November 3, 2025

Introduction to Mathematical Analysis
Homework 7 Due November 7 (Friday), 2025
Please submit your homework online in PDF format.

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1. (15 pts)

   Assume that (S, d) is a metric space, and let $f_n, f : S \to \mathbb{R}$ be real-valued functions. Suppose that $f_n \to f$ uniformly on S, and there exists a constant $M > 0$ such that

   $$|f_n(x)| \le M \quad \text{for all } x \in S \text{ and all } n.$$

   Let $g : \overline{B(0; M)} \to \mathbb{R}$ be continuous, where

   $$B(0; M) = \{y \in \mathbb{R} : |y| < M\}.$$

   Define

   $$h_n(x) = g(f_n(x)), \qquad h(x) = g(f(x)), \quad x \in S.$$

   Prove that $h_n \to h$ uniformly on S.

2. (15 pts) Let $f_n(x) = x^n$. The sequence $\{f_n\}$ converges pointwise but not uniformly on $[0, 1]$. Let g be continuous on $[0, 1]$ with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on $[0, 1]$.

3. (15 pts) Assume that $g_{n+1}(x) \le g_n(x)$ for each x in T and each $n = 1, 2, \ldots$, and suppose that $g_n \to 0$ uniformly on T. Prove that

   $$\sum (-1)^{n+1} g_n(x)$$

   converges uniformly on T.

4. (15 pts)

   $$f_n(x) = \frac{x}{1 + nx^2}, \quad x \in \mathbb{R},\ n = 1, 2, \ldots$$

   Find the limit function f of the sequence $\{f_n\}$ and the limit function g of the sequence $\{f_n'\}$.

   (a) Prove that $f'(x)$ exists for every x but that $f'(0) \ne g(0)$. For what values of x is $f'(x) = g(x)$?

   (b) In what subintervals of $\mathbb{R}$ does $f_n \to f$ uniformly?

   (c) In what subintervals of $\mathbb{R}$ does $f_n' \to g$ uniformly?

5. (15 pts) Prove that

   $$\sum x^n (1 - x)$$

   converges pointwise but not uniformly on $[0, 1]$, whereas

   $$\sum (-1)^n x^n (1 - x)$$

   converges uniformly on $[0, 1]$. This illustrates that uniform convergence of $\sum f_n(x)$ along with pointwise convergence of $\sum |f_n(x)|$ does not necessarily imply uniform convergence of $\sum |f_n(x)|$.

6. (15 pts) Let

   $$f_n(x) = \frac{1}{n} e^{-n^2 x^2}, \quad x \in \mathbb{R},\ n = 1, 2, \ldots$$

   Prove that $f_n \to 0$ uniformly on $\mathbb{R}$, that $f_n' \to 0$ pointwise on $\mathbb{R}$, but that the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

7. (10 pts) Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on $[0,1]$ and assume that $f_n \to f$ uniformly on $[0,1]$. Prove or disprove

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

You can do the following problems to practice. You don't have to submit the following problems.

1. Prove that the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges uniformly on every half-infinite interval

$$1 + h \le s < +\infty,$$

where $h > 0$. Show that the equation

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

is valid for each $s > 1$, and obtain a similar formula for the kth derivative $\zeta^{(k)}(s)$.

2. If r is the radius of convergence of

$$\sum a_n (x - x_0)^n,$$

where each $a_n \neq 0$, show that

$$\liminf_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \le r \le \limsup_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

3. Prove that the series

$$\sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2}\right)$$

converges pointwise but not uniformly on $[0,1]$.

4. Prove that

$$\sum_{n=1}^{\infty} a_n \sin nx \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \cos nx$$

are uniformly convergent on $\mathbb{R}$ if

$$\sum_{n=1}^{\infty} |a_n|$$

converges.

5. Let $\{a_n\}$ be a decreasing sequence of positive terms. Prove that the series

$$\sum a_n \sin nx$$

converges uniformly on $\mathbb{R}$ if and only if $na_n \to 0$ as $n \to \infty$.