Introduction to Mathematical Analysis
Homework 12 Due December 12 (Friday), 2025
Please submit your homework online in PDF format.

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1. (20 pts) **Exercise 5.4.1** Show that if $f : \mathbb{R} \to \mathbb{C}$ is both compactly supported and $\mathbb{Z}$–periodic, then it is identically zero.

   Hint: A function $f : \mathbb{R} \to \mathbb{C}$ is said to be *compactly supported* if the set

   $$\operatorname{supp}(f) := \overline{\{\, x \in \mathbb{R} : f(x) \neq 0 \,\}}$$

   is a compact subset of $\mathbb{R}$. Equivalently, f is compactly supported if there exists a bounded closed interval $[a, b] \subset \mathbb{R}$ such that

   $$f(x) = 0 \qquad \text{whenever } x \notin [a, b].$$

2. (20 pts) (Exercise 5.5.1) Let f be a function in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, and define the *trigonometric Fourier coefficients* a_n, b_n for $n = 0, 1, 2, \ldots$ by

   $$a_n := 2 \int_0^1 f(x) \cos(2\pi n x)\, dx, \qquad b_n := 2 \int_0^1 f(x) \sin(2\pi n x)\, dx.$$

   (a) Show that the series

   $$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \big)$$

   converges to f in the L^2–metric.

   (b) Show that if $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ are absolutely convergent, then the above series actually converges *uniformly* to f (and not just in L^2).

3. (20 pts) (Exercise 5.5.2) Let $f(x)$ be the function defined by $f(x) = (1 - 2x)^2$ when $x \in [0, 1]$, and extended to be $\mathbf{Z}$–periodic on $\mathbf{R}$.

   (a) Using Exercise 5.5.1, show that the series

   $$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x)$$

   converges uniformly to f. (You may use the fact that

   $$\int_0^1 x\, e^{-2\pi i n x}\, dx = -\frac{1}{2\pi i n}, \qquad (n \neq 0),$$

   $$\int_0^1 x^2\, e^{-2\pi i n x}\, dx = -\frac{1}{2\pi i n} + \frac{2}{(2\pi n)^2}, \qquad (n \neq 0).$$

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   (b) Conclude that

   $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

   (c) Conclude that

   $$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

   (Hint: expand the cosines in terms of exponentials and use Plancherel's theorem.)

4. (20 pts) (Exercise 5.5.3) If $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ and P is a trigonometric polynomial, show that

$$\widehat{f * P}(n) = \widehat{f}(n)\, c_n = \widehat{f}(n)\, \widehat{P}(n)$$

for all integers n, where c_n are the Fourier coefficients of P. More generally, if $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, show that

$$\widehat{f * g}(n) = \widehat{f}(n)\, \widehat{g}(n) \quad \text{for all } n \in \mathbf{Z}.$$

5. (20 pts) (Exercise 5.5.4) Let $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ be differentiable, and assume its derivative f' is also continuous. Show that

$$\sum_{n=-\infty}^{\infty} |n\, \widehat{f}(n)|^2 < \infty$$

and that the Fourier coefficients of f' satisfy

$$\widehat{f'}(n) = 2\pi i n\, \widehat{f}(n) \qquad \text{for all } n \in \mathbf{Z}.$$

You can do the following problem for practice. You don't have to turn in the following problems.

1. (5.5.5) Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Prove the Parseval identity

$$\Re \int_0^1 f(x)\overline{g(x)}\, dx = \Re \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \overline{\widehat{g}(n)}.$$

(Hint: apply the Plancherel theorem to $f+g$ and $f-g$, and subtract the two.) Then conclude that the real parts can be removed, i.e.

$$\int_0^1 f(x)\overline{g(x)}\, dx = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \overline{\widehat{g}(n)}.$$

(Hint: apply the first identity with f replaced by if.)

2. (5.5.6) In this exercise we develop Fourier series for functions of an arbitrary period $L > 0$.

Let $L > 0$ and let $f : \mathbb{R} \to \mathbb{C}$ be a continuous L-periodic function. For each integer n define

$$c_n := \frac{1}{L} \int_0^L f(x)\, e^{-2\pi i n x/L}\, dx.$$

(a) Show that the series

$$\sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n x/L}$$

converges to f in L^2-metric. More precisely, prove that

$$\lim_{N\to\infty} \int_0^L \left| f(x) - \sum_{n=-N}^{N} c_n\, e^{2\pi i n x/L} \right|^2 dx = 0.$$

(Hint: apply the Fourier theorem to the function $f(Lx)$.)

(b) If the series $\sum_{n=-\infty}^{\infty} |c_n|$ is absolutely convergent, show that

$$\sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n x/L}$$

converges *uniformly* to f.

(c) Show that

$$\frac{1}{L} \int_0^L |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

(Hint: apply the Plancherel theorem to the function $f(Lx)$.)