Linear Algebra I

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Abstract

The lecture note of Linear Algebra I by professor 余正道.

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Chapter 1

Introduction

Lecture 1: First Lecture

1.1 Vector

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In high school, our vectors are in $\mathbb{R}^2$ and $\mathbb{R}^3$, and we have define the addition and scalar multiplication of vectors.

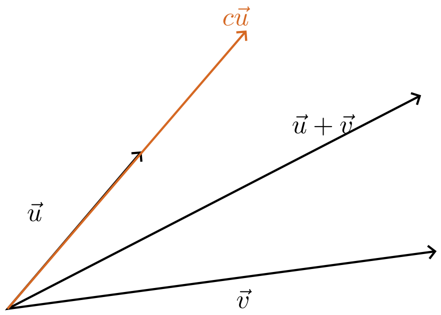

Figure 1.1: Vectors in $\mathbb{R}^2$

Example. $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n \mid a_n \in \mathbb{R})\}$

With this type of space, we can define addition and multiplication as

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = \{a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\}$$
$$\alpha \cdot (a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$$

Also, if we define a space:

Example. $V = \{\text{function } f : (a, b) \to \mathbb{R}\}$, where (a, b) is an open interval.

then this can also be a vector space after defining addtion and multiplication.

Note. In a vector space, we have to make sure the existence of 0-element, which means $0(x) = 0$.

Now we give a more abstract example:

Example. Suppose S is any set, then define $V = \{\text{all functions from } S \text{ to } \mathbb{R}\}$

If we define $(f + g)(s) = f(s) + g(s)$ and $(\alpha \cdot f)(s) = \alpha \cdot f(s)$, and $0(s) = 0$, then this is also a vector space.

Put some linear conditions

Example. In $\mathbb{R}^n$, fix $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, if we define

$$W = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\},$$

then this is also a vector space.

However, if we have

$$W' = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = 1\},$$

then this is not a vector space because it is not close.

Example. In $V = \{(a, b) \to \mathbb{R}\}$ or $W_1 = \{\text{polynomial defined on } (a, b)\}$, these are both vector space.

Remark. In the later course, we will learn that W_1 is a subspace of V.

Example. If we furtherly defined $W_1^{(k)} = \{\text{polynomial degree } \le k\}$, then this is also a vector space.

Remark. $W_1^{(k)}$ is actually isomorphic to $\mathbb{R}^{k+1}$ since

$$a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \leftrightarrow (a_0, a_1, a_2, \dots, a_n).$$

Example. $W_2 = \{\text{continuous function on } (a, b)\}$ and $W_3 = \{\text{differentiable functions}\}$ are also both vector spaces.

Example. $W_4 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0\right\}$ and $W_5 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -f\right\}$ are both vector spaces.

Proof.

$$\begin{aligned} W_4 &= \{a_0 + a_1x\} \\ W_5 &= \{a_1 \cos x + a_2 \sin x\} \end{aligned}$$

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Chapter 2

Formal Introduction to Vector Spaces

2.1 Vector Spaces Over $\mathbb{R}$

Definition 2.1.1. Suppose V is a non-empty set equipped with

- addition: $V \times V \to V$, that is, given $u, v \in V$, defining $u + v \in V$
- scalare multiplication: $\mathbb{R} \times V \to V$, that is, given $\alpha \to \mathbb{R}$ and $v \in V$, we need to have $\alpha v \in V$

Also, we need some good properties or conditions

- For addition,
  - $u + v = v + u$
  - $(u + v) + w = u + (v + w)$
- There exists $0 \in V$ such that $u + 0 = u = 0 + u$
- Given $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0 = (-v) + v$
- For scalar multiplication,
  - $1 \cdot v = v$ for all $v \in V$
  - $(\alpha\beta)v = \alpha \cdot (\beta v)$ for all $\alpha, \beta \in \mathbb{R}$ and $v \in V$.
- For addition and multiplication,
  - $\alpha(u + v) = \alpha u + \alpha v$
  - $(\alpha + \beta)u = \alpha u + \beta u$

Lecture 2: Second Lecture

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Appendix