

# Linear Algebra I HW1

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**Problem 0.0.1.** Let  $W$  be a set of all  $(x_1, x_2, x_3, x_4, x_5)$  in  $\mathbb{R}^5$  which satisfy

$$\begin{array}{rrrrr} 2x_1 - x_2 & +\frac{4}{3}x_3 - x_4 & & & = 0 \\ x_1 & +\frac{2}{3}x_3 & & -x_5 & = 0 \\ 9x_1 - 3x_2 & +6x_3 - 3x_4 & -3x_5 & & = 0 \end{array}$$

Find a finite set of vectors which spans  $W$ .

**Proof.** We can first write the system of equations into the matrix form and then use Gaussian elimination.

$$\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 9 & -3 & 6 & -3 & -3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & \frac{2}{3} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, we can need to solve

$$\begin{cases} x_1 - \frac{1}{2}x_2 + \frac{2}{3}x_3 - \frac{1}{2}x_4 = 0 \\ \frac{1}{2}x_2 + \frac{1}{2}x_4 - x_5 = 0. \end{cases}$$

So we know  $(x_1, x_2, x_3, x_4, x_5) = (t - \frac{2}{3}a, b, a, 2t - b, t)$  for some  $a, b, t \in \mathbb{R}$ , and thus we know the set

$$S = \left\{ (1, 0, 0, 2, 1), \left(-\frac{2}{3}, 0, 1, 0, 0\right), (0, 1, 0, -1, 0) \right\}$$

spans  $W$ . ■

**Problem 0.0.2.** Prove that a subspace of  $\mathbb{R}^2$  is  $\mathbb{R}^2$ , or the zero subspace, or consists of all scalar multiples of some fixed vector in  $\mathbb{R}^2$ . (The last type of subspace is, intuitively, a straight line through the origin.)

**Proof.** We first give a claim:

**Claim 0.0.1.** Suppose  $V$  is a vector space, and if  $W$  is a subspace of  $V$ , then  $\dim W \leq \dim V$ .

**Proof.** Since  $W \subseteq V$ , so suppose  $k = \dim V$  and  $B_1$  is a basis of  $V$ , then  $W \subseteq V = \text{span } B_1$ , which means if there is a basis of  $W$ , say  $B_2$ , then  $|B_2| \leq |B_1|$ , which means  $\dim W \leq \dim V$ . ⊗

Now also we know  $\dim \mathbb{R}^2 = 2$  since  $\{(0, 1), (1, 0)\}$  is a linearly independent set and spans  $\mathbb{R}^2$ . Thus, if there is a subspace of  $\mathbb{R}^2$ , say  $W$ , then  $\dim W = 0, 1, 2$ . If  $\dim W = 0$ , then  $W$  is the zero subspace. If  $\dim W = 1$ , then  $W$  consists of all scalar multiples of some fixed vector in  $\mathbb{R}^2$ . If  $\dim W = 2$ , then we can give a claim first:

**Claim 0.0.2.** Suppose  $W \subseteq V$  and they are both vector spaces, then if  $\dim V = \dim W$ , then  $V = W$ .

**Proof.** Suppose by contradiction, there exists  $v \in V \setminus W$ , and suppose  $B$  is a basis of  $W$ , then we know  $B \cup \{v\}$  is linearly independent in  $V$ . However,  $|B \cup \{v\}| > \dim V$ , which is a contradiction. ⊗

By this claim, we know  $W = \mathbb{R}^2$  if  $\dim W = 2$ . ■

**Alternative proof (not sure about if the above one is legal).** Suppose  $W$  is a subspace of  $\mathbb{R}^2$ . If  $W = \{(0, 0)\}$ , then it is the zero subspace. If not, there is some  $v \neq (0, 0)$  s.t.  $v \in W$ . However, since  $W$  is a vector space, so  $\text{span } \{v\} \subseteq W$ . If the equal sign holds, then  $W$  consists of all scalar multiples of some fixed vector in  $\mathbb{R}^2$ . If the equal sign does not hold, then there exists  $v, w \in W$  s.t.  $w$  is not a scalar multiple of  $v$ .

**Claim 0.0.3.**  $\text{span}\{v, w\} = \mathbb{R}^2$ .

**Proof.** It is trivial that  $\text{span}\{v, w\} \subseteq \mathbb{R}^2$ . Now we show that  $\mathbb{R}^2 \subseteq \text{span}\{v, w\}$ . First assume  $v = (v_1, v_2)$  and  $w = (w_1, w_2)$ . For all  $r = (c_1, c_2) \in \mathbb{R}^2$ , we know the systems of equations

$$\begin{cases} v_1x + w_1y = c_1 \\ v_2x + w_2y = c_2 \end{cases}$$

has a unique solution by Cramer's rule learnt in high school. Hence, we know  $r = xv + yw \in \text{span}\{v, w\}$ , and we're done.  $\otimes$

Besides, it is trivial that a subspace cannot be bigger than the original vector space, that is, there does not exist  $v \in W$  s.t.  $v \notin \mathbb{R}^2$ . Hence,  $W$  cannot be bigger, and thus we have concluded all the cases of  $W$ .  $\blacksquare$

**Problem 0.0.3.** Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$  such that the set-theoretic union of  $W_1$  and  $W_2$  is also a subspace. Prove that one of the spaces  $W_i$  is contained in the other.

**Proof.** Suppose each  $W_i$  is not contained in the other, then there exists  $u, v$  s.t.

$$u \in W_2 \setminus W_1 \quad v \in W_1 \setminus W_2.$$

Thus, we know  $u + v \in W_1 \cup W_2$  since  $W_1 \cup W_2$  is a vector space. Also, we know  $u + v \in W_1$  or  $u + v \in W_2$ . Now if  $u + v \in W_1$ , then  $u = u + v + (-v) \in W_1$ , which is a contradiction, and if  $u + v \in W_2$ , we have  $v = u + v + (-u) \in W_2$ , which is also a contradiction. Hence, we know either  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$  should happen.  $\blacksquare$

**Problem 0.0.4.** Let  $V$  be the vector space of all functions from  $\mathbb{R}$  into  $\mathbb{R}$ ; let  $V_e$  be the subset of even functions,  $f(-x) = f(x)$ ; let  $V_o$  be the subset of odd functions,  $f(-x) = -f(x)$ .

- Prove that  $V_e$  and  $V_o$  are subspaces of  $V$ .
- Prove that  $V_e + V_o = V$ .
- Prove that  $V_e \cap V_o = \{0\}$ .

**Proof.**

- First note that  $V_e \subseteq V$  and  $V_o \subseteq V$ , and they are both non-empty (since zero function is contained in both  $V_e$  and  $V_o$ ), then

- For all  $f, g \in V_e$  and  $\alpha \in F$ , we define  $h(x) = \alpha f(x) + g(x)$ , and we know  $h \in V_e$  since

$$h(-x) = \alpha f(-x) + g(-x) = \alpha f(x) + g(x) = h(x).$$

Hence,  $V_e$  is a subspace of  $V$ .

- For all  $f, g \in V_o$  and  $\alpha \in F$ , we define  $h(x) = \alpha f(x) + g(x)$ , and we know  $h \in V_o$  since

$$h(-x) = \alpha f(-x) + g(-x) = -\alpha f(x) - g(x) = -h(x).$$

Hence,  $V_o$  is a subspace of  $V$ .

- We first show that  $V \subseteq V_e + V_o$ . Since for all  $f \in V$ , we know

$$f(x) = \left( \frac{f(x) + f(-x)}{2} \right) + \left( \frac{f(x) - f(-x)}{2} \right),$$

where

$$f_e(x) = \left( \frac{f(x) + f(-x)}{2} \right) \in V_e \quad f_o(x) = \left( \frac{f(x) - f(-x)}{2} \right) \in V_o$$

since

$$f_e(-x) = \left( \frac{f(-x) + f(x)}{2} \right) = f_e(x)$$
$$f_o(-x) = \left( \frac{f(-x) - f(x)}{2} \right) = -f_o(x).$$

Now we show that  $V_e + V_o \subseteq V$ . This is trivial since for all  $f \in V_e + V_o$  we have  $f = g + h$  for some  $g \in V_e \subseteq V$  and  $h \in V_o \subseteq V$ , so  $f = g + h \in V$  since  $V$  is a vector space. Hence,  $V_e + V_o \subseteq V$ .

Now we have  $V \subseteq V_e + V_o$  and  $V_e + V_o \subseteq V$ , so we can conclude that  $V = V_e + V_o$ .

(c)

■

**Problem 0.0.5.** Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$  such that  $W_1 + W_2 = V$  and  $W_1 \cap W_2 = \{0\}$ . Prove that for each vector  $\alpha$  in  $V$  there are unique vectors  $\alpha_1$  in  $W_1$  and  $\alpha_2$  in  $W_2$  such that  $\alpha = \alpha_1 + \alpha_2$ .

**Proof.** Since  $W_1 + W_2 = V$ , so we know for each vector  $\alpha \in V$ , it can be representend as  $\alpha = \alpha_1 + \alpha_2$  for some  $\alpha_1 \in W_1$  and  $\alpha_2 \in W_2$ . If there are two differet  $(\alpha_1, \alpha_2)$ -pairs to represent  $\alpha$ , say  $\alpha = \alpha_1 + \alpha_2 = \alpha'_1 + \alpha'_2$  where  $\alpha_1, \alpha'_1 \in W_1$  and  $\alpha_2, \alpha'_2 \in W_2$ , then we know  $\alpha_1 - \alpha'_1 = \alpha'_2 - \alpha_2$ . However, since  $\alpha_1 - \alpha'_1 \in W_1$  and  $\alpha'_2 - \alpha_2 \in W_2$ , so  $\alpha_1 - \alpha'_1 = \alpha'_2 - \alpha_2 \in W_1 \cap W_2 = \{0\}$ , which means  $\alpha_1 = \alpha'_1$  and  $\alpha_2 = \alpha'_2$  and thus the  $(\alpha_1, \alpha_2)$ -pair is unique. ■