

Math 2213 Introduction to Analysis
Total =110 pts, Homework 4 Due October 3 (Friday), 2025
Please submit your homework online in PDF format.

- (1) (16 pts)
 (a) Let

$$X := \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

be the space of absolutely convergent sequences. Define the ℓ^1 and ℓ^∞ metrics on this space by

$$d_{\ell^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sum_{n=0}^{\infty} |a_n - b_n|,$$

$$d_{\ell^\infty}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sup_{n \in \mathbb{N}} |a_n - b_n|.$$

Show that these are both metrics on X , but show that there exist sequences

$$x^{(1)}, x^{(2)}, \dots$$

of elements of X (i.e. sequences of sequences) which are convergent with respect to the d_{ℓ^∞} metric but not with respect to the d_{ℓ^1} metric. Conversely, show that any sequence which converges in the d_{ℓ^1} metric automatically converges in the d_{ℓ^∞} metric.

- (b) Let (X, d_{ℓ^1}) be the metric space from part (a). For each natural number n , let $e^{(n)} = (e_j^{(n)})_{j=0}^{\infty}$ be the sequence in X such that

$$e_j^{(n)} := \begin{cases} 1, & \text{if } n = j, \\ 0, & \text{if } n \neq j. \end{cases}$$

Show that the set

$$\{e^{(n)} : n \in \mathbb{N}\}$$

is a closed and bounded subset of X , but is not compact.

(This is despite the fact that (X, d_{ℓ^1}) is even a complete metric space—a fact which we will not prove here. The problem is not that X is incomplete, but rather that it is “infinite-dimensional,” in a sense that we will not discuss here.)

- (2) (24 pts) A metric space (X, d) is called *totally bounded* if for every $\varepsilon > 0$, there exists a natural number n and a finite number of balls

$$B(x^{(1)}, \varepsilon), B(x^{(2)}, \varepsilon), \dots, B(x^{(n)}, \varepsilon)$$

which cover X (i.e. $X = \bigcup_{i=1}^n B(x^{(i)}, \varepsilon)$).

- (a) Show that every totally bounded space is bounded.

- (b) Show the following stronger version of Proposition 1.5.5: if (X, d) is compact, then it is complete and totally bounded. *Hint:* if X is not totally bounded, then there is some $\varepsilon > 0$ such that X cannot be covered by finitely many ε -balls. Then use Exercise 8.5.20 (on page 182 of Analysis I) to find an infinite sequence of balls $B(x^{(n)}, \varepsilon/2)$ which are disjoint from each other. Use this to construct a sequence which has no convergent subsequence.
- (c) Conversely, show that if X is complete and totally bounded, then X is compact. *Hint:* if $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X , use the total boundedness hypothesis to recursively construct a sequence of subsequences $(x^{(n;j)})_{n=1}^{\infty}$ of $(x^{(n)})_{n=1}^{\infty}$ for each positive integer j , such that for each j the elements of the sequence $(x^{(n;j)})_{n=1}^{\infty}$ are contained in a single ball of radius $1/j$. Also ensure that each sequence $(x^{(n;j+1)})_{n=1}^{\infty}$ is a subsequence of the previous one $(x^{(n;j)})_{n=1}^{\infty}$. Then show that the “diagonal” sequence $(x^{(n;n)})_{n=1}^{\infty}$ is a Cauchy sequence, and then use the completeness hypothesis.
- (3) (16 pts)
- (a) A metric space (X, d) is compact if and only if every sequence in X has at least one limit point in X .
- (b) Let (X, d) have the property that every open cover of X has a finite subcover. Show that X is compact.
Hint: If X is not compact, then by part (a) there is a sequence $(x^{(n)})_{n=1}^{\infty}$ with no limit points. Then for every $x \in X$ there exists a ball $B(x, \varepsilon)$ containing x which contains at most finitely many elements of this sequence. Now use the hypothesis.
- (4) (10 pts) Let (X, d) be a compact metric space. Suppose that $(K_{\alpha})_{\alpha \in I}$ is a collection of closed sets in X with the property that any finite subcollection of these sets necessarily has non-empty intersection, thus

$$\bigcap_{\alpha \in F} K_{\alpha} \neq \emptyset \quad \text{for all finite } F \subseteq I.$$

(This property is known as the *finite intersection property*.)

Show that the entire collection has non-empty intersection, thus

$$\bigcap_{\alpha \in I} K_{\alpha} \neq \emptyset.$$

Show by counterexample that this statement fails if X is not compact.

- (5) (24 pts)
- (a) Let (X, d) be a metric space, and let $(E, d|_{E \times E})$ be a subspace of (X, d) . Let $\iota_{E \rightarrow X} : E \rightarrow X$ be the inclusion map, defined by setting

$$\iota_{E \rightarrow X}(x) := x \quad \text{for all } x \in E.$$

Show that $\iota_{E \rightarrow X}$ is continuous.

- (b) Let $f : X \rightarrow Y$ be a function from one metric space (X, d_X) to another (Y, d_Y) . Let E be a subset of X (which we give the induced metric $d_X|_{E \times E}$), and let $f|_E : E \rightarrow Y$ be the restriction of f to E , thus

$$f|_E(x) := f(x) \quad \text{when } x \in E.$$

If $x_0 \in E$ and f is continuous at x_0 , show that $f|_E$ is also continuous at x_0 . (Is the converse of this statement true? Explain.)

Conclude that if f is continuous, then $f|_E$ is continuous. Thus restriction of the domain of a function does not destroy continuity.

Hint: use part (a).

- (c) Let $f : X \rightarrow Y$ be a function from one metric space (X, d_X) to another (Y, d_Y) . Suppose that the image $f(X)$ of X is contained in some subset $E \subseteq Y$ of Y . Let $g : X \rightarrow E$ be the function which is the same as f but with the codomain restricted from Y to E , thus $g(x) = f(x)$ for all $x \in X$.

Note on codomain: The *codomain* of a function is the declared target set of the function, in contrast to the *image* (or range), which is the set of values the function actually takes. So while f is originally defined with codomain Y , its values all lie in the smaller set $E \subseteq Y$. Therefore, one can equivalently regard f as a function $g : X \rightarrow E$. The metric on E is the one *induced from* Y , i.e. $d_Y|_{E \times E}$.

Show that for any $x_0 \in X$, f is continuous at x_0 if and only if g is continuous at x_0 . Conclude that f is continuous if and only if g is continuous.

(Thus the notion of continuity is not affected if one restricts the codomain of the function.)

- (6) (20 pts)

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \rightarrow Y$ is a function from X to Y .

- (a) Prove that f is continuous on X if, and only if,

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset A of X .

- (b) Prove that f is continuous on X if, and only if, f is continuous on every compact subset of X .

Hint: If $x_n \rightarrow p$ in X , the set $\{p, x_1, x_2, \dots\}$ is compact.