Linear Algebra I HW2

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September 15, 2025

Problem 0.0.1. Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove that this vector space is not finite-dimensional.

Proof. Consider the set $S = \{\log 2, \log 3, \dots\}$, which is the set

$$\{\log p \mid p \text{ is prime}\}.$$

Now suppose there are rational numbers α_i s.t.

$$\alpha_1 \log 2 + \alpha_2 \log 3 + \alpha_3 \log 5 + \cdots = 0.$$

Then we know

$$\log\left(2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} \dots\right) = 0.$$

However, the logarithm is strictly increasing, so we know $\alpha_i = 0$ for all i, and thus S is linearly independent. Also, since S is an infinite set, so if there is a basis b of V, then b must also be an infinite set, which means V is not finite-dimensional. ■

Problem 0.0.2. Consider the differentiation transformation on $V = \mathbb{R}[x]$, which is defined by

$$D(f(x)) = f'(x) \quad \forall f(x) \in V.$$

Find the range and null space of this transformation.

Proof. Since for all $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, we know

$$F(x) = C + a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_i}{i+1} x^{i+1} + \cdots + \frac{a_n}{n+1} x^{n+1}$$

has $F'(x) = f(x)$, where C can be any element in $\mathbb{R}$. Also, we have $F(x) \in \mathbb{R}[x]$ and $D(F(x)) = f(x)$, so $f(x) \in \operatorname{Im}(D)$, which means $\operatorname{Im}(D) = \mathbb{R}[x]$.

Now if for some $f(x) \in \mathbb{R}[x]$, we have $D(f(x)) = 0$, then we know $f(x)$ is a constant function, so

$$\ker D = \{f(x) \in \mathbb{R}[x] \mid f(x) = c \text{ for some } c \in \mathbb{R}\}.$$

■

Problem 0.0.3. Let V be the vector space of all $n \times n$ matrices over the field F, and let B be a fixed $n \times n$ matrix. If

$$T(A) = AB - BA,$$

verify that T is a linear transformation from V into V.

Proof. It is trivial that T is a map from V into V. Now we show that T is a linear map. Note that for all $A, C \in M_{n\times n}(F)$, we have

$$\begin{aligned} T(\alpha A + C) &= (\alpha A + C)B - B(\alpha A + C) \\ &= \alpha AB + CB - \alpha BA - BC \\ &= \alpha(AB - BA) + (CB - BC) \\ &= \alpha T(A) + T(C). \end{aligned}$$

Hence, T is a linear map. ■

Problem 0.0.4. Let V be the set of all complex numbers regarded as a vector space over the field of real numbers (usual operations). Find a function from V into V which is a linear transformation on the above vector space, but which is not a linear transformation on C^1, i.e., which is not complex linear.

Proof. We can consider $T : V \to V$, where

$$T(a + bi) = (a + b) + bi \quad \forall a + bi \in V \text{ with } a, b \in \mathbb{R}.$$

We first show that it is a linear transformation on V. For any $\alpha \in \mathbb{R}$, and $a' + b'i \in V$ with $a', b' \in \mathbb{R}$,

$$\begin{aligned} T(\alpha(a + bi) + (a' + b'i)) &= T((\alpha a + a') + (\alpha b + b')i) \\ &= (\alpha(a + b) + a' + b') + (\alpha b + b')i \\ &= (\alpha(a + b) + \alpha bi) + (a' + b' + b'i) \\ &= \alpha T(a + bi) + T(a' + b'i). \end{aligned}$$

Hence, T is linear on V. Now we show that T is not complex linear. For $\alpha = p + qi \in \mathbb{C}$, where $p, q \in \mathbb{R}$, we know for any $a + bi \in V$ with $a, b \in \mathbb{R}$, we have

$$\begin{aligned} T((p + qi)(a + bi)) &= T((pa - qb) + (pb + qa)i) \\ &= (pa - qb + pb + qa) + (pb + qa)i, \end{aligned}$$

but we also have

$$\begin{aligned} (p + qi)T(a + bi) &= (p + qi)((a + b) + bi) \\ &= (p + qi)(a + b) + (p + qi)bi \\ &= pa + pb + qai + qbi + pbi - qb \\ &= (pa + pb - qb) + (qa + qb + pb)i \end{aligned}$$

Note that

$$T((p + qi)(a + bi)) - (p + qi)T(a + bi) = qa - qbi = q(a - bi).$$

If we pick $a \neq b$ and $q \neq 0$, then $T((p + qi)(a + bi)) \neq (p + qi)T(a + bi)$. Thus, T is not complex linear. ∎

Problem 0.0.5. Let V be a vector space and T a linear transformation from V into V. Prove that the following two statements about T are equivalent.

(a) The intersection of the range of T and the null space of T is the zero subspace of V.

(b) If $T(T(\alpha)) = 0$, then $T(\alpha) = 0$.

Proof.

$$\begin{aligned} &\ker T \cap \operatorname{Im} T = \{0\} \\ \Leftrightarrow &\text{If } w \in \ker T \text{ for some } w \in \operatorname{Im} T, \text{ then } w = 0. \\ \Leftrightarrow &\text{If } T(w) = 0 \text{ for some } w \in \operatorname{Im} T, \text{ then } w = 0. \\ \Leftrightarrow &\text{If } T(T(\alpha)) = 0 \text{ for some } \alpha \in V, \text{ then } T(\alpha) = 0. \end{aligned}$$

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