

# Linear Algebra I HW8

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**Problem 0.0.1.** Let  $T$  be the linear operator on  $\mathbb{R}^4$  which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}.$$

Under what conditions on  $a, b$ , and  $c$  is  $T$  diagonalizable?

**Proof.** Suppose

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix},$$

then  $\det(xI - A) = x^4$ , so if  $A$  is diagonalizable, then we must have  $\dim \ker(A - 0I) = \dim \ker A = 4$ , which means  $\text{rank } A = 0$  by rank and nullity theorem, so  $a = b = c = 0$ .  $\blacksquare$

**Problem 0.0.2.** Let  $A$  and  $B$  be  $n \times n$  matrices over the field  $F$ . Prove that if  $(I - AB)$  is invertible, then  $I - BA$  is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

**Proof.** Since  $I - AB$  is not invertible, so  $\det(I - AB) \neq 0$ , and thus we know 1 is not an eigenvalue of  $AB$ , so there does not exist  $v \neq 0$  s.t.  $ABv = v$ . Now suppose by contradiction,  $\det(I - BA) = 0$ , then we know 1 is an eigenvalue of  $BA$ , so there exists  $w \neq 0$  s.t.  $BAw = w$ , so  $AB(Aw) = Aw$ . Now note that  $Aw \neq 0$ , otherwise  $w = BAw = B0 = 0$ , which is impossible. Hence, if we suppose  $v = Aw$ , then  $ABv = v$  and  $v \neq 0$ , which is a contradiction, so  $\det(I - BA) \neq 0$ , and thus  $I - BA$  is invertible.

Now suppose  $X = (I - AB)^{-1}$ , so we have  $(I - AB)X = X(I - AB) = I$ , which gives

$$\begin{aligned} X - XAB &= I \text{ and } X - ABX = I, \\ \Rightarrow XAB &= ABX = X - I. \end{aligned}$$

Thus, we know

$$\begin{aligned} (I - BA)(I + BXA) &= I + BXA - BA - BABXA \\ &= I + BXA - BA - B(X - I)A \\ &= I + BXA - BA - (BXA - BA) = I, \end{aligned}$$

so  $(I - BA)^{-1} = I + BXA = I + B(I - AB)^{-1}A$ , and we're done.  $\blacksquare$

**Problem 0.0.3 (Exercise 9).** Use the result of Exercise 8 to prove that, if  $A$  and  $B$  are  $n \times n$  matrices over the field  $F$ , then  $AB$  and  $BA$  have precisely the same characteristic values in  $F$ .

**Remark 0.0.1 (Exercise 8).** Let  $A$  and  $B$  be  $n \times n$  matrices over the field  $F$ . Prove that if  $(I - AB)$  is invertible, then  $I - BA$  is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

**Proof.** If  $x$  is an eigenvalue of  $AB$ , then

- Case 1:  $x \neq 0$ , then there exists  $v \neq 0$  s.t.  $ABv = xv$ , and thus  $BA(Bv) = xBv$ . Now we claim that  $Bv \neq 0$ . If not, then  $xv = ABv = A0 = 0$ , and since  $x \neq 0$ , so  $v = 0$ , which is a contradiction. Now since  $Bv \neq 0$ , so  $Bv$  is an eigenvector for  $x$  of  $BA$ , so  $x$  is an eigenvalue of  $BA$ .
- Case 2:  $x = 0$ , then  $ABv = 0$  for some  $v \neq 0$  and we have two subcases:

- Subcase 1:  $A$  is invertible, then we know  $Bv = 0$  for  $v \neq 0$ , and since  $A$  is surjective, so there exists  $p$  s.t.  $Ap = v \neq 0$ , and thus  $BAp = Bv = 0$ . Note that  $p \neq 0$  otherwise  $v = Ap = 0$  and it is a contradiction, so we know 0 is an eigenvalue of  $BA$ .
- Subcase 2:  $A$  is not invertible, so there exists  $w \neq 0$  s.t.  $Aw = 0$ , and thus  $BAw = B0 = 0$ , which means 0 is an eigenvalue of  $BA$ .

Thus, we have shown that all eigenvalues of  $AB$  are eigenvalues of  $BA$ . Similarly, we can use same arguments to show all eigenvalues of  $BA$  are eigenvalues of  $AB$ , and thus  $AB$  and  $BA$  have precisely the same characteristic values in  $F$ . ■

**Problem 0.0.4 (Exercise 12).** Use the result of Exercise 11 to prove the following: If  $A$  is a  $2 \times 2$  matrix with complex entries, then  $A$  is similar over  $\mathbb{C}$  to a matrix of one of the two types

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}.$$

**Remark 0.0.2 (Exercise 11).** Let  $N$  be a  $2 \times 2$  complex matrix such that  $N^2 = 0$ . Prove that either  $N = 0$  or  $N$  is similar over  $\mathbb{C}$  to

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

**Proof.** Since  $\mathbb{C}$  is algebraically closed, so we can always get 2 eigenvalues of  $A$ . If two eigenvalues of  $A$  are distinct, then we know  $A$  is diagonalizable and thus it is similar to a matrix of type  $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ . If  $A$ 's two eigenvalues are same, say they are both  $\lambda$ , then if  $A$  is diagonalizable, then  $A$  is also similar to a matrix of the type  $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ . Now we consider the case that  $A$  is not diagonalizable. In this case, we know  $\dim E(\lambda) = 1$ , say  $v \neq 0$  and  $v \in E(\lambda)$ , then we can extend  $\{v\}$  to  $B = \{w, v\}$ , which is a basis of  $\mathbb{C}^2$ . Then, we know

$$A \sim \begin{pmatrix} x & 0 \\ y & \lambda \end{pmatrix},$$

for some  $x, y \in \mathbb{C}$ . Then, we know  $A(w) = xw + yv$ , so if we pick  $w' = \frac{1}{y}w$ , we know

$$A(w') = A\left(\frac{1}{y}w\right) = \frac{1}{y}(xw + yv) = xw' + v,$$

and note that  $B' = \{w', v\}$  is still a basis of  $\mathbb{C}^2$ , so we know

$$A \sim \begin{pmatrix} x & 0 \\ 1 & \lambda \end{pmatrix}.$$

Note that

$$\lambda + \lambda = \text{Tr}(A) = \text{Tr}\begin{pmatrix} x & 0 \\ 1 & \lambda \end{pmatrix} = x + \lambda,$$

so we know  $x = \lambda$ , and we're done. ■

**Problem 0.0.5.** Let  $V$  be the space of  $n \times n$  matrices over  $F$ . Let  $A$  be a fixed  $n \times n$  matrix over  $F$ . Let  $T$  be the linear operator “left multiplication by  $A$ ” on  $V$ . Is it true that  $A$  and  $T$  have the same characteristic values?

**Proof.** The answer is true. If  $\lambda$  is an eigenvalue of  $A$ , then  $\exists v \neq 0$  s.t.  $Av = \lambda v$ , so if we construct

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a matrix  $M \in V$  by all the  $n$  the columns of  $M$  are  $v$ , then we know

$$AM = A(v, v, \dots, v) = (Av, Av, \dots, Av) = (\lambda v, \lambda v, \dots, \lambda v) = \lambda(v, v, \dots, v) = \lambda M,$$

and  $M \neq 0$  is trivial since  $v \neq 0$ . Hence,  $\lambda$  is an eigenvalue of  $T$ .

Now if  $\lambda$  is an eigenvalue of  $T$ , then there exists  $M \neq 0$  s.t.  $AM = \lambda M$ . Suppose the  $i$ -th column of  $M$  is not zero column, say this column is called  $c_i$ , then we know  $Ac_i = \lambda c_i$  since

$$AM = A(\dots, c_i, \dots) = (\dots, Ac_i, \dots) = \lambda M = \lambda(\dots, c_i, \dots) = (\dots, \lambda c_i, \dots).$$

Hence,  $\lambda$  is an eigenvalue of  $A$ .

By above arguments, we know  $A$  and  $T$  have same eigenvalues. ■