4. We'll prove it by induction.

As $n=1$, $\phi \subseteq [1]$ is a symmetry chain partition of B_1.

As $n=k$, say $B_k = C_1 \dot{\cup} C_2 \dot{\cup} \cdots \dot{\cup} C_m$, where C_i is a symmetry chain $\forall i \in [m]$. Now, as $n=k+1$, we will construct two kinds of symmetry chain.

For any $i \in [m]$, say $C_i = \{S_{i,j}, S_{i,j+1}, \cdots, S_{i,k-j}\}$, where $|S_{i,l}| = l$ for all l and $S_{i,j} \subseteq S_{i,j+1} \subseteq \cdots \subseteq S_{i,k-j}$.

Now we start our construction.

- First kind: $C_i' = C_i \cup \{S_{i,k-j} \cup \{k+1\}\}$.

  Since $S_{i,k-j} \subseteq S_{i,k-j} \cup \{k+1\}$ and $|S_{i,k-j} \cup \{k+1\}| = (k+1)-j$, C_j' is also a symmetry chain.

- Second kind: $C_i'' = \{S \cup \{k+1\} : S \in C_i \setminus \{S_{i,k-j}\}\}$.

  Let we first assume C_i'' is nonempty (i.e. $|C_i| > 1$).

  Since $S_{i,j} \cup \{k+1\} \subseteq S_{i,j+1} \cup \{k+1\} \subseteq \cdots \subseteq S_{i,k-1-j} \cup \{k+1\}$ and $|S_{i,l} \cup \{k+1\}| = l+1$ for $l = j, j+1, \cdots, k-1-j$, C_i'' is also a symmetry chain (the size are $j+1, j+2, \cdots, k-j = (k+1)-(j+1)$).

  Now, W.L.O.G, say $C_1'', C_2'', \cdots, C_r''$ are nonempty, $r \leq m$.

Since $C_i \cap C_j = \phi$ $\forall i,j$, by our construction, for all i,j, we have $C_i' \cap C_j' = C_i' \cap C_j'' = C_i'' \cap C_j'' = \phi$.

Also, for any $A \in B_k$, $A \in C_i$ for unique i by the induction hypothesis. For A and $A \cup \{k+1\} \in B_{k+1}$, consider our construction, if A is not the maximum element of C_i, then $A \in C_i'$ and $A \cup \{k+1\} \in C_i''$; if A is the maximum element of C_i, then $A, A \cup \{k+1\} \in C_i'$. Hence, $B_{k+1} = \left(\dot{\bigcup}_{i \in [m]} C_i'\right) \dot{\cup} \left(\dot{\bigcup}_{j \in [r]} C_j''\right)$.

By induction, B_n can be partitioned into symmetry chains for any positive integer n.

6. (a) Let's consider M_ζ first.

Recall that $\zeta_P(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y \end{cases}$. By the Hasse diagram, we have

$$M_\zeta = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Since $M_\zeta \times M_\mu = M_\delta = I$, by computing the inverse of M_ζ, we have

$$M_\mu = M_\zeta^{-1} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Hence, we have

$\mu(i,j) = 0 \quad \forall i \not\leq j$, $\mu(k,k) = 1 \quad \forall k \in [6]$,

$\mu(1,3) = \mu(1,4) = \mu(2,4) = \mu(2,5) = \mu(4,6) = \mu(5,6) = -1$,

and $\mu(1,6) = \mu(2,6) = 1$.

(b) Since M_ζ is an upper triangular matrix, its determinant is the product of its diagonal entries.

$\Rightarrow \det M_\zeta = \prod_{i \in P} \zeta(i,i) = \prod_{i \in P} 1 = 1$

$\Rightarrow M_\mu = \frac{1}{\det M_\zeta} \cdot \operatorname{adj} M_\zeta = \operatorname{adj} M_\zeta$.

Note that $(\operatorname{adj} M_\zeta)_{ij} := (-1)^{i+j} \cdot M_{ji}$, where M_{ji} is the determinant of the $(n-1)\times(n-1)$ matrix obtained by removing the j-th row and the i-th column of M_ζ. Since the entries of M_ζ are neither 0 or 1, M_{ji} is an integer.

Hence, $M_\mu = \operatorname{adj} M_\zeta$ are the matrix with all entries being integers. That is, $\mu(i,j) \in \mathbb{Z} \quad \forall i,j \in P$.