Introduction to Analysis HW11

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Problem 0.0.1 (20 pts **Exercise 5.2.6**). Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$.

(a) Show that if f_n converges uniformly to f, then f_n also converges to f in the L^2 metric.

(b) Give an example where f_n converges to f in the L^2 metric, but does *not* converge to f uniformly.
(Hint: take $f = 0$. Try to make the functions f_n large in sup norm.)

(c) Give an example where f_n converges to f in the L^2 metric, but does *not* converge to f pointwise.
(Hint: take $f = 0$. Try to make the functions f_n large at one point.)

(d) Give an example where f_n converges to f pointwise, but does *not* converge to f in the L^2 metric.
(Hint: take $f = 0$. Try to make the functions f_n large in L^2 norm.)

proof of (a). If $f_n \to f$ uniformly, then for all $\varepsilon > 0$, there exists $N > 0$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0, 1]$. Thus,

$$\begin{aligned} \|f_n - f\|_2 &= \sqrt{\langle f_n - f, f_n - f\rangle} = \sqrt{\int_0^1 (f_n - f)(x)\overline{f_n - f}(x)\,\mathrm{d}x} \\ &= \sqrt{\int_0^1 |(f_n - f)(x)|^2\,\mathrm{d}x} = \sqrt{\int_0^1 |f_n(x) - f(x)|^2\,\mathrm{d}x} < \sqrt{\int_0^1 \varepsilon^2\,\mathrm{d}x} = \varepsilon. \end{aligned}$$

This shows $\lim_{n\to\infty}\|f_n - f\|_2 = 0$, i.e. f_n also converges to f in the L^2 metric. ■

proof of (b). Let $f(x) = 0$ and

$$f_n(x) = \begin{cases} 1 - nx, & \text{if } 0 \le x \le \dfrac{1}{n}; \\ 0, & \text{if } \dfrac{1}{n} < x \le 1. \end{cases},$$

then we have

$$\|f_n - f\|_2 = \sqrt{\int_0^1 |f_n(x)|^2\,\mathrm{d}x} = \sqrt{\int_0^{\frac{1}{n}} (1 - nx)^2\,\mathrm{d}x} = \sqrt{\frac{1}{3n}} \to 0 \text{ as } n \to \infty.$$

Hence, $f_n \to f$ in the L^2 metric. However, suppose $f_n \to f$ uniformly, then for $\varepsilon = \frac{1}{3}$, there exists $N > 0$ s.t. $n \ge N$ implies

$$|f_n(x) - f(x)| = |f_n(x)| < \varepsilon = \frac{1}{3} \quad \forall x \in [0, 1]$$

but $f_n(0) = 1$ for all n and $1 > \frac{1}{3}$, so this is impossible. Hence, f_n does not converge to f uniformly. ■

proof of (c). We use the same f_n and f as in (b), then we have shown that $f_n \to f$ in the L^2 metric, and now if $f_n \to f$ pointwise, then for $x = 0$ and $\varepsilon = \frac{1}{3}$, there exists $N > 0$ s.t. $n \ge N$ implies

$$1 = |f_n(0) - f(0)| < \varepsilon = \frac{1}{3},$$

which is impossible, so f_n does not converge to f pointwise. ■

proof of (d). Let $f(x) = 0$ and

$$f_n(x) = \begin{cases} n^3 x, & \text{if } 0 \le x \le \dfrac{1}{n}; \\ -n^3\left(x - \dfrac{2}{n}\right), & \text{if } \dfrac{1}{n} < x \le \dfrac{2}{n}; \\ 0, & \text{if } \dfrac{2}{n} < x \le 1. \end{cases}$$

Then, for all $x \in (0,1]$ and for all $\varepsilon > 0$, we know there exists $N > 0$ s.t. $\frac{2}{N} < x$, and thus for all $n \ge N$, we have $\frac{2}{n} \le \frac{2}{N} < x$ and thus $f_n(x) = 0$, which gives $|f_n(x) - f(x)| < \varepsilon$. As for $x = 0$, since $f_n(0) = 0$ for all n, so $|f_n(0) - f(0)| < \varepsilon$ for all $\varepsilon > 0$ and all $n \ge 1$. Thus, we can conclude that $f_n \to f$ pointwise. Now since

$$\begin{aligned} \|f_n - f\|_2 &= \sqrt{\int_0^1 |f_n(x)|^2 \,\mathrm{d}x} = \sqrt{\int_0^{\frac{1}{n}} n^6 x^2 \,\mathrm{d}x + \int_{\frac{1}{n}}^{\frac{2}{n}} n^6 \left(x - \frac{2}{n}\right)^2 \mathrm{d}x} \\ &\ge \sqrt{\int_0^{\frac{1}{n}} n^6 x^2 \,\mathrm{d}x} = \sqrt{\frac{n^3}{3}} \to \infty \quad \text{as } n \to \infty, \end{aligned}$$

so we know f_n does not converge to f under L^2 metric. ■

Problem 0.0.2 (20 pts). Let $\{\phi_N\} : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous, periodic functions on $\mathbb{R}$ (with period 1) which satisfy

$$\int_0^1 \phi_N(t)\,dt = 1 \qquad \text{and} \qquad \int_0^1 |\phi_N(t)|\,dt \le M < \infty$$

for all $N \in \mathbb{N}$, and

$$\lim_{N\to\infty} \int_\delta^{1-\delta} |\phi_N(t)|\,dt = 0$$

for each $0 < \delta < 1$.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period 1. Prove that

$$\lim_{N\to\infty} \int_0^1 f(x-t)\,\phi_N(t)\,dt = f(x)$$

uniformly for $x \in \mathbb{R}$.

Problem 0.0.3 (15pts **Exercise 5.2.3.**). If $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ is a non-zero function, show that

$$0 < \|f\|_2 \;\le\; \|f\|_\infty.$$

Conversely, if $0 < A \le B$ are real numbers, show that there exists a non-zero function $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ such that

$$\|f\|_2 = A \qquad \text{and} \qquad \|f\|_\infty = B.$$

(Hint: let g be a non-constant non-negative real-valued function in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, and consider functions of the form $f = (c + dg)^{1/2}$ for some constant real numbers $c, d > 0$.)

Proof. Since f is a non-zero continuous function, so we know

$$\|f\|_2 = \sqrt{\int_0^1 |f(x)|^2 \,\mathrm{d}x} > 0,$$

and since

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|,$$

so we know

$$\|f\|_2 = \sqrt{\int_0^1 |f(x)|^2 \, \mathrm{d}x} \le \sqrt{\int_0^1 \|f\|_\infty^2 \, \mathrm{d}x} = \sqrt{\|f\|_\infty^2} = |f|_\infty.$$

Hence, we can conclude $0 < \|f\|_2 \le |f|_\infty$. Now we prove that for any $0 < A \le B$ where A, B are real numbers, there exists $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ s.t. $\|f\|_2 = A$ and $\|f\|_\infty = B$. Suppose $\delta = B^2 - A^2 \ge 0$, we pick positive (M, μ) pairs s.t.

$$M > \max\left\{2\mu, \left(\frac{\delta}{A^2} + 1\right)\mu\right\}$$

so that if we let

$$\begin{cases} c = B^2 - \dfrac{M}{M-\mu}\delta \\ d = \dfrac{\delta}{M-\mu} \end{cases}$$

then we can show that $c > 0$ and $d \ge 0$. For d, since $\delta \ge 0$ and $M > \mu$, so $d \ge 0$. As for c, if $\delta = 0$, then $c = B^2 > 0$, otherwise we have

$$\begin{aligned} c = B^2 - \frac{M}{M-\mu}\delta &= B^2 - \left(1 + \frac{\mu}{M-\mu}\right)(B^2 - A^2) = B^2\left(\frac{-\mu}{M-\mu}\right) + \left(1 + \frac{\mu}{M-\mu}\right)A^2 \\ &= A^2 + \left(A^2 - B^2\right)\left(\frac{\mu}{M-\mu}\right) = A^2 - \frac{\delta\mu}{M-\mu} > A^2 - \frac{\delta\mu}{\left(\frac{\delta}{A^2}+1\right)\mu - \mu} = A^2 - \frac{\delta\mu}{\frac{\delta}{A^2}\mu} = A^2 - A^2 = 0. \end{aligned}$$

Now if we let

$$g(x) = \begin{cases} \dfrac{M^2}{\mu}x, & \text{if } 0 \le x \le \dfrac{\mu}{M}; \\ -\dfrac{M^2}{\mu}\left(x - \dfrac{2\mu}{M}\right), & \text{if } \dfrac{\mu}{M} < x \le \dfrac{2\mu}{M}; \\ 0, & \text{if } \dfrac{2\mu}{M} < x \le 1. \end{cases}$$

Note that since $M > 2\mu$, so this function is well-defined. Also, we know

$$\sup_{x\in[0,1]} g(x) = M \qquad \int_0^1 g(x)\,\mathrm{d}x = \mu$$

since if we observe $g(x)$ on $[0,1]$, it is a triangle above the x-axis with base on the x-axis and of area μ, and the height of this triangle is M, while the parts not being the triangle have $y = 0$. Now if we let

$$f(x) = (c + dg(x))^{\frac{1}{2}},$$

then we know

$$\begin{aligned} \|f\|_2 &= \sqrt{\int_0^1 |c + dg(x)|\,\mathrm{d}x} = \sqrt{c + d\int_0^1 g(x)\,\mathrm{d}x} = \sqrt{c + d\mu} \\ &= \sqrt{B^2 - \frac{M}{M-\mu}\delta + \frac{\delta}{M-\mu}\mu} = \sqrt{B^2 - \frac{\delta}{M-\mu}(M-\mu)} = \sqrt{B^2 - \delta} = \sqrt{A^2} = A. \end{aligned}$$

since $c, d, g(x)$ are all real and $c > 0$ and $d, g(x) \ge 0$ on $[0,1]$. Besides,

$$\|f\|_\infty = \sup_{x\in[0,1]} |(c + dg(x))|^{\frac{1}{2}} = (c + dM)^{\frac{1}{2}} = \left(B^2 - \frac{M}{M-\mu}\delta + \frac{\delta}{M-\mu}M\right)^{\frac{1}{2}} = \left(B^2\right)^{\frac{1}{2}} = B,$$

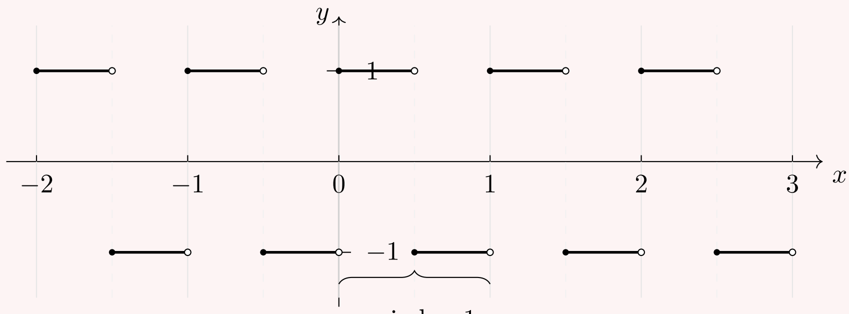
and we're done. ∎

Problem 0.0.4 (15 pts). A *square wave function* is a $\mathbb{Z}$-periodic function defined by

$$f(x) = \begin{cases} 1, & x \in [k,\, k+\frac{1}{2}), \\ -1, & x \in [k+\frac{1}{2},\, k+1), \end{cases} \qquad k \in \mathbb{Z}.$$

Thus f alternates between 1 and -1 on each half-interval, repeating the same pattern on every interval of length 1.

Find a sequence of continuous periodic functions which converges in L^2 to the square wave function.



Problem 0.0.5 (15 pts).

(a) Evaluate

$$S_n(\theta) = \sum_{k=1}^{n} \sin(k\theta).$$

(b) Show that

$$|S_n(\theta)| \leq \pi\varepsilon^{-1} \qquad \text{on } [\varepsilon,\, 2\pi - \varepsilon] \text{ for all } n \geq 1.$$

Problem 0.0.6 (15 pts). Let $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{R})$. We define their *periodic convolution* $f * g : \mathbb{R} \to \mathbb{R}$ by

$$(f * g)(x) := \int_0^1 f(y)\, g(x-y)\, dy.$$

Prove that $(f * g)$ is smooth whenever f is smooth. (Remark: A function is called smooth if it has derivatives of all orders.)

Proof. Fix $x \in \mathbb{R}$, then

$$\begin{aligned} (f * g)(x) &= \int_0^1 f(y)g(x-y)\,\mathrm{d}y = \int_x^{x-1} f(x-t)g(t)\,(-\mathrm{d}t) = \int_{x-1}^{x} f(x-t)g(t)\,\mathrm{d}t \\ &= \int_0^1 f(x-t)g(t)\,\mathrm{d}t. \end{aligned}$$

Now consider

$$\begin{aligned} \lim_{h\to 0} \frac{(f*g)(x+h) - (f*g)(x)}{h} &= \lim_{h\to 0} \int_0^1 \frac{f(x+h-t)g(t) - f(x-t)g(t)}{h}\,\mathrm{d}t \\ &= \lim_{h\to 0} \int_0^1 \left(\frac{f(x+h-t) - f(x-t)}{h} \right) g(t)\,\mathrm{d}t. \end{aligned}$$

Suppose $\phi_h : \mathbb{R} \to \mathbb{R}$ where

$$\phi_h(t) = \left(\frac{f(x+h-t) - f(x-t)}{h} \right) g(t),$$

then we claim that $\left\{ \phi_{\frac{1}{n}}(t) \right\}_{n=1}^{\infty} \to f'(x-t)g(t)$ uniformly, and if we can prove this, then we know

$$\begin{aligned}(f*g)(x)' &= \lim_{h\to 0} \frac{(f*g)(x+h) - (f*g)(x)}{h} = \lim_{h\to 0} \int_0^1 \frac{f(x+h-t)g(t) - f(x-t)g(t)}{h} \, \mathrm{d}t \\ &= \lim_{h\to 0} \int_0^1 \phi_h(t) \, \mathrm{d}t = \lim_{n\to\infty} \int_0^1 \phi_{\frac{1}{n}}(t) \, \mathrm{d}t = \int_0^1 \lim_{n\to\infty} \phi_{\frac{1}{n}}(t) \, \mathrm{d}t = \int_0^1 f'(x-t)g(t) \, \mathrm{d}t \\ &= \left(f' * g\right)(x).\end{aligned}$$

Now if we have this, then since the above argument is true for all $x \in \mathbb{R}$, so $f*g$ is differentiable at every $x \in \mathbb{R}$, and since f' is also smooth, so we can repeat this argument to show that $f*g$ is smooth. Now we prove the claim. For any $\varepsilon > 0$, we want to show that there exists $N > 0$ s.t. $n \geq N$ implies

$$\left| \left(\frac{f\left(x + \frac{1}{n} - t\right) - f(x-t)}{\frac{1}{n}} - f'(x-t) \right) g(t) \right| = \left| \phi_{\frac{1}{n}}(t) - f'(x-t)g(t) \right| < \varepsilon \quad \forall t \in [0,1].$$

However, since

$$f'(x-t) = \lim_{h\to 0} \frac{f(x-t+h) - f(x-t)}{h},$$

so there exists $\delta > 0$ s.t. $|h| < \delta$ implies

$$\left| \frac{f(x-t+h) - f(x-t)}{h} - f'(x-t) \right| < \frac{\varepsilon}{M},$$

where $M = \sup_{t\in[0,1]} |g(t)|$ (here we suppose $M > 0$). Now since there exists $N_1 > 0$ s.t. $\frac{1}{N_1} < \delta$, so for all $n \geq N_1$ we have $\frac{1}{n} \leq \frac{1}{N_1} < \delta$ and we have

$$\left| \left(\frac{f\left(x + \frac{1}{n} - t\right) - f(x-t)}{\frac{1}{n}} - f'(x-t) \right) g(t) \right| < \left| \frac{\varepsilon}{M} g(t) \right| \leq \left| \frac{\varepsilon}{M} \cdot M \right| = \varepsilon,$$

and we're done. Now if $M = 0$, then $g(t) = 0$ on $[0,1]$ and thus we also have

$$\left| \left(\frac{f\left(x + \frac{1}{n} - t\right) - f(x-t)}{\frac{1}{n}} - f'(x-t) \right) g(t) \right| = 0 < \varepsilon.$$

Hence, in either case we can show the claim is true. ■