Linear Algebra I HW7

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Problem 0.0.1. Let A be a 2×2 matrix over a field F. Then the set of all matrices of the form $f(A)$, where f is a polynomial over F, is a commutative ring K with identity. If B is a 2×2 matrix over K, the determinant of B is then a 2×2 matrix over F, of the form $f(A)$. Suppose I is the 2×2 identity matrix over F and that B is the 2×2 matrix over K

$$B = \begin{bmatrix} A - A_{11}I & -A_{12}I \\ -A_{21}I & A - A_{22}I \end{bmatrix}.$$

Show that $\det B = f(A)$, where $f = x^2 - (A_{11} + A_{22})x + \det A$, and also that $f(A) = 0$.

Proof. Note that

$$\begin{aligned} \det B &= (A - A_{11}I)(A - A_{22}I) - A_{12}A_{21}I = A^2 - (A_{11} + A_{22})A + (A_{11}A_{22} - A_{12}A_{21})I \\ &= A^2 - (A_{11} + A_{22})A + (\det A) \cdot I, \end{aligned}$$

so we know $\det B = f(A)$, where $f(x) = x^2 - (A_{11} + A_{22})x + \det A$. Also, since we know

$$\begin{aligned} A^2 &= \begin{pmatrix} A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + A_{22}^2 \end{pmatrix} \\ (A_{11} + A_{22})A &= \begin{pmatrix} A_{11}^2 + A_{11}A_{22} & A_{11}A_{12} + A_{22}A_{12} \\ A_{11}A_{21} + A_{22}A_{21} & A_{11}A_{22} + A_{22}^2 \end{pmatrix} \\ (\det A) \cdot I &= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix}, \end{aligned}$$

so we know $f(A) = A^2 - (A_{11} + A_{22})A + (\det A) \cdot I = 0$. ■

Problem 0.0.2. If σ is a permutation of degree n and A is an $n \times n$ matrix over the field F with row vectors $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$, let $\sigma(A)$ denote the $n \times n$ matrix with row vectors

$$\boldsymbol{\alpha}_{\sigma 1}, \ldots, \boldsymbol{\alpha}_{\sigma n}.$$

(a) Prove that $\sigma(AB) = \sigma(A)B$, and in particular that $\sigma(A) = \sigma(I)A$.

(b) If T is the linear operator of Exercise 9, prove that the matrix of T in the standard ordered basis is $\sigma(I)$.

(c) Is $\sigma^{-1}(I)$ the inverse matrix of $\sigma(I)$?

(d) Is it true that $\sigma(A)$ is similar to A?

Note 0.0.1. In Exercise 9, we define

$$T : F^n \to F^n, \quad T(x_1, x_2, \ldots, x_n) = \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right)$$

for a permutation $\sigma \in S_n$.

Proof.

(a) Suppose AB's rows are $r_1, r_2, \ldots, r_n$ and $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$, then we know

$$r_i = \left(\sum_{k=1}^n a_{ik}b_{k1}, \sum_{k=1}^n a_{ik}b_{k2}, \ldots, \sum_{k=1}^n a_{ik}b_{kn}\right) \quad \forall 1 \leq i \leq n.$$

Thus, we know the p-th row of $\sigma(AB)$ is

$$r'_p = \left(\sum_{k=1}^n a_{\sigma(p)k}b_{k1}, \sum_{k=1}^n a_{\sigma(p)k}b_{k2}, \ldots, \sum_{k=1}^n a_{\sigma(p)k}b_{kn}\right)$$

for all $1 \le p \le n$. Note that $\sigma(A)$'s rows are $\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(n)}$, then if we suppose $\sigma(A)B$'s rows are $r''_1, r''_2, \ldots, r''_n$, then we know

$$r''_i = \left(\sum_{k=1}^{n} a_{\sigma(p)k}b_{k1}, \sum_{k=1}^{n} a_{\sigma(p)k}b_{k2}, \ldots, \sum_{k=1}^{n} a_{\sigma(p)k}b_{kn}\right) = r'_i \quad \forall 1 \le i \le n,$$

so $\sigma(AB) = \sigma(A)B$. Thus, we have

$$\sigma(A) = \sigma(IA) = \sigma(I)A.$$

(b) Suppose b is the standard ordered basis, then if $\sigma(j) = i$, we have $T(e_i) = e_j$. Now if $[T]_b = A = (a_{ij})_{n\times n}$, then if $a_{rc} = 1$, we must have $T(e_c) = e_r$ since every row and every column of A has exactly one 1, while the other entries in the row/column are 0. Hence, we have $c = \sigma(r)$, which means $[T]_b = \sigma(I)$.

(c) Suppose $\sigma^{-1}(I)\sigma(I) = (c_{ij})_{n\times n}$, then for c_{ij}:

- Case 1: $i = j$, we know

$$c_{ii} = \sum_{k=1}^{n} \sigma^{-1}(I)_{ik}\sigma(I)_{ki} = \sigma^{-1}(I)_{i,\sigma^{-1}(i)}\sigma(I)_{\sigma^{-1}(i),i} = \sigma(I)_{\sigma^{-1}(i),i} = \sigma(I)_{w,\sigma(w)} = 1$$

if we suppose $w = \sigma^{-1}(i)$. Note that this is true since $k = \sigma^{-1}(i)$ is the only k s.t. $\sigma^{-1}(I)_{ik} = 1$, otherwise it is equal to 0.

- Case 2: $i \ne j$, then

$$c_{ij} = \sum_{k=1}^{n} \sigma^{-1}(I)_{ik}\sigma(I)_{kj} = \sigma^{-1}(I)_{i,\sigma^{-1}(i)}\sigma(I)_{\sigma^{-1}(i),j}.$$

Note that $\sigma(\sigma^{-1}(i)) = i \ne j$, so we must have $\sigma(I)_{\sigma^{-1}(i),j} = 0$, and thus $c_{ij} = 0$.

Hence, we know $\sigma^{-1}(I)\sigma(I) = I$, which means $\sigma^{-1}(I)$ is the inverse matrix of $\sigma(I)$.

(d) The answer is: not necessarily true.

Claim 0.0.1. If $P \sim I$, then $P = I$.

Proof. If $P \sim I$, then $Q^{-1}PQ = I$ for some Q, so $PQ = Q$, which means $P = PQQ^{-1} = QQ^{-1} = I$. ⊛

With this claim, if we pick some $\sigma \in S_n$ s.t. σ is not identity permutation, then $\sigma(I) \ne I$, and thus $\sigma(I)$ is not similar to I.

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Problem 0.0.3. Let A be an $n \times n$ matrix over K, a commutative ring with identity. Suppose A has the block form

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix}$$

where A_i is an $r_i \times r_i$ matrix. Prove

$$\det A = (\det A_1)(\det A_2)\cdots(\det A_k).$$

Proof. We first do a easier case: If $A = \begin{pmatrix} A_1 & 0 \\ 0 & B \end{pmatrix}$, where $A_1 \in M_{r_1}(K)$ and B is a square matrix, then we show that $\det(A) = \det(A_1)\det(B)$. We do induction on r_1.

- For $r_1 = 1$, we know $A = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}$, where $A = (a)$, then we know

  $$\det(A) = a\det(B) = \det(A)\det(B)$$

  by expanding along the first row.
- Now suppose for all $r_1 \le p - 1$ this is true.
- Then for $r_1 = p$, we know

  $$\det(A) = \sum_{j=1}^{p} (-1)^{1+j} a_{ij} \det\left(A(1 \mid j)\right) = \sum_{j=1}^{p} (-1)^{1+j} a_{ij} \det\begin{pmatrix} A_1(1 \mid j) & 0 \\ 0 & B \end{pmatrix},$$

  by expanding along the first row, and by induction hypothesis, we know

  $$\det\begin{pmatrix} A_1(1 \mid j) & 0 \\ 0 & B \end{pmatrix} = \det\left(A_1(1 \mid j)\right)\det(B),$$

  so we know

  $$\begin{aligned} \det(A) &= \sum_{j=1}^{p} (-1)^{1+j} a_{1j} \det(A_1(1 \mid j))\det(B) = \det(B) \cdot \left(\sum_{j=1}^{p} (-1)^{1+j} a_{1j} \det(A_1(1 \mid j)) \right) \\ &= \det(B) \cdot \det(A), \end{aligned}$$

  so we're done.

By this case, we can first suppose

$$B_1 = \begin{pmatrix} A_2 & 0 & \cdots & 0 \\ 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix},$$

then we know $\det(A) = \det\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} = \det(A_1)\det(B_1)$, and similarly defines $B_2, B_3, \dots, B_{k-1}$, then we know $\det(B_i) = \det(A_{i+1})\det(B_{i+1})$ for all $1 \le i \le k-2$, and thus

$$\det(A) = \det(A_1)\det(A_2)\dots\det(A_k).$$

■

Problem 0.0.4. Let A be an $n \times n$ matrix over a field, $A \neq 0$. If r is any positive integer between 1 and n, an $r \times r$ **submatrix** of A is any $r \times r$ matrix obtained by deleting $(n - r)$ rows and $(n - r)$ columns of A. The **determinant rank** of A is the largest positive integer r such that some $r \times r$ submatrix of A has a **non-zero determinant**. Prove that the determinant rank of A is equal to the **row rank** of A (= **column rank** A).

Proof. Note that if the determinant rank of A is larger or equal to v, then this means we can find an $v \times v$ submatrix of A s.t. all rows of this submatrix are linearly independent. Suppose $\operatorname{rank} A = l$, then we can pick at most l linearly independent rows, so the determinant rank of $A \le l$. Now if we pick l linearly independent rows of A, say they are the $r_1, r_2, \dots, r_l$-th rows of A, and suppose

$$r_i = (a_{r_i 1}, a_{r_i 2}, \dots, a_{r_i n}) \quad \forall 1 \le i \le n,$$

then consider the matrix $R = \begin{pmatrix} r_1 \\ \vdots \\ r_l \end{pmatrix}$, we know $\operatorname{rank} R = l$ since all l rows of R are linearly independent. Now since row rank is equal to column rank, so there are l columns of R are linearly independent, say they are the $c_1, c_2, \ldots, c_l$-th columns, then if we pick $S = (s_{ij})_{l\times l}$ with $s_{ij} = a_{r_i c_j}$ for all $1 \le i, j \le l$, we know S is invertible, and thus the determinant rank of $A \ge l$. Hence, the determinant rank of A is equal to the row rank of A. ■

Problem 0.0.5. Let A, B, C, D be commuting $n \times n$ matrices over the field F. Show that the determinant of the $2n \times 2n$ matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is $\det(AD - BC)$.

Proof. Note that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = \begin{pmatrix} AD-BC & 0 \\ 0 & AD-BC \end{pmatrix},$$

so we know $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\det\begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = (\det(AD-BC))^2$ by previous exercise. Also, notice that

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix},$$

and thus

$$\det\begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\det\left(\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right) = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\det(I) = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Also, since we know

$$\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}^t = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix},$$

so we know

$$\det\begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = \det\begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

so we have

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\det\begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = \left(\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)^2 = (\det(AD-BC))^2,$$

so $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD-BC)$. ■