Introduction to Analysis I

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Abstract

The lecture note of 2025 Fall Introduction to Analysis I by professor 崔茂培.

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Chapter 1

Basic Things

Lecture 1

1.1 Natural Numbers

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The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, \ldots\}$. There exists an addition operation

$$1+1=2 \quad 1+1+1=3 \quad \underbrace{1+1+\cdots+1}_{n \text{ times}} = n.$$

1.2 Integers

The set of integers is $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. There is a zero element 0 such that $z + 0 = z$ for any $z \in \mathbb{Z}$. Also, for $n \in \mathbb{N}$, we have $n + (-n) = 0$ and $n - m = n + (-m)$ for all $n, m \in \mathbb{N}$.

$$\mathbb{Z} \xRightarrow[\text{introduce division}]{} \mathbb{Q} \xRightarrow[\text{Completeness axiom}]{} \mathbb{R}$$

1.3 Field

Next, we introduce the concept of field.

Definition 1.3.1 (Fields). A field is a set F together with two binary operations, called addition($+$) and multiplication($*$), such that the following properties hold:

(a) $a + b = b + a$, $a \cdot b = b \cdot a$ for $a, b \in F$.

(b) $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in F$.

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$.

(d) There are distince element 0 and 1 such that $a + 0 = a$, $a \cdot 1 = a$ for $a \in F$.

(e) For each $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$. If $a \neq 0$, there is an element $\frac{1}{a}$ or a^{-1} in F such that $a \cdot \frac{1}{a} = 1$, or $a \cdot a^{-1} = 1$.

Remark. If $a \in F$, then $a + a \in F$. We denote $a + a$ by $2 \cdot a$. Similarly,

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = n \cdot a,$$

and

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$

if $a \in F$ and $n \in \mathbb{N}$.

Remark. In a field, we have subtraction and division $a - b = a + (-b)$ for $a, b \in F$. If $b \neq 0$, then $\frac{a}{b} = a \cdot b^{-1}$ for $a, b \in F$.

In a field F, we have

$$\begin{aligned}(a+b)^2 &= (a+b) \cdot (a+b) \\ &= (a+b) \cdot a + (a+b) \cdot b \\ &= a \cdot a + b \cdot a + a \cdot b + b \cdot b \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2.\end{aligned}$$

Example.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

if $b \neq 0$ and $d \neq 0$.

Proof.

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= a \cdot b^{-1} + c \cdot d^{-1} \\ &= ab^{-1}dd^{-1} + cd^{-1}bb^{-1} \\ &= adb^{-1}d^{-1} + cbd^{-1}b^{-1} \\ &= \frac{ad + bc}{bd}.\end{aligned}$$

Notice that this is true since we have commutativity in multiplication and

$$d^{-1}b^{-1} = (bd)^{-1} = \frac{1}{bd}.$$

■

Example. The set of rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$ is a field.

Example. The set of real numbers is also a field.

Example. $F_2 = \{0, 1\}$ is also a field since we can define addition and multiplication like $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$, and $0 \cdot 0 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$.

1.4 Order Relation

Next, we introduce the order relation. The real number system is ordered by the relation $<$, which has the following properties.

(f) For each pair of real numbers a and b, exactly one of the follwing is true: $a = b, a < b, b < a$.

(g) If $a < b$ and $b < c$, then $a < c$.

(h) If $a < b$, then $a + c < b + c$ for any c, and if $0 < c$, then $a \cdot c < b \cdot c$.

Definition 1.4.1. A field with an order relation satisfy (f) to (h) is called an ordered field.

Example. The set of rational numbers is an ordered field.

Example. F_2 is not an ordered field.

Proof. If $0 < 1$, then $1 = 0 + 1 < 1 + 1 = 0$, which is a contradiction. If $1 < 0$, then $0 = 1 + 1 < 0 + 1 = 1$, which is also a contradiction. ■

Notation. In an ordered field, we use $a \leq b$ to denote either $a < b$ or $a = b$.

1.5 Absolute Value and Triangle Inequality

Next, we define the absolute value of a real number

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a \leq 0; \end{cases}$$

Theorem 1.5.1 (Triangle Inequality).

$$|a + b| \leq |a| + |b|$$

for all $a, b \in \mathbb{R}$.

Corollary 1.5.1.

$$||a| - |b|| \leq |a - b| \quad \text{and} \quad ||a| - |b|| \leq |a + b|$$

Proof. We write

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Similarly we have

$$|b| \leq |b - a| + |a|.$$

So

$$-|b - a| \leq |a| - |b| \leq |a - b|.$$

Thus,

$$||a| - |b|| \leq |a - b|.$$

■

1.6 Supremum and Infimum

Next, we introduce the notion of supremum of a subset of real numbers.

Definition 1.6.1. Let S be a subset of $\mathbb{R}$,

(1) we say b is an upper bound of S if $x \leq b$ for all $x \in S$.

(2) If B is an upper bound of S, and no number smaller than B is an upper bound of S, then B is called the supremum or the least upper bound of S. We write $B = \sup S$.

Corollary 1.6.1. If $B = \sup S$, then

(1) $x \in S$ implies $x \leq B$

(2) If $b < B$, then b is not an upper bound of S, i.e. there exists $x_1 \in S$ such that $b < x_1$.

Definition 1.6.2. Let S be a subset of $\mathbb{R}$,

(1) we say b is an lower bound of S if $x \geq b$ for all $x \in S$.

(2) If α is an lower bound of S, and no number bigger than α is an lower bound of S, then α is called the infimum or the greatest lower bound of S. We write $\alpha = \inf S$.

Corollary 1.6.2. If $\alpha = \inf S$, then

(1) $x \in S$ implies $x \geq \alpha$

(2) If $\alpha < a$, then a is not an lower bound of S, i.e. there exists $x_1 \in S$ such that $x_1 < a$.

Notation (Interval Notation).

$$(a, b) = \{x \mid a < x < b\}$$
$$(a, b] = \{x \mid a < x \leq b\}$$
$$[a, b) = \{x \mid a \leq x < b\}$$

Example. $S = \{x \mid x < 0\} = (-\infty, 0)$, then $\sup S = 0$ but $\inf S$ does not exists.

Example. $S_1 = \{-1, -2, -3, -4, \dots\} = \{-n \mid n \in \mathbb{N}\}$, then $\sup S = -1$, but $\inf S$ does not exist.

Definition 1.6.3 (Nonempty Sets)**.** A nonempty set is that a set has at least one element. The empty set, written as $\varnothing$, is the set has no elements at all.

Example. $S = \left\{x \mid x \in \mathbb{Q}, x < \sqrt{2}\right\}$

In $\mathbb{Q}$, $\sup S$ does not exist. In $\mathbb{R}$, $\sup S = \sqrt{2}$.

Theorem 1.6.1 (Completeness axiom)**.** If a nonempty set of real numbers (an ordered field) is bounded above, then it has a least upper bound or $\sup S$ exists.

Remark. This is an extra axiom that can't be derived from the properties of ordered field.

Remark. Up to "isomorphism", there is exactly one complete ordered field: the field of real numbers.

Remark. From now, we assume $\mathbb{R}$ satisfies the completeness axiom. Thus, any nonempty subset $S \subseteq \mathbb{R}$ that is bounded above, we have $\sup S$ exists.

We can prove the following property of $\sup S$.

Theorem 1.6.2. If $S \subseteq \mathbb{R}$ is bounded above, then $\sup S$ is the unique real number B such that

(i) $x \leq B$ for all $x \in S$

(ii) for every $\varepsilon > 0$, there exist an $x_0 \in S$ such that $B - \varepsilon < x_0$.

Proof. (i), (ii) follows from the definition. We prove the uniqueness. Suppose $B_1 = \sup S = B_2$. We want to show $B_1 = B_2$. Suppose $B_1 \neq B_2$. Then either $B_1 < B_2$ or $B_2 < B_1$. However, if either one is true, then the other one cannot be $\sup S$. ■

Theorem 1.6.3 (Archimedean Property). If $p > 0$ and $\varepsilon > 0$, then there exists an $n \in \mathbb{N}$ such that $p < n\varepsilon$.

Proof. We prove this contradiction. Suppose it is not true. This implies $n\varepsilon \leq p$ for all $n \in \mathbb{N}$. Consider $S = \{n\varepsilon \mid n \in \mathbb{N}\}$, then p is an upper bound of S, so S is bounded above by p, so we know $B = \sup S$ exists. Hence, $n\varepsilon \leq B$ for all $n \in \mathbb{N}$, so we have $(n+1)\varepsilon \leq B$, which means

$$n\varepsilon \leq B - \varepsilon$$

for all $n \in \mathbb{N}$. This implies $B - \varepsilon$ is also an upper bound of S, which is a contradiction. ■

1.7 Density of other number system

Theorem 1.7.1. Every nonempty subset of the integers that is bounded below has a least element.

Proof. We first introduce an axiom:

Theorem 1.7.2 (Well-Ordering principle). Every non-empty subset of the natural numbers has a least element.

Note. Here, $\mathbb{N}$ can be $\{0, 1, 2, \dots\}$ or $\{1, 2, 3, \dots\}$, which is not that important.

Now we call this subset of integers as S, and suppose we have m as a lower bound of S, then define $S' = \{s - m \mid s \in S\}$, then we know S' is a nonempty subset of $\mathbb{N}$, then by well-ordering principle we know there is a least element in S' and thus there is also a least element in S. ■

Corollary 1.7.1. Every nonempty subset of the integers that is bounded above has a greatest element.

Proof. Suppose M is an upper bound, then define a set $S' = \{M - s \mid s \in S\}$, then by well-ordering principle we know $M - a$ is the least element of S' for some $a \in S$, so we have $M - x \geq M - a$ for all $x \in S$, which means $a \geq x$ for all $x \in S$ and since $a \in S$, so a is the greatest element of S. ■

Theorem 1.7.3. The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{p}{q}$ such that $a < \frac{p}{q} < b$.

Proof. Let $a, b \in \mathbb{R}$, $a < b$. By Archimedean Property, $\exists q \in \mathbb{N}$ such that $q(b - a) > 1$. Let $S = \{m \mid m \text{ is an integer with } m > qa\}$, since we know $S \neq \varnothing$ and S is bounded below. Hence, $p = \inf S$ exists and is an integer by the last theorem. So $qa < p$ and $p - 1 \leq qa$, which means $qa < p \leq qa + 1 < qb$, so we have $a < \frac{p}{q} < b$. ■

Lecture 2

Definition 1.7.1 (Floor Function). For any real number x, the floor function of x is denoted by $\lfloor x \rfloor$, and is defined by the formula $\lfloor n \rfloor$ if $n \leq x < n + 1$ where $n \in \mathbb{Z}$. 4 Sep. 10:20

Corollary 1.7.2.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

Example. $\lfloor 3.7 \rfloor = 3$, $\lfloor -1.2 \rfloor = -2$.

Now by floor function, we can reprove Theorem 1.7.3.

Theorem 1.7.4 (Density of rational number in real number Again)**.** The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{q}{p}$ such that $a < \frac{q}{p} < b$.

Reprove Theorem 1.7.3. Since $a < b$, so we know $b - a > 0$. Now by Archimedean Property, we know there exists $q \in \mathbb{N}$ such that $q(b-a) > 1$. Let $p = \lfloor qa \rfloor + 1$, we have

$$\lfloor qa \rfloor \leq qa < \lfloor qa \rfloor + 1 = p.$$

From our construction, $qb > qa + 1$, so we have

$$p = \lfloor qa \rfloor + 1 \leq qa + 1 < qb,$$

hence we have

$$qa \leq p \leq qb.$$

∎

Note. For some reason, p, q in Theorem 1.7.3 and Theorem 1.7.4 are reversed.

Definition 1.7.2 (irrational number)**.** x is called irrational if x is not rational.

Example. $\sqrt{2}$ is irrational.

Theorem 1.7.5. Let $r \in \mathbb{Q}$ and $x \in \mathbb{R} \backslash \mathbb{Q}$, then

1. $r + x$ is irrational.
2. If $r \neq 0$, then rx is irrational.

sketch of proof.

1. If $r + x = q \in \mathbb{Q}$, then $x = q - r \in \mathbb{Q}$, contradiction.
2. If $rx = q \in \mathbb{Q}$, then $x = \frac{q}{r} \in \mathbb{Q}$ since $r \neq 0$.

∎

Theorem 1.7.6 (irrational number dense in real number)**.** The set of irrational number is dense in real number. That is, if $a, b \in \mathbb{R}$ and $a < b$, then there exists a irrational number t such that $a < t < b$.

Proof. By density of rational number, we can find $a < r_1 < r_2 < b$ where $r_1, r_2 \in \mathbb{Q}$, and then let $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$, then we know

$$a < r_1 < t < r_2 < b.$$

Note. We should use Theorem 1.7.5 and the fact that $\sqrt{2}$ is irrational.

∎

Definition 1.7.3 (bounded set)**.** A set $S \subseteq \mathbb{R}$ is bounded if there are numbers a, b s.t. $a \leq x \leq b$ for all $x \in S$.

Corollary 1.7.3. A bounded non-empty set in $\mathbb{R}$ has a unique supremum and a unique infimum and $\inf S \leq \sup S$.

1.8 Extended real number system

The real number system, together with ∞ and $-\infty$, then we have the following properties:

(a) If $a \in \mathbb{R}$, then $a + \infty = \infty + a = \infty$ and $a - \infty = -\infty + a = -\infty$, and $\frac{a}{\infty} = \frac{a}{-\infty} = 0$.

(b) If $a > 0$, then $a \cdot \infty = \infty \cdot a = \infty$ and $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$

(c) If $a < 0$, then $a \cdot \infty = \infty \cdot a = -\infty$ and $a \cdot -\infty = -\infty \cdot a = \infty$ and $\infty + \infty = \infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$ and $-\infty - \infty = \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$ and $|-\infty| = |\infty| = \infty$

However, there are some indeterminate form:

Theorem 1.8.1. The following things are not defined:

$$\infty - \infty, \ 0 \cdot \infty, \ \frac{\infty}{\infty}, \text{ and } \frac{0}{0}.$$

1.9 Mathematical Induction

Theorem 1.9.1 (Peano's Postulate)**.** The natural numbers satisfy the following properties

(a) $\mathbb{N}$ is nonempty.

(b) For each natural number n, there exists a unique rational number n called the successor of n.

(c) There exists a natural number $\overline{n}$ that is not the sucessor of any natural number.

(d) Different natural numbers have different sucessors, that is, $n \neq m$ implies $n' \neq m'$.

(e) The only subset of $\mathbb{N}$ that contains $\overline{n}$ and also contains the sucessor of every one of its element is $\mathbb{N}$.

Theorem 1.9.2 (Principle of Mathematical Induction)**.** Let $p_1, p_2, \ldots, p_n$ be propositions, one for each positive integers, such that

(a) p_1 is true.

(b) for each positive integer n, p_n implies p_{n+1}.

then p_n is true for each $n \in \mathbb{N}$.

Proof. Let $M = \{n \mid n \in \mathbb{N} \text{ and } p_n \text{ is true}\}$, then from (a) we know $1 \in M$ and from (b) we know $n \in M$ implies $n + 1 \in M$. Hence, from (e) of Peano's Postulate, we know $M = \mathbb{N}$. ∎

Chapter 2

Metric Space

2.1 Definition and examples

Definition 2.1.1. Suppose $x_n \in \mathbb{R}$ for $n \geq m$. We use the notation $(x_n)_{n=m}^{\infty}$ to denote the sequence of numbers

$$x_m, x_{m+1}, \ldots$$

We first recall the definition of a convergent sequence.

Definition 2.1.2 (Convergent Sequence)**.** We say that a sequence $(x_n)_{n=m}^{\infty}$ of real numbers converges to x if for every $\varepsilon > 0$, there exists an $N \geq m$ s.t. $|x_n - x| \leq \varepsilon$ for all $n \geq N$.

Notation. We write $\lim_{n\to\infty} x_n = x$.

On $\mathbb{R}$, we can define the distance function between two points $x, y \in \mathbb{R}$ by $d(x, y) = |x - y|$. We'll discuss this more later.

Lemma 2.1.1. Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number, then $(x_n)_{n=m}^{\infty}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Proof. Assume $(x_n)_{n=m}^{\infty}$ converges to x. Let $\varepsilon > 0$ be arbitrary real number. By definition, there exists an $N \geq m$ such that $|x_n - x| \leq \varepsilon$ for all $n \geq N$. But $d(x_n, x) = |x_n - x|$ by the definition. Hence, $\forall \varepsilon > 0$, $\exists N \geq m$ such that $d(x_n, x) \leq \varepsilon$ fpr all $n \geq N$. This implies that $\forall \varepsilon > 0$, $\exists N \geq m$ such that $|d(x_n, x) - 0| \leq \varepsilon$ for all $n \geq N$. This implies $\lim_{n\to\infty} d(x_n, x) = 0$.

The proof of the other side is the same but writing the above proof from bottom to top again. ■

Definition 2.1.3 (Metric Space)**.** A metric space (X, d) is the space of X of objects(called points), together with a distance function or metric $d : X \times X \to [0, \infty)$ which associates to each x, y of points in X a nonnegative number $d(x, y) \geq 0$, the following. Furthermore, the metric must satisfy 4 axioms.

(a) For any $x \in X$, $d(x, x) = 0$.

(b) (Positivity) For any distinct $x, y \in X$, we have $d(x, y) > 0$.

(c) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.

(d) (Triangle inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Example. On $\mathbb{R}$, we can define $d(x, y) = |x - y|$.

Proof.

- $d(x, y) = |x - y| \geq 0$.

- $d(x, y) = 0$ iff $|x - y| = 0$ iff $x = y$.
- $|x - y| = |y - x|$, so $d(x, y) = d(y, x)$
- $|x - z| \leq |x - y| + |y - z|$ for all $x, y, z \in \mathbb{R}$.

⊛

Example. Let (X, d) be a metric space and $Y \subseteq X$, then Y inherits a natural distance function

$$d|_{Y \times Y} : Y \times Y \to [0, \infty)$$

defined by $d|_{Y \times Y}(\alpha, \beta) = d(\alpha, \beta)$ for all $\alpha, \beta \in Y$.

Note. $(Y, d|_{Y \times Y})$ is called a metric subspace of (X, d). It is obvious that $d|_{Y \times Y}$ is a metric on Y.

Recall $\mathbb{R}^n$. Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

Definition 2.1.4 (l^2-metric)**.** The l^2-metric is defined by

$$d_2(x, y) = \left(\sum_{i=1}^{n} (x_n - y_n)^2 \right)^{\frac{1}{2}} \text{ (or we called } d_{l_2}(x, y)).$$

Definition 2.1.5 (l^1-metric(taxicab metric))**.** The l^1-metric is defined by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| (\text{or we called } d_{l_1}(x, y))$$

Definition 2.1.6 (l^∞-metric)**.** The l^∞-metric is defined by

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

Exercise. Verify they are all metrics.

Note. Actually we have to define inner product and norm first and then we can use the triangle inequality of norm to prove d_2 is a metric. (See lecture notes by professor)

Lecture 3

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Definition 2.1.7 (Cartesian Product)**.** Let A, B be sets. The cartesian product of A and B is defined by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Similarly, the cartesian product of $X_1, X_2, \ldots, X_n$ is

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i \ \forall 1 \leq i \leq n\}.$$

Definition 2.1.8 (Functions)**.** Let $X_1, X_2, \ldots, X_n$ be sets and let Y be another set. A fuction of n variables with codomains is a map $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ which assigns each n-tuple $(x_1, x_2, \ldots, x_n)$ with $x_i \in X_i$ a unique element $f(x_1, x_2, \ldots, x_n)$.

Definition. We talk about the definition of domain, codomain, and range:

Definition 2.1.9. The domain of f is $X_1 \times X_2 \times \cdots \times X_n$ and Y is the codomain of f.

Definition 2.1.10. The range of f is

$$\{f(x_1, x_2, \ldots, x_n) \in Y \mid x_i \in X_i \ \forall i\}.$$

In the definition of metric space, we write (X, d) to emphasize our set X and d is a distance function defined on $X \times X$, i.e.

$$d : X \times X \to [0, \infty) \subseteq \mathbb{R},$$

where

$$d : (x, y) \mapsto d(x, y)$$

for $x, y \in X$. Let (X, d) be a metric space and $Y \subseteq X$. Then $(Y, d|_{Y \times Y})$ is also a metric space with distance function defined by

$$d|_{Y \times Y} \to [0, \infty)$$

and

$$d|_{Y \times Y} : (\alpha, \beta) \mapsto d(\alpha, \beta) \text{ for } \alpha, \beta \in Y.$$

Example. Recall the Taxi-cab metric, it can be used in cryptography. For example, for two binary strings, we know

$$d_1((10010), (10101)) = 3 = \text{the number of mismatched bits.}$$

Example. Recall the l^∞-metric. Suppose two jobs where each consists of 3 tasks, and the time (in hours) to complete each task is represented by a vector

$$x = (2, 4, 6), \ \ y = (3, 7, 5),$$

so

$$d_\infty(x, y) = \max\{|2 - 3|, |4 - 7|, |6 - 5|\} = 3.$$

Definition 2.1.11 (Lipschitz equivalent metrics)**.** Let (X, d_1) and (X, d_2) be two metrics on X. We say d_1 and d_2 are Lipschitz equivalent if $\exists c_1, c_2 > 0$ s.t.

$$c_1 d_1(x, y) \le d_2(x, y) \le c_2 d_1(x, y) \quad \forall x, y \in X$$

Remark. They will have same topology (defined later).

Proposition 2.1.1. For all $x, y \in \mathbb{R}^n$,

$$d_2(x, y) \le d_1(x, y) \le \sqrt{n} d_2(x, y) \tag{2.1}$$
$$d_\infty(x, y) \le d_2(x, y) \le \sqrt{n} d_\infty(x, y) \tag{2.2}$$

Remark.

$$\begin{aligned} d_\infty(x, y) &\ge \frac{1}{\sqrt{n}} d_2(x, y) \\ &\ge \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} d_1(x, y) = \frac{1}{n} d_1(x, y). \end{aligned}$$

Also,

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y).$$

Remark. d_1, d_2, d_∞ are all Lipschitz equivalent.

proof of Proposition 2.1.1 . Recall $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, then

$$d_1(x,y) = \sum_{i=1}^n |x_i - y_i|, \quad d_2(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

By Cauchy-Schurwatz inequality,

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^n |x_i - y_i| \\ &\le \left(\sum_{i=1}^n |x_i - y_i|\right)^{\frac{1}{2}} \left(\sum_{i=1}^n 1^2\right)^{\frac{1}{2}} = \sqrt{n} d_2(x,y). \end{aligned}$$

Now we show that $d_1(x,y) \ge d_2(x,y)$.

$$\begin{aligned} (d_1(x,y))^2 &= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\ &= \sum_{i=1}^n |x_i - y_i|^2 + 2 \sum_{1 \le i < j \le n} |x_i - y_i||x_j - y_j| \\ &\ge \sum_{i=1}^n |x_i - y_i|^2 = d_2(x,y)^2. \end{aligned}$$

Hence, we have $d_1(x,y) \ge d_2(x,y)$.

Now we show that $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$. Note that

$$d_2(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}, \quad d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|.$$

For each i, we know

$$|x_i - y_i| \le d_\infty(x,y),$$

so

$$d_2(x,y)^2 \le \sum_{i=1}^n d_\infty(x,y)^2 = n d_\infty(x,y)^2,$$

so $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$. ■

Definition 2.1.12 (Discrete metric). Let X be any set, define the discrete metric:

$$d_{\text{disc}} : X \times X \to \{0, 1\}$$

where

$$d_{\text{disc}}(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \ne y. \end{cases}$$

Why this is a metric? Because

- $d_{\text{disc}}(x,y) \ge 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$.
- $d_{\text{disc}}(x,y) = d_{\text{disc}}(y,x)$ by definition.
- $d_{\text{disc}}(x,z) \le d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z)$?

proof of triangle inequality in discrete metric. We first consider the case that $x = z$, then

$$d_{\text{disc}}(x, z) = 0,$$

so it is obviously that the triangle inequality is true.

Now if $x \neq z$, then either $y \neq z$ or $y \neq x$ must happen, so the triangle inequality must be true. ■

Example. We can define

$$d(x, x) = 0, \quad d(x, y) = \text{minimal length of a path from } x \text{ to } y,$$

then this is also a metric.

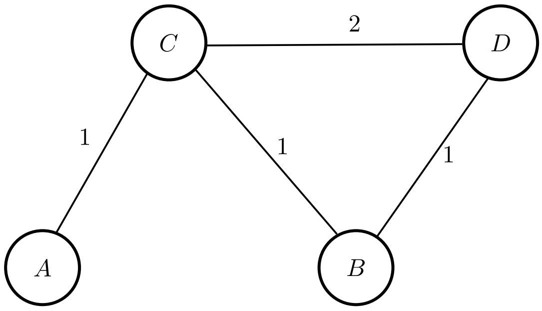

Figure 2.1: Graph metrics

Definition 2.1.13 (Convergence in metric space)**.** Let m be an integer, (X, d) be a metric space, and let $\left(X^{(n)}\right)_{n=m}^{\infty}$ be a sequence of points in X. Let $x \in X$. We say that $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges to x with respect to d iff

$$\lim_{n\to\infty} d\left(X^{(n)}, x\right) = 0,$$

where $\lim_{n\to\infty} d\left(X^{(n)}, x\right) = 0$ iff for every $\varepsilon > 0$, $\exists N \geq m$ s.t. $d\left(X^{(n)}, x\right) \leq \varepsilon$ for all $n \geq N$.

Notation. We also write $\lim_{n\to\infty} X^{(n)} = x$ in (X, d).

Remark. Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges to x in (X, d), then $\left(X^{(n)}\right)_{n=m_1}^{\infty}$ also converges to x in (X, d) if $m_1 \geq m$.

Example. Let $\left(X^{(n)}\right)_{n=1}^{\infty}$ denote the sequence $X^{(n)} = \left(\frac{1}{n}, \frac{1}{n}\right)$ in $\mathbb{R}^2$, then what will this sequence converges to for different metric?

Proof.

- If the metric is d_1, then

$$d_1\left(X^{(n)}, (0, 0)\right) = \left|\frac{1}{n} - 0\right| + \left|\frac{1}{n} - 0\right| = \frac{2}{n},$$

so

$$\lim_{n\to\infty} d_1\left(X^{(n)}, (0, 0)\right) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

- If the metric is d_2, then

$$d_2\left(X^{(d)},(0,0)\right)=\sqrt{\left(\frac{1}{n}-0\right)^2+\left(\frac{1}{n}-0\right)^2}=\frac{\sqrt{2}}{n}.$$

  Hence, under l_2-metric $\left\{X^{(n)}\right\}$ also converges to 0.

- If the metric is d_∞, then

$$d_\infty\left(X^{(n)},(0,0)\right)=\max\left\{\left|\frac{1}{n}\right|,\left|\frac{1}{n}\right|\right\}=\frac{1}{n},$$

  so it also converges to 0.

- If the metric is discrete metric, then however, it will not converges to $(0,0)$ since

$$\lim_{n\to\infty} d_{\text{disc}}\left(X^{(n)},(0,0)\right)=\lim_{n\to\infty} d_{\text{disc}}\left(\left(\frac{1}{n},\frac{1}{n}\right),(0,0)\right)=1.$$

⊛

Definition. Let $f: X\to Y$ be a function with domain X and codomain Y. The range of $f=\{f(x)\mid x\in X\}\subseteq Y$.

Definition 2.1.14 (injective)**.** We say f is injective or one-to-one if for all $x_1,x_2\in X$, $f(x_1)=f(x_2)$ implies $x_1=x_2$.

Definition 2.1.15 (surjective)**.** We say f is surjective or onto if for every $y\in Y$, $\exists x\in X$ s.t. $f(x)=y$.

Definition 2.1.16 (bijective)**.** We say f is bijective if f is injective and surjective.

Corollary 2.1.1. If f is bijective, then there exists $f^{-1}: Y\to X$ defined by $f^{-1}(y)=x$ if $f(x)=y$. We also have

$$\begin{aligned} f\left(f^{-1}(y)\right)&=y\ \forall y\in Y\\ f^{-1}\left(f(x)\right)&=x\ \forall x\in X.\end{aligned}$$

Example. $\lim_{n\to\infty}\frac{1}{n}=0$ in $(\mathbb{R},d)$, where d is the standard metric in $\mathbb{R}$, which is defined by

$$d(x,y)=|x-y|.$$

But in different metric, $\lim_{n\to\infty}\frac{1}{n}$ may not be 0.

Proof. Define $f:[0,1]\to[0,1]$ defined by

$$f(x)=\begin{cases} x, & \text{if } 0<x<1;\\ 1, & \text{if } x=0;\\ 0, & \text{if } x=1.\end{cases}$$

f is bijective on $[0,1]$ to $[0,1]$

Define another metric d^1 on $[0,1]$ by

$$d^1(x,y) = d(f(x), f(y)).$$

We want to show that d^1 is also a metric on $[0,1]$.

- $d^1(x,y) = d(f(x), f(y)) = |f(x) - f(y)| \geq 0$
- $d^1(x,y) = 0$ iff $f(x) = f(y)$ iff $x = y$ since f is injective.
- The triangle inequality is trivially true since we can just use the triangle inequality in d.

In fact, $\lim_{n\to\infty} \frac{1}{n} = 1$ in $\left([0,1], d^1\right)$ since

$$\lim_{n\to\infty} d^1\left(\frac{1}{n}, 1\right) = \lim_{n\to\infty} d\left(\frac{1}{n}, 0\right) = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0.$$

⊛

2.2 Some point set topology of metric space

Definition 2.2.1 (ball)**.** Let (X,d) be a metric space. let $x_0 \in X$ and $r > 0$. We define the ball $B_{(X,d)}(x_0, r)$ in X, centered at x_0 and with radius r in the metric d, to the set

$$B_{(X_0,d)}(X_0, Y) \coloneqq \{x \in X \mid d(x_0, x) < r\}.$$

Sometimes, we write it as $B_X(x_0, r)$ or $B(x_0, r)$.

Example. In $\mathbb{R}^2$,

$$B_{(\mathbb{R}^2, d_2)}\left((0,0), 1\right) = \left\{(x,y) \mid d_2((x,y),(0,0)) = \sqrt{x^2+y^2} < 1\right\},$$

and

$$B_{(\mathbb{R}^2, d_1)}((0,0), 1) = \{(x,y) \mid d_1((x,y),(0,0)) = |x| + |y| < 1\},$$

and

$$B_{(\mathbb{R}^2, d_\infty)}((0,0), 1) = \{(x,y) \mid d_\infty((x,y),(0,0)) = \max\{|x|, |y|\} < 1\},$$

also we can consider the d_{disc} case but I am too lazy to write it down.

Notation. Let $E \subseteq X$, we will write

$$X \setminus E \coloneqq \{x \in X \mid x \notin E\}.$$

Definition. Let (X,d) be a metric space and $E \subseteq X$. For a point $x_0 \in X$,

Definition 2.2.2 (interior point)**.** x_0 is an interior point of E if $\exists r > 0$ s.t. $B(x_0, r) \subseteq E$.

Definition 2.2.3 (exterior point)**.** x_0 is an exterior point of E if $\exists r > 0$ s.t. $B(x_0, r) \subseteq X \setminus E$.

Definition 2.2.4 (boundary point)**.** x_0 is a boundary point of E if it is neither an interior point nor an exterior point of E.

Proposition 2.2.1. x_0 is a boundary point of E iff for all $r > 0$, $B(x_0, r) \cap E \neq \varnothing$ and $B(x_0, r) \cap (X \setminus E) \neq \varnothing$.

Lecture 4

11 Sep. 10:20

Theorem 2.2.1. Let (X, d_1) and (X, d_2) be metrics on X, and suppose d_1 and d_2 are Lipschitz equivalent, then for any sequence $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$, then for any $x \in X$

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_1) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_2).$$

Proof. Since d_1, d_2 are Lipschitz equivalent, so there exists $c_1, c_2 > 0$ s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y).$$

$(\Rightarrow)$ Given $\frac{\varepsilon}{c_2} > 0$, since $\lim_{n\to\infty} x^{(n)} = x$ in (X, d_1), so there exists N s.t. $N \geq m$ and

$$d_1(x^{(n)}, x) \leq \frac{\varepsilon}{c_2} \text{ for } n \geq N.$$

This implies $d_2(x^{(n)}, x) \leq c_2 d_1(x^{(n)}, x) \leq \varepsilon$ for $n \geq N$, which means

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_2).$$

$(\Leftarrow)$ Similar.

■

Remark. On $\mathbb{R}^n$, the metrics d_1, d_2, d_∞ are Lipschitz equivalent, that is,

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_1) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_2) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_\infty)$$

Proposition 2.2.2. Let (X, d_{disc}) be a discrete metric space, and $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$. Then

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_{\text{disc}}) \Leftrightarrow \exists N \geq m \text{ s.t. } x^{(n)} = x \text{ for } n \geq N.$$

Proof. $(\Leftarrow)$ Easy.

$(\Rightarrow)$ Given $\frac{1}{2} > 0$, there exists $N \geq m$ s.t. $d(x_n, x) < \frac{1}{2}$ for $n \geq N$, but $d(x_n, x) < \frac{1}{2}$ implies $d(x_n, x) = 0$, which means $x_n = x$ for all $n \geq N$.

■

Definition. We define the interior, exterior, and boundary point again.

Definition 2.2.5. The set of interior points is denoted by

$$\text{Int}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq E\}.$$

Definition 2.2.6. The set of exterior points is denoted by

$$\text{Ext}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq X \setminus E\}.$$

Definition 2.2.7. A point is a boundary points if it is neithe an interior point nor an exterior point, and we define

$$\partial E = \{x \in X \mid x \notin \text{Int}(E) \text{ and } x \notin \text{Ext}(E)\}.$$

Remark.

1. $$x_0 \notin \text{Int}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (X \setminus E) \neq \varnothing.$$

2. $$x_0 \notin \text{Ext}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (E) \neq \varnothing.$$

3. $\text{Int}(X \setminus E) = \text{Ext}(E)$.

4. $\partial E = \partial(X \setminus E)$ since

$$x_0 \in \partial E \Leftrightarrow x \notin \text{Int}(E) \text{ and } \text{Ext}(E) \Leftrightarrow x_0 \notin \text{Int}(E) \text{ and } x_0 \notin \text{Int}(X \setminus E).$$

Also,

$$x_0 \in \partial(X \setminus E) \Leftrightarrow x \notin \text{Int}(X \setminus E) \text{ and } \text{Ext}(X \setminus E) \Leftrightarrow x_0 \notin \text{Int}(X \setminus E) \text{ and } x_0 \notin \text{Int}(E).$$

Hence, acutually $\partial E = \partial(X \setminus E)$.

Proposition 2.2.3.

$$x_0 \in \partial E \Leftrightarrow \text{ For any } r > 0, B_X(x_0, r) \cap E \neq \varnothing \text{ and } B_X(x_0, r) \cap (X \setminus E) \neq \varnothing$$

Example. Let $(\mathbb{R}, d)$ be the usual metric on $\mathbb{R}$, where

$$d(x, y) = |x - y|.$$

Then, we know in this space,

$$\begin{aligned} B_{\mathbb{R}}(x_0, r) &= \{x \in \mathbb{R} \mid d(x, x_0) < r\} \\ &= \{x \in \mathbb{R} \mid |x - x_0| < r\} \\ &= \{x \in \mathbb{R} \mid -r + x_0 < x < r + x_0\}. \end{aligned}$$

Hence, suppose $E = [1, 2)$, then $\text{Int}(E) = (1, 2)$ since we know $B(x_0, r) = (x_0 - r, x_0 + r)$, so for all $x \in (1, 2)$, we know there is an open ball $B(x_0, r) \subseteq [1, 2)$ for some $r > 0$. Also, consider the endpoint $1, 2$, we can verify that these two points are not interior points. Besides, consider the points not in $[1, 2]$, it is trivial that they cannot be interior points.

Example. We consider (X, d_{disc}). Let $E \subseteq X$. If $x \in E$, we know

$$B\left(x, \frac{1}{2}\right) = \left\{y \mid d(y, x) < \frac{1}{2}\right\} = \{x\} \subseteq E.$$

Hence, $E \subseteq \text{Int}(E)$. Besides, for all $x \in \text{Int}(E)$, we know there exists $r > 0$ s.t. $B(x_0, r) \subseteq E$, also we know $x_0 \in B(x_0, r) \subseteq E$, so $x_0 \in E$, and thus $\text{Int}(E) \subseteq E$. Hence, $E = \text{Int}(E)$. Similarly, $\text{Int}(X \setminus E) = X \setminus E$. Suppose there is a $x \in X$ s.t. $x \in \partial E$, then $x \notin \text{Int}(E) = E$ and $x \notin \text{Ext}(E) = \text{Int}(X \setminus E) = X \setminus E$, so such x does not exist.

Definition 2.2.8 (Closure). Let (X, d) be a metric space, and let $E \subseteq X$ and $x_0 \in X$. We say x_0 is a adherent point of E if for every $r > 0$, $B(x_0, r) \cap E \neq \varnothing$. The set of adeherent points is called the closure of E, and denoted by $\overline{E}$.

Proposition 2.2.4 (TFAE).

(a) x_0 is an adherent point of E.

(b) x_0 is either an interior point or a boundary point of E.

(c) $\exists$ a sequence $\left\{X^{(n)}\right\}_{n=1}^{\infty}$ in E which converges to x_0 in (X, d).

proof from (a) to (b). Suppose $x_0 \in \overline{E}$, then $B(x_0, r) \cap E \neq \varnothing$ for all $r > 0$. If $\exists s > 0$ s.t. $B(x_0, s) \subseteq E$, then $x_0 \in \operatorname{Int}(E)$. If such s does not exists, then we know

$$B(x_0, r) \cap E \neq \varnothing \text{ and } B(x_0, r) \cap (X \setminus E) \neq \varnothing \text{ for all } r > 0,$$

so we can use Proposition 2.2.1 to conclude that x_0 must be a boundary point. ■

proof from (b) to (c). Since either $x_0 \in \operatorname{Int}(E)$ or $x_0 \in \partial E$. If $x_0 \in \operatorname{Int}(E)$, then $x_0 \in E$, then we can choose $X^{(n)} = x_0$ for all $n \geq 1$. If $x_0 \in \partial E$, then given $n \in \mathbb{N}$, $\exists x_n \in B\left(x_0, \frac{1}{n}\right) \cap E \neq \varnothing$. Hence, $x_n \in E$ and $d(x_n, x_0) < \frac{1}{n}$. Pick such x_n to form $\left\{X^{(n)}\right\}_{n=1}^{\infty}$, then we know this sequence converges to x_0. ■

proof from (c) to (a). Suppose $\left\{X^{(n)}\right\} \subseteq E$ s.t. $\lim_{n\to\infty} d\left(X^{(n)}, x_0\right) = 0$, then we want to show $x_0 \in \overline{E}$. Given any $r > 0$, choose $N \geq 1$ s.t.

$$d\left(X^{(n)}, x_0\right) < r \text{ when } n \geq N.$$

This implies for $n \geq N$, $X^{(n)} \in E$ and $X^{(n)} \in B(x_0, r)$, so we know $E \cap B(x_0, r) \neq \varnothing$ for all $r > 0$, which means $x_0 \in \overline{E}$. ■

Remark. The equation (a) and (b) implies $\overline{E} = \operatorname{Int}(E) \cup \partial E$.

Corollary 2.2.1. $\overline{E} = \operatorname{Int}(E) \cup \partial E$.

Theorem 2.2.2. Let (X, d) be a metric space and $E \subseteq X$. Then,

$$X = \operatorname{Int}(E) \uplus \operatorname{Ext}(E) \uplus \partial E$$

Remark. ∂E could be empty. (See previous example.)

Corollary 2.2.2. Let (X, d) be a metric space and $E \subseteq X$. Then

$$\overline{E} = \operatorname{Int}(E) \cup \partial E = X \setminus \operatorname{Ext}(E).$$

Lemma 2.2.1. $\overline{E} = E \cup \partial E$

Proof. We first show that $E \cup \partial E \subseteq \overline{E}$. For every point $x \in E$, we know $x \in B(x, r)$ for all $r > 0$, so $B(x, r) \cap E \neq \varnothing$. Also, by definition, we know $\partial E \subseteq \overline{E}$, so we're done.

Next, we show that $\overline{E} \subseteq E \cup \partial E$. For every $x \in \overline{E}$, if $x \in E$, then $x \in E \cup \partial E$. If not, since $x \in \overline{E}$, so $B(x, r) \cap E \neq \varnothing$ for all $r > 0$. Also, since $x \notin E$, and $x \in B(x, r)$, so $B(x, r) \cap (X \setminus E) \neq \varnothing$, otherwise $x \in B(x, r) \subseteq E$, which is a contradiction. Now we know for every $r > 0$, $B(x, r) \cap E \neq \varnothing$ and $B(x, r) \cap (X \setminus E) \neq \varnothing$, so $x \in \partial E$. ■

Lemma 2.2.2 (Discarded)**.** If $x \in \operatorname{Int}(E)$, then $x \in E$. In other words, $\operatorname{Int}(E) \subseteq E$.

Proof. If $x \in \operatorname{Int}(E)$, then there exists $r > 0$ s.t. $B(x, r) \subseteq E$, and thus $x \in B(x, r) \subseteq E$, which means $x \in E$. ■

Note. I thought we need Lemma 2.2.2 to prove Theorem 2.2.3, but I found it needless. Nevertheless, I still want to keep it since I think it is useful in some elsewhere.

Definition 2.2.9. Let (X, d) be a metric space and $E \subseteq X$. We say E is closed if $\partial E \subseteq E$. We say E is open if it doesn't contain any boundary points i.e. $\partial E \cap E = \varnothing$.

Theorem 2.2.3. E is closed if and only if $\overline{E} = E$.

Proof.

$$E \text{ is closed } \Rightarrow \partial E \subseteq E \Rightarrow \overline{E} = E \cup \partial E = E.$$
$$E = \overline{E} = E \cup \partial E \Rightarrow \partial E \subseteq E \Rightarrow E \text{ is closed.}$$

■

Theorem 2.2.4. E is open. $\Leftrightarrow \operatorname{Int}(E) = E$.

proof of ($\Rightarrow$) **.** E is open means $\partial E \cap E = \varnothing$. Fix $x \in E$, since $x \notin \partial E$, so $\exists r > 0$ s.t. $B(x, r) \cap E = \varnothing$ or $B(x, r) \cap (X \setminus E) = \varnothing$. Since $x \in E$ and $x \in B(x, r)$, so $B(x, r) \cap (X \setminus E) = \varnothing$, which means $B(x, r) \subseteq E$, so $x \in \operatorname{Int}(E)$. Now we know $E \subseteq \operatorname{Int}(E)$. Also, we know $\operatorname{Int}(E) \subseteq E$ by Lemma 2.2.2. Hence, $\operatorname{Int}(E) = E$. ■

proof of ($\Leftarrow$) **.** If $\operatorname{Int}(E) = E$, then given any $x \in E = \operatorname{Int}(E)$, there exists $r > 0$ s.t. $B(x, r) \subseteq E$. Hence, $B(x, r) \cap (X \setminus E) = \varnothing$, so $x \notin \partial E$, and thus $E \cap \partial E = \varnothing$. ■

Appendix