Linear Algebra I

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Abstract

The lecture note of Linear Algebra I by professor 余正道.

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Chapter 1

Vector Space

Lecture 1

1.1 Introduction to vector and vector space

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In high school, our vectors are in $\mathbb{R}^2$ and $\mathbb{R}^3$, and we have define the addition and scalar multiplication of vectors.

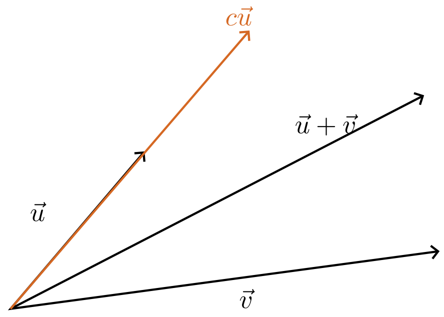

Figure 1.1: Vectors in $\mathbb{R}^2$

Example. $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n \mid a_i \in \mathbb{R})\}$

With this type of space, we can define addition and multiplication as

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = \{a_1 + b_1, a_2 + b_2, \dots, a_n + b_n\}$$
$$\alpha \cdot (a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$$

Also, if we define a space:

Example. $V = \{\text{function } f : (a, b) \to \mathbb{R}\}$, where (a, b) is an open interval.

then this can also be a vector space after defining addtion and multiplication.

Note. In a vector space, we have to make sure the existence of 0-element, which means $0(x) = 0$.

Now we give a more abstract example:

Example. Suppose S is any set, then define $V = \{\text{all functions from } S \text{ to } \mathbb{R}\}$

If we define $(f + g)(s) = f(s) + g(s)$ and $(\alpha \cdot f)(s) = \alpha \cdot f(s)$, and $0(s) = 0$, then this is also a vector space.

Put some linear conditions

Example. In $\mathbb{R}^n$, fix $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, if we define

$$W = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\},$$

then this is also a vector space.

However, if we have

$$W' = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = 1\},$$

then this is not a vector space because it is not close.

Example. In $V = \{(a, b) \to \mathbb{R}\}$ or $W_1 = \{\text{polynomial defined on } (a, b)\}$, these are both vector space.

Remark. In the later course, we will learn that W_1 is a subspace of V.

Example. If we furtherly defined $W_1^{(k)} = \{\text{polynomial degree } \leq k\}$, then this is also a vector space.

Remark. $W_1^{(k)}$ is actually isomorphic to $\mathbb{R}^{k+1}$ since

$$a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \leftrightarrow (a_0, a_1, a_2, \dots, a_n).$$

Example. $W_2 = \{\text{continuous function on } (a, b)\}$ and $W_3 = \{\text{differentiable functions}\}$ are also both vector spaces.

Example. $W_4 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0\right\}$ and $W_5 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -f\right\}$ are both vector spaces.

Proof.

$$\begin{aligned} W_4 &= \{a_0 + a_1x\} \\ W_5 &= \{a_1 \cos x + a_2 \sin x\} \end{aligned}$$

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1.2 Formal definition of vector spaces

1.2.1 Vector Spaces Over $\mathbb{R}$

Definition 1.2.1. Suppose V is a non-empty set equipped with

- addition: $V \times V \to V$, that is, given $u, v \in V$, defining $u + v \in V$
- scalare multiplication: $\mathbb{R} \times V \to V$, that is, given $\alpha \to \mathbb{R}$ and $v \in V$, we need to have $\alpha v \in V$

Also, we need some good properties or conditions

- For addition,
  - $u + v = v + u$
  - $(u + v) + w = u + (v + w)$
- There exists $0 \in V$ such that $u + 0 = u = 0 + u$
- Given $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0 = (-v) + v$

- For scalar multiplication,
  - $1 \cdot v = v$ for all $v \in V$
  - $(\alpha\beta)v = \alpha \cdot (\beta v)$ for all $\alpha, \beta \in \mathbb{R}$ and $v \in V$.
- For addition and multiplication,
  - $\alpha(u + v) = \alpha u + \alpha v$
  - $(\alpha + \beta)u = \alpha u + \beta u$

Lecture 2

1.3 Vector Space over general field

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Now we introduce the concept of field.

Definition 1.3.1 (Field). A set F with $+$ and $\cdot$ is called a **field** if

- $\alpha + \beta = \beta + \alpha$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- There exists $0 \in F$ such that $\alpha + 0 = 0 + \alpha = \alpha$.
- For $\alpha \in F$, there exists $-\alpha$ such that $\alpha + (-\alpha) = 0$.
- $\alpha\beta = \beta\alpha$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
- $\exists 1$ such that $1 \neq 0$ and $1 \cdot \alpha = \alpha$.
- For $\alpha \neq 0$, $\exists \alpha^{-1} \in F$ such that $\alpha\alpha^{-1} = 1$.
- $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Example. $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are all fields but $\mathbb{Z}$ is not.

Example. $\{0, 1\}$ is also a field.

Now we know the concept of filed, so we can make a vector space over a field.

Theorem 1.3.1 (Cancellation law). Suppose $v_1, v_2, w \in V$, a vector space, then if $v_1 + w = v_2 + w$, then $v_1 = v_2$.

Proof.

$$v_1 = v_1 + (w + (-w)) = (v_1 + w) + (-w) = (v_2 + w) + (-w) = v_2 + (w + (-w)) = v_2.$$

■

Theorem 1.3.2. The zero vector 0 is unique.

Proof. Suppose we have $0, 0'$ both zero vector, then for some $0 = 0 + 0' = 0'$. ■

Theorem 1.3.3. For any $v \in V$, $0 \cdot u = 0$.

Proof. $0 \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u$, so $0 = 0 \cdot u$ by cancellation law. ■

Theorem 1.3.4. $(-1) \cdot u = -u$.

Theorem 1.3.5. Given any $u \in V$ is unique, $-u$ is unique.

1.4 Subspaces

Definition 1.4.1 (subspace)**.** Let V be a vector space. A non-empty subset $W \subseteq V$ is called a subspace of V if W is itself a vector space under $+$ and $\cdot$ on V.

Example. $M_n(F) = \{n \times n \text{ matrix with entries in } F\}$ is a vector space, and

$$U_n(F) = \left\{ \begin{pmatrix} a_{11} & & & \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \right\}$$

is a subspace of $M_n(F)$.

Proposition 1.4.1. Suppose V is a vector space, and $W \subseteq V$ is non-empty, then

$$W \text{ is a subspace} \Leftrightarrow \text{For } u, v \in W, \alpha \in F, \text{ we have } u + v \in W \text{ and } \alpha \cdot u \in W.$$

proof of $\Rightarrow$. Clear. ■

proof of $\Leftarrow$. First, we would want to check $0 \in W$, and we can pick any $u \in W$, and pick $\alpha = -1$, so we know $-u \in W$, and thus $0 = u + (-u) \in W$. ■

Corollary 1.4.1. If we want to check W is a subspace, we just need to check for $u, v \in W$, $\alpha \in F$, $u + \alpha v \in W$ or not.

1.5 Linear Combination

Definition 1.5.1 (Linear combinaiton)**.** Given $v_1, v_2, \ldots, v_n \in V$, a linear combination of them is a vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Proposition 1.5.1. Given $v_1, v_2, \ldots, v_n \in V$,

1. $W = \{\text{all linear combinations of } v_, \ldots, v_n\}$ is a subspace.
2. This subspace is the smallest subspace containing $v_1, \ldots, v_n$. That is, if $W' \subseteq V$ is a subspace containing $v_1, \ldots, v_n$, then $W \subseteq W'$.

Notation. $\operatorname{span}\{v_1, v_2, \ldots, v_n\} = \{\text{all linear cominations of } v_1, v_2, \ldots, v_n\}$

1.6 Linearly independent

Definition. Now we talk about the linearly dependence and linearly independence.

Definition 1.6.1 (Linearly dependent). $v_1, v_2, \dots, v_n$ are linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

for some $\alpha_1, \alpha_2, \dots, \alpha_n$ not all zeros.

Definition 1.6.2 (Linearly independent). $v_1, v_2, \dots, v_n$ are called linearly independent if they are not linearly dependet.

Corollary 1.6.1. Say $\alpha_i \neq 0$, then $v_i \in \operatorname{span}\{\hat{v_1}, \hat{v_2}, \dots, \hat{v_k}\}$ suppose the corresponding α_i of $\hat{v_1}, \dots, \hat{v_k}$ are not zeros.

Corollary 1.6.2. Linearly independent means if $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Corollary 1.6.3. Linearly independent meeans if $\sum \alpha_i v_i = \sum \beta_i v_i$, then $\alpha_i = \beta_i$ for all i.

Example.

- $v \in V$ is linearly independent iff $v \neq 0$.
- $v, w \in V$ are linearly independent iff v is not a scaler of w and w is not a scalar of v.

Lemma 1.6.1. $v_1, \dots, v_n$ are linearly independent iff $v_i \notin \operatorname{span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$.

1.7 Basis

Definition. We now talking about basis

Definition 1.7.1 (Basis). $B = \{v_1, v_2, \dots, v_n\}$ is called a basis of V if B spans V and B is linearly independent.

Definition 1.7.2 (Dimension). In this case, n is called the dimension of V, and denoted by $\dim V$.

Notation. $\operatorname{span}\{v_1, v_2, \dots, v_n\} = \langle v_1, v_2, \dots, v_n \rangle$

Notation. $\operatorname{span}(S) = \langle S \rangle$

Theorem 1.7.1. For any $v \in V$, it has a unique expression $v = \sum_{i=1}^{n} \alpha_i v_i$.

Appendix