# 微積分一

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**Abstract**

This note is the lecture note of the 齊震宇微積分一 on NTU OCW.

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# Chapter 1

# Some Basics

## 1.1 Introduction

### Lecture 1: Introduction

8 July. 21:00

**Definition 1.1.1.** Suppose we have a function $f(x)$, then the signed area between $x = a$ and $x = b$ is called the definite integral of the function $f$ on the interval $[a, b]$.

**Notation.** $\int_a^b f(x)\,\mathrm{d}x$ is the definite integral of $f$ on the interval $[a, b]$.

Let $A(x)$ be the signed area of the $f(x)$ on the interval $[a, x]$ of the below figure,

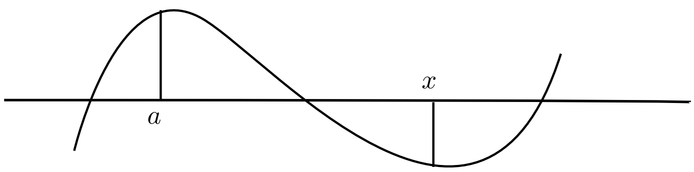


Figure 1.1: $y = f(x)$

then we can draw the figure of $A(x)$.

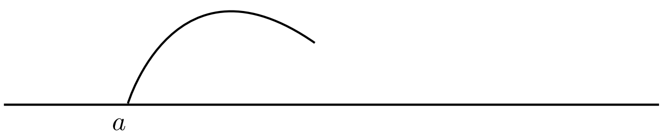


Figure 1.2: A part of $y = A(x)$.

Now we want to show that $y = A'(x)$ is the graph of $y = f(x)$. Compute

$$A'(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{\text{signed area of } f(x) \text{ on } [x, x+h]}{h} \approx f(x)$$

since $h \to 0$, so the area divided by $h$ is approximately equal to $f(x)$.

Also, we know $\int_a^b f(x)\,\mathrm{d}x = A(b)$, but we only know $A(a) = 0$ and $A'(x) = f(x)$.

Suppose that we have found a function $g(x)$ such that $g'(x) = f(x)$. What is the relation between $g$ and $A$?

Consider the function $g(x) - A(x)$, we know its derivative is 0. Hence, we know $g(x) - A(x) = \text{constant}$. Therefore, we have

$$g(b) - g(a) = (A(b) + C) - (A(a) + C) = A(b) - A(a) = A(b).$$

By this, we have $\int_a^b f(x)\,\mathrm{d}x = A(b) = g(b) - g(a)$.

**Theorem 1.1.1.** If $g' = f$, then $g(b) - g(a) = \int_a^b f(x)\,\mathrm{d}x$.

**Remark.** Such a function $g$ is called a primitive (function) (原函數) of $f$.

**Remark.** We have to believe that if $g'(x) = 0$ for some function $g$, then $g$ is a constant function to finish the above proof.

Now we talk about continuity. If we give a function $f$ on $[0, \frac{\pi}{2})$. We should notice that not every $f$ has maximum or minimum value. For example, $y = \tan(x)$ has no maximum. We say that a function $f$ defind on an interval $I$ is continuous at $a \in I$ if $\lim_{x\to a} f(x) = f(a)$. We may think of that if $f$ is a continuous function defined on a closed interval, then it must have maximum and minimum.

Now if we extend the concept of continuity to a parametrized curve $f(t) = (x(t), y(t))$, then we say that the parametrized curve is continuous if both $x(t)$ and $y(t)$ are, and we may think that whether there is a continuous parametrized curve $f$ mapping $\mathbb{R}$ to $\mathbb{R}^2$ such that $f(\mathbb{R}) = \mathbb{R}^2$. Or, if we think $\mathbb{R}$ is too scared, then we can think that whether there is a continuous $f$ mapping $[0,1]$ to a triangle on $\mathbb{R}^2$. This result is called Peano's curve.

## 1.2 Upper Bound and Lower Bound

### Lecture 2

**Definition 1.2.1.** Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$. We say that 9 July. 08:00

- $r$ is a upper (resp. lower) bound of $S$ if $\forall s \in S$, $r \geq$ (resp. $\leq$ ) $s$.
- $r$ is the greatest/largest/highest (resp. least/smallest/lowest) element of $S$ if $r$ is a upper (resps lower) bound of $S$ and $r \in S$.

  **Notation.** $r = \max S$ (resp. $\min S$ ).

- $r$ is the least upper (greatest lower) bound of $S$ if $r = \min\{u \in \mathbb{R} \mid u \text{ is a upper bound of } S\}$ (resp. $r = \max\{l \in \mathbb{R} \mid l \text{ is a lower bound of } S\}$ ).

  **Notation.** $r = \sup S$ (resp. $\inf S$.)

**Remark.** Every $r \in \mathbb{R}$ is a upper bound of $\varnothing$ .

**Note.**
- We write $\sup S = \infty$ (resp. $\inf S = \infty$ ) if and only if $S$ has no upper (resp. lower) bound. If this is the case, we say $\sup S$ (resp. $\inf S$ ) doesn't exist.
- $S$ is bounded above (resp. below) iff $S$ has a upper (resp. lower) bound.

**Definition 1.2.2** (Dedekind cut). Let $A, B \subseteq \mathbb{R}$. We say that $(A, B)$ is a Dedekind cut (of $\mathbb{R}$ ) if

1. $A \neq \varnothing \neq B$
2. $A \cup B = \mathbb{R}$
3. $\forall a \in A,\ b \in B$ we have $a < b$.

We usually call $A$ (resp. $B$) the lower (resp. upper) part of $(A, B)$.

**Theorem 1.2.1.** From now on (until Professor Chi say stop), we assume that $\mathbb{R}$ has the following property (Dedekind's gapless property): If $(A, B)$ is a D-cut of $\mathbb{R}$, then exactly one of the following happens:

1. $\max A$ exists but $\min B$ doesn't;
2. $\min B$ exists but $\max A$ doesn't.

We call the $\max A$ in 1. (resp. $\min B$ in 2. ) the cutting of $(A, B)$.

**Exercise.** We may define Dedekind cuts of $\mathbb{Q}$, $\mathbb{Z}$ similarly. Does the Dedekind gapless property hold for $\mathbb{Q}$ or $\mathbb{Z}$ ?

Hint: Consider $B = \{x \in \mathbb{Q} \mid x > 0, x^2 > 2\}$. We are allowed to use the fact that there does not exist a rational number $x$ such that $x^2 = 2$.

**Theorem 1.2.2** (Weierstrass). Let $\varnothing \neq S \subseteq \mathbb{R}$. If $S$ has a upper bound, then $\sup S$ exists.

**Proof.** Let $B$ be the set of every upper bound of $S$ and suppose $A := \mathbb{R} \backslash B$. We need to show that $\min B$ exists. We first show that $(A, B)$ is a Dedekind cut of $\mathbb{R}$.

Since we know $S$ is not empty, so we know $\forall s \in S$, $s - 1$ is not an upper bound, and thus $s - 1 \in A$, which means $A$ is not empty, and we know $B$ is not empty by the description of the problem.

Also, it is trivial that $A \cup B = \mathbb{R}$.

For $a \in A$ and $b \in B$, we need to show that $a < b$. Were this false, $a \geq b$, and hence $a$ is a upper bound of $S$ since $b$ is a upper bound, so $a \in B$. However, $A \cap B = \varnothing$. Hence, we have shown that $(A, B)$ is a Dedekind cut of $\mathbb{R}$.

Now we want to use Dedekind's gapless property to say that we must have $\min B$ exists. Were this false, we know $\max A$ exists, denoted by $a_0$. Note that $a_0 \in A$, so $a_0 \notin B$, and thus $a_0$ is not an upper bound of $S$. Hence, there exists $s_0 \in S$ such that $s_0 > a_0$, but this implies $s_0 \notin A$, so $s_0 \in B$, and thus $s_0$ is a upper bound of $S$. Now choose some $x$ such that $a_0 < x < s_0$, so $x \in B$ and thus $x$ is an upper bound of $S$, but we have $x < s_0 \in S$, which is a contradiction. ■

**Theorem 1.2.3** (the Archimedean Property). Prove the following statement: $\forall r \in \mathbb{R}$, then $r > 0$ implies that $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < r$.

Hint: Rephrase this statement in a way linking it to the upper bound of the set $S = \mathbb{N} \subseteq \mathbb{R}$.

## 1.3 Sequence

**Definition.** We can define the limit of the sequence as follow.

**Definition 1.3.1.** Let $a_n (n \in \mathbb{N})$ or $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ and $L \in \mathbb{R}$. We say that $a_n$ converges to $L$ (as $n \to \infty$) if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \varepsilon$.

**Notation.** $\lim_{n \to \infty} a_n = L$.

**Note.** If such $L$ exists, we call it the limit of $a_n$ and we call $\{a_n\}$ a convergent sequence, otherwise we call it a divergent sequence.

**Definition 1.3.2.** $\lim_{n\to\infty} a_n = \infty(-\infty)$ means $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $n > N$ implies $a_n \geq (\leq) M$.

**Exercise.**

1. $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$ implies $L = M$.
2. $\{a_n\}$ is convergent implies $\{a_n\}$ is bounded.
3. Suppose we have $\{a_n\}$ , $\{b_n\}$, and $a_n \leq b_n$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, prove that $L \leq M$. What if $\leq$ is replaced by $<$, is this statement still correct?

**Remark.** Changing or removing finitely many terms in $a_n$ does not affect $\{a_n\}$ is convergent or not (and its limit).

**Proposition 1.3.1.** If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1. $\lim_{n\to\infty}(a_n \pm b_n) = L \pm M$.
2. $\lim_{n\to\infty} a_n \cdot b_n = LM$.
3. If $M \neq 0$, then $b_n \neq 0$ for all but finitely many $n$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$.

**proof of 1.** Consider $|(a_n \pm b_n) - (L \pm M)|$. We can see that

$$|(a_n \pm b_n) - (L \pm M)| = |(a_n - L) \pm (b_n - M)| \leq |a_n - L| + |b_n - M| .$$

Also, we know $\forall \varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow |a_n - L| < \varepsilon$ and $n \geq N_2 \Rightarrow |b_n - M| < \varepsilon$. Let $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$|(a_n \pm b_n) - (L \pm M)| \leq |a_n - L| + |b_n - L| < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence, we can pick $\varepsilon_1 = \frac{\varepsilon}{2} = \varepsilon_2$ so that $n \geq N_1' \Rightarrow |a_n - L| < \varepsilon_1 = \frac{\varepsilon}{2}$ and $n \geq N_2' \Rightarrow |b_n - M| < \varepsilon_2 = \frac{\varepsilon}{2}$, and thus we know $n \geq \max\{N_1', N_2'\}$ implies that

$$|(a_n \pm b_n) - (L \pm M)| \leq |a_n - L| + |b_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

**proof of 2.** Consider $|a_n b_n - LM|$. Notice that

$$|a_n b_n - LM| = |a_n b_n - Lb_n + Lb_n - LM| \leq |a_n - L||b_n| + |L||b_n - M|.$$

If we can choose $C > 0$ such that $|b_n| \leq C$ for all $n \in \mathbb{N}$ and $|L| \leq C$, then we have

$$|a_n - L||b_n| + |L||b_n - M| \leq C|a_n - L| + C|b_n - M|.$$

Hence, we want to find some $N_1, N_2$ such that $\forall \varepsilon > 0$, we have $n \geq \max\{N_1, N_2\}$ implies $|a_n - L| \leq \frac{\varepsilon}{2C}$ and $|b_n - M| \leq \frac{\varepsilon}{2C}$, and then we're done.

Now since $\{b_n\}$ is convergent, so it is bounded and thus we can pick such $C$ by a previous exercise. ■

**proof of 3.** We only need to prove $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$, and removing the terms being zeros will not affect the convergence of this sequence. ■

**Note.** What if $L, M = \pm\infty$?

**Exercise.** Suppose $a_n = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$, then $\lim_{n\to\infty} a_n = \frac{1}{1-\frac{1}{2}}$.

## Lecture 3

10 July. 18:00

**As previously seen.** We can think a sequence as a function mapping from $\mathbb{N}$ to $\mathbb{R}$, and the sequence converges means when $N$ is big enough, then the value of every term after $a_N$ will be located in a closed interval $[L-\varepsilon, L+\varepsilon]$.

**Example.** If $a > 1$, then $\lim_{n\to\infty} \frac{1}{a^n} = 0$.

**Proof.** First, we know

$$\frac{1}{a^n} = \frac{1}{(1+(a-1))^n} \le \frac{1}{1+n(a-1)} \le \frac{1}{n(a-1)}.$$

Then, we can use the deduction that $\lim_{n\to\infty} \frac{1}{n} = 0$ to prove that this is correct. (We may need to use the following argument.)

**Exercise.** Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L$, now if $a_n \le c_n \le b_n$ for all $n \in \mathbb{N}$, show that $\lim_{n\to\infty} c_n = L$.

Now we know $0 \le \frac{1}{a^n} \le \frac{1}{n(a-1)}$, so we can prove $\lim_{n\to\infty} \frac{1}{a^n} = 0$. ⊛

**Definition 1.3.3.** A sequence $a_n$ in $\mathbb{R}$ is

1. nondecreasingly monotone / increasing if $a_n \le a_{n+1} \forall n \in \mathbb{N}$.
2. nonincreasingly monotone / decreasing if $a_n \ge a_{n+1} \forall n \in \mathbb{N}$.
3. strictly increasing (resp. strictly decreasing) if $\forall n \in \mathbb{N}$, we have $a_n < a_{n+1}$ (resp. $a_n > a_{n+1}$).

**Theorem 1.3.1.** If $a_n$ is nondecreasing and $\{a_n \mid n \in \mathbb{N}\}$ has a upper bound, then $a_n$ converges to $\sup\{a_n \mid n \in \mathbb{N}\}$ .

**Proof.** $\{a_n \mid n \in \mathbb{N}\}$ has an upper bound, so $L := \sup\{a_n \mid n \in \mathbb{N}\}$ exists. Now we claim that $\lim_{n\to\infty} a_n = L$.

First, we know $\forall\varepsilon > 0$, $L-\varepsilon < L$, and hence $\exists N \in \mathbb{N}$ such that $L - \varepsilon < a_N$, otherwise $L-\varepsilon$ is an upper bound. Since $a_n$ is nondecreasing, so $\forall n \ge N$, we have

$$L-\varepsilon < a_N \le a_n \le L < L+\varepsilon.$$

Hence, $|a_n - L| < \varepsilon$. ■

**Example.** A decimal expression gives a real number, say it is $0.d_1d_2d_3\ldots$, and suppose

$$a_n = 0.d_1d_2d_3\cdots = \frac{d_1}{10} + \cdots + \frac{d_n}{10^n}.$$

Since we know

$$a_n < \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = \frac{9}{10}\frac{1-\left(\frac{1}{10}\right)^{n+1}}{1-\frac{1}{10}} < 1,$$

and since $\{a_n\}$ is nondecreasing, so $\lim_{n\to\infty} a_n = 0.d_1d_2\ldots$.

**Note.** Hence, a real number may have more than one way of representation. For example, 0.1 and $0.09999\ldots$ are same, this can be seen by the limits of both of the decimal representation.

**Example** (The natural base $e$). We first define

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

but we have to show that this limit exists.

**Proof.** Suppose

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{i=0}^{n}\binom{n}{i}\left(\frac{1}{n}\right)^i,$$

we know

$$\begin{aligned}\binom{n}{i}\left(\frac{1}{n}\right)^i &= \frac{n!}{r!(n-r)!}\frac{1}{n^i} = \frac{1}{i!}\left(\frac{n(n-1)\ldots(n-i+1)}{n\cdot n\cdots n}\right)\\ &= \frac{1}{i!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{i-1}{n}\right).\end{aligned}$$

Hence,

$$a_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)\cdots+\frac{1}{i!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{i-1}{n}\right)+\ldots.$$

By this, we can see that $a_n < a_{n+1}$ since we can see that by replacing $n$ with $n+1$ in the above equation then it becomes $a_{n+1}$. Also, we can see that

$$\begin{aligned}a_n &< 1+1+\frac{1}{2!}+\cdots+\frac{1}{i!}+\cdots+\frac{1}{n!}\\ &< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{i-1}}+\cdots+\frac{1}{2^{n-1}}\\ &= 1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < 1+\frac{1}{\frac{1}{2}} = 3,\end{aligned}$$

so it has upper bound and nondecreasing, and thus it converges and $e$ is well-defined.

**Note.** In the near future, we will see that $e = \lim_{n\to\infty} 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}$.

⊛

**Definition 1.3.4.** A sequence of intervals $I_n$ $(n\in\mathbb{N})$ is nested if $I_n \neq \varnothing$ and $I_{n+1}\subseteq I_n$ for all $n\in\mathbb{N}$. $(I_1\supseteq I_2\supseteq\ldots)$.

Now we want to know $\bigcap_{n\in\mathbb{N}}^{\infty} I_n \neq \varnothing$?

Here is some counterexamples. Consider $I_n = (0,\frac{1}{n})$, $n\in\mathbb{N}$. We can show that $\bigcap_{n=1}^{\infty} I_n = \varnothing$ by Archimedean Property. Besides, if $I_n = [n,\infty)$, $n\in\mathbb{N}$, this is trivial that $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

**Theorem 1.3.2** (Theorem of nested intervals). If $I_n$ $(n\in\mathbb{N})$ is a sequence of bounded closed nested intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

**Proof.** Write $I_n = [a_n, b_n]$ for all $n\in\mathbb{N}$. First, we know $I_n$ is nested iff $a_n \le b_n$ and $a_n$ is nondecreasing and $b_n$ is nonincreasing. Hence, $\forall n, m\in\mathbb{N}$, we have $a_n \le a_{\max\{n,m\}} \le b_{\max\{n,m\}} \le b_m$. In other words, for every $m\in\mathbb{N}$, $b_m$ is a upper bound of $\{a_n\}$. Hence, we know $c = \lim_{n\to\infty} a_n = \sup\{a_n\}$.exists. Then, $c\le b_m$ for all $m\in\mathbb{N}$. Also, $c\ge a_n$ for all $n\in\mathbb{N}$. Hence, $a_n\le c\le b_n$ for all $n\in\mathbb{N}$, and thus we know $c\in I_n$ for all $n\in\mathbb{N}$. Thus, $c\in\bigcap_{n=1}^{\infty} I_n$. ■

**Exercise.** What if $I_n = (a_n, b_n)$ is nested but $a_n$ is strictly increasing and $b_n$ is strictly decreasing, is theorem of nested interval still correct?

**Exercise.** $I_n = (a_n, \infty)$ is nested and $\{a_n\}$ bounded from above, is theorem of nested interval still correct?

**Exercise.** We can use Theorem 1.3.2 and Theorem 1.2.3 to show Theorem 1.2.1.

Now we have a neq question, if we have a sequence $\{a_n\}$ in $\mathbb{R}$, can we determine whether $\{a_n\}$ converges or not without referring a limit candidate $L$ but concluding according to the mutual behaviour of the terms of $\{a_n\}$.

**Definition 1.3.5** (Cauchy Sequence)**.** A sequence $\{a_n\}$ in $\mathbb{R}$ is a Cauchy Sequence if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n, m \geq \mathbb{N}$ implies $|a_n - a_m| < \varepsilon$.

**Exercise.** $a_n$ is convergent implies $a_n$ is a Cauchy sequence.

**Exercise.** If $a_n$ is a Cauchy sequence, then $a_n$ is bounded.

**Theorem 1.3.3.** Let $a_n$ be a sequence in $\mathbb{R}$, then

$$a_n \text{ is convergent} \Leftrightarrow a_n \text{ is Cauchy.}$$

**proof from Cauchy to convergent.** We first give a part of the proof as exercise.

**Definition.** Let $a_n$ be a bounded sequence in $\mathbb{R}$.

**Definition.** $u_n \coloneqq \sup\{a_m \mid m \geq n\}$.

**Definition.** $l_n \coloneqq \inf\{a_m \mid m \geq n\}$.

Now we know $l_n \leq a_m \leq u_n$ for all $m, n \in \mathbb{N}$ and $m \geq n$ . Also, we know $l_n$ is increasing and $u_n$ is decreasing.

**Exercise.** $a_n$ converge if and only if $\lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n$, and if any of both sides holds, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n.$$

Let $a_n, b_n$ be two bounded seuences, then

$$\lim_{n\to\infty} u_{a_n+b_n} \leq \lim_{n\to\infty} u_{an} + \lim_{n\to\infty} u_{bn}, \quad \lim_{n\to\infty} l_{a_n+b_n} \geq \lim_{n\to\infty} l_{an} + \lim_{n\to\infty} l_{bn}.$$

(Why?)

Now we start the proof. Assume that $a_n$ is Cauchy. We claim that $\lim_{n\to\infty}(u_n - l_n) = 0$. For $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|a_n - a_m| < \varepsilon$. In particular, $n \geq N$ implies

$$a_N - \varepsilon < a_n < a_N + \varepsilon,$$

which also implies

$$a_N - \varepsilon \leq l_N \leq u_N \leq a_N + \varepsilon,$$

so we have

$$0 \leq u_n - l_n \leq u_N - l_N \leq (a_N + \varepsilon) - (a_N - \varepsilon) = 2\varepsilon.$$

By adjusting the coefficient of $\varepsilon$ we can show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $0 \leq u_n - l_n \leq \varepsilon$, which means $\lim_{n\to\infty}(u_n - l_n) = 0$.

Hence, by the exercise above, we know $\lim_{n\to\infty} a_n = \lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n$, and $a_n$ is convergent. ■

## Lecture 4

We first finish the proof above. 11 July. 9:00

**Exercise.** Let $S \subseteq \mathbb{R}$. Prove that if $|s - s'| \leq 3$ for all $s, s' \in S$, then

1. $S$ is bounded.
2. $\sup S - \inf S \leq 3$.

**Exercise.** If we change the $\leq$ to $<$ in the above exercise, is it still correct?

## 1.4 Series

**Definition 1.4.1.** Let $a_n$ be a sequence in $\mathbb{R}$. We say that the series $\sum_{i=1}^{\infty}$ converges to / has sum a real number $S$ if $\lim_{n\to\infty} s_n = S$, where $s_n := \sum_{i=1}^{n} a_i$, the $n$-th partial sum of $\sum_i a_i$. If such $S$ exists (resp. does not exist), we say that the series $\sum a_n$ is convergent (resp. divergent).

**Example.** Suppose $a_n = \frac{1}{2^n}$, then $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$, so $\lim_{n\to\infty} s_n = \frac{1}{1-\frac{1}{2}} = 2$.

**Note.** However, given a sequence $a_n$, it is not always possible to write down $s_n$ explicitly.

**Example.** $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} =?$, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} = ?$

**Notation.** For a series $\sum_n a_n$ and $l, m \in \mathbb{N}$, $l < m$, we let

$$s_{l,m} := \sum_{i=l}^{m} a_i,$$

the $(l, m)$-tail.

**Exercise.** $\sum_n a_n = 0$ implies $\lim_{n\to\infty} a_n = 0$.

**Proof.** We know $a_n = s_n - s_{n-1}$ and also $\lim_{n\to\infty} s_n - s_{n-1} = 0$ if $\sum_n a_n$ is convergent. ■

**As previously seen.** $\sum_n a_n$ converges iff $s_n$ converges iff $s_n$ is Cauchy, i.e. $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n, m \geq N$ implies $|s_n - s_m| \leq \varepsilon$.

Now if $n > m$, then $s_n - s_m = a_{m+1} + \cdots + a_n$, so we can rewrite the stamtement as: $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $k > N$ and $l \geq 0$ implies $|a_k + \cdots + a_{k+l}| < \varepsilon$. Hence, there is another equivalent condition of converging, which is: $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $k > N$ and $l \geq 0$ implies $|a_k| + \cdots + |a_{k+1}| < \varepsilon$. If we have this, we can deduce that $\sum_n a_n$ is convergent since by triangle inequality this implies the above Cauchy condition. Also, notice that "$\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $k > N$ and $l \geq 0$ implies $|a_k| + \cdots + |a_{k+1}| < \varepsilon$" is equivalent to $\sum_n |a_n|$ is convergent. Besides, notice that

$$\sum_n |a_n| \text{ converges if and only if } \exists M > 0 \forall n \in \mathbb{N} \text{ we have } |a_1| + |a_2| + \cdots + |a_n| \leq M,$$

since $\{s_n\}$ is nondecreasing.

This tells us

**Corollary 1.4.1.** If $b_n$ is a positive sequence, then $\sum_n b_n$ converges if and only if $\exists M > 0$ such that $\forall n \in \mathbb{N}$ we have $b_1 + b_2 + \cdots + b_n \leq M$.

and the important result is

**Corollary 1.4.2.** If $\sum_n |a_n|$ converges, then $\sum_n a_n$ converges.

**Notation.** If $a_n \geq 0$ for all $n$, then we write $\sum_n a_n < \infty$ (resp. $\infty$) to mean that $\sum_n a_n$ converges (resp. diverges).

**Example.** Is $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent?

**Proof.** Notice that

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}.$$

Hence, we have $\forall m \in \mathbb{N}$,

$$s_{2^m} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^{m-1}+2^{m-1}} > 1 + \frac{m}{2},$$

so $s_n$ is unbounded, and thus $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. ⊛

**Example.** Is $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent?

**Proof.** Since

$$\frac{1}{n^2} < \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n},$$

so we know

$$s_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2.$$

Hence, we know $s_n$ is bounded above and nondecreasing, and thus $s_n$ is convergent. ⊛

**Example.** Is $\sum_{n=1}^{\infty} \frac{\sin n}{n^k}$ convergent for some $k \geq 2$ and $k \in \mathbb{N}$?

**Proof.** Since we know

$$\left|\frac{\sin 1}{1^k}\right| + \cdots + \left|\frac{\sin n}{n^k}\right| < 1 + \cdots + \frac{1}{n^k} < 1 + \cdots + \frac{1}{n^2} < 2 \quad \forall n \in \mathbb{N}.$$

Hence, we know $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^k}\right| < \infty$, and thus $\sum_{n=1}^{\infty} \frac{\sin n}{n^k}$ converges. ⊛

**Exercise.** For $a > 1$ and $k \in \mathbb{N}$, show that $\sum_{n=1}^{\infty} \frac{n^k}{a^n} < \infty$.

**Definition 1.4.2.** Given a sequence $a_n$, we say that

1. $\sum_n a_n$ converges absolutely if $\sum_n |a_n| < \infty$ (and thus $\sum_n a_n$ converges).
2. $\sum_n a_n$ converges conditionally if $\sum_n |a_n| = \infty$ but $\sum_n a_n$ converges.

### 1.4.1 Comparison Test

**Theorem 1.4.1** (Comparison Test)**.** Given $a_n, b_n$ are two non-negative sequences, then if "$\exists C > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq Cb_n$", then "$\sum_n b_n < \infty$ implies $\sum_n a_n < \infty$".

**Proof.** For $n \geq N$, we have

$$\begin{aligned} a_1 + \cdots + a_n &= a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n \leq a_1 + \cdots + a_N + C\left(b_{N+1} + \cdots + b_n\right) \\ &\leq a_1 + \cdots + a_N + CM \quad \text{for some } M > 0. \end{aligned}$$

■

**Claim.** If $a_n, b_n$ are two non-negative sequences, then $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists implies $\exists C > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq Cb_n$.

**Proof.** Let $l := \lim_{n\to\infty} \frac{a_n}{b_n}$ and $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\frac{a_n}{b_n} - l\right| \leq 1$. Hence,

$$\frac{a_n}{b_n} \leq l + 1 \Leftrightarrow a_n \leq (l+1)b_n.$$

⊛

**Corollary 1.4.3.** If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then

$$\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty.$$

**Exercise.** If $\lim_{n\to\infty} u_n$ exists, where $u_n = \sup\left\{\frac{a_m}{b_m} \mid m \geq n\right\}$, then "$\exists C > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq Cb_n$".

**Exercise** (The ratio test)**.** Let $a_n$ be a non-negative sequence. Then show that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \exists N \in \mathbb{N} \text{ and } C < 1 \text{ such that } n \geq N \text{ implies } a_{n+1} \leq Ca_n,$$

and thus $\sum_n a_n$ converges. Besides, prove that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \exists N \in \mathbb{N} \text{ and } C > 1 \text{ such that } n \geq N \text{ implies } a_{n+1} \geq Ca_n,$$

and thus $\sum_n a_n = \infty$. However, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then we cannot conclude anything.

**Exercise** (The root test)**.** If $a_n$ is a non-negative sequence, then

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} < 1 \Rightarrow \exists N \in \mathbb{N} \text{ and } C < 1 \text{ such that } n \geq N \text{ implies } a_n \leq C^n,$$

and thus $\sum_n a_n$ converges. Besides, prove that

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} > 1 \Rightarrow \exists N \in \mathbb{N} \text{ and } C > 1 \text{ such that } n \geq N \text{ implies } a_n \geq C^n,$$

and thus $\sum_n a_n = \infty$.

Now we have a question: Does $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges (conditionally)?

**Definition 1.4.3.** A series $\sum_n a_n$ is an alternating series if $\exists b_n > 0$ $(n \in \mathbb{N})$ such that

$$a_n = (-1)^{n-1} b_n (n \in \mathbb{N}).$$

**Theorem 1.4.2** (Leibnize's criterion). If $\sum_n a_n$ is an alternating series and $b_n = |a_n|$ is decreasing and converging to 0 as $n \to \infty$, then $\sum_n a_n$ converges.

**Proof.** Suppose $b_n = (-1)^{n-1} a_n$

$$\begin{aligned} |a_k + \cdots + a_{k+l}| &= \left|(-1)^{k-1}\left(b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}\right)\right| \\ &= b_k - b_{k+1} + \cdots + (-1)^l b_{k+l} \\ &= \begin{cases} b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-1} - b_{k+l}), & \text{if } 2 \mid l; \\ b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-2} - b_{k+l-1}) - b_{k+l}, & \text{otherwise.} \end{cases} \\ &\leq b_k = |a_k| \end{aligned}$$

Since $\lim_{n\to\infty} b_n = 0$, so $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |b_n| < \varepsilon$. Hence, if $k \geq N$, we have

$$|a_k + \cdots + a_{k+l}| \leq b_k \leq \varepsilon \quad \forall l > 0.$$

■

Given a sequence $a_n$, we can separate all terms into two sequences. The first sequence is

$$a_{n_1}, a_{n_2}, \ldots,$$

while the second one is

$$a_{n'_1}, a_{n'_2}, \ldots$$

with $n_1 < n_2 < \ldots$ and $n'_1 < n'_2 < \ldots$ and $\{n_1, n_2, \ldots\} \cup \{n'_1, n'_2, \ldots\} = \mathbb{N}$ such that $a_{n_j} \geq 0$ and $a_{n'_k} \leq 0$ for all $j, k \in \mathbb{N}$. Let $p_j := a_{n_j}$ and $q_k := -a_{n'_k}$ to construct two non-negative sequences.

**Exercise.** Show that $\sum_n |a_n| < \infty$ iff $\sum_j p_j$ nad $\sum_k q_k$ are both convergent. Moreover, if any side holds, then

$$\sum_n |a_n| = \sum_j p_j + \sum_k q_k \text{ and } \sum_n a_n = \sum_j p_j - \sum_k q_k.$$

**Exercise.** Suppose $\sum_n a_n$ and $\sum_n b_n$ are both convergent, then

$$\sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n.$$

**Exercise.** Inserting 0s to a series will not affect its convergence / divergence and its sum if the sum exists.

### 1.4.2 Rearrangement

**Definition 1.4.4.** We know a sequence is a map from $\mathbb{N}$ to $\mathbb{R}$, and a subsequence is a double-map frome $\mathbb{N}$ to $\mathbb{N}$ to $\mathbb{R}$, where the first map from $\mathbb{N}$ to $\mathbb{N}$ is increasing. Moreover, a rearrangement is also a double-map from $\mathbb{N}$ to $\mathbb{N}$ to $\mathbb{R}$, where the first map is a bijection.

Now we have a question. If $\sum_n a_n$ converges, then

1. is $\sum_m a_{n(m)}$ converges where $a_{n(m)}$ is a rearrangement of $a_n$?
2. $\sum_m a_{n(m)} = \sum_n a_n$?

**Note.** We will prove that if the series is absolutely convergent, then rearrangement does not affect the sum of the infinite series.

1. If $a_n \geq 0$ for $n \in \mathbb{N}$, the answers are affirmative.
2. If $\sum_n |a_n| < \infty$ , then the answers are affirmative.

In fact, 1. implies 2 (by the 4-th exercise in subsection 1.4.1). (Why?)

Why does 1. hold? Actually $\sum_n a_n = \sup\{a_{n_1} + a_{n_2} + \cdots + a_{n_k} \mid n_1 < \cdots < n_k,\ k \in \mathbb{N}\}$ (including the case $\infty$) (Why?), and hence 1. holds.

**Theorem 1.4.3** (Dirichlet's Rearrangement theorem (1829)). If $\sum_n a_n$ converges absolutely, then for every rearrangement $a_{n(m)}$ we have $\sum_{m=1}^{\infty} a_{n(m)} = \sum_{n=1}^{\infty} a_n$.

**Theorem 1.4.4** (Riemann's rearrangement theorem (1852)). If $\sum_n a_n$ converges conditionally, then for every $L \in \mathbb{R}$ there exists a rearrangement $a_{n(m)}$ of $a_n$ such that $\sum_{m=1}^{\infty} a_{n(m)} = L$.

**Proof.** We will only use two known facts given by the conditional convergence of $\sum_n a_n$:

1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$.
2. $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 0$. (Since $p_n, q_n$ are subsequence of $a_n$.)

Our thought is to add up $p_1 + p_2 + \cdots + p_{m_1}$ so that

$$p_1 + p_2 + \cdots + p_{m_1 - 1} < L < p_1 + p_2 + \cdots + p_{m_1}.$$

Now we start to minus $q_1 + q_2 + \cdots + q_{m_1'}$ so that

$$\sum_{i=1}^{m_1} p_i - \sum_{i=1}^{m_1'} q_i < L < \sum_{i=1}^{m_1} p_i - \sum_{i=1}^{m_1'-1} q_i,$$

and we continuew to add up some $p_i$ to make the series bigger than $L$, and jump back by minusing some $q_i$. This is feasible since $L$ is not a upper bound of $\sum_n p_n$ and not a lower bound of $\sum_n q_n$. Note that this method construct many partial sums of some rearrangement, say the partial sum of the rearrangement is $s_n$, we want to show $\lim_{n\to\infty} s_n = L$.

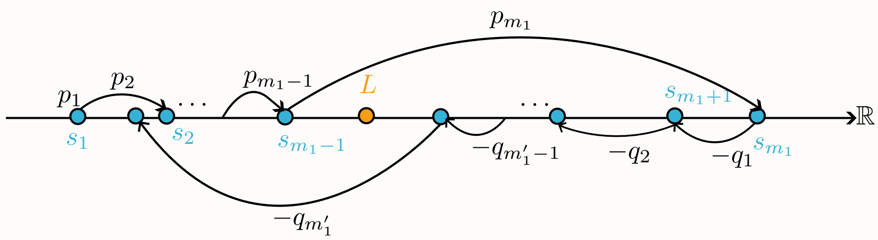


Figure 1.3: Riemann jump

Since $\exists$ natural numbers $m_1 < m_2 < \ldots$ and $m_1' < m_2' < \ldots$ such that

$$\left| s_{m_{l-1}' + m_l + k'} - L \right| < p_{m_l} \quad \text{if } 0 \leq k' < m_l' - m_{l-1}'.$$

Similarly,

$$\left|s_{m_l+m'_l+k} - L\right| < q_{m'_l} \quad \text{if } 0 \le k < m_{l+1} - m_l$$

if we start jumping from $q$. Since we know $\lim_{n\to\infty} p_{m_l} = \lim_{n\to\infty} q_{m'_l} = 0$, so $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $l \ge N_0$ implies $p_{m_l}$ and $q_{m'_l} < \varepsilon$. Let $N = m'_{N_0-1} + m_{N_0}$. Then $n \ge N$ implies $|s_n - L| < \varepsilon$. ■

**Remark.** In 1827, Dirichlet made the following observation:

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \\ 2S &= 2 - 1 + \frac{2}{3} - \frac{2}{4} - \frac{2}{5} + \frac{2}{6} - \dots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \end{aligned}$$

by combining some terms.

### Multiplying absolutely convergent series

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge absolutely.

**Theorem 1.4.5.** Let $c_n = a_n b_0 + \dots + a_0 b_n = \sum_{\substack{i+k=n \\ i,k\ge 0}} a_i b_k$. Then $\sum_n |c_n| < \infty$ and

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{k=0}^{\infty} b_k\right).$$

**Proof.** First, for all $n \in \mathbb{N}$ we know

$$|c_1| + |c_2| + \dots + |c_n| = \sum_{m=0}^{n} \left| \sum_{\substack{j+k=m \\ j,k\ge 0}} a_j b_k \right| \le \sum_{m=0}^{n} \sum_{\substack{j+k=m \\ j,k\ge 0}} |a_j||b_k| \le \left(\sum_{j=0}^{n} |a_j|\right)\left(\sum_{k=0}^{n} |b_k|\right) \le MN$$

for some $M, N$. Hence, $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

Let $A_n := a_0 + \dots + a_n$, $B_n := b_0 + \dots + b_n$, $C_n := c_0 + \dots + c_n$. Hence, we want to show

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} A_n \cdot \lim_{n\to\infty} B_n.$$

**Claim.** $\lim_{n\to\infty} A_n B_n - C_n = 0$.

**Proof.** First, we know

$$\begin{aligned} 0 \le |A_n B_n - C_n| = \sum_{\substack{j+k>n \\ 0\le j,k\le n}} |a_j b_k| &\le \left(\sum_{j=\lfloor \frac{n}{2} \rfloor}^{n} |a_j|\right)\left(\sum_{k=0}^{n} |b_k|\right) + \left(\sum_{j=0}^{n} |a_j|\right)\left(\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n} |b_k|\right) \\ &\to 0 * M + N * 0 \end{aligned}$$

when $n \to \infty$. The last part is derived by convergence implies Cauchy, and the former part can be derived by drawing grids in a square representing the choice of $a_j$ and $b_k$. ⊛

■

**Theorem 1.4.6.** If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely and $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection such that $n \mapsto (j(n), k(n))$, and $c_n \coloneqq a_{j(n)} b_{k(n)}$ for all $n \in \mathbb{N}$, then $\sum_n |c_n| < \infty$ and

$$\sum_n c_n = \left(\sum_n a_n\right)\left(\sum_n b_n\right).$$

**Proof.** $\forall n \in \mathbb{N}$, let $l = \max\{j(1), j(2), \dots, j(n), k(1), k(2), \dots, k(n)\}$. Then

$$\begin{aligned} |c_1| + |c_2| + \cdots + |c_n| &= |a_{j(1)} b_{k(1)}| + |a_{j(2)} b_{k(2)}| + \cdots + |a_{j(n)} b_{k(n)}| \\ &\leq \left(\sum_{j=1}^{l} |a_j|\right)\left(\sum_{k=1}^{l} |b_k|\right) \leq M \times N. \end{aligned}$$

Hence, $\sum_n |c_n|$ converges absolutely. Note that since $\sum_n |c_n|$ converges absolutely, so we can change the order of every term if we want. That is, we can use other bijection to get same value of sum.

Let $A_n = a_1 + \cdots + a_n$, $B_n = b_1 + \cdots + b_n$, and $C_n = c_1 + \cdots + c_n$, then we know

$$A_n B_n = (a_1 + \cdots + a_n)(b_1 + \cdots + b_n) = \sum_{1 \leq j,k \leq n} a_j b_k.$$

Replace the bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by:

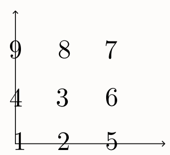


Figure 1.4: New bijection

then we know

$$\sum_{1 \leq j,k \leq n} a_j b_k = C_{n^2}.$$

Hence, we know

$$\lim_{n\to\infty} A_n B_n = \lim_{n\to\infty} C_{n^2} = \lim_{n\to\infty} C_n.$$

■

# Lecture 5: Metric Space

16. July 2025

## 1.5 Metric Space

Recall that we use some concepts of distance, that is, the absolute value, which has the following properties:

1. $|x| \geq 0$ for all $x \in \mathbb{R}$ and the equal sign holds only when $x = 0$.
2. $|x| = |-x|$ for all $x \in \mathbb{R}$.
3. $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

In the definition of limit, we use absolute value, and we want to abstract the concept of absolute value. In $\mathbb{R}^m$, we know $x = (x_1, x_2, \dots x_m)$ has $|x| = \sqrt{\sum_{j=1}^{m} |x_j|^2}$, and $|x|$ also has the properties above. Hence, we want to introduce Distance function / metric space (賦距空間).

Let $X$ be a set.

**Definition 1.5.1.** A function $X \times X \xrightarrow{d} \mathbb{R}$ is called a distance /metric (function) on $X$ if

1. $\forall x, y \in X$ we have $d(x, y) \geq 0$ and the equal sign holds only when $x = y$.
2. $\forall x, y \in X$, we have $d(x, y) = d(y, x)$.
3. $\forall x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

**Example.** In $\mathbb{R}^m$, if $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$, and define

$$d_2(x, y) \coloneqq \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_m - y_m|^1},$$

then $d_2$ is a metric on $\mathbb{R}^m$ by Cauchy inequality.

**Example.** Suppose we define

$$d_1(x, y) \coloneqq |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|$$

in $\mathbb{R}^n$, then $d_1$ is a metric.

**Note.** On a set we can define more than one way of defining metric. For example, $d_1(x, y)$ and $d_2(x, y)$.

**Example.**

$$d_\infty \coloneqq \max\left\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\right\}$$

is also a metric on $\mathbb{R}^n$.

**Example.** Suppose $X$ is a set and $x, y \in X$, then let

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

This is called the discrete metric.

**Example.** Suppose $p$ is a prime number, $x, y \in \mathbb{Q}$, then

$$|x|_{p\text{-adic}} \coloneqq p^{-m} \text{ if } x = \frac{a}{b} p^m \text{ with } a, b, m \in \mathbb{Z} \text{ and } \gcd(a, p) = \gcd(b, p) = 1$$

and

$$d_{p\text{-adic}}(x, y) \coloneqq |x - y|_{p\text{-adic}},$$

which is also a metric.

Actually now we have

1. $|x + y|_{p\text{-adic}} \leq \max\left\{|x|_{p\text{-adic}}, |y|_{p\text{-adic}}\right\}$
2. $|x + y|_{p\text{-adic}} = |x|_{p\text{-adic}}$ if $|x|_{p\text{-adic}} > |y|_{p\text{-adic}}$.

**Intuition.** All triangles in $\mathbb{Q}$ are isosceles if we define the distance function as $p$-adic distance.

**Exercise.** Suppose $(X, d)$ is a metric space. Show that $\forall x, y, z \in X$,

$$|d(x, y) - d(y, z)| \leq d(x, y).$$

We may generalize the definitions about limits and convergence to metric spaces.

**Definition 1.5.2.** Let $(X, d)$ be a metric space, $a_n$ $(n \in \mathbb{N})$ be a sequence in $X$, and $L$ is an element in $X$, we say that $\lim_{n\to\infty} a_n = L$ or $a_n$ converges to $L$ or $L$ is the limit of $a_n$ as $n \to \infty$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow d(a_n, L) < \varepsilon.$$

**Note.** Notice that

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow d(a_n, L) < \varepsilon.$$

is equivalent to

$$\lim_{n\to\infty} d(a_n, L) = 0.$$

($d(a_n, L)$ is a real number.)

**Intuition.** Given any open ball centered at $L$, we can find $N$ such that $n \geq N$ implies $a_n$ is in this open ball.

**Exercise.** Can we prove that the limit is unique in a metric space?

**Exercise.** In $\mathbb{R}^m$, can we prove the basic limit properties hold when the metric is $d_2$? (The basic limit properties are something like $\lim_{n\to\infty} a_n \pm b_n = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.)

**Definition 1.5.3.** Let $(X, d)$ is a metric space and $S \subseteq X$.

1. For $r > 0$ and $x_0 \in X$, we let

   $$B_r(x_0) = B(x_0, r) = B_{x_0}(r) := \{x \in X \mid d(x, x_0) < r\},$$

   an open ball.

2. $S$ is an open set (of $(X, d)$) if

   $$\forall x_0 \in S,\ \exists r > 0 \text{ such that } B_r(x_0) \subseteq S.$$

**Definition 1.5.4.** For $r > 0$ and $x_0 \in X$, we let

$$\overline{B_r(x_0)} = \overline{B(x_0, r)} = \overline{B_{x_0}(r)} := \{x \in X \mid d(x, x_0) \leq r\},$$

a closed ball.

**Definition 1.5.5.** Let $(X, d)$ be a metric space and $S \subseteq X$.

1. $S$ is a *closed set* (of $(X, d)$) if $X \setminus S$ is open; that is,

   $$S \text{ is closed } \Leftrightarrow X \setminus S \text{ is open.}$$

2. Equivalently, $S$ is closed if for every sequence $\{x_n\} \subseteq S$ that converges to some $x \in X$, we have $x \in S$; that is,

   $$\forall \{x_n\} \subseteq S,\ x_n \to x \in X \Rightarrow x \in S.$$

**Exercise.** Suppose $(X, d)$ is a metric space, $x_0 \in X$, and $r > 0$.

1. Show that $B_r(x_0)$ is open.
2. $\{x \in X \mid d(x, x_0) > r\} = X \setminus \overline{B_r(x_0)}$ is open.

# Lecture 6: Open Set and Closed Set

18. July 17:00

> **As previously seen.** $S(\subseteq X)$ is called an open set of $X$ (with respect to $d$) if $\forall x_0 \in S$, $\exists r > 0$ such that $B_r(x_0) \subseteq X$.

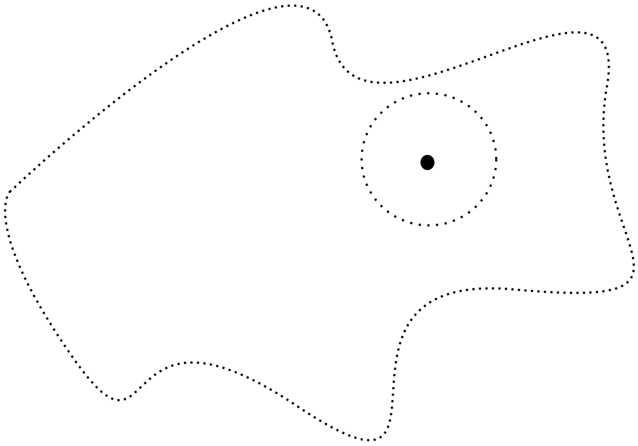

Figure 1.5: open set

> **As previously seen.** $F(\subseteq X)$ is a closed set of $X$ if $X \backslash F \subseteq X$ is open.

**Example.** A set is either an open set or a closed set?

**Proof.** No. Consider $(1, 3]$, and the metric is the absolute value. ⊛

**Example.** A set can not be open and closed at the same time?

**Proof.** No. Consider $\varnothing$. By definition, it is open. Also, the universal space is open, so $\varnothing$ is closed. ⊛

**Note.** Closed ball is close.

**Proof.** ⊛

**Exercise.** Let $(X, d)$ be a metric space. Show that

1. $X$ and $\varnothing$ are open.
2. $O_1$ and $O_2$ are open implies $O_1 \cap O_2$ is open.
3. Suppose we have $O_\alpha \subseteq X$ for all $\alpha \in A$, and they are all open, then $\bigcup_{\alpha \in A} O_\alpha$ is open in $X$.
4. What if we change "open" to "closed" in the above 3 statements? Should we make some modification to the statements to make them true?

**Example.** Intersection of infinitely many open set is still open?

**Proof.** Consider

$$\bigcap_{i=1}^{\infty}\left(-\frac{1}{i},\frac{1}{i}\right).$$

⊛

**Exercise.** Show that

$$\begin{aligned} &\lim_{n\to\infty} a_n = L \\ \Leftrightarrow &\lim_{n\to\infty} d(a_n, L) = 0 \\ \Leftrightarrow &\forall \text{ open } U\subseteq X \text{ such that } L\in U,\ \exists N\in\mathbb{N} \text{ such that } n\geq N\Rightarrow a_n\in U \end{aligned}$$

**Example.** If we have a metric space and a subspace of this space, can we restrict the metric on this subspace, and this space is still a metric space?

**Proof.** Yes. This does not violate the definition of metric space. ⊛

**Definition 1.5.6.** $S\subseteq X$ is bounded with respect to $d$ if $\exists r>0$ and $x_0\in X$ such that $S\subseteq B_r(x_0)$.

**Theorem 1.5.1** (Bolzano-Weierstrass theorem)**.** Suppose we have a bounded sequence $a_n\in\mathbb{R}^m$, then $\exists$ a subsequence $a_{n(m)}$ such that $a_{n(m)}$ is convergent.

**Proof.** We just talk about the case $m=2$, and the higher case is similar. Choose $M>0$ such that $a_n\in[-M,M]\times[-M,M]$ for all $n\in\mathbb{N}$. Suppose $a_n\in[-M,M]\times[-M,M]$ is called $Q$. Divide $Q$ into 4 squares with equal size, and choose one, say $Q_1$ such that $|\{n\mid a_n\in Q_1\}=\infty|$. Select $n_1\in\mathbb{N}$ such that $a_{n_1}\in Q_1$. Repeat this step, that is, divide $Q_1$ into 4 subparts, then says the one subpart with infinite many $a_n$ in it is $Q_2$ ($Q_2$ must exists). Select $n_2\in\mathbb{N}$ such that $a_{n_2}\in Q_2$ and $n_2>n_1$. Keep repeating this step, then by Theorem 1.3.2 we know

$$\bigcap_{n=1}^{\infty} Q_n\neq\varnothing.$$

**Note.** Just think of the nested intervals are in $x$ and $y$ directions.

Actually, $\bigcap_{n=1}^{\infty} Q_n=\{a\}$ for some $a\in\mathbb{R}^2$, otherwise if there are two points in the intersection, then at some moment we will divide them into different subpart, which is a contradiction. It can been seen that $\lim_{k\to\infty} a_{n_k}=a$. ■

**Exercise.** Suppose $(X,d)$ is a metric space, $F\subseteq X$. Show that

$$F \text{ is closed} \Leftrightarrow \text{If } a_n\in F \text{ and } \lim_{n\to\infty} a_n = a\in X, \text{ then } a\in F.$$

**Definition 1.5.7** (Open Cover)**.** Suppose $(X,d)$ is a metric space, and $S\subseteq X$. If we have $O_\alpha\subseteq X$ for $\alpha\in A$ and they are all open. We say that all $O_\alpha$s form an open cover of $S$ if $S\subseteq\bigcup_{\alpha\in A} O_\alpha$.

**Definition 1.5.8** (Compact Set)**.** $S$ is called a compact set if $\forall$ open cover $O_\alpha(\alpha\in A)$ of $S$,

$\exists \alpha_1, \alpha_2, \ldots, \alpha_m \in A$ such that

$$S \subseteq \bigcup_{i=1}^{m} O_{\alpha_i}, \quad \text{where} \bigcup_{i=1}^{m} O_{\alpha_i} \text{is a finite subcover.}$$

**Example.** Is $(0,1)$ with normal metric a compact set?

**Proof.** No. Consider

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right),$$

we cannot pick finite many interval to cover $(0,1)$. ⊛

**Example.** Is $(1,\infty)$ a compact set?

**Proof.** No. Consider

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{2}, n\right).$$

⊛

**Theorem 1.5.2** (The Heine Borel theorem)**.** Let $S \subseteq \mathbb{R}^m$, then

$$S \text{ is compact} \Leftrightarrow S \text{ is bounded and closed.}$$

**Proof of $\Leftarrow$.** Suppose that $S$ is bounded and closed, and there is an open cover $O_\alpha$ of $S$, which admits no finite subcover.

First, choose a cube $Q$ containing $S$. Divide $Q$ into 4 equal-sized cubes and select one of them, say $Q_1$, such that $Q_1 \cap S$ cannot be covered by finitely many $O_\alpha$. Keep repeating this step and get $Q_2, Q_3, \ldots$. Note that we have

$$Q_1 \supseteq Q_2 \supseteq \ldots.$$

Hence,

$$\bigcap_{n=1}^{\infty} Q_n = \{a\} \quad \text{for some } a.$$

Choose $s_1 \in Q_1 \cap S$, $s_2 \in Q_2 \cap S$ and so on, then we know $\lim_{n\to\infty} s_n = a$ (think of this is also nested intervals). However, the sequence $s_i$ is in $S$ and $S$ is closed, so by the previous exercise we know $a \in S$. Hence, $\exists \alpha$ such that $a \in O_\alpha$ since $\bigcup_{i=1}^{\infty} O_i$ is a cover of $S$. Since $O_\alpha$ is open, so there exists an open ball $B_r(a) \subseteq Q_\alpha$. However, $a$ is in many subcubes, so $\exists n \in \mathbb{N}$ such that $Q_n \subseteq B_r(a) \subseteq O_\alpha$, and since $O_n \cap S \subseteq Q_n$, so we know $Q_n \cap S \subseteq O_\alpha$, which is a contradiction since we suppose that $Q_i \cap S$ cannot be covered by finitely many $O_\alpha$.

> **Note.** Note that we need "bounded" and "closed" since we need to use Theorem 1.3.2.

■

**Exercise.** How to prove if $S$ is compact, then $S$ is bounded and close?