Exercise Sheet 1

Due date: 15:30, Sep 23rd, to be submitted on COOL.

Working with your partner, you should try to solve all of the exercises below. You should then submit solutions to four of the problems, with each of you writing two, clearly indicating the author of each solution. Note that each problem is worth 10 points, and starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting on COOL, please send your solutions by e-mail.

Exercise 1 In a game of Scrabble, there is a bag containing fourteen letter tiles, namely 'ABCDEFGHIJKLMN'. A player reaches in, pulls out seven of the tiles, and then arranges them in some order on their rack.

(a) How many different strings of seven letters can the player have on their rack?

(b) What if, instead, the tiles in the bag were 'REARRANGEMENTS'?

Exercise 2 Let q be a prime power, and let $\mathbb{F}_q$ be the finite field of order q. Let $V = \mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. We denote by $\begin{bmatrix} V \\ k \end{bmatrix}_q$ the set of k-dimensional vector subspaces of V, and by $\begin{bmatrix} n \\ k \end{bmatrix}_q = \left| \begin{bmatrix} V \\ k \end{bmatrix}_q \right|$ the number of such subspaces.

(a) By double-counting, or otherwise, give a formula for $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

(b) Let $\vec{v} \in V$ be a non-zero vector. How many k-dimensional subspaces of V contain $\vec{v}$?

Exercise 3 Let X be a set of n elements, and call a sequence $(x_1, x_2, \ldots, x_\ell) \in X^\ell$ *non-repetitive* if we have $x_{i+1} \neq x_i$ for all $1 \leq i \leq \ell - 1$.

(a) How many non-repetitive sequences of length ℓ are there?

We call the sequence *cyclically non-repetitive* if we also have $x_1 \neq x_\ell$. Let $N_{n,\ell}$ denote the number of cyclically non-repetitive sequences of length ℓ.

(b) For $n \geq 1$ and $\ell \geq 3$, prove that $N_{n,\ell-1} + N_{n,\ell} = n(n-1)^{\ell-1}$.

(c) Prove that $N_{n,\ell} = (n-1)^\ell + (-1)^\ell(n-1)$ for all $n \geq 1$ and $\ell \geq 2$.

Exercise 4 Prove the Binomial Theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

by induction.

Exercise 5 Prove the following binomial identities.

(a) $\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$.

(b) $\sum_{j=0}^{k} (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k}$. (Note that the case $k = n$ generalises our identity from lectures about odd- and even-sized subsets.)

(c) $\sum_{j=0}^{n} \binom{n}{j} j = n2^{n-1}$.