October 9, 2025

Introduction to Mathematical Analysis
Homework 6 Due October 17 (Friday), 2025
Please submit your homework online in PDF format.

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1. (20 pts)

   Definition (Totally ordered set). A *totally ordered set* (or *linearly ordered set*) is a pair $(X, \leq)$ consisting of a nonempty set X together with a binary relation $\leq$ on X satisfying the following properties:

   1. **Reflexivity:** For all $x \in X$, $x \leq x$.
   2. **Antisymmetry:** For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.
   3. **Transitivity:** For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
   4. **Totality (or Comparability):** For all $x, y \in X$, either $x \leq y$ or $y \leq x$.

   A relation $\leq$ satisfying only (1)–(3) is called a *partial order*. If, in addition, (4) holds, the order is said to be *total*, meaning that any two elements of X can be compared.

   Definition (Hausdorff space). A topological space $(X, \mathcal{F})$ is called a *Hausdorff space* (or T_2 space) if for every pair of distinct points $x, y \in X$ there exist neighborhoods $U, V \in \mathcal{F}$ such that

   $$x \in U, \quad y \in V, \quad \text{and } U \cap V = \varnothing.$$

   (a) Given any totally ordered set X with order relation $\leq$, declare a set $V \subseteq X$ to be open if for every $x \in V$ there exists a set I, which is an interval $\{y \in X : a < y < b\}$ for some $a, b \in X$, or $\{y \in X : a < y\}$ for some $a \in X$, or $\{y \in X : y < b\}$ for some $b \in X$, or the whole space X, which contains x and is contained in V. Let $\mathcal{F}$ be the set of all open subsets of X. Show that $(X, \mathcal{F})$ is a topology (this is the *order topology* on the totally ordered set $(X, \leq)$ which is Hausdorff in the sense of Definition 2.5.4-2 or the definition above).

   (b) Show that on the real line $\mathbb{R}$ (with the standard ordering $\leq$), the order topology matches the standard topology (i.e., the topology arising from the standard metric).

   (c) If instead one defines V to be open if the extended real line $\mathbb{R} \cup \{\pm\infty\}$ has an open set with boundary $\{\pm\infty\}$, then $(X, \mathcal{F})$ is a sequence of numbers in $\mathbb{R}$ (and hence in $\mathbb{R}$), show that $x_n \to +\infty$ if and only if $\inf_{n \geq N} x_n \to +\infty$, and $x_n \to -\infty$ if and only if $\sup_{n \geq N} x_n \to -\infty$.

2. (15 pts)

   Definition (Metrizable space). A topological space $(X, \mathcal{F})$ is said to be *metrizable* if there exists a metric $d : X \times X \to [0, \infty)$ such that the topology $\mathcal{F}$ coincides with the topology $\mathcal{F}_d$ induced by d. That is,

   $$\mathcal{F} = \mathcal{F}_d := \{\, U \subseteq X : \forall x \in U, \exists\, \varepsilon > 0 \text{ such that } B_d(x, \varepsilon) \subseteq U \,\},$$

   where $B_d(x, \varepsilon) := \{\, y \in X : d(x, y) < \varepsilon \,\}$ denotes the open ball centered at x with radius ε.

   If no such metric d exists, then $(X, \mathcal{F})$ is said to be *not metrizable*. In other words, its topology cannot arise from any metric on X.

   (a) Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or cofinite (which means that $X \setminus E$ is finite). Show that $(X, \mathcal{F})$ is a topology (this is called the *cofinite topology* on X) which is not Hausdorff and is compact.

   (b) Show that if $\{V_i : i \in I\}$ is any countable collection of open sets containing x, then $\bigcap_i V_i \neq \varnothing$. Use this to show that the cofinite topology cannot be derived from any metric (i.e., $(X, \mathcal{F})$ is not metrizable). (Hint: what is the set $\bigcap_{n=1}^{\infty} B(x, 1/n)$ equal to in a metric space?)

3. (15 pts) Let $(X, \mathcal{F})$ be a compact topological space. Assume that this space is first countable, which means that for every $x \in X$ there exist countable collections of open sets $V_1, V_2, \ldots$ of neighborhoods of x, such that every neighborhood of x contains one of the V_n. Show that every sequence in X has a convergent subsequence (see Exercise 1.5.11).

4. (15 pts) Let $(X, \mathcal{F})$ be a compact topological space and $(Y, \mathcal{G})$ be a Hausdorff topological space. If $f : X \to Y$ is continuous, then f is a *closed map*; i.e., for every closed subset $F \subseteq X$, the image $f(F)$ is closed in Y.

5. (20 pts) Let $\{f_n\}$ be a sequence of continuous functions real-valued defined on a compact metric space S and assume that $\{f_n\}$ converges pointwise on S to a limit function f. Prove that $f_n \to f$ uniformly on S if, and only if, the following two conditions hold:

   (i) The limit function f is continuous on S.

   (ii) For every $\varepsilon > 0$, there exist $m > 0$ and $\delta > 0$ such that $n > m$ and
   $$|f_k(x) - f(x)| < \delta \implies |f_{k+n}(x) - f(x)| < \varepsilon$$
   for all $x \in S$ and all $k = 1, 2, \ldots$.

   Hint. To prove the sufficiency of (i) and (ii), show that for each $x_0 \in S$ there is a neighborhood $B(x_0, R)$ and an integer k (depending on x_0) such that
   $$|f_k(x) - f(x)| < \delta \quad \text{if } x \in B(x_0, R).$$
   By compactness, a finite set of integers, say $A = \{k_1, \ldots, k_r\}$, has the property that for each $x \in S$, some $k \in A$ satisfies $|f_k(x) - f(x)| < \delta$. Uniform convergence is an easy consequence of this fact.

6. (15 pts) The purpose of this exercise is to demonstrate a concrete relationship between continuity and pointwise convergence, and between uniform continuity and uniform convergence.

   Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For any $a \in \mathbb{R}$, let $f_a : \mathbb{R} \to \mathbb{R}$ be the shifted function defined by
   $$f_a(x) := f(x - a).$$

   (a) Show that f is continuous if and only if, whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge pointwise to f.

   (b) Show that f is uniformly continuous if and only if, whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge uniformly to f.

You can do the following problems to practice. You don't have to submit the following problems.

1. Let $(X, \mathcal{F})$ be a topological space and let B be a subsets of X. Prove the following set equality:
   $$\overline{X \backslash B} = X \setminus Int(B).$$

2. Let $(X, \mathcal{F})$ be a topological space and $(Y, \mathcal{G})$ be a Hausdorff topological space. Suppose $f, g : X \to Y$ are continuous maps. Show that the set $Z = \{x \in X | f(x) = g(x)\}$ is closed in X. Give a counterexample if Y is not Hausdorff. Hint: Show $X \setminus Z$ is open.

3. Suppose X is a topological space, and for every $p \in X$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{pre}(0) = \{p\}$. Show that X is Hausdorff.

4. Define two sequences $\{f_n\}$ and $\{g_n\}$ as follows:
   $$f_n(x) = x\left(1 + \frac{1}{n}\right), \qquad x \in \mathbb{R}, \quad n = 1, 2, \ldots$$
   and
   $$g_n(x) = \begin{cases} \dfrac{1}{n}, & \text{if } x = 0 \text{ or } x \text{ is irrational,} \\ b + \dfrac{1}{n}, & \text{if } x \text{ is rational, say } x = \dfrac{a}{b},\ b > 0. \end{cases}$$
   Let $h_n(x) = f_n(x)g_n(x)$.

(a) Prove that both $\{f_n\}$ and $\{g_n\}$ converge uniformly on every bounded interval.

(b) Prove that $\{h_n\}$ does not converge uniformly on any bounded interval.

5. Let (X, d_X) be a metric space, and for every integer $n \geq 1$, let $f_n : X \to \mathbb{R}$ be a real-valued function. Suppose that f_n converges pointwise to another function $f : X \to \mathbb{R}$ on X (in this question we give $\mathbb{R}$ the standard metric $d(x, y) = |x - y|$).

   Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the functions $h \circ f_n$ converge pointwise to $h \circ f$ on X, where $h \circ f_n : X \to \mathbb{R}$ is defined by $h \circ f_n(x) := h(f_n(x))$, and similarly for $h \circ f$.

6. (a) Use Problem 5 in the first part to prove the following theorem of Dini:

   Dini's Theorem. If $\{f_n\}$ is a sequence of real-valued continuous functions converging pointwise to a continuous limit function f on a compact set S in a metric space, and if

   $$f_n(x) \geq f_{n+1}(x) \quad \text{for each } x \in S \text{ and every } n = 1, 2, \ldots,$$

   then $f_n \to f$ uniformly on S.

   (b) Let

   $$f_n(x) = \frac{1}{nx + 1}, \qquad 0 < x < 1, \quad n = 1, 2, \ldots$$

   Prove that $\{f_n\}$ converges pointwise but not uniformly on $(0, 1)$.

   (c) Use the sequence in part (b) to show that compactness of S is essential in Dini's theorem.