Introduction to Analysis I HW3

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September 26, 2025

Problem 0.0.1. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set

$$S = \{x^{(n)} : n \geq m\}.$$

Is the converse true?

Proof. Suppose L is a limit point of the sequence. By definition,

$$\forall N \geq m, \forall \varepsilon > 0, \ \exists n \geq N \text{ such that } d(x^{(n)}, L) \leq \varepsilon.$$

This implies that for every $\varepsilon > 0$, there exists $n \geq m$ with

$$x^{(n)} \in B(L, \varepsilon) \cap S \neq \varnothing.$$

Hence,

$$\forall \varepsilon > 0, \ B(L, \varepsilon) \cap S \neq \varnothing \quad \Rightarrow \quad L \text{ is an adherent point of } S.$$

Now, we check the converse. The converse statement is **NOT** true. Consider $X = \mathbb{R}$ with the standard metric, and let $m = 1$. Define

$$x^{(n)} = \frac{1}{n}, \quad \text{so } S = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\}.$$

It is clear that 1 is an adherent point of S, since for every $\varepsilon > 0$,

$$1 \in (1 - \varepsilon, 1 + \varepsilon), \quad \Rightarrow \quad B(1, \varepsilon) \cap S \neq \varnothing.$$

However, 1 is not a limit point of the sequence. Indeed, if $N \geq 2$, then for all $n \geq N$,

$$d(x^{(n)}, 1) = \left| \frac{1}{n} - 1 \right| \geq \frac{1}{2}.$$

So if we take $\varepsilon = 0.48763$ and $N = 2$, there is no $n \geq N$ such that $d(x^{(n)}, 1) \leq \varepsilon$. Therefore, 1 is not a limit point of $(x^{(n)})_{n=1}^{\infty}$, even though it is an adherent point of S. ■

Problem 0.0.2. The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

(a) Given any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, we introduce the *formal limit*

$$\mathrm{LIM}_{n\to\infty} x_n.$$

We say that two formal limits $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Show that this equality relation obeys the reflexive, symmetry, and transitive axioms, i.e. that it is an equivalence relation.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, modulo the above equivalence relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to [0, \infty)$ by

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) := \lim_{n\to\infty} d(x_n, y_n).$$

Show that this function is well-defined (the limit exists and does not depend on the choice of representatives) and that it satisfies the axioms of a metric. Thus $(\overline{X}, d_{\overline{X}})$ is a metric space.

(c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.

(d) We identify an element $x \in X$ with the corresponding constant Cauchy sequence $(x, x, x, \dots)$, i.e. with the formal limit $\mathrm{LIM}_{n\to\infty} x$. Show that this is legitimate: for $x, y \in X$,

$$x = y \iff \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y.$$

With this identification, show that

$$d(x, y) = d_{\overline{X}}(x, y),$$

and thus (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) Show that the closure of X in $\overline{X}$ is $\overline{X}$ itself. (This explains the choice of notation.)

(f) Finally, show that the formal limit agrees with the actual limit: if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X that converges in X, then

$$\lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n \quad \text{in } \overline{X}.$$

a. We verify the following properties:

- Reflexive: $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} x_n$ are equal since d is metric, so $\forall n, d(x_n, x_n) = 0$.
- Symmetry: If $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal, this mean $\lim_{n\to\infty} d(x_n, y_n) = 0$. And since d is metric, so $\lim_{n\to\infty} d(y_n, x_n) = 0$, hence $\mathrm{LIM}_{n\to\infty} y_n$ and $\mathrm{LIM}_{n\to\infty} x_n$ are equal.
- Transitive: If $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal and $\mathrm{LIM}_{n\to\infty} y_n$ and $\mathrm{LIM}_{n\to\infty} z_n$ are equal, then we have $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, z_n) = 0$. By definition, there exists $N_1, N_2 > 0$ s.t. for all $n \geq N_1$, we have $d(x_n, y_n) < \frac{\varepsilon}{2}$ and for all $n \geq N_2$ we have $d(y_n, z_n) < \frac{\varepsilon}{2}$. Thus, for all $n \geq \max\{N_1, N_2\}$, we have

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

  which means $\lim_{n\to\infty} d(x_n, z_n) = 0$, and thus $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} z_n$.

■

b. We first show that the limit exists. Note that $\lim_{n\to\infty} d(x_n, y_n) \in \mathbb{R}_{\geq 0}$ for all Cauchy sequence $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ in X. We already know $(\mathbb{R}, |\cdot|)$ is complete, so we know $(\mathbb{R}_{\geq 0}, |\cdot|)$ is also complete as it is a closed subspace of $(\mathbb{R}, |\cdot|)$. Now we define $u_n := d(x_n, y_n)$ for all $n \geq 1$, then we want to show that $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}_{\geq 0}$, and then we can conclude that $\{u_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}_{\geq 0}$, and thus $\lim_{n\to\infty} d(x_n, y_n)$ exists.

Claim 0.0.1. Suppose (X, d) is a metric space, then for all $a, b, c, d \in X$ we have

$$|d(a, b) - d(c, d)| \leq d(a, c) + d(b, d)$$

Proof. Since

$$\begin{cases} d(a, b) \leq d(a, c) + d(c, b) \leq d(a, c) + d(c, d) + d(d, b) \\ d(c, d) \leq d(c, a) + d(a, d) \leq d(c, a) + d(a, b) + d(b, d), \end{cases}$$

so we have

$$\begin{cases} d(a, b) - d(c, d) \leq d(a, c) + d(d, b) \\ -d(c, a) - d(b, d) \leq d(a, b) - d(c, d), \end{cases}$$

so we can conbine these two equations and get the result. ⊛

By Claim 0.0.1, we know for all $p, q \geq 1$, we have

$$|u_p - u_q| = |d(x_p, y_p) - d(x_q, y_q)| \leq d(x_p, x_q) + d(y_p, y_q).$$

Now since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy, so for every $\varepsilon > 0$, there exists $N_1, N_2 > 0$ s.t.

$$\begin{cases} d(x_p, x_q) < \dfrac{\varepsilon}{2} & \forall p, q \geq N_1 \\ d(y_p, y_q) < \dfrac{\varepsilon}{2} & \forall p, q \geq N_2. \end{cases}$$

Thus, for all $p, q \geq \max\{N_1, N_2\}$, we know

$$|u_p - u_q| \leq d(x_p, x_q) + d(y_p, y_q) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we know $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}_{\geq 0}, |\cdot|$.

Now we show that $d_{\overline{X}}$ is well-defined. In other words, if $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} z_n$, then we want to show

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) = d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} z_n, \mathrm{LIM}_{n\to\infty} y_n) \quad \forall \text{ Cauchy } \{y_n\}_{n=1}^{\infty} \text{ in } (X, d).$$

Equivalently, we want to show $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(z_n, y_n)$. Note that we have

$$\lim_{n\to\infty} d(x_n, z_n) = 0 \text{ and } d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n),$$

so we know

$$\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(z_n, y_n).$$

Also, we have $d(z_n, y_n) \leq d(z_n, x_n) + d(x_n, y_n)$, so we know

$$\lim_{n\to\infty} d(z_n, y_n) \leq \lim_{n\to\infty} d(z_n, x_n) + \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n, y_n),$$

and thus we can conclude that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(z_n, y_n)$.

Finally, we want to show that $\left(\overline{X}, d_{\overline{X}}\right)$ is a metric space.

- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$, we have

  $$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}, x_n, \mathrm{LIM}_{n\to\infty}, y_n) = \lim_{n\to\infty} d(x_n, y_n) \geq 0$$

  since d is a metric.
- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty} \in X$, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} x_n) = \lim_{n\to\infty} d(x_n, x_n) = 0$.
- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \in X$,

  $$\begin{aligned} d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) = \lim_{n\to\infty} d(x_n, y_n) &= \lim_{n\to\infty} d(y_n, x_n) \\ &= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} y_n, \mathrm{LIM}_{n\to\infty} x_n) \end{aligned}$$

- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \in X$,

  $$\begin{aligned} d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} z_n) &= \lim_{n\to\infty} d(x_n, z_n) \\ &\leq \lim_{n\to\infty} (d(x_n, y_n) + d(y_n, z_n)) = \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\ &= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) + d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} y_n, \mathrm{LIM}_{n\to\infty} z_n). \end{aligned}$$

Hence, we know $\left(\overline{X}, d_{\overline{X}}\right)$ is a metric space. ■

c. We want to show that for all $\{u_n\}_{n=1}^{\infty} \subseteq \overline{X}$, there exists $\{z_n\}_{n=1}^{\infty} \subseteq X$ s.t. $\lim_{n\to\infty} u_n = \mathrm{LIM}_{n\to\infty} z_n$. Since $\{u_n\}_{n=1}^{\infty}$ is a sequence of formal limit of Cauchy sequences in X, so we can define $u_k = \mathrm{LIM}_{n\to\infty} x_n^{(k)}$ for all $k \geq 1$. Now we construct $\{z_n\}_{n=1}^{\infty}$. Since we know for all $k \geq 1$, $\left\{x_n^{(k)}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in X, so for all $k \geq 1$, there exists $N_k > 0$ s.t. $n \geq N_k$ implies

$d\left(x_n^{(k)}, x_{N_k}^{(k)}\right) < \frac{1}{k}$. Now we let $z_k = x_{N_k}^{(k)}$ for all $k \geq 1$.

Claim 0.0.2. $\{z_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X.

Proof. For all $\varepsilon > 0$, we know there exists $K \geq 0$ s.t. $\frac{1}{K} < \frac{\varepsilon}{3}$. Also, since $\{u_n\}_{n=1}^{\infty}$ is Cauchy, so there exists $N > 0$ s.t. $i, j \geq N$ implies $d_{\overline{X}}(u_i, u_j) < \frac{\varepsilon}{3}$, which can be writen as $\lim_{n\to\infty} d\left(x_n^{(i)}, x_n^{(j)}\right) < \frac{\varepsilon}{3}$. To be more precise, there exists $N > 0$ and $N' > 0$ s.t. if $i, j \geq N$ and $n \geq N'$, then $d\left(x_n^{(i)}, x_n^{(j)}\right) < \frac{\varepsilon}{3}$. Now for all $p, q \geq \max\{N, K\}$ and $n \geq \max\{N_p, N_q, N'\}$, we have

$$\begin{aligned} d(z_p, z_q) = d\left(x_{N_p}^{(p)}, x_{N_q}^{(q)}\right) &\leq d\left(x_{N_p}^{(p)}, x_n^{(p)}\right) + d\left(x_n^{(p)}, x_{N_q}^{(q)}\right) \\ &\leq d\left(x_{N_p}^{(p)}, x_n^{(p)}\right) + d\left(x_n^{(p)}, x_n^{(q)}\right) + d\left(x_n^{(q)}, x_{N_q}^{(q)}\right) \\ &< \frac{1}{p} + \frac{\varepsilon}{3} + \frac{1}{q} < \frac{1}{K} + \frac{\varepsilon}{3} + \frac{1}{K} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, we know $\{z_k\}_{k=1}^{\infty}$ is Cauchy. ⊛

Claim 0.0.3. $\lim_{n\to\infty} u_n = \mathrm{LIM}_{n\to\infty} z_n$.

Proof. Suppose $L = \mathrm{LIM}_{n\to\infty} z_n$. For all $\varepsilon > 0$, we want to show there exists $N > 0$ s.t. $m \geq N$ implies $d_{\overline{X}}(u_m, L) < \varepsilon$, which is equivalent to $\lim_{n\to\infty} d\left(x_n^{(m)}, z_n\right) < \varepsilon$. To be more precise, we want to show there exists $N \geq 0$ and $N' > 0$ s.t. if $m \geq N$ and $n \geq N'$, then $d\left(x_n^{(m)}, z_n\right) < \varepsilon$. Note that $d\left(x_n^{(m)}, z_n\right) \leq d\left(x_n^{(m)}, z_m\right) + d(z_m, z_n)$. Suppose $K > 0$ has $\frac{1}{K} < \frac{\varepsilon}{2}$, we know such K exists. Also, since $\{z_n\}_{n=1}^{\infty}$ is Cauchy, so we know there exists $N_1' > 0$ s.t. for all $p, q \geq N_1'$, we have $d(z_p, z_q) < \frac{\varepsilon}{2}$. Hence, if we pick $m \geq \max\{K, N_1'\}$ and $n \geq \max\{N_m, N_1'\}$, then

$$\begin{aligned} d\left(x_n^{(m)}, z_n\right) &\leq d\left(x_n^{(m)}, z_m\right) + d(z_m, z_n) < \frac{1}{m} + \frac{\varepsilon}{2} \\ &\leq \frac{1}{K} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and we're done. ⊛

By Claim 0.0.2 and Claim 0.0.3, we know every Cauchy sequence in $\overline{X}$ converges to a formal limit of a Cauchy sequence of X, which means it converges in $\overline{X}$, and thus $\left(\overline{X}, d_{\overline{X}}\right)$ is complete. ■

d. We first show that $x = y \Leftrightarrow \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$. If $x = y$, then we know

$$\lim_{n\to\infty} d(x, y) = \lim_{n\to\infty} d(x, x) = 0,$$

which means $\mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$. Now we prove the converse, if $\mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$, then we know $\lim_{n\to\infty} d(x, y) = d(x, y) = 0$, so $x = y$.

Now we show that $d(x, y) = d_{\overline{X}}(x, y)$. Note that

$$d_{\overline{X}}(x, y) = \lim_{n\to\infty} d(x, y) = d(x, y),$$

so this is true. ■

e. Since we know $\mathrm{cl}_{\overline{X}}(X) \subseteq \overline{X}$, we only need to show $\overline{X} \subseteq \mathrm{cl}_{\overline{X}}(X)$. Suppose $x \in \overline{X}$, then $x = \mathrm{LIM}_{n\to\infty} x_n$ where $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Now we want to show that $x \in \mathrm{cl}_{\overline{X}}(X)$, which is equivalent to show for all $\varepsilon > 0$, there exists $y \in X$ s.t. $y \in B_{\overline{X}}(x, \varepsilon)$. If such y exists, then $d_{\overline{X}}(x, y) < \varepsilon$, which means $\lim_{n\to\infty} d(x_n, y) < \varepsilon$. However, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, so there exists $N > 0$ s.t. $i, j \geq N$ implies $d(x_i, x_j) < \frac{\varepsilon}{2}$. Thus, we can pick $y = x_N$, and then we have for all $n \geq N$, $d(x_n, y) < \frac{\varepsilon}{2} < \varepsilon$ Hence, we have $\lim_{n\to\infty} d(x_n, y) < \varepsilon$, and we're done. ■

f. Since $\{x_n\}_{n=1}^{\infty}$ can be seen as a sequence of elements in $\overline{X}$, and notice that $\{x_n\}_{n=1}^{\infty}$ is still Cauchy in $\overline{X}$ since for all $\varepsilon > 0$, we know there exists $N > 0$ s.t. $p, q \geq N$ implies $d(x_p, x_q) < \varepsilon$, so under same circumstances, we know

$$d_{\overline{X}}(x_p, x_q) = \lim_{n\to\infty} d(x_p, x_q) < \varepsilon,$$

and we're done. Now since we have proved $\overline{X}$ is complete in (c), so we know there exists $L \in \overline{X}$ s.t. $\lim_{n\to\infty} x_n = L$. Also, since $L \in \overline{X}$, so $L = \mathrm{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ in X. Now we want to show $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} x_n$. Hence, we want to show $\lim_{n\to\infty} d(a_n, x_n) = 0$, which is equivalent to prove $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $n \geq N$ implies $d(a_n, x_n) < \varepsilon$.

- Notice that since $\lim_{n\to\infty} x_n = L \in \overline{X}$, so $\forall \varepsilon > 0$, $\exists N_1 > 0$ s.t. $p \geq N_1$ implies $d_{\overline{X}}(x_p, L) < \frac{\varepsilon}{2}$, and thus $\lim_{n\to\infty} d(x_p, a_n) < \frac{\varepsilon}{2}$. Hence, there exists $M > 0$ s.t. if $p \geq N_1$ and $n \geq M$, then $d(x_p, a_n) < \frac{\varepsilon}{2}$.
- Also, since $\{x_n\}_{n=1}^{\infty}$ is Cauchy in X, so there exists $N_2 > 0$ s.t. $p, q \geq N_2$ implies $d(x_p, x_q) < \frac{\varepsilon}{2}$.

Use the above two properties, we know for all $n \geq \max\{M, N_2\}$ we can choose $s \geq \max\{N_1, N_2\}$ so that

$$d(a_n, x_n) \leq d(a_n, x_s) + d(x_s, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and we're done. ∎

Problem 0.0.3. In the following, all the sets are subsets of a metric space (X, d).

(a) If $\overline{A} \cap \overline{B} = \varnothing$, then

$$\partial(A \cup B) = \partial A \cup \partial B.$$

(b) For a finite family $\{A_i\}_{i=1}^{n} \subseteq X$, show that

$$\operatorname{int}\Big(\bigcap_{i=1}^{n} A_i\Big) = \bigcap_{i=1}^{n} \operatorname{int}(A_i).$$

(c) For an arbitrary (possibly infinite) family $\{A_\alpha\}_{\alpha\in F} \subseteq X$, prove that

$$\operatorname{int}\Big(\bigcap_{\alpha\in F} A_\alpha\Big) \subseteq \bigcap_{\alpha\in F} \operatorname{int}(A_\alpha).$$

(d) Give an example where the inclusion in part (c) is strict (i.e., equality fails).

(e) For any family $\{A_\alpha\}_{\alpha\in F} \subseteq M$, prove that

$$\bigcup_{\alpha\in F} \operatorname{int}(A_\alpha) \subseteq \operatorname{int}\Big(\bigcup_{\alpha\in F} A_\alpha\Big).$$

(f) Give an example of a finite collection F in which equality does not hold in part (e).

a. If $x \in \partial(A \cup B)$, then for all $r > 0$, we have

$$\begin{cases} B_X(x,r) \cap (A \cup B) = (B_X(x,r) \cap A) \cup (B_X(x,r) \cap B) \neq \varnothing. \\ B_X(x,r) \cap (X \setminus (A \cup B)) = B_X(x,r) \cap (X \setminus A) \cap (X \setminus B) \neq \varnothing. \end{cases}$$

Hence, either $B_X(x,r) \cap A$ or $B_X(x,r) \cap B$ is not empty. Also, we have $B_X(x,r) \cap (X \setminus A) \neq \varnothing$ and $B_X(x,r) \cap (X \setminus B) \neq \varnothing$. Thus, $x \in \partial A \cup \partial B$, which means $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

Now we show that $\partial A \cup \partial B \subseteq \partial(A \cup B)$. If $x \in \partial A \cup \partial B$, then we first give a claim:

Claim 0.0.4. If $x \in \partial A$, then $x \notin \partial B$, and vice versa.

Proof. If $x \in \partial A \cap \partial B$, then since $x \in \partial A \subseteq \overline{A}$ and $x \in \partial B \subseteq \overline{B}$, so $x \in \overline{A} \cap \overline{B} = \varnothing$, which is a contradiction. ⊛

Without lose of generality, we can suppose $x \in \partial A$ and $x \notin \partial B$, then we know

$$\forall r > 0 \text{ we have } \begin{cases} B_X(x, r) \cap A \neq \varnothing \\ B_X(x, r) \cap (X \setminus A) \neq \varnothing \end{cases},$$

$$\exists r' > 0 \text{ s.t. exactly one of } \begin{cases} B_X(x, r') \subseteq B \\ B_X(x, r') \subseteq (X \setminus B) \end{cases} \text{ occurs.}$$

However, if $B_X(x, r') \subseteq B$, then $x \in B_X(x, r') \subseteq B \subseteq \overline{B}$. However, $x \in \partial A \subseteq \overline{A}$, so $x \in \overline{A} \cap \overline{B} = \varnothing$, which is a contradiction. Thus, we know $B_X(x, r') \subseteq B$. Now since $x \in \partial A$, so $\forall r > 0$, we have $\varnothing \neq B_X(x, r) \cap A \subseteq B_X(x, r) \cap (A \cup B)$. Now we want to show $B_X(x, r) \cap (X \setminus A) \cap (X \setminus B) \neq \varnothing$.

- Case 1: $r \geq r'$, then we have $B_X(x, r) \subseteq B_X(x, r') \subseteq X \setminus B$ and thus

$$B_X(x, r) \cap (X \setminus A) \subseteq X \setminus B \Rightarrow B_X(x, r) \cap (X \setminus A) \cap (X \setminus B) \neq \varnothing$$

  since $B_X(x, r) \cap (X \setminus A) \neq \varnothing$.

- Case 2: $r' < r$, then we know $B_X(x, r') \subseteq (X \setminus B)$ and $B_X(x, r') \subseteq B_X(x, r)$. Now if we can show $B_X(x, r') \cap (X \setminus A) \cap (X \setminus B) \neq \varnothing$, then since $B_X(x, r') \subseteq B_X(x, r)$, so we know

$$\varnothing \neq B_X(x, r') \cap (X \setminus A) \cap (X \setminus B) \subseteq B_X(x, r) \cap (X \setminus A) \cap (X \setminus B).$$

  Now we show that $B_X(x, r') \cap (X \setminus A) \cap (X \setminus B) \neq \varnothing$. Note that since $B_X(x, r') \subseteq (X \setminus B)$, so in fact

$$B_X(x, r') \cap (X \setminus A) \cap (X \setminus B) = B_X(x, r') \cap (X \setminus A) \neq \varnothing$$

  since $x \in \partial A$, and thus we're done.

■

b. If $x \in \operatorname{Int}\left(\bigcap_{i=1}^n A_i\right)$, then $\exists r_1 > 0$ s.t. $B_X(x, r_1) \subseteq \bigcap_{i=1}^n A_i$. Hence, $B_X(x, r_1) \subseteq A_i$ for all $1 \leq i \leq n$, which means $x \in \operatorname{Int}(A_i)$ for al $1 \leq i \leq n$, and thus $x \in \bigcap_{i=1}^n \operatorname{Int}(A_i)$. This shows $\operatorname{Int}\left(\bigcap_{i=1}^n A_i\right) \subseteq \bigcap_{i=1}^n \operatorname{Int}(A_i)$. This shows $\operatorname{Int}\left(\bigcap_{i=1}^n A_i\right) \subseteq \bigcap_{i=1}^n \operatorname{Int}(A_i)$. Now we show that $\bigcap_{i=1}^n \operatorname{Int}(A_i) \subseteq \operatorname{Int}\left(\bigcap_{i=1}^n A_i\right)$. Suppose $x \in \bigcap_{i=1}^n \operatorname{Int}(A_i)$, for each i s.t. $1 \leq i \leq n$, we know there exists $r_i > 0$ s.t. $B_X(x, r_i) \subseteq A_i$, so if we pick $r' = \min\{r_1, r_2, \ldots, r_n\}$, then $B_X(x, r') \subseteq \bigcap_{i=1}^n A_i$, and thus $x \in \operatorname{Int}\left(\bigcap_{i=1}^n A_i\right)$. ■

c. If $x \in \operatorname{Int}\left(\bigcap_{\alpha \in F} A_\alpha\right)$, then $\exists r_1 > 0$ s.t. $B_X(x, r_1) \subseteq \bigcap_{\alpha \in F} A_\alpha$. Hence, $B_X(x, r_1) \subseteq A_\alpha$ for all $\alpha \in F$, which means $x \in \operatorname{Int}(A_\alpha)$ for all $\alpha \in F$, and thus $x \in \bigcap_{\alpha \in F} \operatorname{Int}(A_\alpha)$. This shows $\operatorname{Int}\left(\bigcap_{\alpha \in F} A_\alpha\right) \subseteq \bigcap_{\alpha \in F} \operatorname{Int}(A_\alpha)$. ■

d. Suppose $\{A_\alpha\}_{\alpha \in F} = \left\{\left(1 - \frac{1}{n}, 2 + \frac{1}{n}\right)\right\}_{n \in \mathbb{N}}$, then $\bigcap_{\alpha \in F} A_\alpha = [1, 2]$, and $\operatorname{Int}([1, 2]) = (1, 2)$. Besides, $\operatorname{Int}\left(1 - \frac{1}{n}, 2 + \frac{1}{n}\right) = \left(1 - \frac{1}{n}, 2 + \frac{1}{n}\right)$, and $\bigcap_{n \in \mathbb{N}}\left(1 - \frac{1}{n}, 2 + \frac{1}{n}\right) = [1, 2]$. Hence, in this case, the equality fails. ■

e. If $x \in \bigcup_{\alpha \in F} \operatorname{Int}(A_\alpha)$, then $x \in \operatorname{Int}(A_i)$ for some $i \in F$, and thus there exists $r_i > 0$ s.t. $B(x, r_i) \subseteq A_i$. Hence, $B(x, r_i) \subseteq \bigcup_{\alpha \in F} A_i$, and thus $x \in \operatorname{Int}\left(\bigcup_{\alpha \in F} A_i\right)$. ■

f. Suppose the family is $\{[1, 2], [2, 3]\}$, then

$$\operatorname{Int}[1, 2] \cup \operatorname{Int}[2, 3] = (1, 2) \cup (2, 3).$$

Also, $[1, 2] \cup [2, 3] = [1, 3]$, so $\operatorname{Int}([1, 2] \cup [2, 3]) = \operatorname{Int}[1, 3] = (1, 3)$. This is the case the equality fails. ■

Problem 0.0.4. Let (X, d) be a metric space and $Y \subset X$ be an open subset. For any subset $A \subset Y$, show that A is open in Y if and only if it is open in X.

Proof.

($\Rightarrow$) Since A is open in Y, so there exists open $O \subseteq X$ s.t. $A = O \cap Y$. Since O and Y are both open sets in X, so there exists $r_1, r_2 > 0$ s.t.

$$B_X(x, r_1) \subseteq O \quad \text{and} \quad B_X(x, r_2) \subseteq Y.$$

Now let $r_3 = \min\{r_1, r_2\}$, then $B_X(x, r_3) \subseteq O \cap Y = A$, which shows A is open in X.

($\Leftarrow$) Now if A is open in X, then for all $x \in X$, there exists $B_X(x, r) \subseteq A$, but $B_Y(x, r) \subseteq B_X(x, r)$, so we have $B_Y(x, r) \subseteq A$, and thus A is open in Y.

$\blacksquare$

Problem 0.0.5. On the space $(0, 1]$, we may consider the topology induced by the metric space $(\mathbb{R}, d)$ defined by $d(x, y) = |x - y|$. Alternatively, we may also define a distance d' on $(0, 1]$, given by

$$d'(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \qquad \forall x, y \in (0, 1].$$

(a) Show that d' is a metric on $(0, 1]$

(b) Let $x \in (0, 1]$ and $\varepsilon > 0$. Let $B = B_d(x, \varepsilon) = \{y \,|\, |y - x| < \varepsilon\} \cap (0, 1]$ be the open ball centered at x of radius ε for the metric d in $(0, 1]$. Show that for any $y \in B$, we may find $\varepsilon' > 0$ such that

$$B_{d'}(y, \varepsilon') \subseteq B = B_d(x, \varepsilon).$$

(c) Show that an open ball in $((0, 1], d')$ is also an open ball in $((0, 1], d)$.

(d) Conclude that the metric spaces $((0, 1], d)$ and $((0, 1], d')$ are topologically equivalent, that is, a set A is open in one space if and only if it is also open in the other one.

(e) Is $((0, 1], d')$ a complete metric space? How about $((0, 1], d)$?

(a). We verify the following properties:

- $d'(x, y) \geq 0$ since $|\cdot| \geq 0$.
- $d'(x, y) = 0 \Leftrightarrow |\frac{1}{x} - \frac{1}{y}| \Leftrightarrow x = y$.
- $d'(x, y) = |\frac{1}{x} - \frac{1}{y}| = |\frac{1}{y} - \frac{1}{x}| = d'(y, x)$.
- We know if $a, b \in \mathbb{R}$, the triangular inequality $|a| + |b| \geq |a + b|$ holds. Then we can plug $a = \frac{1}{x} - \frac{1}{y}$ and $b = \frac{1}{y} - \frac{1}{z}$ in. Then we can get $|\frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}| \leq |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{y} - \frac{1}{z}|$. Hence $d'(x, z) \leq d'(x, y) + d'(y, z)$.

Hence, d' is indeed a metric on $(0.1]$. $\blacksquare$

(b). We assume that $d(x, y) = \varepsilon'' < \varepsilon$.

Claim 0.0.5. We claim that if we pick $\varepsilon' = \varepsilon - \varepsilon'' > 0$, then $B_{d'}(y, \varepsilon') \subseteq B_d(x, \varepsilon)$

Proof. $\forall z \in B_{d'}(y, \varepsilon')$, $|\frac{1}{y} - \frac{1}{z}| < \varepsilon'$. Then $|\frac{z-y}{zy}| < \varepsilon'$, $|z - y| < \varepsilon' zy$ (Since $z > 0$ and $y > 0$).
Hence we know that $d(y, z) = |y - z| < \varepsilon' zy < \varepsilon'$.
Since d is metric, so $d(x, z) \leq d(x, y) + d(y, z) < \varepsilon'' + (\varepsilon - \varepsilon'') = \varepsilon$.
Hence $z \in B_d(x, \varepsilon)$. ⊛

By the claim above, we're done. $\blacksquare$

(c). Suppose $E = B_{d'}(x, r)$, we first analyze the properties of the elements in E first, $y \in E$ if $|\frac{1}{y} - \frac{1}{x}| < r$, we can solve $\frac{1}{x} - r < \frac{1}{y} < \frac{1}{x} + r$:

- Condition 1: $\frac{1}{y} < \frac{1}{x} + r$, so $y > \frac{x}{1+rx} > 0$
- Condition 2 (Case 1): If $\frac{1}{y} > \frac{1}{x} - r$ and $\frac{1}{x} - r \leq 0$, then any $y \in (0, 1]$ satisfy the condition.
- Condition 2 (Case 2): If $\frac{1}{y} > \frac{1}{x} - r$, $\frac{1}{x} - r > 0$ and $x \geq 1 - rx$, then $y \leq 1 < \frac{x}{1-rx}$
- Condition 2 (Case 3): If $\frac{1}{y} > \frac{1}{x} - r$, $\frac{1}{x} - r > 0$ and $x < 1 - rx$, then $y < \frac{x}{1-rx} < 1$

Then we can construct open ball which is also equal to E in another metric space by selecting center point and radius:
For case 1 and 2, we may choose $c = 1 \in (0, 1]$ and $r' = 1 - \frac{x}{1+rx}$, since

$$B_{((0,1],d)}(c, r') = B_{(\mathbb{R},d)}(c, r') \cap (0, 1] = \{z \in \mathbb{R} \mid 1 - r' < z < 1 + r'\} \cap (0, 1] = \{z \in (0, 1] \mid 1 - r' < z\}$$

$$B_{((0,1],d)}(c, r) = \{z \in (0, 1] \mid \frac{x}{1 + rx} < z\}$$

This is indeed same as the condition requirement.
For case 3, let $a = \frac{x}{1+rx}$, $b = \frac{x}{1-rx}$, we choose $c = \frac{a+b}{2}$ and $r' = \frac{b-a}{2}$, since

$$B_{((0,1],d)}(c, r') = B_{(\mathbb{R},d)}(c, r') \cap (0, 1] = \{z \in \mathbb{R} \mid c - r' < z < c + r'\} \cap (0, 1] = \{z \in (0, 1] \mid a < z < b\}$$

$$B_{((0,1],d)}(c, r) = \{z \in (0, 1] \mid \frac{x}{1 + rx} < z < \frac{x}{1 - rx}\}$$

This is also same as the condition requirement. Hence we proved.

■

(d).

(⇒) Suppose A is open in $((0, 1], d)$, then $\forall x \in A, \exists r_x > 0$ such that $B_d(x, r_x) \subseteq A$. So $A = \bigcup_{x \in A} B_d(x, r_x)$, and from (b), we know any open ball in $((0, 1], d)$ is also open in $((0, 1], d')$, so A is also the union of infinitely many open set in $((0, 1], d')$. By the proposition we have shown in class, we conclude A is also a open set in $((0, 1], d')$.

(⇐) Suppose A is open in $((0, 1], d')$, then $\forall x \in A, \exists r_x > 0$ such that $B_{d'}(x, r_x) \subseteq A$. So $A = \bigcup_{x \in A} B_{d'}(x, r_x)$. and from (c), we know for any open ball in $((0, 1], d')$ is also open in $((0, 1], d)$, so A is also the union of infinitely many open set in $((0, 1], d)$. By the proposition we have shown in class, we conclude A is also a open set in $((0, 1], d)$.

Hence, we proved that A is open set in $((0, 1], d')$ if and only if it is open in $((0, 1], d)$. ■

(e). We first show that $((0, 1], d')$ is complete metric space.
Given Cauchy sequence $(x_n)_{n=m}^{\infty}$ in $((0, 1], d')$, $\forall \varepsilon > 0, \exists N$ such that $\forall n, m \geq N$, $d'(x_n, x_m) = |\frac{1}{x_n} - \frac{1}{x_m}| < \varepsilon$.
Then we can construct another sequence $(y_n)_{n=m}^{\infty}$ such that $y_n = \frac{1}{x_n}$ and since $x_n \in (0, 1]$, $y_n \in [1, \inf)$.
(y_n) is Cauchy sequence since $\forall \varepsilon > 0, \exists N' = N$ such that $\forall n, m \geq N'$, $d(y_n, y_m) = |y_n - y_m| = |\frac{1}{x_n} - \frac{1}{x_m}| < \varepsilon$ by previous proof.
Notice that $[1, \infty)$ is a closed subset in $\mathbb{R}$ and we know $(\mathbb{R}, d)$ is complete, so $([1, \infty), d)$ is also complete. Hence (y_n) converge to some $L \in [1, \infty)$, and this will imply that (x_n) converge to $\frac{1}{L} \in (0, 1]$, so given any Cauchy sequence in $((0, 1], d')$, it converges to some $L' \in (0, 1]$, so $((0, 1], d')$ is complete.

We show that $((0, 1], d)$ is **NOT** complete by showing an example. Consider $(z_n)_{n=m}^{\infty}$, $z_n = \frac{1}{n}$. It is Cauchy sequence since $\forall \varepsilon = \frac{1}{k} > 0, \exists N = \lceil k \rceil$ such that $\forall n, m \geq N$, WLOG suppose $n \geq m$,

$$d(x_n, x_m) = x_n - x_m < x_n \leq \frac{1}{\lceil k \rceil} \leq \frac{1}{k} = \varepsilon.$$

However, it converges to 0, which is not in $(0,1]$, so the series (z_n) doesn't converge in $((0,1],d)$, hence $((0,1],d)$ is not complete.

■

Problem 0.0.6. (a) We say that a family of closed balls

$$\big(\overline{B}(x_n, r_n)\big)_{n\geq 1}$$

is a *decreasing sequence of closed balls* if the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Give an example of a decreasing sequence of closed balls in a complete metric space with empty intersection.

(b) We say that a family of closed balls

$$\big(\overline{B}(x_n, r_n)\big)_{n\geq 1}$$

is a *decreasing sequence of closed balls with radii tending to zero* if

$$r_n \to 0 \quad \text{as } n \to \infty,$$

and the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Show that a metric space (M,d) is complete if and only if every decreasing sequence of closed balls with radii going to zero has a nonempty intersection.

(a). Consider the following example: $X = \mathbb{N}$, and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1 + \dfrac{1}{\min\{x,y\}}, & \text{if } x \neq y. \end{cases}$$

and $C_n = \overline{B}_d\big(n,\, 1+\frac{1}{n}\big) = \{\, x \in X \mid d(x,n) \leq 1 + \frac{1}{n} \,\}$.

Claim 0.0.6. $\forall n \in \mathbb{N}, C_n = \{n, n+1, n+2...\}$

Proof. $\forall m < n$, $d(n,m) = 1 + \frac{1}{m} > 1 + \frac{1}{n}$, so $m \notin C_n$. $\forall m \geq n$, $d(n,m) = 1 + \frac{1}{n} \leq 1 + \frac{1}{n}$, so $m \in C_n$ So C_n is indeed $\{n, n+1, ...\}$. ⊛

$\forall n, C_{n+1} \subseteq C_n$, So $(C_n)_{n\geq 1}$ is a decreasing sequence of closed balls. However, $\forall n, n \notin C_{n+1}$, so $\bigcap_{n=1}^{\infty} C_n = \emptyset$.

Then we show that (X,d) is complete metric space. For every Cauchy sequence $(x_n)_{n=1}^{\infty}$ in (X,d), $\forall \varepsilon > 0, \exists N$ such that $\forall n,m \geq N$, $d(x_n, x_m) < \varepsilon$.
Then we can take $\varepsilon = 0.48763$, by definition, exists N such that $\forall n,m \geq N$, $d(x_n, x_m) < 0.48763$. However if $x_n \neq x_m$, then $d(x_n, x_m) = 1 + \dfrac{1}{\min\{x_n, x_m\}} > 1$, so we know $\forall n \geq N, x_n = x_N$, and this will make the sequence coverage to $x_N \in \mathbb{N}$ since this is a constant sequence.

Hence, every Cauchy sequence in X coverage to some point in $X = \mathbb{N}$, so (X,d) is complete metric space.

■

(b). First, we show that if the nested condition is satisfied and the radii goes to 0, then $(x_n)_{n=1}^{\infty}$ is Cauchy sequence.

Since $r_n \to 0$ as $n \to \infty$, $\forall \varepsilon' > 0, \exists N'$ such that $\forall n \geq N', r_n < \frac{\varepsilon}{2}$.
And since $\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n)$ for all $n \in \mathbb{N}$, so $\{x_{n+1}, x_{n+2}, ...\} \subseteq \overline{B}(x_n, r_n)$ hence $\forall n, m \geq N', d(x_n, x_m) \leq d(x_n, x_{N'}) + d(x_{N'}, x_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So by definition. $(x_n)_{n=1}^{\infty}$ is Cauchy sequence.

Then we can start our proof:

($\Rightarrow$) Since (M, d) is complete and (x_n) is Cauchy sequence, (x_n) will converge to some $x' \in M$.

Claim 0.0.7. $x' \in \overline{B}(x_m, r_m)$ for all $m \in \mathbb{N}$

Proof. Since $(x_n)_{n=1}^{\infty}$ is Cauchy sequence, the subsequence $(x_n)_{n=m}^{\infty}$ is also Cauchy sequence and also converge to x'.
And since $\overline{B}(x_m, r_m)$ is an close ball in (M, d), $x' \in \overline{B}(x_m, r_m)$ by properties we have shown in class. ⊛

Since $x' \in \overline{B}(x_m, r_m)$ for all $m \in \mathbb{N}$, the intersection of these ball are not empty, hence we proved.

($\Leftarrow$) First we do a big claim below:

Claim 0.0.8. For every Cauchy sequence $(x_n)_{n=1}^{\infty}$, we can construct $(r_n)_{n=1}^{\infty}$ such that there is a subsequence $(x'_n, r'_n)_{n=1}^{\infty}$. $\overline{B}(x'_n, r'_n)$ satisfy nested condition. And further more, $r'_n \to 0$ as $n \to \infty$.

Proof. Since (x_n) is Cauchy sequence, $\forall \varepsilon > 0, \exists N$ such that $\forall n, m \geq N, d(x_n, x_m) < \varepsilon$.
Then we might take $\varepsilon = 1$ and discard those term that $(x_n)_{n=1}^{n=N}$, and do the following construction:
For those $r_n, n \geq N$ terms, let $T_n = \{x_n, x_{n+1}, ...\}$.
We define $d_n = \sup\{\, d(x_j, x_k) \mid x_j, x_k \in T_n \,\}$, since given any pair, the distance is smaller than 1 by the definition, so d_n has a upper bound so supremum exists.
The let $r_n = 2d_n$, then $T_n \subseteq \overline{B}(x_n, d_n) \subseteq \overline{B}(x_n, r_n)$.
Since x_n is Cauchy sequence, $d_n \to 0$ as $n \to \infty$, hence $r_n \to 0$ as $n \to \infty$. Then we show that $\overline{B}(x_k, r_k) \subseteq \overline{B}(x_n, r_n)$, for some $k > n$.
First, we need to carefully pick k. Since x_n is Cauchy, again, for $\varepsilon = \frac{d_n}{2}$, exist some $k = N'$ such that $d_k < \varepsilon$.
Then $\forall z \in \overline{B}(x_k, r_k), d(x_n, z) \leq d(x_n, x_k) + d(x_k, z) \leq d_n + r_k \leq d_n + d_n = r_n$.
Then we start from N-th terms and recursively do the above construction, we can get a sequence $(x'_n, r'_n)_{n=1}^{\infty}$, which is the subsequence of $(x_n, r_n)_{n=1}^{\infty}$, satisfy the nest condition and $r'_n \to 0$ as $n \to \infty$ since $r_n \to 0$. ⊛

Then we know the intersection of $\overline{B}(x'_n, r'_n)$ is not empty by hypothesis, let it x', we know that $\lim_{n\to\infty} x'_n$ lie in infinitely many nested close ball, so $\lim_{n\to\infty} x'_n \in \lim_{n\to\infty} \overline{B}(x'_n, r'_n)$ $0 \leq \lim_{n\to\infty} d(x'_n, x') = 0 \leq r'_n$ since $\lim_{n\to\infty} r'_n = 0$, and this implies $x'_n \to x'$. Hence, we show that (x'_n) converge to x', and we know if a Cauchy sequence's subsequence converge to some point x', the original Cauchy sequence also converge to x', so (x_n) converge to x.
since for every Cauchy sequence converge, so (M, d) is complete.

■