Linear Algebra I

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Abstract

The lecture note of Linear Algebra I by professor 余正道.

Contents

Chapter 1

Vector Space

Lecture 1

1.1 Introduction to vector and vector space

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In high school, our vectors are in $\mathbb{R}^2$ and $\mathbb{R}^3$, and we have define the addition and scalar multiplication of vectors.

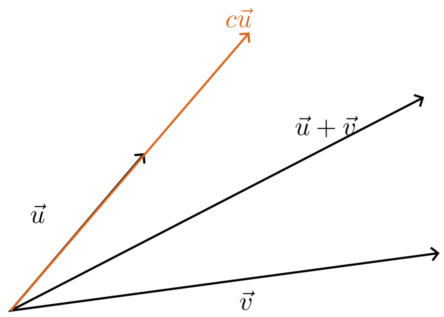

Figure 1.1: Vectors in $\mathbb{R}^2$

Example 1.1.1. $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n \mid a_i \in \mathbb{R})\}$

With this type of space, we can define addition and multiplication as

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = \{a_1 + b_1, a_2 + b_2, \dots, a_n + b_n\}$$
$$\alpha \cdot (a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$$

Also, if we define a space:

Example 1.1.2. $V = \{\text{function } f : (a, b) \to \mathbb{R}\}$, where (a, b) is an open interval.

then this can also be a vector space after defining addtion and multiplication.

Note 1.1.1. In a vector space, we have to make sure the existence of 0-element, which means $0(x) = 0$.

Now we give a more abstract example:

Example 1.1.3. Suppose S is any set, then define $V = \{\text{all functions from } S \text{ to } \mathbb{R}\}$

If we define $(f + g)(s) = f(s) + g(s)$ and $(\alpha \cdot f)(s) = \alpha \cdot f(s)$, and $0(s) = 0$, then this is also a vector space.

Put some linear conditions

Example 1.1.4. In $\mathbb{R}^n$, fix $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, if we define

$$W = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \dots + a_nx_n = 0\},$$

then this is also a vector space.

However, if we have

$$W' = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = 1\},$$

then this is not a vector space because it is not close.

Example 1.1.5. In $V = \{(a, b) \to \mathbb{R}\}$ or $W_1 = \{\text{polynomial defined on } (a, b)\}$, these are both vector space.

Remark 1.1.1. In the later course, we will learn that W_1 is a subspace of V.

Example 1.1.6. If we furtherly defined $W_1^{(k)} = \{\text{polynomial degree } \leq k\}$, then this is also a vector space.

Remark 1.1.2. $W_1^{(k)}$ is actually isomorphic to $\mathbb{R}^{k+1}$ since

$$a_0 + a_1x + a_2x^2 + \dots + a_kx^k \leftrightarrow (a_0, a_1, a_2, \dots, a_n).$$

Example 1.1.7. $W_2 = \{\text{continuous function on } (a, b)\}$ and $W_3 = \{\text{differentiable functions}\}$ are also both vector spaces.

Example 1.1.8. $W_4 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0\right\}$ and $W_5 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -f\right\}$ are both vector spaces.

Proof.

$$\begin{aligned} W_4 &= \{a_0 + a_1x\} \\ W_5 &= \{a_1 \cos x + a_2 \sin x\} \end{aligned}$$

⊛

1.2 Formal definition of vector spaces

1.2.1 Vector Spaces Over $\mathbb{R}$

Definition 1.2.1. Suppose V is a non-empty set equipped with

- addition: $V \times V \to V$, that is, given $u, v \in V$, defining $u + v \in V$
- scalare multiplication: $\mathbb{R} \times V \to V$, that is, given $\alpha \to \mathbb{R}$ and $v \in V$, we need to have $\alpha v \in V$

Also, we need some good properties or conditions

- For addition,
  - $u + v = v + u$
  - $(u + v) + w = u + (v + w)$
- There exists $0 \in V$ such that $u + 0 = u = 0 + u$

- Given $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0 = (-v) + v$
- For scalar multiplication,
  - $1 \cdot v = v$ for all $v \in V$
  - $(\alpha\beta)v = \alpha \cdot (\beta v)$ for all $\alpha, \beta \in \mathbb{R}$ and $v \in V$.
- For addition and multiplication,
  - $\alpha(u + v) = \alpha u + \alpha v$
  - $(\alpha + \beta)u = \alpha u + \beta u$

Lecture 2

1.3 Vector Space over general field

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Now we introduce the concept of field.

Definition 1.3.1 (Field). A set F with $+$ and $\cdot$ is called a **field** if

- $\alpha + \beta = \beta + \alpha$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- There exists $0 \in F$ such that $\alpha + 0 = 0 + \alpha = \alpha$.
- For $\alpha \in F$, there exists $-\alpha$ such that $\alpha + (-\alpha) = 0$.
- $\alpha\beta = \beta\alpha$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
- $\exists 1$ such that $1 \neq 0$ and $1 \cdot \alpha = \alpha$.
- For $\alpha \neq 0$, $\exists \alpha^{-1} \in F$ such that $\alpha\alpha^{-1} = 1$.
- $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Example 1.3.1. $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are all fields but $\mathbb{Z}$ is not.

Example 1.3.2. $\{0, 1\}$ is also a field.

Now we know the concept of filed, so we can make a vector space over a field.

Theorem 1.3.1 (Cancellation law). Suppose $v_1, v_2, w \in V$, a vector space, then if $v_1 + w = v_2 + w$, then $v_1 = v_2$.

Proof.

$$v_1 = v_1 + (w + (-w)) = (v_1 + w) + (-w) = (v_2 + w) + (-w) = v_2 + (w + (-w)) = v_2.$$

■

Theorem 1.3.2. The zero vector 0 is unique.

Proof. Suppose we have $0, 0'$ both zero vector, then for some $0 = 0 + 0' = 0'$. ■

Theorem 1.3.3. For any $v \in V$, $0 \cdot u = 0$.

Proof. $0 \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u$, so $0 = 0 \cdot u$ by cancellation law. ■

Theorem 1.3.4. $(-1) \cdot u = -u$.

Theorem 1.3.5. Given any $u \in V$ is unique, $-u$ is unique.

1.4 Subspaces

Definition 1.4.1 (subspace). Let V be a vector space. A non-empty subset $W \subseteq V$ is called a subspace of V if W is itself a vector space under $+$ and $\cdot$ on V.

Example 1.4.1. $M_n(F) = \{n \times n \text{ matrix with entries in } F\}$ is a vector space, and

$$U_n(F) = \left\{ \begin{pmatrix} a_{11} & & & \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \right\}$$

is a subspace of $M_n(F)$.

Proposition 1.4.1. Suppose V is a vector space, and $W \subseteq V$ is non-empty, then

$$W \text{ is a subspace} \Leftrightarrow \text{For } u, v \in W, \alpha \in F, \text{ we have } u + v \in W \text{ and } \alpha \cdot u \in W.$$

proof of $\Rightarrow$. Clear. ■

proof of $\Leftarrow$. First, we would want to check $0 \in W$, and we can pick any $u \in W$, and pick $\alpha = -1$, so we know $-u \in W$, and thus $0 = u + (-u) \in W$. ■

Corollary 1.4.1. If we want to check W is a subspace, we just need to check for $u, v \in W$, $\alpha \in F$, $u + \alpha v \in W$ or not.

1.5 Linear Combination

Definition 1.5.1 (Linear combinaiton). Given $v_1, v_2, \ldots, v_n \in V$, a linear combination of them is a vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Proposition 1.5.1. Given $v_1, v_2, \ldots, v_n \in V$,

1. $W = \{\text{all linear combinations of } v_, \ldots, v_n\}$ is a subspace.
2. This subspace is the smallest subspace containing $v_1, \ldots, v_n$. That is, if $W' \subseteq V$ is a subspace containing $v_1, \ldots, v_n$, then $W \subseteq W'$.

> **Notation.** $\operatorname{span}\{v_1, v_2, \ldots, v_n\} = \{\text{all linear combinations of } v_1, v_2, \ldots, v_n\}$

1.6 Linearly independent

Definition. Now we talk about the linear dependence and linear independence.

Definition 1.6.1 (Linearly dependent). $v_1, v_2, \ldots, v_n$ are linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zeros.

Definition 1.6.2 (Linearly independent). $v_1, v_2, \ldots, v_n$ are called linearly independent if they are not linearly dependent.

Corollary 1.6.1. Say $\alpha_i \neq 0$, then $v_i \in \operatorname{span}\{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_k\}$ suppose the corresponding α_i of $\hat{v}_1, \ldots, \hat{v}_k$ are not zeros.

Corollary 1.6.2. Linearly independent means if $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Corollary 1.6.3. Linearly independent meeans if $\sum \alpha_i v_i = \sum \beta_i v_i$, then $\alpha_i = \beta_i$ for all i.

Example 1.6.1.

- $v \in V$ is linearly independent iff $v \neq 0$.
- $v, w \in V$ are linearly independent iff v is not a scaler of w and w is not a scalar of v.

Lemma 1.6.1. $v_1, \ldots, v_n$ are linearly independent iff $v_i \notin \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$.

1.7 Basis

Definition. We now talking about basis

Definition 1.7.1 (Basis). $B = \{v_1, v_2, \ldots, v_n\}$ is called a basis of V if B spans V and B is linearly independent.

Definition 1.7.2 (Dimension). In this case, n is called the dimension of V, and denoted by $\dim V$.

Notation. $\operatorname{span}\{v_1, v_2, \ldots, v_n\} = \langle v_1, v_2, \ldots, v_n \rangle$

Notation. $\operatorname{span}(S) = \langle S \rangle$

Theorem 1.7.1. For any $v \in V$, it has a unique expression $v = \sum_{i=1}^{n} \alpha_i v_i$.

Lecture 3

As previously seen. A basis of a vector space V is a set $\{v_1, v_2, \ldots, v_n\}$ that is linearly independent and simultaneously spans V. That is, suppose we have $\sum a_i v_i = 0$ for some scalars a_i, then $a_i = 0$ for all i. Also, we call the number n, the dimension of V. 10 Sep 10:20

Example 1.7.1. Suppose we have $V = F^n = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \mid \alpha_i \in F\}$, then we have a **standard basis**, which is

$$\begin{aligned} e_1 &= (1, 0, \dots, 0) \\ e_2 &= (0, 1, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, \dots, 1) \end{aligned}$$

since $\{e_i\}_{i=1}^n$ is linearly independent and for every $\vec{a} = (a_1, \dots, a_n)$, we know

$$\vec{a} = \sum_{i=1}^n a_i e_i.$$

Example 1.7.2. Suppose

$$V = M_{n\times n}(F) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \ddots & & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \dots & & \alpha_{nn} \end{pmatrix} \right\},$$

then we know

$$\{e_{ij}\}_{1\leq i,j\leq n} = \begin{pmatrix} 0 & 0 & & & \\ 0 & & & & \\ & & 1 & & \\ 0 & & & 0 & \\ 0 & & & & 0 \end{pmatrix},$$

where the 1 is in the i-th row and j-th column.

Theorem 1.7.2. Suppose V is a vector space, and $V = \langle v_1, v_2, \dots, v_n\rangle$ and $\{w_1, w_2, \dots, w_m\}$ is linearly independent, then $m \leq n$. Furthermore, one can make

$$\langle w_1, w_2, \dots, w_m, v_{m+1}, \dots, v_n\rangle = V$$

after rearrangement of $v_1, \dots, v_n$.

Proof. We can do induction on m. It is trivial that $m = 0$ is true. Suppose the statement holds for a fixed m with $m \leq n$. Let $w_1, w_2, \dots, w_{m+1}$ be linearly independent. In particular, $w_1, w_2, \dots, w_m$ is linearly independent.

Claim 1.7.1. $m + 1 \leq n$.

Proof. Otherwise, if $m + 1 > n$, then since $m \leq n$, so $m = n$. Hence, by induction hypothesis, we know $\langle w_1, w_2, \dots, w_m\rangle = V$. However, by Lemma 1.7.1 and the note following it, we know

$$\{w_1, w_2, \dots, w_m\} \cup \{w_{m+1}\}$$

can not be linearly indepndent since $w_{m+1} \in V = \langle w_1, \dots, w_m\rangle$. ⊛

Now we know $m + 1 \leq n$. By induction hypothesis, we know

$$\langle w_1, w_2, \dots, w_m, v_{m+1}, \dots, v_n\rangle = V$$

Claim 1.7.2. One of $v_{m+1}, \dots, v_n$ can be replaced by w_{m+1}.

Proof. Since

$$w_{m+1} = \sum_{i=1}^{m} \alpha_i w_i + \sum_{j=m+1}^{n} \beta_j v_j.$$

Trivially, one of $\beta_j \neq 0$, say $\beta_{m+1} \neq 0$. Check

$$\langle w_1, \ldots, w_m, w_{m+1}, v_{m+2}, \ldots, v_n \rangle = V.$$

⊛

■

Corollary 1.7.1. If $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ are bases of V, then $n = m$.

Remark 1.7.1. Corollary 1.7.1 tells us $\dim V$ is well-defined, which means the size of the bases of a vector space is unique.

Corollary 1.7.2. Suppose $\dim V = n$, then if $\langle v_1, v_2, \ldots, v_m \rangle = V$, then $m \geq n$. If $\{w_1, w_2, \ldots, w_m\}$ is linearly independent, then $m \leq n$. Also, any $\{v_i\}_{i=1}^{m}$ with $m > n$ is linearly dependent.

Lemma 1.7.1. Suppose $v_1, v_2, \ldots, v_n$ is linearly independent. If $w \notin \langle v_1, v_2, \ldots, v_n \rangle$, then

$$\{v_1, v_2, \ldots, v_n, w\}$$

is linearly independent.

Proof. Suppose $\sum_{i=1}^{n} \alpha_i v_i + \alpha_{i+1} w = 0$, then if $\alpha_{i+1} = 0$, we know $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ since $\{v_i\}_{i=1}^{n}$ is linearly independent. If $\alpha_{i+1} \neq 0$, then $w = \frac{1}{\alpha_{i+1}} \sum_{i=1}^{n} \alpha_i v_i \in \langle v_1, v_2, \ldots, v_n \rangle$, which is a contradiction. ■

Note 1.7.1. The reverse of Lemma 1.7.1 is still correct and is trivial. That is, if $w \notin \{v_1, \ldots, v_n\}$ and $\{v_1, v_2, \ldots, v_n, w\}$ is linearly independent, then $\{v_1, \ldots, v_n\}$ is linearly independent.

Corollary 1.7.3. If $W \subseteq V$ is a subspace of V, then $\dim W \leq \dim V$.

Proof. If $\dim V = n$, and $\{w_i\}_{i=1}^{m}$ is a basis of W, then this basis is linearly independent in V, which means $m \leq n$ by Theorem 1.7.2. ■

Corollary 1.7.4. If $v_1, v_2, \ldots, v_m$ is linearly independent, then $\{v_1, v_2, \ldots, v_m\}$ forms a basis after adding some $v_{m+1}, \ldots, v_n$ to it.

Theorem 1.7.3 (Dual version)**.** If $\langle v_1, v_2, \ldots, v_n \rangle = V$, then $\{v_1, v_2, \ldots, v_m\}$ forms a basis after rearrangement, where $m \leq n$.

Remark 1.7.2. Most of the time, we consider finite-dimensional vector spaces.

Remark 1.7.3 (Examples of ∞-dim vector space)**.**

- $$V = \{\text{all polynomials over } F\} = F[x] = \{a_0 + a_1 x + \cdots + a_n x^n \text{ for some } n \text{ where } a_i \in F\}.$$

- $$W = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\}.$$

  Notice that

  $$W' = \{\text{convergent sequence}\} \subseteq W.$$

  and

  $$W'' = l^2 = \left\{ (a_i) \mid \sum_{i=0}^{\infty} a_i^2 \text{ finite} \right\} \subseteq W'$$

Remark 1.7.4. We define $\dim\{0\} = 0$, which is the only vector space with dimension 0, and we define $\langle \varnothing \rangle = \{0\}$, which means $\varnothing$ is the basis of $\{0\}$.

Note 1.7.2. We call a subspace $W \subsetneq V$ is proper.

1.8 More on subspaces

Theorem 1.8.1. If W_1 and W_2 are subspace of V, then $W_1 \cap W_2$ is a subspace.

Theorem 1.8.2. If W_1, W_2 are subspaces of V, then $W_1 + W_2$ is still a subspace of V.

Remark 1.8.1. If W_1, W_2 are subspaces of V, then $W_1 \cup W_2$ may not be a subspace. (See HW1).

Remark 1.8.2. In fact, $W_1 \cap W_2$ is the largest subspaces contained in W_1 and W_2.

Remark 1.8.3. In fact, $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2.

Corollary 1.8.1. Suppose S is the index set, and for all $i \in S$, W_i is a subspace of V, then

$$\bigcap_{i \in S} W_i = \{v \in V \mid v \in W_i \ \forall i\}$$

is also a subspace of V.

Corollary 1.8.2. Suppose S is the index set, and for all $i \in S$, W_i is a subspace of V, then

$$\sum_{i \in S} W_i = \{w_{i_1} + w_{i_2} + \cdots + w_{i_n} \text{ for some } i_j \in S\}$$

is also a subspace of V.

Proposition 1.8.1 (Dimension theorem)**.** Suppose $W_1, W_2 \subseteq V$ are subspaces of V, then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Lecture 4

In calculus, $f : \mathbb{R} \to \mathbb{R}$ is called continuous if $f(\lim_{x \to a} x) = \lim_{x \to a} f(x)$. 12 Sep 10:20

Definition 1.8.1 (Linear transformation). Suppose V, W are vector spaces over F. A function

$$\begin{aligned} T : V &\to W \\ v &\mapsto T(v) \end{aligned}$$

is called a linear transformation or a linear map if

$$T(u+v) = T(u) + T(v) \quad T(\alpha v) = \alpha T(v),$$

or equivalently,

$$T(\alpha u + v) = \alpha T(u) + T(v).$$

Corollary 1.8.3. Suppose T is a linear transformation, then

$$T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i T(u_i).$$

Example 1.8.1. Suppose $V = \{\text{functions from } (-1,1) \text{ to } \mathbb{R}\}$, and define $T_a(f) = f(a)$, then T_a is a linear transformation.

Example 1.8.2. Consider the space of column vectors,

$$F^n = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \mid \alpha_i \in F \right\},$$

and define $A = (a_{ij}) \in M_{n \times n}(F)$ by

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

then if we have $T_A : F^n \to F^m$ where

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

then T_A is a linear map.

Note 1.8.1.

$$\begin{pmatrix} & \vdots & \\ \alpha_{i1} & \cdots & \alpha_{in} \\ & \vdots & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vdots \\ \sum_{j=1}^{n} a_{ij} x_j \\ \vdots \end{pmatrix}$$

Example 1.8.3. Consider row of vector space,

$$F^m = \{(\alpha_1, \ldots, \alpha_m) \mid \alpha_i \in F\},$$

and $A \in M_{m\times n}(F)$, then if $T_A : F^m \to F^n$ where

$$T_A : u = (u_1, \dots, u_m) \mapsto (u_1, \dots, u_m) \cdot A$$

is a linear map.

Obeserve that a linear map $T : V \to W$ is determined by $T(v_i)$, where $\{v_1, \dots, v_n\}$ is a basis of V.

Proposition 1.8.2. Suppose $\{v_1, v_2, \dots, v_n\}$ is a basis of V, then pick any $w_1, \dots, w_n \in W$. Then there is a unique linear map $T : V \to W$ satisfying $T(v_i) = w_i$.

Proof. Since any $v \in V$ has a unique representation $v = \sum_{i=1}^n \alpha_i v_i$. Hence, for a linear map $T : V \to W$, and for any $v \in V$, we know

$$T(v) = T\left(\sum_{i=1} \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i T(v_i) = \sum_{i=1}^n \alpha_i w_i.$$

Hence, if such map exists, then it must be unique. Now we have to show the existence of this map. Now if we define a map

$$T\left(\sum_{i=1}^n \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i w_i,$$

then we can check this is a linear map. ■

Example 1.8.4. Suppose F^n is the span of column vectors, and $A \in M_{m\times n}(F)$, and define $T_A(v) = Av$, then we can check $T_A(e_i) = c_i$, where c_i is the i-th column of A. This is the linear map that sends e_i to $c_i \in F^m$. If we pick $c_1, c_2, \dots, c_n \in F^m$, then there is a unique map sending e_i to c_i. In fact, this map is

$$T_A : v \mapsto Av$$

, where the i-th column of A is c_i.

Definition. Given $T : V \to W$, where T is linear.

Definition 1.8.2 (Kernel)**.** The kernel/nullspace of T is defined as

$$\ker(T) = \{v \in V \mid T(v) = 0\} \subseteq V.$$

Definition 1.8.3 (Image)**.** The image/range of T is defined as

$$\mathrm{Im}(T) = \{T(v) \mid v \in V\} \subseteq W.$$

Remark 1.8.4. Kernel and Image are subspaces.

Lecture 5

As previously seen. Given such a linear map $T : V \to W$, we define 17 Sep. 10:20

$$\begin{aligned} \ker T = T^{-1}(0) &\quad \text{kernel/null space of } T \\ \mathrm{Im}\, T = T(V) &\quad \text{image/range of } T, \end{aligned}$$

and $\ker T$ is a subspaces of V, and $\mathrm{Im}\, T$ is a subspace of W.

Definition. Now we define the nullity and rank of a linear map.

Definition 1.8.4 (nullity). The nullity of T is the number

$$\nu(T) = \dim \ker T.$$

Definition 1.8.5 (rank). The rank of T is the number $\operatorname{rank} T = \dim \operatorname{Im} T$.

Example 1.8.5. Suppose $T : F^n \to F^m$, where F^n is the column space of dimension n, then $T = T_A$ for a matrix $A \in M_{m\times n}(F)$ and $T_A(v) = Av$.

Proof. Suppose $A = (c_1, c_2, \dots, c_n)$, where c_i is the i-th column vector of A. Consider the standard basis $\{e_1, e_2, \dots, e_n\}$ of F^n, where e_i is the column vector with i-th position 1 and the other entries are all 0's. Then, $T_A(e_i) = c_i \in F^m$. Explicitly,

$$T_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1c_1 + \dots + x_nc_n$$

since we know

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_i x_ie_i.$$

and $T_A(e_i) = c_i$. In this case,

$$\begin{aligned} \ker T_A &= \text{all linear relations among } c_1, \dots, c_n \subseteq F^n \\ \operatorname{Im} T_A &= \operatorname{span}\{c_1, \dots, c_n\} \subseteq F^m. \end{aligned}$$

If we want to solve $\ker T_A$, then we need to solve

$$0 = x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Hence, we have to solve

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = 0. \end{cases}$$

Given $A = (c_1, \dots, c_n)_{m\times n}$, then the column rank is $\dim\langle c_1, \dots, c_m\rangle$. If we rewrite $A = (r_1, \dots, r_m)^t$, where r_i is the i-th row of A, then the row rank is $\dim\langle r_1, r_2, \dots, r_m\rangle$. Since we can define $S_A : F^m \to F^n$, where

$$v = (x_1, \dots, x_m) \mapsto vA.$$

Remark 1.8.5. In fact, column rank is equal to row rank in a matrix, and we will prove it later.

⊛

Theorem 1.8.3 (rank and nullity theorem)**.** Suppose $T : V \to W$ is a linear map, then

$$\nu(T) + \operatorname{rank} T = \dim V.$$

Proof. Since $\ker T \subseteq V$, so take a basis $\{v_1, \ldots, v_\nu\}$ of $\ker T$, and $\operatorname{Im} T \subseteq W$, so take a basis $\{w_1, \ldots, w_r\}$ of $\operatorname{Im} T$. Take u_j s.t. $T(u_j) = w_j$.

Claim 1.8.1. $S = \{v_1, \ldots, v_\nu, u_1, \ldots, u_r\}$ forms a basis of V.

Proof. We first show that S is linearly independent. Suppose $\sum \alpha_i v_i + \sum \beta_j u_j = 0$. Apply T on it, we get

$$0 = \sum \alpha_i T(v_i) + \sum \beta_j T(u_j) = \sum \alpha_i T(v_i) + \sum \beta_j w_j = \sum \beta_j w_j.$$

However, $\{w_j\}$ is linearly independent, so $\beta_j = 0$ for all j. Now we know $\sum \alpha_i v_i = 0$, which means $\alpha_i = 0$ for all i, so S is linearly independent. Now we want to show $\langle S \rangle = V$. Given $v \in V$, we know $T(v) \in \operatorname{Im} T$, and thus we can represent it as $T(v) = \sum \beta_j w_j$. We want to show

$$v = \sum \alpha_i v_i + \sum \beta_j u_j.$$

Thus, we want to show $v - \sum \beta_j u_j \in \ker T$, but note that

$$T\left(v - \sum \beta_j u_j\right) = T(v) - \sum \beta_j w_j = \sum \beta_j w_j - \sum \beta_j w_j = 0,$$

so we're done, and thus we have

$$v - \sum \beta_j u_j = \sum \alpha_i v_i$$

for some α_i's, and we're done. ⊛

Hence, $\dim V = |S| = \nu T + \operatorname{rank} T$. ■

Remark 1.8.6. If $\dim V > \dim W$, then $\nu(T) > 0$. Since, $\operatorname{rank} T \leq \dim W$, so if $\dim V > \dim W$, then we have $\nu(T) = \dim V - \operatorname{rank} T \geq \dim V - \dim W > 0$.

As previously seen. A map $f : X \to Y$ is called one-to-one or 1-1 or injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. f is called onto, surjective if $f(X) = Y$. f is called bijective if it is both 1-1 and onto. In this case, there is the inverse map $f^{-1} : Y \to X$ with $y \mapsto x$ if $f(x) = y$.

Proposition 1.8.3. Let $T : V \to W$ be linear, then T is injective iff $\ker T = \{0\}$.

Proof.

- ($\Rightarrow$) If $v \in \ker T$, then since $T(0) = 0$, so $v = 0$.
- ($\Leftarrow$) If $T(v_1) = T(v_2)$, then $T(v_1 - v_2) = 0$, which means $v_1 - v_2 \in \ker T = \{0\}$, so $v_1 = v_2$, which means T is linear.

■

Proposition 1.8.4. If $T : V \to W$ is a linear map, and if b is a basis of V, then T is injective if and only if $T(b)$ is linearly independent.

Proof.

- ($\Rightarrow$) Suppose $v_1, v_2, \ldots, v_n$ is a basis of V and we want to show $T(v_1), \ldots, T(v_n)$ is linearly independent. Suppose $\sum \alpha_i T(v_i) = 0$, then $T\left(\sum \alpha_i v_i\right) = 0$, so $\sum \alpha_i v_i = 0$, and thus $\alpha_i = 0$ for all i.

($\Leftarrow$) T sends one particular basis $v_1, \ldots, v_n$ to a linearly independent set. We want to show $\ker T = \{0\}$. Suppose $v \in \ker T$, then if $v = \sum \alpha_i v_i$, we have

$$0 = T\left(\sum \alpha_i v_i\right) = \sum \alpha_i T(v_i),$$

but since $\{T(v_i)\}$ is linearly independent, so $\alpha_i = 0$ for all i, which means $v = 0$.

■

Proposition 1.8.5. If $T : V \to W$ is a linear map, then TFAE

(a) T is surjective

(b) T sends any basis to a generating set.

(c) T sends one basis to a generating set.

Theorem 1.8.4 (isomorphism). Suppose $T : V \to W$ is linear and bijective, then there is the inverse map $T^{-1} : W \to V$, and T^{-1} is also linear. In this case, $T : V \to W$ is called an isomorphism.

Definition 1.8.6. If T is both injective and surjective, then T is an isomorphism.

Remark 1.8.7. If there is an isomorphism from V to W, we say V is isomorphic to W, or V and W are isomorphic.

Example 1.8.6 (Coordinates). If $\dim V = n$, then V is isomorphic to F^n, we write $V \simeq F^n$.

Proof. In fact, given an order basis $B = \{v_1, \ldots, v_n\}$ of V, then we know $v = \sum_{i=1}^n \alpha_i v_i$, where

$$v = \sum_{i=1}^n \alpha_i v_i \mapsto [v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix},$$

and this is a bijection. Note that this map is well-defined since any v has unique coordinate under B. Hence, we have $v_i \mapsto [v_i]_B = e_i$. ⊛

Hence, if $T : V \to W$, and we know $V \simeq F^n$ and $W \simeq F^m$, and we know there is a matrix sends F^n to F^m, called $[T]_{B'}^B$, and we can use it to represent the transformation from V to W, which is T.

Exercise 1.8.1. $T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$.

Proof. Suppose $T(v_3) = w_1 + w_2$, we want to show $v_3 = v_1 + v_2$. Hence, we need to check

$$w_1 + w_2 = T\left(T^{-1}(w_1) + T^{-1}(w_2)\right) = T\left(T^{-1}(w_1)\right) + T\left(T^{-1}(w_2)\right) = w_1 + w_2,$$

which is true. ■

Lecture 6

As previously seen. T is called an isomorphism if T is both injective and surjective. 19 Sep. 10:20

Proposition 1.8.6. Suppose $\dim V = \dim W = n$, then TFAE

(i) T is an isomorphism.

(ii) T is injective.

(iii) T is surjective.

(iv) T sends any basis of V to a basis of W.

(v) T sends one basis to a basis.

Example 1.8.7. Suppose $A \in M_{m\times n}(F)$, say $A = (c_1, c_2, \ldots, c_n)$, then T_A is injective if and only if $\{c_1, \ldots, c_n\}$ is linearly independent. (which means $n \leq m$).

Proof. Since $T_A(e_i) = c_i$ and $\{e_i\}_{i=1}^n$ forms a basis. ⊛

Example 1.8.8. Following the last example, T_A is surjective if and only if $\{c_1, c_2, \ldots, c_n\}$ spans W. (which means $n \geq m$).

1.9 Space of linear maps

Consider

$$\{f : V \to W\},$$

and then we can define addition and multiplication by

$$(f+g)(v) = f(v) + g(v) \quad (\alpha \cdot f)(v) = \alpha f(v).$$

Hence, we know it is a vector space. Now if we collect all linear maps, say

$$\mathcal{L}(V, W) = \{\text{linear } T : V \to W\}.$$

Observe that $\mathcal{L}(V, W)$ is a vector space since we can similarly define the addition and multiplication.

Now if we have U, V, W, three vector spaces, and $f : U \to V$ is a linear map, then if we define a map

$$\begin{aligned} R_f : \mathcal{L}(V, W) &\to \mathcal{L}(U, W) \\ T &\mapsto T \circ f, \end{aligned}$$

then this map is linear. Similarly,

$$\begin{aligned} L_f : \mathcal{L}(W, U) &\to \mathcal{L}(W, V) \\ T &\mapsto f \circ T, \end{aligned}$$

then this is also a linear map.

Note 1.9.1. We just need to check something like

$$R_f(T+S) = R_f(T) + R_f(S) \quad R_f(\alpha T) = \alpha R_f(T).$$

Now if we consider

$$\begin{aligned} \mathcal{L}(V, W) \times \mathcal{L}(U, V) &\to \mathcal{L}(U, W) \\ (T, S) &\mapsto T \circ S, \end{aligned}$$

then this is also a linear map.

Example 1.9.1. $\mathcal{L}(F^n, F^m) = M_{m\times n}(F)$.

Proof. Check that
$$T_A + T_B = T_{A+B}.$$

Note 1.9.2. More precisely, they are isomorphic, that is, $\mathcal{L}(F^n, F^m) \cong M_{m\times n}(F)$.

⊛

Example 1.9.2. Consider
$$\mathcal{L}(F^n, F^m) \times \mathcal{L}(F^p, F^n) \to \mathcal{L}(F^p, F^m),$$
we know this is a linear map, and by Example 1.9.1, we know
$$M_{m\times n}(F) \times M_{n\times p}(F) \to M_{m\times p}(F)$$
is a linear map.

Proof. Check
$$(T_A \circ T_B)(v) = T_{AB}(v) \Leftrightarrow A(Bv) = (AB)(v).$$

⊛

Definition 1.9.1. We call
$$V \cong F^n$$
a basic isomorphisms if $\dim V = n$.

Corollary 1.9.1. $\mathcal{L}(F^n, F^m) \cong M_{m\times n}(F)$.

Remark 1.9.1. If you change F^n to V and F^m to W, then this is also correct since $F^n \cong V$ and $F^m \cong W$. (We suppose $\dim V = n$ and $\dim W = m$.)

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$$\begin{tikzcd}
U \arrow[d, "f"'] \arrow[dr, "T\circ F"] & \\
V \arrow[r, "T"'] & W
\end{tikzcd}$$

linear

There is a special case,
$$\mathcal{L}(V, V) \coloneqq \mathcal{L}(V) = \{\text{linear } T : V \to V\},$$
which is the space of linear operators on V.

Now consider linear $T_A : F^n \to F^m, T_B : F^p \to F^m$, then we can define a map $T_{AB} = T_A \circ T_B$, and it will be a linear map.

$$\begin{tikzcd} F^p \arrow[d, "T_B"'] \arrow[dr, "T_A \circ T_B = T_{AB}"] & \\ F^n \arrow[r, "T_A"'] & F^m \end{tikzcd}$$

Also, note that T_A, T_B corresponds to two matrices A, B, respectively, and it turns out that T_{AB} corresponds to the matrix AB. (Check)

Hence, $\mathcal{L}(F^n) = M_n(F)$.

A matrix P is called invertible if T_P is bijective. In this case,

$$\begin{tikzcd} F^n \arrow[r, shift left, "T_p"] & F^m \arrow[l, shift left, "T_Q"] \end{tikzcd}$$

Hence, there exists $Q \in M_n(F)$ s.t. $QP = PQ = I_n$ since we know $T_P \circ T_Q = T_Q \circ T_P = I$.

Thus, we have

$$P = (c_1, c_2, \ldots, c_n) \text{ invertible} \Leftrightarrow \{c_1, \ldots, c_n\} \text{ is a basis.}$$

by Proposition 1.8.6.

1.10 Map/matrix correspondence

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\cdot]_B}"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "{[\cdot]_{B'}}"] \\ F^n \arrow[r, "\text{What is this?}"'] & F^m \end{tikzcd}$$

Take an ordered basis $B = \{v_1, v_2, \ldots, v_n\}$ and $B' = \{w_1, \ldots, w_m\}$, and says

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i \mapsto \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}.$$

Now consider the matrix

$$A = (\alpha_{ij}) = \left([T(v_1)]_{B'}, [T(v_2)]_{B'}, \ldots\right),$$

and then we called A the martix of T relative to B and B'. (matrix representative of T), and we denote this by $[T]_{B'}^{B}$.

Theorem 1.10.1.

$$[T(v)]_{B'} = [T]_{B'}^{B}[v]_B.$$

Theorem 1.10.2. We have $[\cdot]_{B'}^{B} : \mathcal{L}(V, W) \to M_{m\times n}(F)$, ans this matrix representative $[\cdot]_{B'}^{B}$ is an isomorphism, which means

- $[T+S]_{B'}^{B} = [T]_{B'}^{B} + [S]_{B'}^{B}$.
- It is bijective.

Corollary 1.10.1. if $\dim V = n$ and $\dim W = m$, then

$$\dim(\mathcal{L}(V,W)) = \dim V \cdot \dim W.$$

Theorem 1.10.3.

$$[T]^{B}_{B'}[S]^{B''}_{B} = [T \circ S]^{B''}_{B'}.$$

$$\begin{tikzcd}
v_j \arrow[dd, leftrightarrow] & V \arrow[r] \arrow[d] & W \arrow[d] & \sum_{i=1}^{n} \alpha_{ij} w_i \arrow[dd, leftrightarrow] \\
 & F^n \arrow[r] & F^m & \\
e_j \arrow[rrr, mapsto] & & & c_j = (\alpha_{1j}, \cdots, \alpha_{mj})^t
\end{tikzcd}$$

Special case:

$$\mathcal{L}(V) \to M_n(F).$$

Take an ordered basis $B = \{v_1, \dots, v_n\}$. If $T \in \mathcal{L}(V)$, then we can define $[T]_B = [T]^B_B$.

Corollary 1.10.2. Given $T : V \to W$. There are $B = \{v_1, \dots, v_n\}$ and $B' = \{w_1, \dots, w_m\}$ where B is a basis of V and B' is a basis of W and

$$[T]^B_{B'} = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

where $p = \operatorname{rank}(T)$.

Proof. We can let $B = \{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$, where $\{v_{r+1}, \dots, v_n\}$ is a basis of $\ker T$ and $T(v_1), \dots, T(v_r)$ is a basis of $\operatorname{Im}(T)$, (Recall the proof in Theorem 1.8.3), then we can let $B' = \{T(v_1), \dots, T(v_r), \dots\}$. ■

Example 1.10.1. Suppose $V = \{\text{polynomials with degree} \le k\}$ and W is the space of polynomials with degree $\le k+1$, then if $T : V \to W$ and $p(x) \mapsto \int_0^x p(t)\,\mathrm{d}t$, then we know an ordered basis $B = \left\{1, x, x^2, \dots, x^k\right\}$ and $B' = \left\{1, x, x^2, \dots, x^{k+1}\right\}$, and then

$$[T]^B_{B'} = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ 0 & \frac{1}{2} & & \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & \frac{1}{k+1} \end{pmatrix}.$$

Example 1.10.2. Suppose V is the space of polynomials of degree $\le k$, and $B = \left\{1, x, x^j, \dots, x^k\right\}$, and $B' = \left\{1, y, y^2, \dots, y^k\right\}$ with $y = x - 1$. Then, if T is the identity transformation, note that

$$x^j = (y+1)^j = 1 + j \cdot y + \binom{j}{2} y^2 + \cdots + \binom{j}{j} y^j.$$

Hence, we have

$$[T]_{B'}^{B} = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & & \\ 0 & \binom{1}{1} & \binom{2}{1} & & \\ 0 & 0 & \binom{2}{2} & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \end{pmatrix}$$

Question. Given V, and B, B' are ordered basis, then what is the relation between $[v]_B$ and $[v]_{B'}$?

Answer. Change of bases. ⊛

Corollary 1.10.3.

$$[id]_{B'}^{B}[v]_B = [v]_{B'}.$$

Corollary 1.10.4.

$$[id]_{B'}^{B}[id]_{B}^{B'} = [id]_{B'}^{B'}.$$

Corollary 1.10.5. Given any $A \in M_{m\times n}(F)$. There are invertible matrices $P \in M_m(F)$ and $Q \in M_n(F)$ s.t.

$$PAQ = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

where p is the row rank of A.

Proof. Suppose $A = [T]_{B}^{B'}$, and by Corollary 1.10.2, we know there exists b, b' s.t. $[T]_{b}^{b'}$ is the matrix we want, then we can let $Q = [id]_{b'}^{B'}$ and $P = [id]_{b}^{B}$, and we're done. ■

Lecture 8

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Lemma 1.10.1. Consider

$$V' \xrightarrow{\quad f \quad} V \xrightarrow{\quad T \quad} W \xrightarrow{\quad g \quad} W'$$

- Suppose g is injective, then $\ker(g \circ T) = \ker T$.
- Suppose f is surjective, then $\operatorname{Im}(T \circ f) = \operatorname{Im} T$.

Definition 1.10.1 (Matrix Equivalence). Let $A, B \in M_{m\times n}(\mathbb{F})$. We say that A and B are *equivalent* if there exist invertible matrices $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$ such that

$$B = PAQ.$$

Remark 1.10.1. Matrix equivalence means that one can obtain B from A by a sequence of invertible row and column operations.

Equivalently, if A represents a linear map $T : \mathbb{F}^n \to \mathbb{F}^m$, then B represents the same linear map with respect to different bases of the domain and codomain.

Theorem 1.10.4 (Row Rank Equals Column Rank). Let $A \in M_{m\times n}(\mathbb{F})$ be any matrix over a field $\mathbb{F}$. Then

$$\text{row rank}(A) = \text{column rank}(A).$$

Proof. We prove this using invertible row and column operations.

Step 1: Reduce A to canonical form.

It is a standard fact that any matrix $A \in M_{m\times n}(\mathbb{F})$ can be transformed into a block matrix of the form

$$C = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times n},$$

by multiplying on the left and right by invertible matrices $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$:

$$C = PAQ.$$

Here $r = \text{rank}(A)$ and I_r is the $r \times r$ identity matrix. This uses Gaussian elimination (invertible row operations) and invertible column operations.

Step 2: Row and column ranks of C.

- The first r rows of C are linearly independent, and the remaining $m-r$ rows are zero. So

$$\text{row rank}(C) = r.$$

- The first r columns of C are linearly independent, and the remaining $n-r$ columns are zero. So

$$\text{column rank}(C) = r.$$

Step 3: Equivalence preserves row and column ranks.

We have $C = PAQ$.

1. *Left multiplication by P (row operations):* Multiplying A on the left by invertible P corresponds to invertible row operations. Row operations do not change the linear independence of the rows. Hence

$$\text{row rank}(PA) = \text{row rank}(A).$$

2. *Right multiplication by Q (column operations):* Each row of AQ is obtained by multiplying the corresponding row of A by Q:

$$\text{row}_i(AQ) = \text{row}_i(A) \cdot Q.$$

Since Q is invertible, this is an invertible linear transformation on $\mathbb{F}^n$, which preserves linear independence of the rows. Therefore

$$\text{row rank}(AQ) = \text{row rank}(A).$$

Note 1.10.1.

$$\sum_{i\in I} \alpha_i \text{row}_i(A) \cdot Q = 0 \Leftrightarrow \sum_{i\in I} \alpha_i \text{row}_i(A) = 0$$

since Q is invertible.

Combining the above, for $C = PAQ$ we get

$$\text{row rank}(C) = \text{row rank}(A) = r,$$

and similarly

$$\text{column rank}(C) = \text{column rank}(A) = r.$$

Step 4: Conclusion.

From Step 2 and Step 3, we have

$$\text{row rank}(A) = \text{row rank}(C) = r = \text{column rank}(C) = \text{column rank}(A).$$

Hence, the row rank of A equals the column rank of A. ∎

Theorem 1.10.5. Two matrices A and B of same sizes are equivalent if and only if $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Proof. Suppose A, B equivalent, then $A = PBQ$ for some invertible P, Q. By Lemma 1.10.1, we know $\operatorname{Im}(BQ) = \operatorname{Im} B$, which gives $\operatorname{rank}(BQ) = \operatorname{rank} B$. Also, since $\ker(P(BQ)) = \ker(BQ)$, so $\operatorname{rank}(P(BQ)) = \operatorname{rank}(BQ)$ by rank and nullity theorem. Hence, we have $\operatorname{rank} A = \operatorname{rank}(PBQ) = \operatorname{rank}(BQ) = \operatorname{rank} B$.

Now if $\operatorname{rank} A = \operatorname{rank} B$, then we know

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = P'BQ',$$

so $A = P^{-1}P'BQ'Q^{-1}$, which means A, B are equivalent. ∎

Theorem 1.10.6. Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces over a field $\mathbb{F}$. Let $B = \{v_1, \dots, v_n\}$ be a basis for V and $C = \{w_1, \dots, w_m\}$ be a basis for W. Let

$$A = [T]_{B,C} \in M_{m\times n}(\mathbb{F})$$

be the matrix of T with respect to the bases B and C. Then

$$\operatorname{rank}(A) = \dim(\operatorname{Im}(T)).$$

Proof. **Step 1: Express the image of T in terms of the basis.**

The matrix A is given by

$$A = \big[[T(v_1)]_C \; [T(v_2)]_C \; \dots \; [T(v_n)]_C\big],$$

where $[T(v_j)]_C$ denotes the coordinate vector of $T(v_j)$ with respect to C.

Since B is a basis for V, any vector $v \in V$ can be written as

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

for some scalars $c_1, \dots, c_n \in \mathbb{F}$. By linearity of T,

$$T(v) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

Thus, every vector in $\operatorname{Im}(T)$ is a linear combination of

$$\{T(v_1), T(v_2), \dots, T(v_n)\},$$

and hence

$$\operatorname{Im}(T) = \operatorname{span}\{T(v_1), T(v_2), \dots, T(v_n)\}.$$

Step 2: Relate $\operatorname{Im}(T)$ to the column space of A.

The column space of A, denoted $\operatorname{Col}(A)$, is

$$\operatorname{Col}(A) = \operatorname{span}\{[T(v_1)]_C, [T(v_2)]_C, \dots, [T(v_n)]_C\}.$$

The coordinate mapping $[\cdot]_C : W \to \mathbb{F}^m$ is a linear isomorphism. In particular, it preserves linear independence and spanning sets. Therefore, the map

$$T(v_j) \longmapsto [T(v_j)]_C$$

establishes a linear isomorphism between $\operatorname{Im}(T)$ and $\operatorname{Col}(A)$:

$$\operatorname{Im}(T) \cong \operatorname{Col}(A).$$

Step 3: Compare dimensions.

Since isomorphic vector spaces have the same dimension,

$$\dim(\operatorname{Im}(T)) = \dim(\operatorname{Col}(A)).$$

By definition, the rank of A is the dimension of its column space:

$$\operatorname{rank}(A) = \dim(\operatorname{Col}(A)).$$

Combining these equalities, we obtain

$$\operatorname{rank}(A) = \dim(\operatorname{Im}(T)),$$

as desired.

This shows that the rank of a matrix representing a linear transformation is independent of the choice of bases B and C, since $\dim(\operatorname{Im}(T))$ depends only on T itself. ■

Appendix