

# Linear Algebra I

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### **Abstract**

The lecture note of Linear Algebra I by professor 余正道.

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# Chapter 1

## Vector Space

### Lecture 1

#### 1.1 Introduction to vector and vector space

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In high school, our vectors are in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , and we have define the addition and scalar multiplication of vectors.

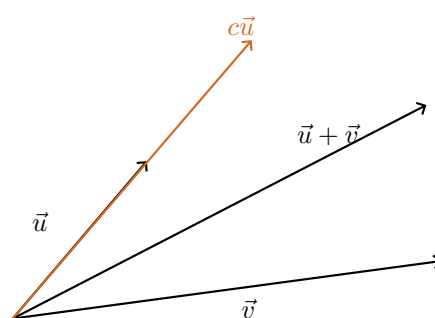


Figure 1.1: Vectors in  $\mathbb{R}^2$

**Example.**  $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}\}$

With this type of space, we can define addition and multiplication as

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = \{a_1 + b_1, a_2 + b_2, \dots, a_n + b_n\}$$
$$\alpha \cdot (a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$$

Also, if we define a space:

**Example.**  $V = \{\text{function } f : (a, b) \rightarrow \mathbb{R}\}$ , where  $(a, b)$  is an open interval.

then this can also be a vector space after defining addition and multiplication.

**Note.** In a vector space, we have to make sure the existence of 0-element, which means  $0(x) = 0$ .

Now we give a more abstract example:

**Example.** Suppose  $S$  is any set, then define  $V = \{\text{all functions from } S \text{ to } \mathbb{R}\}$

If we define  $(f + g)(s) = f(s) + g(s)$  and  $(\alpha \cdot f)(s) = \alpha \cdot f(s)$ , and  $0(s) = 0$ , then this is also a vector space.

### Put some linear conditions

**Example.** In  $\mathbb{R}^n$ , fix  $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ , if we define

$$W = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \dots + a_nx_n = 0\},$$

then this is also a vector space.

However, if we have

$$W' = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = 1\},$$

then this is not a vector space because it is not close.

**Example.** In  $V = \{(a, b) \rightarrow \mathbb{R}\}$  or  $W_1 = \{\text{polynomial defined on } (a, b)\}$ , these are both vector space.

**Remark.** In the later course, we will learn that  $W_1$  is a subspace of  $V$ .

**Example.** If we furtherly defined  $W_1^{(k)} = \{\text{polynomial degree} \leq k\}$ , then this is also a vector space.

**Remark.**  $W_1^{(k)}$  is actually isomorphic to  $\mathbb{R}^{k+1}$  since

$$a_0 + a_1x + a_2x^2 + \dots + a_kx^k \leftrightarrow (a_0, a_1, a_2, \dots, a_k).$$

**Example.**  $W_2 = \{\text{continuous function on } (a, b)\}$  and  $W_3 = \{\text{differentiable functions}\}$  are also both vector spaces.

**Example.**  $W_4 = \left\{\frac{d^2f}{dx^2} = 0\right\}$  and  $W_5 = \left\{\frac{d^2f}{dx^2} = -f\right\}$  are both vector spaces.

**Proof.**

$$\begin{aligned} W_4 &= \{a_0 + a_1x\} \\ W_5 &= \{a_1 \cos x + a_2 \sin x\} \end{aligned}$$

⊛

## 1.2 Formal definition of vector spaces

### 1.2.1 Vector Spaces Over $\mathbb{R}$

**Definition 1.2.1.** Suppose  $V$  is a non-empty set equipped with

- addition:  $V \times V \rightarrow V$ , that is, given  $u, v \in V$ , defining  $u + v \in V$
- scalare multiplication:  $\mathbb{R} \times V \rightarrow V$ , that is, given  $\alpha \in \mathbb{R}$  and  $v \in V$ , we need to have  $\alpha v \in V$

Also, we need some good properties or conditions

- For addition,
  - $u + v = v + u$
  - $(u + v) + w = u + (v + w)$
- There exists  $0 \in V$  such that  $u + 0 = u = 0 + u$
- Given  $v \in V$ , there exists  $-v \in V$  such that  $v + (-v) = 0 = (-v) + v$

- For scalar multiplication,
  - $1 \cdot v = v$  for all  $v \in V$
  - $(\alpha\beta)v = \alpha \cdot (\beta v)$  for all  $\alpha, \beta \in \mathbb{R}$  and  $v \in V$ .
- For addition and multiplication,
  - $\alpha(u + v) = \alpha u + \alpha v$
  - $(\alpha + \beta)u = \alpha u + \beta u$

## Lecture 2

### 1.3 Vector Space over general field

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Now we introduce the concept of field.

**Definition 1.3.1 (Field).** A set  $F$  with  $+$  and  $\cdot$  is called a **field** if

- $\alpha + \beta = \beta + \alpha$  and  $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ .
- There exists  $0 \in F$  such that  $\alpha + 0 = 0 + \alpha = \alpha$ .
- For  $\alpha \in F$ , there exists  $-\alpha$  such that  $\alpha + (-\alpha) = 0$ .
- $\alpha\beta = \beta\alpha$  and  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
- $\exists 1$  such that  $1 \neq 0$  and  $1 \cdot \alpha = \alpha$ .
- For  $\alpha \neq 0$ ,  $\exists \alpha^{-1} \in F$  such that  $\alpha\alpha^{-1} = 1$ .
- $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

**Example.**  $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$  are all fields but  $\mathbb{Z}$  is not.

**Example.**  $\{0, 1\}$  is also a field.

Now we know the concept of field, so we can make a vector space over a field.

**Theorem 1.3.1 (Cancellation law).** Suppose  $v_1, v_2, w \in V$ , a vector space, then if  $v_1 + w = v_2 + w$ , then  $v_1 = v_2$ .

**Proof.**

$$v_1 = v_1 + (w + (-w)) = (v_1 + w) + (-w) = (v_2 + w) + (-w) = v_2 + (w + (-w)) = v_2.$$

■

**Theorem 1.3.2.** The zero vector  $0$  is unique.

**Proof.** Suppose we have  $0, 0'$  both zero vector, then for some  $0 = 0 + 0' = 0'$ . ■

**Theorem 1.3.3.** For any  $v \in V$ ,  $0 \cdot u = 0$ .

**Proof.**  $0 \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u$ , so  $0 = 0 \cdot u$  by [cancellation law](#). ■

**Theorem 1.3.4.**  $(-1) \cdot u = -u$ .

**Theorem 1.3.5.** Given any  $u \in V$  is unique,  $-u$  is unique.

## 1.4 Subspaces

**Definition 1.4.1 (subspace).** Let  $V$  be a vector space. A non-empty subset  $W \subseteq V$  is called a subspace of  $V$  if  $W$  is itself a vector space under  $+$  and  $\cdot$  on  $V$ .

**Example.**  $M_n(F) = \{n \times n \text{ matrix with entries in } F\}$  is a vector space, and

$$U_n(F) = \left\{ \begin{pmatrix} a_{11} & & & \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \right\}$$

is a subspace of  $M_n(F)$ .

**Proposition 1.4.1.** Suppose  $V$  is a vector space, and  $W \subseteq V$  is non-empty, then

$W$  is a subspace  $\Leftrightarrow$  For  $u, v \in W, \alpha \in F$ , we have  $u + v \in W$  and  $\alpha \cdot u \in W$ .

**proof of  $\Rightarrow$ .** Clear. ■

**proof of  $\Leftarrow$ .** First, we would want to check  $0 \in W$ , and we can pick any  $u \in W$ , and pick  $\alpha = -1$ , so we know  $-u \in W$ , and thus  $0 = u + (-u) \in W$ . ■

**Corollary 1.4.1.** If we want to check  $W$  is a subspace, we just need to check for  $u, v \in W, \alpha \in F$ ,  $u + \alpha v \in W$  or not.

## 1.5 Linear Combination

**Definition 1.5.1 (Linear combination).** Given  $v_1, v_2, \dots, v_n \in V$ , a linear combination of them is a vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

**Proposition 1.5.1.** Given  $v_1, v_2, \dots, v_n \in V$ ,

1.  $W = \{\text{all linear combinations of } v_1, \dots, v_n\}$  is a subspace.
2. This subspace is the smallest subspace containing  $v_1, \dots, v_n$ . That is, if  $W' \subseteq V$  is a subspace containing  $v_1, \dots, v_n$ , then  $W \subseteq W'$ .

**Notation.**  $\text{span}\{v_1, v_2, \dots, v_n\} = \{\text{all linear combinations of } v_1, v_2, \dots, v_n\}$

## 1.6 Linearly independent

**Definition.** Now we talk about the linear dependence and linear independence.

**Definition 1.6.1 (Linearly dependent).**  $v_1, v_2, \dots, v_n$  are linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

for some  $\alpha_1, \alpha_2, \dots, \alpha_n$  not all zeros.

**Definition 1.6.2 (Linearly independent).**  $v_1, v_2, \dots, v_n$  are called linearly independent if they are not linearly dependent.

**Corollary 1.6.1.** Say  $\alpha_i \neq 0$ , then  $v_i \in \text{span}\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_k\}$  suppose the corresponding  $\alpha_i$  of  $\hat{v}_1, \dots, \hat{v}_k$  are not zeros.

**Corollary 1.6.2.** Linearly independent means if  $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ , then  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ .

**Corollary 1.6.3.** Linearly independent means if  $\sum \alpha_i v_i = \sum \beta_i v_i$ , then  $\alpha_i = \beta_i$  for all  $i$ .

**Example.**

- $v \in V$  is linearly independent iff  $v \neq 0$ .
- $v, w \in V$  are linearly independent iff  $v$  is not a scalar of  $w$  and  $w$  is not a scalar of  $v$ .

**Lemma 1.6.1.**  $v_1, \dots, v_n$  are linearly independent iff  $v_i \notin \text{span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ .

## 1.7 Basis

**Definition.** We now talking about basis

**Definition 1.7.1 (Basis).**  $B = \{v_1, v_2, \dots, v_n\}$  is called a basis of  $V$  if  $B$  spans  $V$  and  $B$  is linearly independent.

**Definition 1.7.2 (Dimension).** In this case,  $n$  is called the dimension of  $V$ , and denoted by  $\dim V$ .

**Notation.**  $\text{span}\{v_1, v_2, \dots, v_n\} = \langle v_1, v_2, \dots, v_n \rangle$

**Notation.**  $\text{span}(S) = \langle S \rangle$

**Theorem 1.7.1.** For any  $v \in V$ , it has a unique expression  $v = \sum_{i=1}^n \alpha_i v_i$ .

## Lecture 3

**As previously seen.** A basis of a vector space  $V$  is a set  $\{v_1, v_2, \dots, v_n\}$  that is linearly independent and simultaneously spans  $V$ . That is, suppose we have  $\sum a_i v_i = 0$  for some scalars  $a_i$ , then  $a_i = 0$  for all  $i$ . Also, we call the number  $n$ , the dimension of  $V$ .

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**Example.** Suppose we have  $V = F^n = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \mid \alpha_i \in F\}$ , then we have a **standard basis**, which is

$$\begin{aligned} e_1 &= (1, 0, \dots, 0) \\ e_2 &= (0, 1, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, \dots, 1) \end{aligned}$$

since  $\{e_i\}_{i=1}^n$  is linearly independent and for every  $\vec{a} = (a_1, \dots, a_n)$ , we know

$$\vec{a} = \sum_{i=1}^n a_i e_i.$$

**Example.** Suppose

$$V = M_{n \times n}(F) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \ddots & & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \dots & & \alpha_{nn} \end{pmatrix} \right\},$$

then we know

$$\{e_{ij}\}_{1 \leq i, j \leq n} = \begin{pmatrix} 0 & 0 & & \\ 0 & & & \\ & & 1 & \\ 0 & & & 0 \\ 0 & & & 0 \end{pmatrix},$$

where the 1 is in the  $i$ -th row and  $j$ -th column.

**Theorem 1.7.2.** Suppose  $V$  is a vector space, and  $V = \langle v_1, v_2, \dots, v_n \rangle$  and  $\{w_1, w_2, \dots, w_m\}$  is linearly independent, then  $m \leq n$ . Furthermore, one can make

$$\langle w_1, w_2, \dots, w_m, v_{m+1}, \dots, v_n \rangle = V$$

after rearrangement of  $v_1, \dots, v_n$ .

**Proof.** We can do induction on  $m$ . It is trivial that  $m = 0$  is true. Suppose the statement holds for a fixed  $m$  with  $m \leq n$ . Let  $w_1, w_2, \dots, w_{m+1}$  be linearly independent. In particular,  $w_1, w_2, \dots, w_m$  is linearly independent.

**Claim.**  $m + 1 \leq n$ .

**Proof.** Otherwise, if  $m + 1 > n$ , then since  $m \leq n$ , so  $m = n$ . Hence, by induction hypothesis, we know  $\langle w_1, w_2, \dots, w_m \rangle = V$ . However, by [Lemma 1.7.1](#) and the note following it, we know

$$\{w_1, w_2, \dots, w_m\} \cup \{w_{m+1}\}$$

can not be linearly independent since  $w_{m+1} \in V = \langle w_1, \dots, w_m \rangle$ . ⊗

Now we know  $m + 1 \leq n$ . By induction hypothesis, we know

$$\langle w_1, w_2, \dots, w_m, v_{m+1}, \dots, v_n \rangle = V$$

**Claim.** One of  $v_{m+1}, \dots, v_n$  can be replaced by  $w_{m+1}$ .

**Proof.** Since

$$w_{m+1} = \sum_{i=1}^m \alpha_i w_i + \sum_{j=m+1}^n \beta_j v_j.$$

Trivially, one of  $\beta_j \neq 0$ , say  $\beta_{m+1} \neq 0$ . Check

$$\langle w_1, \dots, w_m, w_{m+1}, v_{m+2}, \dots, v_n \rangle = V.$$

⊛

■

**Corollary 1.7.1.** If  $\{v_1, v_2, \dots, v_n\}$  and  $\{w_1, w_2, \dots, w_m\}$  are bases of  $V$ , then  $n = m$ .

**Remark.** Corollary 1.7.1 tells us  $\dim V$  is well-defined, which means the size of the bases of a vector space is unique.

**Corollary 1.7.2.** Suppose  $\dim V = n$ , then if  $\langle v_1, v_2, \dots, v_m \rangle = V$ , then  $m \geq n$ . If  $\{w_1, w_2, \dots, w_m\}$  is linearly independent, then  $m \leq n$ . Also, any  $\{v_i\}_{i=1}^m$  with  $m > n$  is linearly dependent.

**Lemma 1.7.1.** Suppose  $v_1, v_2, \dots, v_n$  is linearly independent. If  $w \notin \langle v_1, v_2, \dots, v_n \rangle$ , then

$$\{v_1, v_2, \dots, v_n, w\}$$

is linearly independent.

**Proof.** Suppose  $\sum_{i=1}^n \alpha_i v_i + \alpha_{n+1} w = 0$ , then if  $\alpha_{n+1} = 0$ , we know  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$  since  $\{v_i\}_{i=1}^n$  is linearly independent. If  $\alpha_{n+1} \neq 0$ , then  $w = \frac{1}{\alpha_{n+1}} \sum_{i=1}^n \alpha_i v_i \in \langle v_1, v_2, \dots, v_n \rangle$ , which is a contradiction. ■

**Note.** The reverse of Lemma 1.7.1 is still correct and is trivial. That is, if  $w \notin \{v_1, \dots, v_n\}$  and  $\{v_1, v_2, \dots, v_n, w\}$  is linearly independent, then  $\{v_1, \dots, v_n\}$  is linearly independent.

**Corollary 1.7.3.** If  $W \subseteq V$  is a subspace of  $V$ , then  $\dim W \leq \dim V$ .

**Proof.** If  $\dim V = n$ , and  $\{w_i\}_{i=1}^m$  is a basis of  $W$ , then this basis is linearly independent in  $V$ , which means  $m \leq n$  by Theorem 1.7.2. ■

**Corollary 1.7.4.** If  $v_1, v_2, \dots, v_m$  is linearly independent, then  $\{v_1, v_2, \dots, v_m\}$  forms a basis after adding some  $v_{m+1}, \dots, v_n$  to it.

**Theorem 1.7.3 (Dual version).** If  $\langle v_1, v_2, \dots, v_n \rangle = V$ , then  $\{v_1, v_2, \dots, v_m\}$  forms a basis after rearrangement, where  $m \leq n$ .

**Remark.** Most of the time, we consider finite-dimensional vector spaces.

**Remark (Examples of  $\infty$ -dim vector space).**

•

$$V = \{\text{all polynomials over } F\} = F[x] = \{a_0 + a_1x + \dots + a_nx^n \text{ for some } n \text{ where } a_i \in F\}.$$

•

$$W = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\}.$$

Notice that

$$W' = \{\text{convergent sequence}\} \subseteq W.$$

and

$$W'' = l^2 = \left\{ (a_i) \mid \sum_{i=0}^{\infty} a_i^2 \text{ finite} \right\} \subseteq W'$$

**Remark.** We define  $\dim \{0\} = 0$ , which is the only vector space with dimension 0, and we define  $\langle \emptyset \rangle = \{0\}$ , which means  $\emptyset$  is the basis of  $\{0\}$ .

**Note.** We call a subspace  $W \subsetneq V$  is proper.

## 1.8 More on subspaces

**Theorem 1.8.1.** If  $W_1$  and  $W_2$  are subspace of  $V$ , then  $W_1 \cap W_2$  is a subspace.

**Theorem 1.8.2.** If  $W_1, W_2$  are subspaces of  $V$ , then  $W_1 + W_2$  is still a subspace of  $V$ .

**Remark.** If  $W_1, W_2$  are subspaces of  $V$ , then  $W_1 \cup W_2$  may not be a subspace. (See HW1).

**Remark.** In fact,  $W_1 \cap W_2$  is the largest subspaces contained in  $W_1$  and  $W_2$ .

**Remark.** In fact,  $W_1 + W_2$  is the smallest subspace containing both  $W_1$  and  $W_2$ .

**Corollary 1.8.1.** Suppose  $S$  is the index set, and for all  $i \in S$ ,  $W_i$  is a subspace of  $V$ , then

$$\bigcap_{i \in S} W_i = \{v \in V \mid v \in W_i \forall i\}$$

is also a subspace of  $V$ .

**Corollary 1.8.2.** Suppose  $S$  is the index set, and for all  $i \in S$ ,  $W_i$  is a subspace of  $V$ , then

$$\sum_{i \in S} W_i = \{w_{i_1} + w_{i_2} + \dots + w_{i_n} \text{ for some } i_j \in S\}$$

is also a subspace of  $V$ .

**Proposition 1.8.1 (Dimension theorem).** Suppose  $W_1, W_2 \subseteq V$  are subspaces of  $V$ , then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

# Appendix