Math 2213 Introduction to Analysis
Total =100 pts, Homework 5 Due October 10 (Friday), 2025
Please submit your homework online in PDF format.

(1) (15 pts)
   (a) Let (X, d_{disc}) be a metric space with the discrete metric. Let E be a subset of X which contains at least two elements. Show that E is disconnected.
   (b) Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous if and only if it is constant. *(Hint: use part (a))*

(2) (15 pts) Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha \in I}$ be a collection of connected sets in X with I non-empty. Suppose also that $\bigcap_{\alpha \in I} E_\alpha$ is non-empty. Show that $\bigcup_{\alpha \in I} E_\alpha$ is connected.

(3) (20 pts) Let (X, d) be a metric space, and let E be a subset of X. We say that E is *path-connected* iff, for every $x, y \in E$, there exists a continuous function

$$\gamma : [0, 1] \to E$$

from the unit interval $[0, 1]$ to E such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every non-empty path-connected set is connected. (The converse is false, but is a bit tricky to show and will not be detailed here.)

(4) (15 pts) Let (X, d) be a metric space, and let E be a subset of X. Show that if E is connected, then the closure $\overline{E}$ of E is also connected. Is the converse true?

(5) (20 pts) Let (X, d) be a metric space. Let us define a relation $x \sim y$ on X by declaring $x \sim y$ iff there exists a connected subset of X which contains both x and y. Show that this is an equivalence relation (i.e., it obeys the reflexive, symmetric, and transitive axioms). Also, show that the equivalence classes of this relation (i.e., the sets of the form

$$\{y \in X : y \sim x\} \quad \text{for some } x \in X$$

are all closed and connected. *(Hint: use Problem 2 and Problem 4)* These sets are known as the *connected components* of X. You can read a note about equivalence relation in the file at NTU cool.

(6) (15 pts) Let $f : S \to T$ be a function from a metric space S to another metric space T. Assume f is uniformly continuous on a subset A of S and that T is complete. Prove that there is a unique extension of f to $\overline{A}$ which is uniformly continuous on $\overline{A}$.

You can do the following problems to practice. You don't have to submit the following problems.

(a) Assume $f : S \to T$ is uniformly continuous on S, where S and T are metric spaces. If $\{x_n\}$ is any Cauchy sequence in S, prove that $\{f(x_n)\}$ is a Cauchy sequence in T.

(b) Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is one-to-one and continuous on $\mathbb{R}^n$. If A is open and disconnected in $\mathbb{R}^n$, prove that $f(A)$ is open and disconnected in $f(\mathbb{R}^n)$.

(c) Let S be an open connected set in $\mathbb{R}^n$. Let T be a connected component of $\mathbb{R}^n - S$. Prove that $\mathbb{R}^n - T$ is connected.

(d) Let (S, d) be a connected metric space which is not bounded. Prove that for every $a \in S$ and every $r > 0$, the set

$$\{x : d(x, a) = r\}$$

is nonempty.