

# Introduction to Analysis I

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### Abstract

The lecture note of 2025 Fall Introduction to Analysis I by professor 崔茂培. In this note, we will write  $(X^{(n)})_{n=m}^{\infty}$  and  $\{X^{(n)}\}_{n=m}^{\infty}$  to express a sequence, they are identical, but 崔茂培 use both during lectures, so I follow him.

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# Chapter 1

## Basic Things

### Lecture 1

#### 1.1 Natural Numbers

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The set of natural numbers is denoted by  $\mathbb{N} = \{1, 2, \dots\}$ . There exists an addition operation

$$1 + 1 = 2 \quad 1 + 1 + 1 = 3 \quad \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n.$$

#### 1.2 Integers

The set of integers is  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ . There is a zero element 0 such that  $z + 0 = z$  for any  $z \in \mathbb{Z}$ . Also, for  $n \in \mathbb{N}$ , we have  $n + (-n) = 0$  and  $n - m = n + (-m)$  for all  $n, m \in \mathbb{N}$ .

$$\mathbb{Z} \xrightarrow[\text{introduce division}]{} \mathbb{Q} \xrightarrow[\text{Completeness axiom}]{} \mathbb{R}$$

#### 1.3 Field

Next, we introduce the concept of field.

**Definition 1.3.1 (Fields).** A field is a set  $F$  together with two binary operations, called addition(+) and multiplication(\*), such that the following properties hold:

- (a)  $a + b = b + a$ ,  $a \cdot b = b \cdot a$  for  $a, b \in F$ .
- (b)  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for  $a, b, c \in F$ .
- (c)  $a \cdot (b + c) = a \cdot b + a \cdot c$ .
- (d) There are distinct element 0 and 1 such that  $a + 0 = a$ ,  $a \cdot 1 = a$  for  $a \in F$ .
- (e) For each  $a \in F$ , there exists  $-a \in F$  such that  $a + (-a) = 0$ . If  $a \neq 0$ , there is an element  $\frac{1}{a}$  or  $a^{-1}$  in  $F$  such that  $a \cdot \frac{1}{a} = 1$ , or  $a \cdot a^{-1} = 1$ .

**Remark 1.3.1.** If  $a \in F$ , then  $a + a \in F$ . We denote  $a + a$  by  $2 \cdot a$ . Similarly,

$$\underbrace{a + a + \dots + a}_{n \text{ times}} = n \cdot a,$$

and

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

if  $a \in F$  and  $n \in \mathbb{N}$ .

**Remark 1.3.2.** In a field, we have subtraction and division  $a - b = a + (-b)$  for  $a, b \in F$ . If  $b \neq 0$ , then  $\frac{a}{b} = a \cdot b^{-1}$  for  $a, b \in F$ .

In a field  $F$ , we have

$$\begin{aligned} (a + b)^2 &= (a + b) \cdot (a + b) \\ &= (a + b) \cdot a + (a + b) \cdot b \\ &= a \cdot a + b \cdot a + a \cdot b + b \cdot b \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2. \end{aligned}$$

**Example 1.3.1.**

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

if  $b \neq 0$  and  $d \neq 0$ .

**Proof.**

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= a \cdot b^{-1} + c \cdot d^{-1} \\ &= ab^{-1}dd^{-1} + cd^{-1}bb^{-1} \\ &= adb^{-1}d^{-1} + cbd^{-1}b^{-1} \\ &= \frac{ad + bc}{bd}. \end{aligned}$$

Notice that this is true since we have commutativity in multiplication and

$$d^{-1}b^{-1} = (bd)^{-1} = \frac{1}{bd}.$$

■

**Example 1.3.2.** The set of rational numbers  $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$  is a field.

**Example 1.3.3.** The set of real numbers is also a field.

**Example 1.3.4.**  $F_2 = \{0, 1\}$  is also a field since we can define addition and multiplication like  $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$ , and  $0 \cdot 0 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$ .

## 1.4 Order Relation

Next, we introduce the order relation. The real number system is ordered by the relation  $<$ , which has the following properties.

- (f) For each pair of real numbers  $a$  and  $b$ , exactly one of the following is true:  $a = b, a < b, b < a$ .
- (g) If  $a < b$  and  $b < c$ , then  $a < c$ .
- (h) If  $a < b$ , then  $a + c < b + c$  for any  $c$ , and if  $0 < c$ , then  $a \cdot c < b \cdot c$ .

**Definition 1.4.1.** A field with an order relation satisfy (f) to (h) is called an ordered field.

**Example 1.4.1.** The set of rational numbers is an ordered field.

**Example 1.4.2.**  $F_2$  is not an ordered field.

**Proof.** If  $0 < 1$ , then  $1 = 0 + 1 < 1 + 1 = 0$ , which is a contradiction. If  $1 < 0$ , then  $0 = 1 + 1 < 0 + 1 = 1$ , which is also a contradiction. ■

**Notation.** In an ordered field, we use  $a \leq b$  to denote either  $a < b$  or  $a = b$ .

## 1.5 Absolute Value and Triangle Inequality

Next, we define the absolute value of a real number

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a \leq 0; \end{cases}$$

**Theorem 1.5.1** (Triangle Inequality).

$$|a + b| \leq |a| + |b|$$

for all  $a, b \in \mathbb{R}$ .

**Corollary 1.5.1.**

$$||a| - |b|| \leq |a - b| \quad \text{and} \quad ||a| - |b|| \leq |a + b|$$

**Proof.** We write

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Similarly we have

$$|b| \leq |b - a| + |a|.$$

So

$$-|b - a| \leq |a| - |b| \leq |a - b|.$$

Thus,

$$||a| - |b|| \leq |a - b|. \quad \blacksquare$$

## 1.6 Supremum and Infimum

Next, we introduce the notion of supremum of a subset of real numbers.

**Definition 1.6.1.** Let  $S$  be a subset of  $\mathbb{R}$ ,

- (1) we say  $b$  is an upper bound of  $S$  if  $x \leq b$  for all  $x \in S$ .
- (2) If  $B$  is an upper bound of  $S$ , and no number smaller than  $B$  is an upper bound of  $S$ , then  $B$  is called the supremum or the least upper bound of  $S$ . We write  $B = \sup S$ .

**Corollary 1.6.1.** If  $B = \sup S$ , then

- (1)  $x \in S$  implies  $x \leq B$

(2) If  $b < B$ , then  $b$  is not an upper bound of  $S$ , i.e. there exists  $x_1 \in S$  such that  $b < x_1$ .

**Definition 1.6.2.** Let  $S$  be a subset of  $\mathbb{R}$ ,

- (1) we say  $b$  is a lower bound of  $S$  if  $x \geq b$  for all  $x \in S$ .
- (2) If  $\alpha$  is a lower bound of  $S$ , and no number bigger than  $\alpha$  is a lower bound of  $S$ , then  $\alpha$  is called the infimum or the greatest lower bound of  $S$ . We write  $\alpha = \inf S$ .

**Corollary 1.6.2.** If  $\alpha = \inf S$ , then

- (1)  $x \in S$  implies  $x \geq \alpha$
- (2) If  $\alpha < a$ , then  $a$  is not a lower bound of  $S$ , i.e. there exists  $x_1 \in S$  such that  $x_1 < a$ .

**Notation** (Interval Notation).

$$(a, b) = \{x \mid a < x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

**Example 1.6.1.**  $S = \{x \mid x < 0\} = (-\infty, 0)$ , then  $\sup S = 0$  but  $\inf S$  does not exist.

**Example 1.6.2.**  $S_1 = \{-1, -2, -3, -4, \dots\} = \{-n \mid n \in \mathbb{N}\}$ , then  $\sup S = -1$ , but  $\inf S$  does not exist.

**Definition 1.6.3 (Nonempty Sets).** A nonempty set is that a set has at least one element. The empty set, written as  $\emptyset$ , is the set has no elements at all.

**Example 1.6.3.**  $S = \{x \mid x \in \mathbb{Q}, x < \sqrt{2}\}$

In  $\mathbb{Q}$ ,  $\sup S$  does not exist. In  $\mathbb{R}$ ,  $\sup S = \sqrt{2}$ .

**Theorem 1.6.1 (Completeness axiom).** If a nonempty set of real numbers (an ordered field) is bounded above, then it has a least upper bound or  $\sup S$  exists.

**Remark 1.6.1.** This is an extra axiom that can't be derived from the properties of ordered field.

**Remark 1.6.2.** Up to "isomorphism", there is exactly one complete ordered field: the field of real numbers.

**Remark 1.6.3.** From now, we assume  $\mathbb{R}$  satisfies the completeness axiom. Thus, any nonempty subset  $S \subseteq \mathbb{R}$  that is bounded above, we have  $\sup S$  exists.

We can prove the following property of  $\sup S$ .

**Theorem 1.6.2.** If  $S \subseteq \mathbb{R}$  is bounded above, then  $\sup S$  is the unique real number  $B$  such that

- (i)  $x \leq B$  for all  $x \in S$
- (ii) for every  $\varepsilon > 0$ , there exist an  $x_0 \in S$  such that  $B - \varepsilon < x_0$ .

**Proof.** (i), (ii) follows from the definition. We prove the uniqueness. Suppose  $B_1 = \sup S = B_2$ . We want to show  $B_1 = B_2$ . Suppose  $B_1 \neq B_2$ . Then either  $B_1 < B_2$  or  $B_2 < B_1$ . However, if either one is true, then the other one cannot be  $\sup S$ . ■

**Theorem 1.6.3 (Archimedean Property).** If  $p > 0$  and  $\varepsilon > 0$ , then there exists an  $n \in \mathbb{N}$  such that  $p < n\varepsilon$ .

**Proof.** We prove this contradiction. Suppose it is not true. This implies  $n\varepsilon \leq p$  for all  $n \in \mathbb{N}$ . Consider  $S = \{n\varepsilon \mid n \in \mathbb{N}\}$ , then  $p$  is an upper bound of  $S$ , so  $S$  is bounded above by  $p$ , so we know  $B = \sup S$  exists. Hence,  $n\varepsilon \leq B$  for all  $n \in \mathbb{N}$ , so we have  $(n+1)\varepsilon \leq B$ , which means

$$n\varepsilon \leq B - \varepsilon$$

for all  $n \in \mathbb{N}$ . This implies  $B - \varepsilon$  is also an upper bound of  $S$ , which is a contradiction. ■

## 1.7 Density of other number system

**Theorem 1.7.1.** Every nonempty subset of the integers that is bounded below has a least element.

**Proof.** We first introduce an axiom:

**Theorem 1.7.2 (Well-Ordering principle).** Every non-empty subset of the natural numbers has a least element.

**Note 1.7.1.** Here,  $\mathbb{N}$  can be  $\{0, 1, 2, \dots\}$  or  $\{1, 2, 3, \dots\}$ , which is not that important.

Now we call this subset of integers as  $S$ , and suppose we have  $m$  as a lower bound of  $S$ , then define  $S' = \{s - m \mid s \in S\}$ , then we know  $S'$  is a nonempty subset of  $\mathbb{N}$ , then by well-ordering principle we know there is a least element in  $S'$  and thus there is also a least element in  $S$ . ■

**Corollary 1.7.1.** Every nonempty subset of the integers that is bounded above has a greatest element.

**Proof.** Suppose  $M$  is an upper bound, then define a set  $S' = \{M - s \mid s \in S\}$ , then by well-ordering principle we know  $M - a$  is the least element of  $S'$  for some  $a \in S$ , so we have  $M - x \geq M - a$  for all  $x \in S$ , which means  $a \geq x$  for all  $x \in S$  and since  $a \in S$ , so  $a$  is the greatest element of  $S$ . ■

**Theorem 1.7.3.** The set of rational numbers is dense in the real number. That is, if  $a$  and  $b$  are real numbers with  $a < b$ , then there exists a rational number  $\frac{p}{q}$  such that  $a < \frac{p}{q} < b$ .

**Proof.** Let  $a, b \in \mathbb{R}$ ,  $a < b$ . By [Archimedean Property](#),  $\exists q \in \mathbb{N}$  such that  $q(b - a) > 1$ . Let  $S = \{m \mid m \text{ is an integer with } m > qa\}$ , since we know  $S \neq \emptyset$  and  $S$  is bounded below. Hence,  $p = \inf S$  exists and is an integer by the last theorem. So  $qa < p$  and  $p - 1 \leq qa$ , which means  $qa < p \leq qa + 1 < qb$ , so we have  $a < \frac{p}{q} < b$ . ■

## Lecture 2

**Definition 1.7.1 (Floor Function).** For any real number  $x$ , the floor function of  $x$  is denoted by  $\lfloor x \rfloor$ , and is defined by the formula  $\lfloor n \rfloor$  if  $n \leq x < n + 1$  where  $n \in \mathbb{Z}$ .

**Corollary 1.7.2.**

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

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**Example 1.7.1.**  $\lfloor 3.7 \rfloor = 3$ ,  $\lfloor -1.2 \rfloor = -2$ .

Now by floor function, we can reprove [Theorem 1.7.3](#).

**Theorem 1.7.4** (Density of rational number in real number Again). The set of rational numbers is dense in the real number. That is, if  $a$  and  $b$  are real numbers with  $a < b$ , then there exists a rational number  $\frac{q}{p}$  such that  $a < \frac{q}{p} < b$ .

**Reprove Theorem 1.7.3.** Since  $a < b$ , so we know  $b - a > 0$ . Now by [Archimedean Property](#), we know there exists  $q \in \mathbb{N}$  such that  $q(b - a) > 1$ . Let  $p = \lfloor qa \rfloor + 1$ , we have

$$\lfloor qa \rfloor \leq qa < \lfloor qa \rfloor + 1 = p.$$

From our construction,  $qb > qa + 1$ , so we have

$$p = \lfloor qa \rfloor + 1 \leq qa + 1 < qb,$$

hence we have

$$qa \leq p \leq qb.$$

■

**Note 1.7.2.** For some reason,  $p, q$  in [Theorem 1.7.3](#) and [Theorem 1.7.4](#) are reversed.

**Definition 1.7.2** (irrational number).  $x$  is called irrational if  $x$  is not rational.

**Example 1.7.2.**  $\sqrt{2}$  is irrational.

**Theorem 1.7.5.** Let  $r \in \mathbb{Q}$  and  $x \in \mathbb{R} \setminus \mathbb{Q}$ , then

1.  $r + x$  is irrational.
2. If  $r \neq 0$ , then  $rx$  is irrational.

**sketch of proof.**

1. If  $r + x = q \in \mathbb{Q}$ , then  $x = q - r \in \mathbb{Q}$ , contradiction.
2. If  $rx = q \in \mathbb{Q}$ , then  $x = \frac{q}{r} \in \mathbb{Q}$  since  $r \neq 0$ .

■

**Theorem 1.7.6** (irrational number dense in real number). The set of irrational number is dense in real number. That is, if  $a, b \in \mathbb{R}$  and  $a < b$ , then there exists a irrational number  $t$  such that  $a < t < b$ .

**Proof.** By [density of rational number](#), we can find  $a < r_1 < r_2 < b$  where  $r_1, r_2 \in \mathbb{Q}$ , and then let  $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$ , then we know

$$a < r_1 < t < r_2 < b.$$

**Note 1.7.3.** We should use [Theorem 1.7.5](#) and the fact that  $\sqrt{2}$  is irrational.

■

**Definition 1.7.3 (bounded set).** A set  $S \subseteq \mathbb{R}$  is bounded if there are numbers  $a, b$  s.t.  $a \leq x \leq b$  for all  $x \in S$ .

**Corollary 1.7.3.** A bounded non-empty set in  $\mathbb{R}$  has a unique supremum and a unique infimum and  $\inf S \leq \sup S$ .

## 1.8 Extended real number system

The real number system, together with  $\infty$  and  $-\infty$ , then we have the following properties:

- (a) If  $a \in \mathbb{R}$ , then  $a + \infty = \infty + a = \infty$  and  $a - \infty = -\infty + a = -\infty$ , and  $\frac{a}{\infty} = \frac{a}{-\infty} = 0$ .
- (b) If  $a > 0$ , then  $a \cdot \infty = \infty \cdot a = \infty$  and  $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$
- (c) If  $a < 0$ , then  $a \cdot \infty = \infty \cdot a = -\infty$  and  $a \cdot (-\infty) = -\infty \cdot a = \infty$  and  $\infty + \infty = \infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$  and  $-\infty - \infty = \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$  and  $|\infty| = |-\infty| = \infty$

However, there are some indeterminate form:

**Theorem 1.8.1.** The following things are not defined:

$$\infty - \infty, 0 \cdot \infty, \frac{\infty}{\infty}, \text{ and } \frac{0}{0}.$$

## 1.9 Mathematical Induction

**Theorem 1.9.1 (Peano's Postulate).** The natural numbers satisfy the following properties

- (a)  $\mathbb{N}$  is nonempty.
- (b) For each natural number  $n$ , there exists a unique rational number  $n$  called the successor of  $n$ .
- (c) There exists a natural number  $\bar{n}$  that is not the successor of any natural number.
- (d) Different natural numbers have different successors, that is,  $n \neq m$  implies  $n' \neq m'$ .
- (e) The only subset of  $\mathbb{N}$  that contains  $\bar{n}$  and also contains the successor of every one of its element is  $\mathbb{N}$ .

**Theorem 1.9.2 (Principle of Mathematical Induction).** Let  $p_1, p_2, \dots, p_n$  be propositions, one for each positive integers, such that

- (a)  $p_1$  is true.
- (b) for each positive integer  $n$ ,  $p_n$  implies  $p_{n+1}$ .

then  $p_n$  is true for each  $n \in \mathbb{N}$ .

**Proof.** Let  $M = \{n \mid n \in \mathbb{N} \text{ and } p_n \text{ is true}\}$ , then from (a) we know  $1 \in M$  and from (b) we know  $n \in M$  implies  $n + 1 \in M$ . Hence, from (e) of [Peano's Postulate](#), we know  $M = \mathbb{N}$ . ■

# Chapter 2

## Metric Space

### 2.1 Definition and examples

**Definition 2.1.1.** Suppose  $x_n \in \mathbb{R}$  for  $n \geq m$ . We use the notation  $(x_n)_{n=m}^{\infty}$  to denote the sequence of numbers

$$x_m, x_{m+1}, \dots$$

We first recall the definition of a convergent sequence.

**Definition 2.1.2 (Convergent Sequence).** We say that a sequence  $(x_n)_{n=m}^{\infty}$  of real numbers converges to  $x$  if for every  $\varepsilon > 0$ , there exists an  $N \geq m$  s.t.  $|x_n - x| \leq \varepsilon$  for all  $n \geq N$ .

**Notation.** We write  $\lim_{n \rightarrow \infty} x_n = x$ .

On  $\mathbb{R}$ , we can define the distance function between two points  $x, y \in \mathbb{R}$  by  $d(x, y) = |x - y|$ . We'll discuss this more later.

**Lemma 2.1.1.** Let  $(x_n)_{n=m}^{\infty}$  be a sequence of real numbers, and let  $x$  be another real number, then  $(x_n)_{n=m}^{\infty}$  converges to  $x$  if and only if  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ .

**Proof.** Assume  $(x_n)_{n=m}^{\infty}$  converges to  $x$ . Let  $\varepsilon > 0$  be arbitrary real number. By definition, there exists an  $N \geq m$  such that  $|x_n - x| \leq \varepsilon$  for all  $n \geq N$ . But  $d(x_n, x) = |x_n - x|$  by the definition. Hence,  $\forall \varepsilon > 0, \exists N \geq m$  such that  $d(x_n, x) \leq \varepsilon$  for all  $n \geq N$ . This implies that  $\forall \varepsilon > 0, \exists N \geq m$  such that  $|d(x_n, x) - 0| \leq \varepsilon$  for all  $n \geq N$ . This implies  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ .

The proof of the other side is the same but writing the above proof from bottom to top again.

**Definition 2.1.3 (Metric Space).** A metric space  $(X, d)$  is the space of  $X$  of objects (called points), together with a distance function or metric  $d : X \times X \rightarrow [0, \infty)$  which associates to each  $x, y$  of points in  $X$  a nonnegative number  $d(x, y) \geq 0$ , the following. Furthermore, the metric must satisfy 4 axioms.

- (a) For any  $x \in X$ ,  $d(x, x) = 0$ .
- (b) (Positivity) For any distinct  $x, y \in X$ , we have  $d(x, y) > 0$ .
- (c) (Symmetry) For any  $x, y \in X$ , we have  $d(x, y) = d(y, x)$ .
- (d) (Triangle inequality) For any  $x, y, z \in X$ , we have  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Example 2.1.1.** On  $\mathbb{R}$ , we can define  $d(x, y) = |x - y|$ .

**Proof.** •  $d(x, y) = |x - y| \geq 0$ .

- $d(x, y) = 0$  iff  $|x - y| = 0$  iff  $x = y$ .
- $|x - y| = |y - x|$ , so  $d(x, y) = d(y, x)$
- $|x - z| \leq |x - y| + |y - z|$  for all  $x, y, z \in \mathbb{R}$ .

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**Example 2.1.2.** Let  $(X, d)$  be a metric space and  $Y \subseteq X$ , then  $Y$  inherits a natural distance function

$$d|_{Y \times Y} : Y \times Y \rightarrow [0, \infty)$$

defined by  $d|_{Y \times Y}(\alpha, \beta) = d(\alpha, \beta)$  for all  $\alpha, \beta \in Y$ .

**Note 2.1.1.**  $(Y, d|_{Y \times Y})$  is called a metric subspace of  $(X, d)$ . It is obvious that  $d|_{Y \times Y}$  is a metric on  $Y$ .

Recall  $\mathbb{R}^n$ . Let  $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ .

**Definition 2.1.4** ( $l^2$ -metric). The  $l^2$ -metric is defined by

$$d_2(x, y) = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}} \quad (\text{or we called } d_{l_2}(x, y)).$$

**Definition 2.1.5** ( $l^1$ -metric (taxicab metric)). The  $l^1$ -metric is defined by

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i| \quad (\text{or we called } d_{l_1}(x, y))$$

**Definition 2.1.6** ( $l^\infty$ -metric). The  $l^\infty$ -metric is defined by

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

**Exercise 2.1.1.** Verify they are all metrics.

**Note 2.1.2.** Actually we have to define inner product and norm first and then we can use the triangle inequality of norm to prove  $d_2$  is a metric. (See lecture notes by professor)

## Lecture 3

**Definition 2.1.7** (Cartesian Product). Let  $A, B$  be sets. The cartesian product of  $A$  and  $B$  is defined by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Similarly, the cartesian product of  $X_1, X_2, \dots, X_n$  is

$$X_1 \times X_2 \times \dots \times X_n = \{(x_1, x_2, \dots, x_n) \mid x_i \in X_i \forall 1 \leq i \leq n\}.$$

**Definition 2.1.8** (Functions). Let  $X_1, X_2, \dots, X_n$  be sets and let  $Y$  be another set. A function of  $n$  variables with codomains is a map  $f : X_1 \times X_2 \times \dots \times X_n \rightarrow Y$  which assigns each  $n$ -tuple  $(x_1, x_2, \dots, x_n)$  with  $x_i \in X_i$  a unique element  $f(x_1, x_2, \dots, x_n)$ .

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**Definition.** We talk about the definition of domain, codomain, and range:

**Definition 2.1.9.** The domain of  $f$  is  $X_1 \times X_2 \times \cdots \times X_n$  and  $Y$  is the codomain of  $f$ .

**Definition 2.1.10.** The range of  $f$  is

$$\{f(x_1, x_2, \dots, x_n) \in Y \mid x_i \in X_i \ \forall i\}.$$

In the definition of metric space, we write  $(X, d)$  to emphasize our set  $X$  and  $d$  is a distance function defined on  $X \times X$ , i.e.

$$d : X \times X \rightarrow [0, \infty) \subseteq \mathbb{R},$$

where

$$d : (x, y) \mapsto d(x, y)$$

for  $x, y \in X$ . Let  $(X, d)$  be a metric space and  $Y \subseteq X$ . Then  $(Y, d|_{Y \times Y})$  is also a metric space with distance function defined by

$$d|_{Y \times Y} \rightarrow [0, \infty)$$

and

$$d|_{Y \times Y} : (\alpha, \beta) \mapsto d(\alpha, \beta) \text{ for } \alpha, \beta \in Y.$$

**Example 2.1.3.** Recall the **Taxi-cab metric**, it can be used in cryptography. For example, for two binary strings, we know

$$d_1((10010), (10101)) = 3 = \text{the number of mismatched bits.}$$

**Example 2.1.4.** Recall the  **$l^\infty$ -metric**. Suppose two jobs where each consists of 3 tasks, and the time (in hours) to complete each task is represented by a vector

$$x = (2, 4, 6), \ y = (3, 7, 5),$$

so

$$d_\infty(x, y) = \max\{|2 - 3|, |4 - 7|, |6 - 5|\} = 3.$$

**Definition 2.1.11 (Lipschitz equivalent metrics).** Let  $(X, d_1)$  and  $(X, d_2)$  be two metrics on  $X$ . We say  $d_1$  and  $d_2$  are Lipschitz equivalent if  $\exists c_1, c_2 > 0$  s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y) \quad \forall x, y \in X$$

**Remark 2.1.1.** They will have same topology (defined later).

**Proposition 2.1.1.** For all  $x, y \in \mathbb{R}^n$ ,

$$d_2(x, y) \leq d_1(x, y) \leq \sqrt{n} d_2(x, y) \tag{2.1}$$

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} d_\infty(x, y) \tag{2.2}$$

**Remark 2.1.2.**

$$\begin{aligned} d_\infty(x, y) &\geq \frac{1}{\sqrt{n}} d_2(x, y) \\ &\geq \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} d_1(x, y) = \frac{1}{n} d_1(x, y). \end{aligned}$$

Also,

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y).$$

**Remark 2.1.3.**  $d_1, d_2, d_\infty$  are all Lipschitz equivalent.

**proof of Proposition 2.1.1** . Recall  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ , then

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|, \quad d_2(x, y) = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

By Cauchy-Schurwatz inequality,

$$\begin{aligned} d_1(x, y) &= \sum_{i=1}^n |x_i - y_i| \\ &\leq \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n 1^2 \right)^{\frac{1}{2}} = \sqrt{n} d_2(x, y). \end{aligned}$$

Now we show that  $d_1(x, y) \geq d_2(x, y)$ .

$$\begin{aligned} (d_1(x, y))^2 &= \left( \sum_{i=1}^n |x_i - y_i| \right)^2 \\ &= \sum_{i=1}^n |x_i - y_i|^2 + 2 \sum_{1 \leq i < j \leq n} |x_i - y_i| |x_j - y_j| \\ &\geq \sum_{i=1}^n |x_i - y_i|^2 = d_2(x, y)^2. \end{aligned}$$

Hence, we have  $d_1(x, y) \geq d_2(x, y)$ .

Now we show that  $d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$ . Note that

$$d_2(x, y) = \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}, \quad d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

For each  $i$ , we know

$$|x_i - y_i| \leq d_\infty(x, y),$$

so

$$d_2(x, y)^2 \leq \sum_{i=1}^n d_\infty(x, y)^2 = n d_\infty(x, y)^2,$$

so  $d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$ . ■

**Definition 2.1.12 (Discrete metric).** Let  $X$  be any set, define the discrete metric:

$$d_{\text{disc}} : X \times X \rightarrow \{0, 1\}$$

where

$$d_{\text{disc}}(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

Why this is a metric? Because

- $d_{\text{disc}}(x, y) \geq 0$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ .
- $d_{\text{disc}}(x, y) = d_{\text{disc}}(y, x)$  by definition.
- $d_{\text{disc}}(x, z) \leq d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z)$ ?

**proof of triangle inequality in discrete metric.** We first consider the case that  $x = z$ , then

$$d_{\text{disc}}(x, z) = 0,$$

so it is obviously that the triangle inequality is true.

Now if  $x \neq z$ , then either  $y \neq z$  or  $y \neq x$  must happen, so the triangle inequality must be true. ■

**Example 2.1.5.** We can define

$$d(x, x) = 0, \quad d(x, y) = \text{minimal length of a path from } x \text{ to } y,$$

then this is also a metric.



Figure 2.1: Graph metrics

**Definition 2.1.13 (Convergence in metric space).** Let  $m$  be an integer,  $(X, d)$  be a metric space, and let  $(X^{(n)})_{n=m}^{\infty}$  be a sequence of points in  $X$ . Let  $x \in X$ . We say that  $(X^{(n)})_{n=m}^{\infty}$  converges to  $x$  with respect to  $d$  iff

$$\lim_{n \rightarrow \infty} d(X^{(n)}, x) = 0,$$

where  $\lim_{n \rightarrow \infty} d(X^{(n)}, x) = 0$  iff for every  $\varepsilon > 0$ ,  $\exists N \geq m$  s.t.  $d(X^{(n)}, x) \leq \varepsilon$  for all  $n \geq N$ .

**Notation.** We also write  $\lim_{n \rightarrow \infty} X^{(n)} = x$  in  $(X, d)$ .

**Remark 2.1.4.** Suppose  $(X^{(n)})_{n=m}^{\infty}$  converges to  $x$  in  $(X, d)$ , then  $(X^{(n)})_{n=m_1}^{\infty}$  also converges to  $x$  in  $(X, d)$  if  $m_1 \geq m$ .

**Example 2.1.6.** Let  $(X^{(n)})_{n=1}^{\infty}$  denote the sequence  $X^{(n)} = (\frac{1}{n}, \frac{1}{n})$  in  $\mathbb{R}^2$ , then what will this sequence converges to for different metric?

**Proof.**

- If the metric is  $d_1$ , then

$$d_1(X^{(n)}, (0, 0)) = \left| \frac{1}{n} - 0 \right| + \left| \frac{1}{n} - 0 \right| = \frac{2}{n},$$

so

$$\lim_{n \rightarrow \infty} d_1(X^{(n)}, (0, 0)) = \lim_{n \rightarrow \infty} \frac{2}{n} = 0.$$

- If the metric is  $d_2$ , then

$$d_2 \left( X^{(d)}, (0, 0) \right) = \sqrt{\left( \frac{1}{n} - 0 \right)^2 + \left( \frac{1}{n} - 0 \right)^2} = \frac{\sqrt{2}}{n}.$$

Hence, under  $l_2$ -metric  $\{X^{(n)}\}$  also converges to 0.

- If the metric is  $d_\infty$ , then

$$d_\infty \left( X^{(n)}, (0, 0) \right) = \max \left\{ \left| \frac{1}{n} \right|, \left| \frac{1}{n} \right| \right\} = \frac{1}{n},$$

so it also converges to 0.

- If the metric is discrete metric, then however, it will not converges to  $(0, 0)$  since

$$\lim_{n \rightarrow \infty} d_{\text{disc}} \left( X^{(n)}, (0, 0) \right) = \lim_{n \rightarrow \infty} d_{\text{disc}} \left( \left( \frac{1}{n}, \frac{1}{n} \right), (0, 0) \right) = 1.$$

⊛

**Definition.** Let  $f : X \rightarrow Y$  be a function with domain  $X$  and codomain  $Y$ . The range of  $f = \{f(x) \mid x \in X\} \subseteq Y$ .

**Definition 2.1.14 (injective).** We say  $f$  is injective or one-to-one if for all  $x_1, x_2 \in X$ ,  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ .

**Definition 2.1.15 (surjective).** We say  $f$  is surjective or onto if for every  $y \in Y$ ,  $\exists x \in X$  s.t.  $f(x) = y$ .

**Definition 2.1.16 (bijective).** We say  $f$  is bijective if  $f$  is injective and surjective.

**Corollary 2.1.1.** If  $f$  is bijective, then there exists  $f^{-1} : Y \rightarrow X$  defined by  $f^{-1}(y) = x$  if  $f(x) = y$ . We also have

$$\begin{aligned} f(f^{-1}(y)) &= y \quad \forall y \in Y \\ f^{-1}(f(x)) &= x \quad \forall x \in X. \end{aligned}$$

**Example 2.1.7.**  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  in  $(\mathbb{R}, d)$ , where  $d$  is the standard metric in  $\mathbb{R}$ , which is defined by

$$d(x, y) = |x - y|.$$

But in different metric,  $\lim_{n \rightarrow \infty} \frac{1}{n}$  may not be 0.

**Proof.** Define  $f : [0, 1] \rightarrow [0, 1]$  defined by

$$f(x) = \begin{cases} x, & \text{if } 0 < x < 1; \\ 1, & \text{if } x = 0; \\ 0, & \text{if } x = 1. \end{cases}$$

$f$  is bijective on  $[0, 1]$  to  $[0, 1]$



Define another metric  $d^1$  on  $[0, 1]$  by

$$d^1(x, y) = d(f(x), f(y)).$$

We want to show that  $d^1$  is also a metric on  $[0, 1]$ .

- $d^1(x, y) = d(f(x), f(y)) = |f(x) - f(y)| \geq 0$
- $d^1(x, y) = 0$  iff  $f(x) = f(y)$  iff  $x = y$  since  $f$  is injective.
- The triangle inequality is trivially true since we can just use the triangle inequality in  $d$ .

In fact,  $\lim_{n \rightarrow \infty} \frac{1}{n} = 1$  in  $([0, 1], d^1)$  since

$$\lim_{n \rightarrow \infty} d^1\left(\frac{1}{n}, 1\right) = \lim_{n \rightarrow \infty} d\left(\frac{1}{n}, 0\right) = \lim_{n \rightarrow \infty} \left|\frac{1}{n}\right| = 0.$$

⊛

## 2.2 Some point set topology of metric space

**Definition 2.2.1 (ball).** Let  $(X, d)$  be a metric space. let  $x_0 \in X$  and  $r > 0$ . We define the ball  $B_{(X, d)}(x_0, r)$  in  $X$ , centered at  $x_0$  and with radius  $r$  in the metric  $d$ , to the set

$$B_{(X, d)}(x_0, r) := \{x \in X \mid d(x_0, x) < r\}.$$

Sometimes, we write it as  $B_X(x_0, r)$  or  $B(x_0, r)$ .

**Example 2.2.1.** In  $\mathbb{R}^2$ ,

$$B_{(\mathbb{R}^2, d_2)}((0, 0), 1) = \{(x, y) \mid d_2((x, y), (0, 0)) = \sqrt{x^2 + y^2} < 1\},$$

and

$$B_{(\mathbb{R}^2, d_1)}((0, 0), 1) = \{(x, y) \mid d_1((x, y), (0, 0)) = |x| + |y| < 1\},$$

and

$$B_{(\mathbb{R}^2, d_\infty)}((0, 0), 1) = \{(x, y) \mid d_\infty((x, y), (0, 0)) = \max\{|x|, |y|\} < 1\},$$

also we can consider the  $d_{\text{disc}}$  case but I am too lazy to write it down.

**Notation.** Let  $E \subseteq X$ , we will write

$$X \setminus E := \{x \in X \mid x \notin E\}.$$

**Definition.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . For a point  $x_0 \in X$ ,

**Definition 2.2.2 (interior point).**  $x_0$  is an interior point of  $E$  if  $\exists r > 0$  s.t.  $B(x_0, r) \subseteq E$ .

**Definition 2.2.3 (exterior point).**  $x_0$  is an exterior point of  $E$  if  $\exists r > 0$  s.t.  $B(x_0, r) \subseteq X \setminus E$ .

**Definition 2.2.4 (boundary point).**  $x_0$  is a boundary point of  $E$  if it is neither an interior point nor an exterior point of  $E$ .

**Proposition 2.2.1.**  $x_0$  is a boundary point of  $E$  iff for all  $r > 0$ ,  $B(x_0, r) \cap E \neq \emptyset$  and  $B(x_0, r) \cap (X \setminus E) \neq \emptyset$ .

## Lecture 4

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**Theorem 2.2.1.** Let  $(X, d_1)$  and  $(X, d_2)$  be metrics on  $X$ , and suppose  $d_1$  and  $d_2$  are Lipschitz equivalent, then for any sequence  $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$ , then for any  $x \in X$

$$\lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (X, d_1) \Leftrightarrow \lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (X, d_2).$$

**Proof.** Since  $d_1, d_2$  are Lipschitz equivalent, so there exists  $c_1, c_2 > 0$  s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y).$$

( $\Rightarrow$ ) Given  $\frac{\varepsilon}{c_2} > 0$ , since  $\lim_{n \rightarrow \infty} x^{(n)} = x$  in  $(X, d_1)$ , so there exists  $N$  s.t.  $N \geq m$  and

$$d_1(x^{(n)}, x) \leq \frac{\varepsilon}{c_2} \text{ for } n \geq N.$$

This implies  $d_2(x^{(n)}, x) \leq c_2 d_1(x^{(n)}, x) \leq \varepsilon$  for  $n \geq N$ , which means

$$\lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (X, d_2).$$

( $\Leftarrow$ ) Similar. ■

**Remark 2.2.1.** On  $\mathbb{R}^n$ , the metrics  $d_1, d_2, d_{\infty}$  are Lipschitz equivalent, that is,

$$\lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_1) \Leftrightarrow \lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_2) \Leftrightarrow \lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_{\infty})$$

**Proposition 2.2.2.** Let  $(X, d_{\text{disc}})$  be a discrete metric space, and  $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$ . Then

$$\lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (X, d_{\text{disc}}) \Leftrightarrow \exists N \geq m \text{ s.t. } x^{(n)} = x \text{ for } n \geq N.$$

**Proof.** ( $\Leftarrow$ ) Easy.

( $\Rightarrow$ ) Given  $\frac{1}{2} > 0$ , there exists  $N \geq m$  s.t.  $d(x_n, x) < \frac{1}{2}$  for  $n \geq N$ , but  $d(x_n, x) < \frac{1}{2}$  implies  $d(x_n, x) = 0$ , which means  $x_n = x$  for all  $n \geq N$ . ■

**Definition.** We define the interior, exterior, and boundary point again.

**Definition 2.2.5.** The set of interior points is denoted by

$$\text{Int}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq E\}.$$

**Definition 2.2.6.** The set of exterior points is denoted by

$$\text{Ext}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq X \setminus E\}.$$

**Definition 2.2.7.** A point is a boundary points if it is neither an interior point nor an exterior point, and we define

$$\partial E = \{x \in X \mid x \notin \text{Int}(E) \text{ and } x \notin \text{Ext}(E)\}.$$

**Remark 2.2.2.**

1.

$$x_0 \notin \text{Int}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (X \setminus E) \neq \emptyset.$$

2.

$$x_0 \notin \text{Ext}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (E) \neq \emptyset.$$

3.  $\text{Int}(X \setminus E) = \text{Ext}(E)$ .4.  $\partial E = \partial(X \setminus E)$  since

$$x_0 \in \partial E \Leftrightarrow x \notin \text{Int}(E) \text{ and } \text{Ext}(E) \Leftrightarrow x_0 \notin \text{Int}(E) \text{ and } x_0 \notin \text{Int}(X \setminus E).$$

Also,

$$x_0 \in \partial(X \setminus E) \Leftrightarrow x \notin \text{Int}(X \setminus E) \text{ and } \text{Ext}(X \setminus E) \Leftrightarrow x_0 \notin \text{Int}(X \setminus E) \text{ and } x_0 \notin \text{Int}(E).$$

Hence, acutually  $\partial E = \partial(X \setminus E)$ .**Proposition 2.2.3.**

$$x_0 \in \partial E \Leftrightarrow \text{For any } r > 0, B_X(x_0, r) \cap E \neq \emptyset \text{ and } B_X(x_0, r) \cap (X \setminus E) \neq \emptyset$$

**Example 2.2.2.** Let  $(\mathbb{R}, d)$  be the usual metric on  $\mathbb{R}$ , where

$$d(x, y) = |x - y|.$$

Then, we know in this space,

$$\begin{aligned} B_{\mathbb{R}}(x_0, r) &= \{x \in \mathbb{R} \mid d(x, x_0) < r\} \\ &= \{x \in \mathbb{R} \mid |x - x_0| < r\} \\ &= \{x \in \mathbb{R} \mid -r + x_0 < x < r + x_0\}. \end{aligned}$$

Hence, suppose  $E = [1, 2]$ , then  $\text{Int}(E) = (1, 2)$  since we know  $B(x_0, r) = (x_0 - r, x_0 + r)$ , so for all  $x \in (1, 2)$ , we know there is an open ball  $B(x_0, r) \subseteq [1, 2]$  for some  $r > 0$ . Also, consider the endpoint 1, 2, we can verify that these two points are not interior points. Besides, consider the points not in  $[1, 2]$ , it is trivial that they cannot be interior points.

**Example 2.2.3.** We consider  $(X, d_{\text{disc}})$ . Let  $E \subseteq X$ . If  $x \in E$ , we know

$$B\left(x, \frac{1}{2}\right) = \left\{y \mid d(y, x) < \frac{1}{2}\right\} = \{x\} \subseteq E.$$

Hence,  $E \subseteq \text{Int}(E)$ . Besides, for all  $x \in \text{Int}(E)$ , we know there exists  $r > 0$  s.t.  $B(x_0, r) \subseteq E$ , also we know  $x_0 \in B(x_0, r) \subseteq E$ , so  $x_0 \in E$ , and thus  $\text{Int}(E) \subseteq E$ . Hence,  $E = \text{Int}(E)$ . Similarly,  $\text{Int}(X \setminus E) = X \setminus E$ . Suppose there is a  $x \in X$  s.t.  $x \in \partial E$ , then  $x \notin \text{Int}(E) = E$  and  $x \notin \text{Ext}(E) = \text{Int}(X \setminus E) = X \setminus E$ , so such  $x$  does not exist.

**Definition 2.2.8 (Closure).** Let  $(X, d)$  be a metric space, and let  $E \subseteq X$  and  $x_0 \in X$ . We say  $x_0$  is a adherent point of  $E$  if for every  $r > 0$ ,  $B(x_0, r) \cap E \neq \emptyset$ . The set of adherent points is called the closure of  $E$ , and denoted by  $\overline{E}$ .

**Proposition 2.2.4 (TFAE).**(a)  $x_0$  is an adherent point of  $E$ .

- (b)  $x_0$  is either an interior point or a boundary point of  $E$ .  
 (c)  $\exists$  a sequence  $\{X^{(n)}\}_{n=1}^{\infty}$  in  $E$  which converges to  $x_0$  in  $(X, d)$ .

**proof from (a) to (b).** Suppose  $x_0 \in \overline{E}$ , then  $B(x_0, r) \cap E \neq \emptyset$  for all  $r > 0$ . If  $\exists s > 0$  s.t.  $B(x_0, s) \subseteq E$ , then  $x_0 \in \text{Int}(E)$ . If such  $s$  does not exist, then we know

$$B(x_0, r) \cap E \neq \emptyset \text{ and } B(x_0, r) \cap (X \setminus E) \neq \emptyset \text{ for all } r > 0,$$

so we can use [Proposition 2.2.1](#) to conclude that  $x_0$  must be a boundary point. ■

**proof from (b) to (c).** Since either  $x_0 \in \text{Int}(E)$  or  $x_0 \in \partial E$ . If  $x_0 \in \text{Int}(E)$ , then  $x_0 \in E$ , then we can choose  $X^{(n)} = x_0$  for all  $n \geq 1$ . If  $x_0 \in \partial E$ , then given  $n \in \mathbb{N}$ ,  $\exists x_n \in B(x_0, \frac{1}{n}) \cap E \neq \emptyset$ . Hence,  $x_n \in E$  and  $d(x_n, x_0) < \frac{1}{n}$ . Pick such  $x_n$  to form  $\{X^{(n)}\}_{n=1}^{\infty}$ , then we know this sequence converges to  $x_0$ . ■

**proof from (c) to (a).** Suppose  $\{X^{(n)}\} \subseteq E$  s.t.  $\lim_{n \rightarrow \infty} d(X^{(n)}, x_0) = 0$ , then we want to show  $x_0 \in \overline{E}$ . Given any  $r > 0$ , choose  $N \geq 1$  s.t.

$$d(X^{(n)}, x_0) < r \text{ when } n \geq N.$$

This implies for  $n \geq N$ ,  $X^{(n)} \in E$  and  $X^{(n)} \in B(x_0, r)$ , so we know  $E \cap B(x_0, r) \neq \emptyset$  for all  $r > 0$ , which means  $x_0 \in \overline{E}$ . ■

**Remark 2.2.3.** The equation (a) and (b) implies  $\overline{E} = \text{Int}(E) \cup \partial E$ .

**An alternative proof.** Since we know  $X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$  by [Theorem 2.2.2](#), and  $\overline{E} \subseteq X$ , so

$$\begin{aligned} \overline{E} &= \overline{E} \cap X = \overline{E} \cap (\text{Int}(E) \cup \text{ext}(E) \cup \partial E) \\ &= (\overline{E} \cap \text{Int}(E)) \cup (\overline{E} \cap \text{Ext}(E)) \cup (\overline{E} \cap \partial E). \end{aligned}$$

Also, notice that

$$\overline{E} \cap \text{Int}(E) = \text{Int}(E) \quad \overline{E} \cap \text{Ext}(E) = \emptyset \quad \overline{E} \cap \partial E = \partial E,$$

so  $\overline{E} = \text{Int}(E) \cup \partial E$ . ■

**Corollary 2.2.1.**  $\overline{E} = \text{Int}(E) \cup \partial E$ .

**Theorem 2.2.2.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . Then,

$$X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$$

**Remark 2.2.4.**  $\partial E$  could be empty. (See previous example.)

**Corollary 2.2.2.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . Then

$$\overline{E} = \text{Int}(E) \cup \partial E = X \setminus \text{Ext}(E).$$

**Lemma 2.2.1.**  $\overline{E} = E \cup \partial E$

**Proof.** We first show that  $E \cup \partial E \subseteq \overline{E}$ . For every point  $x \in E$ , we know  $x \in B(x, r)$  for all  $r > 0$ , so  $B(x, r) \cap E \neq \emptyset$ . Also, by definition, we know  $\partial E \subseteq \overline{E}$ , so we're done.

Next, we show that  $\overline{E} \subseteq E \cup \partial E$ . For every  $x \in \overline{E}$ , if  $x \in E$ , then  $x \in E \cup \partial E$ . If not, since  $x \in \overline{E}$ , so  $B(x, r) \cap E \neq \emptyset$  for all  $r > 0$ . Also, since  $x \notin E$ , and  $x \in B(x, r)$ , so  $B(x, r) \cap (X \setminus E) \neq \emptyset$ ,

otherwise  $x \in B(x, r) \subseteq E$ , which is a contradiction. Now we know for every  $r > 0$ ,  $B(x, r) \cap E \neq \emptyset$  and  $B(x, r) \cap (X \setminus E) \neq \emptyset$ , so  $x \in \partial E$ . ■

**Lemma 2.2.2 (Discarded).** If  $x \in \text{Int}(E)$ , then  $x \in E$ . In other words,  $\text{Int}(E) \subseteq E$ .

**Proof.** If  $x \in \text{Int}(E)$ , then there exists  $r > 0$  s.t.  $B(x, r) \subseteq E$ , and thus  $x \in B(x, r) \subseteq E$ , which means  $x \in E$ . ■

**Note 2.2.1.** I thought we need [Lemma 2.2.2](#) to prove [Theorem 2.2.3](#), but I found it needless. Nevertheless, I still want to keep it since I think it is useful in some elsewhere.

**Definition 2.2.9.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . We say  $E$  is closed if  $\partial E \subseteq E$ . We say  $E$  is open if it doesn't contain any boundary points i.e.  $\partial E \cap E = \emptyset$ .

**Theorem 2.2.3.**  $E$  is closed if and only if  $\overline{E} = E$ .

**Proof.**

$$\begin{aligned} E \text{ is closed} &\Rightarrow \partial E \subseteq E \Rightarrow \overline{E} = E \cup \partial E = E. \\ E = \overline{E} = E \cup \partial E &\Rightarrow \partial E \subseteq E \Rightarrow E \text{ is closed.} \end{aligned}$$

■

**Theorem 2.2.4.**  $E$  is open.  $\Leftrightarrow \text{Int}(E) = E$ .

**proof of  $(\Rightarrow)$ .**  $E$  is open means  $\partial E \cap E = \emptyset$ . Fix  $x \in E$ , since  $x \notin \partial E$ , so  $\exists r > 0$  s.t.  $B(x, r) \cap E = \emptyset$  or  $B(x, r) \cap (X \setminus E) = \emptyset$ . Since  $x \in E$  and  $x \in B(x, r)$ , so  $B(x, r) \cap (X \setminus E) = \emptyset$ , which means  $B(x, r) \subseteq E$ , so  $x \in \text{Int}(E)$ . Now we know  $E \subseteq \text{Int}(E)$ . Also, we know  $\text{Int}(E) \subseteq E$  by [Lemma 2.2.2](#). Hence,  $\text{Int}(E) = E$ . ■

**proof of  $(\Leftarrow)$ .** If  $\text{Int}(E) = E$ , then given any  $x \in E = \text{Int}(E)$ , there exists  $r > 0$  s.t.  $B(x, r) \subseteq E$ . Hence,  $B(x, r) \cap (X \setminus E) = \emptyset$ , so  $x \notin \partial E$ , and thus  $E \cap \partial E = \emptyset$ . ■

**Theorem 2.2.5.** If  $E \subseteq X$ , then  $E$  is open  $\Leftrightarrow X \setminus E$  is closed.

**proof of  $(\Rightarrow)$ .** Since we can write  $X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$ , and  $E$  is open, so

$$X \setminus E = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus E = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus \text{Int}(E) = \text{Ext}(E) \cup \partial E.$$

by [Theorem 2.2.4](#). Now we want to show that  $\partial(X \setminus E) \subseteq X \setminus E$ , and we know

$$X \setminus E = \text{Ext}(E) \cup \partial E = \text{Ext}(E) \cup \partial(X \setminus E)$$

since  $\partial E = \partial(X \setminus E)$ . Hence, we have  $\partial(X \setminus E) \subseteq X \setminus E$ . ■

**proof of  $\Leftarrow$ .** Suppose  $X \setminus E$  is closed, then  $\partial(X \setminus E) \subseteq X \setminus E$ , and since  $\partial E = \partial(X \setminus E)$ , so  $\partial E \subseteq X \setminus E$ , and thus  $\partial E \cap E = \emptyset$ , which means  $E$  is open. ■

## Lecture 5

**Definition 2.2.10.** Let  $(X, d)$  be a metric space,  $E \subseteq X$  and  $x_0 \in E$ . We say  $x_0$  is an adherent point if for every  $r > 0$ ,  $B(x_0, r) \cap E \neq \emptyset$ , and we denote  $\overline{E}$  to the set of all adherent points.

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**Remark 2.2.5.**  $E \subseteq \overline{E}$ , since given any  $x_0 \in E$  and  $r > 0$ ,  $x_0 \in B(x_0, r)$ , so  $B(x_0, r) \cap E \neq \emptyset$ , and thus  $E \subseteq \overline{E}$ .

**Remark 2.2.6.**  $\partial E \subseteq \overline{E}$ . Given  $x_0 \in \partial E$ , we know for any  $r > 0$ ,  $B(x_0, r) \cap E \neq \emptyset$ , so  $x_0 \in \overline{E}$ .

**Proposition 2.2.5.**  $x_0 \in \overline{E}$  if and only if there exists  $(X^{(n)})_{n=1}^{\infty} \subseteq E$  s.t.  $\lim_{n \rightarrow \infty} X^{(n)}$  exists and  $\lim_{n \rightarrow \infty} X^{(n)} = x_0$ .

**proof of  $(\Rightarrow)$ .** Given  $n \in \mathbb{N}$ . Consider  $B(x_0, \frac{1}{n})$ . We know  $B(x_0, \frac{1}{n}) \cap E \neq \emptyset$ . Choose  $X^{(n)} \in B(x_0, \frac{1}{n}) \cap E$ , then  $d(x_0, X^{(n)}) < \frac{1}{n}$ , which means  $\lim_{n \rightarrow \infty} d(x_0, X^{(n)}) = 0$ . Hence, there exists  $(X^{(n)}) \subseteq E$  s.t.  $\lim_{n \rightarrow \infty} X^{(n)} = x_0$ . ■

**proof of  $(\Leftarrow)$ .** There exists  $N$  s.t.  $X^{(n)} \in B(x_0, r)$  when  $n \geq N$ . Given any  $r > 0$ , since  $\lim_{n \rightarrow \infty} X^{(n)} = x_0$ , so  $\lim_{n \rightarrow \infty} d(X^{(n)}, x_0) = 0$ . Hence, there exists  $N$  s.t.  $d(X^{(n)}, x_0) < r$  when  $n \geq N$ . Hence, when  $n \geq N$ , we have  $X^{(n)} \subseteq B(x_0, r)$ . Since we know  $X^{(n)} \in E$  for all  $n$ , so we know  $B(x_0, r) \cap E \neq \emptyset$ , so  $x_0 \in \overline{E}$ . ■

**Proposition 2.2.6.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ , then

$$X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E.$$

**Corollary 2.2.3.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . Then,

$$\overline{E} = \text{Int}(E) \cup \partial E = X \setminus \text{Ext}(E) = E \cup \partial E.$$

**Proof.** Since

$$\begin{aligned} \overline{E} &= \overline{E} \cap X = \overline{E} \cap (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \\ &= (\overline{E} \cap \text{Int}(E)) \cup (\overline{E} \cap \text{Ext}(E)) \cup (\overline{E} \cap \partial E) = \text{Int}(E) \cup \partial E. \end{aligned}$$

Also,

$$X \setminus \text{Ext}(E) = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus \text{Ext}(E) = \text{Int}(E) \cup \partial E = \overline{E}.$$

Besides, we know  $\text{Int}(E) \subseteq E \subseteq \overline{E}$ , so

$$\overline{E} = \text{Int}(E) \cup \partial E \subseteq E \cup \partial E.$$

Also, by Remark 2.2.5 and Remark 2.2.6, we know  $E \cup \partial E \subseteq \overline{E}$ , so we know  $\overline{E} = E \cup \partial E$ . ■

**Definition 2.2.11.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . We say  $E$  is open iff  $\partial E \cap E = \emptyset$ . We say  $E$  is closed iff  $\partial E \subseteq E$ .

**Proposition 2.2.7.**

$$E \text{ is open} \Leftrightarrow \text{Int}(E) = E \Leftrightarrow X \setminus E \text{ is closed.}$$

**proof of  $E \text{ is open} \Leftrightarrow \text{Int}(E) = E$ .**

$(\Rightarrow)$  Since  $E$  is open, so  $\partial E \cap E = \emptyset$ . Hence,

$$\begin{aligned} E &= E \cap X = E \cap (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \\ &= (E \cap \text{Int}(E)) \cup (E \cap \text{Ext}(E)) \cup (E \cap \partial E) = \text{Int}(E) \cup (E \cap \partial E) = \text{Int}(E) \end{aligned}$$

since  $E \cap \text{Ext}(E) = \emptyset$  and we know  $\partial E \cap E = \emptyset$ .

$(\Leftarrow)$  Since  $\text{Int}(E) = E$ , and  $\text{Int}(E) \cap \partial E = \emptyset$ , so  $E \cap \partial E = \emptyset$ , and thus  $E$  is open.

**proof of  $E$  is open  $\Leftrightarrow X \setminus E$  is closed.**

( $\Rightarrow$ )  $X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$ , so

$$X \setminus E = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus \text{Int}(E) = \text{Ext}(E) \cup \partial E = \text{Int}(X \setminus E) \cup \partial(X \setminus E).$$

Hence,  $\partial(X \setminus E) \subseteq X \setminus E$ , which means  $X \setminus E$  is closed.

( $\Leftarrow$ )  $X \setminus E$  is closed, then  $\partial(X \setminus E) \subseteq X \setminus E$ , but  $\partial E = \partial(X \setminus E)$ , so  $\partial E \subseteq X \setminus E$ , and thus  $\partial E \cap E = \emptyset$ .

**Remark 2.2.7.** If  $\partial E = \emptyset$ , then  $E$  is open and closed.

**Definition 2.2.12 (Clopen).** If a set  $S$  is closed and open, then  $S$  is clopen.

**Remark 2.2.8.** Let  $(X, d)$  be a metric space, then  $\emptyset$  is clopen, and we can deduce that  $X$  is also clopen since  $X$  is the complement of  $\emptyset$  and we know  $S$  is open iff  $X \setminus S$  is closed.

**Remark 2.2.9.** In  $(\mathbb{R}, d)$ , where  $d$  is the standard metric, then the only clopen set is  $\mathbb{R}$  or  $\emptyset$ .

**Remark 2.2.10.** Let  $(X, d_{\text{disc}})$  be the discrete metric space on  $X$ . Let  $E$  be any set, then  $E$  is open and closed. Given  $x_0 \in E$ , we know  $B_{\text{disc}}(x_0, \frac{1}{2}) \subseteq E$ , so  $x_0 \in \text{Int}(E)$ , which means  $E = \text{Int}(E)$ , so  $E$  is open. Now since  $X \setminus E$  is also open, so  $E$  is closed. Thus,  $E$  is clopen.

**Proposition 2.2.8.** The following hold:

- (a)  $E$  is open iff  $E = \text{Int}(E)$ .
- (b)  $E$  is closed iff every convergent sequence  $(X^{(n)})_{n=1}^{\infty}$  in  $E$ , then the limit  $\lim_{n \rightarrow \infty} X^{(n)} \in E$ .
- (c) Let  $r > 0$ , then
  - (i)  $\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$  is closed.
  - (ii)  $B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$  is open.
- (d) Any singleton  $\{x_0\}$  where  $x_0 \in X$  is closed.
- (e)  $E$  is open iff  $X \setminus E$  is closed.
- (f)
  - (i) If  $E_1, \dots, E_n$  are open sets in  $X$ , then  $E_1 \cap E_2 \cap \dots \cap E_n$  is open.
  - (ii) If  $F_1, \dots, F_n$  are closed, then  $F_1 \cup \dots \cup F_n$  is closed.
- (g)
  - (i) If  $\{E_\alpha\}_{\alpha \in I}$  is any collection of open sets in  $X$ , then  $\bigcup_{\alpha \in I} E_\alpha$  is open.
  - (ii) If  $\{F_\alpha\}_{\alpha \in I}$  is any collection of closed sets in  $X$ , then  $\bigcap_{\alpha \in I} F_\alpha$  is closed.
- (h)
  - (i) If  $E \subseteq X$ , then  $\text{Int}(E)$  is the largest open set that contained in  $E$  i.e.  $\text{Int}(E)$  is open and if  $V \subseteq E$  and  $V$  is open, then  $V \subseteq \text{Int}(E)$ .
  - (ii) If  $E \subseteq X$ , then  $\overline{E}$  is the smallest closed set containing  $E$  i.e.  $\overline{E}$  is closed and if  $E \subseteq K$  and  $K$  is closed, then  $\overline{E} \subseteq K$ .

**proof of (b).**

( $\Rightarrow$ ) Since  $E$  is closed, so  $\overline{E} = E$ , and we know every convergent sequence  $(X^{(n)})_{n=1}^{\infty}$  converges to  $x_0$  with  $x_0 \in \overline{E}$  by [Proposition 2.2.4](#). Thus, we have  $x_0 \in E$ .

( $\Leftarrow$ ) Assume that every convergent sequence in  $E$  has its limit in  $E$ . We want to prove that  $E$  is closed, i.e. that  $X \setminus E$  is open.

Take any point  $y \in X \setminus E$ . Suppose, for contradiction, that every ball around  $y$  meets  $E$ . That is, for each  $k \in \mathbb{N}$  there exists a point

$$x^{(k)} \in E \cap B(y, \frac{1}{k}).$$

Then, by construction, we have  $x^{(k)} \rightarrow y$ .

By our assumption, the limit of any convergent sequence from  $E$  must lie in  $E$ . Hence  $y \in E$ , contradicting the fact that  $y \in X \setminus E$ .

Therefore, there must exist some radius  $r > 0$  such that

$$B(y, r) \cap E = \emptyset,$$

which means  $B(y, r) \subseteq X \setminus E$ . Thus every point of  $X \setminus E$  is an interior point, so  $X \setminus E$  is open. Hence  $E$  is closed. ■

**proof of (c).**

(i) To show that  $\overline{B}(x_0, r)$  is closed, it suffices to show that  $X \setminus \overline{B}(x_0, r)$  is open. Note that

$$X \setminus \overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) > r\}.$$

Let  $y \in X \setminus \overline{B}(x_0, r)$ , then define  $\varepsilon = d(x_0, y) - r > 0$ , then we can similarly prove that  $B(y, \varepsilon) \subseteq X \setminus \overline{B}(x_0, r)$ . Hence,  $X \setminus \overline{B}(x_0, r) = \text{Int}(X \setminus \overline{B}(x_0, r))$ , and thus it is open.

(ii) If  $y \in B(x_0, r)$ , then  $d(x_0, y) < r$ . Let  $\varepsilon = r - d(x_0, y) > 0$ , then we claim that  $B(y, \varepsilon) \subseteq B(x_0, r)$ . Given  $z \in B(y, \varepsilon)$ , then  $d(z, y) < \varepsilon$ , then use triangle inequality we know  $z \in B(x_0, r)$ . ■

**proof of (d).** It suffices to show that  $X \setminus \{x_0\}$  is open. Given  $y \in X \setminus \{x_0\}$ , so we can show that

$$B\left(y, \frac{d(y, x_0)}{2}\right) \subseteq X \setminus \{x_0\}.$$

Hence,  $y \in \text{Int}(X \setminus \{x_0\})$ , and thus  $X \setminus \{x_0\}$  is open. ■

**proof of (f).**

(i) Given  $x_0 \in E_1 \cap E_2 \cap \cdots \cap E_n$ , then  $x_0 \in E_i$  for all  $1 \leq i \leq n$ . Thus, there exists  $r_i > 0$  s.t.

$$B(x_0, r_i) \subseteq E_i \quad \text{for each } 1 \leq i \leq n.$$

Let  $r = \min\{r_1, \dots, r_n\} > 0$ , then we know  $B(x_0, r) \subseteq B(x_0, r_i) \subseteq E_i$  for all  $1 \leq i \leq n$ . Hence,  $B(x_0, r) \subseteq E_1 \cap E_2 \cap \cdots \cap E_n$ , and thus  $E_1 \cap \cdots \cap E_n$  is open.

(ii) Now if  $F_1, \dots, F_n$  are closed, then  $X \setminus F_1, \dots, X \setminus F_n$  are open. Since we know  $\bigcap_{i=1}^n (X \setminus F_i)$  is open, and

$$\bigcap_{i=1}^n (X \setminus F_i) = X \setminus \left( \bigcup_{i=1}^n F_i \right),$$



so  $X \setminus (\bigcup_{i=1}^n F_i)$  is open, which means  $\bigcup_{i=1}^n F_i$  is closed. ■

**proof of (g).**

- (i) Suppose  $x_0 \in \bigcup_{\alpha \in I} E_\alpha$ , then there exists  $\mathcal{B} \in I$  s.t.  $x_0 \in E_{\mathcal{B}}$ . Now since  $E_{\mathcal{B}}$  is open, so there exists  $r_{x_0} > 0$  s.t.

$$B(x_0, r_{x_0}) \subseteq E_{\mathcal{B}} \subseteq \bigcup_{i \in \alpha} E_\alpha.$$

Hence,  $\bigcup_{\alpha \in I} E_\alpha$  is open.

- (ii)

$$\left( X \setminus \left( \bigcap_{\alpha \in I} F_\alpha \right) \right) = \bigcup_{\alpha \in I} (X \setminus F_\alpha)$$

is open since  $X \setminus F_\alpha$  is open for all  $\alpha \in I$ , so we have  $\bigcap_{\alpha \in I} F_\alpha$  is closed.

**Remark 2.2.11.**

- (1)  $\bigcap_{\alpha \in I} E_\alpha$  may NOT be open. For example,

$$\bigcap_{i=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = \{0\},$$

which is closed.

- (2)  $\bigcup_{\alpha \in I} F_\alpha$  may NOT be closed. For example,

$$\bigcup_{i=1}^{\infty} \left[ -1 + \frac{1}{n}, 1 - \frac{1}{n} \right] = (-1, 1),$$

which is open.

**Note 2.2.2.** In the proof of (f), if the index set  $I$  is infinite, then we can not pick  $\min \{r_1, \dots, r_n\}$ , so we can not deduce that (f) is correct when there infinitely many open sets or closed sets. ■

**proof of (h).**

- (i) We first claim that  $\text{Int}(E)$  is open.

**Proof.** Since for all  $x \in \text{Int}(E)$ ,  $\exists r_x > 0$  s.t.  $B(x, r_x) \subseteq E$ , so

$$\text{Int}(E) = \bigcup_{x \in \text{Int}(E)} B(x, r_x),$$

and by (ii) of (c) and (i) of (g) in [Proposition 2.2.8](#), we know  $\text{Int}(E)$  is open. ■

Now if we have  $V \subseteq E$  and  $V$  is open, then  $y \in V$  implies there exists  $s > 0$  s.t.  $B(y, s) \subseteq V$ , and thus  $B(y, s) \subseteq E$  since  $V \subseteq E$ . Hence, we know  $y \in \text{Int}(E)$ , and thus  $V \subseteq \text{Int}(E)$ .

- (ii) To show  $\overline{E}$  is closed, it suffices to show that  $X \setminus \overline{E}$  is open. Note that

$$\overline{E} = X \setminus \text{Ext}(E) = X \setminus \underbrace{\text{Int}(X \setminus E)}_{\text{open}},$$

so  $\overline{E}$  is closed. Now if  $E \subseteq K$  and  $K$  is closed, then if  $x \in \overline{E}$ , we have  $B(x, r) \cap E \neq \emptyset$  for all  $r > 0$ . Hence,  $B(x, r) \cap K \neq \emptyset$  since  $E \subseteq K$ , so  $x \in \overline{K} = K$  (since  $K$  is closed). Thus,  $\overline{E} \subseteq K$ . ■

## Lecture 6

### 2.3 Relative topology

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Let  $(X, d)$  be a metric space and  $Y \subseteq X$ , then  $(Y, d|_{Y \times Y})$  is also a metric space.

**Example 2.3.1.** Consider  $(\mathbb{R}^2, d_2)$  and  $X = \{(x, 0) \mid x \in \mathbb{R}\}$ , then on  $(X, d_2|_{X \times X}) = (X, d)$ , it is also a metric space.

**Proof.** Since

$$d((x, 0), (y, 0)) = \sqrt{(x - y)^2 + 0^2} = |x - y|,$$

so it is obvious that  $d$  is a metric.

Note that  $X$  is not open in  $\mathbb{R}^2$ . Also, if  $E = \{(x, 0) \mid -1 < x < 1\}$ , then  $E$  is not open in  $\mathbb{R}^2$ , but  $E$  is open in  $(X, d_2|_{X \times X})$ . ⊛

**Example 2.3.2.** Suppose  $X = (-1, 1) \subseteq \mathbb{R}$ , then  $(X, d|_{X \times X})$  is a metric space. Consider  $E = [0, 1)$ , then we know  $E$  is not closed in  $(\mathbb{R}, d)$  since  $1 \notin \overline{E}$ . But  $E$  is closed in  $(X, d|_{X \times X})$  since  $\overline{E} = X$  in  $(X, d|_{X \times X})$ .

**Definition 2.3.1 (relatively open/close).** Let  $(X, d)$  be a metric space and  $Y \subseteq X$ . We say  $E$  is relatively open (resp. closed) in  $Y$  if  $E$  is open (resp. closed) in  $(Y, d|_{Y \times Y})$ .

**Note 2.3.1.** In the following context, if we say  $E$  is open in  $Y$ , then we mean  $E$  is "relatively" open, and if we say  $E$  is closed in  $Y$ , then we mean  $E$  is relatively closed in  $Y$ .

**Note 2.3.2.** If  $Y$  is open/closed in  $E$ , then  $Y \subseteq E$ . Otherwise, we cannot define  $d|_{Y \times Y}(a, b)$  for  $a, b \in E \setminus Y$ .

**Remark 2.3.1.** If  $Y \subseteq X$ , and  $(X, d), (Y, d|_{Y \times Y})$  are both metric spaces, then

$$B_Y(x, r) = \{y \in Y \mid d(y, x) < r\} = B_X(x, r) \cap Y.$$

**Remark 2.3.2.** If  $E$  is relatively open in  $Y$ , then given  $x_0 \in E$ ,  $\exists r_0 > 0$  s.t.  $B_X(x_0, r_0) \cap Y \subseteq E$ . This is because by [Remark 2.3.1](#), we have

$$B_X(x_0, r_0) \cap Y = B_Y(x_0, r_0) \subseteq E.$$

**Remark 2.3.3.** A set  $E \subseteq Y$  is relatively closed in  $Y$  if given any  $r > 0$  and  $x_0 \in Y$ ,

$$B_Y(x_0, r) \cap E \neq \emptyset,$$

then  $x_0 \in E$ . This is because "closed" gives  $E = \overline{E}_Y$ . Note that this statement is equivalent to

$$\text{If } x_0 \in \overline{E}_Y, \text{ then } x_0 \in E = E_Y.$$

**Proposition 2.3.1.** Let  $(X, d)$  be a metric space, and  $Y \subseteq X$  and  $E \subseteq Y$ , then

- (1)  $E$  is relatively open in  $Y$  iff  $\exists$  open set  $V$  in  $(X, d)$  s.t.  $E = V \cap Y$ .
- (2)  $E$  is relatively closed in  $Y$  iff  $\exists$  closed set  $K$  in  $(X, d)$  s.t.  $E = K \cap Y$ .

**proof of (1).**

- ( $\Rightarrow$ ) Given any  $x \in E$ ,  $\exists r_x > 0$  s.t.  $B_X(x, r_x) \cap Y \subseteq E$ . Let  $V = \bigcup_{x \in E} B_X(x, r_x)$ . Obviously,  $V \cap Y = E$  and  $V$  is open.
- ( $\Leftarrow$ ) Suppose  $E = V \cap Y$ , then given any  $x \in E$ , since  $V$  is open, so there exists  $r > 0$  s.t.  $B_X(x, r) \subseteq V$ , and then  $B_X(x, r) \cap Y \subseteq V \cap Y = E$ . Since  $x$  is an interior point of  $E$  in  $Y$ , so  $\text{Int}_Y(E) = E$ , and thus  $E$  is open in  $Y$ . ■

**proof of (2).**

- ( $\Rightarrow$ )  $E$  is relatively closed in  $Y$ , then  $Y \setminus E$  is relatively open, so there exists  $V$  open in  $X$  s.t.  $Y \setminus E = V \cap Y$ . Hence,

$$\begin{aligned} E &= Y \setminus (Y \setminus E) = (X \setminus (Y \setminus E)) \cap Y = (X \setminus (V \cap Y)) \cap Y \\ &= ((X \setminus V) \cup (X \setminus Y)) \cap Y \\ &= ((X \setminus V) \cap Y) \cup ((X \setminus Y) \cap Y) \\ &= (X \setminus V) \cap Y \end{aligned}$$

Let  $E = (X \setminus V) \cap Y = K \cap Y$ , then since  $K = X \setminus V$  is closed in  $X$ , so we're done.

- ( $\Leftarrow$ ) Suppose  $E = K \cap Y$  for some closed  $K$ , then  $Y \setminus E = (X \setminus K) \cap Y$ , which means  $Y \setminus E$  is relatively open in  $Y$  since  $X \setminus K$  is open and by (a), so  $E$  is closed in  $Y$ . ■

**Example 2.3.3.** Let  $X = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$  with the standard metric  $d(x, y) = |x - y|$  with  $x, y \in X$ , then

- (i)  $[0, 1]$  is open and closed in  $X$ .
- (ii)  $\partial_X[0, 1] = \emptyset$ .

**Proof.**

- (i) We want to find  $V$  open in  $\mathbb{R}$  s.t.

$$[0, 1] = V \cap \overbrace{([0, 1] \cup [2, 3])}^X,$$

we can choose  $V = (-\frac{1}{2}, \frac{3}{2})$ , so  $[0, 1]$  is open in  $X$ .

We want to find  $K$  closed in  $\mathbb{R}$  and

$$[0, 1] = K \cap ([0, 1] \cup [2, 3]),$$

and we can choose  $K = [-\frac{1}{2}, \frac{3}{2}]$ , so  $[0, 1]$  is closed in  $X$ .

- (ii) If  $x \in \partial_X[0, 1]$ , then  $B_X(x, r) \cap [0, 1]$  and  $B_X(x, r) \cap [2, 3]$  are both nonempty for any  $r > 0$ . However, this is impossible for any  $x$  in  $X$ , so  $\partial_X[0, 1] = \emptyset$ . ⊛

## 2.4 Cauchy sequence and complete metric space

**Definition 2.4.1 (subsequence).** Suppose  $(X^{(n)})_{n=m}^{\infty}$  is a sequence in  $(X, d)$ . Suppose  $m \leq n_1 < n_2 < \dots$ , then  $(X^{(n_j)})_{j=1}^{\infty}$  is called a subsequence of  $(X^{(n)})_{n=m}^{\infty}$ .

**Example 2.4.1.**  $X^{(n)} = (-1)^n$  for all  $n \in \mathbb{N}$ .

**Proof.**

$$\{X^{(2n)}\}_{n=1}^{\infty}$$

is a subsequence of  $\{X^{(n)}\}_{n=1}^{\infty}$ . ⊛

**Lemma 2.4.1.** Let  $\{X^{(n)}\}_{n=m}^{\infty}$  be a convergent sequence with  $\lim_{n \rightarrow \infty} X^{(n)} = x$ , then every subsequence of  $\{X^{(n)}\}_{n=m}^{\infty}$  also converges to  $x$ .

**Definition 2.4.2 (limit points).** Suppose  $(X^{(n)})_{n=m}^{\infty}$  is a sequence in  $(X, d)$ , then we say  $L$  is a limit point of  $(X^{(n)})_{n=m}^{\infty}$  if for every  $N \geq m$  and every  $\varepsilon > 0$ , there exists  $n \geq N$  s.t.  $d(X^{(n)}, L) \leq \varepsilon$ .

**Proposition 2.4.1.**  $L$  is a limit point of  $(X^{(n)})_{n=m}^{\infty}$  iff there exists a subsequence

$$(X^{(n_j)})_{j=1}^{\infty}$$

converges to  $L$ .

**Proof.**

( $\Rightarrow$ ) Assume  $L$  is a limit point, now we build a subsequence converges to  $L$  by an inductive method. Our goal is to build a subsequence  $\{X^{(n_j)}\}_{j=1}^{\infty}$  so that

$$d(X^{(n_j)}, L) < \frac{1}{j} \quad \forall 1 \leq j.$$

For  $j = 1$ , pick  $N = m$ , and pick  $\varepsilon < \frac{1}{1}$  to pick  $n_1 \geq N$  s.t.

$$d(X^{(n_1)}, L) \leq \varepsilon < \frac{1}{1}.$$

Now suppose  $n_1, n_2, \dots, n_{k-1}$  are all chosen, then now we can pick  $N = n_{k-1} + 1$  and  $\varepsilon < \frac{1}{k}$ , so that we can pick  $n_k \geq N$  s.t.  $d(X^{(n_k)}, L) \leq \varepsilon < \frac{1}{k}$ , so we're done. Now we show that this subsequence converges to  $L$ . For every  $\varepsilon > 0$ , we know there exists  $0 < \frac{1}{k} < \varepsilon$ , so for all  $K \geq k$ , we have

$$d(X^{(K)}, L) < \frac{1}{K} \leq \frac{1}{k} < \varepsilon,$$

so we're done.

( $\Leftarrow$ ) Left as exercise to the reader. ■

**Proposition 2.4.2.**  $L$  is a limit point iff  $L \in \bigcap_{N=1}^{\infty} \overline{S_N}$  where  $S_N = \{X^{(K)}\}_{K \geq N}$ .

**Definition 2.4.3 (Cauchy sequence).** Let  $(X^{(n)})_{n=m}^{\infty}$  be a sequence in  $(X, d)$ . We say this sequence is a Cauchy sequence if for every  $\varepsilon > 0$ , there exists  $N \geq m$  s.t.  $d(X^{(j)}, X^{(k)}) < \varepsilon$  for all  $j, k \geq N$ .

**Lemma 2.4.2.** Suppose  $(X^{(n)})_{n=m}^{\infty}$  converges in  $(X, d)$ , then  $(X^{(n)})_{n=m}^{\infty}$  is a Cauchy sequence in  $(X, d)$ .

**Proof.** Suppose  $\lim_{n \rightarrow \infty} X^{(n)} = X_0$ , then for every  $\frac{\varepsilon}{2} > 0$ , there exists  $N \geq m$  s.t.  $d(X^{(n)}, X_0) < \frac{\varepsilon}{2}$  for all  $n \geq N$ . If  $j, k \geq N$ , then

$$d(X^{(j)}, X^{(k)}) \leq d(X^{(j)}, X_0) + d(X^{(k)}, X_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

**Example 2.4.2.** A sequence in  $\mathbb{Q}$  may not converges in  $\mathbb{Q}$ .

**Proof.** See teacher's note.

⊛

**Definition 2.4.4 (Complete space).** A metric space  $(X, d)$  is complete iff every Cauchy sequence converges to some points in  $X$ .

**Remark 2.4.1.**  $\mathbb{Q} \subseteq \mathbb{R}$ , then  $(\mathbb{Q}, d)$  is not complete.

**Remark 2.4.2.** The limit of a convergent sequence in metric space is unique. If

$$\lim_{n \rightarrow \infty} x^{(n)} = y \quad \text{and} \quad \lim_{n \rightarrow \infty} x^{(n)} = z,$$

then suppose by contradiction,  $y \neq z$ . Then,

$$0 \leq d(y, z) \leq d(y, x^{(n)}) + d(x^{(n)}, z).$$

By squeeze theorem, we know  $d(y, z) = 0$  and thus  $y = z$ .

**Proposition 2.4.3.** Let  $(X, d)$  be a metric space and let  $(Y, d|_{Y \times Y})$  be a subspace of  $(X, d)$ . If  $(Y, d|_{Y \times Y})$  is complete, then  $Y$  is closed in  $X$ .

**Proof.** We want to show that  $Y = \overline{Y}$ , so we want to show for all  $y \in \overline{Y}$ , we have  $y \in Y$ . Now for every  $y \in \overline{Y}$ , then by [Proposition 2.2.4](#), we know there exists a convergent sequence  $\{Y^{(n)}\}_{n=1}^{\infty}$  in  $Y$  and converges to  $y$ . However, every convergent sequence is Cauchy, and since  $(Y, d|_{Y \times Y})$  is complete, so  $\{Y^{(n)}\}_{n=1}^{\infty}$  converges in  $Y$ , which means  $y \in Y$ , and we're done. ■

**Proposition 2.4.4.** If  $(X, d)$  is complete and  $Y \subseteq X$  is closed, then  $(Y, d|_{Y \times Y})$  is complete.

**Proof.** Given a Cauchy sequence  $(X^{(n)})_{n=1}^{\infty}$  in  $Y$ , so this is also a Cauchy sequence in  $X$ , so it converges in  $X$ . If  $\exists x_0 \in X$  s.t.  $\lim_{n \rightarrow \infty} X^{(n)} = x_0$ . Since  $Y$  is closed, so  $Y = \overline{Y}$ , and by [Proposition 2.2.4](#), we know  $x_0 \in \overline{Y} = Y$ , so  $x_0 \in Y$ , and thus  $(X^{(n)})_{n=1}^{\infty}$  also converges in  $Y$ . ■

# Appendix