Linear Algebra I HW6

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Problem 0.0.1. Let W_1, W_2 be subspaces of a finite dimensional vector space V.

(a) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(b) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

Proof.

(a) If $f \in (W_1 + W_2)^0$, then $f(w_1 + w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. Now since $0 \in W_1$ and $0 \in W_2$, so we can pick $w_2 = 0$ so obtain $f(w_1) = 0$ for all $w_1 \in W_1$ and similarly we can obtain $f(w_2) = 0$ for all $w_2 \in W_2$. Hence, $f \in W_1^0 \cap W_2^0$. This means $(W_1 + W_2)^0 \subseteq W_1^0 \cap W_2^0$. Now if $g \in W_1^0 \cap W_2^0$, then since $g(w_1) = 0$ and $g(w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$, so we know $g(w_1 + w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$, and this means $g(w) = 0$ for all $w \in W_1 + W_2$. Hence, $g \in (W_1 + W_2)^0$, which gives $W_1^0 \cap W_2^0 \subseteq (W_1 + W_2)^0$. Hence, $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(b) We first claim that $\dim(W_1 \cap W_2)^0 = \dim\left(W_1^0 + W_2^0\right)$:

$$\begin{aligned}
\dim\left(W_1 \cap W_2\right)^0 &= \dim V - \dim(W_1 \cap W_2) \\
\dim\left(W_1^0 + W_2^0\right) &= \dim W_1^0 + \dim W_2^0 - \dim\left(W_1^0 \cap W_2^0\right) \\
&= (\dim V - \dim W_1) + (\dim V - \dim W_2) - \dim\left(W_1 + W_2\right)^0 \quad \text{(by (a))} \\
&= 2\dim V - \dim W_1 - \dim W_2 - (\dim V - \dim(W_1 + W_2)) \\
&= \dim V + \dim(W_1 + W_2) - \dim W_1 - \dim W_2 \\
&= \dim V - \dim(W_1 \cap W_2).
\end{aligned}$$

Hence, we've prove it. Now we prove that $W_1^0 + W_2^0 \subseteq (W_1 \cap W_2)^0$. If $f \in W_1^0 + W_2^0$, then $f = g + h$ for some $g \in W_1^0$ and $h \in W_2^0$. Hence, we know for all $w \in W_1 \cap W_2$, $f(w) = g(w) + h(w) = 0$, which means $f \in (W_1 \cap W_2)^0$, so $W_1^0 + W_2^0 \subseteq (W_1 \cap W_2)^0$.

Now since we know

$$\begin{cases} \dim\left(W_1 \cap W_2\right)^0 = \dim\left(W_1^0 + W_2^0\right) \\ W_1^0 + W_2^0 \subseteq \left(W_1 \cap W_2\right)^0, \end{cases}$$

so we know $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

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Problem 0.0.2. Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. If f is a linear functional on W, prove that there is a linear functional g on V such that $g(\alpha) = f(\alpha)$ for each α in the subspace W.

Proof. Suppose $B = \{w_1, \ldots, w_n\}$ is a basis of W, and extend it to

$$C = \{w_1, \ldots, w_n, v_{n+1}, \ldots, v_m\},$$

and makes C a basis of V, then if we take dual of C, say

$$C^* = \left\{w_1^*, \ldots, w_n^*, v_{n+1}^*, \ldots, v_m^*\right\},$$

then we know $f = \sum_{i=1}^n \alpha_i w_i^*$ for some α_i's in F since

$$\{w_1^*, w_2^*, \ldots, w_n^*\}$$

is a basis W^* and $f \in W^*$, and thus if we pick $g = \sum_{i=1}^n \alpha_i w_i^* + \sum_{i=n+1}^m v_i^*$, then since we know for all $w \in W$, $v_j^*(w) = 0$ for all $n+1 \le j \le m$, so

$$g(w) = \sum_{i=1}^{n} \alpha_i w_i^*(w) = f(w).$$

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Problem 0.0.3. Let S be a set, F a field, and $V(S;F)$ the space of all functions from S into F:

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x).$$

Let W be any n-dimensional subspace of $V(S;F)$. Show that there exist points $x_1, \ldots, x_n$ in S and functions $f_1, \ldots, f_n$ in W such that $f_i(x_j) = \delta_{ij}$.

Proof. Suppose $B = \{g_1, g_2, \ldots, g_n\}$ is a basis of W, then we define

$$L_x : W \to F, \quad L_x(g) = g(x)$$

where $x \in S$.

Claim 0.0.1. $\exists x_1, x_2, \ldots, x_n \in S$ s.t. $\mathcal{L} = \{L_{x_1}, L_{x_2}, \ldots, L_{x_n}\}$ is linearly independent in W^*.

Proof. Suppose by contradiction, for all $x_1, x_2, \ldots, x_n$, $\{L_{x_1}, L_{x_2}, \ldots, L_{x_n}\}$ is linearly dependent, then we know

$$\dim(\operatorname{span}\{L_x : x \in S\}) < n,$$

otherwise, we can pick $\left\{\sum_{x\in S} \alpha_{ji} L_x\right\}_{j=1}^n$ s.t. this set is linearly independent, but notice that

$$\sum_{x\in S} \alpha_{ji} L_x = L_{\sum_{x\in S} \alpha_{ji} x}$$

by the definition of L_x, and this means we can pick n points $\left\{y_j = \sum_{x\in S} \alpha_{ji} x\right\}_{j=1}^n$ s.t. $\left\{L_{y_j}\right\}_{j=1}^n$ is linearly independent, which is a contradiction.

Now since $\dim(\operatorname{span}\{L_x : x \in S\}) < n$, and $\dim W^* = \dim W = n$, so we know

$$\dim(\operatorname{span}\{L_x : x \in S\})^0 = \dim W^* - \dim(\operatorname{span}\{L_x : x \in S\}) \ge 1,$$

so we can pick $T \neq 0$ s.t. $T \in (\operatorname{span}\{L_x : x \in S\})^0$. Now since we know

$$\mathcal{J} : W \to W^{**}, \quad \mathcal{J}(w)(\varphi) = \varphi(w) \quad \varphi \in W^*$$

is an isomorphism, so we know there exists $w \in W$ s.t. $\mathcal{J}(w) = T$, and since $T \neq 0$, so $w \neq 0$. Also, since $T \in (\operatorname{span}\{L_x : x \in S\})^0$, so for all $x \in S$ we have

$$0 = T(L_x) = J(w)(L_x) = L_x(w) = w(x),$$

which means w is the zero function in W, which is a contradiction. Hence, there must exists $x_1, x_2, \ldots, x_n \in S$ s.t. $\{L_{x_1}, L_{x_2}, \ldots, L_{x_n}\}$ is linearly independent. ⊛

By the claim above, we can pick $\mathcal{L} = \{L_{x_1}, L_{x_2}, \ldots, L_{x_n}\}$ s.t. $\mathcal{L}$ is linearly independent. Now suppose

$$A = \begin{pmatrix} L_{x_1}(g_1) & L_{x_1}(g_2) & \cdots & L_{x_1}(g_n) \\ L_{x_2}(g_1) & L_{x_2}(g_2) & \cdots & L_{x_2}(g_n) \\ \vdots & \vdots & \ddots & \vdots \\ L_{x_n}(g_1) & L_{x_n}(g_2) & \cdots & L_{x_n}(g_n) \end{pmatrix},$$

and we will show that A is invertible. Suppose

$$v_i = \left(L_{x_i}(g_1), L_{x_i}(g_2), \ldots, L_{x_i}(g_n)\right) = \left(g_1(x_i), g_2(x_i), \ldots, g_n(x_i)\right), \quad \forall 1 \leq i \leq n,$$

and suppose $\sum_{i=1}^{n} v_i = 0$, then we have

$$\alpha_1 g_i(x_1) + \alpha_2 g_i(x_2) + \cdots + \alpha_n g_i(x_n) = 0 \quad \forall 1 \leq i \leq n.$$

However, since we know $\mathcal{L}$ is linearly independent, so

$$\begin{aligned} \beta_1, \beta_2, \ldots, \beta_n = 0 &\Leftrightarrow \beta_1 L_{x_1} + \beta_2 L_{x_2} + \cdots + \beta_n L_{x_n} = 0 \\ &\Leftrightarrow \beta_1 p(x_1) + \beta_2 p(x_2) + \cdots + \beta_n p(x_n) = 0 \quad \forall p \in W \\ &\Leftrightarrow \beta_1 g_i(x_1) + \beta_2 g_i(x_2) + \cdots + \beta_n g_i(x_n) = 0 \quad \forall 1 \leq i \leq n. \end{aligned}$$

Hence, we know $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, and thus $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, which shows all rows of A is linearly independent, so A is invertible.

Now since A is invertible, so we can do row operations to make A becomes I_n, and suppose

$$I_n = A' = E_1 E_2 \ldots E_k A,$$

where $E_1, E_2, \ldots, E_k$ are some elementary matrices, then if $A' = \left(a'_{ij}\right)_{n \times n}$, then

$$a'_{ij} = \sum_{k=1}^{n} \beta_{ki} L_{x_k}(g_j) \quad \text{for some constants } \beta_{ki}\text{'s } \forall 1 \leq i \leq n.$$

Hence, we know

$$a'_{ij} = L_{\sum_{k=1}^{n} \beta_{ki} x_k}(g_j),$$

and since $A' = I_n$, so $a'_{ij} = \delta_{ij}$, which means if we pick $y_i = \sum_{k=1}^{n} \beta_{ki} x_k$ and then we have

$$g_i(y_j) = \delta_{ij}.$$

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