Linear Algebra I HW1

B13902024 張沂魁

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Problem 0.0.1. Let W be a set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$\begin{aligned} 2x_1 - x_2 \quad &+\tfrac{4}{3}x_3 - x_4 \quad &= 0 \\ x_1 \quad &+\tfrac{2}{3}x_3 \quad -x_5 &= 0 \\ 9x_1 - 3x_2 \quad &+6x_3 - 3x_4 \quad -3x_5 &= 0 \end{aligned}$$

Find a finite set of vectors which spans W.

Proof. We can first write the system of equations into the matrix form and then use Gaussian elimination.

$$\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 9 & -3 & 6 & -3 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & \frac{2}{3} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, we can need to solve

$$\begin{cases} x_1 - \dfrac{1}{2}x_2 + \dfrac{2}{3}x_3 - \dfrac{1}{2}x_4 = 0 \\ \dfrac{1}{2}x_2 + \dfrac{1}{2}x_4 - x_5 = 0. \end{cases}$$

So we know $(x_{1,2}, x_3, x_4, x_5) = \left(t - \frac{2}{3}a, b, a, 2t - b, t\right)$ for some $a, b, t \in \mathbb{R}^5$, and thus we know the set

$$S = \left\{(1,0,0,2,1), \left(-\frac{2}{3}, 0, 1, 0, 0\right), (0,1,0,-1,0)\right\}$$

spans W. ■

Problem 0.0.2. Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$, or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. (The last type of subspace is, intuitively, a straight line through the origin.)

Proof. We first give a claim:

Claim. Suppose V is a vector space, and if W is a subspace of V, then $\dim W \leq \dim V$.

Proof. Since $W \subseteq V$, so suppose $k = \dim V$ and B_1 is a basis of V, then $W \subseteq V = span B_1$, which means if there is a basis of W, say B_2, then $|B_2| \leq |B_1|$, which means $\dim W \leq \dim V$. ⊛

Now also we know $\dim \mathbb{R}^2 = 2$ since $\{(0,1),(1,0)\}$ is a linearly independent set and spans $\mathbb{R}^2$. Thus, if there is a subspace of $\mathbb{R}^2$, say W, then $\dim W = 0, 1, 2$. If $\dim W = 0$, then W is the zero subspace. If $\dim W = 1$, then W consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. If $\dim W = 2$, then we can give a claim first:

Claim. Suppose $W \subseteq V$ and they are both vector spaces, then if $\dim V = \dim W$, then $V = W$.

Proof. Suppose by contradiction, there exists $v \in V \setminus W$, and suppose B is a basis of W, then we know $B \cup \{v\}$ is linearly independent in V. However, $|B \cup \{v\}| > \dim V$, which is a contradiction. ⊛

By this claim, we know $W = \mathbb{R}^2$ if $\dim W = 2$. ■