Introduction to Analysis I HW3

B13902024 張沂魁

September 20, 2025

Problem 0.0.1. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set

$$S = \{x^{(n)} : n \geq m\}.$$

Is the converse true?

Problem 0.0.2. The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

(a) Given any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, we introduce the *formal limit*

$$\mathrm{LIM}_{n\to\infty} x_n.$$

We say that two formal limits $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Show that this equality relation obeys the reflexive, symmetry, and transitive axioms, i.e. that it is an equivalence relation.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, modulo the above equivalence relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to [0, \infty)$ by

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) := \lim_{n\to\infty} d(x_n, y_n).$$

Show that this function is well-defined (the limit exists and does not depend on the choice of representatives) and that it satisfies the axioms of a metric. Thus $(\overline{X}, d_{\overline{X}})$ is a metric space.

(c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.

(d) We identify an element $x \in X$ with the corresponding constant Cauchy sequence $(x, x, x, \dots)$, i.e. with the formal limit $\mathrm{LIM}_{n\to\infty} x$. Show that this is legitimate: for $x, y \in X$,

$$x = y \quad \Longleftrightarrow \quad \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y.$$

With this identification, show that

$$d(x, y) = d_{\overline{X}}(x, y),$$

and thus (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) Show that the closure of X in $\overline{X}$ is $\overline{X}$ itself. (This explains the choice of notation.)

(f) Finally, show that the formal limit agrees with the actual limit: if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X that converges in X, then

$$\lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n \quad \text{in } \overline{X}.$$

Problem 0.0.3. In the following, all the sets are subsets of a metric space (X, d).

(a) If $\overline{A} \cap \overline{B} = \varnothing$, then

$$\partial(A \cup B) = \partial A \cup \partial B.$$

(b) For a finite family $\{A_i\}_{i=1}^{n} \subseteq X$, show that

$$\mathrm{int}\Big(\bigcap_{i=1}^{n} A_i\Big) = \bigcap_{i=1}^{n} \mathrm{int}(A_i).$$

(c) For an arbitrary (possibly infinite) family $\{A_\alpha\}_{\alpha\in F} \subseteq X$, prove that

$$\operatorname{int}\Big(\bigcap_{\alpha\in F} A_\alpha\Big) \;\subseteq\; \bigcap_{\alpha\in F} \operatorname{int}(A_\alpha).$$

(d) Give an example where the inclusion in part (c) is strict (i.e., equality fails).

(e) For any family $\{A_\alpha\}_{\alpha\in F} \subseteq M$, prove that

$$\bigcup_{\alpha\in F} \operatorname{int}(A_\alpha) \;\subseteq\; \operatorname{int}\Big(\bigcup_{\alpha\in F} A_\alpha\Big).$$

(f) Give an example of a finite collection F in which equality does not hold in part (e).

Problem 0.0.4. Let (X,d) be a metric space and $Y \subset X$ be an open subset. For any subset $A \subset Y$, show that A is open in Y if and only if it is open in X.

Problem 0.0.5. On the space $(0,1]$, we may consider the topology induced by the metric space $(\mathbb{R},d)$ defined by $d(x,y) = |x-y|$. Alternatively, we may also define a distance d' on $(0,1]$, given by

$$d'(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \qquad \forall x,y \in (0,1].$$

(a) Show that d' is a metric on $(0,1]$

(b) Let $x \in (0,1]$ and $\varepsilon > 0$. Let $B = B_d(x,\varepsilon) = \{y \,|\, |y-x| < \varepsilon\} \cap (0,1]$ be the open ball centered at x of radius ε for the metric d in $(0,1]$. Show that for any $y \in B$, we may find $\varepsilon' > 0$ such that

$$B_{d'}(y,\varepsilon') \subseteq B = B_d(x,\varepsilon).$$

(c) Show that an open ball in $((0,1],d')$ is also an open ball in $((0,1],d)$.

(d) Conclude that the metric spaces $((0,1],d)$ and $((0,1],d')$ are topologically equivalent, that is, a set A is open in one space if and only if it is also open in the other one.

(e) Is $((0,1],d')$ a complete metric space? How about $((0,1],d)$?

Problem 0.0.6. (a) We say that a family of closed balls

$$\big(\overline{B}(x_n,r_n)\big)_{n\geq 1}$$

is a *decreasing sequence of closed balls* if the nesting condition

$$\overline{B}(x_{n+1},r_{n+1}) \;\subseteq\; \overline{B}(x_n,r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Give an example of a decreasing sequence of closed balls in a complete metric space with empty intersection.

(b) We say that a family of closed balls

$$\big(\overline{B}(x_n,r_n)\big)_{n\geq 1}$$

is a *decreasing sequence of closed balls with radii tending to zero* if

$$r_n \;\to\; 0 \quad \text{as } n \to \infty,$$

and the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Show that a metric space (M, d) is complete if and only if every decreasing sequence of closed balls with radii going to zero has a nonempty intersection.