4. Since the start and end are in the same city, we can regard the trip as the circular permutation of $2n$ items. We arrange $2n$ cities into $2n$ blocks injectively with each block being numbered. The number 1 block is the start and end of the trip, Then, number the blocks in clockwise order. The city in the number i block is the i-th city that she visited in the trip.

Number the countries from 1 to n and name the city as $C_{i,j}$, where $i \in [n]$ representing that $C_{i,j}$ is in the country i and $j = 1, 2$.

Let S be the set of permutations without the restriction that the two cities from each country should not be visited consecutively.

We have $|S| = (2n)!$.

Define $A_i = \{P \in S : C_{i,1} \text{ and } C_{i,2} \text{ are consecutively in } P\}$.

Our goal is to compute $|S \setminus \bigcup_{i=1}^{n} A_i|$.

$$|S \setminus \bigcup_{i=1}^{n} A_i| = |S| - |\bigcup_{i=1}^{n} A_i| = (2n)! - \sum_{m=1}^{n} \sum_{I \in \binom{[n]}{m}} (-1)^{m+1} |\bigcap_{i \in I} A_i|$$
$$= (2n)! + \sum_{m=1}^{n} (-1)^m \sum_{I \in \binom{[n]}{m}} |\bigcap_{i \in I} A_i|.$$

Note that $\sum_{I \in \binom{[n]}{m}} |\bigcap_{i \in I} A_i|$ is computing there has at least m countries s.t. cities in these countries are visited consecutively. We can first choose m countries from n countries, and define I to be the set of these m countries ($\binom{n}{m}$ ways). Then, we regard $C_{i,1}$ and $C_{i,2}$ as one item C_i no matter the order is $C_{i,1} C_{i,2}$ or $C_{i,2} C_{i,1}$ in the permutation, so now there has $2n-m$ items, and we have $(2n-m)!$ ways to arrange them. Note that $C_{i,1} C_{i,2}$ and $C_{i,2} C_{i,1}$ are actually 2 different order, so by product rule, $\sum_{I \in \binom{[n]}{m}} |\bigcap_{i \in I} A_i| = \binom{n}{m} 2^m (2n-m)!$

$$\Rightarrow |S \setminus \bigcup_{i=1}^{n} A_i| = (2n)! + \sum_{m=1}^{n} (-1)^m \binom{n}{m} 2^m (2n-m)! = \sum_{i=0}^{n} (-1)^i \binom{n}{i} 2^i (2n-i)!.$$

5. Let $m(x) = |\{i : x \in A_i\}|$.

We know that every element $x \in \bigcup_{i=1}^{r} A_i$ will be counted excatly once in $|\bigcup_{i=1}^{r} A_i|$. However, x will be counted $\binom{m(x)}{k}$ times in $\sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i|$, so it will be counted $\sum_{k=1}^{k_0} (-1)^{k+1} \binom{m(x)}{k}$ times in $\sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i|$.

Recall that we have proven that $\sum_{j=0}^{k} (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k}$ in HW1.

Now we consider that

$$1 - \sum_{k=1}^{k_0} (-1)^{k+1} \binom{m(x)}{k} = (-1)^0 \binom{m(x)}{0} + \sum_{k=1}^{k_0} (-1)^k \binom{m(x)}{k} = \sum_{k=0}^{k_0} (-1)^k \binom{m(x)}{k} = (-1)^{k_0} \binom{m(x)-1}{k_0}.$$

Hence, if k_0 is even, then $(-1)^{k_0} \binom{m(x)-1}{k} \geq 0$, which means that the number of times x counted in $|\bigcup_{i=1}^{n} A_i|$ will not less than that in $\sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i|$. Hence,

$$|\bigcup_{i=1}^{n} A_i| \geq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i| \quad \text{for } k_0 \text{ even}.$$

Similarly,

$$|\bigcup_{i=1}^{n} A_i| \leq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i| \quad \text{for } k_0 \text{ odd}.$$