Linear Algebra I HW11

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Section 6.8

Problem. 1. Let T be a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix

$$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}.$$

Express the minimal polynomial p for T in the form $p = p_1p_2$, where p_1 and p_2 are monic and irreducible over the field of real numbers. Let W_i be the null space of $p_i(T)$. Find bases $\mathcal{B}_i$ for the spaces W_1 and W_2. If T_i is the operator induced on W_i by T, find the matrix of T_i in the basis $\mathcal{B}_i$ (above).

Proof. Note that

$$\operatorname{ch}_T(x) = \det \begin{pmatrix} x-6 & 3 & 2 \\ -4 & x+1 & 2 \\ -10 & 5 & x+3 \end{pmatrix} = (x-2)\left(x^2+1\right).$$

If we regard this linear operator is over the field $\mathbb{C}$, then we know $\operatorname{ch}_T(x)$ splits and is $(x-2)(x-i)(x+i)$, and since $m_T(x)$ share same roots as $\operatorname{ch}_T(x)$ and $m_T(x) \mid \operatorname{ch}_T(x)$, so $m_T(x) = (x-2)(x-i)(x+i)$, but here the field is $\mathbb{R}$, so $m_T(x) = (x-2)\left(x^2+1\right)$. Let $p_1(x) = x-2$ and $p_2(x) = x^2+1$, then $W_1 = \ker p_1(T) = \ker(T-2I)$ and $W_2 = \ker p_2(T) = \ker\left(T^2+I\right)$. We first handle W_1. Suppose b is the standard basis of $\mathbb{R}^3$. Note that

$$[T-2I]_b = \begin{pmatrix} 4 & -3 & -2 \\ 4 & -3 & -2 \\ 10 & -5 & -5 \end{pmatrix},$$

so if $[T-2I]_b v = 0$, then we find that $v \in \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$, so $\ker(T-2I) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$ and thus we can let

$$\mathcal{B}_1 = \left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}.$$

Now we handle W_2. Note that

$$\left[T^2+I\right]_b = \begin{pmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 10 & -10 & 0 \end{pmatrix},$$

and we can find that $\ker\left(T^2+I\right) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$, so we can choose

$$\mathcal{B}_2 = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Now since

$$T_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

so $[T_1]_{\mathcal{B}_1} = (2)$. Also,

$$T_2\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}6 & -3 & -2\\4 & -1 & -2\\10 & -5 & -3\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\3\\5\end{pmatrix} = 3\begin{pmatrix}1\\1\\0\end{pmatrix} + 5\begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$T_2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}6 & -3 & -2\\4 & -1 & -2\\10 & -5 & -3\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\-2\\-3\end{pmatrix} = -2\begin{pmatrix}1\\1\\0\end{pmatrix} - 3\begin{pmatrix}0\\0\\1\end{pmatrix},$$

so we know

$$[T_2]_{\mathcal{B}_2} = \begin{pmatrix}3 & -2\\5 & -3\end{pmatrix}.$$

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Problem. 4. Let T be a linear operator on the finite-dimensional space V with characteristic polynomial

$$f = (x-c_1)^{d_1}\cdots(x-c_k)^{d_k}$$

and minimal polynomial

$$p = (x-c_1)^{r_1}\cdots(x-c_k)^{r_k}.$$

Let W_i be the null space of $(T-c_iI)^{r_i}$.

(a) Prove that W_i is the set of all vectors α in V such that $(T-c_iI)^m\alpha = 0$ for some positive integer m (which may depend upon α).

(b) Prove that the dimension of W_i is d_i. (*Hint*: If T_i is the operator induced on W_i by T, then $T_i - c_iI$ is nilpotent; thus the characteristic polynomial for $T_i - c_iI$ must be x^{e_i}, where e_i is the dimension of W_i (proof?); thus the characteristic polynomial of T_i is $(x-c_i)^{e_i}$; now use the fact that the characteristic polynomial for T is the product of the characteristic polynomials of the T_i to show that $e_i = d_i$.)

Proof.

(a) Suppose

$$K_{c_i}(T) = \{\alpha \in V : (T-c_iI)^m\alpha = 0 \text{ for some } m \in \mathbb{N}\},$$

then $W_i \subseteq K_{c_i}(T)$ since $W_i = \ker(T-c_iI)^{r_i}$. Now we show that $K_{c_i}(T) \subseteq W_i$. If we have $(T-c_iI)^m\alpha = 0$ for some $m \in \mathbb{N}$, then we can consider

$$\mathrm{Ann}_T(\alpha) = \{f(x) : f(T)\alpha = 0\},$$

and we know $(x-c_i)^m \in \mathrm{Ann}_T(\alpha)$, so if $\mathrm{Ann}_T(\alpha) = (h(x))$, then $h(x) \mid (x-c_i)^m$. Suppose $h(x) = (x-c_i)^p$ for some $p \in \mathbb{N}$, then we know $p \le m$. Also, since $m_T(x) \in \mathrm{Ann}_T(\alpha)$, so $h(x) \mid m_T(x) = (x-c_i)^{r_i}q_i(x)$, and thus $p \le r_i$. Note that this means

$$0 = h(T)\alpha = (T-c_iI)^p\alpha \text{ for some } p \le r_i,$$

so $(T-c_iI)^{r_i}\alpha = 0$, and thus $\alpha \in \ker(T-c_iI)^{r_i} = W_i$. Hence, $W_i = K_{c_i}(T)$.

(b) If $T_i = T|_{W_i}$, then

$$(T_i - c_iI)^{r_i}\alpha = 0 \text{ for all } \alpha \in W_i = \ker(T-c_iI)^{r_i},$$

so $T - c_iI$ is nilpotent. Suppose $P_i = T_i - c_iI$, then $P_i^{r_i}\alpha = 0$ for all $\alpha \in W_i$, and thus

$$m_{P_i}(x) \mid x^{r_i},$$

so $m_{P_i}(x)$ has only 0 as its root, and thus

$$\mathrm{ch}_{P_i}(x) = x^{e_i} \text{ for some } e_i \in \mathbb{Z}.$$

Note that

$$e_i = \operatorname{degch}_{P_i}(x) = \dim W_i$$

since P_i is a linear operator on W_i. Note that

$$\begin{aligned}
\mathrm{ch}_{P_i}(x) &= \det\left(x[I]_b - [T_1 - c_i I]_b\right) = \det((x + c_i)[I]_b - [T_i]_b) = x^{e_i} \\
\mathrm{ch}_{T_i}(x) &= \det(x[I]_b - [T_i]_b) = \mathrm{ch}_{P_i}(x - c_i) = (x - c_i)^{e_i}.
\end{aligned}$$

Now since $V = \bigoplus_{i=1}^k W_i$ by primary decomposition theorem, so we know

$$\prod_{i=1}^k (x - c_i)^{d_i} = \mathrm{ch}_T(x) = \prod_{i=1}^k \mathrm{ch}_{T_i}(x) = \prod_{i=1}^k (x - c_i)^{e_i},$$

so we must have $d_i = e_i = \dim W_i$ for all $1 \leq i \leq k$.

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Problem. 8. Let V be the space of $n \times n$ matrices over a field F, and let A be a fixed $n \times n$ matrix over F. Define a linear operator T on V by $T(B) = AB - BA$. Prove that if A is a nilpotent matrix, then T is a nilpotent operator.

Proof. If A is nilpotent, then $A^m = 0$ but $A^{m-1} \neq 0$ for some $m \geq 1$. Define $L_A(B) = AB$ and $R_A(B) = BA$, then $T(B) = L_A(B) - R_A(B)$. Then, we know

$$L_A^m(B) = A^m B = 0 \text{ and } R_A^m(B) = BA^m = 0 \text{ for all } B \in V.$$

Hence, L_A and R_A are both nilpotent. Also, since

$$L_A\left(R_A(B)\right) = ABA = R_A\left(L_A(B)\right) \text{ for all } B \in V,$$

so L_A and R_A commute. Hence, we know for all $N \in \mathbb{N}$ we have

$$T^N = (L_A - R_A)^N = \sum_{j=0}^N \binom{N}{j} (-1)^{N-j} L_A^j R_A^{N-j},$$

so if we pick $N = 2m - 1$, then for all $0 \leq j \leq N$ we know either $j \geq m$ or $N - j \geq m$, i.e.

$$\binom{N}{j} (-1)^{N-j} L_A^j R_A^{N-j} \text{ for all } 0 \leq j \leq N.$$

Hence, $T^{2m-1} = 0$ and thus T is nilpotent. ■

Problem. 10. Let T be a linear operator on the finite-dimensional space V, let $p = p_1^{r_1} \cdots p_k^{r_k}$ be the minimal polynomial for T, and let $V = W_1 \oplus \cdots \oplus W_k$ be the primary decomposition for T, i.e., W_j is the null space of $p_j(T)^{r_j}$. Let W be any subspace of V which is invariant under T. Prove that

$$W = (W \cap W_1) \oplus (W \cap W_2) \oplus \cdots \oplus (W \cap W_k).$$

Proof. For all $1 \leq j \leq k$, we define

$$m_T(x) = p_j(x)^{r_j} q_j(x),$$

and note that $p_j(x)^{r_j}$ and $q_j(x)$ are coprime. Thus,

$$\exists a_j(x), b_j(x) \text{ s.t. } a_j(x) q_j(x) + b_j(x) p_j(x)^{r_j} = 1.$$

Let $E_j = a_j(T)q_j(T)$, then we know

$$E_j + b_j(T)p_j(T)^{r_j} = I.$$

Hence, for all $x \in W_j$, we have

$$x = E_j x + b_j(T)p_j(T)^{r_j} x = E_j x,$$

so $E_j x = x$. Also, for $i \neq j$ and $y \in W_i$ we know

$$E_j y = a_j(T)q_j(T)y = a_j(T)\left(\frac{q_j(T)}{p_i(T)^{r_i}}\right)p_i(T)^{r_i}y = 0.$$

Now since for all $x \in V$ we have $x = w_1 + w_2 + \cdots + w_k$ where $w_i \in W_i$, so

$$x = w_1 + w_2 + \cdots + w_k = \sum_{i=1}^{k} E_i x,$$

i.e. $1 = \sum_{i=1}^{k} E_i$. Now since $E_j = a_j(T)q_j(T)$ and W is T-invariant, so $E_j(W) \subseteq W$ and we have shown that $E_j(W) \subseteq W_j$, so

$$W \subseteq \sum_{i=1}^{k} E_i(W) \subseteq \sum_{i=1}^{k} W \cap W_i,$$

and since

$$\sum_{i=1}^{k} W \cap W_i \subseteq W,$$

so

$$W = \sum_{i=1}^{k} W \cap W_i.$$

Now we show $\sum_{i=1}^{k} W \cap W_i$ is in fact a direct product, i.e.

$$W \cap (W_i \cap W_j) = (W \cap W_i) \cap (W \cap W_j) = \{0\} \text{ for any } i \neq j.$$

Suppose $w \in W \cap W_i \cap W_j$, then $p_i(T)^{r_i}(w) = p_j(T)^{r_j}(w) = 0$. Since $p_i(x)^{r_i}$ and $p_j(x)^{r_j}$ are coprime, so there exists $u(x), v(x)$ s.t.

$$u(x)p_i(x)^{r_i} + v(x)p_j(x)^{r_j} = 1,$$

but this means

$$0 = u(T)p_i(T)^{r_i}(w) + v(T)p_j(T)^{r_j}(w) = w,$$

so $w = 0$, and we're done. ■

Section 7.3

Problem. 11. Let N_1 and N_2 be 6×6 nilpotent matrices over the field F. Suppose that N_1 and N_2 have the same minimal polynomial and the same nullity. Prove that N_1 and N_2 are similar. Show that this is not true for 7×7 nilpotent matrices.

Proof. For the 6×6 cases. Since N_1 is nilpotent, we know

$$N_1 \sim J_{\lambda_1}(0) \oplus J_{\lambda_2}(0) \oplus \cdots \oplus J_{\lambda_r}(0)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ are the size of different Jordan blocks and $\sum_{i=1}^{r} \lambda_i = 6$. Note that

$$m_{N_1}(x) = x^{\text{size of largest Jordan block}} = x^{\lambda_1}$$

since N_1 is nilpotent and $J_{\lambda_i}(0)^{\lambda_i} = 0$ for all i. Also, since

$$J_{\lambda_i}(0) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

so every Jordan block contributes exactly 1 to the nullity of N_1 since only the last row is a zero row. Now since N_1 and N_2 has same minimal polynomial and same nullity, so N_1 and N_2's Jordan form has same number of Jordan blocks and same size of largest Jordan block. Consider all the cases:

- Case 1: $\lambda_1 = 6$, then $6 = 6$.
- Case 2: $\lambda_1 = 5$, then $6 = 5 + 1$.
- Case 3: $\lambda_1 = 4$, then $6 = 4 + 2 = 4 + 1 + 1$.
- Case 4: $\lambda_1 = 3$, then $6 = 3 + 3 = 3 + 2 + 1 = 3 + 1 + 1 + 1$.
- Case 5: $\lambda_1 = 2$, then $6 = 2 + 2 + 2 = 2 + 2 + 1 + 1 = 2 + 1 + 1 + 1$.
- Case 6: $\lambda_1 = 1$, then $6 = 1 + 1 + 1 + 1 + 1 + 1$.

Hence, if we fix λ_1 and the number of Jordan blocks, i.e. the number of parts of partition of 6, then the partition is unique, i.e. N_1 and N_2 must have same Jordan form. Hence, $N_1 \sim J \sim N_2$ where J is their mutual Jordan form matrix.

For the 7×7 cases. Since $7 = 3 + 3 + 1 = 3 + 2 + 2$, so N_1 and N_2 may have different Jordan forms. In fact, consider

$$N_1 = \left(\begin{array}{ccc|ccc|c} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & & & & \\ \hline & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & \\ \hline & & & & & & 1 \end{array}\right), \quad N_2 = \left(\begin{array}{ccc|cc|cc} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & & & & \\ \hline & & & 0 & 1 & & \\ & & & & 0 & & \\ \hline & & & & & 0 & 1 \\ & & & & & & 0 \end{array}\right),$$

then if $N_1 \sim N_2$, we will have $N_1 = P^{-1} N_2 P$ for some P and thus

$$\operatorname{rank} N_1^2 = \operatorname{rank}\left(P^{-1} N_2^2 P\right) = \operatorname{rank} N_2^2$$

since left or right multiplication of invertible matrices will not change the rank. However, $\operatorname{rank} N_1^2 = 2$ and $\operatorname{rank} N_2^2 = 1$, so N_1 is not similar to N_2. ■