4. (a) Consider even number n. There will have $\frac{n}{2}$ games. For the first game, there has $\binom{n}{2}$ ways to choose two people to play. For the second game, since everyone shall play exactly one game, two players played in the first game cannot play the second game, so there has $\binom{n-2}{2}$ ways to choose two new players for the second game. Doing the derivation inductively, for the i-th game, there has $\binom{n-2i+2}{2}$ ways to choose players.

Hence, by product rule, $p_n = \prod_{i=1}^{n/2} \binom{n-2i+2}{2} = \frac{n!}{2^{n/2}}$.

In conclusion, $p_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n!}{2^{n/2}} & \text{if } n \text{ is even} \end{cases}$.

(b) $\hat{P}(x) = \sum_{n=0}^{\infty} p_n \cdot \frac{x^n}{n!} = \sum_{k=0}^{\infty} p_{2k} \cdot \frac{x^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k}$

$= \sum_{k=0}^{\infty} (x^2/2)^k = \frac{1}{1-x^2/2} = \frac{2}{2-x^2}$.

5. (a) Consider the size of the second subset. Say it equals k.
So, the has $\binom{n}{k}$ ways to partition n children into two disjoint subsets with the size of the second subset equaling k.
For each of the $n-k$ children in the first subset, there has 2 choices, to learn combinatorics or functional analysis.
Hence, there has 2^{n-k} ways to finish the task of the first subset.
For the second subset, there obviously has $\binom{k}{2}$ ways to finish the task here.
Let a_n be the number of ways to finish the requirement of the problem when there has n children. Then by the product rule, we have $a_n = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{k}{2} \cdot 2^{n-k}$.
Let $b_i = \binom{i}{2}$, $c_j = 2^j$. Note that $a_n = \sum_{k=0}^{\infty} \binom{n}{k} \cdot b_k \cdot c_{n-k}$, we have

$$\hat{A}(x) = \left(\sum_{i=0}^{\infty} b_i \cdot \frac{x^i}{i!}\right) \cdot \left(\sum_{j=0}^{\infty} c_j \cdot \frac{x^j}{j!}\right) = \left(\sum_{i=2}^{\infty} \binom{i}{2} \cdot \frac{x^i}{i!}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{(2x)^j}{j!}\right)$$

$$= \left(\sum_{i=2}^{\infty} \frac{x^i}{2!(i-2)!}\right) \cdot e^{2x} = \frac{x^2 e^{2x}}{2} \cdot \sum_{i=2}^{\infty} \frac{x^{i-2}}{(i-2)!}$$

$$= \frac{x^2 e^{2x}}{2} \cdot \sum_{i=0}^{\infty} \frac{x^i}{i!} = \frac{x^2 e^{2x}}{2} \cdot e^x = \frac{x^2 e^{3x}}{2}.$$

(b)

$$a_5 = \sum_{k=0}^{5} \binom{5}{k} \cdot \binom{k}{2} \cdot 2^{5-k} = \sum_{k=2}^{5} \binom{5}{k} \cdot \binom{k}{2} \cdot 2^{5-k}$$

$$= \binom{5}{2} \cdot \binom{2}{2} \cdot 2^3 + \binom{5}{3} \cdot \binom{3}{2} \cdot 2^2 + \binom{5}{4} \cdot \binom{4}{2} \cdot 2^1 + \binom{5}{5} \cdot \binom{5}{2} \cdot 2^0$$

$$= 80 + 120 + 60 + 10 = 270.$$