Combinatorics I (Math 7701) Fall 2025
Shagnik Das TA: Lu Yun-Chi

Exercise Sheet 3

Due date: 15:30, Oct 21st, to be submitted on COOL.

Working with your partner, you should try to solve all of the exercises below. You should then submit solutions to four of the problems, with each of you writing two, clearly indicating the author of each solution. Note that each problem is worth 10 points, and starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting on COOL, please send your solutions by e-mail.

Exercise 1 In this exercise you will practice building generating functions and decoding their sequences.

(a) Determine a closed form for the generating functions, $a(x)$, of the following sequences.

(i) $a_n = n^3$ for all $n \geq 0$.

(ii) $a_n = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 2^n + 3^{n/2} & \text{if } n \text{ is divisible by 4} \\ 2^n - 3^{n/2} & \text{if } n \text{ is even, but not divisible by 4} \end{cases}$.

(b) Given the following generating functions, find a closed form for the nth term, a_n, of the corresponding sequences.

(I) $a(x) = -\log(1 - 3x^2)$

(II) $a(x) = \cos(x^2)$

Solution: (黃子恆) See last few pages.

Exercise 2 We have a corridor that is 3 metres wide and n metres long, and we want to cover the floor entirely with carpets. We have n identical carpets, measuring 3×1 in metres, each of which can be placed horizontally or vertically. Let c_n denote the number of different ways of covering the corridor.

(a) Find a recurrence relation for c_n, and provide sufficient initial conditions (starting from c_0) to compute the sequence.

(b) Find a closed form for the generating function $c(x) = \sum_{n \geq 0} c_n x^n$.

(c) Derive a closed formula for c_n.

Solution: (黃子恆) See last few pages.

Exercise 3 Using the definitions of the derivatives and products of formal power series, show that Leibniz's Rule (otherwise known as the product rule) also holds for formal power series. That is,

$$(F(x) \cdot G(x))' = F'(x) \cdot G(x) + F(x) \cdot G'(x).$$

Solution: (張沂魁) Suppose $F(x) = \sum_{n=0}^{\infty} f_n x^n$ and $G(x) = \sum_{n=0}^{\infty} g_n x^n$, then we know

$$F(x) \cdot G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k g_{n-k} \right) x^n.$$

Thus,

$$(F(x) \cdot G(x))' = \sum_{n=1}^{\infty} n \left(\sum_{k=0}^{n} f_k g_{n-k} \right) x^{n-1} = \sum_{n=0}^{\infty} (n+1) \left(\sum_{k=0}^{n+1} f_k g_{n+1-k} \right) x^n.$$

Also, since we know

$$F'(x) = \sum_{n=0}^{\infty} (n+1) f_{n+1} x^n \text{ and } \quad G'(x) = \sum_{n=0}^{\infty} (n+1) g_{n+1} x^n,$$

so

$$\begin{cases} F'(x) \cdot G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (k+1) f_{k+1} g_{n-k} \right) x^n \\ F(x) \cdot G'(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k (n-k+1) g_{n-k+1} \right) x^n. \end{cases}$$

Thus, we have

$$\begin{aligned} F'(x) \cdot G(x) + F(x) \cdot G'(x) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (k+1) f_{k+1} g_{n-k} + \sum_{k=0}^{n} (n-k+1) f_k g_{n-k+1} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{u=1}^{n+1} u f_u g_{n-u+1} + \sum_{k=0}^{n} (n-k+1) f_k g_{n-k+1} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{u=0}^{n+1} u f_u g_{n-u+1} + \sum_{k=0}^{n+1} (n-k+1) f_k g_{n-k+1} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n+1} (k+n-k+1) f_k g_{n-k+1} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n+1} (n+1) f_k g_{n-k+1} \right) x^n = (F(x) \cdot G(x))'. \end{aligned}$$

Exercise 4 One of the reasons that the Catalan numbers are so loved by combinators is that they pop up all over the place.[1] Show that the following sequences are equal to the Catalan sequence $(c_n)_{n\geq 0}$.

(a) $(b_n)_{n\geq 0}$, where b_n is the number of rooted full binary trees with $n+1$ leaves. A rooted full binary tree starts from a root node, and every node either has two descendents (a left child and a right child), or none. If a node has no descendents, it is called a leaf. See Figure **??** for the case $n = 3$.

(b) $(t_n)_{n\geq 0}$, where t_n is the number of triangulations of a convex $(n+2)$-gon; that is, the number of ways a convex polygon with $n+2$ sides can be cut into triangles by connecting its vertices with straight non-crossing lines.[2] See Figure **??** for the case $n = 4$.

(c) $(p_n)_{n\geq 0}$, where p_n is the number of permutations in S_n that do not contain an increasing subsequence of length 3; that is, the number of bijections $\pi : [n] \to [n]$ such that there are no $1 \leq i < j < k \leq n$ with $\pi(i) < \pi(j) < \pi(k)$.

Exercise 5 A diagonal lattice path is a path in the grid $\mathbb{Z}^2$ with steps of the form $(1,1)$ or $(1,-1)$. Recall that the Catalan number c_n counts the number of diagonal lattice paths of length $2n$ from $(0,0)$ to $(2n,0)$ that never go below the x-axis. In this exercise you will give an alternative proof to show $c_n = \frac{1}{n+1}\binom{2n}{n}$.

(a) Determine the total number of diagonal lattice paths from $(0,0)$ to $(2n,k)$ for any integer $k \in \mathbb{Z}$.

(b) Call a diagonal lattice path from $(0,0)$ to $(2n,0)$ *illegal* if it goes below the x-axis. Show there is a bijection between these illegal lattice paths and diagonal lattice paths from $(0,0)$ to $(2n,-2)$.

(c) Deduce that the number of Dyck paths of length $2n$ is $\frac{1}{n+1}\binom{2n}{n}$.

Solution: (張沂魁)

(a) If we want to go from $(0,0)$ to $(2n,k)$, then we must have

$$\begin{cases} (\#\text{ of }(1,1)) - (\#\text{ of }(1,-1)) = k \\ (\#\text{ of }(1,1)) + (\#\text{ of }(1,-1)) = 2n \end{cases},$$

[1] Indeed, Richard Stanley's *Enumerative Combinatorics: Volume 2* famously has a set of exercises with no fewer than 66 different Catalan structures!

[2] Note that since there are no convex polygons with 2 sides, we take $t_0 = 1$, since we have nothing to do, and there is one way of doing nothing.

so we know # of $(1,1)$ is $n+\frac{k}{2}$ and # of $(1,-1)$ is $n-\frac{k}{2}$. Hence, suppose the total number of diagonal lattice paths from $(0,0)$ to $(2n,k)$ is $a_{n,k}$, then

$$a_{n,k} = \begin{cases} 0, & \text{if } 2 \nmid k; \\ \begin{pmatrix} 2n \\ n+\frac{k}{2} \end{pmatrix}, & \text{if } 2 \mid k. \end{cases}$$

since any permutation of $n+\frac{k}{2}$ $(1,1)$'s and $n-\frac{k}{2}$ $(1,-1)$'s form a diagonal lattice paths from $(0,0)$ to $(2n,k)$.

(b) Suppose a illegal path P from $(0,0)$ to $(2n,0)$ is illegal, then we suppose P reaches from $(m,0)$ to $(m+1,-1)$ for some $m \in \mathbb{N}$ where this is the first time P intersects $y=-1$. Now suppose the subpath of P from $(m+1,-1)$ to $(2n,0)$ is called P_2, then $P = P_1 + P_2$ where P_1 is the subpath from $(0,0)$ to $(m+1,-1)$. Note that the number of $(1,1)$ step in P_2 is more than the number of $(1,-1)$ step in P_2 by 1 since P_2 goes from $y=-1$ to $y=0$. Hence, if we reflect P_2 with respect to $y=-1$, then P_2 goes from $y=-1$ to $y=-2$, then we define a map that sends any illegal P to this P_2-reflected map. Now we claim that this map is a bijection between the set of illegal diagonal lattice path from $(0,0)$ to $(2n,0)$ and the set of diagonal lattice paths from $(0,0)$ to $(2n,-2)$. We first show that this map is an injection. If illegal P, P' is mapped to same paths, then P, P' have the same P_1 and P_2 part, but P_1 part is the same as it is in the P, P', so P, P' share same part from $(0,0)$ to $(m+1,-1)$. Also, P, P' share same P_2 part after the map, so we can reflect P_2 back, and thus P, P' have same part from $(m+1,-1)$ to $(2n,0)$. Thus, this map is an injection. Also, this map is a surjection since for any diagonal lattice path from $(0,0)$ to $(2n,-2)$, there must be a first intersection with $y=-1$, and thus we can reflect the part after this intersection to get an illegal diagonal lattice path from $(0,0)$ to $(2n,0)$, and obviously this path is in the preimage of the original given path which goes from $(0,0)$ to $(2n,-2)$. Hence, this map is a bijection.

(c) Since the number of diagonal lattice path from $(0,0)$ to $(2n,0)$ is $\binom{2n}{n}$ (any permutation of n $(1,1)$'s and n $(1,-1)$'s gives a choice), and by (b) we know the number of illegal lattice path from $(0,0)$ to $(2n,0)$ is the number of diagonal lattice paths from $(0,0)$ to $(2n,-2)$, which is $\binom{2n}{n-1}$ by (a), so

$$\begin{aligned}
\#\text{ of Dyck paths of length } 2n &= \binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \\
&= \frac{(2n)!}{n!n!} - \frac{(2n)!n}{n!n!(n+1)} \\
&= \frac{(2n)!}{n!n!} - \frac{n}{n+1}\frac{(2n)!}{n!n!} \\
&= \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

1. (a)(i) Let $A(x) = \sum_{n=0}^{\infty} n^3 x^n$ and $B(x) = \sum_{n=0}^{\infty} x^n$.

$\Rightarrow B'(x) = \sum_{n=1}^{\infty} n x^{n-1} \Rightarrow xB'(x) = \sum_{n=0}^{\infty} n x^n$

$\Rightarrow (xB'(x))' = \sum_{n=1}^{\infty} n^2 x^{n-1} \Rightarrow x(xB'(x))' = \sum_{n=0}^{\infty} n^2 x^n$

$\Rightarrow (x(xB'(x))')' = \sum_{n=1}^{\infty} n^3 x^{n-1} \Rightarrow x(x(xB'(x))')' = \sum_{n=0}^{\infty} n^3 x^n = A(x)$

We know that $B(x) = \frac{1}{1-x}$, so

$$A(x) = x(x(xB'(x))')' = x\left(x\left(\frac{x}{(1-x)^2}\right)'\right)'$$

$$= x \times \left(x \times \frac{(1-x)^2 - x \times 2 \times (1-x) \times (-1)}{(1-x)^4}\right)' = x \cdot \left(\frac{x+x^2}{(1-x)^3}\right)'$$

$$= x \times \frac{(1-x)^2(1+4x+x^2)}{(1-x)^6} = \frac{x+4x^2+x^3}{(1-x)^4}.$$

(ii) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n + (3^{0/2}x^0 + 3^{4/2}x^4 + 3^{8/2}x^8 + \dots) - (3^{2/2}x^2 + 3^{6/2}x^6 + 3^{10/2}x^{10} + \dots)$

We know that $\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$. Now, we try to compute other two series. Let $P = 3^{0/2}x^0 + 3^{4/2}x^4 + 3^{8/2}x^8 + \dots$ and $Q = 3^{2/2}x^2 + 3^{6/2}x^6 + 3^{10/2}x^{10} + \dots$. Then, we have

$$\begin{aligned} P &= 1 + 3^2x^4 + 3^4x^8 + \dots \\ -)\ 9x^4 P &= \phantom{1 + {}} 3^2x^4 + 3^4x^8 + \dots \\ \hline (1-9x^4)P &= 1 \Rightarrow P = \frac{1}{1-9x^4} \end{aligned}$$

and

$$\begin{aligned} Q &= 3x^2 + 3^3x^6 + 3^5x^{10} + \dots \\ -)\ 9x^4 Q &= \phantom{3x^2 + {}} 3^3x^6 + 3^5x^{10} + \dots \\ \hline (1-9x^4)Q &= 3x^2 \Rightarrow Q = \frac{3x^2}{1-9x^4} \end{aligned}$$

In conclusion,

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-2x} + \frac{1}{1-9x^4} - \frac{3x^2}{1-9x^4} = \frac{1}{1-2x} + \frac{1}{1+3x^2}$$

(b) By Taylor series, we have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for any function f.
Also, by Taylor series, we have $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

(I) $\ln(1-y) = -\sum_{n=1}^{\infty} \frac{y^n}{n}$. Let $y = 3x^2$, then we have

$$-\ln(1-3x^2) = \sum_{n=1}^{\infty} \frac{3^n x^{2n}}{n} \Rightarrow a_n = \begin{cases} \frac{3^{n/2}}{n/2} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}.$$

(II) $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$. Let $z = x^2$, then we have
$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} \Rightarrow a_n = \begin{cases} \frac{(-1)^{n/4}}{(n/2)!} & \text{if } 4 \mid n \\ 0 & \text{otherwise} \end{cases}$.

2. (a) Let we number every "block" in $(3 \times n)$ corridor as below:

n metres long

3 metres wide

$a_{1,1}$	$a_{2,1}$	$a_{3,1}$		$a_{(n-1),1}$	$a_{n,1}$
$a_{1,2}$	$a_{2,2}$	$a_{3,2}$	$\cdots$	$a_{(n-1),2}$	$a_{n,2}$
$a_{1,3}$	$a_{2,3}$	$a_{3,2}$		$a_{(n-1),3}$	$a_{n,3}$

To cover $a_{n,1}$, there has only two ways:
Method 1. Put the carpet horizontally to cover $a_{(n-2),1}$, $a_{(n-1),1}$, and $a_{n,1}$ at the same time.
Method 2. Put the carpet vertically to cover $a_{n,1}$, $a_{n,2}$, and $a_{n,3}$ at the same time.
For the Method 1., we must need to put two carpets to cover $a_{(n-2),2}$, $a_{(n-1),2}$, $a_{n,2}$ and $a_{(n-2),3}$, $a_{(n-1),3}$, $a_{n,3}$, respectively. Now, the corridor without covering by carpet is 3 metres wide and $(n-3)$ metres long. By definition, we have C_{n-3} ways to cover it.
Also, for the Method 2., we have C_{n-1} ways to cover the corridor without carpet.
Note that Method 1. and Method 2. are disjoint.
By sum rule, $c_n = c_{n-1} + c_{n-3}$ for $n \geq 3$, and we have $c_0 = c_1 = c_2 = 1$ as the initial conditions.

(b) $(1 - x - x^3)\, C(x) = C(x) - xC(x) - x^3 C(x)$
$= C_0 + (C_1 - C_0)x + (C_2 - C_1)x^2 + \sum_{n=3}^{\infty} (C_n - C_{n-1} - C_{n-3}) x^n = 1$.
$\Rightarrow c(x) = \frac{1}{1 - x - x^3}$.

(c) The characteristic polynomial is $p(z) = z^3 - z^2 - 1$.

Consider $p(z) = 0$. Let $z = y + \frac{1}{3}$, then we have

$$\left(y+\tfrac{1}{3}\right)^3 - \left(y+\tfrac{1}{3}\right)^2 - 1 = y^3 - \tfrac{1}{3}y - \tfrac{29}{27} = 0.$$

By Cardano's formula, we have

$$u = \sqrt[3]{\frac{\left(\frac{29}{27}\right)}{2} + \sqrt{\left(\frac{\left(\frac{29}{27}\right)}{2}\right)^2 - \left(\frac{\left(\frac{1}{3}\right)}{3}\right)^3}} = \sqrt[3]{\frac{29}{54} + \sqrt{\frac{31}{108}}}$$

and

$$v = \sqrt[3]{\frac{\left(\frac{29}{27}\right)}{2} - \sqrt{\left(\frac{\left(\frac{29}{27}\right)}{2}\right)^2 - \left(\frac{\left(\frac{1}{3}\right)}{3}\right)^3}} = \sqrt[3]{\frac{29}{54} - \sqrt{\frac{31}{108}}}$$

s.t. the three roots of y is $u+v$, $\omega u + \omega^2 v$, $\omega^2 u + \omega v$

where $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

Hence, $z_1 = \frac{1}{3} + u + v$, $z_2 = \frac{1}{3} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)u - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)v$,

and $z_3 = \frac{1}{3} - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)u + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)v$ is roots of $p(z)$.

So, $c_n = Az_1^n + Bz_2^n + Cz_3^n$ for some constants A, B, C that satisfy $1 = A + B + C$, $1 = Az_1 + Bz_2 + Cz_3$, and $1 = Az_1^2 + Bz_2^2 + Cz_2^2$.

By cramer's rule, since we have

$$\det\begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{pmatrix} = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1),$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & z_2 & z_3 \\ 1 & z_2^2 & z_3^2 \end{pmatrix} = \begin{vmatrix} z_2 & z_3 \\ z_2^2 & z_3^2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ z_2^2 & z_3^2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ z_2 & z_3 \end{vmatrix}$$

$$= z_2 z_3 (z_3 - z_2) - (z_3 + z_2)(z_3 - z_2) + (z_3 - z_2)$$

$$= (z_3 - z_2)(z_2 - 1)(z_3 - 1),$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ z_1 & 1 & z_3 \\ z_1^2 & 1 & z_3^2 \end{pmatrix} = (z_1 - z_3)(z_1 - 1)(z_3 - 1), \text{ and } \det\begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & 1 \\ z_1^2 & z_2^2 & 1 \end{pmatrix} = (z_2 - z_1)(z_2 - 1)(z_1 - 1),$$

$$A = -\frac{(z_2 - 1)(z_3 - 1)}{(z_1 - z_2)(z_3 - z_1)}, \quad B = -\frac{(z_1 - 1)(z_3 - 1)}{(z_1 - z_2)(z_2 - z_3)}, \text{ and } C = -\frac{(z_1 - 1)(z_2 - 1)}{(z_2 - z_3)(z_3 - z_1)}.$$