# Introduction to Analysis I HW3

B13902024 張沂魁

September 27, 2025

**Problem 0.0.1** (16pts).

(a) Let

$$X := \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

be the space of absolutely convergent sequences. Define the $\ell^1$ and $\ell^\infty$ metrics on this space by

$$d_{\ell^1}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) := \sum_{n=0}^{\infty} |a_n - b_n|,$$
$$d_{\ell^\infty}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) := \sup_{n\in\mathbb{N}} |a_n - b_n|.$$

Show that these are both metrics on $X$, but show that there exist sequences

$$x^{(1)}, x^{(2)}, \dots$$

of elements of $X$ (i.e. sequences of sequences) which are convergent with respect to the $d_{\ell^\infty}$ metric but not with respect to the $d_{\ell^1}$ metric. Conversely, show that any sequence which converges in the $d_{\ell^1}$ metric automatically converges in the $d_{\ell^\infty}$ metric.

(b) Let $(X, d_{\ell^1})$ be the metric space from part (a). For each natural number $n$, let $e^{(n)} = (e_j^{(n)})_{j=0}^{\infty}$ be the sequence in $X$ such that

$$e_j^{(n)} := \begin{cases} 1, & \text{if } n = j, \\ 0, & \text{if } n \neq j. \end{cases}$$

Show that the set

$$\{e^{(n)} : n \in \mathbb{N}\}$$

is a closed and bounded subset of $X$, but is not compact.

(This is despite the fact that $(X, d_{\ell^1})$ is even a complete metric space—a fact which we will not prove here. The problem is not that $X$ is incomplete, but rather that it is "infinite-dimensional," in a sense that we will not discuss here.)

**Problem 0.0.2** (24pts). A metric space $(X, d)$ is called *totally bounded* if for every $\varepsilon > 0$, there exists a natural number $n$ and a finite number of balls

$$B(x^{(1)}, \varepsilon),\ B(x^{(2)}, \varepsilon),\ \dots,\ B(x^{(n)}, \varepsilon)$$

which cover $X$ (i.e. $X = \bigcup_{i=1}^{n} B(x^{(i)}, \varepsilon)$).

(a) Show that every totally bounded space is bounded.

(b) Show the following stronger version of Proposition 1.5.5: if $(X, d)$ is compact, then it is complete and totally bounded. *Hint:* if $X$ is not totally bounded, then there is some $\varepsilon > 0$ such that $X$ cannot be covered by finitely many $\varepsilon$-balls. Then use Exercise 8.5.20 (on page 182 of Analysis I)to find an infinite sequence of balls $B(x^{(n)}, \varepsilon/2)$ which are disjoint from each other. Use this to construct a sequence which has no convergent subsequence.

(c) Conversely, show that if $X$ is complete and totally bounded, then $X$ is compact. *Hint:* if $(x^{(n)})_{n=1}^{\infty}$ is a sequence in $X$, use the total boundedness hypothesis to recursively construct a sequence of subsequences $(x^{(n;j)})_{n=1}^{\infty}$ of $(x^{(n)})_{n=1}^{\infty}$ for each positive integer $j$, such that for each $j$ the elements of the sequence $(x^{(n;j)})_{n=1}^{\infty}$ are contained in a single ball of radius $1/j$. Also ensure that each sequence $(x^{(n;j+1)})_{n=1}^{\infty}$ is a subsequence of the previous one $(x^{(n;j)})_{n=1}^{\infty}$. Then show that the "diagonal" sequence $(x^{(n;n)})_{n=1}^{\infty}$ is a Cauchy sequence, and then use the completeness hypothesis.

**Problem 0.0.3** (16pts).

(a) A metric space $(X, d)$ is compact if and only if every sequence in $X$ has at least one limit point in $X$.

(b) Let $(X, d)$ have the property that every open cover of $X$ has a finite subcover. Show that $X$ is compact.

*Hint:* If $X$ is not compact, then by part (a) there is a sequence $(x^{(n)})_{n=1}^{\infty}$ with no limit points. Then for every $x \in X$ there exists a ball $B(x, \varepsilon)$ containing $x$ which contains at most finitely many elements of this sequence. Now use the hypothesis.

**Problem 0.0.4** (10pts). Let $(X, d)$ be a compact metric space. Suppose that $(K_\alpha)_{\alpha \in I}$ is a collection of closed sets in $X$ with the property that any finite subcollection of these sets necessarily has non-empty intersection, thus

$$\bigcap_{\alpha \in F} K_\alpha \neq \varnothing \quad \text{for all finite } F \subseteq I.$$

(This property is known as the *finite intersection property.*)

Show that the entire collection has non-empty intersection, thus

$$\bigcap_{\alpha \in I} K_\alpha \neq \varnothing.$$

Show by counterexample that this statement fails if $X$ is not compact.

**Problem 0.0.5** (24pts).

(a) Let $(X, d)$ be a metric space, and let $(E, d|_{E \times E})$ be a subspace of $(X, d)$. Let $\iota_{E \to X} : E \to X$ be the inclusion map, defined by setting

$$\iota_{E \to X}(x) := x \quad \text{for all } x \in E.$$

Show that $\iota_{E \to X}$ is continuous.

(b) Let $f : X \to Y$ be a function from one metric space $(X, d_X)$ to another $(Y, d_Y)$. Let $E$ be a subset of $X$ (which we give the induced metric $d_X|_{E \times E}$), and let $f|_E : E \to Y$ be the restriction of $f$ to $E$, thus

$$f|_E(x) := f(x) \quad \text{when } x \in E.$$

If $x_0 \in E$ and $f$ is continuous at $x_0$, show that $f|_E$ is also continuous at $x_0$. (Is the converse of this statement true? Explain.)

Conclude that if $f$ is continuous, then $f|_E$ is continuous. Thus restriction of the domain of a function does not destroy continuity.

*Hint: use part (a).*

(c) Let $f : X \to Y$ be a function from one metric space $(X, d_X)$ to another $(Y, d_Y)$. Suppose that the image $f(X)$ of $X$ is contained in some subset $E \subseteq Y$ of $Y$. Let $g : X \to E$ be the function which is the same as $f$ but with the codomain restricted from $Y$ to $E$, thus $g(x) = f(x)$ for all $x \in X$.

**Note on codomain:** The *codomain* of a function is the declared target set of the function, in contrast to the *image* (or range), which is the set of values the function actually takes. So while $f$ is originally defined with codomain $Y$, its values all lie in the smaller set $E \subseteq Y$. Therefore, one can equivalently regard $f$ as a function $g : X \to E$. The metric on $E$ is the one *induced from* $Y$, i.e. $d_Y|_{E \times E}$.

Show that for any $x_0 \in X$, $f$ is continuous at $x_0$ if and only if $g$ is continuous at $x_0$. Conclude that $f$ is continuous if and only if $g$ is continuous.

(Thus the notion of continuity is not affected if one restricts the codomain of the function.)

**Problem 0.0.6** (20pts). Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \mapsto Y$ is a function from $X$ to $Y$.

(a) Prove that $f$ is continuous on $X$ if, and only if,

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset $A$ of $X$.

(b) Prove that $f$ is continuous on $X$ if, and only if, $f$ is continuous on every compact subset of $X$.

*Hint:* If $x_n \to p$ in $X$, the set $\{p, x_1, x_2, \dots\}$ is compact.