Introduction to Analysis II

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Abstract

Lecture note of Introduction to Analysis II.

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Chapter 1

Several Variable Differential Calculus

Lecture 1

In this chapter, we want to approximate non-linear functions by linear maps. If we consider 24 Feb.

$$f : \mathbb{R}^n \to \mathbb{R}^m \quad f(\underbrace{x_1, x_2, \ldots, x_n}_{x}) = (f_1(x), f_2(x), \ldots, f_m(x)),$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ for all i. Now given a real-valued function $F : \mathbb{R}^n \to \mathbb{R}$ which is differentiable at a point x_0. We know that near $x_0 \in \mathbb{R}^n$ we can approximate $F(x)$ in the following way:

$$F(x) \approx F(x_0) + \nabla F(x_0) \cdot (x - x_0)$$

where

$$\nabla F(x_0) = \left(\frac{\partial F(x_0)}{\partial x_1}, \frac{\partial F(x_0)}{\partial x_2}, \ldots, \frac{\partial F(x_0)}{\partial x_n} \right) \in \mathbb{R}^n \text{ with } x_0 = (x_1, x_2, \ldots, x_n)$$

and thus

$$\begin{aligned} \nabla F(x_0) \cdot (x - x_0) &= \left\langle \frac{\partial F(x_0)}{\partial x_1}, \frac{\partial F(x_0)}{\partial x_2}, \ldots, \frac{\partial F(x_0)}{\partial x_n} \right\rangle \cdot \langle x_1, x_2, \ldots, x_n \rangle \\ &= \sum_{i=1}^{n} \frac{\partial F(x_0)}{\partial x_i} x_i. \end{aligned}$$

Hence,

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix} \approx \begin{pmatrix} f_1(x_0) + \nabla f_1(x)(x - x_0) \\ f_2(x_0) + \nabla f_2(x)(x - x_0) \\ \vdots \\ f_n(x_0) + \nabla f_n(x)(x - x_0) \end{pmatrix},$$

which gives

$$f(x) - f(x_0) \approx \begin{pmatrix} \nabla f_1(x)(x - x_0) \\ \nabla f_2(x)(x - x_0) \\ \vdots \\ \nabla f_n(x)(x - x_0) \end{pmatrix} = \begin{pmatrix} \nabla f_1(x) \\ \nabla f_2(x) \\ \vdots \\ \nabla f_n(x) \end{pmatrix} \cdot \underbrace{(x - x_0)}_{\text{column vector}}.$$

1.1 Linear Transformation

Definition 1.1.1 (Row vectors)**.** Let $n \geq 1$ be an integer. We refer to elements of $\mathbb{R}^n$ an n-dimensional row vectors. A typical row vector is $x = (x_1, x_2, \ldots, x_n)$ which we abbreviate as $(x_i)_{1 \leq i \leq m}$. The components $x_1, x_2, \ldots, x_n$ are real numbers. If x and y are two row vectors in $\mathbb{R}^n$, we can define vector sum by

$$x + y = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \coloneqq (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

If $c \in \mathbb{R}$ is any real number, we define scalar multiplications by

$$cx = c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n).$$

Remark 1.1.1.

(1) $-x \coloneqq (-1) \cdot x = (-x_1, -x_2, \ldots, -x_n)$.

(2) zero vector is denoted by 0, i.e. $(0, 0, \ldots, 0)$.

Lemma 1.1.1 ($\mathbb{R}^n$ is a vector space)**.** Let x, y, z be vectors in $\mathbb{R}^n$, and let $c, d \in \mathbb{R}$. Then the following properties hold:

(a) $x + y = y + x$.

(b) $(x + y) + z = x + (y + z)$.

(c) $x + 0 = 0 + x = x$.

(d) $x + (-x) = (-x) + x = 0$.

(e) $(c \cdot d)x = c \cdot (dx)$.

(f) $c(x + y) = cx + cy$.

(g) $(c + d)x = cx + dx$.

(h) $1x = x$.

Definition 1.1.2. Let $x = (x_1, x_2, \ldots, x_n)$ be row vector. Its transpose is the n-dimensional column vector

$$x^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Definition 1.1.3. The standard basis of $\mathbb{R}^n$ consists of $e_1, e_2, \ldots, e_n$, where e_j has 1 in the j-th position and 0 elsewhere:

$$e_j = (0, \ldots, 0, \underbrace{1}_{j\text{-th}}, 0, \ldots, 0).$$

Every row vector

$$x = (x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} x_j e_j.$$

Similarly,

$$x^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} x_j e_j^T.$$

Definition 1.1.4 (Linear transformation). A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is any function from one Euclidean space to another that satisfies the following two properties:

(a) Additivity: For $x, y \in \mathbb{R}^n$, $T(x+y) = T(x) + T(y)$.

(b) Homogenity: For $x \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$, $T(cx) = cT(x)$.

Remark 1.1.2. This definition is equivalent to the following:

$$T(c_1 v_1 + \cdots + c_k v_k) = c_1 T(v_1) + \cdots + c_k T(v_k)$$

where $v_1, \ldots, v_k \in \mathbb{R}^n$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Definition 1.1.5. Let $m, n \geq 1$ be integers. An $m \times n$ ordered matrix is an ordered rectangular array of real numbers

$$A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

consisting of m rows and n columns, where

(a) The entry a_{ij} denote the number in the i-th row and j-th column.

(b) We denote the set of all $m \times n$ matrices by $\mathbb{R}^{m \times n}$.

(c) In particular, a row vector is a $1 \times n$ matrix, a column vector is a $n \times 1$ vector.

Definition 1.1.6 (Matrix multiplication). Given an $m \times n$ matrix $A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ and an $n \times p$ matrix $B = (b_{jk})_{\substack{1 \leq j \leq n \\ 1 \leq k \leq p}}$, we define AB to be the $m \times p$ matrix $(c_{ik})_{\substack{1 \leq i \leq m \\ 1 \leq k \leq p}}$ where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Definition 1.1.7 (Matrix-vector multiplication). Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ be a column vector. We define

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}$$

Remark 1.1.3. In our class, we just treat $\mathbb{R}^n, \mathbb{R}^m$ as column vector spaces, and $L_A(x) = Ax$ is a $m \times 1$ column vector.

Theorem 1.1.1. Let A be a $m \times n$ matrix, then $L_A(x) = Ax$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proof. ■ DIY

Proposition 1.1.1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. For each $j = 1, 2, \dots, n$, let e_j denote the j-th standard basis vector in $\mathbb{R}^n$ and write $T(e_j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$. Define the matrix $A = (a_{ij})$, then $T(x) = Ax$.

Proof. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. We can write $x = \sum_{j=1}^n x_j e_j$, then we know

$$T(x) = T\left(\sum_{j=1}^n x_j e_j\right) = \sum_{j=1}^n x_j T(e_j) = \sum_{j=1}^n x_j \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} = Ax.$$

■

Lemma 1.1.2. Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix. Then $L_A \circ L_B = L_(AB)$.

Proof. It suffices to show that $(L_A \circ L_B)(x) = L_{AB}(x)$ for $x \in \mathbb{R}^p$, and the rest is easy. ■

As previously seen. $f : E \to \mathbb{R}$ where E is a subset of $\mathbb{R}$, then

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x \in E \setminus \{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0}.$$

Suppose now $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$, we can't define

$$f'(x) = \lim_{\substack{x \to x_0 \\ x \in E \setminus \{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0}$$

since the denominator and the numerator are vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$.

1.2 Derivatives in Several Variable Calculus

Lemma 1.2.1. Let $E \subseteq \mathbb{R}$, let $f : E \to \mathbb{R}$ be a function and let $L \in \mathbb{R}$ and x_0 be a limit point of E. Then the following two statements are equivalent:

(a) f is differentiable at x_0 and $f'(x_0) = L$.

(b) $\lim_{\substack{x \to x_0 \\ x \in E \setminus \{x_0\}}} \frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} = 0$.

Proof. Note that

$$\frac{f(x) - f(x_0)}{x - x_0} = L + \frac{f(x) - f(x_0) - L(x - x_0)}{x - x_0} \text{ if } x \neq x_0,$$

so we have

$$\frac{f(x) - f(x_0)}{x - x_0} - L = \frac{f(x) - f(x_0) - L(x - x_0)}{x - x_0} \text{ if } x \neq x_0,$$

and thus

$$0 = \lim_{\substack{x \to x_0 \\ x \in E \setminus \{x_0\}}} \left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| = \lim_{\substack{x \to x_0 \\ x \in E \setminus \{x_0\}}} \left| \frac{f(x) - f(x_0) - L(x - x_0)}{x - x_0} \right|.$$

$\blacksquare$

Appendix