Homework 3

Linear Algebra (II), Spring 2025
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Deadline: 3/12 (Wed.) 12:10

Exercise 1. Let $T : V \to V$ be a linear operator with $\mathrm{ch}_T(x) = (x-\lambda)^n$ and

$$V = \bigoplus_{i=1}^{r} Z(v_i;T)$$

be the decomposition of V into a direct sum of T-cyclic subspaces according to Theorem 3.2. Let $s_i = \dim Z(v_i;T)$ and assume that $s_1 \geq s_2 \geq \cdots \geq s_r$. Determine $\mathrm{m}_T(x)$ and $\dim E_\lambda$. (Express them in terms of r and s_i.)

Solution: Suppose $m_T(x) = (x-\lambda)^d$ and note that

$$V = \ker(ch_T(T)) = \ker(T-\lambda I)^n = K_\lambda$$

so we have $V = K_\lambda$, and we know $m_T(T)(v) = 0$ for all $v \in V$. Since we must have

$$\begin{aligned}(T-\lambda I)^d(v_1) &= 0\\ (T-\lambda I)^d(v_2) &= 0\\ &\vdots\\ (T-\lambda I)^d(v_r) &= 0\end{aligned}$$

so we know $d = \max_{1\leq i\leq r} p_i$ such that p_i is the smallest positive integer with $(T-\lambda I)^{p_i}(v_i) = 0$ since if d is smaller than this number, then there exists some v_i such that $(T-\lambda I)^d(v_i) \neq 0$, and if $d = \max_{1\leq i\leq r} p_i$, then $(T-\lambda I)^d(v_i) = 0$ for all i such that $1 \leq i \leq r$. Now note that in the process of constructing V in Theorem 3.2, we have $(T-\lambda I)^{s_i}(v_i) = 0$ for all i, and since we know

$$B_i = \{\, v_i, (T-\lambda I)(v_i), (T-\lambda I)^2(v_i), \cdots, (T-\lambda I)^{s_i-1}(v_i)\,\}$$

is a basis of $Z(v_i;T)$, so $(T-\lambda I)^{x_i}(v_i) \neq 0$ for all $x_i \leq s_i - 1$, so s_i is the smallest positive integer p_i such that $(T-\lambda I)^{p_i}(v_i) = 0$. That is, $d = \max_{1\leq i\leq r} s_i = s_1$, so $m_T(x) = (x-\lambda)^{s_1}$.
Now we consider the Jordan form of T being

$$T_J = \begin{pmatrix} J_1 & 0 & \cdots & 0 & 0\\ 0 & J_2 & \cdots & 0 & 0\\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & J_r \end{pmatrix}$$

where J_i is a $s_i \times s_i$ Jordan block for all i. Now we claim that for every Jordan block J_i, it has only λ for its eigenvalue and the dimension of its eigenspace is 1. First, since we can find a basis β_i of $Z(v_i;T)$ so that $T|_{Z(v_i;T)}$'s matrix representation with respect to β_i is J_i, and since $Z(v_i;T)$ is a T-invariant space, so $T|_{Z(v_i;T)}$'s characteristic polynomial divides the characteristic polynomial of T, and because T only has λ for its eigenvalue, so $T|_{Z(v_i;T)}$ also only have λ for its eigenvalue. Note that

$$J_i = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1\\ & & & \lambda \end{pmatrix}$$

and if

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix}$$

we must have $a_2 = a_3 = \cdots = a_r = 0$ and a_1 can be any number, so the dimension of the eigenspace of J_i is one. Now suppose we pick some v_i such that v_i is an eigenvector of J_i, then we know $\{ v_i \}$ is a basis of the eigenspace of J_i, and we know

$$\begin{pmatrix} v_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ v_3 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ v_r \end{pmatrix}$$

are all the eigenvectors of T_J, which can be easily verified by block matrix multiplication. Now we call the set consisting of these eigenvectors is called U, and the element of U with v_i embedded in it is called u_i, then we know all elements in U are linearly independent, so $\dim E_\lambda \geq r$. Now we claim that U is a basis of E_λ. First, it is easy to verify $\operatorname{span} U \subseteq E_\lambda$, now we show that $E_\lambda \subseteq \operatorname{span} U$. Suppose

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{pmatrix} \in E_\lambda$$

with each w_i a $s_i \times s_i$ block matrix, so we must have

$$T_J w = \begin{pmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & J_r \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{pmatrix} = \begin{pmatrix} J_1 w_1 \\ J_2 w_2 \\ \vdots \\ J_r w_r \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{pmatrix} = \begin{pmatrix} \lambda w_1 \\ \lambda w_2 \\ \vdots \\ \lambda w_r \end{pmatrix}$$

we can notice that $J_i w_i = \lambda w_i$ for all i, that is, w_i is an eigenvector of J_i for all i, so $w_i = c_i v_i$ for some scalar c_i, so

$$w = \sum_{i=1}^{r} c_i u_i$$

which means $w \in \operatorname{span} U$, so now we know $E_\lambda = \operatorname{span} U$, and thus $\dim E_\lambda = \dim U = r$.

Exercise 2. Let T be a linear operator on a finite-dimensional vector space V with Jordan canonical form

$$\left(\begin{array}{ccc|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right).$$

(a) Determine $\mathrm{ch}_T(x)$ and $\mathrm{m}_T(x)$.

(b) Determine $\dim E_\lambda$ for each eigenvalue λ of T.

(c) For each eigenvalue λ of T, determine the smallest positive integer p for which $K_\lambda = \ker((T - \lambda I)^p)$.

Solution: First note that T has $2, 3$ as its eigenvalues.

(a) Since $V = K_2 \oplus K_3$, and the submatrix with 2 on its diagonal line is with size 5×5, while the submatrix with 3 on its diagonal line has size 2×2, so $ch_T(x) = (x-2)^5(x-3)^2$. Now suppose $m_T(x) = (x-2)^{d_2}(x-3)^{d_3}$, then by Exercise 1 we know d_2 is the largest size of the Jordan block with respect to eigenvalue 2, while d_3 is the largest size of the Jordan block with respect to eigenvalue 3, so $d_2 = 3$ and $d_3 = 1$. Therefore, $m_T(x) = (x-2)^3(x-3)$.

(b) By Exercise 1, we know $\dim E_\lambda$ is the number of Jordan block with respect to eigenvalue λ, so $\dim E_2 = 2$, and $\dim E_3 = 2$.

(c) Claim: For every eigenvalue λ, if $K_\lambda = \bigoplus_{i=1}^r Z(v_i;T)$, where $Z(v_i;T)$ is the T-cyclic subspace generated by v_i(We use the method in Theorem 3.2 to determine v_i), and $s_i = \dim Z(v_i;T)$, and without lose of generacity we suppose $s_1 \geq s_2 \geq \cdots \geq s_r$, now if p is the smallest positive integer such that $K_\lambda = \ker((T-\lambda I)^p)$, then $p = s_1$.
Proof: It is easy to prove that $\ker((T-\lambda I)^s_1) \subseteq K_\lambda$. Now note that

$$B = \bigcup_{i=1}^r \{ v_i, T(v_i), T^2(v_i), \cdots, T^{s_i-1}(v_i) \}$$

is a basis of $\bigoplus_{i=1}^r Z(v_i;T)$, so for all $u \in K_\lambda$, we can write $u = \sum_{i=1}^r \sum_{j=0}^{s_i-1} \alpha_{ij} T^j(v_i)$ for some scalar α_{ij}. Besides, note that $(T-\lambda I)^{s_1}(v_i) = 0$ for all $1 \leq i \leq r$. The reason has been explained in Exercise 1. Therefore, we have

$$(T-\lambda I)^{s_1}\left(\sum_{i=1}^r \sum_{j=0}^{s_i-1} \alpha_{ij} T^j(v_i)\right) = \sum_{i=1}^r \sum_{j=0}^{s_i-1} \alpha_{ij} T^j (T-\lambda I)^{s_1}(v_i) = 0$$

for all α_{ij}. In other words, $K_\lambda \subseteq \ker((T-\lambda I)^{s_1})$, so $K_\lambda = \ker((T-\lambda I)^{s_1})$. Now we have to prove the minimality of s_1. If there is a positive integer k such that $K_\lambda = \ker((T-\lambda I)^{s_1-k})$, then $(T-\lambda I)^{s_1-k}(v_1) = 0$, but since s_1 is the smallest positive integer p such that $(T-\lambda I)^p(v_1) = 0$, so it is impossible to have such k, and we are done.
By this claim, if we say the smallest positive integer p such that $K_\lambda = \ker((T-\lambda I)^p)$ is p_λ, then $p_2 = 3, p_3 = 1$.

Exercise 3. Let

$$A = \begin{pmatrix} 11 & -26 & -11 & 14 & 9 & -7 \\ 4 & -10 & -3 & 4 & 3 & -2 \\ 11 & -22 & -13 & 12 & 9 & -6 \\ 4 & -8 & -4 & 2 & 4 & -2 \\ 2 & -4 & -2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

It is known that $\mathrm{ch}_A(x) = (x+2)^6$.

(a) Determine $\mathrm{m}_A(x)$ and $\dim E_{-2}$.

(b) Determine the Jordan form of A.
Hint: What do the properties proved in Exercise 1 tell you?

Solution:

(a) Suppose $m_A(x) = (x+2)^d$, then by calculation we know 3 is the smallest positive integer p such that $(A+2I)^p = 0$, that is, $d = 3$. Besides, we know $\dim ker(A+2I) = 3$, which can also be done by easy computation.

(b) By (a) and Exercise 1, we know in the Jordan form of A, the biggest Jordan block has the size 3 since $m_A(x) = (x+2)^3$, and the number of Jordan blocks is 3, now since the sum of the size of the Jordan block is 6, which is the size of A, and the size of every Jordan block must be greater than 0, so there are 3 Jordan blocks of the size $3, 2, 1$, respectively, in the Jordan form of A, and since the only eigenvalue of A is -2, so the Jordan form of A is

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Figure 1: Me after doing HW3