

Exercise Sheet 5

Due date: 15:30, Nov 25th, to be submitted on COOL.

Working with your partner, you should try to solve all of the exercises below. You should then submit solutions to four of the problems, with each of you writing two, clearly indicating the author of each solution. Note that each problem is worth 10 points, and starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting on COOL, please send your solutions by e-mail.

Exercise 1 Let $\pi(n) = |\{p \in [n] : p \text{ is prime}\}|$ be the prime number function, counting the number of primes in $[n]$. In this exercise you will determine the order of magnitude of $\pi(n)$.¹

- (a) Show that for every $m \in \mathbb{N}$ and every prime $p \in [m+1, 2m]$, $p \mid \binom{2m}{m}$.
- (b) Deduce $\pi(n) = O\left(\frac{n}{\ln n}\right)$.
- (c) Show that if p^k is a prime power such that $p^k \mid \binom{2m}{m}$, then $p^k \leq 2m$.
- (d) Deduce $\pi(n) = \Omega\left(\frac{n}{\ln n}\right)$.

Bonus (0 points) Define $\text{Li}(x) = \int_2^x \frac{dt}{\ln t}$. Prove that $|\pi(n) - \text{Li}(n)| = O\left(n^{\frac{1}{2}} \ln n\right)$.

Exercise 2* Let $p_{od}(n)$ denote the number of partitions of n into odd, distinct parts; that is, $(\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$ with $\lambda_i \in 2\mathbb{N} + 1$ for all i and $\lambda_1 > \lambda_2 > \dots > \lambda_k \geq 1$. Show that there are positive constants c, C such that for sufficiently large n we have

$$e^{c\sqrt{n}} \leq p_{od}(n) \leq e^{C\sqrt{n}}.$$

¹You are asked to show $\pi(n) = \Theta\left(\frac{n}{\ln n}\right)$. However, more is known. The distribution of the prime numbers has long been central to number theory. Indeed, it was around 1800 that the legendary Legendre conjectured $\pi(n) \approx \frac{n}{\ln n - 1.08366}$. A similar conjecture was made by Gauss around the same time (when he was no older than 16). A few years later, Dirichlet offered the $\text{Li}(n)$ approximation mentioned in the bonus problem.

In 1850, Chebyshev proved that $\frac{n}{\ln n}$ was the correct order of magnitude, and in 1896, Hadamard and de la Vallée Poussin independently extended the work of Riemann and proved the Prime Number Theory, which gives the asymptotics of $\pi(n)$. As conjectured, $\pi(n) \sim \frac{n}{\ln n}$. These proofs all made use of complex analysis.

Since then, several other proofs have been found. Around 1950, Selberg and Erdős found elementary (i.e. not using analysis) proofs. (There was a rather bitter dispute between the two regarding who should get credit for the result.) The simplest proof currently known is due to Newman, although this also uses some complex analysis.

Exercise 3 When studying the twelvefold ways of counting, we determined that the number of surjective divisions of n distinct items into r distinct parts is $r!S(n, r)$, where $S(n, r)$ is the Stirling number of the second kind. Use the Inclusion-Exclusion Principle to find an expression for $r!S(n, r)$ not involving the Stirling numbers.

Exercise 4 Someone is planning a round-the-world trip that involves visiting $2n$ cities, with two cities from each of n different countries. She can choose a city to start and end the journey in, with the other $2n - 1$ cities being visited exactly once. However, she has the restriction that the two cities from each country should not be visited consecutively.² How many different trips are possible?

Exercise 5 Suppose we have finite sets A_1, A_2, \dots, A_r . Prove that when k_0 is even,

$$|\cup_{i=1}^r A_i| \geq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\cap_{i \in I} A_i|,$$

and when k_0 is odd,

$$|\cup_{i=1}^r A_i| \leq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\cap_{i \in I} A_i|.$$

That is, the partial sums in the Inclusion-Exclusion Principle alternate between upper and lower bounds on the size of the union.

²For example, suppose $n = 3$ and the $2n$ cities are $\{\text{Berlin, Frankfurt, Taipei, Kaohsiung, New York, Los Angeles}\}$. Berlin \rightarrow Taipei \rightarrow Los Angeles \rightarrow Kaohsiung \rightarrow Frankfurt \rightarrow New York \rightarrow Berlin is acceptable, but Berlin \rightarrow Kaohsiung \rightarrow Los Angeles \rightarrow Taipei \rightarrow New York \rightarrow Frankfurt \rightarrow Berlin is not, as the final flight is a domestic one.