Homework 1

Linear Algebra (II), Spring 2025

Deadline: 2/26 (Wed.) 12:10

Exercise 1 (Exercise 1.4)**.** Prove that if I_1 and I_2 are two ideals of $F[x]$, then the set

$$\{f_1(x) + f_2(x) : f_j(x) \in I_j\}$$

is also an ideal of $F[x]$.

Exercise 2 (Exercise 1.5)**.** Let $T : V \to V$ be a linear operator on V. Check the following sets are ideals of $F[x]$:

(i) The set $I_T := \{f(x) \in F[x] : f(T) = 0\}$.

(ii) The set $I_T(v) := \{f(x) \in F[x] : f(T)(v) = 0\}$, where $v \in V$ is a fixed given vector.

(iii) The set $I_T(v, W) := \{f(x) \in F[x] : f(T)(v) \in W\}$, where W is a T-invariant subspace of V and $v \in V$ is a given vector.

(iv) In part (iii), if W is only a subspace of V but not T-invariant, does the statement still hold? Prove it or disprove it by giving a counterexample.

Remark. If V is finite-dimensional, then we know that the first set

$$\{f(x) \in F[x] : f(T) = 0\} = (\mathrm{m}_T(x))$$

is a principal ideal generated by the minimal polynomial of T.

Exercise 3 (Exercise 1.6)**.** Prove that if W is a T-invariant subspace of V and $v_1 - v_2 \in W$, then $I_T(v_1, W) = I_T(v_2, W)$.

Exercise 4 (Exercise 1.9)**.** Prove that $(f(x)) = (g(x))$ if and only if $f(x) = cg(x)$ for some nonzero c in F.

Exercise 5. Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let $v \in V$ be a nonzero vector in V. Denote $W = Z(v; T)$ to be the T-cyclic subspace generated by v.

(a) Show that the T-annihilator of v defined in Definition 1.10 is the minimal polynomial of $T|_W$ and that its degree is equal to $\dim W$.

(b) Deduce that the T-annihilator of v is equal to the characteristic polynomial of $T|_W$.

(c) Show that the degree of the T-annihilator of v is 1 if and only if v is an eigenvector of T.

(There are extra exercises in the next page.)

Extra Exercises

You aren't asked to hand in extra exercises, and solving them will NOT affect your grade.

Definition. For a matrix $A \in M_{n\times n}(F)$, one can show that

$$\{f(x) \in F[x] : f(A) = O\}$$

is an ideal in $F[x]$. Hence it is principal, say, generated by a monic polynomial $g(x)$. Then we define the **minimal polynomial** of A to be $g(x)$ and denote it by $\mathrm{m}_A(x)$.

Exercise 6. Given a nonzero matrix $A \in M_{n\times n}(F)$, show that the sequence of matrices

$$I_n, A, A^2, A^3, A^4, \ldots$$

spans a subspace of $M_{n\times n}(F)$ of dimension k, where k is the degree of minimal polynomial of A.

Exercise 7. Let $A \in M_{n\times n}(F)$ and let $\mathrm{m}_A(x) = \displaystyle\prod_{i=1}^{k}(x-\lambda_i)^{m_i}$ be its minimal polynomial. Find the minimal polynomial of $2n \times 2n$ matrix

$$B = \begin{pmatrix} A & I_n \\ O & A \end{pmatrix}$$

in terms of k, λ_i, and m_i.