Remark 5.2.11. Define, for each $n \in \mathbb{N}$, a function $f_n : [0,1] \to \mathbb{R}$ by restricting it to $[0,1]$ and then extending periodically. On $[0,1]$ set

$$f_n(x) := \begin{cases} 0, & 0 \le x \le \dfrac{1}{2} - \dfrac{1}{n}, \\ \dfrac{n}{2}\left(x - \left(\dfrac{1}{2} - \dfrac{1}{n}\right)\right), & \dfrac{1}{2} - \dfrac{1}{n} < x < \dfrac{1}{2} + \dfrac{1}{n}, \\ 1, & \dfrac{1}{2} + \dfrac{1}{n} \le x \le 1. \end{cases}$$

Each f_n is continuous on $[0,1]$. As $n \to \infty$,

$$f_n \longrightarrow f \quad \text{in } L^2([0,1]),$$

where

$$f(x) := \begin{cases} 0, & 0 \le x < \dfrac{1}{2}, \\ \frac{1}{2}, & x = \dfrac{1}{2}, \\ 1, & \dfrac{1}{2} < x \le 1. \end{cases}$$

(Any choice of value at $x = \frac{1}{2}$ gives the same L^2 class.) The limit f has a jump discontinuity at $x = \frac{1}{2}$, hence $f \notin C([0,1], \mathbb{R})$. Therefore $C([0,1], \mathbb{R})$ is not complete with respect to the L^2-metric.

To verify the L^2 convergence, note that

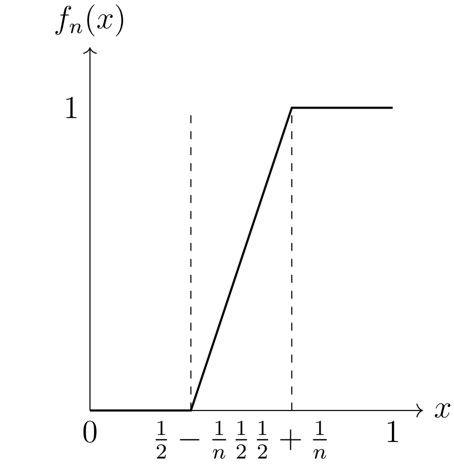
$$(f_n - f)(x) = \begin{cases} 0, & 0 \le x \le \dfrac{1}{2} - \dfrac{1}{n}, \\ \dfrac{n}{2}\left(x - \left(\dfrac{1}{2} - \dfrac{1}{n}\right)\right), & \dfrac{1}{2} - \dfrac{1}{n} < x < \dfrac{1}{2}, \\ 0, & x = \dfrac{1}{2}, \\ \dfrac{n}{2}\left(x - \left(\dfrac{1}{2} - \dfrac{1}{n}\right)\right) - 1, & \dfrac{1}{2} < x < \dfrac{1}{2} + \dfrac{1}{n}, \\ 0, & \dfrac{1}{2} + \dfrac{1}{n} \le x \le 1. \end{cases}$$

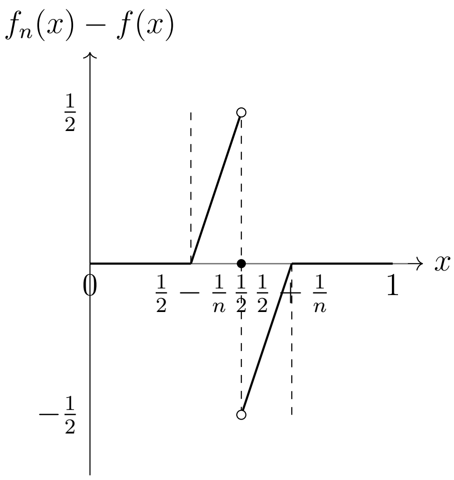
In particular, $|f_n(x) - f(x)| \le \frac{1}{2}$ for all $x \in [0,1]$. Thus

$$\int_0^1 |f_n(x) - f(x)|^2 \, dx \;\le\; \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \frac{1}{4} \, dx \;+\; \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \frac{1}{4} \, dx \;=\; \frac{1}{2n}.$$

Hence

$$\|f_n - f\|_{L^2} \longrightarrow 0 \qquad \text{as } n \to \infty.$$





We now modify the previous example by producing an *even*, 2-periodic version of the functions. Define g_n on $[-1, 1]$ by taking the even extension of f_n.

Definition 0.1. *For each $n \in \mathbb{N}$, define*

$$g_n : [-1, 1] \to \mathbb{R}$$

by

$$g_n(x) := \begin{cases} f_n(x), & 0 \le x \le 1, \\ f_n(-x), & -1 \le x < 0, \end{cases}$$

and extend periodically with period 2*, i.e.*

$$g_n(x+2) = g_n(x) \qquad \textit{for all } x \in \mathbb{R}.$$

Here $f_n : [0,1] \to \mathbb{R}$ is the piecewise linear function

$$f_n(x) := \begin{cases} 0, & 0 \le x \le \dfrac{1}{2} - \dfrac{1}{n}, \\ \dfrac{n}{2}\left(x - \left(\dfrac{1}{2} - \dfrac{1}{n}\right)\right), & \dfrac{1}{2} - \dfrac{1}{n} < x < \dfrac{1}{2} + \dfrac{1}{n}, \\ 1, & \dfrac{1}{2} + \dfrac{1}{n} \le x \le 1. \end{cases}$$

Since $g_n(x) = f_n(|x|)$ for all $x \in [-1,1]$, the pointwise limit $g := \lim_{n\to\infty} g_n$ is the even extension of the limit f, namely,

$$g(x) := \begin{cases} 0, & |x| < \dfrac{1}{2}, \\ \dfrac{1}{2}, & |x| = \dfrac{1}{2}, \\ 1, & \dfrac{1}{2} < |x| \le 1, \end{cases}$$

and extended 2-periodically to all of $\mathbb{R}$.

To verify convergence in $L^2([-1,1])$, note that $g_n(x) = f_n(|x|)$ and $g(x) = f(|x|)$. Hence

$$\|g_n - g\|_{L^2([-1,1])}^2 = 2\,\|f_n - f\|_{L^2([0,1])}^2.$$

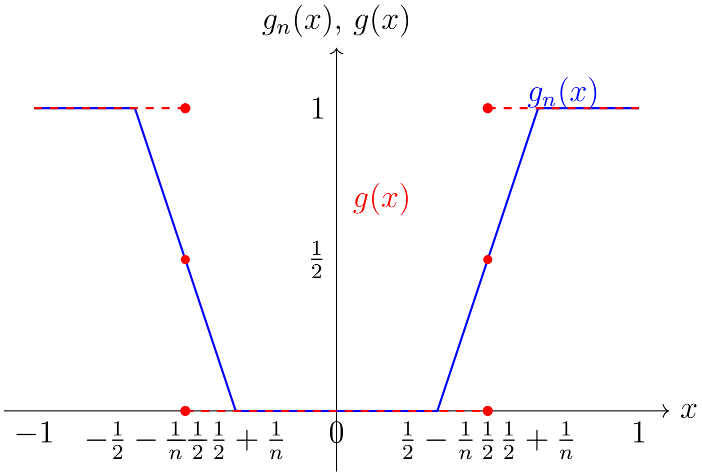
From the previous estimate,

$$\|f_n - f\|_{L^2([0,1])}^2 \le \frac{1}{2n},$$

we obtain

$$\|g_n - g\|_{L^2([-1,1])}^2 \le \frac{1}{n} \qquad \Longrightarrow \qquad \|g_n - g\|_{L^2([-1,1])} \to 0.$$

Thus (g_n) converges in $L^2([-1,1])$ to the discontinuous, even, 2-periodic limit g.



Let $h_n(x) = g_n(2x)$ and $h(x) = g(2x)$. Then for every $x \in \mathbb{R}$,

$$h_n(x+1) = g_n\big(2(x+1)\big) = g_n(2x+2) = g_n(2x) = h_n(x),$$

because each g_n is continuous and 2-periodic. Hence h_n is a continuous, 1-periodic function.

Moreover,

$$\begin{aligned}
\|h_n - h\|_{L^2([0,1])} &= \int_0^1 |g_n(2x) - g(2x)|^2\,dx \\
&= \frac{1}{2}\int_0^2 |g_n(y) - g(y)|^2\,dy \\
&= \frac{1}{2}\int_{-1}^1 |g_n(y) - g(y)|^2\,dy \\
&= \tfrac{1}{2}\,\|g_n - g\|_{L^2([-1,1])} \longrightarrow 0.
\end{aligned}$$

since $g_n \to g$ in $L^2([-1,1])$. Thus (h_n) converges in $L^2([0,1])$ to the discontinuous, 1-periodic limit function h.