Introduction to Analysis I HW3

B13902024 張沂魁

September 23, 2025

Problem 0.0.1. Let $(x^{(n)})_{n=m}^\infty$ be a sequence of points in a metric space (X, d), and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^\infty$, then L is an adherent point of the set

$$S = \{x^{(n)} : n \geq m\}.$$

Is the converse true?

Problem 0.0.2. The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

(a) Given any Cauchy sequence $(x_n)_{n=1}^\infty$ in X, we introduce the *formal limit*

$$\mathrm{LIM}_{n \to \infty} x_n.$$

We say that two formal limits $\mathrm{LIM}_{n \to \infty} x_n$ and $\mathrm{LIM}_{n \to \infty} y_n$ are equal if

$$\lim_{n \to \infty} d(x_n, y_n) = 0.$$

Show that this equality relation obeys the reflexive, symmetry, and transitive axioms, i.e. that it is an equivalence relation.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, modulo the above equivalence relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to [0, \infty)$ by

$$d_{\overline{X}}(\mathrm{LIM}_{n \to \infty} x_n, \mathrm{LIM}_{n \to \infty} y_n) := \lim_{n \to \infty} d(x_n, y_n).$$

Show that this function is well-defined (the limit exists and does not depend on the choice of representatives) and that it satisfies the axioms of a metric. Thus $(\overline{X}, d_{\overline{X}})$ is a metric space.

(c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.

(d) We identify an element $x \in X$ with the corresponding constant Cauchy sequence $(x, x, x, \ldots)$, i.e. with the formal limit $\mathrm{LIM}_{n \to \infty} x$. Show that this is legitimate: for $x, y \in X$,

$$x = y \iff \mathrm{LIM}_{n \to \infty} x = \mathrm{LIM}_{n \to \infty} y.$$

With this identification, show that

$$d(x, y) = d_{\overline{X}}(x, y),$$

and thus (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) Show that the closure of X in $\overline{X}$ is $\overline{X}$ itself. (This explains the choice of notation.)

(f) Finally, show that the formal limit agrees with the actual limit: if $(x_n)_{n=1}^\infty$ is a Cauchy sequence in X that converges in X, then

$$\lim_{n \to \infty} x_n = \mathrm{LIM}_{n \to \infty} x_n \quad \text{in } \overline{X}.$$

a. We verify the following properties:

- Reflexive: $\mathrm{LIM}_{n \to \infty} x_n$ and $\mathrm{LIM}_{n \to \infty} x_n$ are equal since d is metric, so $\forall n, d(x_n, x_n) = 0$.

- Symmetry: If $\mathrm{LIM}_{n \to \infty} x_n$ and $\mathrm{LIM}_{n \to \infty} y_n$ are equal, this mean $\lim_{n \to \infty} d(x_n, y_n) = 0$. And since d is metric, so $\lim_{n \to \infty} d(y_n, x_n) = 0$, hence $\mathrm{LIM}_{n \to \infty} y_n$ and $\mathrm{LIM}_{n \to \infty} x_n$ are equal.

- Transitive: If $\mathrm{LIM}_{n \to \infty} x_n$ and $\mathrm{LIM}_{n \to \infty} y_n$ are equal and $\mathrm{LIM}_{n \to \infty} y_n$ and $\mathrm{LIM}_{n \to \infty} z_n$ are equal, then we have $\lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(y_n, z_n) = 0$. By definition, there exists $N_1, N_2 > 0$ s.t. for all $n \geq N_1$, we have $d(x_n, y_n) < \frac{\varepsilon}{2}$ and for all $n \geq N_2$ we have $d(y_n, z_n) < \frac{\varepsilon}{2}$.

Thus, for all $n \geq \max\{N_1, N_2\}$, we have

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means $\lim_{n \to \infty} d(x_n, z_n) = 0$, and thus $\mathrm{LIM}_{n \to \infty} x_n = \mathrm{LIM}_{n \to \infty} z_n$.

∎

b. We first show that the limit exists. Note that $\lim_{n \to \infty} d(x_n, y_n) \in \mathbb{R}_{\geq 0}$ for all Cauchy sequence $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ in X. We already know $(\mathbb{R}, |\cdot|)$ is complete, so we know $(\mathbb{R}_{\geq 0}, |\cdot|)$ is also complete as it is a closed subspace of $(\mathbb{R}, |\cdot|)$. Now we define $u_n := d(x_n, y_n)$ for all $n \geq 1$, then we want to show that $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}_{\geq 0}$, and then we can conclude that $\{u_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}_{\geq 0}$, and thus $\lim_{n \to \infty} d(x_n, y_n)$ exists.

> **Claim 0.0.1.** Suppose (X, d) is a metric space, then for all $a, b, c, d \in X$ we have
>
> $$|d(a, b) - d(c, d)| \leq d(a, c) + d(b, d)$$
>
> **Proof.** Since
>
> $$\begin{cases} d(a, b) \leq d(a, c) + d(c, b) \leq d(a, c) + d(c, d) + d(d, b) \\ d(c, d) \leq d(c, a) + d(a, d) \leq d(c, a) + d(a, b) + d(b, d), \end{cases}$$
>
> so we have
>
> $$\begin{cases} d(a, b) - d(c, d) \geq d(a, c) + d(d, b) \\ -d(c, a) - d(b, d) \leq d(a, b) - d(c, d), \end{cases}$$
>
> so we can conbine these two equations and get the result. ⊛

By Claim 0.0.1, we know for all $p, q \geq 1$, we have

$$|u_p - u_q| = |d(x_p, y_p) - d(x_q, y_q)| \leq d(x_p, x_q) + d(y_p, y_q).$$

Now since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy, so for every $\varepsilon > 0$, there exists $N_1, N_2 > 0$ s.t.

$$\begin{cases} d(x_p, x_q) < \dfrac{\varepsilon}{2} & \forall p, q \geq N_1 \\ d(y_p, y_q) < \dfrac{\varepsilon}{2} & \forall p, q \geq N_2. \end{cases}$$

Thus, for all $p, q \geq \max\{N_1, N_2\}$, we know

$$|u_p - u_q| \leq d(x_p, x_q) + d(y_p, y_q) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we know $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}_{\geq 0}, |\cdot|$.

Now we show that $d_{\overline{X}}$ is well-defined. In other words, if $\mathrm{LIM}_{n \to \infty} x_n = \mathrm{LIM}_{n \to \infty} z_n$, then we want to show

$$d_{\overline{X}}(\mathrm{LIM}_{n \to \infty} x_n, \mathrm{LIM}_{n \to \infty} y_n) = d_{\overline{X}}(\mathrm{LIM}_{n \to \infty} z_n, \mathrm{LIM}_{n \to \infty} y_n) \quad \forall \text{ Cauchy } \{y_n\}_{n=1}^{\infty} \text{ in } (X, d).$$

Equivalently, we want to show $\lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(z_n, y_n)$. Note that we have

$$\lim_{n \to \infty} d(x_n, z_n) = 0 \text{ and } d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n),$$

so we know

$$\lim_{n \to \infty} d(x_n, y_n) \leq \lim_{n \to \infty} d(x_n, z_n) + \lim_{n \to \infty} d(z_n, y_n) = \lim_{n \to \infty} d(z_n, y_n).$$

Also, we have $d(z_n, y_n) \leq d(z_n, x_n) + d(x_n, y_n)$, so we know

$$\lim_{n \to \infty} d(z_n, y_n) \leq \lim_{n \to \infty} d(z_n, x_n) + \lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(x_n, y_n),$$

and thus we can conclude that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(z_n, y_n)$.

Finally, we want to show that $\left(\overline{X}, d_{\overline{X}}\right)$ is a metric space.

- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty} \in X$, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} x_n) = \lim_{n\to\infty} d(x_n, x_n) = 0$.

- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \in X$,

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n)$$
$$= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} y_n, \mathrm{LIM}_{n\to\infty} x_n)$$

- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \in X$,

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} z_n) = \lim_{n\to\infty} d(x_n, z_n)$$
$$\leq \lim_{n\to\infty} (d(x_n, y_n) + d(y_n, z_n)) = \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n)$$
$$= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) + d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} y_n, \mathrm{LIM}_{n\to\infty} z_n).$$

Hence, we know $\left(\overline{X}, d_{\overline{X}}\right)$ is a metric space. $\blacksquare$

c. We want to show that for all $\{u_n\}_{n=1}^{\infty} \subseteq \overline{X}$, there exists $\{z_n\}_{n=1}^{\infty} \subseteq X$ s.t. $\lim_{n\to\infty} u_n = \mathrm{LIM}_{n\to\infty} z_n$. Since $\{u_n\}_{n=1}^{\infty}$ is a sequence of formal limit of Cauchy sequences in X, so we can define $u_k = \mathrm{LIM}_{n\to\infty} x_n^{(k)}$ for all $k \geq 1$. Now we construct $\{z_n\}_{n=1}^{\infty}$. Since we know for all $k \geq 1$, $\left\{x_n^{(k)}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in X, so for all $k \geq 1$, there exists $N_k > 0$ s.t. $n \geq N_k$ implies $d\left(x_n^{(k)}, x_{N_k}^{(k)}\right) < \frac{1}{k}$. Now we let $z_k = x_{N_k}^{(k)}$ for all $k \geq 1$.

> **Claim 0.0.2.** $\{z_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X.

> **Proof.** For all $\varepsilon > 0$, we know there exists $K \geq 0$ s.t. $\frac{1}{K} < \frac{\varepsilon}{3}$. Also, since $\{u_n\}_{n=1}^{\infty}$ is Cauchy, so there exists $N > 0$ s.t. $i, j \geq N$ implies $d_{\overline{X}}(u_i, u_j) < \frac{\varepsilon}{3}$, which can be written as $\lim_{n\to\infty} d\left(x_n^{(i)}, x_n^{(j)}\right) < \frac{\varepsilon}{3}$. To be more precise, there exists $N > 0$ and $N' > 0$ s.t. if $i, j \geq N$ and $n \geq N'$, then $d\left(x_n^{(i)}, x_n^{(j)}\right) < \frac{\varepsilon}{3}$. Now for all $p, q \geq \max\{N, K\}$ and $n \geq \max\{N_p, N_q, N'\}$, we have
>
> $$d(z_p, z_q) = d\left(x_{N_p}^{(p)}, x_{N_q}^{(q)}\right) \leq d\left(x_{N_p}^{(p)}, x_n^{(p)}\right) + d\left(x_n^{(p)}, x_{N_q}^{(q)}\right)$$
> $$\leq d\left(x_{N_p}^{(p)}, x_n^{(p)}\right) + d\left(x_n^{(p)}, x_n^{(q)}\right) + d\left(x_n^{(q)}, x_{N_q}^{(q)}\right)$$
> $$< \frac{1}{p} + \varepsilon + \frac{1}{q} < \frac{1}{K} + \frac{\varepsilon}{3} + \frac{1}{K} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

> Hence, we know $\{z_k\}_{k=1}^{\infty}$ is Cauchy. $\circledast$

> **Claim 0.0.3.** $\lim_{n\to\infty} u_n = \mathrm{LIM}_{n\to\infty} z_n$.

Proof. Suppose $L = \text{LIM}_{n \to \infty} z_n$. For all $\varepsilon > 0$, we want to show there exists $N > 0$ s.t. $m \geq N$ implies $d_{\overline{X}}(u_m, L) < \varepsilon$, which is equivalent to $\lim_{n \to \infty} d\left(x_n^{(m)}, z_n\right) < \varepsilon$. To be more precise, we want to show there exists $N \geq 0$ and $N' > 0$ s.t. if $m \geq N$ and $n \geq N'$, then $d\left(x_n^{(m)}, z_n\right) < \varepsilon$. Note that $d\left(x_n^{(m)}, z_n\right) \leq d\left(x_n^{(m)}, z_m\right) + d(z_m, z_n)$. Suppose $K > 0$ has $\frac{1}{K} < \frac{\varepsilon}{2}$, we know such K exists. Also, since $\{z_n\}_{n=1}^{\infty}$ is Cauchy, so we know there exists $N_1' > 0$ s.t. for all $p, q \geq N_1'$, we have $d(z_p, z_q) < \frac{\varepsilon}{2}$. Hence, if we pick $m \geq \max\{K, N_1'\}$ and $n \geq \max\{N_m, N_1'\}$, then

$$d\left(x_n^{(m)}, z_n\right) \leq d\left(x_n^{(m)}, z_m\right) + d(z_m, z_n) < \frac{1}{m} + \frac{\varepsilon}{2}$$
$$\leq \frac{1}{K} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and we're done. ⊛

By Claim 0.0.2 and Claim 0.0.3, we know every Cauchy sequence in $\overline{X}$ converges to a formal limit of a Cauchy sequence of X, which means it converges in $\overline{X}$, and thus $\left(\overline{X}, d_{\overline{X}}\right)$ is complete. ∎

d. We first show that $x = y \Leftrightarrow \text{LIM}_{n \to \infty} x = \text{LIM}_{n \to \infty} y$. If $x = y$, then we know

$$\lim_{n \to \infty} d(x, y) = \lim_{n \to \infty} d(x, x) = 0,$$

which means $\text{LIM}_{n \to \infty} x = \text{LIM}_{n \to \infty} y$. Now we prove the converse, if $\text{LIM}_{n \to \infty} x = \text{LIM}_{n \to \infty} y$, then we know $\lim_{n \to \infty} d(x, y) = d(x, y) = 0$, so $x = y$.

Now we show that $d(x, y) - d_{\overline{X}}(x, y)$. Note that

$$d_{\overline{X}}(x, y) = \lim_{n \to \infty} d(x, y) = d(x, y),$$

so this is true. ∎

e. Since we know $\text{cl}_{\overline{X}}(X) \subseteq \overline{X}$, we only need to show $\overline{X} \subseteq \text{cl}_{\overline{X}}(X)$. Suppose $x \in \overline{X}$, then $x = \text{LIM}_{n \to \infty} x_n$ where $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Now we want to show that $x \in \text{cl}_{\overline{X}}(X)$, which is equivalent to show for all $\varepsilon > 0$, there exists $y \in X$ s.t. $y \in B_{\overline{X}}(x, \varepsilon)$. If such y exists, then $d_{\overline{X}}(x, y) < \varepsilon$, which means $\lim_{n \to \infty} d(x_n, y) < \varepsilon$. However, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, so there exists $N > 0$ s.t. $i, j \geq N$ implies $d(x_i, x_j) < \frac{\varepsilon}{2}$. Thus, we can pick $y = x_N$, and then we have for all $n \geq N$, $d(x_n, y) < \frac{\varepsilon}{2} < \varepsilon$ Hence, we have $\lim_{n \to \infty} d(x_n, y) < \varepsilon$, and we're done. ∎

f. Since $\{x_n\}_{n=1}^{\infty}$ can be seen as a sequence of elements in $\overline{X}$, and notice that $\{x_n\}_{n=1}^{\infty}$ is still Cauchy in $\overline{X}$ since for all $\varepsilon > 0$, we know there exists $N > 0$ s.t. $p, q \geq N$ implies $d(x_p, x_q) < \varepsilon$, so under same circumstances, we know

$$d_{\overline{X}}(x_p, x_q) = \lim_{n \to \infty} d(x_p, x_q) < \varepsilon,$$

and we're done. Now since we have proved $\overline{X}$ is complete in (c), so we know there exists $L \in \overline{X}$ s.t. $\lim_{n \to \infty} x_n = L$. Also, since $L \in \overline{X}$, so $L = \text{LIM}_{n \to \infty} a_n$ for some Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ in X. Now we want to show $\text{LIM}_{n \to \infty} a_n = \text{LIM}_{n \to \infty} x_n$. Hence, we want to show $\lim_{n \to \infty} d(a_n, x_n) = 0$, which is equivalent to prove $\forall \varepsilon > 0, \exists N > 0$ s.t. $n \geq N$ implies $d(a_n, x_n) < \varepsilon$.

- Notice that since $\lim_{n \to \infty} x_n = L \in \overline{X}$, so $\forall \varepsilon > 0, \exists N_1 > 0$ s.t. $p \geq N_1$ implies $d_{\overline{X}}(x_p, L) < \frac{\varepsilon}{2}$, and thus $\lim_{n \to \infty} d(x_p, a_n) < \frac{\varepsilon}{2}$. Hence, there exists $M > 0$ s.t. if $p \geq N_1$ and $n \geq M$, then $d(x_p, a_n) < \frac{\varepsilon}{2}$.

- Also, since $\{x_n\}_{n=1}^{\infty}$ is Cauchy in X, so there exists $N_2 > 0$ s.t. $p, q \geq N_2$ implies $d(x_p, x_q) < \frac{\varepsilon}{2}$.

Use the above two properties, we know for all $n \geq \max\{M, N_2\}$ we can choose $s \geq \max\{N_1, N_2\}$ so that

$$d(a_n, x_n) \leq d(a_n, x_s) + d(x_s, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and we're done. ∎

Problem 0.0.3. In the following, all the sets are subsets of a metric space (X, d).

(a) If $\overline{A} \cap \overline{B} = \varnothing$, then
$$\partial(A \cup B) = \partial A \cup \partial B.$$

(b) For a finite family $\{A_i\}_{i=1}^n \subseteq X$, show that
$$\text{int}\left(\bigcap_{i=1}^n A_i\right) = \bigcap_{i=1}^n \text{int}(A_i).$$

(c) For an arbitrary (possibly infinite) family $\{A_\alpha\}_{\alpha \in F} \subseteq X$, prove that
$$\text{int}\left(\bigcap_{\alpha \in F} A_\alpha\right) \subseteq \bigcap_{\alpha \in F} \text{int}(A_\alpha).$$

(d) Give an example where the inclusion in part (c) is strict (i.e., equality fails).

(e) For any family $\{A_\alpha\}_{\alpha \in F} \subseteq M$, prove that
$$\bigcup_{\alpha \in F} \text{int}(A_\alpha) \subseteq \text{int}\left(\bigcup_{\alpha \in F} A_\alpha\right).$$

(f) Give an example of a finite collection F in which equality does not hold in part (e).

Problem 0.0.4. Let (X, d) be a metric space and $Y \subset X$ be an open subset. For any subset $A \subset Y$, show that A is open in Y if and only if it is open in X.

Problem 0.0.5. On the space $(0, 1]$, we may consider the topology induced by the metric space $(\mathbb{R}, d)$ defined by $d(x, y) = |x - y|$. Alternatively, we may also define a distance d' on $(0, 1]$, given by
$$d'(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \qquad \forall x, y \in (0, 1].$$

(a) Show that d' is a metric on $(0, 1]$

(b) Let $x \in (0, 1]$ and $\varepsilon > 0$. Let $B = B_d(x, \varepsilon) = \{y \mid |y - x| < \varepsilon\} \cap (0, 1]$ be the open ball centered at x of radius ε for the metric d in $(0, 1]$. Show that for any $y \in B$, we may find $\varepsilon' > 0$ such that
$$B_{d'}(y, \varepsilon') \subseteq B = B_d(x, \varepsilon).$$

(c) Show that an open ball in $((0, 1], d')$ is also an open ball in $((0, 1], d)$.

(d) Conclude that the metric spaces $((0, 1], d)$ and $((0, 1], d')$ are topologically equivalent, that is, a set A is open in one space if and only if it is also open in the other one.

(e) Is $((0, 1], d')$ a complete metric space? How about $((0, 1], d)$?

Problem 0.0.6. (a) We say that a family of closed balls
$$\left(\overline{B}(x_n, r_n)\right)_{n \geq 1}$$
is a *decreasing sequence of closed balls* if the nesting condition
$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$
is satisfied. Give an example of a decreasing sequence of closed balls in a complete metric

space with empty intersection.

(b) We say that a family of closed balls

$$\left(\overline{B}(x_n, r_n)\right)_{n \geq 1}$$

is a *decreasing sequence of closed balls with radii tending to zero* if

$$r_n \ \to \ 0 \quad \text{as } n \to \infty,$$

and the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \ \subseteq \ \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Show that a metric space (M, d) is complete if and only if every decreasing sequence of closed balls with radii going to zero has a nonempty intersection.