# Linear Algebra I

Kon Yi

September 10, 2025

**Abstract**

The lecture note of Linear Algebra I by professor 余正道.

# Contents

# Chapter 1

# Vector Space

## Lecture 1

## 1.1 Introduction to vector and vector space

3 Sep. 10:20

In high school, our vectors are in $\mathbb{R}^2$ and $\mathbb{R}^3$, and we have define the addition and scalar multiplication of vectors.

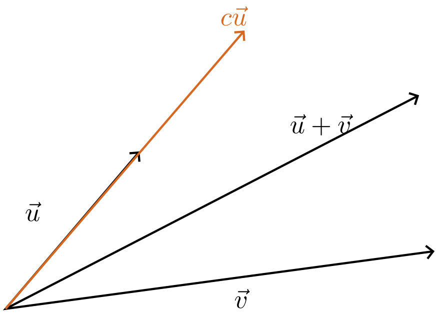


Figure 1.1: Vectors in $\mathbb{R}^2$

**Example.** $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n \mid a_i \in \mathbb{R})\}$

With this type of space, we can define addition and multiplication as

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = \{a_1 + b_1, a_2 + b_2, \dots, a_n + b_n\}$$
$$\alpha \cdot (a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$$

Also, if we define a space:

**Example.** $V = \{\text{function } f : (a, b) \to \mathbb{R}\}$, where $(a, b)$ is an open interval.

then this can also be a vector space after defining addtion and multiplication.

**Note.** In a vector space, we have to make sure the existence of 0-element, which means $0(x) = 0$.

Now we give a more abstract example:

**Example.** Suppose $S$ is any set, then define $V = \{\text{all functions from } S \text{ to } \mathbb{R}\}$

If we define $(f + g)(s) = f(s) + g(s)$ and $(\alpha \cdot f)(s) = \alpha \cdot f(s)$, and $0(s) = 0$, then this is also a vector space.

**Put some linear conditions**

**Example.** In $\mathbb{R}^n$, fix $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, if we define

$$W = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\},$$

then this is also a vector space.

However, if we have

$$W' = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = 1\},$$

then this is not a vector space because it is not close.

**Example.** In $V = \{(a, b) \to \mathbb{R}\}$ or $W_1 = \{\text{polynomial defined on } (a, b)\}$, these are both vector space.

**Remark.** In the later course, we will learn that $W_1$ is a subspace of $V$.

**Example.** If we furtherly defined $W_1^{(k)} = \{\text{polynomial degree } \le k\}$, then this is also a vector space.

**Remark.** $W_1^{(k)}$ is actually isomorphic to $\mathbb{R}^{k+1}$ since

$$a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \leftrightarrow (a_0, a_1, a_2, \ldots, a_n).$$

**Example.** $W_2 = \{\text{continuous function on } (a, b)\}$ and $W_3 = \{\text{differentiable functions}\}$ are also both vector spaces.

**Example.** $W_4 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0\right\}$ and $W_5 = \left\{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -f\right\}$ are both vector spaces.

**Proof.**

$$\begin{aligned} W_4 &= \{a_0 + a_1x\} \\ W_5 &= \{a_1 \cos x + a_2 \sin x\} \end{aligned}$$

⊛

## 1.2 Formal definition of vector spaces

### 1.2.1 Vector Spaces Over $\mathbb{R}$

**Definition 1.2.1.** Suppose $V$ is a non-empty set equipped with

- addition: $V \times V \to V$, that is, given $u, v \in V$, defining $u + v \in V$
- scalare multiplication: $\mathbb{R} \times V \to V$, that is, given $\alpha \to \mathbb{R}$ and $v \in V$, we need to have $\alpha v \in V$

Also, we need some good properties or conditions

- For addition,
  - $u + v = v + u$
  - $(u + v) + w = u + (v + w)$
- There exists $0 \in V$ such that $u + 0 = u = 0 + u$
- Given $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0 = (-v) + v$

- For scalar multiplication,
  - $1 \cdot v = v$ for all $v \in V$
  - $(\alpha\beta)v = \alpha \cdot (\beta v)$ for all $\alpha, \beta \in \mathbb{R}$ and $v \in V$.
- For addition and multiplication,
  - $\alpha(u + v) = \alpha u + \alpha v$
  - $(\alpha + \beta)u = \alpha u + \beta u$

# Lecture 2

## 1.3 Vector Space over general field

5 Sep. 10:20

Now we introduce the concept of field.

**Definition 1.3.1** (Field). A set $F$ with $+$ and $\cdot$ is called a **field** if

- $\alpha + \beta = \beta + \alpha$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- There exists $0 \in F$ such that $\alpha + 0 = 0 + \alpha = \alpha$.
- For $\alpha \in F$, there exists $-\alpha$ such that $\alpha + (-\alpha) = 0$.
- $\alpha\beta = \beta\alpha$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
- $\exists 1$ such that $1 \neq 0$ and $1 \cdot \alpha = \alpha$.
- For $\alpha \neq 0$, $\exists \alpha^{-1} \in F$ such that $\alpha\alpha^{-1} = 1$.
- $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

**Example.** $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are all fields but $\mathbb{Z}$ is not.

**Example.** $\{0, 1\}$ is also a field.

Now we know the concept of filed, so we can make a vector space over a field.

**Theorem 1.3.1** (Cancellation law). Suppose $v_1, v_2, w \in V$, a vector space, then if $v_1 + w = v_2 + w$, then $v_1 = v_2$.

**Proof.**

$$v_1 = v_1 + (w + (-w)) = (v_1 + w) + (-w) = (v_2 + w) + (-w) = v_2 + (w + (-w)) = v_2.$$

■

**Theorem 1.3.2.** The zero vector $0$ is unique.

**Proof.** Suppose we have $0, 0'$ both zero vector, then for some $0 = 0 + 0' = 0'$. ■

**Theorem 1.3.3.** For any $v \in V$, $0 \cdot u = 0$.

**Proof.** $0 \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u$, so $0 = 0 \cdot u$ by cancellation law. ■

**Theorem 1.3.4.** $(-1) \cdot u = -u$.

**Theorem 1.3.5.** Given any $u \in V$ is unique, $-u$ is unique.

## 1.4 Subspaces

**Definition 1.4.1** (subspace)**.** Let $V$ be a vector space. A non-empty subset $W \subseteq V$ is called a subspace of $V$ if $W$ is itself a vector space under $+$ and $\cdot$ on $V$.

**Example.** $M_n(F) = \{n \times n \text{ matrix with entries in } F\}$ is a vector space, and

$$U_n(F) = \left\{ \begin{pmatrix} a_{11} & & & \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \right\}$$

is a subspace of $M_n(F)$.

**Proposition 1.4.1.** Suppose $V$ is a vector space, and $W \subseteq V$ is non-empty, then

$$W \text{ is a subspace } \Leftrightarrow \text{ For } u, v \in W, \alpha \in F, \text{ we have } u + v \in W \text{ and } \alpha \cdot u \in W.$$

**proof of $\Rightarrow$.** Clear. ■

**proof of $\Leftarrow$.** First, we would want to check $0 \in W$, and we can pick any $u \in W$, and pick $\alpha = -1$, so we know $-u \in W$, and thus $0 = u + (-u) \in W$. ■

**Corollary 1.4.1.** If we want to check $W$ is a subspace, we just need to check for $u, v \in W$, $\alpha \in F$, $u + \alpha v \in W$ or not.

## 1.5 Linear Combination

**Definition 1.5.1** (Linear combinaiton)**.** Given $v_1, v_2, \ldots, v_n \in V$, a linear combination of them is a vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

**Proposition 1.5.1.** Given $v_1, v_2, \ldots, v_n \in V$,

1. $W = \{\text{all linear combinations of } v_, \ldots, v_n\}$ is a subspace.
2. This subspace is the smallest subspace containing $v_1, \ldots, v_n$. That is, if $W' \subseteq V$ is a subspace containing $v_1, \ldots, v_n$, then $W \subseteq W'$.

**Notation.** $\operatorname{span}\{v_1, v_2, \ldots, v_n\} = \{\text{all linear cominations of } v_1, v_2, \ldots, v_n\}$

## 1.6 Linearly independent

**Definition.** Now we talk about the linearly dependence and linearly independence.

**Definition 1.6.1** (Linearly dependent). $v_1, v_2, \ldots, v_n$ are linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zeros.

**Definition 1.6.2** (Linearly independent). $v_1, v_2, \ldots, v_n$ are called linearly independent if they are not linearly dependet.

**Corollary 1.6.1.** Say $\alpha_i \neq 0$, then $v_i \in \operatorname{span}\{\hat{v_1}, \hat{v_2}, \ldots, \hat{v_k}\}$ suppose the corresponding $\alpha_i$ of $\hat{v_1}, \ldots, \hat{v_k}$ are not zeros.

**Corollary 1.6.2.** Linearly independent means if $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

**Corollary 1.6.3.** Linearly independent meeans if $\sum \alpha_i v_i = \sum \beta_i v_i$, then $\alpha_i = \beta_i$ for all $i$.

**Example.**

- $v \in V$ is linearly independent iff $v \neq 0$.
- $v, w \in V$ are linearly independent iff $v$ is not a scaler of $w$ and $w$ is not a scalar of $v$.

**Lemma 1.6.1.** $v_1, \ldots, v_n$ are linearly independent iff $v_i \notin \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$.

## 1.7 Basis

**Definition.** We now talking about basis

**Definition 1.7.1** (Basis). $B = \{v_1, v_2, \ldots, v_n\}$ is called a basis of $V$ if $B$ spans $V$ and $B$ is linearly independent.

**Definition 1.7.2** (Dimension). In this case, $n$ is called the dimension of $V$, and denoted by $\dim V$.

**Notation.** $\operatorname{span}\{v_1, v_2, \ldots, v_n\} = \langle v_1, v_2, \ldots, v_n \rangle$

**Notation.** $\operatorname{span}(S) = \langle S \rangle$

**Theorem 1.7.1.** For any $v \in V$, it has a unique expression $v = \sum_{i=1}^{n} \alpha_i v_i$.

# Lecture 3

10 Sep 10:20

**As previously seen.** A basis of a vector space $V$ is a set $\{v_1, v_2, \ldots, v_n\}$ that is linearly independent and simultaneously spans $V$. That is, suppose we have $\sum a_i v_i = 0$ for some scalars $a_i$, then $a_i = 0$ for all $i$. Also, we call the number $n$, the dimension of $V$.

**Example.** Suppose we have $V = F^n = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \in F\}$, then we have a **standard basis**, which is

$$\begin{aligned} e_1 &= (1, 0, \ldots, 0) \\ e_2 &= (0, 1, \ldots, 0) \\ &\vdots \\ e_n &= (0, 0, \ldots, 1) \end{aligned}$$

since $\{e_i\}_{i=1}^n$ is linearly independent and for every $\vec{a} = (a_1, \ldots, a_n)$, we know

$$\vec{a} = \sum_{i=1}^n a_i e_i.$$

**Example.** Suppose

$$V = M_{n\times n}(F) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \ddots & & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \cdots & & \alpha_{nn} \end{pmatrix} \right\},$$

then we know

$$\{e_{ij}\}_{1\leq i,j\leq n} = \begin{pmatrix} 0 & 0 & & & \\ 0 & & & & \\ & & 1 & & \\ 0 & & & 0 & \\ 0 & & & & 0 \end{pmatrix},$$

where the 1 is in the $i$-th row and $j$-th column.

**Theorem 1.7.2.** Suppose $V$ is a vector space, and $V = \langle v_1, v_2, \ldots, v_n\rangle$ and $\{w_1, w_2, \ldots, w_m\}$ is linearly independent, then $m \leq n$. Furtheremore, one can make

$$\langle w_1, w_2, \ldots, w_m, v_{m+1}, \ldots, v_n\rangle = V$$

after rearrangement of $v_1, \ldots, v_n$.

**Proof.** We can do induction on $m$. It is trivial that $m = 0$ is true. Suppose the statement holds for a fixed $m$ with $m \leq n$ . Let $w_1, w_2, \ldots, w_{m+1}$ be linearly independent. In particular, $w_1, w_2, \ldots, w_m$ is linearly independent.

**Claim.** $m + 1 \leq n$.

**Proof.** Otherwise, if $m + 1 > n$, then since $m \leq n$, so $m = n$. Hence, by induction hypothesis, we know $\langle w_1, w_2, \ldots, w_m\rangle = V$. However, by Lemma 1.7.1 and the note following it, we know

$$\{w_1, w_2, \ldots, w_m\} \cup \{w_{m+1}\}$$

can not be linearly indepndent since $w_{m+1} \in V = \langle w_1, \ldots, w_m\rangle$. ⊛

Now we know $m + 1 \leq n$. By induction hypothesis, we know

$$\langle w_1, w_2, \ldots, w_m, v_{m+1}, \ldots, v_n\rangle = V$$

**Claim.** One of $v_{m+1}, \ldots, v_n$ can be replaced by $w_{m+1}$.

**Proof.** Since

$$w_{m+1} = \sum_{i=1}^{n} \alpha_i w_i + \sum_{j=m+1}^{n} \beta_j w_j.$$

Trivially, one of $\beta_j \neq 0$, say $\beta_{m+1} \neq 0$ . Check

$$\langle w_1, \ldots, w_m, w_{m+1}, v_{m+2}, \ldots, v_n \rangle = V.$$

⊛

■

**Corollary 1.7.1.** If $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ are bases of $V$, then $n = m$.

**Remark.** Corollary 1.7.1 tells us $\dim V$ is well-defined, which means the size of the bases of a vector space is unique.

**Corollary 1.7.2.** Suppose $\dim V = n$, then if $\langle v_1, v_2, \ldots, v_m \rangle = V$, then $m \geq n$. If $\{w_1, w_2, \ldots, w_m\}$ is linearly independent, then $m \leq n$. Also, anu $\{v_i\}_{i=1}^{m}$ with $m > n$ is linearly independent.

**Lemma 1.7.1.** Suppose $v_1, v_2, \ldots, v_n$ is linearly independent. If $w \notin \langle v_1, v_2, \ldots, v_n \rangle$, then

$$\{v_1, v_2, \ldots, v_n, w\}$$

is linearly independent.

**Proof.** Suppose $\sum_{i=1}^{n} \alpha_i v_i + \alpha_{i+1} w = 0$, then if $\alpha_{i+1} = 0$, we know $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ since $\{v_i\}_{i=1}^{n}$ is linearly independent. If $\alpha_{i+1} \neq 0$, then $w = \frac{1}{\alpha_{i+1}} \sum_{i=1}^{n} \alpha_i v_i \in \langle v_1, v_2, \ldots, v_n \rangle$, which is a contradiction. ■

**Note.** The reverse of Lemma 1.7.1 is still correct and is trivial. That is, if $w \notin \{v_1, \ldots, v_n\}$ and $\{v_1, v_2, \ldots, v_n, w\}$ is linearly independent, then $\{v_1, \ldots, v_n\}$ is linearly independent.

**Corollary 1.7.3.** If $W \subseteq V$ is a subspace of $V$, then $\dim W \leq \dim V$.

**Proof.** If $\dim V = n$, and $\{w_i\}_{i=1}^{m}$ is a basis of $W$, then this basis is linearly independent in $V$, which means $m \leq n$ by Theorem 1.7.2. ■

**Corollary 1.7.4.** If $v_1, v_2, \ldots, v_m$ is linearly independent, then $\{v_1, v_2, \ldots, v_m\}$ forms a basis for some $v_{m+1}, \ldots, v_n$.

**Theorem 1.7.3** (Dual version)**.** If $\langle v_1, v_2, \ldots, v_n \rangle = V$, then $\{v_1, v_2, \ldots, v_m\}$ forms a basis after rearrangement.

**Remark.** Most of the time, we consider finite-dimensional vector spaces.

**Remark** (Examples of $\infty$-dim vector space)**.**

- $$V = \{\text{all polynomials over } F\} = F[x] = \{a_0 + a_1 x + \cdots + a_n x^n \text{ for some } n \text{ where } a_i \in F\}.$$

- $$W = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\}.$$

  Notice that

  $$W' = \{\text{convergent sequence}\} \subseteq W.$$

  and

  $$W'' = l^2 = \left\{ (a_i) \mid \sum_{i=0}^{\infty} a_i^2 \text{ finite} \right\} \subseteq W'$$

**Remark.** We define $\dim \{0\} = 0$, which is the only vector space with dimension 0, and we define $\langle \varnothing \rangle = \{0\}$, which means $\varnothing$ is the basis of $\{0\}$.

**Note.** We call a subspace $W \subsetneq V$ is proper.

## 1.8 More on subspaces

**Theorem 1.8.1.** If $W_1$ and $W_2$ are subspace of $V$, then $W_1 \cap W_2$ is a subspace.

**Theorem 1.8.2.** If $W_1, W_2$ are subspaces of $V$, then $W_1 + W_2$ is still a subspace of $V$.

**Remark.** If $W_1, W_2$ are subspaces of $V$, then $W_1 \cup W_2$ may not be a subspace. (See HW1).

**Remark.** In fact, $W_1 \cap W_2$ is the largest subspaces contained in $W_1$ and $W_2$.

**Remark.** In fact, $W_1 + W_2$ is the smallest subspace containing both $W_1$ and $W_2$.

**Corollary 1.8.1.** Suppose $S$ is the index set, and for all $i \in S$, $W_i$ is a subspace of $V$, then

$$\bigcap_{i \in S} W_i = \{v \in V \mid v \in W_i \ \forall i\}$$

is also a subspace of $V$.

**Corollary 1.8.2.** Suppose $S$ is the index set, and for all $i \in S$, $W_i$ is a subspace of $V$, then

$$\sum_{i \in S} W_i = \{w_{i_1} + w_{i_2} + \cdots + w_{i_n} \text{ for some } i_j \in S\}$$

is also a subspace of $V$.

**Proposition 1.8.1** (Dimension theorem)**.** Suppose $W_1, W_2 \subseteq V$ are subspaces of $V$, then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

# Appendix