Linear Algebra I HW3

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Sec 3.2

Problem. Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Prove that the transformation UT is not invertible. Generalize the theorem.

Proof. Since by rank and nullity theorem, we know

$$3 = \nu(T) + \operatorname{rank}(T),$$

and $\operatorname{rank} T \leq 2$, so $\nu(T) \geq 1$, which means T is not injective. Hence, there exists $a \neq b$ s.t. $T(a) = T(b)$, and thus $UT(a) = UT(b)$, which means UT is not injective. Hence, UT is not invertible. To generalize the theorem, we can say if $m > n$, and suppose $T : V \to W$ and $U : W \to U$ s.t. $\dim V = \dim U = m$ and $\dim W = n$, then UT is not invertible. ∎

Problem. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

Proof. Suppose

$$U(x, y) = (x + y, 2x + 2y) \quad T(x, y) = (y - 2x, 0),$$

which are two linear operators on $\mathbb{R}^2$, then we know

$$TU(x, y) = (0, 0) \quad UT(x, y) = (y - 2x, 2y - 4x).$$

∎

Problem. Let V be a vector space over the field F and T a linear operator on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator T' on $\mathbb{R}^2$ such that $T'^2 = 0$ but $T' \neq 0$.

Proof. If $T^2 = 0$, then $\operatorname{Im} T \subseteq \ker T$. Consider $T'(x, y) = (x + y, -x - y)$, then

$$T'^2(x, y) = T'(x + y, -x - y) = (0, 0) \quad \forall (x, y) \in \mathbb{R}^2.$$

∎

Problem. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when V is not finite-dimensional. (Hint: Let $T = D$, the differentiation operator on the space of polynomial functions.)

Proof. Since for all $a \in V$, we have $T(U(a)) = a$, so T is surjective, and thus

$$\nu(T) + \operatorname{rank}(T) = \dim V$$

gives $\nu T = \dim V - \operatorname{rank} T = \dim V - \dim V = 0$, whcih means T is injective, so T is bijective and thus invertible. Now we claim $U = T^{-1}$, that is, the inverse is unique. Suppose not, then there exists $b \in V$ s.t. $U(b) \neq T^{-1}(b)$, so we have

$$T(U(b)) = b = T(T^{-1}(b)),$$

but this implies T is not injective, which is a contradiction. Now if $T : \mathbb{R} \to \mathbb{R}$ is a linear operator with $T(f) = D(f)$, where $D(f)$ means differentiating f, then in this case $V = \mathbb{R}[x]$, which is not finite dimensional. Note that we can pick $U : \mathbb{R}[x] \to \mathbb{R}[x]$ by $U(f) = \int f \, \mathrm{d}x$, which is the anti-derivative of f with constant term equal 0, then we know $TU = I$. However, $T(x + 1) = T(x) = 1$, so T is not injective and thus cannot be invertible. Also, notice that there are infinitely many U' s.t. $TU' = I$ since we can let $U' = U + C$ for any constant C. Hence, this statement is not true for infinite-dimensional V. ∎

Sec 3.3

Problem. Let W be the set of all 2×2 complex Hermitian matrices, that is, the set of 2×2 complex matrices A such that $A_{ij} = \overline{A_{ji}}$ (the bar denoting complex conjugation). As we pointed out in Example 6 of Chapter 2, W is a vector space over the field of real numbers, under the usual operations. Verify that the map

$$(x, y, z, t) \longmapsto \begin{bmatrix} t + x & y + iz \\ y - iz & t - x \end{bmatrix}$$

is an isomorphism of $\mathbb{R}^4$ onto W.

Proof. We first show this map is linear. Suppose this map is called T, then we know for all $\alpha \in \mathbb{R}$,

$$T\left(\alpha(x, y, z, t) + (x', y', z', t')\right) = \begin{bmatrix} \alpha t + t' + \alpha x + x' & \alpha y + y' + i(\alpha z + z') \\ \alpha y + y' - i(\alpha z + z') & \alpha t + t' - (\alpha x + x') \end{bmatrix},$$

which is equal to

$$\begin{bmatrix} \alpha t + \alpha x & \alpha y + i\alpha z \\ \alpha y - i\alpha z & \alpha t - \alpha x \end{bmatrix} + \begin{bmatrix} t' + x' & y' + iz' \\ y' - iz' & t' - x' \end{bmatrix} = \alpha T(x, y, z, t) + T\left(x', y', z', t'\right).$$

Hence, T is linear. Now we show that it is bijective. Note that $\dim \mathbb{R}^4 = \dim W = 4$, so we just need to check T is injective. If there exists $(x, y, z, t) \neq (x', y', z', t')$ s.t. $T(x, y, z, t) = T(x', y', z', t')$, then

$$\begin{cases} t + x = t' + x' \\ y + iz = y' + iz' \\ y - iz = y' - iz' \\ t - x = t' - x' \end{cases},$$

and then by adding up the first equation and the last one, we will get $t = t'$ and thus $x = x'$, while adding up the second equation and the third one we have $y = y'$ and $z = z'$, so we know $(x, y, z, t) = (x', y', z', t')$, which means T is injective, and we're done. ■