

# Linear Algebra I HW12

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## Section 7.1

**Problem. 1.** Let  $T$  be a linear operator on  $F^2$ . Prove that any non-zero vector which is not a characteristic vector for  $T$  is a cyclic vector for  $T$ . Hence, prove that either  $T$  has a cyclic vector or  $T$  is a scalar multiple of the identity operator.

**Proof.** Suppose  $v \in F^2$  and  $v \neq 0$  and  $Tv \notin \text{span}\{v\}$ , then  $\{v, Tv\}$  is linearly independent and thus a basis of  $F^2$ , which means  $v$  is a cyclic vector for  $T$ . Now if such  $v$  does not exist, then  $Tv \in \text{span}\{v\}$  for all  $v \in F^2$ . Then if  $\{p, q\}$  form a basis of  $F^2$ , then  $Tp = \lambda p$  and  $Tq = \lambda' q$  for some  $\lambda, \lambda' \in F$ , and we claim that  $\lambda = \lambda'$ . If  $\lambda \neq \lambda'$ , then

$$T(p + q) = T(p) + T(q) = \lambda p + \lambda' q \notin \text{span}\{p + q\},$$

which is impossible. Hence,  $\lambda = \lambda'$ , and thus for all  $v \in V$ , we know  $v = \alpha p + \beta q$  for some  $\alpha, \beta \in F$ , and thus

$$Tv = \alpha Tp + \beta Tq = \alpha \lambda p + \beta \lambda q = \lambda(\alpha p + \beta q) = \lambda v,$$

which shows  $T$  is a scalar multiple of the identity operator. ■

**Problem. 7.** Let  $V$  be an  $n$ -dimensional vector space, and let  $T$  be a linear operator on  $V$ . Suppose that  $T$  is diagonalizable.

- (a) If  $T$  has a cyclic vector, show that  $T$  has  $n$  distinct characteristic values.
- (b) If  $T$  has  $n$  distinct characteristic values, and if  $\{\alpha_1, \dots, \alpha_n\}$  is a basis of characteristic vectors for  $T$ , show that  $\alpha = \alpha_1 + \dots + \alpha_n$  is a cyclic vector for  $T$ .

**Proof.**

- (a) If  $T$  has a cyclic vector, then  $m_T(x) = \text{ch}_T(x)$ , and thus

$$\deg m_T(x) = \deg \text{ch}_T(x) = n,$$

and since  $T$  is diagonalizable, so we know  $m_T(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_n)$  for some  $\lambda_i \in F$ . Thus,

$$\text{ch}_T(x) = m_T(x) = (x - \lambda_1) \dots (x - \lambda_n),$$

which means  $T$  has  $n$  distinct characteristic values.

- (b) Suppose  $T\alpha_i = \lambda_i \alpha_i$  for all  $i = 1, 2, \dots, n$ . Then we know

$$T^i \alpha = T^i \left( \sum_{j=1}^n \alpha_j \right) = \sum_{j=1}^n T^i (\alpha_j) = \sum_{j=1}^n \lambda_j^i \alpha_j.$$

Now we want to show  $\{\alpha, T\alpha, \dots, T^{n-1}\alpha\}$  is a basis of  $V$  so that we know  $\alpha$  is a cyclic vector for  $T$ . Note that

$$\begin{aligned} \alpha &= \alpha_1 + \alpha_2 + \dots + \alpha_n \\ T\alpha &= \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \dots + \lambda_n \alpha_n \\ T^2 \alpha &= \lambda_1^2 \alpha_1 + \lambda_2^2 \alpha_2 + \dots + \lambda_n^2 \alpha_n \\ &\vdots \\ T^{n-1} \alpha &= \lambda_1^{n-1} \alpha_1 + \lambda_2^{n-1} \alpha_2 + \dots + \lambda_n^{n-1} \alpha_n, \end{aligned}$$

so we have

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}}_{A \in M_n(F)} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}_{X \in M_n(F)} = \underbrace{\begin{pmatrix} \alpha \\ T\alpha \\ \vdots \\ T^{n-1}\alpha \end{pmatrix}}_{Y \in M_n(F)},$$

and note that  $\det(A) \neq 0$  since  $\lambda_i \neq \lambda_j$  for all  $i \neq j$  and  $A$  is the Vandermonde matrix. Hence,  $A$  is invertible, and thus

$$\text{rank } Y = \text{rank } AX = \text{rank } X = n$$

since the  $n$  rows of  $X$  are linearly independent, and thus the  $n$  rows of  $Y$  are also linearly independent, which shows  $\{\alpha, T\alpha, \dots, T^{n-1}\alpha\}$  is a basis of  $V$ , and we're done. ■

## Section 7.3

**Problem. 6.** Let  $A$  be the complex matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Find the Jordan form for  $A$ .

**Problem. 10.** Let  $n$  be a positive integer,  $n \geq 2$ , and let  $N$  be an  $n \times n$  matrix over the field  $F$  such that  $N^n = 0$  but  $N^{n-1} \neq 0$ . Prove that  $N$  has no square root, i.e., that there is no  $n \times n$  matrix  $A$  such that  $A^2 = N$ .

**Problem. 13.** If  $N$  is a  $k \times k$  elementary nilpotent matrix, i.e.,  $N^k = 0$  but  $N^{k-1} \neq 0$ , show that  $N^t$  is similar to  $N$ . Now use the Jordan form to prove that every complex  $n \times n$  matrix is similar to its transpose.