Combinatorics I

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Abstract

The lecture note of Combinatorics I by Shagnik Das, where the NTU cool site is https://cool.ntu.edu.tw/courses/55532/.

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Chapter 1

Chatting

Lecture 1

1.1 Prime Numbers

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Theorem 1.1.1 (Euclid $\approx$ 300 BCE). There are infinitely many primes.

proof. (Saidak, 2006).

- Every natural number has at least one prime factor.
- No prime divides n and $n+1$, for any $n \in \mathbb{N}$.

Consider a sequence of pronic number

$$p_1 = 2,\ p_{n+1} = p_n(p_n + 1).$$

Then the number of prime factors of p_n is strictly increasing in n: p_{n+1} has all the factors of p_n together with the (disstinct) ones of $p_n + 1$.

Example. $p_1 = 2, p_2 = 6, p_3 = 42, p_4 = 1806$, where the prime factors of them are $\{2\}$, $\{2,3\}, \{2,3,7\}, \{2,3,7,43\}$.

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1.1.1 How many prime numbers are there?

Definition 1.1.1. We define

$$\pi(n) = |\{p : 1 \leq p \leq n : p \text{ is prime}\}|\,.$$

Note. By Saidak's proof, we know $\pi(p_n) \geq n$. In fact, $\pi(p_n) \geq \log_2 n$.

Theorem 1.1.2 (Legendre, $\approx$ 1800 LE).

$$\pi(n) \approx \frac{n}{\ln n} \Leftrightarrow \lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1$$

Note. Proven by Hadamard and independently de la Vallée Poussin(1896).

Theorem 1.1.3 (Better Approximation)**.**
Dirichlet: $\pi(x) \approx Li(x) = \int_2^x \frac{1}{\ln t} dt$.
Known: $\pi(n) = Li(n) + O\left(ne^{-a\sqrt{\ln n}}\right)$
Believed: $\pi(n) = Li(n) + O\left(\sqrt{n}\ln n\right)$

Chapter 2

Elementary Counting Principles

Fundemental problem: Given a set S, and we want to determine $|S|$.

2.1 Sum Rule

Theorem 2.1.1 (Sum Rule)**.** If $S = \dot\bigcup_{i=1}^{k} S_i$, then $|S| = \sum_{i=1}^{k} |S_i|$.

Note. $\dot\bigcup$ means disjoint union.

Example. A drawer contains 8 pairs of yellow socks, 5 pairs of blue socks, and 3 pairs of red socks. How many socks are there in total.

Informal proof. $2 \times (8 + 5 + 3) = 32$. ■

Proof. Let S be the set of socks in the drawer, then $S = \dot\bigcup_{p \in P} S_p$, where P is the set of pairs of socks, and S_p is the set of two socks in the pair where $p \in P$. By the sum rule,

$$|S| = \sum_{p \in P} |S_p| = \sum_{p \in P} 2 = 2|P| = 32.$$

$P = P_{\text{yellow}} \dot\cup P_{\text{blue}} \dot\cup P_{\text{red}}$. By the sum rule,

$$|P| = |P_{\text{yellow}}| + |P_{\text{blue}}| + |P_{\text{red}}| = 8 + 5 + 3 = 16.$$

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Note. Sum rule is the basis for case analysis arguments. It needs two requirements:

- Cover each case.
- Cover each case exactly once.

Example. Counting subset of a general set.

Notation. If X is a set, and $k \in \mathbb{N} \cup \{0\}$, then

$$\binom{X}{k} = \{T : T \subseteq X, |T| = k\}.$$

We define the binomial coefficient as

$$\binom{|X|}{k} = \left|\binom{X}{k}\right|.$$

i.e. Given $n \geq k \geq 0$, $\binom{n}{k}$ is the number of k-element subsets of a set of size n. ■

Proposition 2.1.1 (Pascal's relation). If $n \geq k \geq 1$, then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof. Proof. Let X be an n-element set (e.g. $X = [n] = \{1, 2, \ldots, n\}$), and let $S = \binom{X}{k} = \{T \subseteq X : |T| = k\}$. Then, by definition, $\binom{n}{k} = |S|$. For each k-element subset, we can ask: "Do you contain n?" Let

$$S_0 = \{T : T \subseteq X, n \notin T, |T| = k\},$$

and

$$S_1 = \{T : T \subseteq X, n \in T, |T| = k\}.$$

Then, $S = S_0 \uplus S_1$. By the sum rule, $|S| = |S_0| + |S_1|$. Observe that

$$\begin{aligned} S_0 &= \{T \subseteq [n], n \notin T, |T| = k\} \\ &= \{T \subseteq [n-1], |T| = k\}, \end{aligned}$$

so by definition,

$$|S_0| = \binom{|[n-1]|}{k} = \binom{n-1}{k}.$$

$$S_1 = \{T \subseteq [n], n \in T, |T| = k\}.$$

Let

$$S_1' = \{T' \subseteq [n-1], |T'| = k-1\},$$

then we know a bijection from S_1 to S_1':

$$T \in S_1 \longleftrightarrow T \backslash \{n\} \in S_1'.$$

Theorem 2.1.2 (bijection rule). Given two sets S and S', if there is a bijection $f : S \to S'$, then $|S| = |S'|$.

By this rule, we know

$$|S_1| = |S_1'| = \binom{|[n-1]|}{k-1} = \binom{n-1}{k-1}.$$

Hence,

$$\binom{n}{k} = |S| = |S_0| + |S_1| = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

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2.1.1 Pascal's Triangle

We can use Pascal's relation to compute $\binom{n}{k}$.

Note. Boundary case: $\binom{n}{0} = 1$, $\binom{n}{n} = 1$. Also, $\binom{n}{k} = 0$ for $k = -1, n+1$.

$$
\begin{tikzcd}[column sep=small]
& & & \binom{0}{0} & & & \\
& & \binom{1}{0} \arrow[dr] & & \binom{1}{1} \arrow[dl] & & \\
& \binom{2}{0} \arrow[dr] & & \binom{2}{1} \arrow[dl] \arrow[dr] & & \binom{2}{2} \arrow[dl] & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{tikzcd}
$$

2.2 Product Rule

Theorem 2.2.1. If $S = S_1 \times S_2 \times \cdots \times S_k = \{(x_1, x_2.., x_k), x_i \in S_i\}$, then $|S| = \prod_{i=1}^{k} |S_i|$.

Proof. Induction on k:
Base case: $k = 1$, trivial.
Induction step: seperate into cases bases on choice of $x_{k+1} \in S_{k+1}$. Let

$$S(x) = \{(x_1, \ldots, x_{k+1}) \in S, x_{k+1} = x \in S_{k+1}\},$$

then

$$S = \dot{\bigcup_{x \in S_{k+1}}} S(x) \to |S| = \sum_{x \in S_{k+1}} |S(x)|.$$

But $S(x) = S_1 \times S_2 \times \cdots \times \{x\}$, which is in bijection with $S_1 \times S_2 \times \cdots \times S_k$. By induction rule,

$$|S(x)| = |S_1 \times S_2 \times \cdots \times S_k| \quad \forall x$$

Hence,

$$\begin{aligned}|S| = \sum_{x \in S_{k+1}} |S(x)| &= \sum_{x \in S_{k+1}} |S_1 \times S_2 \times \ldots S_k| \\ &= |S_1 \times S_2 \times \cdots \times S_k| \times |S_{k+1}| = |S_1| \times |S_2| \times \cdots \times |S_{k+1}|.\end{aligned}$$

■

Example. Consider binary strings of length n.

Proof.

$$S = \{0,1\}^n \Rightarrow |S| = |\{0,1\}^n| = |\{0,1\}|^n = 2^n.$$

■

Definition 2.2.1 (Power Set)**.** Given a finite set X, let 2^X denote the set of all subsets of X (also denoted $\mathcal{P}(x)$), which is called the power set.

Corollary 2.2.1. $\left|2^X\right| = 2^{|X|}$.

Proof. Without lose of generality, $X = [n]$. We build a bijection between $2^{[n]}$ and the set of binary string of length n. Suppose for every $T \in 2^{[n]}$, we have $\chi_T = (x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} 1, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases}$$

Then,

$$\left|2^{[n]}\right| = |\{0,1\}^n| = 2^n.$$

■

2.3 Double-Counting argument

If we count a set in two different ways, the answer should be equal.

Example. Count $2^{[n]}$.

Proof.

1. Product rule $\to 2^n$.
2. Use the sum rule, split the subsets by size.

$$2^{[n]} = \binom{[n]}{0} \cup \binom{[n]}{1} \cup \ldots \cup \binom{[n]}{n}$$

Hence, we have the following proposition:

Proposition 2.3.1. For all $n \geq 0$,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

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2.4 Permutations

Lecture 2

As previously seen. Instead of choosing the subsets all at once, we could pick one element at a time, then we can try to use product rule. 5 Sep. 13:10

Example. Consider

$$\binom{[10]}{3}.$$

Proof. At the choice of the first element, we have 10 choices, the second one has 9 choisce, while the third one has 8 choice, but we didn't consider the order of each picked elements. ⊛

Definition 2.4.1. Given a set X and $k \in \mathbb{N} \cup \{0\}$, a k-permutation of X is

- an ordered choice of k distinct elements from X.
- a k-tuple $(x_1, x_2, \ldots, x_k)$ with $x_i \in X$ and $x_i \neq x_j$ for each $i \neq j$.
- an injection $f : [k] \to X$.

where these 3 statements are equivalent.

Notation. $X^{\underline{k}} = \{k\text{-permutation of } X\} \subseteq X^k$ where $X^k = X \times X \times \cdots \times X$ allows repitition of the elements but $X^{\underline{k}}$ don't allow repitition.

Note. If $|X| = n$, then

$$n^{\underline{k}} = \left|X^{\underline{k}}\right|.$$

Definition 2.4.2.

- a n-permutation is a n-permutation of $[n]$.
- a X-permutation is a $|X|$-permutation of X.

Theorem 2.4.1 (Generalized Product Rule). Suppose we are enumerating S, and can uniquely determine an element $s \in S$ through a series of k questions, if i-th problem always has n_i possible outcomes, independently to the permutation, then

$$|S| = n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i$$

Proof. Can make a bijection from S to

$$[n_1] \times [n_2] \times \cdots \times [n_k].$$

Map each element in S to the index of its answer in the series of answer.

Our moral is when counting we don't care about what the options are but only how many options. ■

Proposition 2.4.1.

$$\begin{aligned} n^{\underline{k}} &= n(n-1)\dots(n-(k-1)) \\ &= \prod_{i=0}^{k-1}(n-i) = \frac{n!}{(n-k)!}. \end{aligned}$$

Proof. Use the generalized product rule.

Question i: What is the i-th element in the k-permutation of $[n]$?

We can choose anything except what we're alreafy chosen, so there are $i-1$ forbidden choices and thus there are $n-(i-1)$ possible choices. ■

Proposition 2.4.2. For all $0 \le k \le n$,

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k^{\underline{k}}} = \frac{\left(\frac{n!}{(n-k)!}\right)}{k!} = \frac{n!}{k!(n-k)!}.$$

Proof. Double-count $[n]^{\underline{k}}$ i.e. k-permutation of $[n]$.

- Direct counting $\left|[n]^{\underline{k}}\right| = n^{\underline{k}}$.
- First choose the k elements to appear in the k-permutation, $\binom{n}{k}$ options, then choose the order in which they appear, $k^{\underline{k}}$ options.

  Then, by the generalized product rule, the number of k-permutation of $[n]$ is $\binom{n}{k} \cdot k^{\underline{k}}$.

Hence,

$$n^{\underline{k}} = \left|[n]^{\underline{k}}\right| = \binom{n}{k} \cdot k^{\underline{k}}.$$

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Corollary 2.4.1. We can then use this result to reprove Pascal's Property again.

Proof. ■

Exercise. 6 players at the tennis club want to have three matches involving all the players? How many ways can we arrange the games.

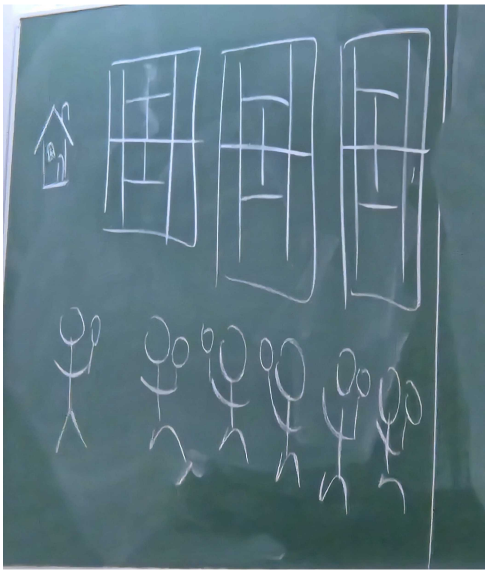

Figure 2.1: Tennis Games

Proof. We only care about who plays against whom, not about which court or who versus first, e.t.c.

The arrangement of games is a set of three disjoint pairs of players.

$$\{\{1,2\},\{3,4\},\{5,6\}\} \neq \{\{1,3\},\{2,4\},\{5,6\}\}.$$

Double-count the arrangements of games where counts do matter.

- Choose a pair of players for Court A: $\binom{6}{2}$
- Choose a pair of players for Court B: $\binom{4}{2}$
- Choose a pair of players for Court C: $\binom{2}{2}$

Generalized product rule tells

$$\text{number of choices} = \binom{6}{2}\binom{4}{2}\binom{2}{2} = 90.$$

Second count: First gets a set of 3 pairs, say there are x possibilities , and assign the three pairs to 3 courts, so there are 3! , so $x \cdot 3! = 90$, and thus $x = \frac{90}{3!} = 15$. ■

Appendix