Combinatorics I (Math 7701) Fall 2025
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Exercise Sheet 5

Due date: 15:30, Nov 25th, to be submitted on COOL.

Working with your partner, you should try to solve all of the exercises below. You should then submit solutions to four of the problems, with each of you writing two, clearly indicating the author of each solution. Note that each problem is worth 10 points, and starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting on COOL, please send your solutions by e-mail.

Exercise 1 Let $\pi(n) = |\{p \in [n] : p \text{ is prime}\}|$ be the prime number function, counting the number of primes in $[n]$. In this exercise you will determine the order of magnitude of $\pi(n)$.[1]

(a) Show that for every $m \in \mathbb{N}$ and every prime $p \in [m+1, 2m]$, $p | \binom{2m}{m}$.

(b) Deduce $\pi(n) = O\left(\frac{n}{\ln n}\right)$.

(c) Show that if p^k is a prime power such that $p^k | \binom{2m}{m}$, then $p^k \le 2m$.

(d) Deduce $\pi(n) = \Omega\left(\frac{n}{\ln n}\right)$.

Solution: (張沂魁)

(a) Since $p \mid \prod_{k=0}^{m-1}(2m-k)$ and $p \nmid m!$, and since

$$\binom{2m}{m} = \frac{\prod_{k=0}^{m-1}(2m-k)}{m!} \in \mathbb{Z},$$

so we have $p \mid \binom{2m}{m}$.

[1]You are asked to show $\pi(n) = \Theta\left(\frac{n}{\ln n}\right)$. However, more is known. The distribution of the prime numbers has long been central to number theory. Indeed, it was around 1800 that the legendary Legendre conjectured $\pi(n) \approx \frac{n}{\ln n - 1.08366}$. A similar conjecture was made by Gauss around the same time (when he was no older than 16). A few years later, Dirichlet offered the $\mathrm{Li}(n)$ approximation mentioned in the bonus problem.

In 1850, Chebyshev proved that $\frac{n}{\ln n}$ was the correct order of magnitude, and in 1896, Hadamard and de la Vallée Poussin independently extended the work of Riemann and proved the Prime Number Theory, which gives the asymptotics of $\pi(n)$. As conjectured, $\pi(n) \sim \frac{n}{\ln n}$. These proofs all made use of complex analysis.

Since then, several other proofs have been found. Around 1950, Selberg and Erdős found elementary (i.e. not using analysis) proofs. (There was a rather bitter dispute between the two regarding who should get credit for the result.) The simplest proof currently known is due to Newman, although this also uses some complex analysis.

(b) Suppose

$$\pi'(i) = \#\left\{p : p \text{ is prime}, 2^i + 1 \leq p \leq 2^{i+1}\right\},$$

then we know

$$\pi(n) \leq \sum_{i=0}^{\lfloor \log_2(n) \rfloor} \pi'(i)$$

since $[1, n] \subseteq \bigcup_{i=0}^{\lfloor \log_2(n) \rfloor} [2^i + 1, 2^{i+1}]$. Also, note that for all $m \in \mathbb{N}$

$$\prod_{\substack{p \text{ is prime} \\ p \in [m+1, 2m]}} p \mid \binom{2m}{m}$$

by (a), so

$$\prod_{\substack{p \text{ is prime} \\ p \in [m+1, 2m]}} p \leq \binom{2m}{m}.$$

Now since $\binom{2m}{m}$ counts the number of subsets of $[2m]$ of size m, so $\binom{2m}{m} \leq 2^{2m}$ because 2^{2m} counts the number of subsets of $[2m]$. Hence, we have

$$\prod_{\substack{p \text{ is prime} \\ p \in [m+1, 2m]}} p \leq \binom{2m}{m} \leq 2^{2m} = 4^m,$$

and if we take ln on the both sides, for all $m \geq 1$ we have

$$\sum_{\substack{p \text{ is prime} \\ p \in [m+1, 2m]}} \ln p \leq m \ln 4 \implies \#\left\{p : p \text{ is prime}, p \in [m+1, 2m]\right\} \cdot \ln(m+1) \leq m \ln 4$$

$$\implies \#\left\{p : p \text{ is prime}, p \in [m+1, 2m]\right\} \leq \frac{m \ln 4}{\ln(m+1)} \leq \frac{m \ln 4}{\ln m} = O\left(\frac{m}{\ln m}\right)$$

$$\implies \#\left\{p : p \text{ is prime}, p \in [m+1, 2m]\right\} = O\left(\frac{m}{\ln m}\right).$$

Hence, we know $\pi'(i) = O\left(\frac{2^i}{\ln 2^i}\right) = O\left(\frac{2^i}{\ln 2 i}\right) = O\left(\frac{2^i}{i}\right)$ for all $i \geq 1$. This gives

$$\pi(n) \leq \sum_{i=0}^{\lfloor \log_2(n) \rfloor} \pi'(i) = 1 + \sum_{i=1}^{\lfloor \log_2(n) \rfloor} O\left(\frac{2^i}{i}\right) = O\left(\sum_{i=1}^{\lfloor \log_2(n) \rfloor} \frac{2^i}{i}\right)$$

$$= O\left(\left(\sum_{i=1}^{\lfloor \log_2(n) \rfloor} \left(\frac{3}{4}\right)^{\lfloor \log_2(n) \rfloor - i}\right) \frac{2^{\lfloor \log_2(n) \rfloor}}{\lfloor \log_2(n) \rfloor}\right)$$

where the last equality holds since

$$\frac{2^i}{i} = \frac{1}{2} \cdot \frac{2^{i+1}}{i+1} \cdot \frac{i+1}{i} = \frac{1}{2}\left(\frac{2^{i+1}}{i+1}\right)\left(1+\frac{1}{i}\right) \le \frac{1}{2}\left(\frac{2^{i+1}}{i+1}\right)\left(1+\frac{1}{i}\right)\frac{3}{2} = \frac{3}{4}\left(\frac{2^{i+1}}{i+1}\right)$$

for all $i \ge 1$. Note that

$$\begin{aligned} O\left(\left(\sum_{i=1}^{\lfloor \log_2(n) \rfloor} \left(\frac{3}{4}\right)^{\lfloor \log_2(n) \rfloor - i}\right) \frac{2^{\lfloor \log_2(n) \rfloor}}{\lfloor \log_2(n) \rfloor}\right) &= O\left(4\left(1 - \left(\frac{3}{4}\right)^{\lfloor \log_2(n) \rfloor}\right) \cdot \frac{2^{\lfloor \log_2(n) \rfloor}}{\lfloor \log_2(n) \rfloor}\right) \\ &= O\left(\frac{2^{\lfloor \log_2(n) \rfloor}}{\lfloor \log_2(n) \rfloor}\right) = O\left(\frac{2^{\log_2(n)}}{\log_2(n)}\right) = O\left(\frac{n}{\frac{\ln n}{\ln 2}}\right) \\ &= O\left(\frac{n}{\ln n}\right). \end{aligned}$$

Hence, $\pi(n) = O\left(\frac{n}{\ln n}\right)$.

(c) We define $\nu_p(x) = k$ for $x \in \mathbb{N}$ if

$$p^k \mid x \text{ but } p^{k+1} \nmid x.$$

Thus, we know

$$\nu_p(n!) = \sum_{i \ge 1} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

This gives

$$\nu_p\left(\binom{2m}{m}\right) = \nu_p\left(\frac{(2m)!}{m!m!}\right) = \nu_p((2m)!) - 2\nu_p(m!) = \sum_{i \ge 1} \left\lfloor \frac{2m}{p^i} \right\rfloor - 2\left\lfloor \frac{m}{p^i} \right\rfloor.$$

Now we define $f(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor$, then

$$f(x) = \begin{cases} 0, & \text{if } \{x\} < \frac{1}{2}; \\ 1, & \text{if } \{x\} \ge \frac{1}{2} \end{cases} \text{ where } \{x\} \text{ is the fractional part of } x.$$

Note that if $1 > \frac{2m}{p^i}$, then we have $1 > \frac{2m}{p^i} > \frac{m}{p^i}$ and thus

$$f\left(\frac{m}{p^i}\right) = 0 - 0 = 0.$$

Hence, $f\left(\frac{m}{p^i}\right) = 1$ only if $\frac{2m}{p^i} \ge 1$. Now suppose L is the maximal integer s.t.

$$p^{L+1} \ge 2m \ge p^L,$$

then for all positive integer $i \leq L$, we have $\frac{2m}{p^i} \geq 1$, while for all positive integer $j > L$ we have $\frac{2m}{p^j} < 1$. Hence,

$$\nu_p\left(\binom{2m}{m}\right) = \sum_{i\geq 1} \left\lfloor \frac{2m}{p^i} \right\rfloor - 2\left\lfloor \frac{m}{p^i} \right\rfloor \leq L.$$

Now since $p^k \mid \binom{2m}{m}$, so

$$k \leq \nu_p\left(\binom{2m}{m}\right) \leq L,$$

which means $p^k \leq p^L \leq 2m$, and we're done.

(d) If

$$\binom{2m}{m} = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

is the prime decomposition of $\binom{2m}{m}$, then we know $p_i^{\alpha_i} \leq 2m$ by (c), and thus

$$\binom{2m}{m} \leq (2m)^k \leq (2m)^{\pi(2m)}.$$

Note that

$$2^{2m} = \sum_{k=0}^{2m} \binom{2m}{k} < (2m+1)\binom{2m}{m}$$

since

$$\frac{\binom{2m}{s}}{\binom{2m}{s-1}} = \frac{2m-s+1}{s} \begin{cases} \geq 1, & \text{if } s \leq m; \\ < 1, & \text{if } s \geq m+1. \end{cases},$$

so we have

$$\frac{2^{2m}}{2m+1} < \binom{2m}{m} \leq (2m)^{\pi(2m)},$$

and if we take ln on both sides, we have

$$(2m)\ln 2 - \ln(2m+1) < \pi(2m) \cdot \ln(2m),$$

which means

$$\begin{aligned}\pi(2m) &> \frac{(2m)\ln 2 - \ln(2m+1)}{\ln(2m)} = \frac{2m}{\ln(2m)} \cdot \ln 2 - \log_{2m}(2m+1) \\ &> \frac{2m}{\ln(2m)} \cdot \ln 2 - 2 = \Omega\left(\frac{2m}{\ln(2m)}\right).\end{aligned}$$

Hence, for all $n \in \mathbb{N}$, if $n = 2m$ for some m, then

$$\pi(n) = \pi(2m) = \Omega\left(\frac{2m}{\ln(2m)}\right) = \Omega\left(\frac{n}{\ln n}\right),$$

while for $n = 2m + 1$ for some m, then

$$\begin{aligned}\pi(n) = \pi(2m+1) \geq \pi(2m) &= \Omega\left(\frac{2m}{\ln(2m)}\right)\\ &= \Omega\left(\frac{2m}{\ln(2m+1)}\right) = \Omega\left(\frac{2m+1}{\ln(2m+1)}\right) = \Omega\left(\frac{n}{\ln n}\right)\end{aligned}$$

Hence,

$$\pi(n) = \Omega\left(\frac{n}{\ln n}\right).$$

Bonus (0 points) Define $\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$. Prove that $|\pi(n) - \mathrm{Li}(n)| = O\left(n^{\frac{1}{2}} \ln n\right)$.

Exercise 2* Let $p_{od}(n)$ denote the number of partitions of n into odd, distinct parts; that is, $(\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ with $\lambda_i \in 2\mathbb{N} + 1$ for all i and $\lambda_1 > \lambda_2 > \ldots > \lambda_k \geq 1$. Show that there are positive constants c, C such that for sufficiently large n we have

$$e^{c\sqrt{n}} \leq p_{od}(n) \leq e^{C\sqrt{n}}.$$

Exercise 3 When studying the twelvefold ways of counting, we determined that the number of surjective divisions of n distinct items into r distinct parts is $r!S(n, r)$, where $S(n, r)$ is the Stirling number of the second kind. Use the Inclusion-Exclusion Principle to find an expression for $r!S(n, r)$ not involving the Stirling numbers.

Solution: (張沂魁) Note that

$$r!S(n, r) = \#\left\{\text{division of } [n] \text{ into } r \text{ non-empty ordered parts}\right\}.$$

Hence,

$$r!S(n, r) = \#\left\{f : [n] \to [r] \mid f \text{ is surjective}\right\}.$$

Suppose

$$A_i = \{f : [n] \to [r] \mid \nexists j \in [n] \text{ s.t. } f(j) = i\}$$

for all $i \in [r]$, then

$$\begin{aligned}r!S(n, r) &= \left|(\{f : [n] \to [r]\}) \setminus \left(\bigcup_{i=1}^{r} A_i\right)\right| = \sum_{I \subseteq [r]} (-1)^{|I|} \cdot \left|\bigcap_{i \in I} A_i\right|\\ &= \sum_{I \subseteq [r]} (-1)^{|I|} (r - |I|)^n = \sum_{k=0}^{r} \binom{r}{k} \cdot (-1)^k \cdot (r - k)^n\end{aligned}$$

by the Inclusion-Exclusion Principle.

Exercise 4 Someone is planning a round-the-world trip that involves visiting $2n$ cities, with two cities from each of n different countries. She can choose a city to start and end the journey in, with the other $2n-1$ cities being visited exactly once. However, she has the restriction that the two cities from each country should not be visited consecutively.[2] How many different trips are possible?

Solution: (黃子恆) See last few pages.

Exercise 5 Suppose we have finite sets $A_1, A_2, \ldots, A_r$. Prove that when k_0 is even,

$$|\cup_{i=1}^{r} A_i| \geq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\cap_{i \in I} A_i| \,,$$

and when k_0 is odd,

$$|\cup_{i=1}^{r} A_i| \leq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\cap_{i \in I} A_i| \,.$$

That is, the partial sums in the Inclusion-Exclusion Principle alternate between upper and lower bounds on the size of the union.

Solution: (黃子恆) See last few pages.

[2]For example, suppose $n = 3$ and the $2n$ cities are {Berlin, Frankfurt, Taipei, Kaohsiung, New York, Los Angeles}. Berlin → Taipei → Los Angeles → Kaohsiung → Frankfurt → New York → Berlin is acceptable, but Berlin → Kaohsiung → Los Angeles → Taipei → New York → Frankfurt → Berlin is not, as the final flight is a domestic one.

4. Since the start and end are in the same city, we can regard the trip as the circular permutation of $2n$ items. We arrange $2n$ cities into $2n$ blocks injectively with each block being numbered. The number 1 block is the start and end of the trip, Then, number the blocks in clockwise order. The city in the number i block is the i-th city that she visited in the trip.

Number the countries from 1 to n and name the city as $C_{i,j}$, where $i \in [n]$ representing that $C_{i,j}$ is in the country i and $j=1,2$.

Let S be the set of permutations without the restriction that the two cities from each country should not be visited consecutively. We have $|S| = (2n)!$.

Define $A_i = \{P \in S : C_{i,1}$ and $C_{i,2}$ are consecutively in $P\}$.

Our goal is to compute $|S \setminus \bigcup_{i=1}^n A_i|$.

$$|S \setminus \bigcup_{i=1}^n A_i| = |S| - |\bigcup_{i=1}^n A_i| = (2n)! - \sum_{m=1}^n \sum_{I \in \binom{[n]}{m}} (-1)^{m+1} |\bigcap_{i \in I} A_i|$$
$$= (2n)! + \sum_{m=1}^n (-1)^m \sum_{I \in \binom{[n]}{m}} |\bigcap_{i \in I} A_i|.$$

Note that $\sum_{I \in \binom{[n]}{m}} |\bigcap_{i \in I} A_i|$ is computing there has at least m Countries s.t. cities in these countries are visited consecutively. We can first choose m countries from n countries, and define I to be the set of these m countries ($\binom{n}{m}$ ways). Then, we regard $C_{i,1}$ and $C_{i,2}$ as one item C_i no matter the order is $C_{i,1} C_{i,2}$ or $C_{i,2} C_{i,1}$ in the permutation, so now there has $2n-m$ items, and we have $(2n-m)!$ ways to arrange them. Note that $C_{i,1} C_{i,2}$ and $C_{i,2} C_{i,1}$ are actually 2 different order, so by product rule, $\sum_{I \in \binom{[n]}{m}} |\bigcap_{i \in I} A_i| = \binom{n}{m} 2^m (2n-m)!$

$$\Rightarrow |S \setminus \bigcup_{i=1}^n A_i| = (2n)! + \sum_{m=1}^n (-1)^m \binom{n}{m} 2^m (2n-m)! = \sum_{i=0}^n (-1)^i \binom{n}{i} 2^i (2n-i)!.$$

5. Let $m(x) = |\{i : x \in A_i\}|$.

We know that every element $x \in \bigcup_{i=1}^{r} A_i$ will be counted excatly once in $|\bigcup_{i=1}^{r} A_i|$. However, x will be counted $\binom{m(x)}{k}$ times in $\sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i|$, so it will be counted $\sum_{k=1}^{k_0} (-1)^{k+1} \binom{m(x)}{k}$ times in $\sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i|$.

Recall that we have proven that $\sum_{j=0}^{k} (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k}$ in HW1.

Now we consider that

$$1 - \sum_{k=1}^{k_0} (-1)^{k+1} \binom{m(x)}{k} = (-1)^0 \binom{m(x)}{0} + \sum_{k=1}^{k_0} (-1)^k \binom{m(x)}{k} = \sum_{k=0}^{k_0} (-1)^k \binom{m(x)}{k} = (-1)^{k_0} \binom{m(x)-1}{k_0}.$$

Hence, if k_0 is even, then $(-1)^{k_0} \binom{m(x)-1}{k} \geq 0$, which means that the number of times x counted in $|\bigcup_{i=1}^{n} A_i|$ will not less than that in $\sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i|$. Hence,

$$|\bigcup_{i=1}^{n} A_i| \geq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i| \quad \text{for } k_0 \text{ even.}$$

Similarly,

$$|\bigcup_{i=1}^{n} A_i| \leq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \in \binom{[r]}{k}} |\bigcap_{i \in I} A_i| \quad \text{for } k_0 \text{ odd.}$$