

Linear Algebra I

Kon Yi

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Abstract

The lecture note of Linear Algebra I by professor 余正道.

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Chapter 1

Vector Space

Lecture 1

1.1 Introduction to vector and vector space

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In high school, our vectors are in \mathbb{R}^2 and \mathbb{R}^3 , and we have define the addition and scalar multiplication of vectors.

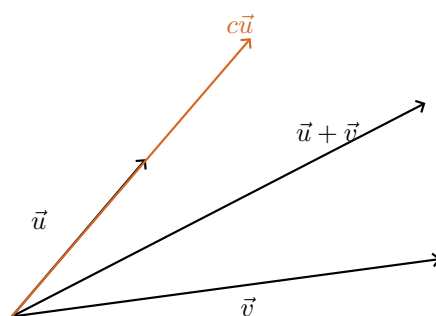


Figure 1.1: Vectors in \mathbb{R}^2

Example. $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}\}$

With this type of space, we can define addition and multiplication as

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = \{a_1 + b_1, a_2 + b_2, \dots, a_n + b_n\}$$
$$\alpha \cdot (a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$$

Also, if we define a space:

Example. $V = \{\text{function } f : (a, b) \rightarrow \mathbb{R}\}$, where (a, b) is an open interval.

then this can also be a vector space after defining addition and multiplication.

Note. In a vector space, we have to make sure the existence of 0-element, which means $0(x) = 0$.

Now we give a more abstract example:

Example. Suppose S is any set, then define $V = \{\text{all functions from } S \text{ to } \mathbb{R}\}$

If we define $(f + g)(s) = f(s) + g(s)$ and $(\alpha \cdot f)(s) = \alpha \cdot f(s)$, and $0(s) = 0$, then this is also a vector space.

Put some linear conditions

Example. In \mathbb{R}^n , fix $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, if we define

$$W = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \dots + a_nx_n = 0\},$$

then this is also a vector space.

However, if we have

$$W' = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = 1\},$$

then this is not a vector space because it is not close.

Example. In $V = \{(a, b) \rightarrow \mathbb{R}\}$ or $W_1 = \{\text{polynomial defined on } (a, b)\}$, these are both vector space.

Remark. In the later course, we will learn that W_1 is a subspace of V .

Example. If we furtherly defined $W_1^{(k)} = \{\text{polynomial degree } \leq k\}$, then this is also a vector space.

Remark. $W_1^{(k)}$ is actually isomorphic to \mathbb{R}^{k+1} since

$$a_0 + a_1x + a_2x^2 + \dots + a_kx^k \leftrightarrow (a_0, a_1, a_2, \dots, a_k).$$

Example. $W_2 = \{\text{continuous function on } (a, b)\}$ and $W_3 = \{\text{differentiable functions}\}$ are also both vector spaces.

Example. $W_4 = \left\{\frac{d^2f}{dx^2} = 0\right\}$ and $W_5 = \left\{\frac{d^2f}{dx^2} = -f\right\}$ are both vector spaces.

Proof.

$$\begin{aligned} W_4 &= \{a_0 + a_1x\} \\ W_5 &= \{a_1 \cos x + a_2 \sin x\} \end{aligned}$$

⊛

1.2 Formal definition of vector spaces

1.2.1 Vector Spaces Over \mathbb{R}

Definition 1.2.1. Suppose V is a non-empty set equipped with

- addition: $V \times V \rightarrow V$, that is, given $u, v \in V$, defining $u + v \in V$
- scalare multiplication: $\mathbb{R} \times V \rightarrow V$, that is, given $\alpha \in \mathbb{R}$ and $v \in V$, we need to have $\alpha v \in V$

Also, we need some good properties or conditions

- For addition,
 - $u + v = v + u$
 - $(u + v) + w = u + (v + w)$
- There exists $0 \in V$ such that $u + 0 = u = 0 + u$
- Given $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0 = (-v) + v$

- For scalar multiplication,
 - $1 \cdot v = v$ for all $v \in V$
 - $(\alpha\beta)v = \alpha \cdot (\beta v)$ for all $\alpha, \beta \in \mathbb{R}$ and $v \in V$.
- For addition and multiplication,
 - $\alpha(u + v) = \alpha u + \alpha v$
 - $(\alpha + \beta)u = \alpha u + \beta u$

Lecture 2

1.3 Vector Space over general field

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Now we introduce the concept of field.

Definition 1.3.1 (Field). A set F with $+$ and \cdot is called a **field** if

- $\alpha + \beta = \beta + \alpha$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- There exists $0 \in F$ such that $\alpha + 0 = 0 + \alpha = \alpha$.
- For $\alpha \in F$, there exists $-\alpha$ such that $\alpha + (-\alpha) = 0$.
- $\alpha\beta = \beta\alpha$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
- $\exists 1$ such that $1 \neq 0$ and $1 \cdot \alpha = \alpha$.
- For $\alpha \neq 0$, $\exists \alpha^{-1} \in F$ such that $\alpha\alpha^{-1} = 1$.
- $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Example. $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are all fields but \mathbb{Z} is not.

Example. $\{0, 1\}$ is also a field.

Now we know the concept of field, so we can make a vector space over a field.

Theorem 1.3.1 (Cancellation law). Suppose $v_1, v_2, w \in V$, a vector space, then if $v_1 + w = v_2 + w$, then $v_1 = v_2$.

Proof.

$$v_1 = v_1 + (w + (-w)) = (v_1 + w) + (-w) = (v_2 + w) + (-w) = v_2 + (w + (-w)) = v_2.$$

■

Theorem 1.3.2. The zero vector 0 is unique.

Proof. Suppose we have $0, 0'$ both zero vector, then for some $0 = 0 + 0' = 0'$. ■

Theorem 1.3.3. For any $v \in V$, $0 \cdot u = 0$.

Proof. $0 \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u$, so $0 = 0 \cdot u$ by [cancellation law](#). ■

Theorem 1.3.4. $(-1) \cdot u = -u$.

Theorem 1.3.5. Given any $u \in V$ is unique, $-u$ is unique.

1.4 Subspaces

Definition 1.4.1 (subspace). Let V be a vector space. A non-empty subset $W \subseteq V$ is called a subspace of V if W is itself a vector space under $+$ and \cdot on V .

Example. $M_n(F) = \{n \times n \text{ matrix with entries in } F\}$ is a vector space, and

$$U_n(F) = \left\{ \begin{pmatrix} a_{11} & & & \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \right\}$$

is a subspace of $M_n(F)$.

Proposition 1.4.1. Suppose V is a vector space, and $W \subseteq V$ is non-empty, then

W is a subspace \Leftrightarrow For $u, v \in W, \alpha \in F$, we have $u + v \in W$ and $\alpha \cdot u \in W$.

proof of \Rightarrow . Clear. ■

proof of \Leftarrow . First, we would want to check $0 \in W$, and we can pick any $u \in W$, and pick $\alpha = -1$, so we know $-u \in W$, and thus $0 = u + (-u) \in W$. ■

Corollary 1.4.1. If we want to check W is a subspace, we just need to check for $u, v \in W, \alpha \in F$, $u + \alpha v \in W$ or not.

1.5 Linear Combination

Definition 1.5.1 (Linear combination). Given $v_1, v_2, \dots, v_n \in V$, a linear combination of them is a vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Proposition 1.5.1. Given $v_1, v_2, \dots, v_n \in V$,

1. $W = \{\text{all linear combinations of } v_1, \dots, v_n\}$ is a subspace.
2. This subspace is the smallest subspace containing v_1, \dots, v_n . That is, if $W' \subseteq V$ is a subspace containing v_1, \dots, v_n , then $W \subseteq W'$.

Notation. $\text{span}\{v_1, v_2, \dots, v_n\} = \{\text{all linear combinations of } v_1, v_2, \dots, v_n\}$

1.6 Linearly independent

Definition. Now we talk about the linearly dependence and linearly independence.

Definition 1.6.1 (Linearly dependent). v_1, v_2, \dots, v_n are linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

for some $\alpha_1, \alpha_2, \dots, \alpha_n$ not all zeros.

Definition 1.6.2 (Linearly independent). v_1, v_2, \dots, v_n are called linearly independent if they are not linearly dependent.

Corollary 1.6.1. Say $\alpha_i \neq 0$, then $v_i \in \text{span}\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_k\}$ suppose the corresponding α_i of $\hat{v}_1, \dots, \hat{v}_k$ are not zeros.

Corollary 1.6.2. Linearly independent means if $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Corollary 1.6.3. Linearly independent means if $\sum \alpha_i v_i = \sum \beta_i v_i$, then $\alpha_i = \beta_i$ for all i .

Example.

- $v \in V$ is linearly independent iff $v \neq 0$.
- $v, w \in V$ are linearly independent iff v is not a scalar of w and w is not a scalar of v .

Lemma 1.6.1. v_1, \dots, v_n are linearly independent iff $v_i \notin \text{span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$.

1.7 Basis

Definition. We now talking about basis

Definition 1.7.1 (Basis). $B = \{v_1, v_2, \dots, v_n\}$ is called a basis of V if B spans V and B is linearly independent.

Definition 1.7.2 (Dimension). In this case, n is called the dimension of V , and denoted by $\dim V$.

Notation. $\text{span}\{v_1, v_2, \dots, v_n\} = \langle v_1, v_2, \dots, v_n \rangle$

Notation. $\text{span}(S) = \langle S \rangle$

Theorem 1.7.1. For any $v \in V$, it has a unique expression $v = \sum_{i=1}^n \alpha_i v_i$.

Lecture 3

As previously seen. A basis of a vector space V is a set $\{v_1, v_2, \dots, v_n\}$ that is linearly independent and simultaneously spans V . That is, suppose we have $\sum a_i v_i = 0$ for some scalars a_i , then $a_i = 0$ for all i . Also, we call the number n , the dimension of V .

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Example. Suppose we have $V = F^n = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \mid \alpha_i \in F\}$, then we have a **standard basis**, which is

$$\begin{aligned} e_1 &= (1, 0, \dots, 0) \\ e_2 &= (0, 1, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, \dots, 1) \end{aligned}$$

since $\{e_i\}_{i=1}^n$ is linearly independent and for every $\vec{a} = (a_1, \dots, a_n)$, we know

$$\vec{a} = \sum_{i=1}^n a_i e_i.$$

Example. Suppose

$$V = M_{n \times n}(F) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \ddots & & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \dots & & \alpha_{nn} \end{pmatrix} \right\},$$

then we know

$$\{e_{ij}\}_{1 \leq i, j \leq n} = \begin{pmatrix} 0 & 0 & & \\ 0 & & & \\ & & 1 & \\ 0 & & & 0 \\ 0 & & & 0 \end{pmatrix},$$

where the 1 is in the i -th row and j -th column.

Theorem 1.7.2. Suppose V is a vector space, and $V = \langle v_1, v_2, \dots, v_n \rangle$ and $\{w_1, w_2, \dots, w_m\}$ is linearly independent, then $m \leq n$. Furthermore, one can make

$$\langle w_1, w_2, \dots, w_m, v_{m+1}, \dots, v_n \rangle = V$$

after rearrangement of v_1, \dots, v_n .

Proof. We can do induction on m . It is trivial that $m = 0$ is true. Suppose the statement holds for a fixed m with $m \leq n$. Let w_1, w_2, \dots, w_{m+1} be linearly independent. In particular, w_1, w_2, \dots, w_m is linearly independent.

Claim. $m + 1 \leq n$.

Proof. Otherwise, if $m + 1 > n$, then since $m \leq n$, so $m = n$. Hence, by induction hypothesis, we know $\langle w_1, w_2, \dots, w_m \rangle = V$. However, by [Lemma 1.7.1](#) and the note following it, we know

$$\{w_1, w_2, \dots, w_m\} \cup \{w_{m+1}\}$$

can not be linearly independent since $w_{m+1} \in V = \langle w_1, \dots, w_m \rangle$. ⊗

Now we know $m + 1 \leq n$. By induction hypothesis, we know

$$\langle w_1, w_2, \dots, w_m, v_{m+1}, \dots, v_n \rangle = V$$

Claim. One of v_{m+1}, \dots, v_n can be replaced by w_{m+1} .

Proof. Since

$$w_{m+1} = \sum_{i=1}^n \alpha_i w_i + \sum_{j=m+1}^n \beta_j w_j.$$

Trivially, one of $\beta_j \neq 0$, say $\beta_{m+1} \neq 0$. Check

$$\langle w_1, \dots, w_m, w_{m+1}, v_{m+2}, \dots, v_n \rangle = V.$$

⊛

■

Corollary 1.7.1. If $\{v_1, v_2, \dots, v_n\}$ and $\{w_1, w_2, \dots, w_m\}$ are bases of V , then $n = m$.

Remark. Corollary 1.7.1 tells us $\dim V$ is well-defined, which means the size of the bases of a vector space is unique.

Corollary 1.7.2. Suppose $\dim V = n$, then if $\langle v_1, v_2, \dots, v_m \rangle = V$, then $m \geq n$. If $\{w_1, w_2, \dots, w_m\}$ is linearly independent, then $m \leq n$. Also, any $\{v_i\}_{i=1}^m$ with $m > n$ is linearly independent.

Lemma 1.7.1. Suppose v_1, v_2, \dots, v_n is linearly independent. If $w \notin \langle v_1, v_2, \dots, v_n \rangle$, then

$$\{v_1, v_2, \dots, v_n, w\}$$

is linearly independent.

Proof. Suppose $\sum_{i=1}^n \alpha_i v_i + \alpha_{n+1} w = 0$, then if $\alpha_{n+1} = 0$, we know $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ since $\{v_i\}_{i=1}^n$ is linearly independent. If $\alpha_{n+1} \neq 0$, then $w = \frac{1}{\alpha_{n+1}} \sum_{i=1}^n \alpha_i v_i \in \langle v_1, v_2, \dots, v_n \rangle$, which is a contradiction. ■

Note. The reverse of Lemma 1.7.1 is still correct and is trivial. That is, if $w \notin \{v_1, \dots, v_n\}$ and $\{v_1, v_2, \dots, v_n, w\}$ is linearly independent, then $\{v_1, \dots, v_n\}$ is linearly independent.

Corollary 1.7.3. If $W \subseteq V$ is a subspace of V , then $\dim W \leq \dim V$.

Proof. If $\dim V = n$, and $\{w_i\}_{i=1}^m$ is a basis of W , then this basis is linearly independent in V , which means $m \leq n$ by Theorem 1.7.2. ■

Corollary 1.7.4. If v_1, v_2, \dots, v_m is linearly independent, then $\{v_1, v_2, \dots, v_m\}$ forms a basis for some v_{m+1}, \dots, v_n .

Theorem 1.7.3 (Dual version). If $\langle v_1, v_2, \dots, v_n \rangle = V$, then $\{v_1, v_2, \dots, v_m\}$ forms a basis after rearrangement.

Remark. Most of the time, we consider finite-dimensional vector spaces.

Remark (Examples of ∞ -dim vector space).

•

$$V = \{\text{all polynomials over } F\} = F[x] = \{a_0 + a_1x + \dots + a_nx^n \text{ for some } n \text{ where } a_i \in F\}.$$

•

$$W = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\}.$$

Notice that

$$W' = \{\text{convergent sequence}\} \subseteq W.$$

and

$$W'' = l^2 = \left\{ (a_i) \mid \sum_{i=0}^{\infty} a_i^2 \text{ finite} \right\} \subseteq W'$$

Remark. We define $\dim \{0\} = 0$, which is the only vector space with dimension 0, and we define $\langle \emptyset \rangle = \{0\}$, which means \emptyset is the basis of $\{0\}$.

Note. We call a subspace $W \subsetneq V$ is proper.

1.8 More on subspaces

Theorem 1.8.1. If W_1 and W_2 are subspace of V , then $W_1 \cap W_2$ is a subspace.

Theorem 1.8.2. If W_1, W_2 are subspaces of V , then $W_1 + W_2$ is still a subspace of V .

Remark. If W_1, W_2 are subspaces of V , then $W_1 \cup W_2$ may not be a subspace. (See HW1).

Remark. In fact, $W_1 \cap W_2$ is the largest subspaces contained in W_1 and W_2 .

Remark. In fact, $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2 .

Corollary 1.8.1. Suppose S is the index set, and for all $i \in S$, W_i is a subspace of V , then

$$\bigcap_{i \in S} W_i = \{v \in V \mid v \in W_i \forall i\}$$

is also a subspace of V .

Corollary 1.8.2. Suppose S is the index set, and for all $i \in S$, W_i is a subspace of V , then

$$\sum_{i \in S} W_i = \{w_{i_1} + w_{i_2} + \dots + w_{i_n} \text{ for some } i_j \in S\}$$

is also a subspace of V .

Proposition 1.8.1 (Dimension theorem). Suppose $W_1, W_2 \subseteq V$ are subspaces of V , then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Appendix