Linear Algebra I HW9

B13902024 張沂魁

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Problem 0.0.1. Let A be the 4×4 real matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

Show that the characteristic polynomial for A is $x^2(x-1)^2$ and that it is also the minimal polynomial.

Proof. For the characteristic polynomial,

$$\begin{aligned}
\operatorname{ch}_A(x) = \det(xI - A) &= \det \begin{pmatrix} x-1 & -1 & 0 & 0 \\ 1 & x+1 & 0 & 0 \\ 2 & 2 & x-2 & -1 \\ -1 & -1 & 1 & x \end{pmatrix} \\
&= (x-1) \det \begin{pmatrix} x+1 & 0 & 0 \\ 2 & x-2 & -1 \\ -1 & 1 & x \end{pmatrix} + \det \begin{pmatrix} 1 & 0 & 0 \\ 2 & x-2 & -1 \\ -1 & 1 & x \end{pmatrix} \\
&= (x-1)(x+1) \det \begin{pmatrix} x-2 & -1 \\ 1 & x \end{pmatrix} + \det \begin{pmatrix} x-2 & -1 \\ 1 & x \end{pmatrix} \\
&= (x-1)(x+1)\left((x-2)x+1\right) + (x-2)x+1 \\
&= \left((x-1)(x+1)+1\right)\left((x-2)x+1\right) = x^2(x-1)^2.
\end{aligned}$$

Now by Cayley-Hamilton theorem, we know $m_A(x) \mid x^2(x-1)^2$, and we know $x(x-1) \mid m_A(x)$.

- Case 1: $x^2 - x$. $\left(A^2 - A\right)_{11} = 1 + (-1) - 1 = -1 \neq 0$, so $x^2 - x$ is not the minimal polynomial.
- Case 2: $x^2(x-1)$. Since

$$A^2 = \begin{pmatrix} & & & \\ & & & \\ -3 & -3 & 3 & 2 \\ & & & \end{pmatrix} \Rightarrow A^2(A-I) = \begin{pmatrix} & & & \\ & & & \\ 3 & -3 & 3 & 2 \\ & & & \end{pmatrix} \begin{pmatrix} 0 & & & \\ -1 & & & \\ -2 & & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} & & & \\ & & & \\ -1 & & & \\ & & & \end{pmatrix},$$

  so $A^2(A-I) \neq 0$, and thus $x^2(x-1)$ is not the minimal polynomial.
- Case 3: $x(x-1)^2$. Since

$$(A-I)^2 = \begin{pmatrix} -1 & & & \\ 2 & & & \\ 1 & & & \\ 0 & & & \end{pmatrix} \Rightarrow A(A-I)^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} -1 & & & \\ 2 & & & \\ 1 & & & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix},$$

  so $A(A-I)^2 \neq 0$.

Hence, we know $m_A(x) = x^2(x-1)^2 = \operatorname{ch}_A(x)$. ■

Problem 0.0.2. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix}.$$

Is A similar over the field of real numbers to a triangular matrix? If so, find such a triangular matrix.

Proof. Since we know

$$\operatorname{ch}_A(x) = \det \begin{pmatrix} x & -1 & 0 \\ -2 & x+2 & -2 \\ -2 & 3 & x-2 \end{pmatrix} = x^3,$$

and $m_A(x) \mid \mathrm{ch}_A(x)$, so $m_A(x)$ must split, and thus A is triangulanizable. Now since the only eigenvalue of A is 0, so we can pick some w_1 in $\ker A$ first. Let's pick

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow Aw_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we want to pick some w_2 s.t. $Aw_2 \in \langle w_1, w_2 \rangle$, so we can pick

$$w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow Aw_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 \cdot w_1 + 0 \cdot w_2.$$

Now we can pick third vector w_3 to be any vector which cannot be represented as the linear combination of w_1 and w_2, suppose we pick

$$W_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow Aw_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = 0 \cdot w_1 + 2 \cdot w_2 + 0 \cdot w_3,$$

so we know $b = \{w_1, w_2, w_3\}$ is a basis of $\mathbb{R}^3$, and

$$[A]_b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

■

Problem 0.0.3. Let T be a diagonalizable linear operator on the n-dimensional vector space V, and let W be a subspace which is **invariant** under T. Prove that the restriction operator T_W is diagonalizable.

Proof. Since we have shown that $m_{T_W}(x) \mid m_T(x)$, and since T is diagonalizable, so

$$m_T(x) = \prod_{i=1}^{r} (x - \lambda_i)$$

where $\lambda_i \neq \lambda_j$ for distinct i, j, so we know

$$m_{T_W}(x) = \prod_{k=1}^{r'} (x - \lambda_{a_k}),$$

for some $\{a_k\}_{k=1}^{r'} \subseteq [r]$. Thus, T_W is diagonalizable. ■

Problem 0.0.4. Let T be a linear operator on V. If every subspace of V is invariant under T, then T is a scalar multiple of the identity operator.

Proof. For all $x \in V$, we know $\langle x \rangle$ is a subspace of V and thus T-invariant. Thus, $Tx = c_x x$ for some constant c_x. Now for $\lambda x \in V$, we know

$$\lambda c_x x = \lambda T(x) = T(\lambda x) = c' \lambda x,$$

so $c_x = c'$, and thus for all $v \in \langle x \rangle$, $Tv = c_x x$ for a fixed constant c_x. Now if $y \notin \langle x \rangle$, then $T(y) = c_y y$, and if $c_y \neq c_x$, then

$$c_{x+y}(x + y) = T(x + y) = T(x) + T(y) = c_x x + c_y y,$$

which gives

$$(c_{x+y} - c_x)\, x + (c_{x+y} - c_y) y = 0,$$

but since $\{x, y\}$ is linearly independent, so $c_x = c_y = c_{x+y}$. Hence, if we pick a basis B of V, then we know

$$T(v_i) = cv_i$$

for a fixed c for all $v_i \in V$ since we can do the same arguments as above. Hence, for all $v \in V$, since it can be written as a linear combination of B, and we have shown that $T(v_i) = cv_i$ for all $v_i \in B$ and $T(\lambda v_i) = c(\lambda v_i)$, so we know T must be a scalar multiple of the identity operator. ■

Problem 0.0.5. Let V be the space of $n \times n$ matrices over F. Let A be a fixed $n \times n$ matrix over F. Let T and U be the linear operators on V defined by

$$\begin{aligned} T(B) &= AB \\ U(B) &= AB - BA. \end{aligned}$$

(a) True or false? If A is diagonalizable (over F), then T is diagonalizable.
(b) True or false? If A is diagonalizable, then U is diagonalizable.

Proof.

(a) True. Suppose $m_A(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$, then

$$m_A(T)(B) = \left(a_m T^m + a_{m-1} T^{m-1} + \cdots + a_0 I\right)(B),$$

and note that $T^i(B) = A^i B$, so

$$\begin{aligned} m_A(T)(B) &= a_m T^m B + a_{m-1} T^{m-1} B + \cdots + a_0 B \\ &= a_m A^m B + a_{m-1} A^{m-1} B + \cdots + a_0 B \\ &= \left(a_m A^m + a_{m-1} A^{m-1} + \cdots + a_0\right) B = m_A(A)(B) = 0. \end{aligned}$$

Hence, $m_T(x) \mid m_A(x)$, and thus if A is diagonalizable, then $m_A(x)$ has all distinct roots, and thus $m_T(x)$ has all distinct roots, which means T is diagonalizable.

(b) True. If A is diagonalizable, then suppose $P^{-1}AP = D = \operatorname{diag}[d_1, d_2, \ldots, d_n]$, and suppose $b = \{E^{p,q}\}_{1 \leq p,q \leq n}$ is the standard basis of V, i.e. $E^{p,q}$ is a matrix with (p,q)-entry equal 1 and all the other entries 0 for all p, q. Then, note that $\beta' = \left\{PE^{p,q}P^{-1}\right\}_{1 \leq p,q \leq n}$ is a basis of V since

- If

$$\sum_{p,q} \alpha_{p,q} \left(PE^{p,q}P^{-1}\right) = 0,$$

then

$$0 = \sum_{p,q} P(\alpha_{p,q} E^{p,q}) P^{-1} = P\left(\sum_{p,q} \alpha_{p,q} E^{p,q}\right) P^{-1},$$

which gives

$$0 = P^{-1} 0 P = P^{-1} P \left(\sum_{p,q} \alpha_{p,q} E^{p,q}\right) P^{-1} P = \sum_{p,q} \alpha_{p,q} E^{p,q},$$

so $\alpha_{p,q} = 0$ for all p, q since $\{E^{p,q}\}_{1 \leq p,q \leq n}$ is a basis of V. Hence, β' is linearly independent.

- Now since for any $M \in V$, $P^{-1}MP$ can be represented as a linear combination of b, say

$$P^{-1}MP = \sum_{p,q} s_{p,q} E^{p,q},$$

so

$$M = P\sum_{p,q} s_{p,q}E^{p,q}P^{-1} = \sum_{p,q} s_{p,q}PE^{p,q}P^{-1},$$

so M is a linear combination of β', and thus β' spans V.

Now note that

$$\begin{aligned}
U\left(PE^{p,q}P^{-1}\right) &= APE^{p,q}P^{-1} - PE^{p,q}P^{-1}A \\
&= \left(PP^{-1}\right)APE^{p,q}P^{-1} - PE^{p,q}P^{-1}A\left(PP^{-1}\right) \\
&= PDE^{p,q}P^{-1} - PE^{p,q}DP^{-1} \\
&= P\left(DE^{p,q} - E^{p,q}D\right)P^{-1}.
\end{aligned}$$

Also, we have

$$\left(DE^{p,q} - E^{p,q}D\right)_{ij} = \sum_{k=1}^{n} D_{ik}E^{p,q}_{kj} - \sum_{k=1}^{n} E^{p,q}_{ik}D_{kj} = \begin{cases} d_p - d_q, & \text{if } (i,j) = (p,q); \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\left(DE^{p,q} - E^{p,q}D\right)_{ij} = (d_p - d_q)E^{p,q},$$

which gives

$$U\left(PE^{p,q}P^{-1}\right) = (d_p - d_q)PE^{p,q}P^{-1},$$

so we know $[U]_{\beta'}$ is diagonal and thus U is diagonalizable.

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