1.(a) Choose s desks from d desks to place the examination paper in advance $\Rightarrow \binom{d}{s}$ ways.

(b) After choosing s desks from d desks, since the papers are distinct, we need to consider its arrangement $\Rightarrow \binom{d}{s} \times s! = \frac{d!}{(d-s)!}$ ways.

(c) Number the students from lefts to right, with the one on the far left is number 1 and the one on the far right is number s. Then suppose there has l_i desks between number i student and number $i+1$ student. Besides, say there has l_0 desks on the left of number 1 student and l_s desks on the right of number s student. Then from the problem's assumption, $l_i \geq 2$ for $i=1,2,3,\cdots,s-1$, $l_j \geq 0$ for $j=0$ or s, and $l_0+l_1+l_2+\cdots+l_s = d-s$.

Now, define $t_i = \begin{cases} l_i & \text{if } i=0 \text{ or } s \\ l_i-2 & \text{if } i\neq 0 \text{ or } s \end{cases}$, so $t_i \geq 0 \ \forall i$ and

$t_0+t_1+\cdots t_s = d-s-2(s-1) = d-3s+2$.

Consider the stars and bars diagram. There has $d-3s+2$ stars and s bars. So there has $\binom{d-3s+2+s}{s} = \binom{d-2s+2}{s}$ ways.

6. (a) Let $M_j(z) = [\prod_{i=1}^{q}(z-\lambda_i)^{k_i}]/(z-\lambda_j)^{k_j}$.

<u>Claim.</u> $\exists !\, F_i(z)$ s.t. $R(z) \equiv F_i(z) \pmod{(z-\lambda_i)^{k_i}}$ and $F_i(z)$ has the factor $M_i(z)$ and $\deg F_i(z) \le \deg R(z)$.

<u>pf.</u> Note that $F_i(z) \equiv \begin{cases} R(z) & \pmod{(z-\lambda_i)^{k_i}} \\ 0 & \pmod{M_i(z)} \end{cases}$.

Since $(z-\lambda_i)^{k_i}$ and $M_i(z)$ are relatively prime, by Euclidean algorithm, there exists polynomial $a_i(z)$ s.t. $a_i(z)M_i(z) \equiv 1 \pmod{(z-\lambda_i)^{k_i}}$. Consider $R(z)a_i(z)$ divided by $(z-\lambda_i)^{k_i}$, say $R(z)a_i(z) = p_i(z)(x-\lambda_i)^{k_i} + q_i(z)$ with $\deg q_i(z) \le k_i - 1$. Let $F_i(z) = q_i(z)M_i(z)$.

Since $\deg q_i(z)M_i(z) \le (k_i-1)+(k-k_i) = k-1 = \deg R(z)$ and $q_i(z)M_i(z) \equiv R(z)a_i(z)M_i(z) \equiv R(z) \pmod{(z-\lambda_i)^{k_i}}$, we have proven the existence. Also, note that $q_i(z)$ is unique and $M_i(z)$ is given, $F_i(z)$ is also unique.

Now, say $F_i(z) = s_i(z)(z-\lambda_i)^{k_i} + t_i(z)$ with $\deg t_i(z) \le k_i - 1$.

Follow the notation above, we have $R(z) \equiv F_i(z) \equiv t_i(z)$ $\forall i$.

By Chinese Remainder theorem, $\exists$ a polynomial $d(z)$ such that

$$R(z) = d(z) \cdot \prod_{i=1}^{q}(z-\lambda_i)^{k_i} + \sum_{i=1}^{q}\{t_i(z)a_i(z)M_i(z)\}$$

where $a_i(z)M_i(z) \equiv 1 \pmod{(z-\lambda_i)^{k_i}}$. Here, d should be unique since $\deg R(z) < k$ and $\deg(\prod_{i=1}^{q}(z-\lambda_i)^{k_i}) = k$.

$\forall i$, say $t_i(z)a_i(z) = g_i(z)(z-\lambda_i)^{k_i} + h_i(z)$ with $\deg h_i(z) \le k_i - 1$.

$\Rightarrow \sum_{i=1}^{q}\{t_i(z)a_i(z)M_i(z)\} = \sum_{i=1}^{q}(h_i(z)M_i(z)) + (\sum_{i=1}^{q} g_i(z)) \cdot \prod_{i=1}^{q}(z-\lambda_i)^{k_i}$.

Note that $\deg(h_i(z)M_i(z)) \le \deg R(z)$. Let $k(z) = -\sum_{i=1}^{q} g_i(z)$.

$\Rightarrow R(z) = \sum_{i=1}^{q}(h_i(z)M_i(z))$

$\Rightarrow R(z)/\prod_{i=1}^{q}(z-\lambda_i)^{k_i} = \sum_{i=1}^{q}(h_i(z)/(z-\lambda_i)^{k_i})$.

Since $h_i(z)$ is unique for all i, let $R_i(z) = h_i(z)$, we are done!

(b) Let $\mathbb{R}[z]_{\le k_i-1}$ be the vector spaces of all polynomials of z of degree at most k_i-1. $\dim(\mathbb{R}[z]_{k_i-1}) = k_i$.

Note that $R_i(z) \in \mathbb{R}[z]_{\le k_i-1}$ and $\{(z-\lambda_i)^s \mid 0 \le s \le k_i-1, s \in \mathbb{Z}\}$ is a basis for $\mathbb{R}[z]_{\le k_i-1}$. So $\exists !$ constant $a_{i,1}, a_{i,2}, \cdots, a_{i,k_i}$ s.t. $R_i(z) = \sum_{t=0}^{k_i-1}\left(a_{i,t+1}(z-\lambda_i)^t\right)$.

$\Rightarrow R_i(z)/(z-\lambda_i)^{k_i} = \sum_{t=0}^{k_i-1}\left(a_{i,t+1}/(z-\lambda_i)^{k_i-t}\right)$.

Let $A_{i,j} = a_{i,k_i-j}$, then we are done!

(c) Claim. If $b(x) = (1-\lambda x)^{-j}$, then $b_n = \binom{j+n-1}{n}\lambda^n$

pf. We'll prove it by induction.

As $j=1$, $b(x) = 1+\lambda x+(\lambda x)^2+(\lambda x)^3+\cdots = \sum_{i=0}^{\infty}(\lambda x)^i$.

$\Rightarrow b_n = \lambda^n = \binom{1+n-1}{n}\lambda^n$. The claim holds.

Assume the statement holds up to $j=k$. Then for $j=k+1$, we first set $a(x) = (1-\lambda x)b(x) = (1-\lambda x)^{-k}$. By the induction hypothesis, $a_n = \binom{k+n-1}{n}\lambda^n$.

Also, $(1-\lambda x)b(x) = \sum_{i=0}^{\infty} b_i x^i - \sum_{i=1}^{\infty}\lambda b_{i-1}x^i = \sum_{i=0}^{\infty} a_i x^i$.

We have a recurrence relation $a_n = b_n - \lambda b_{n-1}$ and $a_0 = b_0 = 1$.

By $b_n = \lambda b_{n-1} + \binom{k+n-1}{n}\lambda^n$ and $\lambda^i b_{n-i} = \lambda^{i+1}b_{n-i-1} + \binom{k+i-1}{i}\lambda^n$

$$
\begin{aligned}
\Rightarrow \quad b_n &= \lambda b_{n-1} + \binom{k+n-1}{n}\lambda^n \\
\lambda b_{n-1} &= \lambda^2 b_{n-2} + \binom{k+n-2}{n-1}\lambda^n \\
&\vdots \\
\lambda^{n-1}b_1 &= \lambda^n b_0 + \binom{k}{1}\lambda^n \\
+)\quad \lambda^n b_0 &= \lambda^n = \binom{k}{0}\lambda^n \\
\hline
b_n &= \lambda^n \times \left(\binom{k+n-1}{n} + \binom{k+n-2}{n-1} + \cdots + \binom{k+1}{2} + \binom{k}{1} + \binom{k}{0}\right) \\
&= \lambda^n \times \left(\binom{k+n-1}{n} + \binom{k+n-2}{n-1} + \cdots + \binom{k+1}{2} + \binom{k+1}{1}\right) \\
&= \cdots = \binom{k+n}{n}\lambda^n = \binom{(k+1)+n-1}{n}\lambda^n.
\end{aligned}
$$

By induction, the claim holds for all integers $j \ge 1$.

(d) Since the characteristic polynomial is $p(z) = \prod_{i=1}^{q} (z-\lambda_i)^{k_i}$, the recurrence relation shall be $a_n = \sum_{i=1}^{k} \alpha_{k-i} a_{n-i}$ where $1 - \alpha_{k-1} x - \alpha_{k-2} x^2 - \cdots - \alpha_0 x^k = \prod_{i=1}^{q} (1-\lambda_i x)^{k_i}$.

From the recurrence relation, we have

$$A(x) := \sum_{i=0}^{\infty} a_i x^i$$

$$-\alpha_{k-1} \cdot x \cdot A(x) = \sum_{i=1}^{\infty} -\alpha_{k-1} a_{i-1} x^i$$

$$-\alpha_{k-2} \cdot x^2 \cdot A(x) = \sum_{i=2}^{\infty} -\alpha_{k-2} a_{i-2} x^i$$

$$\vdots$$

$$+)\quad -\alpha_0 \cdot x^k \cdot A(x) = \sum_{i=k}^{\infty} -\alpha_0 a_{i-k} x^i$$

$$\prod_{i=1}^{q} (1-\lambda_i x)^{k_i} \cdot A(x) = \sum_{i=k}^{\infty} \left(a_i - \sum_{j=1}^{k} \alpha_{k-j} a_{i-j}\right) x^i + r(x) = r(x)$$

where $r(x)$ is a polynomial of degree at most $k-1$.

Hence, $A(x) = \sum_{i=0}^{\infty} a_i x^i = r(x) / \prod_{i=1}^{q} (1-\lambda_i x)^{k_i}$

$$= \left[(-1)^k \cdot \left(\prod_{i=1}^{q} \lambda_i^{-k_i}\right) \cdot r(x)\right] / \prod_{i=1}^{q} \left(x - \frac{1}{\lambda_i}\right)^{k_i}.$$

Let $R(x) = (-1)^k \cdot \left(\prod_{i=1}^{q} \lambda_i^{-k_i}\right) \cdot r(x)$ and $\lambda_i' = \frac{1}{\lambda_i}$. Then apply the conclusions in (a) and (b), we have

$$A(x) = R(x) / \prod_{i=1}^{q} (x-\lambda_i')^{k_i} = \sum_{i=1}^{q} \left(R_i(x) / (x-\lambda_i')^{k_i}\right) = \sum_{i=1}^{q} \sum_{j=1}^{k_i} \frac{B_{i,j}}{(x-\lambda_i')^j}$$

where $R_i(x)$'s are polynomials of degree at most $k_i - 1$ and $B_{i,j}$'s are constants.

$$\Rightarrow A(x) = \sum_{i=1}^{q} \sum_{j=1}^{k_i} \left(B_{i,j} (-\lambda_i)^j / (1-\lambda_i x)^j\right) = \sum_{i=1}^{q} \sum_{j=1}^{k_i} \left(A_{i,j} / (1-\lambda_i x)^j\right)$$

where $A_{i,j} = B_{i,j} (-\lambda_i)^j$ is also constant.

Use the conclusion in (C), we have

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{i=1}^{q} \sum_{j=1}^{k_i} \left(A_{i,j} \times \sum_{n=0}^{\infty} \left[\binom{j+n-1}{n} \lambda_i^n x^n\right]\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=1}^{q} \sum_{j=1}^{k_i} A_{i,j} \binom{j+n-1}{n} \lambda_i^n x^n\right)$$

Hence, $a_n = \sum_{i=1}^{q} \left(\sum_{j=1}^{k_i} A_{i,j} \binom{j+n-1}{n}\right) \lambda_i^n$.