invariant subspace, then
  $$0 < \max_{\alpha} \deg s(\alpha; W) \leq \dim V,$$
  and we can choose a vector β so that $\deg s(\beta; W)$ attains that maximum. Note that $\beta \notin W$ since for any vector w in W, we have $\deg s(w; W) = 0$. By this, we know $W + Z(\beta; T)$ is

then T-invariant and has dimension larger than $\dim W$. Apply this process to $W = W_0$ to obtain β_1. If $W_1 = W_0 + Z(\beta_1; T)$ is still proper, then apply the process to W_1 to obtain β_2. Continue in that manner. Since $\dim W_k > \dim W_{k-1}$ in every step, so we must reach $W_r = V$ in no more than $\dim V$ steps.

- Step 2: Let $\beta_1, \beta_2, \ldots, \beta_r$ be non-zero vectors which satisfy conditions (a) and (b) of Step 1. Fix k for some $1 \leq k \leq r$. Let β be any vector in V and let $f = s(\beta; W_{k-1})$. If

$$f\beta = \beta_0 + \sum_{1 \leq i < k} g_i \beta_i \quad \text{for } \beta_i \in W_i,$$

then f divides each polynomial g_i and $\beta_0 = f\gamma_0$, where $\gamma_0 \in W_0$.

If $k = 1$, then this is just the statement that W_0 is T-admissible. In order to prove the assertion for $k > 1$, apply the division algorithm:

$$g_i = fh_i + r_i, \quad r_i = 0 \text{ or } \deg r_i < \deg f.$$

We wish to show that $r_i = 0$ for each i. Let

$$\gamma = \beta - \sum_{i=1}^{k-1} h_i \beta_i,$$

then since $\gamma - \beta \in W_{k-1}$, so we have

$$s(\gamma; W_{k-1}) = s(\beta; W_{k-1}) = f$$

since $p\gamma \in W_{k-1}$ iff $p\beta \in W_{k-1}$ for any polynomial p. Furthermore,

$$f\gamma = f\beta - \sum_{i=1}^{k-1} fh_i\beta_i = \beta_0 + \sum_{i=1}^{k-1} g_i\beta_i - \sum_{i=1}^{k-1} fh_i\beta_i = \beta_0 + \sum_{i=1}^{k-1} \gamma_i\beta_i.$$

Suppose that some $r_i \neq 0$, then we shall deduce a contradiction. Let j be the largest index i for which $r_i \neq 0$. Then

$$f\gamma = \beta_0 + \sum_{i=1}^{j} r_i\beta_i \quad r_j \neq 0 \text{ and } \deg r_j < \deg f.$$

Let $p = s(\gamma; W_{j-1})$. Since $W_{j-1} \subseteq W_{k-1}$, so the conductor $f = s(\gamma; W_{k-1})$ must divide p, i.e.

$$p = fg.$$

Apply $g(T)$ to the both side of above equation we have

$$p\gamma = gf\gamma = g\beta_0 + gr_j\beta_j + \sum_{1 \leq i < j} gr_i\beta_i.$$

By definition, $p\gamma \in W_{j-1}$, and $g\beta_0$ and $\sum_{1 \leq i < j} gr_i\beta_i$ are in W_{j-1}. Therefore, $gr_j\beta_j \in W_{j-1}$. Now use condition (b) of Step 1:

$$\deg(gr_i) \geq \deg s(\beta_j; W_{j-1}) \geq \deg s(\gamma; W_{j-1}) \geq = \deg p = \deg fg.$$

Thus, $\deg r_j \geq \deg f$, and that contradicts the choice of j. We now know that f divides each g_i and hence that $\beta_0 = f\gamma$. (Recall that $\gamma = \beta - \sum_{i=1}^{k-1} h_i\beta_i$) Since W_0 is T-admissible, so $\beta_0 = f\gamma_0$ where $\gamma_0 \in W_0$.

> **Remark 5.3.1.** Step 2 is a strengthened form of the assertion that each of the subspace $W_1, W_2, \ldots, W_r$ is T-admissible.

- Step 3: There exists non-zero vectors $\alpha_1, \alpha_2, \ldots, \alpha_r \in V$ which satisfy conditions (i) and (ii) of this theorem.

  Start with vectors $\beta_1, \beta_2, \ldots, \beta_r$ as in Step 1. Fix $1 \leq k \leq r$. We apply Step 2 to the vector $\beta = \beta_k$ and the T-conductor $f = p_k$. We obtain

  $$p_k\beta_k = p_k\gamma_0 + \sum_{1\leq i<k} p_k h_i \beta_i$$

  where $\gamma_0 \in W_0$ and $h_1, h_2, \ldots, h_{k-1}$ are polynomials. Let

  $$\alpha_k = \beta_k - \gamma_0 - \sum_{1\leq i<k} h_i\beta_i.$$

  Since $\beta_k - \alpha_k \in W_{k-1}$, so

  $$s(\alpha_k; W_{k-1}) = s(\beta_k; W_{k-1}) = p_k$$

  and since

  $$p_k\alpha_k = p_k\beta_k - p_k\gamma_0 - \sum_{1\leq i<k} p_k h_i\beta_i = 0,$$

  so we have

  $$W_{k-1} \cap Z(\alpha_k; T) = \{0\}.$$

  Because each α_k can satisfy this argument, so

  $$W_k = W_0 \oplus Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_k; T)$$

  and that p_k is the T-annihilator of α_k (Note that $p_k\alpha_k = 0$). Note that

  $$p_k\alpha_k = 0 + p_1\alpha_1 + \cdots + p_{k-1}\alpha_{k-1}.$$

  Apply Step 2 with $\beta_1, \ldots, \beta_k$ replaced by $\alpha_1, \ldots, \alpha_k$ and with $\beta = \alpha_k$. We can conclude that $p_k \mid p_i$ for each $i < k$.

- Step 4: The number r and the polynomials $p_1, \ldots, p_r$ are uniquely determined by the conditions of this theorem.

■

Corollary 5.3.2. If $T \in \mathcal{L}(V)$ and $\dim V < \infty$, then every T-admissible subspace has a complementary subspace which is also invariant under T.

Proof. Let W_0 be an admissible subspace of V. If $W_0 = V$, the complement is $\{0\}$. If W_0 is proper, then apply the previous theorem we know

$$V = W_0 \oplus W_0' \text{ where } W_0' = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T).$$

Note that W_0' is invariant under T. ■

Corollary 5.3.3. Let $T \in \mathcal{L}(V)$ and $\dim V < \infty$, then

(a) There exists a vector $\alpha \in V$ s.t. the T-annihilator of α is $m_T(x)$.

(b) T has a cyclic vector if and only if $\mathrm{ch}_T(x) = m_T(x)$.

Proof. If $V = \{0\}$, the results are trivially true. If $V \neq \{0\}$, let

$$V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T)$$

where the T-annihilators $p_1, \ldots, p_r$ are such that $p_{k+1} \mid p_k$ for $1 \leq k \leq r-1$. Note that this gives

$m_T(x) = p_1(x)$ since for all $v \in V$, we know

$$p_1(T)v = p_1(T)(a_1 + \cdots + a_r) = p_1(T)a_1 + \cdots + p_1(T)a_r = 0$$

where $a_i \in Z(\alpha_i; T)$. Thus, we have shown (a). Now for (b), we have shown that T has a cyclic vector implies $\mathrm{ch}_T(x) = m_T(x)$ in the previous section. Now we show the converse. We claim that if $\mathrm{ch}_T(x) = m_T(x)$, then $V = Z(\alpha_1; T)$. Note that

$$\dim Z(\alpha_1; T) = \deg p_1 = \deg m_T = \deg \mathrm{ch}_T = \dim V,$$

and $Z(\alpha_1; T) \subseteq V$, so we're done. ■

Theorem 5.3.2 (Generalized Cayley-Hamilton Theorem). Let $T \in \mathcal{L}(V)$ and $\dim V < \infty$. Then,

(i) $m_T(x) \mid \mathrm{ch}_T(x)$.

(ii) $m_T(x)$ and $\mathrm{ch}_T(x)$ have the same prime factors, except for multiplicities.

(iii) If $m_T(x) = f_1^{r_1} \dots f_k^{r_k}$ is the prime factorization of $m_T(x)$, then

$$\mathrm{ch}_T(x) = f_1^{d_1} \dots f_k^{d_k}$$

where $d_i = \frac{\nu(f_i(T)^{r_i})}{\deg f_i}$.

Proof. We disregard the trivial case $V = \{0\}$. To prove (i) and (ii), consider a cyclic decomposition of V

$$V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T).$$

As we noted in the proof of last corollary, $p_1 = m_T(x)$. Let $U_i = T|_{Z(\alpha_i;T)}$. Then U_i has a cyclic vector and so p_i is both $m_{U_i}(x)$ and $\mathrm{ch}_{U_i}(x)$. Thus, we know

$$\mathrm{ch}_T(x) = \mathrm{ch}_{U_1}(x) \dots \mathrm{ch}_{U_r}(x) = p_1 p_2 \dots p_r.$$

Clearly, $p_1 = m_T(x)$ divides $\mathrm{ch}_T(x)$. This proves (i). Clearly, any prime divisor of m_T is a prime divisor of ch_T since $m_T \mid \mathrm{ch}_T$. Conversely, a prime divisor of $\mathrm{ch}_T(x)$ must divide one of p_i's, which in turn divides $p_1 = m_T(x)$. Thus, we prove (ii).

Now we show (iii). We employ the primary decomposition theorem. It tells us

$$V = V_1 \oplus \cdots \oplus V_k, \text{ where } V_i = \ker f_i(T)^{r_i}.$$

Also, note that $f_i^{r_i} = m_{T_i}$, where $T_i = T|_{V_i}$. Now apply (ii) of this theorem to T_i, we know

$$\mathrm{ch}_{T_i}(x) = f_i^{d_i} \text{ for some } d_i \geq r_i.$$

By definition, we know

$$\dim V_i = \deg \mathrm{ch}_{T_i}(x) = \deg f_i^{d_i} = \deg f_i \cdot d_i,$$

so we know

$$d_i = \frac{\dim V_i}{\deg f_i} = \frac{\nu\left(f_i(T)^{r_i}\right)}{\deg f_i}.$$

Since T is the direct product of T_i's, so

$$\mathrm{ch}_T(x) = \mathrm{ch}_{T_1}(x) \dots \mathrm{ch}_{T_k}(x) = f_1^{d_1} \dots f_k^{d_k}.$$

■

Corollary 5.3.4. If T is a nilpotent linear operator on a vector space of dimension n, then $\mathrm{ch}_T(x) = x^n$.

Now let us look at the matrix analogue of the cyclic decomposition theorem. If we have the operator

T and the direct sum decomposition

$$V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T),$$

then let $\mathcal{B}_i = \{\alpha_i, T\alpha_i, \ldots, T^{k_i-1}\alpha_i\}$ be the cyclic ordered basis for $Z(\alpha_i; T)$. Here $k_i = \dim Z(\alpha_i; T) = \deg p_i$. The matrix of the induced operator T_i in the ordered basis $\mathcal{B}_i$ is the companion matrix of the polynomial p_i. Thus, if we let $\mathcal{B}$ be the oredered basis for V which is the union of the $\mathcal{B}_i$ arranged in the order $\mathcal{B}_1, \ldots, \mathcal{B}_r$, then the matrix of T in the ordered basis $\mathcal{B}$ will be

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix},$$

where A_i is the $k_1 \times k_i$ companion matrix of p_i. An $n \times n$ matrix A, which is the direct sum of companion matrices of non-scalar monic polynomials $p_1, p_2, \ldots, p_r$ such that $p_{i+1} \mid p_i$ for $i = 1, 2, \ldots, r-1$ will be said to be in **rational form**. The cyclic decomposition theorem tells us the following concerning matrices.

Theorem 5.3.3. Let F be a field and let $B \in M_n(F)$, then B is similar over the field F to one and only one matrix which is in rational form.

Proof. We have shown the existence. Now we show the uniqueness. If C is a rational form of B, where $C = C_1 \oplus \cdots \oplus C_s$ by some square matrix C_i's. Then, C_i is the companion matrix with respect to some polynomial g_i such that $g_{i+1} \mid g_i$ for all $i = 1, 2, \ldots, s-1$. It shows that there exists $\beta_1, \ldots, \beta_s \in V$ with T-annihilators $g_1, \ldots, g_s$ s.t.

$$V = Z(\beta_1; T) \oplus \cdots \oplus Z(\beta_s; T).$$

But then by the uniqueness in cyclic decomposition theorem, we know g_i is unique and s is unique. Thus, the rational form of B is unique. ■

Remark 5.3.2. The polynomials $p_1, \ldots, p_r$ are called the **invariant factors** for the matrix B.

5.4 The Jordan Form

Suppose N is a nilpotent linear operator on the finite-dimensional space V. Then suppose

$$V = Z(\alpha_1; N) \oplus Z(\alpha_2; N) \oplus \cdots \oplus Z(\alpha_r; N)$$

where $p_1, p_2, \ldots, p_r$ are the N-annihilaotrs of $\alpha_1, \ldots, \alpha_r$ and $p_{i+1} \mid p_i$ for $i = 1, 2, \ldots, r-1$ (by the theorem in previous section). Since N is nilpotent, the minimal polynomial is x^k for some $k \leq n$. Thus, each p_i is of the form $p_i = x^{k_i}$ and the divisibility condition says

$$k_1 \geq k_2 \geq \cdots \geq k_r.$$

Note that we must have $k_1 = k$ since $m_N(x) = x^k$ and $m_N(N)u = 0$ for all $u \in Z(\alpha_i; N)$ for all $i = 1, 2, \ldots, r$ and k should be the smallest number satisfying this condition. Also, $k_r \geq 1$. The companion matrix of x^{k_i} is the $k_i \times k_i$ matrix

$$A_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

so we know there exists an ordered basis of V in which the matrix of N is the direct product of A_is for $i = 1, 2, \ldots, r$, where the size of A_is decrease as i increases. Hence, one sees from this that associated with a nilpotent $n \times n$ matrix is a positive integer r and r positive integers $k_1, k_2, \ldots, k_r$ s.t.

$$k_1 + k_2 + \cdots + k_r = n \text{ and } k_i \geq k_{i+1},$$

and these positive integers determine the rational form of the matrix, i.e. determine the matrix up to similarity.

Proposition 5.4.1. $r = \nu(N)$.

Proof. Since $[N] \sim A_1 \oplus A_2 \oplus \cdots \oplus A_r$ where each A_i contributes one to the nullity, so the nullity of N is r. ■

Remark 5.4.1.

$$\left\{N^{k_1-1}\alpha_1, N^{k_2-1}\alpha_2, \ldots, N^{k_r-1}\alpha_r\right\}$$

form a basis of $\ker N$ since $N^{k_i}\alpha_i = 0$ and this set is linearly independent (since V is the direct product of $Z(\alpha_i; N)$).

Hence, suppose $\alpha \in \ker N$, then we know

$$\alpha = f_1(N)\alpha_1 + \cdots + f_r(N)\alpha_r$$

where f_i is a polynomial, the degree of which we may assume is less than k_i. Since $N\alpha = 0$, so we have

$$0 = N\alpha = N(f_1(N))\alpha_1 + \cdots + N(f_r(N))\alpha_r,$$

and since $N(f_i(N))\alpha_i \in Z(\alpha_i; N)$, so we know

$$0 = N(f_i(N))\alpha_i = ((xf_i)(N))\alpha_i$$

for all i. Thus, $x^{k_i} \mid xf_i$, so $\deg(f_i) \geq k_i - 1$, so $f_i = c_i x^{k_i-1}$, where c_i is some scalar. But then

$$\alpha = c_1\left(x^{k_1-1}\alpha_1\right) + \cdots + c_r\left(x^{k_r-1}\alpha_r\right).$$

This gives an alternative way to show that

$$\left\{N^{k_1-1}\alpha_1, \ldots, N^{k_r-1}\alpha_r\right\}$$

form a basis of $\ker N$.

Now what we want to do is to combine our findings about nilpotent operators or matrices with the primary decomposition theorem. The situation is this: Suppose $T \in \mathcal{L}(V)$ and

$$\mathrm{ch}_T(x) = (x - c_1)^{d_1} \ldots (x - c_k)^{d_k},$$

where $c_i \neq c_j$ for all $i \neq j$ and $d_i \geq 1$ for all i. Then

$$m_T(x) = (x - c_1)^{r_1} \ldots (x - c_k)^{r_k},$$

where $1 \leq r_i \leq d_i$. If $W_i = \ker(T - c_i I)^{r_i}$, then the primary decomposition theorem tells us

$$V = W_1 \oplus \cdots \oplus W_k,$$

and that the operaotr T_i induced on W_i by T has

$$m_{T_i}(x) = (x - c_i)^{r_i}.$$

Let $N_i = T_i - c_i I$ where $N_i \in \mathcal{L}(W_i)$, then N_i is nilpotent since

$$0 = m_{T_i}(T_i) = (T_i - c_i I)^{r_i} = N_i^{r_i}$$

and

$$m_{N_i}(x) = x^{r_i}.$$

Hence, we know $T_i = N_i + c_i I$ and if we choose a basis for the subspace W_i corresponding to the cyclic decomposition for the nilpotent operator N_i, then

$$[T_i] \sim \bigoplus \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} + c_i I = \bigoplus \begin{pmatrix} c_i & 0 & \cdots & 0 & 0 \\ 1 & c_i & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_i \end{pmatrix}.$$

Hence, if we pick all the basis for the W_i together, we obtain an ordered basis for V. Let us describe hte matrix A of T in this ordered basis, then

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix}.$$

Each

$$A_i = \begin{pmatrix} J_1(c_i) & & & \\ & J_2(c_i) & & \\ & & \ddots & \\ & & & J_{n_i}(c_i) \end{pmatrix}$$

where each $J_s(c_i)$ is a Jordan block with characteristic value c_i. In this case, we call this A is in Jordan form.

Remark 5.4.2. We can decompose N_i into the direct product of many companion matrix of monic polynomial of the form x^z since we can decompose W_i into the direct product of many N_i-cyclic subspace and suppose

$$W_i = Z(\beta_{i1}; N_i) \oplus \cdots \oplus Z(\beta_{in_i}; N_i),$$

then by Theorem 5.2.2 we know there exists basis B_{il} s.t.

$$[N_i]_{\bigcup_{l=1}^{n_i} B_{il}} = \bigoplus \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

Remark 5.4.3. We have just pointed out that if T is a linear operator for which $\mathrm{ch}_T(x)$ splits over the field, then there is an ordered basis for V in which T is represented by a matrix which is in Jordan form.

We would like to show now that this matrix is something uniquely assosciated with T, up to the order in which the characteristic values of T is written down. In other words, if two matrices both in Jordan form and they are similar, then they can differ only in that the order of scalar c_i is different.

The uniqueness we see as follows. Suppose there is some ordered basis for V in which T is represented by some Jordan matrix A described in the previous paragraph. If A_i is a $d_i \times d_i$ matrix, then we know

$$\mathrm{ch}_A(x) = \mathrm{ch}_{A_1}(x) \dots \mathrm{ch}_{A_k}(x),$$

and note that

$$(A_i - c_i I)^{d_i} = 0$$

so $x^{d_i} \mid \mathrm{m}_{A_i}(x) \mid \mathrm{ch}_{A_i}(x)$, and since $\deg \mathrm{ch}_{A_i}(x) = d_i$, so we know $\mathrm{ch}_{A_i}(x) = (x - c_i)^{d_i}$, which gives

$$\mathrm{ch}_A(x) = (x - c_1)^{d_1} \dots (x - c_k)^{d_k},$$

so the characteristic polynomial is unique given a matrix in Jordan form. Hence, for a given linear operator T with $\mathrm{ch}_T(x) = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$, we know its Jordan form must be the direct product of A_i's where A_i has diagonal entries all c_i and the only place that may not be unique is the size of the Jordan blocks. Hence, if we can prove the following theorem, then we can show that the Jordan form of T is unique.

Theorem 5.4.1. If N is

$$J_{s_1}(0) \oplus J_{s_2}(0) \oplus \cdots \oplus J_{s_r}(0),$$

then $\{s_i\}_{i=1}^r$ is unique up to permutation.

Proof. Note that

$$\dim\ker N^k = \sum_{i=1}^{r} \dim\ker\left(J_i(0)^k\right)$$

by rank and nullity theorem. Also, since $J_i(0)$ is nilpotent, so we can observe that

$$\dim\ker\left(J_i(0)^k\right) = \min\{s_i, k\}$$

since $J_i(0)$ is of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

which shift the 1-line left by 1. Hence,

$$\dim\ker N^k = \sum_{i=1}^{r} \min\{s_i, k\},$$

and thus we have

$$\begin{aligned} \dim\ker N^{k+1} - \dim\ker N^k &= \sum_{i=1}^{r} \min\{s_i, k+1\} - \min\{s_i, k\} \\ &= \#\text{ of Jordan blocks of size } \geq k+1 \\ &= \ell_{k+1}, \end{aligned}$$

so the number of Jordan blocks of size k is $\ell_k - \ell_{k+1}$ and

$$\ell_k - \ell_{k+1} = \left(\dim\ker N^k - \dim\ker N^{k-1}\right) - \left(\dim\ker N^{k+1} - \dim\ker N^k\right),$$

which is unique for any N, so we know $\{s_i\}_{i=1}^k$ is unique. ∎

Remark 5.4.4. We say this theorem shows the uniqueness of Jordan form since

$$A = A_1 \oplus \cdots \oplus A_k,$$

and $A_i = N_i + c_i I$, so if we show N_i is unique, then A_i is unique.

Corollary 5.4.1. Given any linear operator T, then if $\mathrm{ch}_T(x)$ splits, then T has a unique Jordan form.

Now we with to make some further observations about the operator T and the Jordan matrix A which represents T in some ordered basis.

(i) Every characteristic value c_i of T is repeated d_i times on the diagonal line, where d_i is the multiplicity of c_i as a root of $\mathrm{ch}_T(x)$, i.e. $d_i = \dim W_i$.

(ii) For each i, the matrix A_i is the direct sum of n_i Jordan blocks with characteristic value c_i. For n_i, the number of Jordan blocks in A_i, we know it is equal to $\dim\ker(T - c_i I)$. In particular, T is diagonalizable if and only if $n_i = d_i$ for eeach i.

(iii) For each i, the size of the biggest Jordan block in A_i is r_i, where r_i is the multiplicity of c_i as a root of $m_T(x)$. This follows from the fact that the minimal polynomial for the nilpotent operator $(T_i - c_i I)$ is x^{r_i}.

Appendix