Linear Algebra I HW7

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Problem 0.0.1. Let A be a 2×2 matrix over a field F. Then the set of all matrices of the form $f(A)$, where f is a polynomial over F, is a commutative ring K with identity. If B is a 2×2 matrix over K, the determinant of B is then a 2×2 matrix over F, of the form $f(A)$. Suppose I is the 2×2 identity matrix over F and that B is the 2×2 matrix over K

$$B = \begin{bmatrix} A - A_{11}I & -A_{12}I \\ -A_{21}I & A - A_{22}I \end{bmatrix}.$$

Show that $\det B = f(A)$, where $f = x^2 - (A_{11} + A_{22})x + \det A$, and also that $f(A) = 0$.

Proof. Note that

$$\begin{aligned} \det B &= (A - A_{11}I)(A - A_{22}I) - A_{12}A_{21}I = A^2 - (A_{11} + A_{22})A + (A_{11}A_{22} - A_{12}A_{21})I \\ &= A^2 - (A_{11} + A_{22})A + (\det A) \cdot I, \end{aligned}$$

so we know $\det B = f(A)$, where $f(x) = x^2 - (A_{11} + A_{22})x + \det A$. Also, since we know

$$\begin{aligned} A^2 &= \begin{pmatrix} A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + A_{22}^2 \end{pmatrix} \\ (A_{11} + A_{22})A &= \begin{pmatrix} A_{11}^2 + A_{11}A_{22} & A_{11}A_{12} + A_{22}A_{12} \\ A_{11}A_{21} + A_{22}A_{21} & A_{11}A_{22} + A_{22}^2 \end{pmatrix} \\ (\det A) \cdot I &= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix}, \end{aligned}$$

so we know $f(A) = A^2 - (A_{11} + A_{22})A + (\det A) \cdot I = 0$. ■