**Math 2213 Introduction to Analysis**
**Homework 1 Due September 10 (Thursday), 2025**
**Please submit your homework online in PDF format.**

(1) (10 pts) Dyadic density via the Archimedean property] Let $a < b$ be real numbers. Prove that there exists a *dyadic rational*

$$q = \frac{k}{2^n} \in \mathbb{Q} \qquad (k \in \mathbb{Z}, n \in \mathbb{N})$$

such that $a < q < b$. Further show that there are *infinitely many* such dyadic rationals in $(a, b)$.

(2) **A tour of the $p$-adic world.**

The field $\mathbb{Q}$ inherits the Euclidean metric from $\mathbb{R}$, but it also carries a very different metric: the *$p$-adic metric*.

Given a prime number $p$ and an integer $n$, the $p$-adic norm of $n$ is defined as

$$|n|_p = \frac{1}{p^k},$$

where $p^k$ is the largest power of $p$ dividing $n$. (We define $|0|_p := 0$.) The more factors of $p$ appear in $n$, the smaller the $p$-adic norm becomes.

For a rational number $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, we may factor $x$ as

$$x = p^k \cdot \frac{r}{s},$$

where $k \in \mathbb{Z}$ and $p$ divides neither $r$ nor $s$. We then define

$$|x|_p = p^{-k}.$$

The $p$-adic metric on $\mathbb{Q}$ is given by

$$d_p(x, y) := |x - y|_p.$$

**(a)** To compute the 5-adic norm $|x|_5$ of a rational number $x$, we examine how many factors of 5 occur in $x$ (in either numerator or denominator).

- If $x = 5^k \cdot \frac{a}{b}$ with $a, b$ not divisible by 5 and $k \in \mathbb{Z}$, then the 5-adic norm is

$$|x|_5 = 5^{-k}.$$

- **Examples.**
  (i) $30 = 2 \cdot 3 \cdot 5$. There is exactly one factor of 5, so

$$|30|_5 = 5^{-1} = \tfrac{1}{5}.$$

  (ii) $32 = 2^5$. There is no factor of 5, so

$$|32|_5 = 5^0 = 1.$$

(iii) Compute $\left|\frac{1}{250}\right|_5$.

$$250 = 2 \cdot 5^3.$$

So

$$\frac{1}{250} = \frac{1}{2 \cdot 5^3} = 5^{-3} \cdot \frac{1}{2},$$

where $\frac{1}{2}$ has no factor of 5 in numerator or denominator.
Therefore,

$$\left|\tfrac{1}{250}\right|_5 = 5^{-(-3)} = 5^3 = 125.$$

Hence,

$$\boxed{\left|\tfrac{1}{250}\right|_5 = 125.}$$

Now practice computing the following 5-adic norms: (6 pts)
(i) $|75|_5$
(ii) $\left|\frac{10}{9}\right|_5$
(iii) $\left|-\frac{20}{375}\right|_5$

**(b)** (9 pts) Further properties of the 5-adic norm.
(i) Determine all rational numbers $x$ satisfying $|x|_5 \leq 1$.
(ii) Which rational numbers $x$ satisfy $|x|_5 = 1$?
(iii) What is $\lim_{n\to\infty} 5^n$ in $(\mathbb{Q}, d_5)$ (the 5-adic metric)?
*Hint:* Compute $d_5(5^n, 0)$.

**(c)** (15 pts) **Non-Archimedean absolute value and metric.** Prove that $|\cdot|_p$ satisfies

$$|xy|_p = |x|_p|y|_p, \qquad |x+y|_p \leq \max\{|x|_p, |y|_p\},$$

and show that $d_p$ is a metric on $\mathbb{Q}$.

(3) (exercise 1.1.3) (20 pts) Let $X$ be a set, and let $d : X \times X \to [0, \infty)$ be a function.
(a) Give an example of a pair $(X, d)$ which obeys axioms (bcd) of Definition 1.1.2, but not (a). (Hint: modify the discrete metric.)
(b) Give an example of a pair $(X, d)$ which obeys axioms (acd) of Definition 1.1.2, but not (b).
(c) Give an example of a pair $(X, d)$ which obeys axioms (abd) of Definition 1.1.2, but not (c).
(d) Give an example of a pair $(X, d)$ which obeys axioms (abc) of Definition 1.1.2, but not (d). (Hint: try examples where $X$ is a finite set.)

(4) (20 pts) Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be vectors in $\mathbb{R}^n$.

(a) The $\ell^1$ metric is defined by

$$d_1(x,y) := \sum_{i=1}^{n} |x_i - y_i|.$$

Show that $d_1$ is a metric on $\mathbb{R}^n$

(b) The $\ell^\infty$ metric is defined by

$$d_\infty(x,y) := \max_{1\leq i\leq n} |x_i - y_i|.$$

Show that $d_\infty$ is a metric on $\mathbb{R}^n$

(5) (10 pts) A *vector space* $V$ over $\mathbb{R}$ s a set equipped with two operations:

(a) **Vector addition:** $+ : V \times V \to V$, written $(u,v) \mapsto u+v$.

(b) **Scalar multiplication:** $\cdot : \mathbb{R} \times V \to V$, written $(\alpha, v) \mapsto \alpha v$,

such that the following properties hold for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:

(VS1) $(u+v)+w = u+(v+w)$ (associativity of addition)

(VS2) $u+v = v+u$ (commutativity of addition)

(VS3) There exists $0 \in V$ such that $u + 0 = u$ (additive identity)

(VS4) For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = 0$ (additive inverse)

(VS5) $\alpha(u+v) = \alpha u + \alpha v$ (distributivity I)

(VS6) $(\alpha+\beta)u = \alpha u + \beta u$ (distributivity II)

(VS7) $(\alpha\beta)u = \alpha(\beta u)$ (compatibility of scalar multiplication)

(VS8) $1 \cdot u = u$ (identity element of scalar multiplication)

A function $\|\cdot\| : V \to [0,\infty)$ is called a *norm* on $V$ if, for all $u, v \in V$ and $\alpha \in \mathbb{R}$, the following properties hold:

(N1) $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$. (positivity)

(N2) $\|\alpha v\| = |\alpha| \cdot \|v\|$. (homogeneity)

(N3) $\|u+v\| \leq \|u\| + \|v\|$. (triangle inequality)

Given a norm $\|\cdot\|$ on $V$, define $d : V \times V \to [0,\infty)$ by

$$d(u,v) = \|u - v\|.$$

Prove that $d$ is a *metric* on $V$, that is, for all $x, y, z \in V$ show that:

(a) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$.

(b) $d(x,y) = d(y,x)$.

(c) $d(x,z) \leq d(x,y) + d(y,z)$.

(Thus we conclude that every normed vector space $(V, \|\cdot\|)$ is also a metric space with metric $d(u,v) = \|u-v\|$. )

(6) (10 pts) Let $S$ be a bounded nonempty set of real numbers, and let $a$ and $b$ be fixed nonzero real numbers. Define $T = \{as+b | s \in S\}$ Find formulas for $\sup T$ and $\inf T$ in terms of $\sup S$ and $\inf S$. Prove your formulas.