Introduction to Analysis I

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Abstract

The lecture note of 2025 Fall Introduction to Analysis I by professor 崔茂培. In this note, we will write $\left(X^{(n)}\right)_{n=m}^{\infty}$ and $\left\{X^{(n)}\right\}_{n=m}^{\infty}$ to express a sequence, they are identical, but 崔茂培 use both during lectures, so I follow him.

Contents

Chapter 1

Basic Things

Lecture 1

1.1 Natural Numbers

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The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, \dots\}$. There exists an addition operation

$$1 + 1 = 2 \quad 1 + 1 + 1 = 3 \quad \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n.$$

1.2 Integers

The set of integers is $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. There is a zero element 0 such that $z + 0 = z$ for any $z \in \mathbb{Z}$. Also, for $n \in \mathbb{N}$, we have $n + (-n) = 0$ and $n - m = n + (-m)$ for all $n, m \in \mathbb{N}$.

$$\mathbb{Z} \xRightarrow[\text{introduce division}]{} \mathbb{Q} \xRightarrow[\text{Completeness axiom}]{} \mathbb{R}$$

1.3 Field

Next, we introduce the concept of field.

Definition 1.3.1 (Fields)**.** A field is a set F together with two binary operations, called addition($+$) and multiplication($*$), such that the following properties hold:

(a) $a + b = b + a$, $a \cdot b = b \cdot a$ for $a, b \in F$.

(b) $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in F$.

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$.

(d) There are distince element 0 and 1 such that $a + 0 = a$, $a \cdot 1 = a$ for $a \in F$.

(e) For each $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$. If $a \neq 0$, there is an element $\frac{1}{a}$ or a^{-1} in F such that $a \cdot \frac{1}{a} = 1$, or $a \cdot a^{-1} = 1$.

Remark 1.3.1. If $a \in F$, then $a + a \in F$. We denote $a + a$ by $2 \cdot a$. Similarly,

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = n \cdot a,$$

and

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$

if $a \in F$ and $n \in \mathbb{N}$.

Remark 1.3.2. In a field, we have subtraction and division $a - b = a + (-b)$ for $a, b \in F$. If $b \neq 0$, then $\frac{a}{b} = a \cdot b^{-1}$ for $a, b \in F$.

In a field F, we have

$$\begin{aligned}
(a+b)^2 &= (a+b) \cdot (a+b) \\
&= (a+b) \cdot a + (a+b) \cdot b \\
&= a \cdot a + b \cdot a + a \cdot b + b \cdot b \\
&= a^2 + ab + ab + b^2 \\
&= a^2 + 2ab + b^2.
\end{aligned}$$

Example 1.3.1.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

if $b \neq 0$ and $d \neq 0$.

Proof.

$$\begin{aligned}
\frac{a}{b} + \frac{c}{d} &= a \cdot b^{-1} + c \cdot d^{-1} \\
&= ab^{-1}dd^{-1} + cd^{-1}bb^{-1} \\
&= adb^{-1}d^{-1} + cbd^{-1}b^{-1} \\
&= \frac{ad + bc}{bd}.
\end{aligned}$$

Notice that this is true since we have commutativity in multiplication and

$$d^{-1}b^{-1} = (bd)^{-1} = \frac{1}{bd}.$$

■

Example 1.3.2. The set of rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$ is a field.

Example 1.3.3. The set of real numbers is also a field.

Example 1.3.4. $F_2 = \{0, 1\}$ is also a field since we can define addition and multiplication like $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$, and $0 \cdot 0 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$.

1.4 Order Relation

Next, we introduce the order relation. The real number system is ordered by the relation $<$, which has the following properties.

(f) For each pair of real numbers a and b, exactly one of the follwing is true: $a = b, a < b, b < a$.

(g) If $a < b$ and $b < c$, then $a < c$.

(h) If $a < b$, then $a + c < b + c$ for any c, and if $0 < c$, then $a \cdot c < b \cdot c$.

Definition 1.4.1. A field with an order relation satisfy (f) to (h) is called an ordered field.

Example 1.4.1. The set of rational numbers is an ordered field.

Example 1.4.2. F_2 is not an ordered field.

Proof. If $0 < 1$, then $1 = 0 + 1 < 1 + 1 = 0$, which is a contradiction. If $1 < 0$, then $0 = 1 + 1 < 0 + 1 = 1$, which is also a contradiction. ∎

Notation. In an ordered field, we use $a \leq b$ to denote either $a < b$ or $a = b$.

1.5 Absolute Value and Triangle Inequality

Next, we define the absolute value of a real number

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a \leq 0; \end{cases}$$

Theorem 1.5.1 (Triangle Inequality)**.**

$$|a + b| \leq |a| + |b|$$

for all $a, b \in \mathbb{R}$.

Corollary 1.5.1.

$$||a| - |b|| \leq |a - b| \quad \text{and} \quad ||a| - |b|| \leq |a + b|$$

Proof. We write

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Similarly we have

$$|b| \leq |b - a| + |a|.$$

So

$$-|b - a| \leq |a| - |b| \leq |a - b|.$$

Thus,

$$||a| - |b|| \leq |a - b|.$$

∎

1.6 Supremum and Infimum

Next, we introduce the notion of supremum of a subset of real numbers.

Definition 1.6.1. Let S be a subset of $\mathbb{R}$,

(1) we say b is an upper bound of S if $x \leq b$ for all $x \in S$.

(2) If B is an upper bound of S, and no number smaller than B is an upper bound of S, then B is called the supremum or the least upper bound of S. We write $B = \sup S$.

Corollary 1.6.1. If $B = \sup S$, then

(1) $x \in S$ implies $x \leq B$

(2) If $b < B$, then b is not an upper bound of S, i.e. there exists $x_1 \in S$ such that $b < x_1$.

Definition 1.6.2. Let S be a subset of $\mathbb{R}$,

(1) we say b is an lower bound of S if $x \geq b$ for all $x \in S$.

(2) If α is an lower bound of S, and no number bigger than α is an lower bound of S, then α is called the infimum or the greatest lower bound of S. We write $\alpha = \inf S$.

Corollary 1.6.2. If $\alpha = \inf S$, then

(1) $x \in S$ implies $x \geq \alpha$

(2) If $\alpha < a$, then a is not an lower bound of S, i.e. there exists $x_1 \in S$ such that $x_1 < a$.

Notation (Interval Notation).

$$(a, b) = \{x \mid a < x < b\}$$
$$(a, b] = \{x \mid a < x \leq b\}$$
$$[a, b) = \{x \mid a \leq x < b\}$$

Example 1.6.1. $S = \{x \mid x < 0\} = (-\infty, 0)$, then $\sup S = 0$ but $\inf S$ does not exists.

Example 1.6.2. $S_1 = \{-1, -2, -3, -4, \dots\} = \{-n \mid n \in \mathbb{N}\}$, then $\sup S = -1$, but $\inf S$ does not exist.

Definition 1.6.3 (Nonempty Sets). A nonempty set is that a set has at least one element. The empty set, written as $\varnothing$, is the set has no elements at all.

Example 1.6.3. $S = \left\{x \mid x \in \mathbb{Q}, x < \sqrt{2}\right\}$

In $\mathbb{Q}$, $\sup S$ does not exist. In $\mathbb{R}$, $\sup S = \sqrt{2}$.

Theorem 1.6.1 (Completeness axiom). If a nonempty set of real numbers (an ordered field) is bounded above, then it has a least upper bound or $\sup S$ exists.

Remark 1.6.1. This is an extra axiom that can't be derived from the properties of ordered field.

Remark 1.6.2. Up to "isomorphism", there is exactly one complete ordered field: the field of real numbers.

Remark 1.6.3. From now, we assume $\mathbb{R}$ satisfies the completeness axiom. Thus, any nonempty subset $S \subseteq \mathbb{R}$ that is bounded above, we have $\sup S$ exists.

We can prove the following property of $\sup S$.

Theorem 1.6.2. If $S \subseteq \mathbb{R}$ is bounded above, then $\sup S$ is the unique real number B such that

(i) $x \leq B$ for all $x \in S$

(ii) for every $\varepsilon > 0$, there exist an $x_0 \in S$ such that $B - \varepsilon < x_0$.

Proof. (i), (ii) follows from the definition. We prove the uniqueness. Suppose $B_1 = \sup S = B_2$. We want to show $B_1 = B_2$. Suppose $B_1 \neq B_2$. Then either $B_1 < B_2$ or $B_2 < B_1$. However, if either one is true, then the other one cannot be $\sup S$. ■

Theorem 1.6.3 (Archimedean Property)**.** If $p > 0$ and $\varepsilon > 0$, then there exists an $n \in \mathbb{N}$ such that $p < n\varepsilon$.

Proof. We prove this contradiction. Suppose it is not true. This implies $n\varepsilon \leq p$ for all $n \in \mathbb{N}$. Consider $S = \{n\varepsilon \mid n \in \mathbb{N}\}$, then p is an upper bound of S, so S is bounded above by p, so we know $B = \sup S$ exists. Hence, $n\varepsilon \leq B$ for all $n \in \mathbb{N}$, so we have $(n+1)\varepsilon \leq B$, which means

$$n\varepsilon \leq B - \varepsilon$$

for all $n \in \mathbb{N}$. This implies $B - \varepsilon$ is also an upper bound of S, which is a contradiction. ■

1.7 Density of other number system

Theorem 1.7.1. Every nonempty subset of the integers that is bounded below has a least element.

Proof. We first introduce an axiom:

> **Theorem 1.7.2** (Well-Ordering principle)**.** Every non-empty subset of the natural numbers has a least element.

> **Note 1.7.1.** Here, $\mathbb{N}$ can be $\{0, 1, 2, \dots\}$ or $\{1, 2, 3, \dots\}$, which is not that important.

Now we call this subset of integers as S, and suppose we have m as a lower bound of S, then define $S' = \{s - m \mid s \in S\}$, then we know S' is a nonempty subset of $\mathbb{N}$, then by well-ordering principle we know there is a least element in S' and thus there is also a least element in S. ■

Corollary 1.7.1. Every nonempty subset of the integers that is bounded above has a greatest element.

Proof. Suppose M is an upper bound, then define a set $S' = \{M - s \mid s \in S\}$, then by well-ordering principle we know $M - a$ is the least element of S' for some $a \in S$, so we have $M - x \geq M - a$ for all $x \in S$, which means $a \geq x$ for all $x \in S$ and since $a \in S$, so a is the greatest element of S. ■

Theorem 1.7.3. The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{p}{q}$ such that $a < \frac{p}{q} < b$.

Proof. Let $a, b \in \mathbb{R}$, $a < b$. By Archimedean Property, $\exists q \in \mathbb{N}$ such that $q(b - a) > 1$. Let $S = \{m \mid m \text{ is an integer with } m > qa\}$, since we know $S \neq \varnothing$ and S is bounded below. Hence, $p = \inf S$ exists and is an integer by the last theorem. So $qa < p$ and $p - 1 \leq qa$, which means $qa < p \leq qa + 1 < qb$, so we have $a < \frac{p}{q} < b$. ■

Lecture 2

Definition 1.7.1 (Floor Function)**.** For any real number x, the floor function of x is denoted by $\lfloor x \rfloor$, and is defined by the formula $\lfloor n \rfloor$ if $n \leq x < n + 1$ where $n \in \mathbb{Z}$. 4 Sep. 10:20

Corollary 1.7.2.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

Example 1.7.1. $\lfloor 3.7 \rfloor = 3$, $\lfloor -1.2 \rfloor = -2$.

Now by floor function, we can reprove Theorem 1.7.3.

Theorem 1.7.4 (Density of rational number in real number Again)**.** The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{q}{p}$ such that $a < \frac{q}{p} < b$.

Reprove Theorem 1.7.3. Since $a < b$, so we know $b - a > 0$. Now by Archimedean Property, we know there exists $q \in \mathbb{N}$ such that $q(b-a) > 1$. Let $p = \lfloor qa \rfloor + 1$, we have

$$\lfloor qa \rfloor \leq qa < \lfloor qa \rfloor + 1 = p.$$

From our construction, $qb > qa + 1$, so we have

$$p = \lfloor qa \rfloor + 1 \leq qa + 1 < qb,$$

hence we have

$$qa \leq p \leq qb.$$

■

Note 1.7.2. For some reason, p, q in Theorem 1.7.3 and Theorem 1.7.4 are reversed.

Definition 1.7.2 (irrational number)**.** x is called irrational if x is not rational.

Example 1.7.2. $\sqrt{2}$ is irrational.

Theorem 1.7.5. Let $r \in \mathbb{Q}$ and $x \in \mathbb{R} \backslash \mathbb{Q}$, then

1. $r + x$ is irrational.
2. If $r \neq 0$, then rx is irrational.

sketch of proof.

1. If $r + x = q \in \mathbb{Q}$, then $x = q - r \in \mathbb{Q}$, contradiction.
2. If $rx = q \in \mathbb{Q}$, then $x = \frac{q}{r} \in \mathbb{Q}$ since $r \neq 0$.

■

Theorem 1.7.6 (irrational number dense in real number)**.** The set of irrational number is dense in real number. That is, if $a, b \in \mathbb{R}$ and $a < b$, then there exists a irrational number t such that $a < t < b$.

Proof. By density of rational number, we can find $a < r_1 < r_2 < b$ where $r_1, r_2 \in \mathbb{Q}$, and then let $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$, then we know

$$a < r_1 < t < r_2 < b.$$

Note 1.7.3. We should use Theorem 1.7.5 and the fact that $\sqrt{2}$ is irrational.

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Definition 1.7.3 (bounded set). A set $S \subseteq \mathbb{R}$ is bounded if there are numbers a, b s.t. $a \leq x \leq b$ for all $x \in S$.

Corollary 1.7.3. A bounded non-empty set in $\mathbb{R}$ has a unique supremum and a unique infimum and $\inf S \leq \sup S$.

1.8 Extended real number system

The real number system, together with ∞ and $-\infty$, then we have the following properties:

(a) If $a \in \mathbb{R}$, then $a + \infty = \infty + a = \infty$ and $a - \infty = -\infty + a = -\infty$, and $\frac{a}{\infty} = \frac{a}{-\infty} = 0$.

(b) If $a > 0$, then $a \cdot \infty = \infty \cdot a = \infty$ and $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$

(c) If $a < 0$, then $a \cdot \infty = \infty \cdot a = -\infty$ and $a \cdot -\infty = -\infty \cdot a = \infty$ and $\infty + \infty = \infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$ and $-\infty - \infty = \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$ and $|-\infty| = |\infty| = \infty$

However, there are some indeterminate form:

Theorem 1.8.1. The following things are not defined:

$$\infty - \infty,\ 0 \cdot \infty,\ \frac{\infty}{\infty},\ \text{and}\ \frac{0}{0}.$$

1.9 Mathematical Induction

Theorem 1.9.1 (Peano's Postulate). The natural numbers satisfy the following properties

(a) $\mathbb{N}$ is nonempty.

(b) For each natural number n, there exists a unique rational number n called the successor of n.

(c) There exists a natural number $\overline{n}$ that is not the sucessor of any natural number.

(d) Different natural numbers have different sucessors, that is, $n \neq m$ implies $n' \neq m'$.

(e) The only subset of $\mathbb{N}$ that contains $\overline{n}$ and also contains the sucessor of every one of its element is $\mathbb{N}$.

Theorem 1.9.2 (Principle of Mathematical Induction). Let $p_1, p_2, \ldots, p_n$ be propositions, one for each positive integers, such that

(a) p_1 is true.

(b) for each positive integer n, p_n implies p_{n+1}.

then p_n is true for each $n \in \mathbb{N}$.

Proof. Let $M = \{n \mid n \in \mathbb{N} \text{ and } p_n \text{ is true}\}$, then from (a) we know $1 \in M$ and from (b) we know $n \in M$ implies $n + 1 \in M$. Hence, from (e) of Peano's Postulate, we know $M = \mathbb{N}$. ∎

Chapter 2

Metric Space

2.1 Definition and examples

Definition 2.1.1. Suppose $x_n \in \mathbb{R}$ for $n \geq m$. We use the notation $(x_n)_{n=m}^{\infty}$ to denote the sequence of numbers

$$x_m, x_{m+1}, \ldots$$

We first recall the definition of a convergent sequence.

Definition 2.1.2 (Convergent Sequence). We say that a sequence $(x_n)_{n=m}^{\infty}$ of real numbers converges to x if for every $\varepsilon > 0$, there exists an $N \geq m$ s.t. $|x_n - x| \leq \varepsilon$ for all $n \geq N$.

> **Notation.** We write $\lim_{n\to\infty} x_n = x$.

On $\mathbb{R}$, we can define the distance function between two points $x, y \in \mathbb{R}$ by $d(x, y) = |x - y|$. We'll discuss this more later.

Lemma 2.1.1. Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number, then $(x_n)_{n=m}^{\infty}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Proof. Assume $(x_n)_{n=m}^{\infty}$ converges to x. Let $\varepsilon > 0$ be arbitrary real number. By definition, there exists an $N \geq m$ such that $|x_n - x| \leq \varepsilon$ for all $n \geq N$. But $d(x_n, x) = |x_n - x|$ by the definition. Hence, $\forall \varepsilon > 0$, $\exists N \geq m$ such that $d(x_n, x) \leq \varepsilon$ for all $n \geq N$. This implies that $\forall \varepsilon > 0$, $\exists N \geq m$ such that $|d(x_n, x) - 0| \leq \varepsilon$ for all $n \geq N$. This implies $\lim_{n\to\infty} d(x_n, x) = 0$.

The proof of the other side is the same but writing the above proof from bottom to top again.

■

Definition 2.1.3 (Metric Space). A metric space (X, d) is the space of X of objects(called points), together with a distance function or metric $d : X \times X \to [0, \infty)$ which associates to each x, y of points in X a nonnegative number $d(x, y) \geq 0$. Furthermore, the metric must satisfy 4 axioms.

(a) For any $x \in X$, $d(x, x) = 0$.

(b) (Positivity) For any distinct $x, y \in X$, we have $d(x, y) > 0$.

(c) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.

(d) (Triangle inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Example 2.1.1. On $\mathbb{R}$, we can define $d(x, y) = |x - y|$.

Proof.

- $d(x, y) = |x - y| \geq 0$.

- $d(x, y) = 0$ iff $|x - y| = 0$ iff $x = y$.
- $|x - y| = |y - x|$, so $d(x, y) = d(y, x)$
- $|x - z| \leq |x - y| + |y - z|$ for all $x, y, z \in \mathbb{R}$.

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Example 2.1.2. Let (X, d) be a metric space and $Y \subseteq X$, then Y inherits a natural distance function

$$d|_{Y \times Y} : Y \times Y \to [0, \infty)$$

defined by $d|_{Y \times Y}(\alpha, \beta) = d(\alpha, \beta)$ for all $\alpha, \beta \in Y$.

Note 2.1.1. $(Y, d|_{Y \times Y})$ is called a metric subspace of (X, d). It is obvious that $d|_{Y \times Y}$ is a metric on Y.

Recall $\mathbb{R}^n$. Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

Definition 2.1.4 (l^2-metric)**.** The l^2-metric is defined by

$$d_2(x, y) = \left(\sum_{i=1}^{n} (x_n - y_n)^2 \right)^{\frac{1}{2}} \text{ (or we called } d_{l_2}(x, y)).$$

Definition 2.1.5 (l^1-metric(taxicab metric))**.** The l^1-metric is defined by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| (\text{or we called } d_{l_1}(x, y))$$

Definition 2.1.6 (l^∞-metric)**.** The l^∞-metric is defined by

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

Exercise 2.1.1. Verify they are all metrics.

Note 2.1.2. Actually we have to define inner product and norm first and then we can use the triangle inequality of norm to prove d_2 is a metric. (See lecture notes by professor)

Lecture 3

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Definition 2.1.7 (Cartesian Product)**.** Let A, B be sets. The cartesian product of A and B is defined by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Similarly, the cartesian product of $X_1, X_2, \ldots, X_n$ is

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i \ \forall 1 \leq i \leq n\}.$$

Definition 2.1.8 (Functions)**.** Let $X_1, X_2, \ldots, X_n$ be sets and let Y be another set. A fuction of n variables with codomains is a map $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ which assigns each n-tuple $(x_1, x_2, \ldots, x_n)$ with $x_i \in X_i$ a unique element $f(x_1, x_2, \ldots, x_n)$.

Definition. We talk about the definition of domain, codomain, and range:

Definition 2.1.9. The domain of f is $X_1 \times X_2 \times \cdots \times X_n$ and Y is the codomain of f.

Definition 2.1.10. The range of f is

$$\{f(x_1, x_2, \ldots, x_n) \in Y \mid x_i \in X_i \ \forall i\}.$$

In the definition of metric space, we write (X, d) to emphasize our set X and d is a distance function defined on $X \times X$, i.e.

$$d : X \times X \to [0, \infty) \subseteq \mathbb{R},$$

where

$$d : (x, y) \mapsto d(x, y)$$

for $x, y \in X$. Let (X, d) be a metric space and $Y \subseteq X$. Then $(Y, d|_{Y \times Y})$ is also a metric space with distance function defined by

$$d|_{Y \times Y} \to [0, \infty)$$

and

$$d|_{Y \times Y} : (\alpha, \beta) \mapsto d(\alpha, \beta) \text{ for } \alpha, \beta \in Y.$$

Example 2.1.3. Recall the Taxi-cab metric, it can be used in cryptography. For example, for two binary strings, we know

$$d_1((10010), (10101)) = 3 = \text{the number of mismatched bits.}$$

Example 2.1.4. Recall the l^∞-metric. Suppose two jobs where each consists of 3 tasks, and the time (in hours) to complete each task is represented by a vector

$$x = (2, 4, 6), \ \ y = (3, 7, 5),$$

so

$$d_\infty(x, y) = \max\{|2 - 3|, |4 - 7|, |6 - 5|\} = 3.$$

Definition 2.1.11 (Lipschitz equivalent metrics)**.** Let (X, d_1) and (X, d_2) be two metrics on X. We say d_1 and d_2 are Lipschitz equivalent if $\exists c_1, c_2 > 0$ s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y) \quad \forall x, y \in X$$

Remark 2.1.1. They will have same topology (defined later).

Proposition 2.1.1. For all $x, y \in \mathbb{R}^n$,

$$d_2(x, y) \leq d_1(x, y) \leq \sqrt{n} d_2(x, y) \tag{2.1}$$

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} d_\infty(x, y) \tag{2.2}$$

Remark 2.1.2.

$$\begin{aligned} d_\infty(x, y) &\geq \frac{1}{\sqrt{n}} d_2(x, y) \\ &\geq \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} d_1(x, y) = \frac{1}{n} d_1(x, y). \end{aligned}$$

Also,

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y).$$

Remark 2.1.3. d_1, d_2, d_∞ are all Lipschitz equivalent.

proof of Proposition 2.1.1 . Recall $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, then

$$d_1(x,y) = \sum_{i=1}^n |x_i - y_i|, \quad d_2(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

By Cauchy-Schurwatz inequality,

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^n |x_i - y_i| \\ &\le \left(\sum_{i=1}^n |x_i - y_i|\right)^{\frac{1}{2}} \left(\sum_{i=1}^n 1^2\right)^{\frac{1}{2}} = \sqrt{n} d_2(x,y). \end{aligned}$$

Now we show that $d_1(x,y) \ge d_2(x,y)$.

$$\begin{aligned} (d_1(x,y))^2 &= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\ &= \sum_{i=1}^n |x_i - y_i|^2 + 2 \sum_{1 \le i < j \le n} |x_i - y_i||x_j - y_j| \\ &\ge \sum_{i=1}^n |x_i - y_i|^2 = d_2(x,y)^2. \end{aligned}$$

Hence, we have $d_1(x,y) \ge d_2(x,y)$.

Now we show that $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$. Note that

$$d_2(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}, \quad d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|.$$

For each i, we know

$$|x_i - y_i| \le d_\infty(x,y),$$

so

$$d_2(x,y)^2 \le \sum_{i=1}^n d_\infty(x,y)^2 = n d_\infty(x,y)^2,$$

so $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$. ∎

Definition 2.1.12 (Discrete metric). Let X be any set, define the discrete metric:

$$d_{\text{disc}} : X \times X \to \{0,1\}$$

where

$$d_{\text{disc}}(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \ne y. \end{cases}$$

Why this is a metric? Because

- $d_{\text{disc}}(x,y) \ge 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$.
- $d_{\text{disc}}(x,y) = d_{\text{disc}}(y,x)$ by definition.
- $d_{\text{disc}}(x,z) \le d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z)$?

proof of triangle inequality in discrete metric. We first consider the case that $x = z$, then

$$d_{\text{disc}}(x, z) = 0,$$

so it is obviously that the triangle inequality is true.

Now if $x \neq z$, then either $y \neq z$ or $y \neq x$ must happen, so the triangle inequality must be true. ∎

Example 2.1.5. We can define

$$d(x, x) = 0, \quad d(x, y) = \text{minimal length of a path from } x \text{ to } y,$$

then this is also a metric.

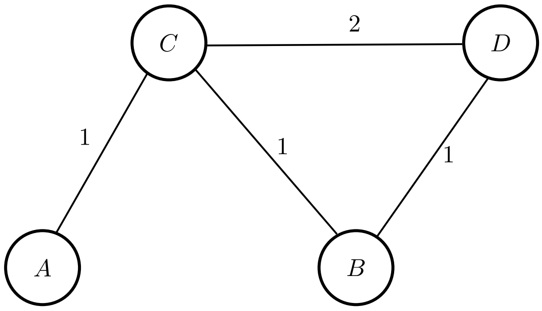

Figure 2.1: Graph metrics

Definition 2.1.13 (Convergence in metric space). Let m be an integer, (X, d) be a metric space, and let $\left(X^{(n)}\right)_{n=m}^{\infty}$ be a sequence of points in X. Let $x \in X$. We say that $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges to x with respect to d iff

$$\lim_{n\to\infty} d\left(X^{(n)}, x\right) = 0,$$

where $\lim_{n\to\infty} d\left(X^{(n)}, x\right) = 0$ iff for every $\varepsilon > 0$, $\exists N \geq m$ s.t. $d\left(X^{(n)}, x\right) \leq \varepsilon$ for all $n \geq N$.

Notation. We also write $\lim_{n\to\infty} X^{(n)} = x$ in (X, d).

Remark 2.1.4. Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges to x in (X, d), then $\left(X^{(n)}\right)_{n=m_1}^{\infty}$ also converges to x in (X, d) if $m_1 \geq m$.

Example 2.1.6. Let $\left(X^{(n)}\right)_{n=1}^{\infty}$ denote the sequence $X^{(n)} = \left(\frac{1}{n}, \frac{1}{n}\right)$ in $\mathbb{R}^2$, then what will this sequence converges to for different metric?

Proof.

- If the metric is d_1, then

$$d_1\left(X^{(n)}, (0,0)\right) = \left|\frac{1}{n} - 0\right| + \left|\frac{1}{n} - 0\right| = \frac{2}{n},$$

so

$$\lim_{n\to\infty} d_1\left(X^{(n)}, (0,0)\right) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

- If the metric is d_2, then

$$d_2\left(X^{(d)}, (0,0)\right) = \sqrt{\left(\frac{1}{n} - 0\right)^2 + \left(\frac{1}{n} - 0\right)^2} = \frac{\sqrt{2}}{n}.$$

  Hence, under l_2-metric $\left\{X^{(n)}\right\}$ also converges to 0.

- If the metric is d_∞, then

$$d_\infty\left(X^{(n)}, (0,0)\right) = \max\left\{\left|\frac{1}{n}\right|, \left|\frac{1}{n}\right|\right\} = \frac{1}{n},$$

  so it also converges to 0.

- If the metric is discrete metric, then however, it will not converges to $(0,0)$ since

$$\lim_{n\to\infty} d_{\text{disc}}\left(X^{(n)}, (0,0)\right) = \lim_{n\to\infty} d_{\text{disc}}\left(\left(\frac{1}{n}, \frac{1}{n}\right), (0,0)\right) = 1.$$

⊛

Definition. Let $f : X \to Y$ be a function with domain X and codomain Y. The range of $f = \{f(x) \mid x \in X\} \subseteq Y$.

Definition 2.1.14 (injective)**.** We say f is injective or one-to-one if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Definition 2.1.15 (surjective)**.** We say f is surjective or onto if for every $y \in Y$, $\exists x \in X$ s.t. $f(x) = y$.

Definition 2.1.16 (bijective)**.** We say f is bijective if f is injective and surjective.

Corollary 2.1.1. If f is bijective, then there exists $f^{-1} : Y \to X$ defined by $f^{-1}(y) = x$ if $f(x) = y$. We also have

$$f\left(f^{-1}(y)\right) = y \ \forall y \in Y$$
$$f^{-1}\left(f(x)\right) = x \ \forall x \in X.$$

Example 2.1.7. $\lim_{n\to\infty} \frac{1}{n} = 0$ in $(\mathbb{R}, d)$, where d is the standard metric in $\mathbb{R}$, which is defined by

$$d(x,y) = |x - y|.$$

But in different metric, $\lim_{n\to\infty} \frac{1}{n}$ may not be 0.

Proof. Define $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} x, & \text{if } 0 < x < 1; \\ 1, & \text{if } x = 0; \\ 0, & \text{if } x = 1. \end{cases}$$

f is bijective on $[0,1]$ to $[0,1]$

Define another metric d^1 on $[0,1]$ by

$$d^1(x,y) = d(f(x), f(y)).$$

We want to show that d^1 is also a metric on $[0,1]$.

- $d^1(x,y) = d(f(x), f(y)) = |f(x) - f(y)| \geq 0$
- $d^1(x,y) = 0$ iff $f(x) = f(y)$ iff $x = y$ since f is injective.
- The triangle inequality is trivially true since we can just use the triangle inequality in d.

In fact, $\lim_{n\to\infty} \frac{1}{n} = 1$ in $\left([0,1], d^1\right)$ since

$$\lim_{n\to\infty} d^1\left(\frac{1}{n}, 1\right) = \lim_{n\to\infty} d\left(\frac{1}{n}, 0\right) = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0.$$

⊛

2.2 Some point set topology of metric space

Definition 2.2.1 (ball)**.** Let (X,d) be a metric space. Let $x_0 \in X$ and $r > 0$. We define the ball $B_{(X,d)}(x_0, r)$ in X, centered at x_0 and with radius r in the metric d, to the set

$$B_{(X_0,d)}(x_0, r) := \{x \in X \mid d(x_0, x) < r\}.$$

Sometimes, we write it as $B_X(x_0, r)$ or $B(x_0, r)$.

Example 2.2.1. In $\mathbb{R}^2$,

$$B_{(\mathbb{R}^2, d_2)}\left((0,0), 1\right) = \left\{(x,y) \mid d_2((x,y),(0,0)) = \sqrt{x^2+y^2} < 1\right\},$$

and

$$B_{(\mathbb{R}^2, d_1)}((0,0), 1) = \{(x,y) \mid d_1((x,y),(0,0)) = |x| + |y| < 1\},$$

and

$$B_{(\mathbb{R}^2, d_\infty)}((0,0), 1) = \{(x,y) \mid d_\infty((x,y),(0,0)) = \max\{|x|, |y|\} < 1\},$$

also we can consider the d_{disc} case but I am too lazy to write it down.

Notation. Let $E \subseteq X$, we will write

$$X \setminus E := \{x \in X \mid x \notin E\}.$$

Definition. Let (X,d) be a metric space and $E \subseteq X$. For a point $x_0 \in X$,

Definition 2.2.2 (interior point)**.** x_0 is an interior point of E if $\exists r > 0$ s.t. $B(x_0, r) \subseteq E$.

Definition 2.2.3 (exterior point)**.** x_0 is an exterior point of E if $\exists r > 0$ s.t. $B(x_0, r) \subseteq X \setminus E$.

Definition 2.2.4 (boundary point)**.** x_0 is a boundary point of E if it is neither an interior point nor an exterior point of E.

Proposition 2.2.1. x_0 is a boundary point of E iff for all $r > 0$, $B(x_0, r) \cap E \neq \varnothing$ and $B(x_0, r) \cap (X \setminus E) \neq \varnothing$.

Lecture 4

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Theorem 2.2.1. Let (X, d_1) and (X, d_2) be metrics on X, and suppose d_1 and d_2 are Lipschitz equivalent, then for any sequence $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$ and any $x \in X$, we have

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_1) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_2).$$

Proof. Since d_1, d_2 are Lipschitz equivalent, so there exists $c_1, c_2 > 0$ s.t.

$$c_1 d_1(x, y) \le d_2(x, y) \le c_2 d_1(x, y).$$

($\Rightarrow$) Given $\frac{\varepsilon}{c_2} > 0$, since $\lim_{n\to\infty} x^{(n)} = x$ in (X, d_1), so there exists N s.t. $N \ge m$ and

$$d_1(x^{(n)}, x) \le \frac{\varepsilon}{c_2} \text{ for } n \ge N.$$

This implies $d_2(x^{(n)}, x) \le c_2 d_1(x^{(n)}, x) \le \varepsilon$ for $n \ge N$, which means

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_2).$$

($\Leftarrow$) Similar.

■

Remark 2.2.1. On $\mathbb{R}^n$, the metrics d_1, d_2, d_∞ are Lipschitz equivalent, that is,

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_1) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_2) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_\infty)$$

Proposition 2.2.2. Let (X, d_{disc}) be a discrete metric space, and $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$. Then

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_{\text{disc}}) \Leftrightarrow \exists N \ge m \text{ s.t. } x^{(n)} = x \text{ for } n \ge N.$$

Proof. ($\Leftarrow$) Easy.

($\Rightarrow$) Given $\frac{1}{2} > 0$, there exists $N \ge m$ s.t. $d(x_n, x) < \frac{1}{2}$ for $n \ge N$, but $d(x_n, x) < \frac{1}{2}$ implies $d(x_n, x) = 0$, which means $x_n = x$ for all $n \ge N$.

■

Definition. We define the interior, exterior, and boundary point again.

Definition 2.2.5. The set of interior points is denoted by

$$\text{Int}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq E\}.$$

Definition 2.2.6. The set of exterior points is denoted by

$$\text{Ext}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq X \setminus E\}.$$

Definition 2.2.7. A point is a boundary points if it is neithe an interior point nor an exterior point, and we define

$$\partial E = \{x \in X \mid x \notin \text{Int}(E) \text{ and } x \notin \text{Ext}(E)\}.$$

Remark 2.2.2.

1. $$x_0 \notin \text{Int}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (X \setminus E) \neq \varnothing.$$
2. $$x_0 \notin \text{Ext}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (E) \neq \varnothing.$$
3. $\text{Int}(X \setminus E) = \text{Ext}(E)$.
4. $\partial E = \partial(X \setminus E)$ since
$$x_0 \in \partial E \Leftrightarrow x \notin \text{Int}(E) \text{ and } \text{Ext}(E) \Leftrightarrow x_0 \notin \text{Ext}(X \setminus E) \text{ and } x_0 \notin \text{Int}(X \setminus E).$$
Also,
$$x_0 \in \partial(X \setminus E) \Leftrightarrow x \notin \text{Int}(X \setminus E) \text{ and } \text{Ext}(X \setminus E) \Leftrightarrow x_0 \notin \text{Ext}(E) \text{ and } x_0 \notin \text{Int}(E).$$
Hence, acutually $\partial E = \partial(X \setminus E)$.

Proposition 2.2.3.

$$x_0 \in \partial E \Leftrightarrow \text{ For any } r > 0, B_X(x_0, r) \cap E \neq \varnothing \text{ and } B_X(x_0, r) \cap (X \setminus E) \neq \varnothing$$

Example 2.2.2. Let $(\mathbb{R}, d)$ be the usual metric on $\mathbb{R}$, where

$$d(x, y) = |x - y|.$$

Then, we know in this space,

$$\begin{aligned} B_{\mathbb{R}}(x_0, r) &= \{x \in \mathbb{R} \mid d(x, x_0) < r\} \\ &= \{x \in \mathbb{R} \mid |x - x_0| < r\} \\ &= \{x \in \mathbb{R} \mid -r + x_0 < x < r + x_0\}. \end{aligned}$$

Hence, suppose $E = [1, 2)$, then $\text{Int}(E) = (1, 2)$ since we know $B(x_0, r) = (x_0 - r, x_0 + r)$, so for all $x \in (1, 2)$, we know there is an open ball $B(x_0, r) \subseteq [1, 2)$ for some $r > 0$. Also, consider the endpoint $1, 2$, we can verify that these two points are not interior points. Besides, consider the points not in $[1, 2]$, it is trivial that they cannot be interior points.

Example 2.2.3. We consider (X, d_{disc}). Let $E \subseteq X$. If $x \in E$, we know

$$B\left(x, \frac{1}{2}\right) = \left\{y \mid d(y, x) < \frac{1}{2}\right\} = \{x\} \subseteq E.$$

Hence, $E \subseteq \text{Int}(E)$. Besides, for all $x \in \text{Int}(E)$, we know there exists $r > 0$ s.t. $B(x_0, r) \subseteq E$, also we know $x_0 \in B(x_0, r) \subseteq E$, so $x_0 \in E$, and thus $\text{Int}(E) \subseteq E$. Hence, $E = \text{Int}(E)$. Similarly, $\text{Int}(X \setminus E) = X \setminus E$. Suppose there is a $x \in X$ s.t. $x \in \partial E$, then $x \notin \text{Int}(E) = E$ and $x \notin \text{Ext}(E) = \text{Int}(X \setminus E) = X \setminus E$, so such x does not exist.

Definition 2.2.8 (Closure)**.** Let (X, d) be a metric space, and let $E \subseteq X$ and $x_0 \in X$. We say x_0 is an adherent point of E if for every $r > 0$, $B(x_0, r) \cap E \neq \varnothing$. The set of adeherent points is called the closure of E, and denoted by $\overline{E}$.

Proposition 2.2.4 (TFAE)**.**

(a) x_0 is an adherent point of E.

(b) x_0 is either an interior point or a boundary point of E.

(c) $\exists$ a sequence $\left\{X^{(n)}\right\}_{n=1}^{\infty}$ in E which converges to x_0 in (X, d).

proof from (a) to (b). Suppose $x_0 \in \overline{E}$, then $B(x_0, r) \cap E \neq \varnothing$ for all $r > 0$. If $\exists s > 0$ s.t. $B(x_0, s) \subseteq E$, then $x_0 \in \mathrm{Int}(E)$. If such s does not exists, then we know

$$B(x_0, r) \cap E \neq \varnothing \text{ and } B(x_0, r) \cap (X \setminus E) \neq \varnothing \text{ for all } r > 0,$$

so we can use Proposition 2.2.1 to conclude that x_0 must be a boundary point. ∎

proof from (b) to (c). Since either $x_0 \in \mathrm{Int}(E)$ or $x_0 \in \partial E$. If $x_0 \in \mathrm{Int}(E)$, then $x_0 \in E$, then we can choose $X^{(n)} = x_0$ for all $n \geq 1$. If $x_0 \in \partial E$, then given $n \in \mathbb{N}$, $\exists x_n \in B\left(x_0, \frac{1}{n}\right) \cap E \neq \varnothing$. Hence, $x_n \in E$ and $d(x_n, x_0) < \frac{1}{n}$. Pick such x_n to form $\left\{X^{(n)}\right\}_{n=1}^{\infty}$, then we know this sequence converges to x_0. ∎

proof from (c) to (a). Suppose $\left\{X^{(n)}\right\} \subseteq E$ s.t. $\lim_{n\to\infty} d\left(X^{(n)}, x_0\right) = 0$, then we want to show $x_0 \in \overline{E}$. Given any $r > 0$, choose $N \geq 1$ s.t.

$$d\left(X^{(n)}, x_0\right) < r \text{ when } n \geq N.$$

This implies for $n \geq N$, $X^{(n)} \in E$ and $X^{(n)} \in B(x_0, r)$, so we know $E \cap B(x_0, r) \neq \varnothing$ for all $r > 0$, which means $x_0 \in \overline{E}$. ∎

Remark 2.2.3. The equation (a) and (b) implies $\overline{E} = \mathrm{Int}(E) \cup \partial E$.

An alternative proof. Since we know $X = \mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E$ by Theorem 2.2.2, and $\overline{E} \subseteq X$, so

$$\begin{aligned}\overline{E} = \overline{E} \cap X &= \overline{E} \cap (\mathrm{Int}(E) \cup \mathrm{ext}(E) \cup \partial E) \\ &= (\overline{E} \cap \mathrm{Int}(E)) \cup (\overline{E} \cap \mathrm{Ext}(E)) \cup (\overline{E} \cap \partial E).\end{aligned}$$

Also, notice that

$$\overline{E} \cap \mathrm{Int}(E) = \mathrm{Int}(E) \quad \overline{E} \cap \mathrm{Ext}(E) = \varnothing \quad \overline{E} \cap \partial E = \partial E,$$

so $\overline{E} = \mathrm{Int}(E) \cup \partial E$. ∎

Corollary 2.2.1. $\overline{E} = \mathrm{Int}(E) \cup \partial E$.

Theorem 2.2.2. Let (X, d) be a metric space and $E \subseteq X$. Then,

$$X = \mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E$$

Remark 2.2.4. ∂E could be empty. (See previous example.)

Corollary 2.2.2. Let (X, d) be a metric space and $E \subseteq X$. Then

$$\overline{E} = \mathrm{Int}(E) \cup \partial E = X \setminus \mathrm{Ext}(E).$$

Lemma 2.2.1. $\overline{E} = E \cup \partial E$

Proof. We first show that $E \cup \partial E \subseteq \overline{E}$. For every point $x \in E$, we know $x \in B(x, r)$ for all $r > 0$, so $B(x, r) \cap E \neq \varnothing$. Also, by definition, we know $\partial E \subseteq \overline{E}$, so we're done.

Next, we show that $\overline{E} \subseteq E \cup \partial E$. For every $x \in \overline{E}$, if $x \in E$, then $x \in E \cup \partial E$. If not, since $x \in \overline{E}$, so $B(x, r) \cap E \neq \varnothing$ for all $r > 0$. Also, since $x \notin E$, and $x \in B(x, r)$, so $B(x, r) \cap (X \setminus E) \neq \varnothing$, otherwise $x \in B(x,r) \subseteq E$, which is a contradiction. Now we know for every $r > 0$, $B(x,r) \cap E \neq \varnothing$ and $B(x,r) \cap (X \setminus E) \neq \varnothing$, so $x \in \partial E$. ■

Lemma 2.2.2 (Discarded)**.** If $x \in \operatorname{Int}(E)$, then $x \in E$. In other words, $\operatorname{Int}(E) \subseteq E$.

Proof. If $x \in \operatorname{Int}(E)$, then there exists $r > 0$ s.t. $B(x,r) \subseteq E$, and thus $x \in B(x,r) \subseteq E$, which means $x \in E$. ■

Note 2.2.1. I thought we need Lemma 2.2.2 to prove Theorem 2.2.3, but I found it needless. Nevertheless, I still want to keep it since I think it is useful in some elsewhere.

Definition 2.2.9. Let (X,d) be a metric space and $E \subseteq X$. We say E is closed if $\partial E \subseteq E$. We say E is open if it doesn't contain any boundary points i.e. $\partial E \cap E = \varnothing$.

Theorem 2.2.3. E is closed if and only if $\overline{E} = E$.

Proof.

$$E \text{ is closed } \Rightarrow \partial E \subseteq E \Rightarrow \overline{E} = E \cup \partial E = E.$$
$$E = \overline{E} = E \cup \partial E \Rightarrow \partial E \subseteq E \Rightarrow E \text{ is closed.}$$

■

Theorem 2.2.4. E is open. $\Leftrightarrow \operatorname{Int}(E) = E$.

proof of $(\Rightarrow)$ **.** E is open means $\partial E \cap E = \varnothing$. Fix $x \in E$, since $x \notin \partial E$, so $\exists r > 0$ s.t. $B(x,r) \cap E = \varnothing$ or $B(x,r) \cap (X \setminus E) = \varnothing$. Since $x \in E$ and $x \in B(x,r)$, so $B(x,r) \cap (X \setminus E) = \varnothing$, which means $B(x,r) \subseteq E$, so $x \in \operatorname{Int}(E)$. Now we know $E \subseteq \operatorname{Int}(E)$. Also, we know $\operatorname{Int}(E) \subseteq E$ by Lemma 2.2.2. Hence, $\operatorname{Int}(E) = E$. ■

proof of $(\Leftarrow)$ **.** If $\operatorname{Int}(E) = E$, then given any $x \in E = \operatorname{Int}(E)$, there exists $r > 0$ s.t. $B(x,r) \subseteq E$. Hence, $B(x,r) \cap (X \setminus E) = \varnothing$, so $x \notin \partial E$, and thus $E \cap \partial E = \varnothing$. ■

Theorem 2.2.5. If $E \subseteq X$, then E is open $\Leftrightarrow X \setminus E$ is closed.

proof of $(\Rightarrow)$**.** Since we can write $X = \operatorname{Int}(E) \uplus \operatorname{Ext}(E) \uplus \partial E$, and E is open, so

$$X \setminus E = (\operatorname{Int}(E) \uplus \operatorname{Ext}(E) \uplus \partial E) \setminus E = (\operatorname{Int}(E) \uplus \operatorname{Ext}(E) \uplus \partial E) \setminus \operatorname{Int}(E) = \operatorname{Ext}(E) \uplus \partial E.$$

by Theorem 2.2.4. Now we want to show that $\partial(X \setminus E) \subseteq X \setminus E$, and we know

$$X \setminus E = \operatorname{Ext}(E) \uplus \partial E = \operatorname{Ext}(E) \uplus \partial(X \setminus E)$$

since $\partial E = \partial(X \setminus E)$. Hence, we have $\partial(X \setminus E) \subseteq X \setminus E$. ■

proof of $(\Leftarrow)$**.** Suppose $X \setminus E$ is closed, then $\partial(X \setminus E) \subseteq X \setminus E$, and since $\partial E = \partial(X \setminus E)$, so $\partial E \subseteq X \setminus E$, and thus $\partial E \cap E = \varnothing$, which means E is open. ■

Lecture 5

Definition 2.2.10. Let (X,d) be a metric space, $E \subseteq X$ and $x_0 \in E$. We say x_0 is an adherent point if for every $r > 0$, $B(x_0,r) \cap E \neq \varnothing$, and we denote $\overline{E}$ to the set of all adherent points. 16 Sep. 10:20

Remark 2.2.5. $E \subseteq \overline{E}$, since given any $x_0 \in E$ and $r > 0$, $x_0 \in B(x_0, r)$, so $B(x_0, r) \cap E \neq \varnothing$, and thus $E \subseteq \overline{E}$.

Remark 2.2.6. $\partial E \subseteq \overline{E}$. Given $x_0 \in \partial E$, we know for any $r > 0$, $B(x_0, r) \cap E \neq \varnothing$, so $x_0 \in \overline{E}$.

Proposition 2.2.5. $x_0 \in \overline{E}$ if and only if there exists $\left(X^{(n)}\right)_{n=1}^{\infty} \subseteq E$ s.t. $\lim_{n\to\infty} X^{(n)}$ exists and $\lim_{n\to\infty} X^{(n)} = x_0$.

proof of ($\Rightarrow$). Given $n \in \mathbb{N}$. Consider $B\left(x_0, \frac{1}{n}\right)$. We know $B\left(x_0, \frac{1}{n}\right) \cap E \neq \varnothing$. Choose $X^{(n)} \in B\left(x_0, \frac{1}{n}\right) \cap E$, then $d\left(x_0, X^{(n)}\right) < \frac{1}{n}$, which means $\lim_{n\to\infty} d\left(x_0, X^{(n)}\right) = 0$. Hence, there exists $\left(X^{(n)}\right) \subseteq E$ s.t. $\lim_{n\to\infty} X^{(n)} = x_0$. ■

proof of ($\Leftarrow$). There exists N s.t. $X^{(n)} \in B(x_0, r)$ when $n \geq N$. Given any $r > 0$, since $\lim_{n\to\infty} X^{(n)} = x_0$, so $\lim_{n\to\infty} d\left(X^{(n)}, x_0\right) = 0$. Hence, there exists N s.t. $d\left(X^{(n)}, x_0\right) < r$ when $n \geq N$. Hence, when $n \geq N$, we have $X^{(n)} \subseteq B(x_0, r)$. Since we know $X^{(n)} \in E$ for all n, so we know $B(x_0, r) \cap E \neq \varnothing$, so $x_0 \in \overline{E}$. ■

Proposition 2.2.6. Let (X, d) be a metric space and $E \subseteq X$, then

$$X = \operatorname{Int}(E) \uplus \operatorname{Ext}(E) \uplus \partial E.$$

Corollary 2.2.3. Let (X, d) be a metric space and $E \subseteq X$. Then,

$$\overline{E} = \operatorname{Int}(E) \cup \partial E = X \setminus \operatorname{Ext}(E) = E \cup \partial E.$$

Proof. Since

$$\begin{aligned}\overline{E} &= \overline{E} \cap X = \overline{E} \cap (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E)\\ &= \left(\overline{E} \cap \operatorname{Int}(E)\right) \cup \left(\overline{E} \cap \operatorname{Ext}(E)\right) \cup \left(\overline{E} \cap \partial E\right) = \operatorname{Int}(E) \uplus \partial E.\end{aligned}$$

Also,

$$X \setminus \operatorname{Ext}(E) = (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E) \setminus \operatorname{Ext}(E) = \operatorname{Int}(E) \cup \partial E = \overline{E}.$$

Besides, we know $\operatorname{Int}(E) \subseteq E \subseteq \overline{E}$, so

$$\overline{E} = \operatorname{Int}(E) \cup \partial E \subseteq E \cup \partial E.$$

Also, by Remark 2.2.5 and Remark 2.2.6, we know $E \cup \partial E \subseteq \overline{E}$, so we know $\overline{E} = E \cup \partial E$. ■

Definition 2.2.11. Let (X, d) be a metric space and $E \subseteq X$. We say E is open iff $\partial E \cap E \neq \varnothing$. We say E is closed iff $\partial E \subseteq E$.

Proposition 2.2.7.

$$E \text{ is open} \Leftrightarrow \operatorname{Int}(E) = E \Leftrightarrow X \setminus E \text{ is closed}.$$

proof of E is open $\Leftrightarrow \operatorname{Int}(E) = E$.

($\Rightarrow$) Since E is open, so $\partial E \cap E = \varnothing$. Hence,

$$\begin{aligned}E &= E \cap X = E \cap (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E)\\ &= (E \cap \operatorname{Int}(E)) \cup (E \cap \operatorname{Ext}(E)) \cup (E \cap \partial E) = \operatorname{Int}(E) \cup (E \cap \partial E) = \operatorname{Int}(E)\end{aligned}$$

since $E \cap \operatorname{Ext}(E) = \varnothing$ and we know $\partial E \cap E = \varnothing$.

($\Leftarrow$) Since $\operatorname{Int}(E) = E$, and $\operatorname{Int}(E) \cap \partial E = \varnothing$, so $E \cap \partial E = \varnothing$, and thus E is open.

■

proof of E is open $\Leftrightarrow X \setminus E$ is closed.

($\Rightarrow$) $X = \operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E$, so

$$X \setminus E = (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E) \setminus \operatorname{Int}(E) = \operatorname{Ext}(E) \cup \partial E = \operatorname{Int}(X \setminus E) \cup \partial(X \setminus E).$$

Hence, $\partial(X \setminus E) \subseteq X \setminus E$, which means $X \setminus E$ is closed.

($\Leftarrow$) $X \setminus E$ is closed, then $\partial(X \setminus E) \subseteq X \setminus E$, but $\partial E = \partial(X \setminus E)$, so $\partial E \subseteq X \setminus E$, and thus $\partial E \cap E = \varnothing$.

■

Remark 2.2.7. If $\partial E = \varnothing$, then E is open and closed.

Definition 2.2.12 (Clopen). If a set S is closed and open, then S is clopen.

Remark 2.2.8. Let (X, d) be a metric space, then $\varnothing$ is clopen, and we can deduce that X is also clopen since X is the complement of $\varnothing$ and we know S is open iff $X \setminus S$ is closed.

Remark 2.2.9. In $(\mathbb{R}, d)$, where d is the standard metric, then the only clopen set is $\mathbb{R}$ or $\varnothing$.

Remark 2.2.10. Let (X, d_{disc}) be the discrete metric space on X. Let E be any set, then E is open and closed. Given $x_0 \in E$, we know $B_{\text{disc}}\left(x_0, \frac{1}{2}\right) \subseteq E$, so $x_0 \in \operatorname{Int}(E)$, which means $E = \operatorname{Int}(E)$, so E is open. Now since $X \setminus E$ is also open, so E is closed. Thus, E is clopen.

Proposition 2.2.8. The following hold:

(a) E is open iff $E = \operatorname{Int}(E)$.

(b) E is closed iff for every convergent sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ in E, then the limit $\lim_{n\to\infty} X^{(n)} \in E$.

(c) Let $r > 0$, then

(i)
$$\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\} \text{ is closed.}$$

(ii)
$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\} \text{ is open.}$$

(d) Any singleton $\{x_0\}$ where $x_0 \in X$ is closed.

(e) E is open iff $X \setminus E$ is closed.

(f) (i) If $E_1, \ldots, E_n$ are open sets in X, then $E_1 \cap E_2 \cap \cdots \cap E_n$ is open.
(ii) If $F_1, \ldots, F_n$ are closed, then $F_1 \cup \cdots \cup F_n$ is closed.

(g) (i) If $\{E_\alpha\}_{\alpha \in I}$ is any collection of open sets in X, then $\bigcup_{\alpha \in I} E_\alpha$ is open.
(ii) If $\{F_\alpha\}_{\alpha \in I}$ is any collection of closed sets in X, then $\bigcap_{\alpha \in I} F_\alpha$ is closed.

(h) (i) If $E \subseteq X$, then $\operatorname{Int}(E)$ is the largest open set that contained in E i.e. $\operatorname{Int}(E)$ is open and if $V \subseteq E$ and V is open, then $V \subseteq \operatorname{Int}(E)$.
(ii) If $E \subseteq X$, then $\overline{E}$ is the smallest closed set containing E i.e. $\overline{E}$ is closed and if $E \subseteq K$ and K is closed, then $\overline{E} \subseteq K$.

proof of (b).

($\Rightarrow$) Since E is closed, so $\overline{E} = E$, and we know every convergent sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ converges to x_0 with $x_0 \in \overline{E}$ by Proposition 2.2.4. Thus, we have $x_0 \in E$.

($\Leftarrow$) Assume that every convergent sequence in E has its limit in E. We want to prove that E is closed, i.e. that $X \setminus E$ is open.

Take any point $y \in X \setminus E$. Suppose, for contradiction, that every ball around y meets E. That is, for each $k \in \mathbb{N}$ there exists a point

$$x^{(k)} \in E \cap B\big(y, \tfrac{1}{k}\big) .$$

Then, by construction, we have $x^{(k)} \to y$.

By our assumption, the limit of any convergent sequence from E must lie in E. Hence $y \in E$, contradicting the fact that $y \in X \setminus E$.

Therefore, there must exist some radius $r > 0$ such that

$$B(y, r) \cap E = \varnothing,$$

which means $B(y, r) \subseteq X \setminus E$. Thus every point of $X \setminus E$ is an interior point, so $X \setminus E$ is open. Hence E is closed.

$\blacksquare$

proof of (c).

(i) To show that $\overline{B}(x_0, r)$ is closed, it sufficies to show that $X \setminus \overline{B}(x_0, r)$ is open. Note that

$$X \setminus \overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) > r\} .$$

Let $y \in X \setminus \overline{B}(x_0, r)$, then define $\varepsilon = d(x_0, y) - r > 0$, then we can similarly prove that $B(y, \varepsilon) \subseteq X \setminus \overline{B}(x_0, r)$. Hence, $X \setminus \overline{B}(x_0, r) = \operatorname{Int}(X \setminus \overline{B}(x_0, r))$, and thus it is open.

(ii) If $y \in B(x_0, r)$, then $d(x_0, y) < r$. Let $\varepsilon = r - d(x_0, y) > 0$, then we claim that $B(y, \varepsilon) \subseteq B(x_0, r)$. Given $z \in B(y, \varepsilon)$, then $d(z, y) < \varepsilon$, then use triangle inequality we know $z \in B(x_0, r)$.

$\blacksquare$

proof of (d). It sufficies to show that $X \setminus \{x_0\}$ is open. Given $y \in X \setminus \{x_0\}$, so we can show that

$$B\left(y, \frac{d(y, x_0)}{2}\right) \subseteq X \setminus \{x_0\} .$$

Hence, $y \in \operatorname{Int}(X \setminus \{x_0\})$, and thus $X \setminus \{x_0\}$ is open. $\blacksquare$

proof of (f).

(i) Given $x_0 \in E_1 \cap E_2 \cap \cdots \cap E_n$, then $x_0 \in E_i$ for all $1 \leq i \leq n$. Thus, there exists $r_i > 0$ s.t.

$$B(x_0, r_i) \subseteq E_i \quad \text{for each } 1 \leq i \leq n.$$

Let $r = \min\{r_1, \ldots, r_n\} > 0$, then we know $B(x_0, r) \subseteq B(x_0, r_i) \subseteq E_i$ for all $1 \leq i \leq n$. Hence, $B(x_0, r) \subseteq E_1 \cap E_2 \cap \cdots \cap E_n$, and thus $E_1 \cap \cdots \cap E_n$ is open.

(ii) Now if $F_1, \ldots, F_n$ are closed, then $X \setminus F_1, \ldots, X \setminus F_n$ are open. Since we know $\bigcap_{i=1}^{n}(X \setminus F_i)$ is open, and

$$\bigcap_{i=1}^{n} (X \setminus F_i) = X \setminus \left(\bigcup_{i=1}^{n} F_i\right) ,$$

so $X \setminus (\bigcup_{i=1}^{n} F_i)$ is open, which means $\bigcup_{i=1}^{n} F_i$ is closed.

■

proof of (g).

(i) Suppose $x_0 \in \bigcup_{\alpha \in I} E_\alpha$, then there exists $\mathcal{B} \in I$ s.t. $x_0 \in E_{\mathcal{B}}$. Now since $E_{\mathcal{B}}$ is open, so there exists $r_{x_0} > 0$ s.t.

$$B(x_0, r_{x_0}) \subseteq E_{\mathcal{B}} \subseteq \bigcup_{i \in \alpha} E_\alpha.$$

Hence, $\bigcup_{\alpha \in I} E_\alpha$ is open.

(ii)

$$\left(X \setminus \left(\bigcap_{\alpha \in I} F_\alpha\right)\right) = \bigcup_{\alpha \in I} (X \setminus F_\alpha)$$

is open since $X \setminus F_\alpha$ is open for all $\alpha \in I$, so we have $\bigcap_{\alpha \in I} F_\alpha$ is closed.

Remark 2.2.11.

(1) $\bigcap_{\alpha \in I} E_\alpha$ may NOT be open. For example,

$$\bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

which is closed.

(2) $\bigcup_{\alpha \in I} F_\alpha$ may NOT be closed. For example,

$$\bigcup_{i=1}^{\infty} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] = (-1, 1),$$

which is open.

Note 2.2.2. In the proof of (f), if the index set I is infinite, then we can not pick $\min\{r_1, \ldots, r_n\}$, so we can not deduce that (f) is correct when there infinitely many open sets or closed sets.

■

proof of (h).

(i) We first claim that $\mathrm{Int}(E)$ is open.

Proof. Since for all $x \in \mathrm{Int}(E)$, $\exists r_x > 0$ s.t. $B(x, r_x) \subseteq E$, so

$$\mathrm{Int}(E) = \bigcup_{x \in \mathrm{Int}(E)} B(x, r_x),$$

and by (ii) of (c) and (i) of (g) in Proposition 2.2.8, we know $\mathrm{Int}(E)$ is open. ■

Now if we have $V \subseteq E$ and V is open, then $y \in V$ implies there exists $s > 0$ s.t. $B(y, s) \subseteq V$, and thus $B(y, s) \subseteq E$ since $V \subseteq E$. Hence, we know $y \in$ Int(E), and thus $V \subseteq \mathrm{Int}(E)$.

(ii) To show $\overline{E}$ is closed, it sufficies to show that $X \setminus \overline{E}$ is open. Note that

$$\overline{E} = X \setminus \mathrm{Ext}(E) = X \setminus \underbrace{\mathrm{Int}(X \setminus E)}_{\text{open}},$$

so $\overline{E}$ is closed. Now if $E \subseteq K$ and K is closed, then if $x \in \overline{E}$, we have $B(x,r) \cap E \neq \varnothing$ for all $r > 0$. Hence, $B(x,r) \cap K \neq \varnothing$ since $E \subseteq K$, so $x \in \overline{K} = K$ (since K is closed). Thus, $\overline{E} \subseteq K$.

■

Lecture 6

2.3 Relative topology

18 Sep. 10:20

Let (X,d) be a metric space and $Y \subseteq X$, then $(Y, d|_{Y\times Y})$ is also a metric space.

Example 2.3.1. Consider $(\mathbb{R}^2, d_2)$ and $X = \{(x,0) \mid x \in \mathbb{R}\}$, then on $(X, d_2|_{X\times X}) = (X,d)$, it is also a metric space.

Proof. Since

$$d((x,0),(y,0)) = \sqrt{(x-y)^2 + 0^2} = |x-y|,$$

so it is obvious that d is a metric.

Note that X is not open in $\mathbb{R}^2$. Also, if $E = \{(x,0) \mid -1 < x < 1\}$, then E is not open in $\mathbb{R}^2$, but E is open in $(X, d_2|_{X\times X})$. ⊛

Example 2.3.2. Suppose $X = (-1,1) \subseteq \mathbb{R}$, then $(X, d|_{X\times X})$ is a metric space. Consider $E = [0,1)$, then we know E is not closed in $(\mathbb{R}, d)$ since $1 \in \overline{E}$ but $1 \notin E$. But E is closed in $(X, d|_{X\times X})$ since $\overline{E} = E$ in $(X, d|_{X\times X})$.

Definition 2.3.1 (relatively open/close)**.** Let (X,d) be a metric space and $Y \subseteq X$. We say E is relatively open (resp. closed) in Y if E is open (resp. closed) in $(Y, d|_{Y\times Y})$.

Note 2.3.1. In the following context, if we say E is open in Y, then we mean E is "relatively" open, and if we say E is closed in Y, then we mean E is relatively closed in Y.

Note 2.3.2. If E is open/closed in Y, then $E \subseteq Y$. Otherwise, we cannot define $d|_{Y\times Y}(a,b)$ for $a,b \in E \setminus Y$.

Remark 2.3.1. If $Y \subseteq X$, and $(X,d), (Y, d|_{Y\times Y})$ are both metric spaces, then

$$B_Y(x,r) = \{y \in Y \mid d(y,x) < r\} = B_X(x,r) \cap Y.$$

Remark 2.3.2. If E is relatively open in Y, then given $x_0 \in E$, $\exists r_0 > 0$ s.t. $B_X(x_0, r_0) \cap Y \subseteq E$. This is because by Remark 2.3.1, we have

$$B_X(x_0,r_0) \cap Y = B_Y(x_0,r_0) \subseteq E.$$

Remark 2.3.3. A set $E \subseteq Y$ is relatively closed in Y if given any $r > 0$ and $x_0 \in Y$,

$$B_Y(x_0,r) \cap E \neq \varnothing,$$

then $x_0 \in E$. This is because "closed" gives $E = \overline{E}_Y$. Note that this statement is equivalent to

$$\text{If } x_0 \in \overline{E}_Y, \text{ then } x_0 \in E = E_Y.$$

Proposition 2.3.1. Let (X, d) be a metric space, and $Y \subseteq X$ and $E \subseteq Y$, then

(1) E is relatively open in Y iff $\exists$ open set V in (X, d) s.t. $E = V \cap Y$.

(2) E is relatively closed in Y iff $\exists$ closed set K in (X, d) s.t. $E = K \cap Y$.

proof of (1).

($\Rightarrow$) Given any $x \in E$, $\exists r_x > 0$ s.t. $B_X(x, r_x) \cap Y \subseteq E$. Let $V = \bigcup_{x \in E} B_X(x, r_x)$. Obviously, $V \cap Y = E$ and V is open.

($\Leftarrow$) Suppose $E = V \cap Y$, then given any $x \in E$, since V is open, so there exists $r > 0$ s.t. $B_X(x, r) \subseteq V$, and then $B_X(x, r) \cap Y \subseteq V \cap Y = E$. Since x is an interior point of E in Y, so $\mathrm{Int}_Y(E) = E$, and thus E is open in Y.

$\blacksquare$

proof of (2).

($\Rightarrow$) E is relatively closed in Y, then $Y \setminus E$ is relatively open, so there exists V open in X s.t. $Y \setminus E = V \cap Y$. Hence,

$$\begin{aligned} E = Y \setminus (Y \setminus E) = (X \setminus (Y \setminus E)) \cap Y &= (X \setminus (V \cap Y)) \cap Y \\ &= ((X \setminus V) \cup (X \setminus Y)) \cap Y \\ &= ((X \setminus V) \cap Y) \cup ((X \setminus Y) \cap Y) \\ &= (X \setminus V) \cap Y \end{aligned}$$

Let $E = (X \setminus V) \cap Y = K \cap Y$, then since $K = X \setminus V$ is closed in X, so we're done.

($\Leftarrow$) Suppose $E = K \cap Y$ for some closed K, then $Y \setminus E = (X \setminus K) \cap Y$, which means $Y \setminus E$ is relatively open in Y since $X \setminus K$ is open and by (a), so E is closed in Y.

$\blacksquare$

Example 2.3.3. Let $X = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ with $x, y \in X$, then

(i) $[0, 1]$ is open and closed in X.

(ii) $\partial_X [0, 1] = \varnothing$.

Proof.

(i) We want to find V open in $\mathbb{R}$ s.t.

$$[0, 1] = V \cap \overbrace{([0, 1] \cup [2, 3])}^{X},$$

we can choosed $V = \left(-\frac{1}{2}, \frac{3}{2}\right)$, so $[0, 1]$ is open in X.
We want to find K closed in $\mathbb{R}$ and

$$[0, 1] = K \cap ([0, 1] \cup [2, 3]),$$

and we can choosed $K = \left[-\frac{1}{2}, \frac{3}{2}\right]$, so $[0, 1]$ is closed in X.

(ii) If $x \in \partial_X [0, 1]$, then $B_X(x, r) \cap [0, 1]$ and $B_X(x, r) \cap [2, 3]$ are both nonempty for any $r > 0$. However, this is impossible for any x in X, so $\partial_X [0, 1] = \varnothing$.

$\circledast$

2.4 Cauchy sequence and complete metric space

Definition 2.4.1 (subsequence). Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ is a sequence in (X, d). Suppose $m \leq n_1 < n_2 < \ldots$, then $\left(X^{(n_j)}\right)_{j=1}^{\infty}$ is called a subsequence of $\left(X^{(n)}\right)_{n=m}^{\infty}$.

Example 2.4.1. $X^{(n)} = (-1)^n$ for all $n \in \mathbb{N}$.

Proof.

$$\left\{X^{(2n)}\right\}_{n=1}^{\infty}$$

is a subsequence of $\left\{X^{(n)}\right\}_{n=1}^{\infty}$. ⊛

Lemma 2.4.1. Let $\left\{X^{(n)}\right\}_{n=m}^{\infty}$ be a convergent sequence with $\lim_{n\to\infty} X^{(n)} = x$, then every subsequence of $\left\{X^{(n)}\right\}_{n=m}^{\infty}$ also converges to x_0.

Definition 2.4.2 (limit points). Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ is a sequence in (X, d), then we say L is a limit point of $\left(X^{(n)}\right)_{n=m}^{\infty}$ if for every $N \geq m$ and every $\varepsilon > 0$, there exists $n \geq N$ s.t. $d\left(X^{(n)}, L\right) \leq \varepsilon$.

Proposition 2.4.1. L is a limit point of $\left(X^{(n)}\right)_{n=m}^{\infty}$ iff there exists a subsequence

$$\left(X^{(n_j)}\right)_{j=1}^{n}$$

converges to L.

Proof.

($\Rightarrow$) Assume L is a limit point, now we build a subsequence converges to L by an inductive method. Our goal is to build a subsequence $\left\{X^{(n_j)}\right\}_{j=1}^{\infty}$ so that

$$d\left(X^{(n_j)}, L\right) < \frac{1}{j} \quad \forall 1 \leq j.$$

For $j = 1$, pick $N = m$, and pick $\varepsilon < \frac{1}{1}$ to pick $n_1 \geq N$ s.t.

$$d\left(X^{(n_1)}, L\right) \leq \varepsilon < \frac{1}{1}.$$

Now suppose $n_1, n_2, \ldots, n_{k-1}$ are all chosen, then now we can pick $N = n_{k-1} + 1$ and $\varepsilon < \frac{1}{k}$, so that we can pick $n_k \geq N$ s.t. $d\left(X^{(n_k)}, L\right) \leq \varepsilon < \frac{1}{k}$, so we're done. Now we show that this subsequence converges to L. For every $\varepsilon > 0$, we know there exists $0 < \frac{1}{k} < \varepsilon$, so for all $K \geq k$, we have

$$d\left(X^{(K)}, L\right) < \frac{1}{K} \leq \frac{1}{k} < \varepsilon,$$

so we're done.

($\Leftarrow$) Left as exercise to the reader.

■

Proposition 2.4.2. L is a limit point iff $L \in \bigcap_{N=1}^{\infty} \overline{S_N}$ where $S_N = \left\{X^{(K)}\right\}_{K \geq N}$.

Definition 2.4.3 (Cauchy sequence). Let $\left(X^{(n)}\right)_{n=m}^{\infty}$ be a sequence in (X, d). We say this sequence is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \geq m$ s.t. $d\left(X^{(j)}, X^{(k)}\right) < \varepsilon$ for all $j, k \geq N$.

Lemma 2.4.2. Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges in (X, d), then $\left(X^{(n)}\right)_{n=m}^{\infty}$ is a Cauchy sequence in (X, d).

Proof. Suppose $\lim_{n\to\infty} X^{(n)} = X_0$, then for every $\frac{\varepsilon}{2} > 0$, there exists $N \geq m$ s.t. $d\left(X^{(n)}, X_0\right) < \frac{\varepsilon}{2}$ for all $n \geq N$. If $j, k \geq N$, then

$$d\left(X^{(j)}, X^{(k)}\right) \leq d\left(X^{(j)}, X_0\right) + d\left(X^{(k)}, X_0\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

Example 2.4.2. A sequence in $\mathbb{Q}$ may not converges in $\mathbb{Q}$.

Proof. See teacher's note. ⊛

Definition 2.4.4 (Complete space). A metric space (X, d) is complete iff every Cauchy sequence converges to some points in X.

Remark 2.4.1. $\mathbb{Q} \subseteq \mathbb{R}$, and $(\mathbb{Q}, d)$ is not complete.

Remark 2.4.2. The limit of a convergent sequence in metric space is unique. If

$$\lim_{n\to\infty} x^{(n)} = y \quad \text{and} \quad \lim_{n\to\infty} x^{(n)} = z,$$

then suppose by contradiction, $y \neq z$. Then,

$$0 \leq d(y, z) \leq d\left(y, x^{(n)}\right) + d\left(z, x^{(n)}\right).$$

By squeeze theorem, we know $d(y, z) = 0$ and thus $y = z$.

Proposition 2.4.3. Let (X, d) be a metric space and let $(Y, d|_{Y\times Y})$ be a subspace of (X, d). If $(Y, d|_{Y\times Y})$ is complete, then Y is closed in X.

Proof. We want to show that $Y = \overline{Y}$, so we want to show for all $y \in \overline{Y}$, we have $y \in Y$. Now for every $y \in \overline{Y}$, then by Proposition 2.2.4, we know there exists a convergent sequence $\left\{Y^{(n)}\right\}_{n=1}^{\infty}$ in Y and converges to y. However, every convergent sequence is Cauchy, and since $(Y, d|_{Y\times Y})$ is complete, so $\left\{Y^{(n)}\right\}_{n=1}^{\infty}$ converges in Y, which means $y \in Y$, and we're done. ■

Proposition 2.4.4. If (X, d) is complete and $Y \subseteq X$ is closed, then $(Y, d|_{Y\times Y})$ is complete.

Proof. Given a Cauchy sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ in Y, so this is also a Cauchy sequence in X, so it converges in X. If $\exists x_0 \in X$ s.t. $\lim_{n\to\infty} X^{(n)} = x_0$. Since Y is closed, so $Y = \overline{Y}$, and by Proposition 2.2.4, we know $x_0 \in \overline{Y} = Y$, so $x_0 \in Y$, and thus $\left(X^{(n)}\right)_{n=1}^{\infty}$ also converges in Y. ■

Lecture 7

Completeness of $\mathbb{R}^n$ with d_2, d_1, d_∞ 23 Sep. 09:10

> **As previously seen.** (X, d_1) and (X, d_2) are Lipschitz equivalent if $\exists c_1, c_2 > 0$s.t.
> $$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_2(x, y) \quad \forall x, y \in X.$$

Theorem 2.4.1. Suppose (X, d_1) and (X, d_2) are Lipschitz equivalent, then

$$(X, d_1) \text{ is complete} \Leftrightarrow (X, d_2) \text{ is complete.}$$

Proof.

($\Rightarrow$) Given any Cauchy sequence $\left(x^{(n)}\right)_{n=1}^{\infty}$ in (X, d_2), then since $d_1(x, y) \leq \frac{1}{c_1} d_2(x, y)$, so $\left(x^{(n)}\right)_{n=1}^{\infty}$ is Cauchy in (X, d_1). Since (X, d_1) is complete, so there exists $x \in X$ s.t. $\lim_{n\to\infty} x_n = x \in (X, d_1)$. However, $x \in (X, d_2)$, so (X, d_2) is complete.

($\Leftarrow$) Similar.

■

Theorem 2.4.2. $(\mathbb{R}^n, d_2)$ is a complete metric space.

Corollary 2.4.1. Since $(\mathbb{R}^n, d_2), (\mathbb{R}^n, d_1), (\mathbb{R}^n, d_\infty)$ are Lipschitz equivalent, so they are all complete by Theorem 2.4.1 and Theorem 2.4.2.

2.5 Compact metric space

Definition 2.5.1 (Compact space). A metric space (X, d) is compact iff every sequence in (X, d) has at least one convergent subsequence convergeing in X. A subset $Y \subseteq X$ is compact if $(Y, d|_{Y\times Y})$ is compact. That is, $(Y, d|_{Y\times Y})$ is compact if for any sequence $\left(y^{(n)}\right)_{n=1}^{\infty} \subseteq Y$, there exists a subsequence $\left(y^{(n_j)}\right)_{j=1}^{\infty}$ and $y \in Y$ s.t. $\lim_{k\to\infty} y^{(n_k)} = y$.

Definition 2.5.2 (Bounded). Let (X, d) be a metric space and let $Y \subseteq X$. We say Y is bounded iff for any $x \in X$, there exists $r > 0$ s.t. $Y \subseteq B_X(x, r)$.

Theorem 2.5.1.

$$Y \text{ is bounded} \Leftrightarrow \exists x_0 \in X \text{ and } R > 0 \text{ s.t. } Y \subseteq B_X(x_0, R).$$

Proof. The "($\Rightarrow$)" is easy, so we just prove the other direction. Given any $x \in X$, we can choose $r_x = R + d(x, x_0)$.

Claim 2.5.1. $Y \subseteq B_X(x, r_x)$.

Proof. Let $y \in Y$, we know

$$d(y, x) \leq d(y, x_0) + d(x_0, x) < R + d(x_0, x).$$

Hence, $y \in B_X(x, r_x)$. ⊛

■

Proposition 2.5.1. Let (X, d) be a compact metric space. Then (X, d) is complete and bounded.

Proof.

- We want to show that (X, d) is complete. Given any Cauchy sequence $\left(x^{(n)}\right)_{n=1}^{\infty}$ in (X, d), then since (X, d) is compact, so there exists a compact subsequence $\left(x^{(n_k)}\right)_{k=1}^{\infty}$ in X s.t. $\lim_{k\to\infty} x^{(n_k)} = x$. Since $\left(x^{(n)}\right)_{n=1}^{\infty}$ is Cauchy sequence and $\left(x^{(n_k)}\right)_{k=1}^{\infty}$ converges to x, so $\lim_{n\to\infty} x^{(n)} = x$. (See Theorem A.1.1)
- Consider $x_0 \in X$. Suppose X is not bounded, then $B(x_0, n)$ will not contain X for all n. For each $n \in \mathbb{N}$,
$$\exists y^{(n)} \in X \text{ and } y^{(n)} \notin B_X(x_0, n) \text{ i.e. } d\left(y^{(n)}, x_0\right) \geq n.$$
Hence, $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ is a sequence in (X, d) with $d\left(y^{(n)}, x_0\right) \geq n$. Since (X, d) is compact, so there exists a convergent sequence $\left\{y^{(n_k)}\right\}_{k=1}^{\infty}$ and $y \in X$ s.t. $\lim_{k\to\infty} y^{(n_k)} = y$. Hence, there exists $R > 0$ s.t. $d\left(y, y^{(n_k)}\right) < R$ for all k which is big enough, but this means
$$n_k \leq d\left(y^{(n_k)}, x_0\right) \leq d\left(y^{(n_k)}, y\right) + d(y, x_0) < R + d(y, x_0),$$
which is a fixed value, but n_k can be arbitrary large, so this is a contradiction.

■

Corollary 2.5.1. Let (X, d) be a metric space and Y be a compact subset, then Y is closed and bounded.

Proof. Since Y is a compact subset, so $(Y, d|_{Y\times Y})$ is compact. Thus, Y is bounded by Proposition 2.5.1. Hence, $\exists y_0 \in Y$ and $R > 0$ s.t.
$$Y \subseteq B_Y(y_0, R) = B_X(y_0, R) \cap Y \subseteq B_X(y_0, R).$$
Let $y \in \overline{Y}$, then $\exists \left(y^{(n)}\right)_{n=1}^{\infty}$ in Y s.t. $\lim_{n\to\infty} y^{(n)} = y$. Also, since Y is compact, so for the convergent sequence $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, there is a subsequence $\left\{y^{(n_k)}\right\}_{k=1}^{\infty}$ and $y_0 \in Y$ s.t. $\lim_{k\to\infty} y^{(n_k)} = y_0 \in Y$. By uniqueness of limit in metric space, we know $y = y_0$, and thus $y \in \overline{Y}$. Hence, $\overline{Y} = Y$. (Actually, by Lemma 2.4.2, we know $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ is Cauchy, and then by Theorem A.1.1, we know $y = y_0$.) ■

Theorem 2.5.2 (Heine-Borel Theorem)**.** Let $(\mathbb{R}^n, d)$ be $\mathbb{R}^n$ with $d = d_2, d_\infty, d_1$, and let $E \subseteq \mathbb{R}^n$, then
$$E \text{ is compact} \Leftrightarrow E \text{ is closed and bounded.}$$

Proof.

($\Rightarrow$) Trivial by the corollary.

($\Leftarrow$) Suppose E is closed and bounded. Given a sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ in E. By Bolzano-Weierstrass Theorem , every bounded sequence has a convergent subsequnce. Since E is closed, so $E = \overline{E}$, and thus the convergent subsequence converges in E. Hence, E is compact.

■

Remark 2.5.1. In a metric space, closed and bounded do not imply compact but compact implies closed and bounded.

Example 2.5.1. Consider $(\mathbb{Z}.d_{\text{disc}})$, then $\mathbb{Z}$ is bounded since $\mathbb{Z} \subseteq B_{\text{disc}}(0, 2)$ and $\mathbb{Z}$ is closed in $\mathbb{Z}$ but $\mathbb{Z}$ is not compact since any subsequence of $\{n\}_{n\in\mathbb{N}}$ does not converge in (Z, d_{disc}).

Theorem 2.5.3. Let (X, d) be a metric space, let Y be a compact subset of X. Let $(V_\alpha)_{\alpha \in A}$ be a collection of open sets in X, and suppose that $Y \subseteq \bigcup_{\alpha \in A} V_\alpha$ (i.e. $(V_\alpha)_{\alpha \in A}$ covers Y). Then, there exists a finite subset $F \subseteq A$ s.t. $Y \subseteq \bigcup_{\alpha \in F} V_\alpha$.

Proof. We prove by contradiction. Suppose there does not exist a finite subset $F \subseteq A$ s.t. $Y \subseteq \bigcup_{\alpha \in F} V_\alpha$. For each $y \in Y \subseteq \bigcup_{\alpha \in A} V_\alpha$. $\exists \alpha \in A$ s.t. $y \in V_\alpha$. Since V_α is open, so there exists $r > 0$ s.t. $B(y, r) \subseteq V_\alpha$. Define

$$r(y) = \sup \{r > 0 : B_X(y, r) \subseteq V_\alpha \text{ for some } \alpha \in A\}.$$

Note that $r(y) > 0$ for all $y \in Y$. Now if we pick $r_0 = \inf\{r(y) : y \in Y\}$, then $r_0 \geq 0$.

- Case 1: $r_0 = 0$, there exists $y^{(n)} \in Y$ s.t. $0 < r\left(y^{(n)}\right) < \frac{1}{n}$. Thus, $\left(y^{(n)}\right)_{n=1}^\infty$ is a sequence in Y, and since Y is compact, so there exists a convergent subsequence $\left(y^{(n_k)}\right)_{k=1}^\infty$ converging to $y_0 \in Y$. Also, there exists $\varepsilon > 0$ and $\alpha \in A$ s.t. $B_X(y_0, \varepsilon) \subseteq V_\alpha$. Since $\lim_{k\to\infty} d\left(y^{(n_k)}, y_0\right) = 0$, so there exists $N > 0$ s.t. $j \geq N$ implies

$$y^{(n_j)} \in B_X\left(y_0, \frac{\varepsilon}{2}\right).$$

Claim 2.5.2. For all $j \geq N$, $B\left(y^{(n_j)}, \frac{\varepsilon}{2}\right) \subseteq B(y_0, \varepsilon)$.

Proof. Suppose $z \in B\left(y^{(n_j)}, \frac{\varepsilon}{2}\right)$, then $d\left(z, y^{(n_j)}\right) < \frac{\varepsilon}{2}$, and thus

$$d(z, y_0) \leq d\left(z, y^{(n_j)}\right) + d\left(y^{(n_j)}, y_0\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

⊛

Now since $B_X(y_0, \varepsilon) \subseteq V_\alpha$, so for $j \geq N$, $B\left(y^{(n_j)}, \frac{\varepsilon}{2}\right) \subseteq V_\alpha$, which means

$$r\left(y^{(n_j)}\right) \geq \frac{\varepsilon}{2} > 0.$$

However, this contradicts to the assumption that $r\left(y^{(n_j)}\right) < \frac{1}{n_j}$ for all j. Hence, Case 1 is impossible.

- Case 2: $\infty > r_0 > 0$. We know $r_0 \leq r(y)$ for all $y \in Y$ by definition. Hence, $0 < \frac{r_0}{2} < r(y)$. This means for each $y \in Y$, there exists $\alpha \in A$ s.t. $B_X\left(y, \frac{r_0}{2}\right) \subseteq V_\alpha$. Choose a point $y^{(1)} \in Y$ s.t. $\exists \alpha_1 \in A$ s.t. $B_X\left(y^{(1)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_1}$. Since V_{α_1} cannot cover Y, so there exists $y^{(2)} \in Y$ and $y^{(2)} \notin B_X\left(y^{(1)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_1}$. Hence, $d\left(y^{(2)}, y^{(1)}\right) \geq \frac{r_0}{2}$. Now we set the induction hypothesis: Suppose there exists $y^{(1)}, \ldots, y^{(k)} \in Y$ and $\alpha_1, \ldots, \alpha_k \in A$ s.t.

$$B_X\left(y^{(j)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_j} \text{ and } d\left(y^{(i)}, y^{(j)}\right) \geq \frac{r_0}{2} \quad \forall i \neq j,$$

and $B_X\left(y^{(1)}, \frac{r_0}{2}\right) \cup \cdots \cup B_X\left(y^{(k)}, \frac{r_0}{2}\right)$ cannot cover Y, then we can find

$$y^{(k+1)} \notin B_X\left(y^{(1)}, \frac{r_0}{2}\right) \cup \cdots \cup B_X\left(y^{(k)}, \frac{r_0}{2}\right),$$

and thus $d\left(y^{(k+1)}, y^{(i)}\right) \geq \frac{r_0}{2}$ for $1 \leq i \leq k$. Also, $\exists \alpha_{k+1}$ s.t. $B\left(y^{(k+1)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_{k+1}}$. Now we know $B\left(y^{(1)}, \frac{r_0}{2}\right) \cup \cdots \cup B\left(y^{(k+1)}, \frac{r_0}{2}\right)$ won't cover Y, then $\left\{y^{(k)}\right\}_{k=1}^\infty$ is a sequence in Y and $d\left(y^{(j)}, y^{(l)}\right) \geq \frac{r_0}{2}$. Since Y is compact, so there exists a subsequence of $\left\{y^{(k)}\right\}_{k=1}^\infty$ which is convergent, but it is impossible, so we have a contradiction.

- Case 3: $r_0 = \infty$. If so, then it means $\inf\{r(y) : y \in Y\} = \infty$, so $r(y) = \infty$ for all $y \in Y$, otherwise if for some $y' \in Y$, $r(y')$ is finite, then $r_0 \leq r(y')$, and will get a contradiction. Now we have $r(y) = \infty$ for all $y \in Y$. This means for all $r > 0$, there exists some $\alpha \in A$ s.t.

$B_X(y, r) \subseteq V_\alpha$. Now since Y is compact, so Y is bounded, which means for all $y \in Y$, there exists r_y s.t. $Y \subseteq B_X(y, r_y)$. However, since $r(y) = \infty$ and by the previous argument, we know $B_X(y, r_y) \subseteq V_{\alpha_y}$ for some $\alpha_y \in A$, and thus $Y \subseteq V_{\alpha_y}$, and thus V_{α_y} covers Y, which is a contradiciton.

■

Lecture 8

25 Sep. 10:20

Theorem 2.5.4 (Review Theorem 2.5.3). Let Y be a compact subset of a metric space (X, d) and let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of Y. Then $\exists$ a finite subcover of $\{V_\alpha\}_{\alpha \in A}$ i.e. $\exists \alpha_1, \dots, \alpha_n \in A$ s.t. $Y \subseteq \bigcup_{i=1}^n V_{\alpha_i}$.

Remark 2.5.2.

$$Y \text{ is compact } \Leftrightarrow \text{ Any open cover of } Y \text{ has a finite subcover.}$$

Proof. The $(\Rightarrow)$ direction is proved. Now we proved the other direction.

Claim 2.5.3. If (X, d) is a metric space and for all open cover of X, there exists finite subcover of X, then X is complete.

Proof. We will prove this by contradiction, using the definition of compactness (the open cover property).

1. Assumption Assume for the sake of contradiction that X is **compact** but **not complete**. Since X is not complete, there exists a **Cauchy sequence** $\{x_n\}_{n=1}^\infty$ in X that **does not converge** to any point $p \in X$.

2. Constructing the Open Cover Since $\{x_n\}$ does not converge to a point $p \in X$, for every $p \in X$, the point p is not the limit of the sequence. This means there exists some $\epsilon_p > 0$ such that the open ball $B_{\epsilon_p}(p)$ contains only a **finite number of terms** of the sequence $\{x_n\}$.

To see why, suppose for a contradiction that there was some $p \in X$ such that for all $\epsilon > 0$, the ball $B_\epsilon(p)$ contains an infinite number of terms of $\{x_n\}$. Let $\{x_{n_k}\}$ be a subsequence with $x_{n_k} \in B_{1/k}(p)$. This subsequence converges to p. Since $\{x_n\}$ is a Cauchy sequence and has a convergent subsequence, the entire sequence $\{x_n\}$ must converge to the same limit p, which contradicts our initial assumption. Therefore, the property holds: for every $p \in X$, there is an $\epsilon_p > 0$ such that $B_{\epsilon_p}(p)$ contains x_n for only finitely many n.

Consider the collection of open balls $\mathcal{U} = \{B_{\epsilon_p}(p) : p \in X\}$. Since the union of these balls covers every point $p \in X$, $\mathcal{U}$ is an **open cover** of X:

$$X \subseteq \bigcup_{p \in X} B_{\epsilon_p}(p).$$

3. Using Compactness to Find a Finite Subcover Since X is **compact**, the open cover $\mathcal{U}$ must have a **finite subcover**. That is, there exist a finite number of points $p_1, p_2, \dots, p_k \in X$ such that

$$X \subseteq B_{\epsilon_{p_1}}(p_1) \cup B_{\epsilon_{p_2}}(p_2) \cup \dots \cup B_{\epsilon_{p_k}}(p_k) = \bigcup_{i=1}^k B_{\epsilon_{p_i}}(p_i).$$

4. Reaching the Contradiction By the definition of ϵ_{p_i}, each ball $B_{\epsilon_{p_i}}(p_i)$ contains x_n for only a **finite number** of indices n. The union of a finite number of finite sets is a finite set. Therefore, the finite union $\bigcup_{i=1}^k B_{\epsilon_{p_i}}(p_i)$ can contain x_n for only a finite number of indices n. However, since this finite union covers all of X (step 3), it must contain **all** terms of the sequence $\{x_n\}_{n=1}^\infty$. Since the sequence $\{x_n\}$ is an infinite set of points, this is a **contradiction**.

The initial assumption that X is not complete must be false. Thus, every compact metric space is complete. ⊛

Suppose any open cover of Y has a finite subcover, then given any sequence $\left(y^{(n)}\right)_{n=1}^{\infty}$. Consider

$$\bigcup_{x\in Y} B_Y(x,1),$$

then this is an open cover of Y, and now we know there is a finite subcover

$$\bigcup_{i=1}^{k} B_Y(x_i,1)$$

of Y where $x_i \in Y$ for all i. Now since $\left(y^{(n)}\right)_{n=1}^{\infty}$ has infinitely many terms, so we know for some i, we have infinitely many terms of $\left(y^{(n)}\right)_{n=1}^{\infty} \subseteq B_Y(x_i,1)$ by Pigeonhole principle. Hence, there are infinitely many terms of $\left(y^{(n)}\right)_{n=1}^{\infty}$ are in

$$\left\{y\in Y : 0 \leq d(y,x_i) < \frac{1}{2}\right\} \cup \left\{y \in Y : \frac{1}{2} \leq d(y,x_i) < 1\right\}.$$

Thus, again, by Pigeonhold principle we know there are infinitely many terms of $\left(y^{(n)}\right)_{n=1}^{\infty}$ are in either one of the above two sets. By repeating split the space into half as what we do above, we know for all $k \geq 0$, there are infinitely many terms of $\left(y^{(n)}\right)_{n=1}^{\infty}$ has the following property: Every two terms of these "infinitely many terms" has distance less than $\frac{1}{2^k}$. Note that this means we can pick a subsequence of $\left(y^{(n)}\right)_{n=1}^{\infty}$ so that it is Cauchy, and since every Cauchy sequence converges in Y (Since Claim 2.5.3), so we're done.

■

Corollary 2.5.2. Let (X,d) be a metric space and let $K_1, K_2, \ldots$ be a sequence of nonempty compact subsets of X s.t. $K_{i+1} \subseteq K_i$ for $i \in \mathbb{N}$, that is, $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ for $i \in \mathbb{N}$, then

$$\bigcap_{i=1}^{\infty} K_i \neq \varnothing.$$

Proof. Suppose $\bigcap_{i=1}^{\infty} K_i = \varnothing$. Since K_i's are compact, so they are closed. Also, we have

$$\bigcup_{i=1}^{\infty} (K_1 \setminus K_n) = K_1 \setminus \left(\bigcap_{i=1}^{\infty} K_n\right) = K_1.$$

Let $V_i = K_1 \setminus K_i = K_1 \cap K_i^C$. Note that K_i^C is open in X. Hence, we have V_i is open in K_1, and thus $\{V_i\}_{i=1}^{\infty}$ is an open cover of K_1 in K_1. ($(K_1, d|_{K_1\times K_1})$ is compact.) By Theorem 2.5.3, we know there exists $\alpha_1, \alpha_2, \ldots, \alpha_l$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_l$ s.t.

$$\begin{aligned} K_1 \subseteq \bigcup_{i=1}^{l} V_{\alpha_i} &= \bigcup_{i=1}^{l} (K_1 \setminus K_{\alpha_i}) \\ &= K_1 \setminus \bigcap_{i=1}^{l} K_{\alpha_i} = K_1 \setminus K_{\alpha_l} \end{aligned}$$

since $K_{\alpha_1} \supseteq K_{\alpha_2} \supseteq \cdots \supseteq K_{\alpha_l}$. However, $K_{\alpha_l} \subseteq K_1$ and $K_{\alpha_l} \neq \varnothing$. Thus, we have a contradiction.
■

Example 2.5.2. Consider $I_1 = [0,1]$, and $I_2 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$, and picking $I_3, I_4, \ldots$ with same method, then $I_{n+1} \subseteq I_n$ for all n and they are compact, so

$$\bigcap_{i=1}^{\infty} I_i \neq \varnothing.$$

Theorem 2.5.5. Let (X, d) be a metric space.

(a) If Y is a compact subset of X, and $Z \subseteq Y$, then Z is compact iff Z is closed.

(b) If $Y_1, \ldots, Y_n$ are a finite collection of compact subsets of X, then $\bigcup_{i=1}^{n} Y_i$ are also compact.

proof of (a). If Z is compact, then by Corollary 2.5.1, we know Z is closed. Now we show that if Z is closed, then Z is comapct. If Z is closed, then $Y \setminus Z$ is open in Y, then we know

$$Y \setminus Z = V \cap Y$$

for some open set $V \subseteq Y$, so note that $(Y \setminus Z) \subseteq V$. Now suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of Z. Hence, we know $\{U_\alpha\}_{\alpha \in A} \cup \{V\}$ is an open cover of Y since the former covers Z and the latter covers $Y \setminus Z$. Now since Y is compact, so we know for some $\alpha_1, \alpha_2, \ldots, \alpha_n$, there is

$$Y \subseteq \left(\bigcup_{i=1}^{n} U_{\alpha_i} \right) \cup V,$$

and thus we can write

$$Z \subseteq Y \subseteq \left(\bigcup_{i=1}^{n} U_{\alpha_i} \right) \cup V.$$

However, note that $Z \cap V = \varnothing$ since

$$Z = Y \setminus (Y \setminus Z) = Y \setminus (V \cap Y) = (Y \setminus V) \cup (Y \setminus Y) = Y \setminus V.$$

Hence, we know

$$Z \subseteq \bigcup_{i=1}^{n} U_{\alpha_i},$$

and thus for any open cover of Z, we know there exists a finite subcover of Z, and we're done. $\blacksquare$

Chapter 3

Continuous functions on metric spaces

Suppose (X, d_X) and (Y, d_Y) are metric spaces. Let $f : X \to Y$ be a function from X to Y. Then we want that if $x \in X$ is close to $y \in X$, then, then $f(x) \in Y$ is close to $f(y) \in Y$.

Definition 3.0.1 (Continuous function). Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function. Suppose $x_0 \in X$, we will say f is continuous at x_0 iff for every $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{whereas } d_X(x, x_0) < \delta.$$

We say f is continuous if f is continuous at every point $x \in X$.

Definition 3.0.2 (Preimage). Let $f : X \to Y$ be a function from $X \to Y$ and $V \subseteq Y$. The preimage (inverse image) of V is

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \subseteq X.$$

Example 3.0.1. Suppose $f(x) = x^2$, then what is the preimage of $(1, \infty)$?

Answer.

$$f^{-1}((1, \infty)) = (-\infty, -1) \cup (1, \infty).$$

⊛

Now we build an equivalent definition of continuity. If f is continuous at x_0, then given any $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$f(x) \in B_Y(f(x_0), \varepsilon) \quad \text{whereas } x \in B_X(x_0, \delta).$$

Also, $f(x) \in B_Y(f(x_0), \varepsilon)$ if and only if

$$x \in f^{-1}(B_Y(f(x_0), \varepsilon)).$$

Hence, we have

Corollary 3.0.1. f is continuous at x_0 if and only if

$$\text{Given any } \varepsilon > 0, \exists \delta > 0 \text{ s.t. } B_X(x_0, \delta) \subseteq f^{-1}(B_Y(f(x_0), \varepsilon))..$$

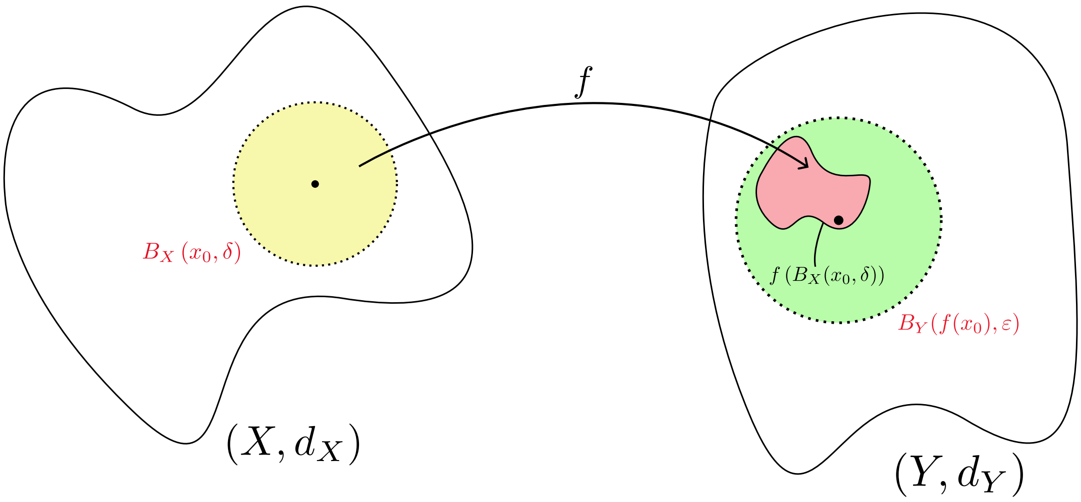

Figure 3.1: Continuous function from (X, d_X) to (Y, d_Y)

Remark 3.0.1. If $f : X \to Y$ is continuous and $K \subseteq X$, then $f|_K : K \to Y$ is continuous.

Proof. Given any point $x_0 \in K \subseteq X$. Since f is continuous at x_0, so $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{if } d_X(x, x_0) < \delta.$$

If $z \in K$ and $d_K(z, x_0) < \delta$, then $d_Y(f(z), f(x_0)) < \varepsilon$ since d_K is actually d_X but intersected to K, so f is continuous on K. ⊛

Theorem 3.0.1. Suppose that (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ is a function and let $x_0 \in X$, then TFAE:

(a) f is continuous at x_0.

(b) Whenever $\left(x^{(n)}\right)_{n=1}^{\infty}$ is a sequence in X converges to x_0, then

$$\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0) \text{ in } (Y, d_Y).$$

(c) For every open set $V \subseteq Y$ that contains $f(x_0)$, $\exists$ an open set $U \subseteq X$ containing x_0 s.t. $f(U) \subseteq V$, or equivalently, $U \subseteq f^{-1}(V)$.

proof of (a) ⇒ (b). Given any $\varepsilon > 0$, since f is continuous at x_0, so $\exists\delta > 0$ s.t. if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$. Now if $\lim_{n\to\infty} x^{(n)} = x_0$. Then there exists $N > 0$ s.t. $n \geq N$ implies $d_X\left(x^{(n)}, x_0\right) < \delta$. Hence, we know $d_Y\left(f\left(x^{(n)}\right), f(x_0)\right) < \varepsilon$. Hence, for this ε, we know there exists N s.t. $n \geq N$ implies $d_Y\left(f\left(x^{(n)}\right), f(x_0)\right) < \varepsilon$, and thus $\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0)$. ■

proof of (b) ⇒ (c). Let $f(x_0) \in V \subseteq Y$ for some open V.

Claim 3.0.1. There exists an open set U s.t. $x_0 \in U \subseteq X$ and $f(U) \subseteq V$.

Proof. If this is not true, then this implies that for every open set U with $x_0 \in U$, consider $U_n = B_X\left(x_0, \frac{1}{n}\right)$ for all n, then $\exists x_n \in U_n$ s.t. $f(x_n) \notin V$, then pick all of this x_n to be $\left\{x^{(n)}\right\}_{n=1}^{\infty}$ with $x^{(n)} \in U_n$ for all n, we know $\forall x^{(n)}$ we have $f\left(x^{(n)}\right) \notin V$. Then, $\left\{x^{(n)}\right\}_{n=1}^{\infty}$ is a sequence converges to x_0. By (b), we know $\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0) \in V$. However, by our choice, $f\left(x^{(n)}\right) \notin V$, so $f\left(x^{(n)}\right) \in Y \setminus V$. Since V is open, so $Y \setminus V$ is closed. Hence, we must have $\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0) \in Y \setminus V$, which is a contradiction. ⊛

■

proof of (c) ⇒ (a). Suppose (c) holds, then we want to show f is continuous at x_0, which means for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. $B_X(x_0, \delta) \subseteq f^{-1}\left(B_Y(f(x_0), \varepsilon)\right)$. Now consider $V = B_Y(f(x_0), \varepsilon) \subseteq Y$, then by (c) we know there exists open $U \subseteq X$ s.t. $x_0 \in U \subseteq X$ s.t. $U \subseteq f^{-1}(V)$. Now since U is open and $x_0 \in U$, so there exists $B_X(x_0, \delta) \subseteq U$, and thus

$$B_X(x_0, \delta) \subseteq U \subseteq f^{-1}(V) = f^{-1}\left(B_Y(f(x_0), \varepsilon)\right),$$

and we're done. ■

Theorem 3.0.2. Suppose $f : X \to Y$, then TFAE

(a) f is continuous.

(b) If $\lim_{n\to\infty} x^{(n)} = x \in (X, d_X)$, then $\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x)$ in (Y, d_Y).

(c) If V is open in Y, then $f^{-1}(V)$ is open in X.

(d) If F is closed in Y, then $f^{-1}(F)$ is closed in X.

Note 3.0.1. (a) says f is continuous, so it is a statement for every point in X and Theorem 3.0.1 is a theorem for a single $x_0 \in X$.

(a) ⇔ (b). By Theorem 3.0.1, we know it is true. ■

(c) holds iff for every $x \in X$, (c) in Theorem 3.0.1 holds. If we have (c) in Theorem 3.0.1 holding for all $x \in X$ and we know V is open in Y, then for each $x \in f^{-1}(V)$, we have $f(x) \in V$, so there exists an open set u_x s.t. $x \in u_x \subseteq f^{-1}(V)$. Hence,

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} u_x.$$

Since u_x is open, so $f^{-1}(V)$ is open.

Now if we have (c), then

If V is open in Y, then $f^{-1}(V)$ is open in X.

Now for all open $V \subseteq Y$ that contains $f(x_0)$ for some $x_0 \in X$, we know $f(x_0) \in V$, so $x_0 \in f^{-1}(V)$, and since $f^{-1}(V)$ is open in X, so we can just pick $U = f^{-1}(V)$, and we're done since this proof is valid for all $x_0 \in X$. ■

(c) ⇔ (d). If F is closed in Y, then $Y \setminus F$ is open in Y, and thus $f^{-1}(Y \setminus F)$ is open in X, and since $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is open, so $f^{-1}(F)$ is closed. This is the proof from (c) to (d), and the proof from (d) to (c) is similar. ■

Note 3.0.2. Theorem 3.0.2 is a stronger version of Theorem 3.0.1 since it states the version holding for all x.

Remark 3.0.2. Theorem 3.0.2 tells us that continuity tells us if the image is open/closed, then the preimage is open/closed. However, if f is continuous, then the image of an open set on f may not be open, and the image of a closed set on f may not be closed.

Example 3.0.2. Consider $f(x) = x^2$, then $f(-1, 1) = [0, 1)$ is not open.

Example 3.0.3. Consider $f(x) = \arctan(x)$, then $f([0, \infty)) = [0, \frac{\pi}{2}]$.

Now if $f : X \to Y$ and $g : X \to Z$, then consider

$$(f, g) : X \to Y \times Z \text{ with } x \mapsto (f(x), g(x)),$$

we know $Y \times Z$ has a natural metric, which is defined as: If (y_1, z_1) and (y_2, z_2) are in $Y \times Z$, then

$$d_{Y \times Z}\left((y_1, z_1), (y_2, z_2)\right) = d_Y(y_1, y_2) + d_Z(z_1, z_2).$$

Lemma 3.0.1. Consider $f : X \to Y$ and $g : X \to Z$ and $(f, g) : X \to Y \times Z$, then f and g are both continuous if and only if (f, g) is continuous.

Proof. Suppose f, g are both continuous, then given $x \in X$, we know $\lim_{n\to\infty} x^{(n)} = x \in (X, d_X)$ implies

$$\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x) \text{ and } \lim_{n\to\infty} g\left(x^{(n)}\right) = g(x).$$

Claim 3.0.2.

$$\lim_{n\to\infty} (f, g)\left(x^{(n)}\right) = (f, g)(x).$$

Proof. Check. ⊛

The other direction is also easy to prove. ∎

Lecture 9

30 Sep. 9:10

Corollary 3.0.2. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces.

(a) If $f : X \to Y$ is continuous at $x_0 \in X$, $g : Y \to Z$ is continuous at $f(x_0)$, then $g \circ f : X \to Z$ is continuous at x_0.

(b) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

proof of (a). Fix $x_0 \in X$. Since f is continuous at x_0, we have

$$\lim_{n\to\infty} x^{(n)} = x_0 \Rightarrow \lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0) \in Y.$$

Recall $(g \circ f)(x) = g(f(x))$. Since g is continuous at $f(x_0) \in Y$. It follows that

$$\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0) \Rightarrow \lim_{n\to\infty} g\left(f\left(x^{(n)}\right)\right) = g(f(x_0)).$$

Note that this means

$$\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0) \Rightarrow \lim_{n\to\infty} (g \circ f)\left(x^{(n)}\right) = (g \circ f)(x_0).$$

Hence,

$$\lim_{n\to\infty} x^{(n)} = x_0 \Rightarrow \lim_{n\to\infty} (g \circ f)\left(x^{(n)}\right) = (g \circ f)(x_0),$$

which means $g \circ f$ is continuous at x_0. ∎

3.1 Continuity and Product Spaces

Given two functions $f : X \to Y$ and $g : X \to Z$, we can define the pairing $(f, g) : X \to Y \times Z$ by

$$(f, g)(x) = (f(x), g(x)).$$

Example 3.1.1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 + 3$ and $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = 4x$, then we can define $(f, g) : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$(f, g)(x) = (x^2 + 3x, 4x).$$

Definition 3.1.1 (Product metric). Let (Y, d_Y) and (Z, d_Z) be metric spaces. Define a metric $d^1_{Y\times Z}$ on $Y \times Z$ by

$$d^1_{Y\times Z}((y_1, z_1), (y_2, z_2)) = d_Y(y_1, y_2) + d_Z(z_1, z_2).$$

Also, we can define $d^\infty_{Y\times Z}$ by

$$d^\infty_{Y\times Z}((y_1, z_1), (y_2, z_2)) = \max\{d_Y(y_1, y_2), d_Z(z_1, z_2)\}.$$

Finally, we can define

$$d^2_{Y\times Z}((y_1, z_1), (y_2, z_2)) = \sqrt{(d_Y(y_1, y_2))^2 + (d_Z(z_1, z_2))^2}.$$

Proposition 3.1.1. Let (Y, d_Y) and (Z, d_Z) be metric spaces, then $d^1_{Y\times Z}, d^2_{Y\times Z}, d^\infty_{Y\times Z}$ are metrics on $Y \times Z$.

Proof. Here we only prove $d^1_{Y\times Z}$ is a metric. For $(y_1, z_1), (y_2, z_2), (y_3, z_3)$ in $Y \times Z$.

- $d^1_{Y\times Z}((y_1, z_1), (y_2, z_2)) = d_Y(y_1, y_2) + d_Z(z_1, z_2) \geq 0$.
- If $d^1_{Y\times Z}((y_1, z_1), (y_2, z_2)) = 0$, then $y_1 = y_2 = z_1 = z_2 = 0$.
- $d^1_{Y\times Z}((y_1, z_1), (y_2, z_2)) = d^1_{Y\times Z}((y_2, z_2), (y_1, z_1))$.
- $$\begin{aligned} d^1_{Y\times Z}((y_1, z_1), (y_3, z_3)) &= d_Y(y_1, y_3) + d_Z(z_1, z_3) \\ &\leq d_Y(y_1, y_2) + d_Y(y_2, y_3) + d_Z(z_1, z_2) + d_Z(z_2, z_3) \\ &= d^1_{Y\times Z}((y_1, z_1), (y_2, z_2)) + d^1_{Y\times Z}((y_2, z_2), (y_3, z_3)). \end{aligned}$$

■

Exercise 3.1.1. Show that $d^2_{Y\times Z}$ is a metric.

Proof. We only show the triangle inequality holds here. We use d instead of $d^2_{Y\times Z}$ here. For $(y_1, z_1), (y_2, z_2), (y_3, z_3)$ here, we have

$$\begin{aligned} & d((y_1, z_1), (y_3, z_3))^2 = d_Y(y_1, y_3)^2 + d_Z(z_1, z_3)^2 \leq (d_Y(y_1, y_2) + d_Y(y_2, y_3))^2 + (d_Z(z_1, z_2) + d_Z(z_2, z_3))^2 \\ & = d_Y(y_1, y_2)^2 + d_Y(y_2, y_3)^2 + d_Z(z_1, z_2)^2 + d_Z(z_2, z_3)^2 + 2(d_Y(y_1, y_2)d_Y(y_2, y_3) + d_Z(z_1, z_2)d_Z(z_2, z_3)) \\ & \leq d_Y(y_1, y_2)^2 + d_Y(y_2, y_3)^2 + d_Z(z_1, z_2)^2 + d_Z(z_2, z_3)^2 + 2\sqrt{d_Y(y_1, y_2)^2 + d_Z(z_1, z_2)^2}\sqrt{d_Y(y_2, y_3)^2 + d_Z(z_2, z_3)^2} \\ & = \left(\sqrt{d_Y(y_1, y_2) + d_Z(z_1, z_2)} + \sqrt{d_Y(y_2, y_3) + d_Z(z_2, z_3)}\right)^2 \\ & = (d((y_1, z_1), (y_2, z_2)) + d((y_2, z_2), (y_3, z_3)))^2. \end{aligned}$$

■

Proposition 3.1.2. $\lim_{n\to\infty}(y_n, z_n) = (y, z)$ in $Y \times Z$ w.r.t. d^1, d^2, d^∞ iff

$$\lim_{n\to\infty} y_n = y \text{ in } (Y, d_Y) \text{ and } \lim_{n\to\infty} z_n = z \text{ in } (Z, d_Z),$$

Proof. We prove the case w.r.t. $d^\infty_{Y\times Z}$ metric. Since

$$\lim_{n\to\infty}(y_n, z_n) = (y, z) \Leftrightarrow \lim_{n\to\infty} d^\infty_{Y\times Z}((y_n, z_n), (y, z)) = 0 \Leftrightarrow \lim_{n\to\infty} \max\{d_Y(y_n, y), d_Z(z_n, z)\} = 0.$$

Also,

$$\begin{aligned} 0 &\le d_Y(y_n, y) \le \max\{d_Y(y_n, y), d_Z(z_n, z)\} \\ 0 &\le d_Z(z_n, z) \le \max\{d_Y(y_n, y), d_Z(z_n, z)\}. \end{aligned}$$

Hence, by squeeze theorem, we must have $\lim_{n\to\infty} d_Y(y_n, y) = \lim_{n\to\infty} d_Z(z, z_n) = 0$.
If $\lim_{n\to\infty} d_Y(y_n, y) = 0$ and $\lim_{n\to\infty} d_Z(z_n, z) = 0$, then

$$\lim_{n\to\infty} d^\infty_{Y\times Z}((y_n, z_n), (y, z)) = \lim_{n\to\infty} \max\{d_Y(y_n, y), d_Z(z_n, z)\} = 0.$$

Hence, $\lim_{n\to\infty}(y_n, z_n) = (y, z)$ in $d^\infty_{Y\times Z}$ metrics. ∎

Theorem 3.1.1. Let (Y, d_Y) and (Z, d_Z) be metric spaces. On $Y \times Z$, we have the metric d^1, d^2, d^∞. The map $(f, g) : X \to Y \times Z$ is continuous iff $f : X \to Y$ and $g : X \to Z$ are both continuous.

Proof. If (f, g) is continuous. Take any sequence $x^{(n)}$ with $\lim_{n\to\infty} x^{(n)} = x$ in (X, d_X). Then,

$$\lim_{n\to\infty}(f, g)\left(x^{(n)}\right) = (f, g)(x) \in Y \times Z.$$

Recall that $(f, g)\left(x^{(n)}\right) = \left(f\left(x^{(n)}\right), g\left(x^{(n)}\right)\right)$. Hence, we have

$$\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x) \in (Y, d_Y) \text{ and } \lim_{n\to\infty} g\left(x^{(n)}\right) = g(x) \in (Z, d_Z)$$

by Proposition 3.1.2. Thus, f, g are both continuous at x. Since x can be arbirary point in (X, d_X), so f, g are both continuous. ∎

Lemma 3.1.1. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions, and define $(f, g) : X \to \mathbb{R}^2$. We give $\mathbb{R}^2$ the Euclidean metric. Then, f, g are both continuous iff (f, g) is continuous.

Proof. By Theorem 3.1.1. Choose d^2 on $\mathbb{R} \times \mathbb{R}$, then $d^2 = d_2 =$ Euclidean metric. ∎

Lemma 3.1.2. The following functions are continuous.

- $(x, y) \mapsto x + y$ on $\mathbb{R}^2$
- $(x, y) \mapsto x - y$ on $\mathbb{R}^2$
- $(x, y) \mapsto x \cdot y$ on $\mathbb{R}^2$
- $(x, y) \mapsto \max\{x, y\}$ on $\mathbb{R}^2$
- $(x, y) \mapsto \frac{x}{y}$ on $\mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}$.
- $x \mapsto cx$ on $\mathbb{R}$ for any $c \in \mathbb{R}$.

Proof. We prove the $(x, y) \mapsto x \cdot y$ case. Define $M(x, y) = x \cdot y$. Choose $(x_0, y_0) \in \mathbb{R}^2$. Given $\varepsilon > 0$, we want to find $\delta > 0$ s.t.

$$d_2((x, y), (x_0, y_0)) = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |M(x, y) - M(x_0, y_0)| < \varepsilon.$$

We will choose δ later. Now suppose we have chosen some appropriate δ, then we have $|x-x_0|, |y-y_0| < \delta$. Then,

$$|M(x,y) - M(x_0,y_0)| = |xy - x_0y_0| = |x(y-y_0) + y_0(x-x_0)|.$$

Now we choose some $\delta \leq 1$. Then since $|x-x_0| < \delta \leq 1$, we have

$$|x| = |x - x_0 + x_0| < |x-x_0| + |x_0| < 1 + |x_0|.$$

Thus, we have

$$\begin{aligned}|M(x,y) - M(x_0,y_0)| &\leq |x||y-y_0| + |y_0||x-x_0| \\ &< (1+|x_0|)|y-y_0| + |y_0||x-x_0| \\ &< (1+|x_0|)\delta + |y_0|\delta \\ &= (1+|x_0|+|y_0|)\delta.\end{aligned}$$

Hence, we can choose

$$\delta = \min\left\{1, \frac{\varepsilon}{1+|x_0|+|y_0|}\right\},$$

and we will have

$$|M(x,y) - M(x_0,y_0)| < \varepsilon \text{ whenever } d_2\left((x,y),(x_0,y_0)\right) < \delta.$$

■

Proof. Here we prove the $(x,y) \mapsto \max\{x,y\}$ case. Note that

$$\max\{x,y\} = \frac{x+y+|x-y|}{2}.$$

Then, we have

$$\begin{aligned}|\max\{x,y\} - \max\{a,b\}| &= \left|\frac{x+y+|x-y|}{2} - \frac{a+b+|a-b|}{2}\right| \\ &= \frac{1}{2}\left|(x-a)+(y-b)+|x-y|-|a-b|\right| \\ &\leq \frac{1}{2}\left(|x-a|+|y-b|+\left||x-y|-|a-b|\right|\right) \\ &\leq \frac{1}{2}\left(|x-a|+|y-b|+|(x-y)-(a-b)|\right) \\ &= \frac{1}{2}\left(|x-a|+|y-b|+|(x-a)+(b-y)|\right) \\ &\leq |x-a|+|b-y|.\end{aligned}$$

Note that if $d_2\left((x,y),(a,b)\right) < \delta$, then $|x-a| < \delta$ and $|y-b| < \delta$. Hence, for every $\varepsilon > 0$, we can just pick $\delta = \frac{\varepsilon}{2}$, and then we can show that

$$|\max\{x,y\} - \max\{a,b\}| < \varepsilon.$$

■

Corollary 3.1.1. Let (X,d) be a metric space, and let $f,g : X \to \mathbb{R}$ be functions. Let $c \in \mathbb{R}$. If f,g are continuous on X, then $f+g, f-g, f\cdot g, \max\{f,g\}, \min\{f,g\}, cf$ are continuous. Also, $\frac{f}{g}$ is also continuous at x_0 if $g(x_0) \neq 0$.

Proof. For example, we know

$$(f+g)(x) = \text{Add} \circ (f,g)(x) = \text{Add}(f(x),g(x)) = f(x)+g(x).$$

Since (f,g) is continuous and Add is continuous, so the composition function Add $\circ\, (f,g)$ is also continuous. ■

3.2 Continuity and Compactness

Theorem 3.2.1. Let $f: X \to Y$ be a continuous function from (X, d_X) to (Y, d_Y). Let $K \subseteq X$ be a compact subset of X. Then

$$f(K) = \{f(x) \mid x \in K\}$$

is also compact in Y.

Proof. Given any sequence $\left(y^{(n)}\right)_{n=1}^{\infty}$ in $f(K)$, then $\exists x^{(n)} \in K$ s.t. $f\left(x^{(n)}\right) = y^{(n)}$. Since K is compact, then there exists a convergent subsequence $\left(x^{(n_k)}\right)_{k=1}^{\infty}$ s.t. $\lim_{k\to\infty} x^{(n_k)} = x_* \in K$. Since f is continuous, so $\lim_{n\to\infty} f\left(x^{(n_k)}\right) = f\left(x_*\right)$ in Y. Hence, $\lim_{n\to\infty} y^{(n_k)} = f(x_*) \in f(K)$. Thus, $f(K)$ is compact. ■

Antoher method. Let $\{V_\alpha : \alpha \in A\}$ be any open cover of $f(K)$ i.e. $f(K) \subseteq \bigcup_{\alpha\in A} V_\alpha$ and V_α is open in Y. Hence,

$$K \subseteq \bigcup_{\alpha \in A} f^{-1}(V_\alpha).$$

Note that since f is continuous and V_α is open, so $f^{-1}(V_\alpha)$ is open by Theorem 3.0.2. Hence, $\left\{f^{-1}(V_\alpha) : \alpha \in A\right\}$ is an open cover of K. Since K is compact, then

$$K \subseteq \bigcup_{i=1}^{n} f^{-1}\left(V_{\alpha_i}\right)$$

Hence, $f(K) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$, which shows $f(K)$ is compact. ■

Lecture 10

2 Oct. 10:20

Proposition 3.2.1. Let (X, d) be a compact metric space and let $f: X \to \mathbb{R}$ be a continuous map. Then

(1) f is bounded on X i.e. $f(X)$ is bounded in $\mathbb{R}$.

(2) $\exists x_{\max}, x_{\min} \in X$ s.t.

$$f\left(x_{\max}\right) = \max_{x\in X} f(x) \quad f\left(x_{\min}\right) = \min_{x\in X} f(x).$$

Proof. Since X is compact and f is continuous, so $f(X)$ is compact in $\mathbb{R}$ and thus $f(X)$ is bounded by Corollary 2.5.1. Now since $\{f(x) \mid x \in X\}$ is bounded in $\mathbb{R}$, so $\exists p_1, p_2$ s.t. $p_1 \leq f(x) \leq p_2$ for all $x \in X$. Let $M = \sup_{x\in X} f(x)$ and $P = \inf_{x\in X} f(x)$. Thus, there exists $\{y_n\}_{n=1}^{\infty} \subseteq f(X)$ s.t. $y_n \leq M$ and $\lim_{n\to\infty} y_n = M$ for all n. Hence, $\exists \left\{x^{(n)}\right\}_{n=1}^{\infty}$ s.t. $y_n = f\left(x^{(n)}\right)$ for all n. Since X is compact, so there exists a subsequence $\left\{x^{(n_k)}\right\}_{k=1}^{\infty}$ s.t.

$$\lim_{k\to\infty} x^{(n_k)} = x_* \in X.$$

Since f is continuous, so $\lim_{k\to\infty} f\left(x^{(n_k)}\right) = f\left(x_*\right)$, which means $\lim_{k\to\infty} y_{n_k} = f\left(x^*\right)$. Since $\lim_{n\to\infty} y_n = M$, so $f(x_*) = M$, and thus $f\left(x_*\right) = \max_{x\in X} f(x)$.

Question. Why we can always find $\{y_n\}_{n=1}^{\infty} \subseteq f(X)$ converges to $M = \sup\{f(X)\}$?

Answer. Recall the definition of sup, we know $\forall \varepsilon > 0$, $M - \varepsilon < y_\varepsilon \leq M$ for some $y_\varepsilon \in f(X)$, so for all $\varepsilon = \frac{1}{N}$, we can pick all y_N to form a sequence converge to M. ⊛

Question. Why $\max_{x\in X} f(x) = \sup_{x\in X} f(x)$ here?

Answer. Since X is compact, so $x_* \in X$, which has been proved, and thus $M = f(x_*) \in f(X)$, which means $\sup_{x\in X} f(x) \in f(X)$. This is equivalent to $\sup_{x\in X} f(X) = \max_{x\in X} f(x)$. ⊛

■

Example 3.2.1. This proposition is false if X is not compact.

Proof. Consider $f : (0,1) \to \mathbb{R}$ and $f(x) = x$, then f can't achieve its sup on $(0,1)$.

Note 3.2.1. $(0,1)$ is not compact since if it is compact, then it is closed, and thus all convergent sequence in $(0,1)$ converging in $(0,1)$, but consider the sequence $\left\{\frac{1}{n}\right\}_{n=2}^{\infty}$, and this sequence converges to $0 \notin (0,1)$.

⊛

3.3 Uniformly Continuous

Definition 3.3.1 (uniformly continuous)**.** Let $f : X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y), we say f is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$d_Y(f(x), f(x')) < \varepsilon \text{ whenever } x, x' \in X \text{ and } d_X(x, x') < \delta.$$

Remark 3.3.1. This δ is independent of x'.

Example 3.3.1. $f(x) = \frac{1}{x}$ is continuous on $(0,1]$, and f is not uniformly continuous on $(0,1]$.

Proof. Let $\varepsilon = 10$. Suppose $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \text{ if } |x-y| < \delta.$$

We may assume $\delta < 1$. Choose $x = \delta$ and $y = \frac{\delta}{11}$, then $|x-y| = \frac{10}{11}\delta < \delta$, and

$$|f(x) - f(y)| = \left|\frac{1}{\delta} - \frac{11}{\delta}\right| = \frac{10}{\delta} > 10.$$

⊛

Theorem 3.3.1. Suppose X is compact, then $f : X \to Y$ is continuous iff f is uniformly continuous.

Proof. If f is uniformly continuous, then f is continuous. Now we show the other direction. If f is continuous, then by contradiction, if it is not uniformly continuous, then $\exists \varepsilon > 0$ s.t. no matter how small δ is, then $\exists p, q$ s.t. $d_X(p,q) < \delta$ and $d_Y(f(p), f(q)) \geq \varepsilon$. Choose $\delta = \frac{1}{n}$, then exists $p^{(n)}, q^{(n)} \in X$ s.t.

$$d_X\left(p^{(n)}, q^{(n)}\right) < \frac{1}{n} \text{ and } d_Y\left(f\left(p^{(n)}\right), f\left(q^{(n)}\right)\right) \geq \varepsilon.$$

Since X is compact, so $\left\{p^{(n)}\right\}$ has a convergent subsequence, say

$$\lim_{k\to\infty} p^{(n_k)} = p \in X.$$

Also, there exsits $\left\{q^{(n_k)}\right\}$ s.t. $\lim_{k\to\infty} q^{(n_k)} = p$ since $\lim_{k\to\infty} d_X\left(p^{(n_k)}, q^{(n_k)}\right) = 0$. (By Theorem A.1.4). Thus, we have

$$\lim_{k\to\infty} d_Y\left(f\left(p^{(n_k)}\right), f\left(q^{(n_k)}\right)\right) = 0$$

since f is continuous. Hence, it is a contradiction, since

$$\lim_{n\to\infty} d_Y\left(f\left(p^{(n)}\right), f\left(q^{(n)}\right)\right) \geq \varepsilon.$$

■

3.4 Connectedness

Definition 3.4.1 (disconnected/connected)**.** Let (X, d) be a metric space. We say X is disconnected iff $\exists$ non-empty open V, W in X s.t. $V \cup W = X$ and $V \cap W = \varnothing$. Also, we called X is connected if it is non-empty and not disconnected.

Remark 3.4.1. X is disconnected iff X has a nonempty proper subset which is both open and closed.

Proof.

($\Rightarrow$) $V \cup W = X$ and $V \cap W = \varnothing$, so $X \setminus V = W$ is open, and thus V is closed. Since we already know V is open, so V is a proper subset of X that is clopen.

($\Leftarrow$) Suppose V is clopen in X and $V \neq X$. Let $W = X \setminus V \neq \varnothing$, then we know W is open since V is closed and thus $V \cup W = X$ and $V \cap W = \varnothing$ and V, W are both non-empty open in X, so X is disconnected.

■

Example 3.4.1. Suppose $X = [1, 2] \cup [3, 4]$, then X is disconnected.

Proof. Since

$$[1, 2] = (-\infty, 2.5) \cap X,$$

so $[1, 2]$ is open in X, and similarly we can show $[3, 4]$ is open in X, and $[1, 2] \cup [3, 4] = X$, and $[1, 2] \cap [3, 4] = \varnothing$, so X is disconnected. ⊛

Definition 3.4.2. Let (X, d) be a metric space and $Y \subseteq X$. We say Y is connected (resp. disconnected) iff the metric space $(Y, d|_{Y\times Y})$ is connected (resp. connected).

Remark 3.4.2. $Y \subseteq X$ is disconnected iff $\exists U, V$ open in X s.t. $Y \subseteq U \cup V$ with $U \cap Y \neq \varnothing$ and $V \cap Y \neq \varnothing$ and $U \cap V \cap Y = \varnothing$.

Proof.

($\Rightarrow$) If Y is disconnected, then $\exists$ open sets O_1, O_2 in Y s.t. $Y \subseteq O_1 \cup O_2$ and $O_1 \neq \varnothing$ and $O_2 \neq \varnothing$ and $O_1 \cap O_2 = \varnothing$. Since O_1 is open in Y, so there exists open set U_1 in X s.t. $O_1 = U_1 \cap Y \neq \varnothing$. Similarly, we know there exists U_2 open in X s.t. $O_2 = U_2 \cap Y \neq \varnothing$. Since

$$Y \subseteq O_1 \cup O_2 = (U_1 \cap Y) \cup (U_2 \cap Y) = (U_1 \cup U_2) \cap Y \subseteq U_1 \cup U_2,$$

and we know

$$\varnothing = O_1 \cap O_2 = (U_1 \cap Y) \cap (U_2 \cap Y) = U_1 \cap U_2 \cap Y.$$

($\Leftarrow$) Choose $O_1 = U \cap Y$ and $O_2 = V \cap Y$. Note that O_1, O_2 are non-empty and open in Y , and we can easily check that $O_1 \cup O_2 = Y$ and $O_1 \cap O_2 = \varnothing$, so Y is disconnected.

■

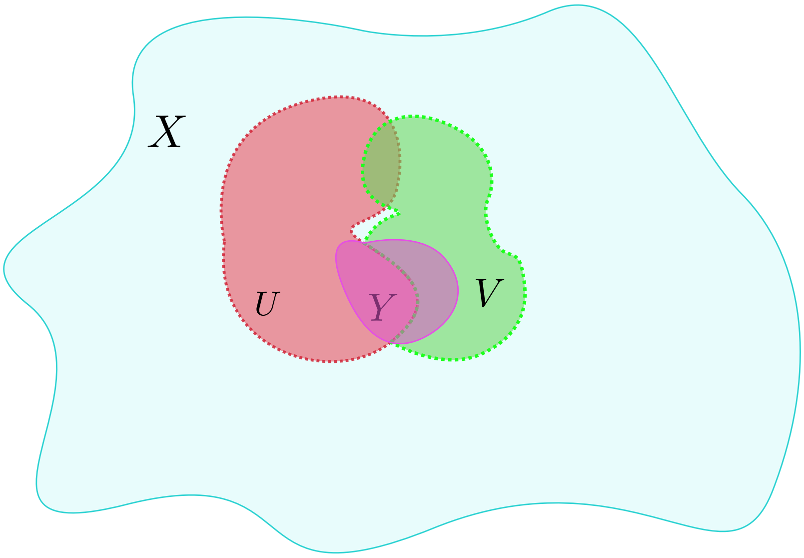

Figure 3.2: Disconnected set condition in Remark 3.4.2

Theorem 3.4.1. Let $f : X \to Y$ be a continuous map between metric spaces. Let E be any connected subset of X. Then $f(E)$ is connected in Y.

Proof. Suppose not, $f(E)$ is disconnected in Y, then $\exists$ open set U, V in Y s.t. $f(E) \subseteq U \cup V$ and $U \cap f(E) \neq \varnothing$ and $V \cap f(E) \neq \varnothing$ and $U \cap V \cap f(E) = \varnothing$. Since f is continuous, so $f^{-1}(U)$ and $f^{-1}(V)$ are both open and non-empty and $\left(E \cap f^{-1}(U)\right) \cap \left(E \cap f^{-1}(V)\right) = \varnothing$, and note that $E \subseteq \left(E \cap f^{-1}(U)\right) \cup \left(E \cap f^{-1}(V)\right)$, so E is disconnected.

Remark 3.4.3. $E \cap f^{-1}(U)$ and $E \cap f^{-1}(V)$ are open in E.

■

Theorem 3.4.2. Let X be a nonempty subset of $\mathbb{R}$, then TFAE:

(a) X is connected.

(b) Whenever $x, y \in X$ and $x < y$, we have $[x, y] \subseteq X$.

(c) X is an interval.

proof from (a) to (b). Suppose not, then there exists $z \notin X$ s.t. $x < z < y$. Hence, we can pick $U = (-\infty, z) \cap X$ and $V = (z, \infty) \cap X$, then $U \neq \varnothing$ and $V \neq \varnothing$ and U, V both open in X and $U \cap V = \varnothing$ and $X \subseteq U \cup V$, so X is disconnected, which is a contradiction. ■

proof from (b) to (a). Suppose not, then X is disconnected, and thus there exists V, W open in X s.t. $X = V \cup W$ and $V \cap W = \varnothing$. Now fix $x \in V$ and $y \in W$. WLOG, suppose $x < y$, then $[x, y] \subseteq X = V \cup W$ by the hypothesis. Now suppose $S = [x, y] \cap V \subseteq X \subseteq \mathbb{R}$, then since $y \geq s$ for all $s \in S$, so S is a subset of $\mathbb{R}$ which is bounded above, and thus $z = \sup S$ exists. Note that $z \leq y$, so $z \in [x, y] \subseteq X = V \cup W$.

- Case 1: $z \in V$, then $z < y$ since $y \in W$ and $V \cap W = \varnothing$. Now since V is open in X, so there exists $\varepsilon > 0$ s.t. $B_X(z, \varepsilon) \subseteq V$, which means $(z - \varepsilon, z + \varepsilon) \cap X \subseteq V$. In particular, we have $(z - \varepsilon, z + \varepsilon) \cap [x, y] \subseteq V$, and thus

$$(z, z + \varepsilon) \cap [x, y] \subseteq V \cap [x, y] = S.$$

Now since $z < y$, so $p \in (z, z+\varepsilon) \cap [x, y]$ for some p, which means $p \in S$. However, $p \in (z, z+\varepsilon)$ gives $p > z$, so S contains a $p > z = \sup S$, which is a contradiction.

- Case 2: $z \in W$, then there exists $\varepsilon > 0$ s.t. $B_X(z, \varepsilon) \subseteq W$ since W is open in X. Hence,

$$(z - \varepsilon, z + \varepsilon) \cap [x, y] \subseteq (z - \varepsilon, z + \varepsilon) \cap X \subseteq W.$$

Hence, we have $(z - \varepsilon, z + \varepsilon) \cap [x, y] \cap V = \varnothing$ since $V \cap W = \varnothing$. Note that this means $(z - \varepsilon, z + \varepsilon) \cap S = \varnothing$. However, we can construct a sequence of S converges to $\sup S$ (See Theorem A.1.5), so there exists $y \in S$ s.t. $y \in \left(z - \frac{\varepsilon}{2}, z + \frac{\varepsilon}{2}\right) \subseteq (z - \varepsilon, z + \varepsilon)$, which means $y \in (z - \varepsilon, z + \varepsilon) \cap S = \varnothing$, so it is a contradiction.

■

Remark 3.4.4. The fact that (b) is equivalent to (c) is trivial, so we don't give a proof.

Lecture 11

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Theorem 3.4.3 (Review of Theorem 3.4.1). Let $f : X \to Y$ be a continuous map and let E be a connected subset of X. Then $f(E)$ is connected in Y.

Corollary 3.4.1 (Intermediate value theorem). Let $f : X \to \mathbb{R}$ be a continuous map from (X, d_X) to $\mathbb{R}$. Let $E \subseteq X$ be any connected subset, and let $a, b \in E$. Suppose y is a real number between $f(a)$ and $f(b)$ i.e.

$$\min\{f(a), f(b)\} \leq y \leq \max\{f(a), f(b)\},$$

then $\exists c \in E$ s.t. $f(c) = y$.

Proof. There are 3 cases:

- Case 1: $f(a) = f(b)$, then trivial.
- Case 2: $f(a) < f(b)$, Since E is connected and f is continuous, so $f(E)$ is connected in $\mathbb{R}$. Hence, for $f(a), f(b) \in f(E)$, we know $(f(a), f(b)) \subseteq f(E)$ by Theorem 3.4.2, so if $f(a) < y < f(b)$, then $\exists c \in E$ s.t. $f(c) = y$.
- Case 3: $f(a) > f(b)$, then let $a' = b$ and $b' = a$, then $f(a') < f(b')$ and use the result of Case 2.

■

3.5 Topological space

In metric space (X, d_X), we define open ball

$$B_X(x, r) = \{y \mid d_X(y, x) < r\},$$

and a set u is open if for any $x \in u$, $\exists r_x > 0$ s.t. $B_X(x, r_x) \subseteq u$, so $u = \bigcup_{x \in u} B_X(x, r_x)$. Hence, in metric space open sets are in fact union of open balls. We also proved that

- $\varnothing$ and X are open.
- If $u_1, \ldots, u_n$ are open in X, then $\bigcap_{i=1}^n u_i$ is open in X.
- If $\{u_i\}_{i \in A}$ are open in X, then $\bigcup_{i \in A} u_i$ is also open.

Now we want to extend this concept.

Definition 3.5.1 (Power sets). For a given set X, we define 2^X the power set of X i.e.

$$2^X := \{A : A \subseteq X\}$$

the collection of all subsets of X.

Example 3.5.1. $X = \{a, b\}$ for $a \neq b$, then

$$2^X = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$

Definition 3.5.2 (Topological space). A topological space is a pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F} \subseteq 2^X$ is a collection of subsets of X, called the open sets. The collection $\mathcal{F}$ must satisfy

- $\varnothing$ and X are all in $\mathcal{F}$.
- If $u_1, \ldots, u_n$ are in $\mathcal{F}$, then $\bigcap_{i=1}^n u_i$ is in $\mathcal{F}$.
- If $\{u_i\}_{i \in A}$ are in $\mathcal{F}$, then $\bigcup_{i \in A} u_i$ is in $\mathcal{F}$.

Remark 3.5.1. In a metric space, let

$$\mathcal{F} = \text{ the set of open sets in } (X, d_X) = \{u \mid \forall x \in u, \exists r_x > 0 \text{ s.t. } B_X(x, r_x) \subseteq u\},$$

then $(X, \mathcal{F})$ is a topological space.

Example 3.5.2. On any set $X \neq \varnothing$, we have a trivial topology on X i.e. $\mathcal{F} = \{\varnothing, X\}$, which means $(X, \mathcal{F})$ is a topological space in X.

Example 3.5.3. Consider $\mathcal{F} = 2^X$, then $(X, 2^X)$ is also a topological space on X.

Definition 3.5.3 (Neighborhood). Let $(X, \mathcal{F})$ be a topological space, and let $x \in X$. A neighborhood of x is any open set $u \in \mathcal{F}$ s.t. $x \in u$.

Example 3.5.4. $X = \{a, b\}$ and $a \neq b$, $\mathcal{F} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$, then $\{a\}, \{a, b\}$ are neighborhoods of a and $\{b\}, \{a, b\}$ are neighborhoods of b.

Definition 3.5.4 (Interior/Exterior/Boundary point). Let $(X, \mathcal{F})$ be a topological space, and $E \subseteq X$ be a subset. We say that

- x_0 is an interior point of E if $\exists$ a neighborhood V of x_0 s.t. $V \subseteq E$.
- x_0 is an exterior point of E if $\exists$ a neighborhood V of x_0 s.t. $V \subseteq X \setminus E$.
- x_0 is a boundary point if it is neither interior or exterior.

Corollary 3.5.1.

$$X = \operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E.$$

Proof. ■ DIY

Definition 3.5.5 (Adherent point). We say x_0 is an adherent point of E if every neighborhood V of x_0 has a nonempty intersection with E, and we called $\overline{E}$ the set of all adherent points.

Corollary 3.5.2. $\overline{E} = \text{Int}(E) \cup \partial E$.

Proof. We first show that $\text{Int}(E) \cup \partial E \subseteq \overline{E}$. Suppose $x_0 \in \text{Int}(E)$, then $x_0 \in E$, so $x_0 \in \overline{E}$ since $E \subseteq \overline{E}$. Now if $x_0 \in \partial E$, then any neighborhood V of x_0 contains points in E, so $x_0 \in \overline{E}$. Now we show that $\overline{E} \subseteq \text{Int}(E) \cup \partial E$. Since $\overline{E} \subseteq X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$, so we want to show for all $x_0 \in \overline{E}$, we have $x_0 \notin \text{Ext}(E)$. By the definition of exterior point, we can easily show this. ■

Definition (Open and Closed Sets). Suppose $(X, \mathcal{F})$ is a topological space. Then:

- A set $O \subseteq X$ is called *open* if $O \in \mathcal{F}$.
- A set $F \subseteq X$ is called *closed* if its complement $X \setminus F$ is open, i.e., $X \setminus F \in \mathcal{F}$.

Corollary 3.5.3. A set $E \subseteq X$ is open if and only if $E = \text{Int}(E)$.

Proof.

($\Rightarrow$) If E is open, then since

$$\text{Int}(E) = \bigcup \{O \subseteq E : O \in \mathcal{F}\}, \tag{3.1}$$

we know $E \subseteq \text{Int}(E)$.

($\Leftarrow$) By the definition of topological space and Equation 3.1, we know $\text{Int}(E)$ is open, so $E = \text{Int}(E)$ implies E is open.

■

Corollary 3.5.4. A set $F \subseteq X$ is closed if and only if $\overline{F} = F$.

Proof. If F is closed, then $X \setminus F$ is open, so $X \setminus F = \text{Int}(X \setminus F)$, so

$$F = X \setminus (X \setminus F) = X \setminus \text{Int}(X \setminus F) = X \setminus \text{Ext}(F) = \text{Int}(F) \cup \partial(F) = \overline{F}.$$

The other direction is similar. ■

Definition 3.5.6 (Topological subspace). Let $(X, \mathcal{F})$ be a topological space and $Y \subseteq X$. We define $\mathcal{F}_Y = \{V \cap Y \mid V \in \mathcal{F}\}$ and call $(Y, \mathcal{F}_Y)$ the topological subspace of $(X, \mathcal{F})$ induced by Y. We can show that $\mathcal{F}_Y$ is a topology on Y.

Definition 3.5.7 (Continuous map). Let $(X, \mathcal{F})$ and (Y, g) be topological spaces ane let $f : X \to Y$ be a function. We say f is continuous at $x_0 \in X$ if for every neighborhood $V \in g$ of $f(x_0)$, there exists a neighborhood $u \in \mathcal{F}$ of x_0 s.t. $u \subseteq f^{-1}(V)$.

Definition 3.5.8 (Convergence). Let m be an integer, $(X, \mathcal{F})$ be a topological space, and let $\left(x^{(n)}\right)_{n=m}^{\infty}$ be a sequence of points in X. We say $x^{(n)} \to x$ iff for every neighborhood V of x, $\exists N \geq m$ s.t. $x^{(n)} \in V$ for all $n \geq N$.

Remark 3.5.2. In a topological space, a sequence may converge to more than one points.

Example 3.5.5. Let $X = \{a, b\}$ and $a \neq b$. Consider the trivial topology $\mathcal{F} = \{\varnothing, X\}$. If we have the constant sequence $\left\{x^{(n)}\right\}_{n=m}^{\infty}$ with $x^{(n)} = a$, then $\lim_{n\to\infty} x^{(n)} = a$ and $\lim_{n\to\infty} x^{(n)} = b$.

Proof. Any neighborhood of a must be $\{a, b\}$, and any neighborhood of b must also be $\{a, b\}$, so $\lim_{n\to\infty} x^{(n)} = a$ and $\lim_{n\to\infty} x^{(n)} = b$. ⊛

Example 3.5.6. On $\mathbb{R}$, we consider

$$\mathcal{F} = \{\varnothing\} \cup \{u : \mathbb{R} \setminus u \text{ is finite points}\},$$

then $\mathcal{F}$ is a topology on $\mathbb{R}$.

Proof. Since $\varnothing, \mathbb{R} \in \mathcal{F}$, so it satisfies the first rule of topology. Now if $u_1 \in \mathcal{F}$ and $u_2 \in \mathcal{F}$, then we want to show $u_1 \cap u_2 \in F$. We assume u_1, u_2 are non-empty, otherwise it is trivial. Consider $\mathbb{R} \setminus (u_1 \cap u_2) = (\mathbb{R} \setminus u_1) \cup (\mathbb{R} \setminus u_2)$, since $\mathbb{R} \setminus u_1$ and $\mathbb{R} \setminus u_2$ are both finite points, so $u_1 \cap u_2$ is finite points. Hence, we know for finitely many $u_1, \ldots, u_n$, $\bigcap_{i=1}^{n} u_i$ is in $\mathcal{F}$. Now we know for $\mathcal{F}_\alpha \in \mathcal{F}$,

$$\mathbb{R} \setminus \left(\bigcup_{\alpha} \mathcal{F}_\alpha \right) = \bigcap_{\alpha} (\mathbb{R} \setminus \mathcal{F}_\alpha) \subseteq \mathbb{R} \setminus \mathcal{F}_{\alpha_i}$$

for some α_i in the index set, so $\mathbb{R} \setminus (\bigcup_\alpha \mathcal{F}_\alpha)$ is also finite points, and we're done. ⊛

Remark 3.5.3. In the topological space induced by the topology in Example 3.5.6. If we consider $\left\{x^{(n)}\right\}_{n=1}^{\infty}$ with $x^{(n)} = n$, then $\lim_{n\to\infty} x^{(n)} = p$ for any $p \in \mathbb{R}$.

Proof. Since any neighborhood u of p has

$$\mathbb{R} \setminus u = \{p_1, \ldots, p_k\}$$

with $p_1 < p_2 < \cdots < p_k$, so we have $x^{(n)} \in u$ for $n > p_k$. ■

Lecture 12

Last time, we show that the sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n = n$ converges to any point p in $\mathbb{R}$ in cofinite topology i.e. 9 Oct. 10:20

$$\mathcal{F} = \{\varnothing\} \cup \{\mathbb{R} \setminus \{\text{finite points}\}\}$$

beacuse each non-empty neighborhood of p is very big. In general, a sequence in a topological space may converge to more than one points.

Definition 3.5.9 (Hausdorff). A topological space $(X, \mathcal{F})$ is called Hausdorff if given any two distinct points $x, y \in X$, there exists open sets $U, V \in \mathcal{F}$ s.t. $x \in U$ and $y \in V$ and $U \cap V = \varnothing$.

Example 3.5.7. A metric space is Hausdorff since given $x \neq y$, $B_X\left(x, \frac{r}{2}\right)$ and $B_X\left(y, \frac{r}{2}\right)$ are open and they seperate x and y where $r = \frac{d(x,y)}{2}$.

Theorem 3.5.1. Suppose $(X, \mathcal{J})$ is a Hausdorff topological space, then the limit of a convergent sequence is unique.

Proof. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ for some $x \neq y$, then since $(X, \mathcal{J})$ is Hausdorff, so there exists neighborhood U of x and V of y and $U \cap V = \varnothing$. Also, there exists $N_1 > 0$ s.t. $x_n \in U$ if $n \geq N_1$, and there exists $N_2 > 0$ s.t. $x_n \in V$ if $n \geq N_2$. Hence, for all $n \geq \max\{N_1, N_2\}$, we know $x_n \in U \cap V = \varnothing$, which is a contradiction. Hence, the limit of a convergence sequence is unique. ■

Definition 3.5.10 (Compact). Let $(X, \mathcal{F})$ be topological space, we say X is compact if for every open cover

$$\{U_\alpha : \alpha \in A\} \subseteq \mathcal{F} \text{ with } X \subseteq \bigcup_{\alpha \in A} U_\alpha,$$

there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ s.t.

$$X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Theorem 3.5.2. Let $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be a continuous map between topological spaces. If $K \subseteq X$ is compact, then $f(K)$ is also compact in $(Y, \mathcal{G})$.

Proof. Let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$ i.e. $\{V_\alpha\}_{\alpha \in A} \subseteq \mathcal{G}$ and $f(K) \subseteq \bigcup_{\alpha \in A} V_\alpha$. Since f is continuous, so $f^{-1}(V_\alpha) \in \mathcal{F}$ and

$$K \subseteq \bigcup_{\alpha \in A} f^{-1}(V_\alpha).$$

Now since K is compact, so there exists finite subcover, which means $K \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$, so $f(K) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$, and thus $f(K)$ has a finite subcover of $\{V_\alpha\}_{\alpha \in A}$, which means $f(K)$ is compact.

Remark 3.5.4. In topological space, we also have: $f : X \to Y$ is continuous if and only if whenever $V \subseteq Y$ is open (resp. closed), we have $f^{-1}(V)$ is open (resp. closed) in X.

Proof. ■ DIY

■

Proposition 3.5.1. Let $(X, \mathcal{F})$ be a compact topological space and $f : X \to \mathbb{R}$ is continuous, then

(1) f is bounded on X.

(2) If $X \neq \varnothing$, then $\exists x_{\min}, x_{\max} \in X$ s.t. $f(x_{\max}) = \max_{x \in X} f(x)$ and $f(x_{\min}) = \min_{x \in X} f(x)$.

Proof.

(1) Since X is compact, so $f(X)$ is compact in $\mathbb{R}$ by Theorem 3.5.2, and since $\mathbb{R}$ is a metric space, so $f(X)$ is closed and bounded in $\mathbb{R}$, which means f is bounded on X.

(2) Now since $f(X)$ is bounded, so $\sup_{x \in X} f(x)$ and $\inf_{x \in X} f(x)$ exists. Thus, we can pick $(y_n) \in f(X)$ and $(z_n) \in f(X)$ s.t. $y_n \to \sup_{x \in X} f(x)$ and $z_n \to \inf_{x \to X} f(x)$ by Theorem A.1.5. Now since $f(X)$ is closed, so

$$\sup_{x \in X} f(x) \in \overline{f(X)} = f(X),$$

so there exists x^* s.t. $f(x^*) = \sup_{x \in X} f(x)$ and similarly the "min" case can be proved.

■

Chapter 4

Uniform Convergence

In a metric space (X, d), we define the convergence of a sequence $\left\{x^{(n)}\right\}_{n=m}^{\infty}$, $\lim_{n\to\infty} x^{(n)} = x$, by "Given any $\varepsilon > 0$, $\exists N \geq m$ s.t. $d\left(x^{(n)}, x\right) < \varepsilon$ for all $n \geq N$ ". Now suppose

$$f^{(n)} : X \to Y \quad \forall n \in \mathbb{N},$$

where X, Y are metric spaces, then if we define the convergence of these functions at some point x to be:

$$f(x) = \lim_{n\to\infty} f^{(n)}(x).$$

Do we have

$$\lim_{n\to\infty} \lim_{x\to x_0} f^{(n)}(x) = \lim_{x\to x_0} \lim_{n\to\infty} f^{(n)}(x)?$$

Short answer: Not always true.

In this chapter, we will discuss the concept of limiting function, that is,

$$\lim_{n\to\infty} f^{(n)} = f.$$

4.1 Limiting values of functions

Definition 4.1.1. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $E \subseteq X$ and $f : E \to Y$ be a function. If $x_0 \in X$ is an adherent point of E, and $L \in Y$, we say that

$$\lim_{x\to x_0, x\in E} f(x) = L$$

if for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. $d_X(x, x_0) < \delta$ and $x \in E$ implies $d_Y\left(f(x), L\right) < \varepsilon$.

Remark 4.1.1. In this definition, we need not $x_0 \in E$, we just need $x_0 \in \overline{E}$.

Remark 4.1.2. In other texboock, if

$$f(x) = \begin{cases} |x|, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0, \end{cases}$$

then $\lim_{x\to 0} f(x) = 0$ because it does not consider $x = 0$. More precisely, the definition of

$$\lim_{x\to x_0, x\in E} f(x) = L$$

in other textbook is "$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y\left(f(x), L\right) < \varepsilon$ for all $x \in E$ and $0 < d_X(x, x_0) < \delta$". Note that it exclude the case $x = x_0$.

However, if $x_0 \in E$, then by Terrence Tao's definition, $f(x_0) = L$ if $d_Y\left(f(x), L\right) < \varepsilon$ for all $\varepsilon > 0$ and for $d_X(x, x_0) < \delta$ for the corresponding δ. Also, if $x_0 \notin E$, then since $x_0 \in \overline{E}$, so $\exists x \in E$ s.t.

$d(x, x_0) < \delta$, so the definition of $\lim_{x\to x_0, x\in E} f(x)$ is well-defined. In our notation, other textbooks' definition is like
$$\lim_{x\to x_0, x\in E\setminus\{x_0\}} f(x) = L.$$

Lemma 4.1.1. If (X, d_X) and (Y, d_Y) are metric spaces, then $f : X \to Y$ is continuous at $x_0 \in X$ is in fact
$$\lim_{x\to x_0, x\in X} f(x) = f(x_0).$$

Proof. Since f is continuous at x_0 means for all $(x_n) \to x_0$, we have $f(x_n) \to f(x_0)$, so this is true. ∎

Proposition 4.1.1. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a function. Let $x_0 \in X$ be an adherent point of $E \subseteq X$ and $L \in Y$, then TFAE:

(a) $\lim_{x\to x_0, x\to E} f(x) = L$.

(b) For every sequence $\left\{x^{(n)}\right\}_{n=1}^{\infty}$ in E converges to x_0, the sequence $\lim_{n\to\infty} f\left(x^{(n)}\right) = L$ in Y.

(c) For every open set $V \subseteq Y$ containing L, there exists an open set $U \subseteq X$ containing x_0 s.t. $U \cap E \subseteq f^{-1}(V)$.

(d) If one define $g : E \cup \{x_0\} \to Y$ by
$$g(x) = \begin{cases} f(x), & \text{if } x \in E \setminus \{x_0\}; \\ L, & \text{if } x = x_0, \end{cases}$$
then g is continuous at x_0 on E.

proof from (a) to (b). We know for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. $x \in E$ and $d_X(x, x_0) < \delta$ implies $d_Y(f(x), L) < \varepsilon$. Also, if we have a sequence $\left\{x^{(n)}\right\}_{n=1}^{\infty} \subseteq E$ converges to x_0, then there exists $N > 0$ s.t. $n \geq N$ implies $d_X\left(x^{(n)}, x_0\right) < \delta$, so for all $n \geq N$, we know $d_Y\left(f\left(x^{(n)}\right), L\right) < \varepsilon$. ∎

proof from (b) to (c). Suppose by contradiction, $V \subseteq Y$ is an open neighborhood of L, and there does not exist an open neighborhood $U \subseteq X$ of x_0 has
$$U \cap E \subseteq f^{-1}(V),$$
then for all $n \in \mathbb{N}$, we know $\exists y_n \in B_X\left(x_0, \frac{1}{n}\right) \cap E$ and $y_n \notin f^{-1}(V)$, and note that $(y_n)_{n=1}^{\infty}$ is a sequence converges to x_0 in E, so by (b) we know $(f(y_n))_{n=1}^{\infty}$ must converges to L, which means for the neighborhood V of L, there exists $N > 0$ s.t. $n \geq N$ implies $f(y_n) \in V$. ∎

proof from (c) to (d). ∎ DIY

proof from (d) to (a). ∎ DIY

4.2 Pointwise and Uniform Convergence

Definition 4.2.1 (Pointwise convergence)**.** Let $\left(f^{(n)}\right)_{n=1}^{\infty}$ be a sequence of functions from (X, d_X) to (Y, d_Y), and let $f : X \to Y$ be another function. We say that $f^{(n)}$ converges pointwise to f on X if for every $x \in X$ and every $\varepsilon > 0$, $\exists N_x > 0$ s.t.
$$d_Y\left(f^{(n)}(x), f(x)\right) < \varepsilon \quad \forall n \geq N_x.$$

Definition 4.2.2 (Uniformly convergence). We say $f^{(n)}$ converges uniformly to f on X if for every $\varepsilon > 0$, $\exists N > 0$ s.t.

$$d_Y\left(f^{(n)}(x), f(x)\right) < \varepsilon$$

for all $n \geq N$ and all $x \in X$. (N is independent of x)

Example 4.2.1. Suppose $f_n(x) = \frac{x}{n}$, then $f_n \to 0$ pointwise (0 is the zero function here). However, $\{f^{(n)}\}$ is not uniformly convergent since given $\varepsilon = 1$, if $\exists N > 0$ s.t.

$$|f_n(x) - 0| < 1$$

for all $n \geq N$ and $x \in \mathbb{R}$, then

$$\left|\frac{x}{n}\right| < 1$$

for all $n \geq N$ and $x \in X$, but if we pick $n = N$ and $x = 2N$, then it gives a contradiction.

Example 4.2.2. $f_n(x) = x^n$ on $[0,1]$, then we know

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ 1, & \text{if } x = 1, \end{cases}$$

so f_n continuous and pointwise convergent but not uniformly convergent.

Example 4.2.3. Suppose $f_n(x) = \frac{x}{n}$ on $[0,1]$, then $\lim_{n\to\infty} f_n(x) = 0$ uniformly.

Proof. ∎ DIY

Example 4.2.4. Consider

$$f_n(x) = \begin{cases} 2n, & \text{if } x \in \left[\frac{1}{2n}, \frac{1}{n}\right]; \\ 0, & \text{if } x \in \mathbb{R} \setminus \left[\frac{1}{2n}, \frac{1}{n}\right], \end{cases}$$

then $\lim_{n\to\infty} f_n(x) = 0$ pointwisely, but if we integrate on both side, then

$$\int_0^1 f^{(n)}(x)\,\mathrm{d}x = 2n\left(\frac{1}{n} - \frac{1}{2n}\right) = 2n \cdot \frac{1}{2n} = 1,$$

but $\int_0^1 0\,\mathrm{d}x = 0$, so we know pointwise convergence will not implies they will be equal after integration.

> **Remark 4.2.1.** We will learn that uniform convergence can ensure integration after taking limit takes same value of taking limit after integration.

Lecture 13

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As previously seen. $\lim_{n\to\infty} f_n = f$ uniformly where $f_n, f : X \to Y$ iff given any $\varepsilon > 0$, $\exists N > 0$ s.t.

$$d_Y(f_n(x), f(x)) < \varepsilon \quad \forall x \in X \text{ and } n \geq N.$$

Theorem 4.2.1 (考試會考). Suppose $\left(f^{(n)}\right)_{n=1}^{\infty}$ is a sequence of functions from one metric space (X, d_X) to (Y, d_Y) and suppose this sequence of functions converge uniformly to another function

$f : X \to Y$. Let $x_0 \in X$. If each $f^{(n)}$ is continuous at x_0, then f is continuous at x_0.

Proof. Since $f_n \to f$ uniformly. Given $\varepsilon > 0$, $\exists N > 0$ s.t.

$$d_Y\left(f^{(n)}(x), f(x)\right) < \frac{\varepsilon}{3} \quad \text{for all } x \in X, n \geq N.$$

Since $f^{(N)}$ is continuous at x_0, so there exists $\exists \delta > 0$ s.t. if $d_X(x, x_0) < \delta$, then

$$d_Y\left(f^{(N)}(x), f^{(N)}(x_0)\right) < \frac{\varepsilon}{3}.$$

Now if $d_X(x, x_0) < \delta$, then

$$\begin{aligned} d_Y(f(x), f(x_0)) &\leq d_Y\left(f(x), f^{(N)}(x)\right) + d_Y\left(f^{(N)}(x), f^{(N)}(x_0)\right) + d_Y\left(f^{(N)}(x_0), f(x_0)\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, f is continuous at x_0. ■

Corollary 4.2.1. Let $f^{(n)}, f : X \to Y$. Suppose $f^{(n)} : X \to Y$ are continuous for all n. If $\lim_{n\to\infty} f^{(n)} = f$ uniformly, then f is also continuous.

Example 4.2.5. Suppose $f_n(x) = x^n$ define on $[0, 1]$ for all $n \in \mathbb{N}$, then we know

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ 1, & \text{if } x = 1. \end{cases}$$

Now suppose

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ 1, & \text{if } x = 1. \end{cases},$$

then $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise. Note that $f_n = x^n$ is continuous on $[0, 1]$ but f is not continuous. Hence, f_n does not converge uniformly to f.

Remark 4.2.2. This example tells us if we change converge uniformly to converge pointwise in Theorem 4.2.1, then it may not be true.

Now we want to know is

$$\lim_{n\to\infty} \lim_{x\to x_0, x\in E} f^{(n)}(x) = \lim_{x\to x_0, x\in E} \lim_{n\to\infty} f^{(n)}(x)?$$

We will later show that the equality holds if $f^{(n)} \to f$ uniformly and Y complete, where we define $f^{(n)}, f : X \to Y$.

Proposition 4.2.1. Let (X, d_X) and (Y, d_Y) be metric spaces with Y complete, and let $E \subseteq X$. Suppose $\left(f^{(n)}\right)_{n=1}^{\infty}$ is a sequence of functions from E to Y that converges uniformly to some function $f : E \to Y$. Let $x_0 \in X$ be adherent point of E, and suppose that for each n, $\lim_{x\to x_0, x\in E} f^{(n)}(x)$ exists. Then the limit $\lim_{x\to x_0, x\in E} f(x)$ also exists, and moreover,

$$\lim_{n\to\infty} \lim_{x\to x_0, x\in E} f^{(n)}(x) = \lim_{x\to x_0, x\in E} \lim_{n\to\infty} f^{(n)}(x).$$

Proof. Since $\lim_{x\to x_0, x\in E} f^{(n)}(x)$ exists, so we can let $L_n := \lim_{x\to x_0, x\in E} f^{(n)}(x) \in Y$. First, we show that $\{L_n\}_{n=1}^{\infty}$ is Cauchy. Since $\lim_{n\to\infty} f^{(n)} = f$ uniformly on E. Given $\varepsilon > 0$, $\exists N > 0$ s.t.

$$d_Y\left(f^{(n)}(x), f(x)\right) < \frac{\varepsilon}{6} \text{ for all } x \in E, n \geq N.$$

Hence, $n, m \geq N$ implies

$$d_Y\left(f^{(n)}(x), f^{(m)}(x)\right) \leq d_Y\left(f^{(n)}(x), f(x)\right) + d_Y\left(f(x), f^{(m)}(x)\right) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

Now since $\lim_{x\to x_0, x\in E} f^{(n)}(x) = L_n$ and $\lim_{x\to x_0, x\in E} f^{(m)}(x) = L_m$, so there exists $\delta_n, \delta_m > 0$ s.t. for all $x \in E$,

$$d(x, x_0) < \delta_n \Rightarrow d_Y\left(f^{(n)}(x), L_n\right) < \frac{\varepsilon}{3}$$
$$d(x, x_0) < \delta_m \Rightarrow d_Y\left(f^{(m)}(x), L_m\right) < \frac{\varepsilon}{3}.$$

Choose $\delta = \min\{\delta_n, \delta_m\}$ and fix $x \in E$ with $d_X(x, x_0) < \delta$ (since $x_0 \in \overline{E}$ so this is possible), then

$$d_Y(L_n, L_m) \leq d_Y\left(L_n, f^{(n)}(x)\right) + d_Y\left(f^{(n)}(x), f^{(m)}(x)\right) + d_Y\left(f^{(m)}(x), L_m\right) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence, $\{L_n\}_{n=1}^{\infty}$ is Cauchy. Since Y is complete, so $\lim_{n\to\infty} L_n = L$ for some $L \in Y$.

Claim 4.2.1. $\lim_{x\to x_0, x\in E} f(x) = L$.

Proof. Fix $\varepsilon > 0$, then $\lim_{n\to\infty} L_n = L$ and $\lim_{n\to\infty} f^{(n)}(x) = f(x)$ uniformly on E, so there exists $N > 0$ s.t.

$$d_Y(L_n, L) < \frac{\varepsilon}{3} \text{ and } d_Y\left(f^{(n)}(x), f(x)\right) < \frac{\varepsilon}{3} \text{ for all } x \in E, n \geq N.$$

Now since $L_n = \lim_{x\to x_0, x\in E} f^{(n)}(x)$, so there exists $\delta > 0$ s.t. for all $x \in E$ and for all n,

$$d_X(x, x_0) < \delta \Rightarrow d_Y\left(f^{(n)}(x), L_n\right) < \frac{\varepsilon}{3}.$$

For this δ, if $d_X(x, x_0) < \delta$ and $x \in E$, then we know

$$d_Y(f(x), L) \leq d_Y\left(f(x), f^{(N)}(x)\right) + d_Y\left(f^{(N)}(x), L_N\right) + d_Y(L_N, L) < \varepsilon.$$

⊛

Hence,

$$\lim_{n\to\infty} \lim_{x\to x_0, x\in E} f^{(n)}(x) = \lim_{n\to\infty} L_n = L$$
$$\lim_{x\to x_0, x\in E} \lim_{n\to\infty} f^{(n)}(x) = \lim_{x\to x_0, x\in E} f(x) = L.$$

This means $\lim_{x\to x_0, x\in E}$ and $\lim_{n\to\infty}$ is exchangable here. ■

Proposition 4.2.2. Let $f^{(n)} : X \to Y$ be a sequence of continuous functions. If $\lim_{n\to\infty} f^{(n)} = f$ uniformly. Let $\lim_{n\to\infty} x^{(n)} = x$ in X, then $\lim_{n\to\infty} f^{(n)}\left(x^{(n)}\right) = f(x)$.

Proof. Since $f^{(n)} \to f$ uniformly, so given $\varepsilon > 0$, there exists $N_1 > 0$ s.t.

$$d_Y\left(f^{(n)}(x), f(x)\right) < \frac{\varepsilon}{2} \text{ for all } x \in X, n \geq N_1.$$

Since $f^{(n)} \to f$ uniformly, so f is also continuous by Corollary 4.2.1, so there exists $\delta_x > 0$ s.t.

$$d_X(y, x) < \delta_x \Rightarrow d_Y(f(y), f(x)) < \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} x^{(n)} = x$, so there exists $N_2 > 0$ s.t. $d_X\left(x^{(n)}, x\right) < \delta_x$ for all $n \geq N_2$. Let

$N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$d_Y\left(f^{(n)}\left(x^{(n)}\right), f(x)\right) \leq d_Y\left(f^{(n)}\left(x^{(n)}\right), f\left(x^{(n)}\right)\right) + d_Y\left(f\left(x^{(n)}\right), f(x)\right) < \varepsilon.$$

■

Definition 4.2.3 (bounded function)**.** A function $f : X \to Y$ from (X, d_X) to (Y, d_Y) is called bounded if its image $f(X)$ is bounded in Y i.e. there exists $y_0 \in Y$ and $R > 0$ s.t.

$$d_Y(f(x), y_0) < R \text{ for all } x \in X.$$

Proposition 4.2.3. Let $\left(f^{(n)}\right)_{n=1}^{\infty}$ be a sequence of functions from (X, d_X) to (Y, d_Y). Suppose $\lim_{n\to\infty} f^{(n)} = f$ uniformly where $f : X \to Y$. If each $f^{(n)}$ is bounded on X, then f is also bounded.

Proof. Since $\lim_{n\to\infty} f^{(n)} = f$ uniformly, so given $\varepsilon = 1$, there exists $N > 0$ s.t.

$$d_Y\left(f^{(n)}(x), f(x)\right) < 1 \text{ for all } x \in X, n \geq N.$$

Since $f^{(N)}$ is bounded, so there exists y_0 and $R_N > 0$ s.t.

$$d_Y\left(f^{(N)}(x), y_0\right) < R_N \text{ for all } x \in X.$$

Hence,

$$\begin{aligned} d_Y(f(x), y_0) &\leq d_Y\left(f(x), f^{(N)}(x)\right) + d_Y\left(f^{(N)}(x), y_0\right) \\ &< 1 + R_N \end{aligned}$$

for all $x \in X$, so f is bounded. ■

4.3 The Metric of Uniform Convergence

Definition 4.3.1. Suppose (X, d_X) and (Y, d_Y) be metric spaces. We let $B(X \to Y)$ denotes the set of all bounded functions from X to Y i.e.

$$B(X \to Y) = \{f \mid f : X \to Y \text{ is bounded}\}.$$

If $X \neq \varnothing$, we define a metric d_∞ on $B(X \to Y)$ by

$$d_\infty(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$$

for all $f, g \in B(X \to Y)$.

Proposition 4.3.1. If X is non-empty, then d_∞ is a metric on $B(X \to Y)$.

Proof. Given $f, g \in B(X \to Y)$, then there exists $y_f, y_g \in Y$ and $M_f, M_g > 0$ s.t.

$$d_Y(f(x), y_f) < M_f \text{ and } d_Y(g(x), y_g) < M_g \text{ for all } x \in X.$$

Hence,

$$d_Y(f(x), g(x)) \leq d_Y(f(x), y_f) + d_Y(y_f, y_g) + d_Y(y_g, g(x)) < M_f + M_g + d_Y(y_f, y_g),$$

which means $\sup_{x\in X} d_Y(f(x), g(x))$ exists and ≥ 0, and thus d_∞ is well-defined.

Now since

(1) $d_\infty(f,g) = \sup_{x\in X} d_Y(f(x), g(x)) \geq 0$.

(2) $d_\infty(f,g) = d_\infty(g,f)$.

(3) For $f, g, h \in B(X \to Y)$,

$$d_Y(f(x), h(x)) \leq d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \leq d_\infty(f,g) + d_\infty(g,h),$$

so $\sup_{x\in X} d_Y(f(x), h(x)) \leq d_\infty(f,g) + d_\infty(g,h)$.

(4) $d_\infty(f,g) = 0$ iff $\sup_{x\in X} d_Y(f(x), g(x)) = 0$ iff $d_Y(f(x), g(x)) = 0$ for all $x \in X$ iff $f(x) = g(x)$ for all $x \in X$.

So d_∞ is a metric. ■

Proposition 4.3.2. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\left(f^{(n)}\right)_{n=1}^{\infty}$ be a sequence of functions in $B(X \to Y)$, and let f be another function in $B(X \to Y)$. Then

$$f^{(n)} \to f \text{ in } d_\infty (= d_{B(X\to Y)}) \Leftrightarrow f^{(n)} \to f \text{ uniformly.}$$

Proof.

($\Rightarrow$) Since $\lim_{n\to\infty} d_\infty\left(f^{(n)}, f\right) = 0$. Given $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$\sup_{x\in X} d_Y\left(f^{(n)}(x), f(x)\right) = d_\infty\left(f^{(n)}, f\right) < \varepsilon,$$

so

$$d_Y\left(f^{(n)}(x), f(x)\right) \leq \sup_{x\in X} d_Y\left(f^{(n)}(x), f\right) < \varepsilon$$

whenever $n \geq N$, so $f^{(n)} \to f$ uniformly.

($\Leftarrow$) DIY

■

Now let $C(X \to Y)$ be the set of bounded and continuous function, so $C(X \to Y) \subseteq B(X \to Y)$.

Theorem 4.3.1. If Y is complete, then $C(X \to Y)$ is a complete metric space. (The metric is d_∞ intersected to $C(X \to Y)$).

Proof. Given any Cauchy sequence $\left\{f^{(n)}\right\}_{n=1}^{\infty}$ in $(C(X \to Y), d_\infty)$, then $f^{(n)} : X \to Y$ is continuous and bounded. Given $\varepsilon > 0$, there exists $N > 0$ s.t.

$$\sup_{x\in X} d_Y\left(f^{(n)}(x), f^{(m)}(x)\right) = d_\infty\left(f^{(n)}, f^{(m)}\right) < \frac{\varepsilon}{2} \text{ for all } n, m \geq N.$$

Now fix $x \in X$. Consider the sequence $\left\{f^{(n)}(x)\right\}_{n=1}^{\infty}$ in Y, so $\left\{f^{(n)}(x)\right\}_{n=1}^{\infty}$ is Cauchy in Y. Now since Y is complete, so $\lim_{n\to\infty} f^{(n)}(x)$ exists. Let $f(x) = \lim_{n\to\infty} f^{(n)}(x)$. If we set up $f(x)$ similarly for all $x \in X$ to construct f, then we give a claim.

Claim 4.3.1. $\lim_{n\to\infty} d_\infty\left(f^{(n)}, f\right) = 0$.

Proof. Fix $\varepsilon > 0$, choose $N > 0$ s.t. $d_\infty\left(f^{(n)}, f^{(m)}\right) < \frac{\varepsilon}{2}$ for all $n, m \geq N$. Then for all $n \geq N$ we know for all $x \in X$

$$d_Y\left(f^{(n)}(x), f(x)\right) = \lim_{m\to\infty} d\left(f^{(n)}(x), f^{(m)}(x)\right) \leq \sup_{m\geq N, x\in X} d\left(f^{(n)}(x), f^{(m)}(x)\right)$$
$$= \sup_{m\geq N} d_\infty\left(f^{(n)}, f^{(m)}\right) \leq \frac{\varepsilon}{2}.$$

Hence, $\lim_{n\to\infty} d_\infty\left(f^{(n)}, f\right) = 0$. ⊛

By this claim, we know $f^{(n)} \to f$, so Y is complete.

Now since $f^{(n)} \to f$ on d_∞, so $f^{(n)} \to f$ uniformly by Proposition 4.3.2, and since $\left\{f^{(n)}\right\}_{n=1}^\infty$ is continuous, so f is continuous by Corollary 4.2.1 we know f is continuous, and by Proposition 4.2.3, we know f is bounded, so $f \in C(X \to Y)$. Thus, we know $\left\{f^{(n)}\right\}_{n=1}^\infty$ converges in $C(X \to Y)$, so $C(X \to Y)$ is complete. ■

Lecture 14

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Lemma 4.3.1 (HW6 P4). Let $(X, \mathcal{F})$ be a topological space, and $K \subseteq X$ and K is compact, then if X is Hausdorff, then K is closed.

Lemma 4.3.2. Let (Y, d_Y) be a metric space and for any $a \in Y$, and we have a map $\varphi : Y \to \mathbb{R}$ defined by $\varphi(y) = d_Y(a, y)$, then φ is continuous. In fact,

$$|\varphi(y) - \varphi(y')| \leq d_Y(y, y').$$

Proof. Since

$$\varphi(y) = d_Y(a, y) \leq d_Y(a, y') + d_Y(y', y) = \varphi(y') + d_Y(y', y),$$

so

$$|\varphi(y) - \varphi(y')| \leq d_Y(y, y').$$

Thus, if given $\varepsilon > 0$, then choose $\delta = \varepsilon$, then we know if $d_Y(y, y') < \delta = \varepsilon$, then

$$|\varphi(y) - \varphi(y')| \leq d_Y(y, y') < \delta = \varepsilon,$$

so φ is continuous. ■

Remark 4.3.1. Suppose $\lim_{n\to\infty} y_n = y$ in Y, then $\lim_{n\to\infty} d_Y(y_n, a) = d_Y(y, a)$.

Lemma 4.3.3. If $\lim_{n\to\infty} a_n = a$ in (Y, d_Y). Given any $N \in \mathbb{N}$, then

$$\inf_{m\geq N} a_m \leq \lim_{n\to\infty} a_n \leq \sup_{m\geq N} a_m.$$

Proof. For any $K \geq N$, we have

$$\inf_{m\geq N} a_m \leq a_K \leq \sup_{m\geq N} a_m,$$

We know that $\lim_{K\to\infty} a_K$ exists by Squeeze Theorem, so we have

$$\inf_{m\geq N} a_m \leq \lim_{k\to\infty} a_k \leq \sup_{m\geq N} a_m.$$

■

Theorem 4.3.2 (Reprove Theorem 4.3.1). Let (X, d_X) be a metric space, and (Y, d_Y) be a complete metric space. Let $C(X \to Y)$ be the space of continuous and bounded function. Then $(C(X \to Y), d_\infty)$ is a complete metric space, where

$$d_\infty(f,g) = \sup_{x\in X} d_Y(f(x), g(x)) \text{ for } f, g \in C(X \to Y).$$

Proof. Given a Cauchy sequence $\left\{f^{(n)}\right\}_{n=1}^\infty$ in $(C(X \to Y), d_\infty)$, then we want to show there exists $f \in C(X \to Y)$ s.t. $\lim_{n\to\infty} d_\infty(f_n, f) = 0$. i.e. $f_n \to f$ uniformly by Proposition 4.3.2. Given $\varepsilon > 0$, then there exists $N > 0$ s.t. $\sup_{x\in X} d_Y(f_n(x), f_m(x)) = d_\infty(f_n, f_m) < \frac{\varepsilon}{2}$ for all $n, m \geq N$. This implies that $\left\{f^{(n)}(x)\right\}_{n=1}^\infty$ is Cauchy in Y for all $x \in X$. Since Y is complete, so $\lim_{n\to\infty} f^{(n)}(x)$ exists in Y for all $x \in X$. Now we define $f(x) := \lim_{n\to\infty} f^{(n)}(x)$ for all $x \in X$, then for any $x \in X$, consider $n \geq N$, we know

$$d_Y(f_n(x), f(x)) = \lim_{m\to\infty} d_Y(f_n(x), f_m(x)) \text{ by Lemma 4.3.2 and } \lim_{m\to\infty} f_m(x) = f(x) \in Y.$$

By Lemma 4.3.3, we know

$$d_Y(f_n(x), f(x)) = \lim_{m\to\infty} d_Y(f_n(x), f_m(x)) \leq \sup_{m\geq N} d_Y(f_n(x), f_m(x)) \leq \sup_{m\geq N} d_\infty(f_n, f_m) \leq \frac{\varepsilon}{2} < \varepsilon,$$

so f_n converges uniformly to f, and by Corollary 4.2.1 and Proposition 4.2.3, we know that $f \in C(X \to Y)$, and we're done. ■

Example 4.3.1. If Y is not complete, then $(C(X \to Y), d_\infty)$ may not be complete.

Proof. Let $X = [0,1]$ with standard metric, and let $Y = \mathbb{Q}$, then note that Y is not complete. Let $r_n \in \mathbb{Q}$ and $\lim_{n\to\infty} r_n = \sqrt{2}$. If we define

$$f_n : [0,1] \to \mathbb{Q}, \quad f_n(x) = r_n \quad \forall x \in [0,1],$$

then

$$d_\infty(f_n, f_m) = \sup_{x\in[0,1]} |f_n(x) - f_m(x)| = \sup |r_n - r_m|.$$

Since $\lim_{n\to\infty} r_n = \sqrt{2}$, so $\{r_n\}_{n=1}^\infty$ is Cauchy, so $\{f_n\}_{n=1}^\infty$ is Cauchy in $(C(X \to Y), d_\infty)$. Note that $\lim_{n\to\infty} f_n = f$ but f is not a function from $[0,1]$ to $\mathbb{Q}$ since $f(x) = \sqrt{2}$ for all $x \in [0,1]$. ⊛

Note 4.3.1. I think unifromly convergent is unique, i.e. if (f_n) converges to f unifromly, then f is unique, but i am not sure.

End of midterm

4.4 Series of functions

Suppose (X, d_X) is a metric space and $f^{(n)} : X \to \mathbb{R}$ and $f : X \to \mathbb{R}$ for all $n \geq 1$. We define the partial sum of $\left\{f^{(n)}\right\}_{n=1}^\infty$ by

$$S_N(x) = \sum_{i=1}^N f^{(n)}(x).$$

Definition 4.4.1. Let (X, d_X) be a metric space, and let $\left\{f^{(n)}\right\}_{n=1}^\infty$ be a sequence of functions from X to $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be another function. We say that the infinite series $\sum_{i=1}^\infty f^{(i)}$ converges pointwise to f if

$$\lim_{n\to\infty} S_N(x) = f(x) \text{ pointwise where } S_N(x) = \sum_{i=1}^N f^{(i)}(x),$$

and we say $\sum_{i=1}^{\infty} f^{(i)}$ converges to f uniformly if

$$\lim_{N\to\infty} S_N(x) = f(x) \text{ uniformly.}$$

Definition 4.4.2 (Sup norm). Let $f : X \to \mathbb{R}$ be a bounded valued function, then we can define

$$\|f\|_\infty := \sup_{x\in X} |f(x)| = d_\infty(f, 0).$$

Theorem 4.4.1 (Weierstrass M-test). Let (X, d_X) be a metric space, and $\left(f^{(n)}\right)_{n=1}^{\infty}$ be a sequence of bounded, real-valued, and continuous functions on X. Suppose $\sum_{n=1}^{\infty} \left\|f^{(n)}\right\|_\infty$ converges, then $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly on X to a bounded and continuous, real-valued functions on X.

Proof. Let $S_N(x) = \sum_{i=1}^{N} f^{(i)}(x)$, then recall that $(C(X \to \mathbb{R}), d_\infty)$ is a complete metric space. (Since $\mathbb{R}$ is complete with standard metric). Let $M_n := \left\|f^{(n)}\right\|_\infty \in \mathbb{R}$. We know that $\sum_{n=1}^{\infty} M_n$ converges, then we know $\left(\sum_{n=1}^{k} M_n\right)_{k=1}^{\infty}$ converges, so we can define $t_N = \sum_{i=1}^{N} M_i$. Thus, we know $\{t_n\}_{n=1}^{\infty}$ is Cauchy, so given $\varepsilon > 0$, we know there exists $N > 0$ s.t. $m > n \geq N$ implies

$$\left|\sum_{i=n+1}^{m} M_i\right| = |t_m - t_n| < \varepsilon.$$

Hence, for all $m > n \geq N$,

$$\begin{aligned} d_\infty(S_m, S_n) &= \sup_{x\in X} \left|\sum_{i=1}^{m} f^{(i)}(x) - \sum_{i=1}^{n} f^{(i)}(x)\right| \\ &= \sup_{x\in X} \left|\sum_{i=n+1}^{m} f^{(i)}(x)\right| \leq \sup_{x\in X} \sum_{i=m+1}^{n} \left|f^{(i)}(x)\right| \leq \sum_{i=m+1}^{n} M_i < \varepsilon. \end{aligned}$$

Thus, $\{S_n\}_{n=1}^{\infty}$ is Cauchy in $(C(X \to \mathbb{R}), d_\infty)$, and since $\mathbb{R}$ is complete, so we know $S_n \to f$ in $(C(X \to \mathbb{R}), d_\infty)$, which means $S_n \to f$ uniformly in $C(X \to \mathbb{R})$ by Theorem 4.3.1 and Proposition 4.3.2.

> **Note 4.4.1.** S_N is bounded and continuous since $f^{(i)}$ is bounded and continuous for all $i \in \mathbb{N}$ and the sum of bounded and continuous function is still bounded and continuous.

■

Example 4.4.1.

- $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ pointwise when $x \in (-1, 1)$.
- $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ uniformly when $x \in [-r, r]$ where $0 < r < 1$.

Proof. Consider $S_N(x) = \sum_{i=1}^{N} x^n$, then

$$s_N(x) - x s_N(x) = x - x^{N+1} \Rightarrow s_N(x) = \frac{x - x^{N+1}}{1-x}.$$

When $|x| < 1$, then $\lim_{N\to\infty} x^{N+1} = 0$, so

$$\lim_{N\to\infty} s_N(x) = \lim_{N\to\infty} \frac{x - x^{N+1}}{1-x} = \frac{x}{1-x}.$$

Thus, $\sum_{i=1}^{\infty} x^i$ converges to $\frac{x}{1-x}$ pointwise.

Now if $x \in [-r, r]$ with $0 < r < 1$, then $|x^i| \leq r^i$ for $x \in [-r, r]$. Thus, $\|x^i\|_\infty = r^i$, so $\sum_{i=1}^{\infty} \|x^i\|_\infty = \sum_{i=1}^{\infty} r^i$ converges to $\frac{r}{1-r}$, and thus $\sum_{i=1}^{\infty} x^i$ converges uniformly to $\frac{x}{1-x}$ by Theorem 4.4.1. ⊛

Lecture 15

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Example 4.4.2. Suppose

$$f_n(x) = \begin{cases} 0, & \text{if } x \leq 0 \text{ or } x \geq \dfrac{1}{n}; \\ 4n^2 x, & \text{if } 0 \leq x \leq \dfrac{1}{2n}; \\ -4n^2 x + 4n, & \text{otherwise} \dfrac{1}{2n} \leq x \leq \dfrac{1}{n}, \end{cases}$$

then $\lim_{n\to\infty} f_n(x) = 0$ pointwise. Also, $f_n(x)$ is continuous and $\int_{-\infty}^{\infty} f_n(x)\,\mathrm{d}x = 1$. However, $\lim_{n\to\infty} f_n(x) = f(x) = 0$ for all $x \in \mathbb{R}$, so

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,\mathrm{d}x \neq \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)\,\mathrm{d}x,$$

and we will talk about when will exchanging lim and integration get same result.

Note 4.4.2. The problem is that $f_n \to f$ only pointwise.

Now we quickly review Riemann integral. Assume f is continuous on $[a, b]$, then we partition the interval $[a, b]$ into subintervals $I_k = [x_{k-1}, x_k]$ for all $1 \leq k \leq n$. On I_k, find the maximum and the minimum of f on I_k, and let $M_k = \sup_{x\in I_k} f(x)$ and $m_k = \inf_{x\in I_k} f(x)$, and use the area of the rectangle with height M_k and base $x_k - x_{k-1} = \Delta X_k$, and similarly use the area of the rectangle with height m_k and base $x_k - x_{k-1} = \Delta X_k$, then we know

$$m_k \cdot \Delta X_k \leq \text{The real area of } f \text{ on } I_k \leq M_k \cdot \Delta X_k,$$

and thus we can define the upper Riemann sum to be $\sum_{k=1}^{n} M_k \cdot \Delta X_k$ and the lower Riemann sum to be $\sum_{k=1}^{n} m_k \cdot \Delta X_k$. Hence, we know

$$\text{Lower Riemann Sum} \leq \text{Area} \leq \text{Upper Riemann Sum}.$$

Also, since upper Riemann sum decreases if one has more partition and lower Riemann sum increases if one has more partition, so we can approach the area by having more partition.

Definition 4.4.3. Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval, and let p be a partition of I, we define the upper Riemann integral $U(f, p)$ and the lower Riemann sum $L(f, p)$ by

$$U(f,p) = \sum_{J\in p, J\neq\varnothing} \left(\sup_{x\in J} f(x)\right)|J|, \quad L(f,p) = \sum_{J\in p, J\neq\varnothing} \left(\inf_{x\in J} f(x)\right)|J|$$

Proposition 4.4.1 (From Analysis I)**.** Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I, then we define

$$\overline{\int_I} f = \inf\{U(f,p) : p \text{ is a partition of } I\} \quad \text{(upper integral)}$$
$$\underline{\int_I} f = \sup\{L(f,p) : p \text{ is a partition of } I\} \quad \text{(lower integral)},$$

and we have

(1) $\underline{\int_I} f \leq \overline{\int_I} f$

(2) Suppose $f \leq g$ on I, then we have

$$\underline{\int_I} f \leq \underline{\int_I} g, \text{ and } \overline{\int_I} f \leq \overline{\int_I} g.$$

Proof.

(1) Since $\underline{\int_I} f \leq$ real area $\leq \overline{\int_I} f$, so this is true. (Not rigorous)

(2) Since we fix any partition p on I, we have

$$L(f,p) \leq L(g,p) \quad U(f,p) \leq U(g,p),$$

so this is true.

> **Remark 4.4.1.** More intuitively, since lower Riemann integral and upper Riemann integral are both approaching the real area (if the function is not weird), so this can be intuitively accepted.

■

Definition 4.4.4. Let $f : I \to \mathbb{R}$ be a bounded function. We say f is Riemann integrable on I if $\underline{\int_I} f = \overline{\int_I} f$, and we denote it by $\int_I f$.

Remark 4.4.2. To prove f is (Riemann) integrable, we need to prove $\underline{\int_I} f = \overline{\int_I} f$.

Example 4.4.3. Suppose $f : I \to \mathbb{R}$ with $I = [0,1]$ and

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1], x \in \mathbb{Q}; \\ 0, & \text{if } x \in [0,1], x \notin \mathbb{Q}, \end{cases}$$

then $\overline{\int_I} f = 1$ and $\underline{\int_I} f = 0$, so f is not integrable.

> **Note 4.4.3.** $f(x) = 0$ almost everywhere on $[0,1]$, and we say f is Lebesgue integrable and its Lebesgue integral is 0. We'll discuss this next semester.

Theorem 4.4.2. Let $f^{(n)} : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions. Suppose $\lim_{n\to\infty} f^{(n)} = f$ uniformly where $f : [a,b] \to \mathbb{R}$, then f is also Riemann integrable and $\lim_{n\to\infty} \int_I f^{(n)} = \int_I f$, or equivalently

$$\lim_{n\to\infty} \int_I f^{(n)} = \int_I \lim_{n\to\infty} f^{(n)}.$$

Proof. First, we want to show that $\overline{\int_I} f = \underline{\int_I}(f)$, then since $f_n \to f$ uniformly on $[a,b]$, so given $\varepsilon > 0$, $\exists N > 0$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in [a,b]$. Thus, we have

$$f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon \quad \forall n \geq N \text{ and } x \in [a,b].$$

Hence, we have

$$\underline{\int_I}(f_n - \varepsilon) \leq \underline{\int_I} f, \quad \overline{\int_I} f \leq \overline{\int_I}(f_n + \varepsilon) \quad \forall n \geq N,$$

which gives

$$\underline{\int_I} f_n - \varepsilon(b-a) \le \underline{\int_I} f, \quad \overline{\int_I} f \le \overline{\int_I} f_n + \varepsilon(b-a) \quad \forall n \ge N.$$

Now since f_n is Riemann integrable, so $\underline{\int_I} f_n = \overline{\int_I} f_n = \int_I f_n$. Hence, we have

$$\int_I f_n - \varepsilon(b-a) \le \underline{\int_I} f \le \overline{\int_I} f \le \int_I f_n + \varepsilon(b-a) \quad \forall n \ge N.$$

Hence,

$$\overline{\int_I} f - \underline{\int_I} f \le 2\varepsilon(b-a) \quad \forall \varepsilon > 0,$$

which gives $\overline{\int_I} f = \underline{\int_I} f$. Hence, f is Riemann integrable.

Also, we have

$$\int_I f_n - \varepsilon(b-a) \le \int_I f \le \int_I f_n + \varepsilon(b-a) \quad \forall n \ge N,$$

so we have

$$\left| \int_I f - \int_I f_n \right| \le \varepsilon(b-a) \quad \forall n \ge N,$$

which gives $\lim_{n\to\infty} \int_I f^{(n)} = \int_I f$. ■

Theorem 4.4.3. Let $f^{(n)} : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable function. Suppose $\sum_{n=1}^{\infty} f^{(n)}(x)$ converges uniformly, and let $f(x) \coloneqq \sum_{n=1}^{\infty} f^{(n)}(x)$, then f is Riemann integrable and

$$\sum_{n=1}^{\infty} \int_I f^{(n)} = \int_I f = \int_I \sum_{n=1}^{\infty} f^{(n)}.$$

Proof. Let $S_k(x) = \sum_{i=1}^{k} f^{(i)}(x)$, then since $\sum_{i=1}^{\infty} f^{(i)}(x)$ converges uniformly, so $\lim_{k\to\infty} S_k(x) = f(x)$ where $f(x) = \sum_{i=1}^{\infty} f^{(n)}(x)$. Now $S_k = \sum_{i=1}^{k} f^{(i)}$ is a sum of Riemann integrable functions, so S_k is also Riemann integrable. By Theorem 4.4.2, we know f is Riemann integrable, and $\lim_{k\to\infty} \int_I S_k = \int_I f$, which means $\lim_{k\to\infty} \sum_{i=1}^{k} \int_I f^{(i)} = \int_I f$ by the linearity of Riemann integral, and thus

$$\sum_{n=1}^{\infty} \int_I f^{(n)} = \int_I f.$$

Note 4.4.4.

$$\int_I S_k = \int_I \sum_{i=1}^{k} f^{(i)} = \sum_{i=1}^{k} \int_I f^{(i)}$$

for finite k.

■

Example 4.4.4. $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ pointwise for $x \in (-1,1)$, and $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ pointwise for $x \in (-1,1)$, but if we fix $r \in (-1,1)$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ uniformly on $[-r,r]$. Now since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ uniformly on $[0,r]$ for $-1 < r < 1$, then by Theorem 4.4.3, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \int_0^r x^n \,\mathrm{d}x &= \int_0^r \frac{1}{1-x} \,\mathrm{d}x \\ \Rightarrow \sum_{n=0}^{\infty} \frac{r^{n+1}}{n+1} &= -\ln(1-r) + \ln 1 = -\ln(1-r). \end{aligned}$$

4.5 Uniform Convergence & Derivatives

We talk about two examples to show that uniform convergence does not preserve the value of derivatives at some point.

Example 4.5.1. Suppose $f_n : [0, 2\pi] \to \mathbb{R}$ and

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}},$$

then $\lim_{n\to\infty} f_n = 0$ uniformly since

$$|f_n(x)| = \left|\frac{\sin(nx)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}},$$

but its derivative is

$$f_n'(x) = \frac{\cos(nx)\cdot n}{\sqrt{n}},$$

so $f_n'(0) = \sqrt{n}$, and thus

$$\lim_{n\to\infty} f_n'(0) = \infty \neq f'(0) = 0.$$

Example 4.5.2. Suppose $f_n(x) = \sqrt{\frac{1}{n^2} + x^2}$, then $\lim_{n\to\infty} f_n(x) = |x|$ uniformly since

$$\begin{aligned} 0 \leq f_n(x) - |x| = \sqrt{\frac{1}{n^2} + x^2} - |x| &= \frac{\left(\sqrt{\frac{1}{n^2}+x^2} - |x|\right)\left(\sqrt{\frac{1}{n^2}+x^2} + |x|\right)}{\sqrt{\frac{1}{n^2}+x^2} + |x|} \\ &= \frac{\frac{1}{n^2}}{\sqrt{\frac{1}{n^2}+x^2}} \leq \frac{\frac{1}{n^2}}{\sqrt{\frac{1}{n^2}}} = \frac{1}{n}. \end{aligned}$$

Let $f(x) = |x|$. We have $\lim_{n\to\infty} f_n = f$ uniformly, and note that $f_n(x) = \sqrt{\frac{1}{n^2} + x^2}$ and $f_n'(x) = \frac{x}{\sqrt{\frac{1}{n^2}+x^2}}$. Note that $f'(0)$ does not exists and $\lim_{n\to\infty} f_n'(0) = 0$, so in this case $f_n \to f$ uniformly and f_n' exists, but $f'(0)$ doesn't exist.

Theorem 4.5.1. Let $[a, b]$ be an interval and for $n \geq 1$, let $f_n : [a, b] \to \mathbb{R}$ be a differentiable function whose derivative $f_n' : [a, b] \to \mathbb{R}$ is continuous. Suppose $f_n' \to g$ uniformly where $g : [a, b] \to \mathbb{R}$. Suppose $\exists x_0 \in [a, b]$ s.t. $\lim_{n\to\infty} f_n(x_0)$ exists, then $\exists$ a differentiable f s.t. $\lim_{n\to\infty} f_n = f$ uniformly and $f' = g$.

Proof. Since f_n' is continuous and $f_n' \to g$ uniformly, then g is Riemann integrable since f_n' is Riemann integrable for all $n \in \mathbb{N}$ and by Theorem 4.4.2. Also, we know

$$\lim_{n\to\infty} \int_{x_0}^{x} f_n'(s)\,\mathrm{d}s = \int_{x_0}^{x} g(s)\,\mathrm{d}s \text{ uniformly} \quad \text{for } x_0, x \in [a, b]$$

since

$$\left|\int_{x_0}^{x} (f_n' - g)(s)\,\mathrm{d}s\right| \leq \int_{x_0}^{x} |f_n'(s) - g(s)|\,\mathrm{d}s \leq \frac{\varepsilon}{b-a}\cdot|x - x_0| \leq \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon \quad \forall \varepsilon > 0, n \geq N$$

for some $N \in \mathbb{N}$, and Fundamental theorem of Calculus tells us $\lim_{n\to\infty}(f_n(x) - f_n(x_0)) = \int_{x_0}^{x} g(s)\,\mathrm{d}s$ uniformly, which means $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x_0) + \int_{x_0}^{x} g(s)\,\mathrm{d}s$ uniformly, and we can let f to be R.H.S. ∎

Remark 4.5.1. Informally, the theorem states that if f_n' continuous and converges uniformly and $f_n(x_0)$ converges for some x_0, then f_n itself converges uniformly, and moreover

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} f_n'(x).$$

Remark 4.5.2. We need f_n' to be continuous to ensure that f_n' is Riemann integrable (Note that we have $f_n'([a,b])$ is bounded and continuous, so f_n' is Riemann integrable). Also, we need continuity to use Fundamental Theorem of Calculus.

Corollary 4.5.1. Let $[a,b]$ be an interval and for $n \geq 1$, let $f_n : [a,b] \to \mathbb{R}$ be a differentiable function whose derivatives $f_n' : [a,b] \to \mathbb{R}$ is continuous. Suppose $\sum_{n=1}^{\infty} \|f_n'\|_\infty$ converges. Suppose also that $\sum_{n=1}^{\infty} f^{(n)}(x_0)$ converges for some $x_0 \in [a,b]$, then $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly to a differentiable function and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} f_n' \text{ for } x \in [a,b].$$

Proof. Suppose $S_k(x) = \sum_{i=1}^{k} f_i(x)$ and $S_k'(x) = \sum_{i=1}^{k} f_i'(x)$, then since $\sum_{i=1}^{\infty} \|f_n'\|_\infty$ converges, then by Weierstrass M-test , we know $\lim_{k\to\infty} S_k' = \sum_{i=1}^{\infty} f_i'$ uniformly. We also know that $S_k(x_0) = \sum_{i=1}^{k} f_i(x_0)$ converges to $\sum_{i=1}^{\infty} f_i(x_0)$ by our conditions. Now by Theorem 4.5.1 (Suppose S_k here is f_n in Theorem 4.5.1), then we know $\lim_{k\to\infty} S_k = S$ uniformly for some function $S : [a,b] \to f$ and $S' = \sum_{i=1}^{\infty} f_i'$, which means

$$\sum_{n=1}^{\infty} f_n' = \frac{\mathrm{d}}{\mathrm{d}x} S = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lim_{k\to\infty} S_k\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} f_n\right).$$

Note 4.5.1. $\sum_{n=1}^{\infty} \|f_n'\|_\infty$ converges implies f_n' is bounded for all n, so we can use Weirestrass M-test. In fact, f_n' is continuous gives f_n' is bounded since its domain is an bounded interval.

■

Example 4.5.3. $f(x) = \sum_{n=1}^{\infty} 4^{-n} \cos(32^n \pi x)$, and $f(x)$ is not differentiable anywhere.

Proof. Since

$$\left|4^{-n} \cos\left(32^{-n}\pi x\right)\right| \leq 4^{-n},$$

and $\sum_{n=1}^{\infty} 4^{-n}$ converges, so $\sum_{n=1}^{\infty} 4^{-n} \cos(32^{-n}\pi x)$ converges uniformly by Weierstrass M-test. Also, we know

$$\left(4^{-n} \cos\left(32^{-n}\pi x\right)\right)' = -8^n \sin\left(32^n \pi x\right),$$

and we will learn that $f'(x)$ does not exist at any point from the exercise. (See Exercise 4.7.10 in the textbook) ⊛

Chapter 5

Formal Power Series

Lecture 16

5.1 Review of series

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Definition 5.1.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers,

(a) The limit superior or (lim sup) of a sequence (a_n) is defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k.$$

Let $S_n = \sup_{k\geq n} a_k$. Note that the index set $\{k \geq n\}$ is larger than $\{k \geq n+1\}$, so $S_{n+1} \leq S_n$. Equivalently, we can define $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} S_n$, and since S_n is decreasing, so $\lim_{n\to\infty} S_n$ exists, but it could be ∞ or $-\infty$.

(b) Similarly we can define

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} \inf a_k.$$

Let $I_n = \inf_{k\geq n} a_k$, then we know $I_n \leq I_{n+1}$, so I_n is increasing, so $\lim_{n\to\infty} I_n = \liminf_{n\to\infty} a_n$ exists, but it could be ∞ or $-\infty$.

Example 5.1.1.

$$\limsup_{n\to\infty} n = \lim_{n\to\infty} \sup_{k\geq n} k = \infty.$$

$$\liminf_{n\to\infty} n = \lim_{n\to\infty} \inf_{k\geq n} k = \lim_{n\to\infty} n = \infty.$$

Definition 5.1.2. Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers, and let $S_N = \sum_{n=1}^{N} a_n$, then we say $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{N\to\infty} S_N$ exists and

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N.$$

Theorem 5.1.1. If (a_n) is a sequence of real numbers, and if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose $S_N = \sum_{n=1}^{N} a_n$, then we know $S_{n+1} - S_n = a_{n+1}$, then we know

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} S_{n+1} - S_n = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = 0.$$

■

Corollary 5.1.1. If (a_n) is a sequence of real numbers, and if $\lim_{n\to\infty} a_n$ doesn't exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Definition 5.1.3. We say a real series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 5.1.2. If (a_n) is a sequence of real numbers, and if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Let $S_n = \sum_{i=1}^{n} a_i$, then suppose $n \geq m$, then we have

$$|S_n - S_m| = \left| \sum_{i=m+1}^{n} a_n \right| \leq \sum_{i=m+1}^{n} |a_i|.$$

Let $T_n = \sum_{i=1}^{n} |a_i|$, then we know $|T_n - T_m| = \sum_{i=m+1}^{n} |a_i|$. Since $\sum_{n=1}^{\infty} a_n$ converges absolutely, so $\lim_{n\to\infty} T_n$ exists, and thus $\{T_n\}$ is Cauchy. Since

$$|S_n - S_m| \leq |T_n - T_m|\,,$$

so $\{S_n\}$ is also Cauchy, which means it is convergent. ∎

5.2 Formal Power Series

Definition 5.2.1. Let $a \in \mathbb{R}$. A formal power series centered at a is any series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where $\{c_n\}_{n=1}^{\infty}$ is a sequence of real numbers. We refer c_n to the n-th coefficient of the series. Each term $c_n(x-a)^n$ is a function of x.

Example 5.2.1. $\sum_{n=0}^{\infty} n!(x-2)^n$ is a formal power series centered at 2 but $\sum_{n=0}^{\infty} 2^x(x-3)^n$ is not a formal power series since 2^x depends on x not on n.

Definition 5.2.2. Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series. We define the radius of convergence R of this series to be the quantity $R = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$ with the convention $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$.

Remark 5.2.1. The radius of convergence must be non-negative.

Lemma 5.2.1. Let (a_n) be a sequence of real number. Suppose $L = \limsup_{n\to\infty} a_n \in [0, \infty)$, then for any $\varepsilon > 0$, $\exists N > 0$ s.t. $a_n < L + \varepsilon$ for all $n \geq N$.

Proof. If $L = \infty$, then this is true. Now if L is a real number, then $L = \lim_{n\to\infty} S_n$ where $S_n = \sup_{k\geq n} a_k$. Given any $\varepsilon > 0$, $\exists N > 0$ s.t. $|S_n - L| < \varepsilon$ for all $n \geq N$, so $S_n < L + \varepsilon$ for all $n \geq N$. Hence,

$$a_n \leq \sup_{k\geq n} a_k = S_n < L + \varepsilon \quad \forall n \geq N.$$

∎

Lemma 5.2.2. Let (a_n) be a sequence of real numbers, and let $p := \limsup_{n\to\infty} a_n$. Then for every $\varepsilon > 0$, $\exists$ infinitely many indices n s.t. $a_n > p - \varepsilon$.

Proof. We know $p = \lim_{n\to\infty} S_n$ where $S_n = \sup_{k\geq n} a_k$, so given $\varepsilon > 0$, $\exists N > 0$ s.t.

$$|S_n - p| < \frac{\varepsilon}{2} \quad \forall n \geq N,$$

so $p - \frac{\varepsilon}{2} < S_n = \sup_{k\geq n} a_k$, which means $\exists k_1 \geq n$ s.t. $p - \frac{\varepsilon}{2} < a_{k_1}$, and let $n = k_1 + 1$ and repeat this step to get k_2 and so on, then we know $\{k_n\}$ is an infinite set. ∎

Theorem 5.2.1 (Ratio Test). Suppose $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \in [0,\infty)$. If $L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely. If $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges. If $L = 1$, then no conclusion.

Proof. If $L < 1$, then we know there exists r s.t. $L < r < 1$, so there exists $N > 0$ s.t. for all $n \geq N$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| \leq r,$$

so we know $|a_{n+1}| \leq r|a_n|$ for all $n \geq N$, which means $|a_{n+k}| \leq r^k |a_n|$, so

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{N-1} |a_n| + \sum_{k=0}^{\infty} |a_{N+k}| = C + \sum_{k=0}^{\infty} r^k |a_N| = C + |a_N| \sum_{k=0}^{\infty} r^k,$$

which means $\sum_{n=0}^{\infty} |a_n|$ is convergent since $r < 1$.

If $L > 1$(including $L = \infty$), then by the definition of limit, we know for some $\varepsilon > 0$, there exists $N > 0$ s.t.

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1 + \varepsilon$$

for all $n \geq N$, and thus $|a_{n+1}| \geq (1+\varepsilon)|a_n|$ for all $n \geq N$. In particular, there must exists some $k > 0$ with $|a_k| > 0$, otherwise L is not well-defined, and thus we know $\lim_{n\to\infty} a_n \neq 0$ since $|a_n|$ is strictly increasing, so by Theorem 5.1.1, we know $\sum_{n=0}^{\infty} a_n$ diverges.

If $L = 1$, then for example, we know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\sum_{n=1}^{\infty} \frac{1}{n^2}$converges. ∎

Theorem 5.2.2. Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series, and let R be its radius of convergence $R = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$, then

(a) $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges if $|x-a| > R$.

(b) $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely if $|x-a| < R$.

(c) $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly on $[a-r, a+r]$ when $0 < r < R$.

(d) Let $f = \sum_{n=0}^{\infty} c_n(x-a)^n$, and if $|x-a| < R$, then f is differentiable and for any $0 < r < R$, $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ converges uniformly to f' on $[a-r, a+r]$.

(e) For any $[y,z] \subset (a-R, a+R)$,

$$\int_y^z f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{c_n(z-a)^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{c_n(y-a)^{n+1}}{n+1}.$$

proof from (c) to (e). Trivial by Theorem 4.4.3. We may have to show the radius of convergence remains the same to show that we can write

$$\sum_{n=0}^{\infty} \frac{c_n(z-a)^{n+1}}{n+1} - \frac{c_n(y-a)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{c_n(z-a)^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{c_n(y-a)^{n+1}}{n+1}.$$

∎

proof of (a). Write $L := \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$, so $L = \frac{1}{R}$. Now if $|x-a| > R$, then let $S = |x-a| > R$, and we have

$$\limsup_{n\to\infty} |c_n(x-a)^n|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}|x-a| = |x-a| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = SL = \frac{S}{R} > 1.$$

Choose $\varepsilon > 0$ s.t. $\frac{S}{R} - \varepsilon > 1$. From Lemma 5.2.2, there exists infinitely many n s.t.

$$|c_n|^{\frac{1}{n}} |x-a| > \frac{S}{R} - \varepsilon > 1,$$

so there are infinitely many n has $|c_n(x-a)^n| > 1$, so $\lim_{n\to\infty} c_n(x-a)^n \neq 0$, and thus we know $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges. ∎

proof of (b). If $|x-a| < R$, then $\frac{|x-a|}{R} < 1$ i.e. $|x-a|L < 1$ (L is same as it is in proof of (a)). Now we have $\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = L$, so we can choose $\varepsilon > 0$ s.t. $|x-a|(L+\varepsilon) < 1$, so there exists $N > 0$ s.t.

$$|c_n|^{\frac{1}{n}} < L + \varepsilon \quad \forall n \geq N \Leftrightarrow |c_n| < (L+\varepsilon)^n \Leftrightarrow |c_n(x-a)^n| \leq (L+\varepsilon)^n |x-a|^n$$

by Lemma 5.2.1, so

$$\sum_{n=0}^{\infty} |c_n(x-a)^n| \leq \sum_{n=0}^{N-1} |c_n(x-a)^n| + \sum_{n=N}^{\infty} |c_n(x-a)^n| \leq \sum_{n=0}^{N-1} |c_n||x-a|^n + \sum_{n=N}^{\infty} |(L+\varepsilon)(x-a)|^n$$

and R.H.S. converges since the left term is finite and for the right term we have $|(L+\varepsilon)(x-a)| < 1$. ∎

proof of (c). Now if $|x-a| \leq r < R$, then we can choose $\varepsilon > 0$ s.t. $q = (L+\varepsilon)r < 1$ since $Lr = \frac{r}{R} < 1$. Note that

$$|(L+\varepsilon)(x-a)|^n \leq ((L+\varepsilon)r)^n = q^n,$$

and this is independent of x. (Note that $L > 0$ so we can get rid of the absolute value) Note that this means there exists $N > 0$ s.t.

$$\begin{aligned}
\sum_{n=0}^{\infty} \|c_n(x-a)^n\|_{\infty} &= \sum_{n=0}^{\infty} |c_n r^n| = \sum_{n=0}^{N-1} |c_n r^n| + \sum_{n=N}^{\infty} |c_n r^n| \\
&= C + \sum_{n=N}^{\infty} \left| c_n^{\frac{1}{n}} r \right|^n \leq C + \sum_{n=N}^{\infty} |(L+\varepsilon)r|^n \leq C + \sum_{n=N}^{\infty} q^n,
\end{aligned}$$

by Lemma 5.2.1, and we know this means $\sum_{n=0}^{\infty} \|c_n(x-a)^n\|_{\infty}$ is convergent since $q < 1$, so by Weierstrass M-test, we know $\sum_{n=0}^{\infty} |c_n||x-a|^n$ converges uniformly. ∎

proof of (d). Suppose $S_N(x) = \sum_{n=0}^{N} c_n(x-a)^n$ and

$$S_N'(x) = \sum_{n=1}^{N} n c_n (x-a)^{n-1} = \sum_{n=0}^{N-1} (n+1) c_{n+1} (x-a)^n.$$

Then, we know S_N' is continuous for all $N > 0$. Now we claim that the radius of convergence of $\sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n$ is R. Note that

$$\limsup_{n\to\infty} |(n+1)c_{n+1}|^{\frac{1}{n}} = \lim_{n\to\infty} \sup_{k\geq n} (k+1)^{\frac{1}{k}} |c_{k+1}|^{\frac{1}{k}} = \lim_{n\to\infty} (n+1)^{\frac{1}{n}} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}}.$$

Note that we can show that $\lim_{n\to\infty}(n+1)^{\frac{1}{n}} = 1$, which is easy, so we skip here, and we have

$$\begin{aligned}
\limsup_{n\to\infty} |(n+1)c_{n+1}|^{\frac{1}{n}} &= \lim_{n\to\infty} (n+1)^{\frac{1}{n}} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} = \lim_{n\to\infty} (n+1)^{\frac{1}{n}} \lim_{n\to\infty} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} \\
&= \lim_{n\to\infty} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} = \limsup_{n\to\infty} |c_{n+1}|^{\frac{1}{n}}.
\end{aligned}$$

Now we show that $\limsup_{n\to\infty} |c_{n+1}|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = L$. For all $\varepsilon > 0$, there exists $N > 0$ s.t. for all $n \geq N$ we have

$$|c_n|^{\frac{1}{n}} < L + \varepsilon \Leftrightarrow |c_{n+1}|^{\frac{1}{n+1}} < L + \varepsilon \Leftrightarrow |c_{n+1}|^{\frac{1}{n}} < (L+\varepsilon)^{\frac{n+1}{n}} = (L+\varepsilon)^{1+\frac{1}{n}}.$$

Hence, we have

$$\sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} \leq (L+\varepsilon)^{1+\frac{1}{n}} \quad \forall n \geq N$$

since $(L+\varepsilon)^{1+\frac{1}{n}} \geq (L+\varepsilon)^{1+\frac{1}{k}} > |c_{k+1}|^{\frac{1}{k}}$ for all $k \geq n$. Hence, we know

$$\lim_{n\to\infty} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} \leq \lim_{n\to\infty} (L+\varepsilon)^{1+\frac{1}{n}} = L + \varepsilon.$$

Since for every $\varepsilon > 0$ this is true, so $\lim_{n\to\infty} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} \leq L$. Now we show

$$\lim_{n\to\infty} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} \geq L,$$

and then we can conclude that

$$\lim_{n\to\infty} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{n}} = L = \lim_{n\to\infty} \sup_{k\geq N} |c_n|^{\frac{1}{n}}.$$

For all $\varepsilon > 0$, we know there exists infinitely many n has

$$|c_n|^{\frac{1}{n}} > L - \varepsilon.$$

Hence, for every $n \in \mathbb{N}$, there exists $s_n > n$ has $|c_{s_n}|^{\frac{1}{s_n}} > L - \varepsilon$, and we collect $\{s_n\}_{n=1}^{\infty}$. Thus, we have

$$|c_{s_n}|^{\frac{1}{s_n - 1}} > (L-\varepsilon)^{\frac{s_n}{s_n - 1}} = (L-\varepsilon)^{1+\frac{1}{s_n-1}}$$

for all s_n. This means

$$\sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} \geq |c_{(s_{n+1}-1)+1}|^{\frac{1}{s_{n+1}-1}} = |c_{s_{n+1}}|^{\frac{1}{s_{n+1}-1}} > (L-\varepsilon)^{1+\frac{1}{s_{n+1}-1}},$$

and thus

$$\lim_{n\to\infty} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} > \lim_{n\to\infty} (L-\varepsilon)^{1+\frac{1}{s_{n+1}-1}} = (L-\varepsilon) \lim_{n\to\infty} (L-\varepsilon)^{\frac{1}{s_{n+1}-1}} = L - \varepsilon$$

since $s_{n+1} - 1 > n$. Note that this is true for all $\varepsilon > 0$, so we know $\lim_{n\to\infty} \sup_{k\geq n} |c_{k+1}|^{\frac{1}{k}} \geq L$, and we're done.

Now since we know the radius of convergence of $\lim_{n\to\infty} S'_n$ is R, so by (c) we know S'_n converges uniformly to $\sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n$ since from the problem condition we have $|x-a| < R$. Also, we know $\lim_{n\to\infty} S_n(a) = 0$, which means the limit exists, so by Theorem 4.5.1, we know there exists f s.t. $S_N \to f$ uniformly and $f' = g$ if we let $S'_n \to g$ uniformly. This means $S'_n \to f'$ uniformly, where $f = \sum_{n=0}^{\infty} c_n(x-a)^n$, and thus

$$S'_n = \sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n \to f'$$

uniformly. ■

Remark 5.2.2. If $|x-a| = R$, then $\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge or diverge, and there is no conclusion.

Appendix

Appendix A

Some Extra proof

A.1 Uncategorized

Theorem A.1.1. For a Cauchy sequence $\{x^{(n)}\}_{n=1}^{\infty}$, if there exists a subsequence $\{x^{(n_j)}\}_{j=1}^{\infty}$ converges to x, then $\{x^{(n)}\}_{n=1}^{\infty}$ also converges to x.

Proof. For all $\varepsilon > 0$, we know there exists $N > 0$ s.t. $j \geq N$ implies

$$d\left(x^{(n_j)}, x\right) < \frac{\varepsilon}{2}.$$

Also, there exists $N' > 0$ s.t. $i, j \geq N'$ implies

$$d\left(x^{(i)}, x^{(j)}\right) < \frac{\varepsilon}{2}.$$

Hence, if we pick some $d \geq N$ and $n_d \geq N'$, then we know for all $n \geq N'$, we have

$$d\left(x^{(n)}, x\right) \leq d\left(x^{(n)}, x^{(n_d)}\right) + d\left(x^{(n_d)}, x\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means $\{x^{(n)}\}_{n=1}^{\infty}$ converges to x. ∎

Definition A.1.1. A sequence of intervals I_n $(n \in \mathbb{N})$ is nested if $I_n \neq \varnothing$ and $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. $(I_1 \supseteq I_2 \supseteq \ldots)$.

Now we want to know $\bigcap_{n\in\mathbb{N}}^{\infty} I_n \neq \varnothing$?

Here is some counterexamples. Consider $I_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$. We can show that $\bigcap_{n=1}^{\infty} I_n = \varnothing$ by Archimedean Property. Besides, if $I_n = [n, \infty)$, $n \in \mathbb{N}$, this is trivial that $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

Theorem A.1.2 (Theorem of nested intervals). If I_n $(n \in \mathbb{N})$ is a sequence of bounded closed nested intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

Proof. Write $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$. First, we know I_n is nested iff $a_n \leq b_n$ and a_n is nondecreasing and b_n is nonincreasing. Hence, $\forall n, m \in \mathbb{N}$, we have $a_n \leq a_{\max\{n,m\}} \leq b_{\max\{n,m\}} \leq b_m$. In other words, for every $m \in \mathbb{N}$, b_m is a upper bound of $\{a_n\}$. Hence, we know $c = \lim_{n\to\infty} a_n = \sup\{a_n\}$.exists. Then, $c \leq b_m$ for all $m \in \mathbb{N}$. Also, $c \geq a_n$ for all $n \in \mathbb{N}$. Hence, $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$, and thus we know $c \in I_n$ for all $n \in \mathbb{N}$. Thus, $c \in \bigcap_{n=1}^{\infty} I_n$. ∎

Theorem A.1.3 (Bolzano Weierstrass Theorem). Suppose we have a bounded infinite sequence $a_n \in \mathbb{R}^m$, then $\exists$ a subsequence $a_{n(m)}$ such that $a_{n(m)}$ is convergent.

Proof. We just talk about the case $m = 2$, and the higher case is similar. Choose $M > 0$ such that $a_n \in [-M, M] \times [-M, M]$ for all $n \in \mathbb{N}$. Suppose $[-M, M] \times [-M, M]$ is called Q. Divide Q into 4 squares with equal size, and choose one, say Q_1 such that $|\{n \mid a_n \in Q_1\} = \infty|$. Select $n_1 \in \mathbb{N}$ such that $a_{n_1} \in Q_1$. Repeat this step, that is, divide Q_1 into 4 subparts, then says the one subpart with infinite many a_n in it is Q_2 (Q_2 must exists). Select $n_2 \in \mathbb{N}$ such that $a_{n_2} \in Q_2$ and $n_2 > n_1$. Keep repeating this step, then by Theorem A.1.2 we know

$$\bigcap_{n=1}^{\infty} Q_n \neq \varnothing.$$

Note A.1.1. Just think of the nested intervals are in x and y directions.

Actually, $\bigcap_{n=1}^{\infty} Q_n = \{a\}$ for some $a \in \mathbb{R}^2$, otherwise if there are two points in the intersection, then at some moment we will divide them into different subpart, which is a contradiction. It can been seen that $\lim_{k\to\infty} a_{n(k)} = a$. ■

Theorem A.1.4. If (X, d) is a metric space and $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subseteq X$. Now if $\lim_{n\to\infty} d(a_n, b_n) = 0$ and $\lim_{n\to\infty} a_n = p$ for some $p \in X$, then $\lim_{n\to\infty} b_n = p$.

Proof. Since we know for all $\varepsilon > 0$, $\exists N > 0$ s.t. $n \geq N$ implies $d(a_n, p) < \varepsilon$, and there exists $N_1, N_2 > 0$ s.t. $n \geq N_1$ implies $d(b_n, a_n) < \frac{\varepsilon}{2}$ and $n \geq N_2$ implies $d(a_n, p) < \frac{\varepsilon}{2}$, so now for $n \geq \max\{N_1, N_2\}$, we know

$$d(b_n, p) = d(b_n, a_n) + d(a_n, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

Theorem A.1.5. If $S \subseteq \mathbb{R}$, and $\sup S$ exists for some set S, then there exists a sequence of S converging to $\sup S$.

Proof. By the definition of sup, we know for all $\varepsilon > 0$, $\exists s \in S$ s.t. $\sup S \geq s > \sup S - \varepsilon$, so pick $\varepsilon = \frac{1}{n}$ for all $n \in \mathbb{N}$, we can form $\left(s^{(n)}\right)_{n=1}^{\infty}$ convergeing to $\sup S$. ■

A.2 The uniqueness of the convergence of function

Theorem A.2.1. If $\left(f^{(n)}\right)_{n=1}^{\infty}$ converges pointwise to f and converges pointwise to g, then $f = g$.

Proof. Write down the definition and use triangle inequality. ■

Theorem A.2.2. If $\left(f^{(n)}\right)_{n=1}^{\infty}$ converges uniformly to f and converges uniformly to g, then $f = g$.

Proof. Since uniform convergence implies pointwise convergence, so $\left(f^{(n)}\right)_{n=1}^{\infty}$ converges uniformly to f and g, so $f = g$ by Theorem A.2.1. ■

Theorem A.2.3. If $\left(f^{(n)}\right)_{n=1}^{\infty}$ converges uniformly to f and converges pointwise to g, then $f = g$.

Proof. Since uniform convergence implies pointwise convergence, so $\left(f^{(n)}\right)_{n=1}^{\infty}$ converges pointwise to f, and by Theorem A.2.1, we know $f = g$. ■

Appendix B

TA Class