Introduction to Analysis I HW7

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Problem 0.0.1 (15pts). Assume that (S, d) is a metric space, and let $f_n, f : S \to \mathbb{R}$ be real-valued functions. Suppose that $f_n \to f$ uniformly on S, and there exists a constant $M > 0$ such that

$$|f_n(x)| \leq M \quad \text{for all } x \in S \text{ and all } n.$$

Let $g : \overline{B(0; M)} \to \mathbb{R}$ be continuous, where

$$B(0; M) = \{y \in \mathbb{R} : |y| < M\}.$$

Define

$$h_n(x) = g(f_n(x)), \qquad h(x) = g(f(x)), \quad x \in S.$$

Prove that $h_n \to h$ uniformly on S.

Proof. Since g is continuous, so for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|x_1 - x_2| < \delta$ implies $|g(x_1) - g(x_2)| < \varepsilon$. Also, since $f_n \to f$ uniformly, so $\exists N' > 0$ s.t. $n \geq N'$ implies $|f_n(x) - f(x)| < \delta$ for all $x \in S$. Hence, we can pick $N = N'$ so that if $n \geq N'$, then $|f_n(x) - f(x)| < \delta$ for all $x \in S$ and thus

$$|g(f_n(x)) - g(f(x))| < \varepsilon$$

for all $x \in S$, so we know $h_n \to h$ uniformly on S. ■

Problem 0.0.2 (15pts). Let $f_n(x) = x^n$. The sequence $\{f_n\}$ converges pointwise but not uniformly on $[0, 1]$. Let g be continuous on $[0, 1]$ with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on $[0, 1]$.

Problem 0.0.3 (15pts). Assume that $g_{n+1}(x) \leq g_n(x)$ for each x in T and each $n = 1, 2, \ldots$, and suppose that $g_n \to 0$ uniformly on T. Prove that

$$\sum (-1)^{n+1} g_n(x)$$

converges uniformly on T.

Proof. We first give a claim:

Claim 0.0.1. $g_n(x) \geq 0$ for all $x \in T$ and $n \in \mathbb{N}$.

Proof. If $-c = g_{n_1}(x_1) < 0$ for some $x_1 \in T$ and $n_1 \in \mathbb{N}$, then for all $n \geq n_1$ we have $g_n(x_1) \leq -c$. If we pick some ε s.t. $0 < \varepsilon < c$, then since $g_n \to 0$ uniformly on T, so there exists $N > 0$ s.t. $n \geq N$ implies $|g_n(x)| < \varepsilon < c$ for all $x \in T$, and thus if we pick $n_2 = \max\{N, n_1\}$, we know $g_{n_2}(x_1) \leq -c$ and thus $|g_{n_2}(x_1)| \geq c > \varepsilon$, which is a contradiction. ⊛

Now we define $S_n(x) = \sum_{k=1}^{n} (-1)^{k+1} g_k(x)$. We first show that

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} (-1)^{k+1} g_k(x)$$

exists.

Claim 0.0.2. If we fix some $x \in T$, then $-g_n(x) \leq \sum_{k=n}^{m} (-1)^{k+1} g_k(x) \leq g_n(x)$ for all $n, m \in \mathbb{N}$ and $x \in T$.

Proof. If $m < n$, then $\sum_{k=n}^{m}(-1)^{k+1}g_k(x) = 0$, so it is true by Claim 0.0.1. If $m \geq n$, then suppose n is odd, and then we have

$$\begin{aligned}\sum_{k=n}^{m}(-1)^{k+1}g_k(x) &= g_n(x) - g_{n+1}(x) + \cdots + (-1)^m g_m(x) \\ &= g_n(x) - (g_{n+1}(x) - g_{n+2}(x)) - \cdots \leq g_n(x)\end{aligned}$$

since $g_i(x) - g_{i+1}(x) \geq 0$. Also, we know

$$\begin{aligned}\sum_{k=n}^{m}(-1)^{k+1}g_k(x) &= g_n(x) - g_{n+1}(x) + \cdots + (-1)^m g_m(x) \\ &= (g_n(x) - g_{n+1}(x)) + (g_{n+2}(x) - g_{n+3}(x)) + \cdots \geq 0,\end{aligned}$$

Thus, for odd n, this statement is true. If n is even, then we can similarly show that

$$\sum_{k=n}^{m}(-1)^{k+1}g_k(x) \geq -g_n(x) \text{ and } \sum_{k=n}^{m}(-1)^{k+1}g_k(x) \leq 0,$$

so this is also true. ⊛

Now by Claim 0.0.2 we know $\{S_n(x)\}_{n=1}^{\infty}$ is bounded for any fixed $x \in T$, and if we fix $x_0 \in T$ and suppose

$$a_k := S_{2k-1}(x_0), \quad b_k := S_{2k}(x_0),$$

then we can check $(a_k)_{k=1}^{\infty}$ is decreasing and $(b_k)_{k=1}^{\infty}$ is increasing, so they are both convergent since they are monotonic and bounded. Also, we know $(|a_k - b_k|)_{k=0}^{\infty}$ converges to 0 since

$$|a_k - b_k| = \left|(-1)^{2k}g_{2k}(x_0)\right| = g_{2k}(x_0)$$

by Claim 0.0.1 and we know $g_n \to 0$ uniformly. Hence, we know $(a_k)_{k=0}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ converges to same point. Thus, $(S_n(x_0))_{n=1}^{\infty}$ converges. Note that this argument holds for all $x_0 \in T$, so we know $\lim_{n\to\infty} S_n$ exists.

Now since for all $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies $g_{n+1}(x) = |g_{n+1}(x)| < \varepsilon$ for all $x \in T$, so $n \geq N$ implies

$$\begin{aligned}\left|\sum_{k=1}^{n}(-1)^{k+1}g_k(x) - \sum_{k=1}^{\infty}(-1)^{k+1}g_k(x)\right| &= \left|\sum_{k=n+1}^{\infty}(-1)^{k+1}g_k(x)\right| \\ &= \left|\lim_{m\to\infty}\sum_{k=n+1}^{m}(-1)^{k+1}g_k(x)\right| \\ &\leq \left|\lim_{m\to\infty} g_{n+1}(x)\right| = g_{n+1}(x) < \varepsilon\end{aligned}$$

for all $x \in T$ by Claim 0.0.1 and Claim 0.0.2, which means $\sum_{n=0}^{\infty}(-1)^{n+1}g_n(x)$ converges uniformly on T. ■

Problem 0.0.4 (15pts).

$$f_n(x) = \frac{x}{1 + nx^2}, \quad x \in \mathbb{R}, \; n = 1, 2, \ldots$$

Find the limit function f of the sequence $\{f_n\}$ and the limit function g of the sequence $\{f_n'\}$.

(a) Prove that $f'(x)$ exists for every x but that $f'(0) \neq g(0)$. For what values of x is $f'(x) = g(x)$?

(b) In what subintervals of $\mathbb{R}$ does $f_n \to f$ uniformly?

(c) In what subintervals of $\mathbb{R}$ does $f_n' \to g$ uniformly?

Problem 0.0.5 (15pts). Prove that
$$\sum x^n(1-x)$$
converges pointwise but not uniformly on $[0,1]$, whereas
$$\sum (-1)^n x^n (1-x)$$
converges uniformly on $[0,1]$. This illustrates that uniform convergence of $\sum f_n(x)$ along with pointwise convergence of $\sum |f_n(x)|$ does not necessarily imply uniform convergence of $\sum |f_n(x)|$.

Proof. Suppose $S_N(x) = \sum_{n=0}^{N} x^n(1-x)$, then we know
$$S_N(x) = (1-x)\sum_{n=0}^{N} x^n = (1-x)\frac{1-x^{N+1}}{1-x} = 1-x^{N+1}.$$
Now suppose
$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1); \\ 0, & \text{if } x = 1. \end{cases}$$
Then, we claim that $\sum_{n=0}^{\infty} x^n(1-x) \to f(x)$ pointwise on $[0,1]$.

- Case 1: $x = 1$, then for all $\varepsilon > 0$, we can pick $N_1 = 1$ so that $n \geq N_1$ implies
$$|S_n(1) - f(1)| = \left|\left(1-1^{n+1}\right) - 0\right| = 0 < \varepsilon.$$
- Case 2: $x \in [0,1)$, then for all $\varepsilon > 0$, we know there exists $N_2 > 0$ s.t. $x^{N_2+1} < \varepsilon$. Hence, $n \geq N_2$ implies
$$|S_n(x) - f(x)| = \left|1 - x^{n+1} - 1\right| = \left|x^{n+1}\right| = x^{n+1} \leq x^{N_2+1} < \varepsilon.$$

Hence, we're done. Now we show that $\sum_{n=0}^{\infty} x^n(1-x)$ does not converge uniformly on $[0,1]$. Suppose by contradiction, $\sum_{n=0}^{\infty} x^n(1-x)$ converges uniformly to f, then for all $\varepsilon > 0$, there exists $N_3 > 0$ s.t. $n \geq N_3$ implies
$$\left|1 - x^{n+1} - f(x)\right| < \varepsilon \quad \forall x \in [0,1].$$
Hence, if we pick some n_1, n_2 s.t. $n_1 > n_2 \geq N_3$, then we have
$$\left|1 - x^{n_1+1} - f(x)\right| < \varepsilon, \quad \left|1 - x^{n_2+1} - f(x)\right| < \varepsilon,$$
so by triangle inequality we have
$$\begin{aligned} \left|x^{n_2+1} - x^{n_1+1}\right| &= \left|1 - x^{n_1+1} - f(x) + \left(-1 + x^{n_2+1} + f(x)\right)\right| \\ &\leq \left|1 - x^{n_1+1} - f(x)\right| + \left|1 - x^{n_2+1} - f(x)\right| < 2\varepsilon \end{aligned}$$
Note that
$$\begin{aligned} x^{n_2+1}(1-x) = x^{n_2+1} - x^{n_2+2} &\leq x^{n_2+1} - x^{n_1+1} \\ &= \left|x^{n_2+1} - x^{n_1+1}\right| < 2\varepsilon. \end{aligned}$$
If we pick $x = 0.5$ and ε s.t. $0 < \varepsilon < \dfrac{0.5^{n_2+2}}{2}$, then we have
$$0.5^{n_2+2} < 2\varepsilon < 0.5^{n_2+2},$$
which is a contradiction. Hence, $\sum_{n=0}^{\infty} x^n(1-x)$ does not converges uniformly.

Now we show that $\sum_{n=0}^{\infty}(-1)^n x^n(1-x)$ converges uniformly on $[0,1]$. Suppose

$$s_N(x)=\sum_{n=0}^{N}(-1)^n x^n(1-x)=(1-x)\sum_{n=0}^{N}(-x)^n=(1-x)\frac{1-(-x)^{N+1}}{1-(-x)}=\frac{(1-x)\left(1-(-x)^{N+1}\right)}{1+x},$$

and $g(x)=\frac{1-x}{1+x}$, then we claim that $\sum_{n=0}^{\infty}(-1)^n x^n(1-x)\to g(x)$ uniformly on $[0,1]$. Note that

$$|s_n(x)-g(x)|=\left|\frac{1-x}{1+x}\left(1-(-x)^{n+1}-1\right)\right|=\left|\frac{1-x}{1+x}x^{n+1}\right|\leq(1-x)x^{n+1}.$$

Suppose $h_n(x)=(1-x)x^{n+1}=x^{n+1}-x^{n+2}$, then

$$h_n'(x)=(n+1)x^n-(n+2)x^{n+1}=x^n\left((n+1)-(n+2)x\right),$$

so we know h_n attains its maximum on $[0,1]$ at $x=\frac{n+1}{n+2}$. Hence, we have

$$\begin{aligned}|s_n(x)-g(x)|\leq(1-x)x^{n+1}&\leq\left(1-\frac{n+1}{n+2}\right)\left(\frac{n+1}{n+2}\right)^{n+1}\\&=\frac{1}{n+2}\left(1-\frac{1}{n+2}\right)^{n+1}=\frac{1}{(n+2)\left(1-\frac{1}{n+2}\right)}\frac{1}{\left(1+\frac{1}{-(n+2)}\right)^{-(n+2)}}\\&=\frac{1}{n+1}\frac{1}{\left(1+\frac{1}{-(n+2)}\right)^{-(n+2)}}\to 0\cdot\frac{1}{e}=0\text{ as }n\to\infty.\end{aligned}$$

Hence, for all $\varepsilon>0$, we can pick some $N>0$ s.t.

$$p(N)=\left(1-\frac{N+1}{N+2}\right)\left(\frac{N+1}{N+2}\right)^{N+1}<\varepsilon,$$

and thus for all $n\geq N$ we have

$$|s_n(x)-g(x)|\leq p(n)\leq p(N)<\varepsilon\quad\forall x\in[0,1],$$

which means $\sum_{n=0}^{\infty}(-1)^n x^n(1-x)\to g(x)$ uniformly on $[0,1]$. ■

Problem 0.0.6 (15pts). Let

$$f_n(x)=\frac{1}{n}e^{-n^2x^2},\quad x\in\mathbb{R},\ n=1,2,\ldots$$

Prove that $f_n\to 0$ uniformly on $\mathbb{R}$, that $f_n'\to 0$ pointwise on $\mathbb{R}$, but that the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

Problem 0.0.7 (10pts). Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on $[0,1]$ and assume that $f_n\to f$ uniformly on $[0,1]$. Prove or disprove

$$\lim_{n\to\infty}\int_0^{1-1/n}f_n(x)\,dx=\int_0^1 f(x)\,dx.$$

Proof. First note that

$$\int_0^1 f(x)\,\mathrm{d}x=\int_0^1\lim_{n\to\infty}f_n(x)\,\mathrm{d}x=\lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x$$

since $f_n\to f$ uniformly on $[0,1]$. Also, since f_n is continuous and defined on $[0,1]$ for all $n\in\mathbb{N}$, so by Extreme value theorem we know $f_n(x) \leq M$ for some $M \in \mathbb{R}$ for all $n \in \mathbb{N}$. Hence, we know

$$\int_0^1 f_n(x)\,\mathrm{d}x - \int_0^{1-\frac{1}{n}} f_n(x)\,\mathrm{d}x = \int_{1-\frac{1}{n}}^1 f_n(x)\,\mathrm{d}x \leq M \cdot \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Hence, for all $\varepsilon > 0$ we can pick some $N > 0$ s.t. $M \cdot \frac{1}{N} < \varepsilon$ so that $n \geq N$ implies

$$\int_0^1 f_n(x)\,\mathrm{d}x - \int_0^{1-\frac{1}{n}} f_n(x)\,\mathrm{d}x \leq M \cdot \frac{1}{n} \leq M \cdot \frac{1}{N} < \varepsilon,$$

which means

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x - \int_0^{1-\frac{1}{n}} f_n(x)\,\mathrm{d}x = 0,$$

so

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^{1-\frac{1}{n}} f_n(x)\,\mathrm{d}x,$$

and thus

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^{1-\frac{1}{n}} f_n(x)\,\mathrm{d}x,$$

and we're done.

Remark 0.0.1. Since f_n is continuous and defined on $[0,1]$, so f_n is Riemann integrable and thus f is Riemann integrable. Hence, we know

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x$$

exists. Hence, we know

$$\begin{aligned} -\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x - \int_0^{1-\frac{1}{n}} f_n(x)\,\mathrm{d}x - \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \left(\int_0^1 f_n(x)\,\mathrm{d}x - \int_0^{1-\frac{1}{n}} f_n(x)\,\mathrm{d}x - \int_0^1 f_n(x)\,\mathrm{d}x \right) \\ &= \lim_{n\to\infty} - \int_0^{1-\frac{1}{n}} f_n(x)\,\mathrm{d}x \end{aligned}$$

exists. These are some details about why we can operate the limit as above.

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