

Introduction to Mathematical Analysis
Homework 11 Due December 5 (Friday), 2025
Please submit your homework online in PDF format.

1. (20 pts) **Exercise 5.2.6** Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, and let $(f_n)_{n=1}^\infty$ be a sequence of functions in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$.
- (a) Show that if f_n converges uniformly to f , then f_n also converges to f in the L^2 metric.
 - (b) Give an example where f_n converges to f in the L^2 metric, but does *not* converge to f uniformly.
(Hint: take $f = 0$. Try to make the functions f_n large in sup norm.)
 - (c) Give an example where f_n converges to f in the L^2 metric, but does *not* converge to f pointwise.
(Hint: take $f = 0$. Try to make the functions f_n large at one point.)
 - (d) Give an example where f_n converges to f pointwise, but does *not* converge to f in the L^2 metric.
(Hint: take $f = 0$. Try to make the functions f_n large in L^2 norm.)

2. (20 pts) Let $\{\phi_N\} : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of continuous, periodic functions on \mathbb{R} (with period 1) which satisfy

$$\int_0^1 \phi_N(t) dt = 1 \quad \text{and} \quad \int_0^1 |\phi_N(t)| dt \leq M < \infty$$

for all $N \in \mathbb{N}$, and

$$\lim_{N \rightarrow \infty} \int_\delta^{1-\delta} |\phi_N(t)| dt = 0$$

for each $0 < \delta < 1$.

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and periodic with period 1. Prove that

$$\lim_{N \rightarrow \infty} \int_0^1 f(x-t) \phi_N(t) dt = f(x)$$

uniformly for $x \in \mathbb{R}$.

3. (15 pts) **Exercise 5.2.3.** If $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ is a non-zero function, show that

$$0 < \|f\|_2 \leq \|f\|_\infty.$$

Conversely, if $0 < A \leq B$ are real numbers, show that there exists a non-zero function $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ such that

$$\|f\|_2 = A \quad \text{and} \quad \|f\|_\infty = B.$$

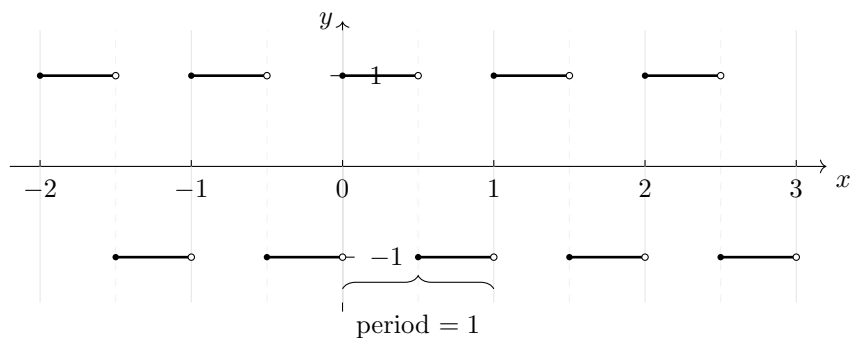
(Hint: let g be a non-constant non-negative real-valued function in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, and consider functions of the form $f = (c + dg)^{1/2}$ for some constant real numbers $c, d > 0$.)

4. (15 pts) A *square wave function* is a \mathbb{Z} -periodic function defined by

$$f(x) = \begin{cases} 1, & x \in [k, k + \frac{1}{2}), \\ -1, & x \in [k + \frac{1}{2}, k + 1), \end{cases} \quad k \in \mathbb{Z}.$$

Thus f alternates between 1 and -1 on each half-interval, repeating the same pattern on every interval of length 1.

Find a sequence of continuous periodic functions which converges in L^2 to the square wave function.



5. (15 pts)

(a) Evaluate

$$S_n(\theta) = \sum_{k=1}^n \sin(k\theta).$$

(b) Show that

$$|S_n(\theta)| \leq \pi \varepsilon^{-1} \quad \text{on } [\varepsilon, 2\pi - \varepsilon] \text{ for all } n \geq 1.$$

6. (15 pts) Let $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{R})$. We define their *periodic convolution* $f * g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(f * g)(x) := \int_0^1 f(y) g(x - y) dy.$$

Prove that $(f * g)$ is smooth whenever f is smooth. (Remark: A function is called smooth if it has derivatives of all orders.)