Abstract Algebra I

Homework 7

Due: 19th November 2025

For a finite group G and a prime p dividing $|G|$, let $n_p(G)$ denote the number of Sylow p-subgroups of G.

Exercise 1 Let G be a group of order 24, and suppose $n_2(G) > 1$.

(i) Prove that G has a normal subgroup of order 4.

(ii) Is it possible that $n_3(G) > 1$?

Exercise 2 Let m be an odd integer and G be a group of order $2m$. Consider the action of G on itself via left multiplication; this induces a group homomorphism

$$\pi : G \to \mathrm{Perm}(G).$$

Recall that we have a group homomorphism

$$\mathrm{sgn} : \mathrm{Perm}(G) \to \{\pm 1\}$$

that sends each permutation to its sign.

(i) Show that the composition $\mathrm{sgn} \circ \pi : G \to \{\pm 1\}$ is surjective. *(Hint: Let h be a generator of a Sylow 2-subgroup of G, and decompose G into right cosets $\{e,h\}g_1, \ldots, \{e,h\}g_m$. Now consider $\pi(h)$.)*

(ii) Deduce that G has a normal subgroup of order m.

For the next two questions, the following fact will be helpful (try to prove it on your own): If N is a normal subgroup of a group G and a Sylow p-subgroup P of N is normal in N, then P is normal in G.

Exercise 3 et G be a group of order 105.

(i) Show that G has a normal subgroup H of order 35.

(ii) Show that H is cyclic.

(iii) Prove that $n_5(G) = n_7(G) = 1$.

Exercise 4 More generally, let G be a group of order pqr, where p, q, r are distinct primes and $p < q < r$. We want to show that $n_r(G) = 1$.

(i) Suppose $n_r(G) > 1$. Show that $n_q(G) = 1$. Thus we deduce already that G is not simple.

(ii) We now suppose $n_q(G) = 1$, and let Q be the unique Sylow q-subgroup of G. Show that G/Q has a normal subgroup of order r.

(iii) Deduce that G has a normal subgroup of order qr and conclude that $n_r(G) = 1$.

Exercise 5 Let R be a commutative ring. We say an element $x \in R$ is a *zero divisor* if there is some nonzero element $y \in R$ such that $xy = 0$. If a ring has no nonzero zero divisors, we say the ring is an *integral domain*, or sometimes just *domain* for short.

(i) Classify all zero divisors of the following rings:

$$\mathbb{Z}, \ \mathbb{Z}/6\mathbb{Z}, \ \mathbb{Q}, \ \mathbb{C}[x].$$

Which of them are integral domains? (You don't need to check they are commutative rings; addition and multiplication are carried out as usual.)

(ii) Show that any element in R cannot be both invertible *and* a zero divisor at the same time.

(iii) However, an element may be neither invertible nor a zero divisor. Find an example.

(iv) Show that the invertible elements of the polynomial ring $R[x]$ coincide with the invertible elements of R.