Introduction to Analysis I HW6

B13902024 張沂魁

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Problem 0.0.1 (20pts).

Definition 0.0.1 (Totally ordered set). A *totally ordered set* (or *linearly ordered set*) is a pair $(X, \leq)$ consisting of a nonempty set X together with a binary relation $\leq$ on X satisfying the following properties:

1. **Reflexivity:** For all $x \in X$, $x \leq x$.
2. **Antisymmetry:** For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.
3. **Transitivity:** For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
4. **Totality (or Comparability):** For all $x, y \in X$, either $x \leq y$ or $y \leq x$.

A relation $\leq$ satisfying only (1)–(3) is called a *partial order*. If, in addition, (4) holds, the order is said to be *total*, meaning that any two elements of X can be compared.

Definition 0.0.2 (Hausdorff space). A topological space $(X, \mathcal{F})$ is called a *Hausdorff space* (or T_2 space) if for every pair of distinct points $x, y \in X$ there exist neighborhoods $U, V \in \mathcal{F}$ such that

$$x \in U, \quad y \in V, \quad \text{and } U \cap V = \varnothing.$$

(a) Given any totally ordered set X with order relation $\leq$, declare a set $V \subseteq X$ to be open if for every $x \in V$ there exists a set I, which is an interval $\{y \in X : a < y < b\}$ for some $a, b \in X$, or $\{y \in X : a < y\}$ for some $a \in X$, or $\{y \in X : y < b\}$ for some $b \in X$, or the whole space X, which contains x and is contained in V. Let $\mathcal{F}$ be the set of all open subsets of X. Show that $(X, \mathcal{F})$ is a topology (this is the *order topology* on the totally ordered set $(X, \leq)$ which is Hausdorff in the sense of Definition 2.5.4-2 or the definition above).

(b) Show that on the real line $\mathbb{R}$ (with the standard ordering $\leq$), the order topology matches the standard topology (i.e., the topology arising from the standard metric).

(c) If instead one defines V to be open if the extended real line $\mathbb{R} \cup \{\pm\infty\}$ has an open set with boundary $\{\pm\infty\}$, then $(X, \mathcal{F})$ is a sequence of numbers in $\mathbb{R}$ (and hence in $\mathbb{R}$), show that $x_n \to +\infty$ if and only if $\inf_{n \geq N} x_n \to +\infty$, and $x_n \to -\infty$ if and only if $\sup_{n \geq N} x_n \to -\infty$.

(a). First note that $\varnothing, X \subseteq \mathcal{F}$, which is trivial by the definition of $\mathcal{F}$. Next, we give a claim:

Claim 0.0.1. If $V_1, V_2 \in \mathcal{F}$, then $V_1 \cap V_2 \in \mathcal{F}$.

Proof. For all $x \in V_1 \cap V_2$, there exists I_1, I_2 s.t. $x \in I_1 \subseteq V_1$ and $x \in I_2 \subseteq V_2$ and $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$ (a_1, b_1, a_2, b_2 may be $\pm\infty$ or some element in X, please see following remark).

Remark 0.0.1. To be convenient, if I_1 or I_2 is $\{y \in X : a < y < b\}$, then we use (a, b) to denote them, and if it is $\{y \in X : a < y\}$, then we use (a, ∞) to denote them, and if it is $\{y \in X : y < b\}$, then we use $(-\infty, b)$ to denote them. Also, if it is the whole X, then we use $(-\infty, \infty)$ to denote. Also, we suppose $-\infty < c$ and $\infty > c$ for all $c \in X$. This is notation may be not formal, but it is useful.

Now we can pick $I_3 = I_1 \cap I_2 = (\max\{a_1, a_2\}, \min\{b_1, b_2\})$ (min, max is similarly defined as when $\leq$ is defined in $\mathbb{R}$.) Hence, we know $x \in I_3$ and $I_3 \subseteq V_1 \cap V_2$, so $V_1 \cap V_2 \subseteq \mathcal{F}$.

Note that I_3 is well-defined since $x \in I_1 \cap I_2$, so I_3 is not empty, and it will not happen that $\min\{b_1, b_2\} \leq \max\{a_1, a_2\}$. ⊛

Now if we have $V_1, V_2, \ldots, V_n \in \mathcal{F}$, then by Claim 0.0.1, we know $V_1 \cap V_2 \cap \mathcal{F}$, and applying

Claim 0.0.1 again, then we know $V_1 \cap V_2 \cap V_3 \in \mathcal{F}$, then repeating this we have

$$\bigcap_{i=1}^{n} V_i \in \mathcal{F}.$$

Now if we have $\{V_\alpha\}_{\alpha \in A}$, then for all $x \in \bigcup_{\alpha \in A} V_\alpha$, we can pick some $\alpha_0 \in A$ s.t. $x \in V_{\alpha_0}$, and we know there exists I_{α_0} s.t. $x \in I_{\alpha_0} \subseteq V_{\alpha_0} \subseteq \bigcup_{\alpha \in A} V_\alpha$ and I_{α_0} is an interval, so we know $\bigcup_{\alpha \in A} V_\alpha \in \mathcal{F}$.

By above arguments, we know $(X < \mathcal{F})$ is a topology. ∎

(b). Suppose $\mathcal{F}'$ is the order topology on $\mathbb{R}$ and $\mathcal{F}$ is the standard topology on $\mathbb{R}$, then if $V \in \mathcal{F}'$, then for all $x \in V$, we know there exists intevral I s.t. $x \in I \subseteq V$, then similarly we use the notation in Remark 0.0.1, which means $I = (a, b)$, and this time, if $I \neq X$, then we know

$$x \in B_{\mathbb{R}}(x, \min\{x - a, b - x\}),$$

where we define $\infty - x$ is still ∞ and $x - \infty$ is $-\infty$ and $-\infty - x$ is $-\infty$ and $x - (-\infty)$ is ∞, then since $I \neq X$, so we know $\min\{x - a, b - x\}$ must be some $r \in \mathbb{R}$, and thus $B_{\mathbb{R}}(x, \min\{x - a, b - x\})$ is well-defined. In this case, $V \in \mathcal{F}$. If $I = X = \mathbb{R}$, then $\mathbb{R} = I \subseteq V \subseteq \mathbb{R}$, so $V = \mathbb{R}$ and thus $V \in \mathcal{F}$. Thus, we know $\mathcal{F}' \subseteq \mathcal{F}$.

Now if $V \in \mathcal{F}$, then for all $x \in V$, we know there exists $r_x > 0$ s.t. $B_{\mathbb{R}}(x, r_x) \subseteq V$, so

$$x \in (x - r_x, x + r_x) \subseteq V,$$

and this means $V \in \mathcal{F}'$ by definition. Thus, $\mathcal{F} \subseteq \mathcal{F}'$.

Thus, we can conclude $\mathcal{F} = \mathcal{F}'$. ∎

(c). We first show the $x_n \to +\infty$ if and only if $\inf_{n \geq N} x_n \to +\infty$ part:

($\Rightarrow$) Now if $x_n \to +\infty$, then for all $(a, +\infty)$, there exists $N > 0$ s.t. $n \geq N$ implies $x_n \in (a, +\infty)$. Hence, we know $a < x_n$ for all $n \geq N$ and thus $a \leq \inf_{n \geq N} x_n$, so we know

$$\inf_{n \geq N} x_n \in (a - 1, +\infty),$$

which means $\inf_{n \geq N} x_n \to +\infty$.

($\Leftarrow$) Now if $\inf_{n \geq N} x_n \to +\infty$, then for all $(a, +\infty)$, we know there exists $N_1 > 0$ s.t. $N \geq N_1$ implies

$$\inf_{n \geq N} x_n \in (a, +\infty),$$

so for all $n \geq N_1$, we have $x_n \in (a, +\infty)$, which means $x_n \to +\infty$.

Next, we show that $x_n \to -\infty$ if and only if $\sup_{n \geq N} \to -\infty$:

($\Rightarrow$) Suppose $x_n \to -\infty$. Then for all intervals $(-\infty, a)$, there exists $N > 0$ such that $n \geq N$ implies $x_n \in (-\infty, a)$. Hence, we know $x_n < a$ for all $n \geq N$, and thus $\sup_{n \geq N} x_n \leq a$. Therefore,

$$\sup_{n \geq N} x_n \in (-\infty, a + 1),$$

which means $\sup_{n \geq N} x_n \to -\infty$.

($\Leftarrow$) Now suppose $\sup_{n \geq N} x_n \to -\infty$. Then for all intervals $(-\infty, a)$, there exists $N_1 > 0$ such that $N \geq N_1$ implies

$$\sup_{n \geq N} x_n \in (-\infty, a).$$

Hence, for all $n \geq N_1$, we have $x_n \in (-\infty, a)$, which means $x_n \to -\infty$.

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Problem 0.0.2 (15pts).

Definition 0.0.3 (Metrizable space). A topological space $(X, \mathcal{F})$ is said to be *metrizable* if there exists a metric $d : X \times X \to [0, \infty)$ such that the topology $\mathcal{F}$ coincides with the topology $\mathcal{F}_d$ induced by d. That is,

$$\mathcal{F} = \mathcal{F}_d := \{ U \subseteq X : \forall x \in U, \exists \varepsilon > 0 \text{ such that } B_d(x, \varepsilon) \subseteq U \},$$

where $B_d(x, \varepsilon) := \{ y \in X : d(x, y) < \varepsilon \}$ denotes the open ball centered at x with radius ε.

If no such metric d exists, then $(X, \mathcal{F})$ is said to be *not metrizable*. In other words, its topology cannot arise from any metric on X.

(a) Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or cofinite (which means that $X \setminus E$ is finite). Show that $(X, \mathcal{F})$ is a topology (this is called the *cofinite topology* on X) which is not Hausdorff and is compact.

(b) Show that if $\{V_i : i \in I\}$ is any countable collection of open sets containing x, then $\bigcap_i V_i \neq \varnothing$. Use this to show that the cofinite topology cannot be derived from any metric (i.e., $(X, \mathcal{F})$ is not metrizable). (Hint: what is the set $\bigcap_{n=1}^{\infty} B(x, 1/n)$ equal to in a metric space?)

Problem 0.0.3 (15pts). Let $(X, \mathcal{F})$ be a compact topological space. Assume that this space is first countable, which means that for every $x \in X$ there exist countable collections of open sets $V_1, V_2, \dots$ of neighborhoods of x, such that every neighborhood of x contains one of the V_n. Show that every sequence in X has a convergent subsequence (see Exercise 1.5.11).

Problem 0.0.4 (15pts). Let $(X, \mathcal{F})$ be a compact topological space and $(Y, \mathcal{G})$ be a Hausdorff topological space. If $f : X \to Y$ is continuous, then f is a *closed map*; i.e., for every closed subset $F \subseteq X$, the image $f(F)$ is closed in Y.

Claim 0.0.2. For all closed $F \subseteq X$, F is compact.

Proof. Suppose $\{V_\alpha\}_{\alpha \in A}$ is an open cover of F, and since $X \setminus F$ is open, so for all $x \in X \setminus F$, there exists a neighborhood of x, U_x s.t. $U_x \subseteq X \setminus F$. Thus, we know

$$X \setminus F = \bigcup_{x \in X \setminus F} U_x,$$

and thus

$$X = F \cup (X \setminus F) = \left(\bigcup_{\alpha \in A} V_\alpha \right) \cup \left(\bigcup_{x \in X \setminus F} U_x \right),$$

so $\{V_\alpha\}_{\alpha \in A} \cup \{U_x\}_{x \in X \setminus F}$ is an open cover of X. Now since X is compact, so there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ and $x_1, x_2, \dots, x_m \in X \setminus F$ s.t.

$$X \subseteq \left(\bigcup_{i=1}^{n} V_{\alpha_i} \right) \cup \left(\bigcup_{i=1}^{m} U_{x_i} \right).$$

Now since $\bigcup_{i=1}^{m} U_{x_i} \subseteq X \setminus F$, so we know $F \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$, which shows F is compact. ⊛

Hence, for all $F \subseteq X$, since F is compact by Claim 0.0.2 and f is continuous, so $f(F)$ is comapct in Y. Now we show that $f(F)$ is closed in Y. Thus, we want to show $Y \setminus f(F)$ is open. Now suppose $y \in Y \setminus f(F)$, then for all $x \in f(F)$, there exists a neighborhood of x, $V_x \in \mathcal{G}$, and a neighborhood of y, $U_x \in \mathcal{G}$ s.t. $U_x \cap V_x = \varnothing$ since $(Y, \mathcal{G})$ is Hausdorff. Note that $U_x \subseteq Y \setminus V_x$ for all $x \in X$. Also, we know

$$f(F) \subseteq \bigcup_{x \in f(F)} V_x,$$

and since $f(F)$ is compact, so there exists $x_1, x_2, \ldots, x_n \in f(F)$ s.t.

$$f(F) \subseteq \bigcup_{i=1}^{n} V_{x_i}.$$

Hence, we have

$$\bigcap_{i=1}^{n} (Y \setminus V_{x_i}) = Y \setminus \bigcup_{i=1}^{n} V_{x_i} \subseteq Y \setminus f(F),$$

and since $U_x \subseteq Y \setminus V_x$ for all $x \in f(F)$, so we know

$$\bigcap_{i=1}^{n} U_{x_i} \subseteq \bigcap_{i=1}^{n} (Y \setminus V_{x_i}) \subseteq Y \setminus f(F),$$

and by the definition of topology, we know $\bigcap_{i=1}^{n} U_{x_i} \in \mathcal{G}$ and it is a neighborhood of y, so $Y \setminus f(F)$ is open, and we're done.

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Problem 0.0.5 (20pts). Let $\{f_n\}$ be a sequence of continuous functions real-valued defined on a compact metric space S and assume that $\{f_n\}$ converges pointwise on S to a limit function f. Prove that $f_n \to f$ uniformly on S if, and only if, the following two conditions hold:

(i) The limit function f is continuous on S.

(ii) For every $\varepsilon > 0$, there exist $m > 0$ and $\delta > 0$ such that $n > m$ and

$$|f_k(x) - f(x)| < \delta \Rightarrow |f_{k+n}(x) - f(x)| < \varepsilon$$

for all $x \in S$ and all $k = 1, 2, \ldots$.

Hint. To prove the sufficiency of (i) and (ii), show that for each $x_0 \in S$ there is a neighborhood $B(x_0, R)$ and an integer k (depending on x_0) such that

$$|f_k(x) - f(x)| < \delta \quad \text{if } x \in B(x_0, R).$$

By compactness, a finite set of integers, say $A = \{k_1, \ldots, k_r\}$, has the property that for each $x \in S$, some $k \in A$ satisfies $|f_k(x) - f(x)| < \delta$. Uniform convergence is an easy consequence of this fact.

Problem 0.0.6 (15pts). The purpose of this exercise is to demonstrate a concrete relationship between continuity and pointwise convergence, and between uniform continuity and uniform convergence.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For any $a \in \mathbb{R}$, let $f_a : \mathbb{R} \to \mathbb{R}$ be the shifted function defined by

$$f_a(x) := f(x - a).$$

(a) Show that f is continuous if and only if, whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge pointwise to f.

(b) Show that f is uniformly continuous if and only if, whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge uniformly to f.

proof of (a).

($\Rightarrow$) If f is continuous and suppose $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to 0, then given any $x \in \mathbb{R}$ and $\varepsilon > 0$, we know there exists $\delta > 0$ s.t. $|a_n| = |(x - a_n) - x| < \delta$ implies

$$|f(x - a_n) - f(x)| < \varepsilon,$$

and since $(a_n)_{n=0}^{\infty}$ converges to 0, so there exists $N > 0$ s.t. $n \geq N$ implies $|a_n| < \delta$. Thus, for all $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f_{a_n}(x) - f(x)| = |f(x - a_n) - f(x)| < \varepsilon,$$

which means f_{a_n} converge pointwise to f.

($\Leftarrow$) Now if we have a sequence in $\mathbb{R}$, $\{b_n\}_{n=0}^{\infty}$, converges to $b \in \mathbb{R}$, then we know $\{c_n = b - b_n\}_{n=0}^{\infty}$ is a sequence converges to 0, so f_{c_n} converge pointwise to f. This means for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f(x - b + b_n) - f(x)| = |f(x - c_n) - f(x)| = |f_{c_n}(x) - f(x)| < \varepsilon,$$

so if we pick $x = b \in \mathbb{R}$, we know for all $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f(b_n) - f(b)| < \varepsilon,$$

which means $\lim_{n\to\infty} f(b_n) = f(b)$, so f is continuous.

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proof of (b).

($\Rightarrow$) If f is uniformly continuous and $(a_n)_{n=0}^{\infty} \to 0$, then for all $\varepsilon > 0$, we know there exists $\delta > 0$ s.t. if $|a_n| = |(x - a_n) - x| < \delta$, then $|f(x - a_n) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$, and since $(a_n)_{n=0}^{\infty}$ converges to 0, so there exists $N > 0$ s.t. $n \geq N$ implies $|a_n| < \delta$. Thus, for all $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f_{a_n}(x) - f(x)| = |f(x - a_n) - f(x)| < \varepsilon \text{ for all } x \in \mathbb{R},$$

so f_{a_n} converges uniformly to f.

($\Leftarrow$) We want to show that for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Now suppose by contradiction, there exists $\varepsilon_1 > 0$ s.t. for all $\delta > 0$, there exists $x, y \in \mathbb{R}$ s.t. $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_1$. Then for all $\delta = \frac{1}{k}$ with $k \in \mathbb{N}$, we can pick x_k, y_k s.t. $|x_k - y_k| < \frac{1}{k}$ and $|f(x_k) - f(y_k)| \geq \varepsilon_1$. Hence, we know $\{c_n = y_n - x_n\}_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ which converges to 0. Thus, f_{c_n} converges uniformly to f. Thus, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f(x - y_n + x_n) - f(x)| = |f(x - c_n) - f(x)| = |f_{c_n}(x) - f(x)| < \varepsilon_1$$

for all $x \in \mathbb{R}$, so if we pick $x = y_N$, then for $n = N$ we know

$$|f(x_N) - f(y_N)| < \varepsilon_1$$

but this is impossible, so this is a contradiction. Hence, f is uniformly continuous.

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