Abstract algebra I

Homework 2

Due: 24th September 2025

1) Let $G = \mathbb{Z}/30\mathbb{Z}$, which is an additive group as we have known. Among the following subsets of G, determine whether each of them is a subgroup. Prove or disprove them.

   (a) $G_1 = \{0, 1, 2, 3, \cdots, 14\}$

   (b) $G_2 = \{0, 2, 4, 6, \cdots, 28\}$

   (c) $G_3 = \{1, 7, 11, 13, 17, 19, 23, 29\}$

2) Recall that a subgroup H of a group G is a subset $H \subset G$ that itself is a group, with the induced operation. We now introduce the *criterion of a subset* and we aimed to prove this handful criterion and give an application.

   (i) (Subgroup criterion) Suppose $(G, *)$ is a group and $H \subset G$. Suppose that

   - H is not empty,
   - (closedness of $*$) for any $a, b \in H$, we have $a * b \in H$, and
   - (closedness of inverse) for all $a \in H$, its inverse $a^{-1} \in H$.

   Prove that H is a subgroup of G.

   (ii) Using the above criterion, check that the *special linear group* $\mathrm{SL}_n(\mathbb{R})$ is a subgroup of *general linear group* $\mathrm{GL}_n(\mathbb{R})$. (*Recall that* $\mathrm{GL}_n(\mathbb{R})$ *is the set of all n by n matrices with real coefficients (entries) so that its determinants are not zero, while* $\mathrm{SL}_n(\mathbb{R})$ *is the subset of* $\mathrm{GL}_n(\mathbb{R})$ *contains those matrices with determinant 1.*)

3) We define the *order* of a finite group G to be $|G|$, i.e., the number of elements (or the *cardinality*) of G. Also, for any $g \in G$, the order of g, denoted by $|g|$ (or sometimes $o(g)$), is the smallest positive integer m such that $g^m = e$.

   (a) For $n \geq 1$, show that the set of bijections $\{1, 2, ..., n\} \to \{1, 2, ..., n\}$, denoted by S_n, is a group of order $n!$.

   (b) Show that the subset given by $\{\sigma \in S_4 : \sigma(1) = 1\}$, i.e., the collection of bijections fixing 1, is a subgroup of S_4. Find its order.

   (c) Consider the multiplicative non-abelian group

   $$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = \pm 1 \right\}.$$

   Find the orders of $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

4) For some positive integer n, let G be a finite cyclic group of order n. *(Recall that a cyclic group is a group generated by one of its elements, i.e., there exists some $g \in G$ of order n.)*

(a) Prove that if g is a generator of G, then g^k is a generator of G if and only if $\gcd(k, n) = 1$.

(b) For any $m|n$, prove that G has exactly one subgroup of order m.

(c) Show that S_3, the symmetric group on 3 letters, is not cyclic.

5) For any subgroup N of a group $(G, *)$, and any $g \in G$, we define

$$gN = \{g * n : n \in N\} \quad \text{and} \quad Ng = \{n * g : n \in N\}.$$

We say that N is a *normal subgroup* of G, denoted $N \lhd G$, if $gN = Ng$ for all $g \in G$.

(a) Show that if $N \lhd G$, then $(G/N, \cdot)$ is a group under the operation given by

$$g_1N \cdot g_2N = (g_1 * g_2)N.$$

(b) Consider the subgroup $H = \{(1), (12)\} \subset S_3$. Show that H is not a normal subgroup of S_3.

(c) Show that every subgroup of an abelian group G is normal.

(d) The converse is false; there exist non-abelian groups with all subgroups normal. Search for one, no justification required.