1. (a) (i) Let $A(x) = \sum_{n=0}^{\infty} n^3 x^n$ and $B(x) = \sum_{n=0}^{\infty} x^n$.

$$\Rightarrow B'(x) = \sum_{n=1}^{\infty} n x^{n-1} \Rightarrow xB'(x) = \sum_{n=0}^{\infty} n x^n$$

$$\Rightarrow (xB'(x))' = \sum_{n=1}^{\infty} n^2 x^{n-1} \Rightarrow x(xB'(x))' = \sum_{n=0}^{\infty} n^2 x^n$$

$$\Rightarrow (x(xB'(x))')' = \sum_{n=1}^{\infty} n^3 x^{n-1} \Rightarrow x(x(xB'(x))')' = \sum_{n=0}^{\infty} n^3 x^n = A(x)$$

We know that $B(x) = \frac{1}{1-x}$, so

$$A(x) = x(x(xB'(x))')' = x\left(x\left(\frac{x}{(1-x)^2}\right)'\right)'$$

$$= x \times \left(x \times \frac{(1-x)^2 - x \times 2 \times (1-x) \times (-1)}{(1-x)^4}\right)' = x \cdot \left(\frac{x+x^2}{(1-x)^3}\right)'$$

$$= x \times \frac{(1-x)^2(1+4x+x^2)}{(1-x)^6} = \frac{x+4x^2+x^3}{(1-x)^4}.$$

(ii) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n + (3^{0/2}x^0 + 3^{4/2}x^4 + 3^{8/2}x^8 + \dots) - (3^{2/2}x^2 + 3^{6/2}x^6 + 3^{10/2}x^{10} + \dots)$

We know that $\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$. Now, we try to compute other two series. Let $P = 3^{0/2}x^0 + 3^{4/2}x^4 + 3^{8/2}x^8 + \dots$ and $Q = 3^{2/2}x^2 + 3^{6/2}x^6 + 3^{10/2}x^{10} + \dots$. Then, we have

$$P = 1 + 3^2x^4 + 3^4x^8 + \dots$$

$$-)\ 9x^4 P = \quad 3^2x^4 + 3^4x^8 + \dots$$

$$(1-9x^4)P = 1 \Rightarrow P = \frac{1}{1-9x^4}$$

and

$$Q = 3x^2 + 3^3x^6 + 3^5x^{10} + \dots$$

$$-)\ 9x^4 Q = \quad 3^3x^6 + 3^5x^{10} + \dots$$

$$(1-9x^4)Q = 3x^2 \Rightarrow Q = \frac{3x^2}{1-9x^4}$$

In conclusion,

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-2x} + \frac{1}{1-9x^4} - \frac{3x^2}{1-9x^4} = \frac{1}{1-2x} + \frac{1}{1+3x^2}$$

(b) By Taylor series, we have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for any function f.
Also, by Taylor series, we have $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

(I) $\ln(1-y) = -\sum_{n=1}^{\infty} \frac{y^n}{n}$. Let $y = 3x^2$, then we have

$$-\ln(1-3x^2) = \sum_{n=1}^{\infty} \frac{3^n x^{2n}}{n} \Rightarrow a_n = \begin{cases} \frac{3^{n/2}}{n/2} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}.$$

(II) $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$. Let $z = x^2$, then we have

$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} \Rightarrow a_n = \begin{cases} \frac{(-1)^{n/4}}{(n/2)!} & \text{if } 4 \mid n \\ 0 & \text{otherwise} \end{cases}$.

2. (a) Let we number every "block" in $(3 \times n)$ corridor as below:

n metres long

3 metres wide

$a_{1,1}$	$a_{2,1}$	$a_{3,1}$		$a_{(n-1),1}$	$a_{n,1}$
$a_{1,2}$	$a_{2,2}$	$a_{3,2}$	...	$a_{(n-1),2}$	$a_{n,2}$
$a_{1,3}$	$a_{2,3}$	$a_{3,2}$		$a_{(n-1),3}$	$a_{n,3}$

To cover $a_{n,1}$, there has only two ways:

Method 1. Put the carpet horizontally to cover $a_{(n-2),1}$, $a_{(n-1),1}$, and $a_{n,1}$ at the same time.

Method 2. Put the carpet vertically to cover $a_{n,1}$, $a_{n,2}$, and $a_{n,3}$ at the same time.

For the Method 1., we must need to put two carpets to cover $a_{(n-2),2}$, $a_{(n-1),2}$, $a_{n,2}$ and $a_{(n-2),3}$, $a_{(n-1),3}$, $a_{n,3}$, respectively. Now, the corridor without covering by carpet is 3 metres wide and $(n-3)$ metres long. By definition, we have C_{n-3} ways to cover it.

Also, for the Method 2., we have C_{n-1} ways to cover the corridor without carpet.

Note that Method 1. and Method 2. are disjoint.

By sum rule, $c_n = c_{n-1} + c_{n-3}$ for $n \geq 3$, and we have $c_0 = c_1 = c_2 = 1$ as the initial conditions.

(b) $(1 - x - x^3) C(x) = C(x) - xC(x) - x^3 C(x)$

$= C_0 + (C_1 - C_0)x + (C_2 - C_1)x^2 + \sum_{n=3}^{\infty} (C_n - C_{n-1} - C_{n-3}) x^n = 1$.

$\Rightarrow c(x) = \frac{1}{1 - x - x^3}$.

(c) The characteristic polynomial is $p(z) = z^3 - z^2 - 1$.
Consider $p(z) = 0$. Let $z = y + \frac{1}{3}$, then we have

$$(y+\tfrac{1}{3})^3 - (y+\tfrac{1}{3})^2 - 1 = y^3 - \tfrac{1}{3}y - \tfrac{29}{27} = 0.$$

By Cardano's formula, we have

$$u = \sqrt[3]{\frac{\left(\frac{29}{27}\right)}{2} + \sqrt{\left(\frac{\left(\frac{29}{27}\right)}{2}\right)^2 - \left(\frac{\left(\frac{1}{3}\right)}{3}\right)^3}} = \sqrt[3]{\frac{29}{54} + \sqrt{\frac{31}{108}}}$$

and

$$v = \sqrt[3]{\frac{\left(\frac{29}{27}\right)}{2} - \sqrt{\left(\frac{\left(\frac{29}{27}\right)}{2}\right)^2 - \left(\frac{\left(\frac{1}{3}\right)}{3}\right)^3}} = \sqrt[3]{\frac{29}{54} - \sqrt{\frac{31}{108}}}$$

s.t. the three roots of y is $u+v$, $wu+w^2v$, w^2u+wv
where $w = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.
Hence, $z_1 = \frac{1}{3} + u + v$, $z_2 = \frac{1}{3} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)u - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)v$,
and $z_3 = \frac{1}{3} - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)u + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)v$ is roots of $p(z)$.
So, $C_n = Az_1^n + Bz_2^n + Cz_3^n$ for some constants A, B, C that satisfy $1 = A+B+C$, $1 = Az_1 + Bz_2 + Cz_3$, and $1 = Az_1^2 + Bz_2^2 + Cz_3^2$.
By cramer's rule, since we have

$$\det\begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{pmatrix} = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1),$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & z_2 & z_3 \\ 1 & z_2^2 & z_3^2 \end{pmatrix} = \begin{vmatrix} z_2 & z_3 \\ z_2^2 & z_3^2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ z_2^2 & z_3^2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ z_2 & z_3 \end{vmatrix}$$

$$= z_2 z_3 (z_3 - z_2) - (z_3 + z_2)(z_3 - z_2) + (z_3 - z_2)$$

$$= (z_3 - z_2)(z_2 - 1)(z_3 - 1),$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ z_1 & 1 & z_3 \\ z_1^2 & 1 & z_3^2 \end{pmatrix} = (z_1 - z_3)(z_1 - 1)(z_3 - 1), \text{ and } \det\begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & 1 \\ z_1^2 & z_2^2 & 1 \end{pmatrix} = (z_2 - z_1)(z_2 - 1)(z_1 - 1),$$

$$A = -\frac{(z_2 - 1)(z_3 - 1)}{(z_1 - z_2)(z_3 - z_1)},\quad B = -\frac{(z_1 - 1)(z_3 - 1)}{(z_1 - z_2)(z_2 - z_3)},\text{ and } C = -\frac{(z_1 - 1)(z_2 - 1)}{(z_2 - z_3)(z_3 - z_1)}.$$