Introduction to Analysis I HW 1

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Problem 0.0.1 (10pts). Dyadic density via the Archimedean property. Let $a < b$ be real numbers. Prove that there exists a dyadic rational

$$q = \frac{k}{2^n} \in \mathbb{Q} \quad (k \in \mathbb{Z}, n \in \mathbb{N})$$

such that $a < q < b$. Further show that there are infinitely many such dyadic rationals in (a, b).

Proof. We first need to show a lemma first:

Lemma 0.0.1. For any real numbers a, b such that $a < b$, there exists $n \in \mathbb{N}$ such that $2^n a > b$.

Proof. By Archimedean Property, we know there exists $q \in \mathbb{N}$ such that $qa > b$, so if we pick $n = q + 2$, then we have

$$2^n = 2^{q+2} > q + 2 > q,$$

so we have $2^n a > qa > b$, and we're done. ⊛

Now using Lemma 0.0.1 , we can get there exists some $n \in \mathbb{N}$ such that $2^n(b - a) > 1$, so if we let $k = \lfloor 2^n a \rfloor + 1$, then we have

$$2^n a < \lfloor 2^n a \rfloor + 1 = k \leq 2^n a + 1 < 2^n b.$$

Hence,

$$a < \frac{k}{2^n} < b$$

here. Note that $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, so we can pick $q = \frac{k}{2^n}$.

Next we'll show that there are infinitely many such dyadic rationals in (a, b). Actually we can just repeat the step above but let a be $q^{(0)}$ that $q^{(0)}$ is the q we found above and then we know there exists another dyadic rationals $q^{(1)}$ in $(q^{(0)}, b)$, and then doing again this step we know there exists another dyadic rationals $q^{(2)}$ in $(q^{(1)}, b)$. and so on. Then, since $q^{(i)} \neq q^{(j)}$ if $i \neq j$, so we know

$$a < q^{(0)} < q^{(1)} < q^{(2)} < \cdots < b,$$

which means there are infinitely many such dyadic rationals in (a, b). ■

Problem 0.0.2 (A tour of the p-adic world.). The field $\mathbb{Q}$ inherits the Euclidean metric from $\mathbb{R}$, but it also carries a very different metric: the *p-adic metric*.

Given a prime number p and an integer n, the p-adic norm of n is defined as

$$|n|_p = \frac{1}{p^k},$$

where p^k is the largest power of p dividing n. (We define $|0|_p := 0$.) The more factors of p appear in n, the smaller the p-adic norm becomes.

For a rational number $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, we may factor x as

$$x = p^k \cdot \frac{r}{s},$$

where $k \in \mathbb{Z}$ and p divides neither r nor s. We then define

$$|x|_p = p^{-k}.$$

The p-adic metric on $\mathbb{Q}$ is given by

$$d_p(x, y) := |x - y|_p.$$

(a) To compute the 5-adic norm $|x|_5$ of a rational number x, we examine how many factors of 5 occur in x (in either numerator or denominator).

- If $x = 5^k \cdot \frac{a}{b}$ with a, b not divisible by 5 and $k \in \mathbb{Z}$, then the 5-adic norm is

$$|x|_5 = 5^{-k}.$$

- **Examples.**

  (a) $30 = 2 \cdot 3 \cdot 5$. There is exactly one factor of 5, so

  $$|30|_5 = 5^{-1} = \tfrac{1}{5}.$$

  (b) $32 = 2^5$. There is no factor of 5, so

  $$|32|_5 = 5^0 = 1.$$

  (c) Compute $\left|\frac{1}{250}\right|_5$.

  $$250 = 2 \cdot 5^3.$$

  So

  $$\frac{1}{250} = \frac{1}{2 \cdot 5^3} = 5^{-3} \cdot \frac{1}{2},$$

  where $\frac{1}{2}$ has no factor of 5 in numerator or denominator. Therefore,

  $$\left|\tfrac{1}{250}\right|_5 = 5^{-(-3)} = 5^3 = 125.$$

  Hence,

  $$\boxed{\left|\tfrac{1}{250}\right|_5 = 125.}$$

Now practice computing the following 5-adic norms: (6 pts)

(a) $|75|_5$

(b) $\left|\frac{10}{9}\right|_5$

(c) $\left|-\frac{20}{375}\right|_5$

(b) (9 pts) Further properties of the 5-adic norm.

(a) Determine all rational numbers x satisfying $|x|_5 \leq 1$.

(b) Which rational numbers x satisfy $|x|_5 = 1$?

(c) What is $\lim_{n\to\infty} 5^n$ in $(\mathbb{Q}, d_5)$ (the 5-adic metric)?
Hint: Compute $d_5(5^n, 0)$.

(c) (15 pts) **Non-Archimedean absolute value and metric.** Prove that $|\cdot|_p$ satisfies

$$|xy|_p = |x|_p|y|_p, \qquad |x + y|_p \leq \max\{|x|_p, |y|_p\},$$

and show that d_p is a metric on $\mathbb{Q}$.

Proof.

(a)

(a) First note that $75 = 5^2 \cdot 3$, so $|75|_5 = 5^{-2} = \frac{1}{25}$.

(b) First note that $\frac{10}{9} = 5 \cdot \frac{2}{9}$, so $\left|\frac{10}{9}\right|_5 = 5^{-1} = \frac{1}{5}$.

(c) First note that $-\frac{4 \cdot 5}{5^3 \cdot 3} = 5^{-2} \cdot \frac{-4}{3}$, so $\left|-\frac{20}{375}\right|_5 = 5^{-(-2)} = 25$.

(b)

(a) Suppose $x = 5^k \cdot \frac{r}{s}$ where $k, r, s \in \mathbb{Z}$ and 5 divides neither r nor s, then we know $|x|_5 = 5^{-k}$, and we want $5^{-k} \leq 1$, which means $k \geq 0$. Hence,

$$\{\text{all rational numbers } x \text{ satisfying } |x|_5 \leq 1\} = \left\{5^k \cdot \frac{r}{s} \mid k, r, s \in \mathbb{Z} \text{ and } k \geq 0 \text{ and } 5 \nmid rs\right\}.$$

(b)

$$\{\text{all rational numbers } x \text{ satisfying } |x|_5 = 1\} = \left\{\frac{r}{s} \mid r, s \in \mathbb{Z} \text{ and } 5 \nmid rs\right\}$$

(c) First notice that $d_5(5^n, 0) = |5^n - 0|_5 = 5^{-n}$. Also, we know

$$0 = \lim_{n\to\infty} 5^{-n} = \lim_{n\to\infty} d_5(5^n, 0),$$

so we know $\lim_{n\to\infty} 5^n = 0$ in $(\mathbb{Q}, d_5)$.

(c) First we consider the case that x, y are both not zero. Now suppose $x = p^{k_1}\frac{r_1}{s_1}$ and $y = p^{k_2}\frac{r_2}{s_2}$, where $p \nmid r_1 s_1 r_2 s_2$. Hence, $xy = p^{k_1+k_2}\frac{r_1 r_2}{s_1 s_2}$, and thus

$$|xy|_p = p^{-(k_1+k_2)}.$$

Also, we know

$$|x|_p = p^{-k_1} \quad |y|_p = p^{-k_2},$$

so

$$|xy|_p = p^{-(k_1+k_2)} = p^{-k_1}p^{-k_2} = |x|_p|y|_p.$$

Now without lose of genrality, suppose $k_1 \geq k_2$, then we know

$$x + y = p^{k_2}\left(\frac{p^{k_1-k_2}r_1 s_2 + r_2 s_1}{s_1 s_2}\right),$$

and thus

$$|x + y|_p \leq p^{-k_2} = |y|_p = \max\{|x|_p, |y|_p\}.$$

> **Note.** When $k_1 = k_2$, it may happen that $|x + y|_p < \max\{|x|_p, |y|_p\}$.

And the case that $k_2 \geq k_1$ is similar.

As for the case that either x or y is zero, we know that $|0|_p = 0$. We first talk about the case that $x = 0$, so

$$|xy|_p = |0|_p = 0 = |x|_p|y|_p$$

and

$$|x + y|_p = |y|_p = \max\{|x|_p, |y|_p\}.$$

Similarly, we know the case that $y = 0$ is also true by repeating the steps above.

Next, we want to show that d_p is a metric on $\mathbb{Q}$. From now on we suppose $x = p^{k_1}\frac{r_1}{s_1}$, $y = p^{k_2}\frac{r_2}{s_2}$, and $z = p^{k_3}\frac{r_3}{s_3}$ for some $x, y, z \in \mathbb{Q}$ and $p \nmid r_i s_i$ for $i = 1, 2, 3$. Hence,

- $d_p(x, x) = |0|_p = 0$.
- $d_p(x, y) = |x - y|_p = \frac{1}{p^z}$ for some $z \in \mathbb{Z}$, so $d_p(x, y) > 0$.

– Without lose of generality, suppose $k_1 \geq k_2$, then

$$x - y = p^{k_2}\left(\frac{p^{k_1-k_2}r_1 s_2 - r_2 s_1}{s_1 s_2}\right)$$

and

$$y - x = -p^{k_2}\left(\frac{p^{k_1-k_2}r_1 s_2 - r_2 s_1}{s_1 s_2}\right),$$

so we know

$$d_p(x,y) = |x-y|_p = k_2 = |y-x|_p = d(y,x).$$

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$$\begin{aligned} d(x,z) &= |x-z|_p = |(x-y)+(y-z)|_p \\ &\leq \max\{|x-y|_p, |y-z|_p\} \leq |x-y|_p + |y-z|_p = d(x,y) + d(y,z). \end{aligned}$$

By the above four properties of d_p, we can conclude that d_p is a metric on $\mathbb{Q}$.

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Problem 0.0.3 (exercise 1.1.3 (20 pts)). Let X be a set, and let $d : X \times X \to [0,\infty)$ be a function.

(a) Give an example of a pair (X,d) which obeys axioms (bcd) of Definition 1.1.2, but not (a). (Hint: modify the discrete metric.)

(b) Give an example of a pair (X,d) which obeys axioms (acd) of Definition 1.1.2, but not (b).

(c) Give an example of a pair (X,d) which obeys axioms (abd) of Definition 1.1.2, but not (c).

(d) Give an example of a pair (X,d) which obeys axioms (abc) of Definition 1.1.2, but not (d). (Hint: try examples where X is a finite set.)

Problem 0.0.4 (20 pts). Let $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ be vectors in $\mathbb{R}^n$.

(a) The ℓ^1 metric is defined by

$$d_1(x,y) := \sum_{i=1}^{n} |x_i - y_i|.$$

Show that d_1 is a metric on $\mathbb{R}^n$

(b) The ℓ^∞ metric is defined by

$$d_\infty(x,y) := \max_{1\leq i\leq n} |x_i - y_i|.$$

Show that d_∞ is a metric on $\mathbb{R}^n$

Problem 0.0.5 (10 pts). A *vector space* V over $\mathbb{R}$ s a set equipped with two operations:

1. **Vector addition:** $+ : V \times V \to V$, written $(u,v) \mapsto u+v$.
2. **Scalar multiplication:** $\cdot : \mathbb{R} \times V \to V$, written $(\alpha, v) \mapsto \alpha v$,

such that the following properties hold for all $u,v,w \in V$ and $\alpha,\beta \in \mathbb{R}$:

(VS1) $(u+v)+w = u+(v+w)$ (associativity of addition)

(VS2) $u+v = v+u$ (commutativity of addition)

(VS3) There exists $0 \in V$ such that $u+0 = u$ (additive identity)

(VS4) For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = 0$ (additive inverse)

(VS5) $\alpha(u + v) = \alpha u + \alpha v$ (distributivity I)

(VS6) $(\alpha + \beta)u = \alpha u + \beta u$ (distributivity II)

(VS7) $(\alpha\beta)u = \alpha(\beta u)$ (compatibility of scalar multiplication)

(VS8) $1 \cdot u = u$ (identity element of scalar multiplication)

A function $\|\cdot\| : V \to [0, \infty)$ is called a *norm* on V if, for all $u, v \in V$ and $\alpha \in \mathbb{R}$, the following properties hold:

(N1) $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$. (positivity)

(N2) $\|\alpha v\| = |\alpha| \cdot \|v\|$. (homogeneity)

(N3) $\|u + v\| \leq \|u\| + \|v\|$. (triangle inequality)

Given a norm $\|\cdot\|$ on V, define $d : V \times V \to [0, \infty)$ by

$$d(u, v) = \|u - v\|.$$

Prove that d is a *metric* on V, that is, for all $x, y, z \in V$ show that:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

(Thus we conclude that every normed vector space $(V, \|\cdot\|)$ is also a metric space with metric $d(u, v) = \|u - v\|$.)

Problem 0.0.6 (10 pts). Let S be a bounded nonempty set of real numbers, and let a and b be fixed nonzero real numbers. Define $T = \{as + b | s \in S\}$ Find formulas for $\sup T$ and $\inf T$ in terms of $\sup S$ and $\inf S$. Prove your formulas.

Proof. We first consider the case that $a > 0$.

Claim. If $a > 0$, then $\sup T = a \sup S + b$.

Proof. First notice that for all $t \in T$, we can write $t = as + b$ for some $s \in S$. Hence,

$$t = as + b \leq a \sup S + b,$$

which means $a \sup S + b$ is an upper bound of T. Now if $a \sup S + b \neq \sup T$, then there exists $\varepsilon > 0$ such that $a \sup S + b - \varepsilon \geq t$ for all $t \in T$, and we can write all $t \in T$ as $as' + b$ for some $s' \in S$, so

$$a \sup S + b - \varepsilon \geq as' + b \Leftrightarrow \sup S - \left(\frac{\varepsilon}{a}\right) \geq s' \quad \forall s' \in S,$$

so $\sup S - \left(\frac{\varepsilon}{a}\right)$ is an upper bound of S and smaller than $\sup S$, which is a contradiction, so $\sup T = a \sup S + b$. ⊛

Claim. If $a > 0$, then $\inf T = a \inf S + b$.

Proof. First notice that for all $t \in T$, we can write $t = as + b$ for some $s \in S$. Hence,

$$t = as + b \geq a \inf S + b,$$

which means $a \inf S + b$ is a lower bound of T. Now if $a \inf S + b \neq \inf T$, then there exists $\varepsilon > 0$ such that $a \inf S + b + \varepsilon \leq t$ for all $t \in T$, and we can write all $t \in T$ as $as' + b$ for some $s' \in S$, so

$$a \inf S + b + \varepsilon \leq as' + b \Leftrightarrow \inf S + \left(\frac{\varepsilon}{a}\right) \leq s' \quad \forall s' \in S,$$

so $\inf S + \left(\frac{\varepsilon}{a}\right)$ is a lower bound of S and bigger than $\inf S$, which is a contradiction, so $\inf T = a \inf S + b$. ⊛

Now we talk about the case $a < 0$, but it is actually very similar.

Claim. If $a < 0$, then $\sup T = a \inf S + b$.

Proof. First notice that for all $t \in T$, we can write $t = as + b$ for some $s \in S$. Hence,

$$t = as + b \leq a \inf S + b,$$

which means $a \inf S + b$ is an upper bound of T. Now if $a \inf S + b \neq \sup T$, then there exists $\varepsilon > 0$ such that $a \inf S + b - \varepsilon \geq t$ for all $t \in T$. Also, we can write every $t \in T$ as $as' + b$ for some $s' \in S$, so

$$a \inf S + b - \varepsilon \geq as' + b \Leftrightarrow a \inf S \geq as' + \varepsilon \Leftrightarrow \inf S \leq s' + \left(\frac{\varepsilon}{a}\right).$$

Note that $\left(\frac{\varepsilon}{a}\right) \leq 0$, so we know

$$\inf S \leq \inf S - \left(\frac{\varepsilon}{a}\right) \leq s' \quad \forall s' \in S,$$

so we can find that $\inf S - \left(\frac{\varepsilon}{a}\right)$ is also a lower bound of S but bigger than $\inf S$, which is a contradiction. Thus, $\sup T = a \inf S + b$ if $a < 0$. ⊛

Claim. If $a < 0$, then $\inf T = a \sup S + b$.

Proof. First notice that for all $t \in T$, we can write $t = as + b$ for some $s \in S$. Hence,

$$t = as + b \geq a \sup S + b,$$

which means $a \sup S + b$ is a lower bound of T. Now if $a \sup S + b \neq \inf T$, then there exists $\varepsilon > 0$ such that $a \sup S + b + \varepsilon \leq t$ for all $t \in T$. Also, we can write every $t \in T$ as $as' + b$ for some $s' \in S$, so

$$a \sup S + b + \varepsilon \leq as' + b \Leftrightarrow a \sup S + \varepsilon \leq as' \Leftrightarrow \sup S + \left(\frac{\varepsilon}{a}\right) \geq s'.$$

Note that $\left(\frac{\varepsilon}{a}\right) \leq 0$, so we know

$$\sup S \geq \sup S + \left(\frac{\varepsilon}{a}\right) \geq s' \quad \forall s' \in S,$$

so we can find that $\sup S + \left(\frac{\varepsilon}{a}\right)$ is also a lower bound of S but smaller than $\sup S$, which is a contradiction. Thus, $\inf T = a \sup S + b$ if $a < 0$. ⊛

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