

Linear Algebra I HW10

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Problem 0.0.1. Find a projection E which projects \mathbb{R}^2 onto the subspace spanned by $(1, -1)$ along the subspace spanned by $(1, 2)$.

Proof. Since $(1, -1) \in \text{span}\{(1, -1)\}$, so $E(1, -1) = (1, -1)$, and $E(1, 2) = (0, 0)$ since E is a projection along $\text{span}\{(1, 2)\}$. Hence, suppose $[E]_b = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where b is the standard basis of \mathbb{R}^2 , then

$$\begin{cases} a - b = 1 \\ c - d = -1 \end{cases}, \quad \begin{cases} c - d = -1 \\ c + 2d = 0 \end{cases},$$

so $(a, b) = (\frac{2}{3}, -\frac{1}{3})$ and $(c, d) = (-\frac{2}{3}, \frac{1}{3})$. Thus,

$$[E]_b = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \Rightarrow E(x, y) = \left(\frac{2}{3}x - \frac{1}{3}y, -\frac{2}{3}x + \frac{1}{3}y \right).$$

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Problem 0.0.2. Let V be a real vector space and E an idempotent linear operator on V , i.e., a projection. Prove that $(I + E)$ is invertible. Find $(I + E)^{-1}$.

Proof. Note that

$$\left(I - \frac{1}{2}E \right) (I + E) = I + E - \frac{1}{2}E - \frac{1}{2}E^2 = I + E - \frac{1}{2}E - \frac{1}{2}E = I + E,$$

so $I + E$ is invertible and $(I + E)^{-1} = I - \frac{1}{2}E$.

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Problem 0.0.3. Let F be a subfield of the complex numbers (or, a field of characteristic zero). Let V be a finite-dimensional vector space over F . Suppose that E_1, \dots, E_k are projections of V and that $E_1 + \dots + E_k = I$. Prove that $E_i E_j = 0$ for $i \neq j$ (Hint: Use the trace function and ask yourself what the trace of a projection is.)

Proof. We first show that if E is a projection, then $\text{Tr}(E) = \text{rank}(E)$. If E is a projection, then $E^2 = E$, which means $E(E - I) = 0$, so if $m_E(x)$ is the minimal polynomial of E , then $m_E(x) \mid x(x - 1)$, which means E must be diagonalizable since $m_E(x)$ has no repeated roots and E 's eigenvalues are 0, 1. Hence, there exists a basis b of V s.t.

$$[E]_b = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix},$$

and since $\text{rank}(E)$ and $\text{Tr}(E)$ does not depend on the choice of matrix representation, so

$$\text{rank}(E) = \text{rank}([E]_b) = \text{Tr}([E]_b) = \text{Tr}(E).$$

Now since $E_1 + \dots + E_k = I$, so for all $i = 1, 2, \dots, k$ we have

$$E_i = E_i(E_1 + \dots + E_k) = E_i^2 + \sum_{j \neq i} E_i E_j = E_i + \sum_{j \neq i} E_i E_j,$$

so $0 = \sum_{j \neq i} E_i E_j$, which gives

$$0 = \sum_{j \neq i} \text{Tr}(E_i E_j) = \sum_{j \neq i} \text{rank}(E_i E_j),$$

and since $\text{rank}(E_i E_j) \geq 0$ for all $j \neq i$, so $\text{rank}(E_i E_j) = 0$ for all $j \neq i$, which means $E_i E_j = 0$ for all $j \neq i$, and since i can be $1, 2, \dots, k$, so we're done. ■

Problem 0.0.4. Let T be the linear operator on \mathbb{R}^2 , the matrix of which in the standard ordered basis is

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Let W_1 be the subspace of \mathbb{R}^2 spanned by the vector $\epsilon_1 = (1, 0)$.

- (a) Prove that W_1 is invariant under T .
- (b) Prove that there is no subspace W_2 which is invariant under T and which is complementary to W_1 :

$$\mathbb{R}^2 = W_1 \oplus W_2.$$

proof of (a). Note that

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \epsilon_1 = 2\epsilon_1,$$

so for all $v \in W_1$, we can write $v = c\epsilon_1$ for some $c \in \mathbb{R}$, and thus

$$T(v) = T(c\epsilon_1) = cT(\epsilon_1) = 2c\epsilon_1 \in W_1.$$

This means W_1 is T -invariant. ■

proof of (b). If such W_2 exists, then

$$2 = \dim \mathbb{R}^2 = \dim W_1 + \dim W_2 = 1 + \dim W_2,$$

so $\dim W_2 = 1$, which means $W_2 = \text{span}\{(a, b)\}$ for some $(a, b) \in \mathbb{R}^2$ and W_2 is T -invariant. This means

$$T(a, b) = c(a, b) \text{ for some } c \in \mathbb{R},$$

and since $(a, b) \neq (0, 0)$, so c is an eigenvalue of T . However, note that

$$\det \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = (x - 2)^2,$$

so $c = 2$. Hence,

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix},$$

which gives $(a, b) \in \text{span}\{(1, 0)\} = W_1$, so $W_2 = W_1$ since $(a, b) \neq (0, 0)$, but $\mathbb{R}^2 = W_1 \oplus W_2$, so this is impossible. Hence, such W_2 does not exist. ■

Problem 0.0.5. Let T be a linear operator on V . Suppose $V = W_1 \oplus \cdots \oplus W_k$, where each W_i is invariant under T . Let T_i be the induced (restriction) operator on W_i .

- (a) Prove that $\det(T) = \det(T_1) \cdots \det(T_k)$.
- (b) Prove that the characteristic polynomial for T is the product of the characteristic polynomials for T_1, \dots, T_k .
- (c) Prove that the minimal polynomial for T is the least common multiple of the minimal polynomials for T_1, \dots, T_k . (Hint: Prove and then use the corresponding facts about direct sums of matrices.)

proof of (a). Suppose b_i is a basis of W_i for $i = 1, 2, \dots, k$, then $B = \bigcup_{i=1}^k b_i$ is a basis of V , and note that

$$[T]_B = \begin{pmatrix} [T_1]_{b_1} & 0 & \cdots & 0 \\ 0 & [T_2]_{b_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [T_k]_{b_k} \end{pmatrix},$$

so $\det([T]_B) = \det([T_1]_{b_1}) \det([T_2]_{b_2}) \dots \det([T_k]_{b_k})$, which gives

$$\det(T) = \det(T_1) \det(T_2) \dots \det(T_k).$$

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proof of (b). Note that

$$xI - [T]_B = \begin{pmatrix} xI - [T_1]_{b_1} & 0 & \cdots & 0 \\ 0 & xI - [T_2]_{b_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & xI - [T_k]_{b_k} \end{pmatrix},$$

so

$$\det(xI - [T]_B) = \det(xI - [T_1]_{b_1}) \det(xI - [T_2]_{b_2}) \dots \det(xI - [T_k]_{b_k}),$$

which gives

$$\text{ch}_T(x) = \text{ch}_{T_1}(x)\text{ch}_{T_2}(x) \dots \text{ch}_{T_k}(x).$$

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proof of (c). Note that for all $i = 1, 2, \dots, k$ we have $m_T(T_i) = 0$ since for any $w_i \in W_i$ we know

$$0 = m_T(T)(w_i) = m_T(T_i)(w_i).$$

This means $m_{T_i}(x) \mid m_T(x)$ for all $i = 1, 2, \dots, k$. Now let $f(x) = \text{lcm}(m_{T_1}(x), m_{T_2}(x), \dots, m_{T_k}(x))$, then note that $f(x)$ is the polynomial with least degree s.t. $m_{T_i}(x) \mid f(x)$ for all $i = 1, 2, \dots, k$ and $f(x)$ is monic since $m_{T_i}(x)$ is monic for all $i = 1, 2, \dots, k$. Note that for all $v \in V$, $v = \sum_{i=1}^k w_i$ where $w_i \in W_i$ for all $i = 1, 2, \dots, k$ and

$$f(T)(v) = \sum_{i=1}^k f(T)(w_i) = \sum_{i=1}^k 0 = 0$$

since for all $i = 1, 2, \dots, k$ we know $m_{T_i}(x) \mid f(x)$, i.e. $f(T) = q_i(T)m_{T_i}(T)$ for some $q_i(x) \in F[x]$, and thus $f(T)(w_i) = q_i(T)m_{T_i}(T)(w_i) = q_i(T)m_{T_i}(T_i)(w_i) = 0$. Hence, $f(T)(v) = 0$ for all $v \in V$ and thus $f(x) = m_T(x)$.

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