Real Analysis 1

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Abstract

Contents

Chapter 1

Basic Things

Lecture 1: Real numbers system, Ordered Field, Completeness axiom, Archimedean property, Density of rational and irrational numbers

1.1 Natural Numbers

2 Sep. 09:10

The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, \dots\}$. There exists an addition operation

$$1 + 1 = 2 \quad 1 + 1 + 1 = 3 \quad \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n.$$

1.2 Integers

The set of integers is $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. There is a zero element 0 such that $z + 0 = z$ for any $z \in \mathbb{Z}$. Also, for $n \in \mathbb{N}$, we have $n + (-n) = 0$ and $n - m = n + (-m)$ for all $n, m \in \mathbb{N}$.

$$\mathbb{Z} \xRightarrow[\text{introduce division}]{} \mathbb{Q} \xRightarrow[\text{Completeness axiom}]{} \mathbb{R}$$

1.3 Field

Next, we introduce the concept of field.

Definition 1.3.1 (Fields). A field is a set F together with two binary operations, called addition($+$) and multiplication($*$), such that the following properties hold:

(a) $a + b = b + a$, $a \cdot b = b \cdot a$ for $a, b \in F$.

(b) $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in F$.

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$.

(d) There are distince element 0 and 1 such that $a + 0 = a$, $a \cdot 1 = a$ for $a \in F$.

(e) For each $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$. If $a \neq 0$, there is an element $\frac{1}{a}$ or a^{-1} in F such that $a \cdot \frac{1}{a} = 1$, or $a \cdot a^{-1} = 1$.

Remark. If $a \in F$, then $a + a \in F$. We denote $a + a$ by $2 \cdot a$. Similarly,

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = n \cdot a,$$

and

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

if $a \in F$ and $n \in \mathbb{N}$.

Remark. In a field, we have subtraction and division $a - b = a + (-b)$ for $a, b \in F$. If $b \neq 0$, then $\frac{a}{b} = a \cdot b^{-1}$ for $a, b \in F$.

In a field F, we have

$$\begin{aligned}(a+b)^2 &= (a+b) \cdot (a+b) \\ &= (a+b) \cdot a + (a+b) \cdot b \\ &= a \cdot a + b \cdot a + a \cdot b + b \cdot b \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2.\end{aligned}$$

Example.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

if $b \neq 0$ and $d \neq 0$.

Proof.

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= a \cdot b^{-1} + c \cdot d^{-1} \\ &= ab^{-1}dd^{-1} + cd^{-1}bb^{-1} \\ &= adb^{-1}d^{-1} + cbd^{-1}b^{-1} \\ &= \frac{ad + bc}{bd}.\end{aligned}$$

Notice that this is true since we have commutativity in multiplication and

$$d^{-1}b^{-1} = (bd)^{-1} = \frac{1}{bd}.$$

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Example. The set of rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$ is a field.

Example. The set of real numbers is also a field.

Example. $F_2 = \{0, 1\}$ is also a field since we can define addition and multiplication like $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$, and $0 \cdot 0 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$.

1.4 Order Relation

Next, we introduce the order relation. The real number system is ordered by the relation $<$, which has the following properties.

(f) For each pair of real numbers a and b, exactly one of the follwing is true: $a = b, a < b, b < a$.

(g) If $a < b$ and $b < c$, then $a < c$.

(h) If $a < b$, then $a + c < b + c$ for any c for any c, and if $0 < c$, then $a \cdot c < b \cdot c$.

Definition 1.4.1. A field with an order relation satisfy (f) to (h) is called an ordered field.

Example. The set of rational numbers is an ordered field.

Example. F_2 is not an ordered field.

Proof. If $0 < 1$, then $1 = 0 + 1 < 1 + 1 = 0$, which is a contradiction. If $1 < 0$, then $0 = 1 + 1 < 0 + 1 = 1$, which is also a contradiction. ■

Notation. In an ordered field, we use $a \leq b$ to denote either $a < b$ or $a = b$.

1.5 Absolute Value and Triangle Inequality

Next, we define the absolute value of a real number

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a \leq 0; \end{cases}$$

Theorem 1.5.1 (Triangle Inequality)**.**

$$|a + b| \leq |a| + |b|$$

for all $a, b \in \mathbb{R}$.

Corollary 1.5.1.

$$||a| - |b|| \leq |a - b| \quad \text{and} \quad ||a| - |b|| \leq |a + b|$$

Proof. We write

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Similarly we have

$$|b| \leq |b - a| + |a|.$$

So

$$-|b - a| \leq |a| - |b| \leq |a - b|.$$

Thus,

$$||a| - |b|| \leq |a - b|.$$

■

1.6 Supremum and Infimum

Next, we introduce the notion of supremum of a subset of real numbers.

Definition 1.6.1. Let S be a subset of $\mathbb{R}$,

(1) we say b is an upper bound of S if $x \leq b$ for all $x \in S$.

(2) If B is an upper bound of S, and no number smaller than B is an upper bound of S, then B is called the supremum or the least upper bound of S. We write $B = \sup S$.

Corollary 1.6.1. If $B = \sup S$, then

(1) $x \in S$ implies $x \leq B$

(2) If $b < B$, then b is not an upper bound of S, i.e. there exists $x_1 \in S$ such that $b < x_1$.

Definition 1.6.2. Let S be a subset of $\mathbb{R}$,

(1) we say b is an lower bound of S if $x \geq b$ for all $x \in S$.

(2) If α is an lower bound of S, and no number bigger than α is an lower bound of S, then α is called the infimum or the greatest lower bound of S. We write $\alpha = \inf S$.

Corollary 1.6.2. If $\alpha = \inf S$, then

(1) $x \in S$ implies $x \geq \alpha$

(2) If $\alpha < a$, then a is not an lower bound of S, i.e. there exists $x_1 \in S$ such that $x_1 < a$.

Notation (Interval Notation).

$$(a, b) = \{x \mid a < x < b\}$$
$$(a, b] = \{x \mid a < x \leq b\}$$
$$[a, b) = \{x \mid a \leq x < b\}$$

Example. $S = \{x \mid x < 0\} = (-\infty, 0)$, then $\sup S = 0$ but $\inf S$ does not exists.

Example. $S_1 = \{-1, -2, -3, -4, \dots\} = \{-n \mid n \in \mathbb{N}\}$, then $\sup S = -1$, but $\inf S$ does not exist.

Definition 1.6.3. A nonempty set if that a set has at least one element. The empty set, written as $\varnothing$, is the set has no elements at all.

Example. $S = \left\{x \mid x \in \mathbb{Q}, x < \sqrt{2}\right\}$

In $\mathbb{Q}$, $\sup S$ does not exist. In $\mathbb{R}$, $\sup S = \sqrt{2}$.

Theorem 1.6.1 (Completeness axiom)**.** If a nonempty set of real numbers (an ordered field) is bounded above, then it has a least upper bound or $\sup S$ exists.

Remark. This is an extra axiom that can't be derived from the properties of ordered field.

Remark. Up to "isomorphism", there is exactly one complete ordered field: the field of real numbers.

Remark. From now, we assume $\mathbb{R}$ satisfies the completeness axiom. Thus, any nonempty subset $S \subseteq \mathbb{R}$, that is bounded above, then $\sup S$ exists.

We can prove the following property of $\sup S$.

Theorem 1.6.2. If $S \subseteq \mathbb{R}$ is bounded above, then $\sup S$ is the unique real number B such that

(i) $x \leq B$ for all $x \in S$

(ii) for every $\varepsilon > 0$, there exist an $x_0 \in S$ such that $B\varepsilon < x_0$.

Proof. (i), (ii) follows from the definition. We prove the uniqueness. Suppose $B_1 = \sup S = B_2$. We want to show $B_1 = B_2$. Suppose $B_1 \neq B_2$. Then either $B_1 < B_2$ or $B_2 < B_1$. However, if either one is true, then the other one cannot be $\sup S$. ■

Theorem 1.6.3 (Archimedean Property)**.** If $p > 0$ and $\varepsilon > 0$, then there exists an $n \in \mathbb{N}$ such that $p < n\varepsilon$.

Proof. We prove this contradiction. Suppose it is not true. This implies $n\varepsilon \leq p$ for all $n \in \mathbb{N}$. Consider $S = \{n\varepsilon \mid n \in \mathbb{N}\}$, then p is an upper bound of S, so S is bounded above by p, so we know $B = \sup S$ exists. Hence, $n\varepsilon \leq B$ for all $n \in \mathbb{N}$, so we have $(n+1)\varepsilon \leq B$, which means

$$n\varepsilon \leq B - \varepsilon$$

for all $n \in \mathbb{N}$. This implies $B - \varepsilon$ is also an upper bound of S, which is a contradiction. ■

Theorem 1.6.4. Every nonempty subset of the integers that is bounded below has a least element.

Proof. We first introduce an axiom:

Theorem 1.6.5 (Well-Ordering principle)**.** Every non-empty subset of the natural numbers has a least element.

Note. Here, $\mathbb{N}$ can be $\{0, 1, 2, \dots\}$ or $\{1, 2, 3, \dots\}$, which is not that important.

Now we call this subset of integers as S, and suppose we have m as a lower bound of S, then define $S' = \{s - m \mid s \in S\}$, then we know S' is a nonempty subset of $\mathbb{N}$, then by well-ordering principle we know there is a least element in S' and thus there is also a least element in S. ■

Corollary 1.6.3. Every nonempty subset of the integers that is bounded above has a greatest element.

Proof. Suppose M is an upper bound, then define a set $S' = \{M - s \mid s \in S\}$, then by well-ordering principle we know $M - a$ is the least element of S' for some $a \in S$, so we have $M - x \geq M - a$ for all $x \in S$, which means $a \geq x$ for all $x \in S$ and since $a \in S$, so a is the greatest element of S. ■

Theorem 1.6.6. The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{p}{q}$ such that $a < \frac{p}{q} < b$.

Proof. Let $a, b \in \mathbb{R}$, $a < b$. By Archimedean Property, $\exists q \in \mathbb{N}$ such that $q(b-a) > 1$. Let $S = \{m \mid m \text{ is an integer with } m > qa\}$, since we know $S \neq \varnothing$ and S is bounded below. Hence, $p = \inf S$ exists and is an integer by the last theorem. So $qa < p$ and $p - 1 \leq qa$, which means $qa < p \leq qa + 1 < qb$, so we have $a < \frac{p}{q} < b$. ■

Lecture 2: Second Lecture in 1st Week

4 Sep. 10:20

Appendix