Linear Algebra I HW4

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Problem 0.0.1. Let V be a two-dimensional vector space over the field F, and let $\mathcal{B}$ be an ordered basis for V, If T is a linear operator on V and

$$[T]_{\mathcal{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

prove that $T^2 - (a+d)T + (ad-cb)I = 0$.

Proof. Note that

$$\left[T^2\right]_{\mathcal{B}} = [T]_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+cd & bc+d^2 \end{bmatrix}.$$

Hence,

$$\begin{aligned}\left[T^2 - (a+d)T + (ad-cb)I\right]_{\mathcal{B}} &= \left[T^2\right]_{\mathcal{B}} - (a+d)[T]_{\mathcal{B}} + (ad-cb)[I]_{\mathcal{B}} \\ &= \begin{bmatrix} a^2+bc & ab+bd \\ ca+cd & bc+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-cb)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\end{aligned}$$

which means $T^2 - (a+d)T + (ad-cb)I = 0$. ■

Problem 0.0.2. Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T(x_1, x_2) = (-x_2, x_1).$$

(a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

(b) What is the matrix of T in the ordered basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1,2)$ and $\alpha_2 = (1,-1)$?

(c) Prove that for every real number c the operator $(T - cI)$ is invertible.

(d) Prove that if $\mathcal{B}$ is any ordered basis for $\mathbb{R}^2$ and $[T]_{\mathcal{B}} = A$, then $A_{12}A_{21} \neq 0$.

Proof.

(a) Suppose the standard ordered basis is $B = \{e_1, e_2\}$, then we know

$$\begin{aligned}T(1,0) &= (0,1) = 0 \cdot e_1 + 1 \cdot e_2 \\ T(0,1) &= (-1,0) = (-1) \cdot e_1 + 0 \cdot e_2.\end{aligned}$$

Hence, we know

$$[T]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(b) Since we have

$$\begin{aligned}T(\alpha_1) &= (-2,1) = -\frac{1}{3}\alpha_1 - \frac{5}{3}\alpha_2 \\ T(\alpha_2) &= (1,1) = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2,\end{aligned}$$

so we know

$$[T]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{bmatrix}.$$

(c) Suppose B is the standard ordered basis, then by (a) we know

$$[T]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and thus for any real number c we have

$$[T - cI]_B = \begin{bmatrix} -c & -1 \\ 1 & -c \end{bmatrix}.$$

Note that for any $c \in \mathbb{R}$ we must have $\operatorname{rank}(T - cI) = \operatorname{rank}[T - cI]_B = 2$, which means $T - cI$ is surjective and thus bijective, so $T - cI$ is invertible.

(d) Suppose $\mathcal{B} = \{(a, b), (c, d)\}$ is a basis of $\mathbb{R}^2$. If $A_{12} = 0$, then since by definition we have

$$\begin{aligned} (-b, a) &= A_{11}(a, b) + A_{21}(c, d) = (A_{11}a + A_{21}c, A_{11}b + A_{21}d) \\ (-d, c) &= A_{12}(a, b) + A_{22}(c, d) = (A_{12}a + A_{22}c, A_{12}b + A_{22}d)\,, \end{aligned}$$

we have

$$(-d, c) = A_{22}(c, d),$$

and this gives $(1 + A_{22})^2 c = 0$, which means $c = 0$, and then this impiles $d = 0$, but this means $(c, d) = (0, 0) \in \mathcal{B}$, which is a basis of $\mathbb{R}^2$, so it is a contradiction, and thus $A_{12} \neq 0$. Similarly, we can get $A_{21} \neq 0$, and thus we have $A_{12}A_{21} \neq 0$.

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Problem 0.0.3. Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

(Hint: Let T be the linear operator on $\mathbb{C}^2$ which is represented by the first matrix in the standard ordered basis. Then find vectors α_1 and α_2 such that $T\alpha_1 = e^{i\theta}\alpha_1$, $T\alpha_2 = e^{-i\theta}\alpha_2$, and $\{\alpha_1, \alpha_2\}$ is a basis.)

Proof. First, we suppose the first matrix is the matrix of a linear operator T on standard basis of $\mathbb{C}^2$, then we know

$$T(ae_1 + be_2) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta).$$

Now since $e^{i\theta} = \cos\theta + i\sin\theta$, and if $\alpha_1 = ae_1 + be_2$, then

$$T(ae_1 + be_2) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta) = e^{i\theta}(ae_1 + be_2) = ((\cos\theta + i\sin\theta)a, (\cos\theta + i\sin\theta)b).$$

This gives $ai = -b$, so we can pick $\alpha_1 = (1, -i)$, and use similar method, we can pick $\alpha_2 = (1, i)$. Note that $\{(1, -i), (1, i)\}$ is a basis of $\mathbb{C}^2$ since if $z_1, z_2 \in \mathbb{C}$ and $z_1(1, -i) + z_2(1, i) = (0, 0)$, then

$$\begin{cases} z_1 + z_2 = 0 \\ i(-z_1 + z_2) = 0 \end{cases},$$

which shows $z_1 = z_2 = 0$, and thus $\{(1, -i), (1, i)\}$ is a linearly independent set with size $2 = \dim \mathbb{C}^2$, so it is a basis, and we're done. ■

Problem 0.0.4. We have seen that the linear operator T on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, 0)$ is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This operator satisfies $T^2 = T$. Prove that if S is a linear operator on $\mathbb{R}^2$ such that $S^2 = S$, then $S = 0$, or $S = I$, or there is an ordered basis $\mathcal{B}$ for $\mathbb{R}^2$ such that $[S]_{\mathcal{B}} = A$ (above).

Proof. Since $\dim \operatorname{Im} S \leq 2$, so we can discuss all cases:

- Case 1: $\dim \operatorname{Im} S = 0$, then $S = 0$.
- Case 2: $\dim \operatorname{Im} S = 1$. Note that for all $w \in \operatorname{Im} S$, we have $S(v) = w$ for some $v \in \mathbb{R}^2$, and thus $w = S(v) = S^2(v) = S(w)$, so we have $S(w) = w$. Now since we know $\dim \ker S = 2 - \dim \operatorname{Im} S = 1$, so there exists $v \in \ker S$ s.t. $v \neq 0$, which means $S(v) = 0$ with $v \neq 0$, so $v \notin \operatorname{Im} S$ since every w in $\operatorname{Im} S$ has $S(w) = w$. Hence, we can pick any $u \in \operatorname{Im} S$, and then $\{v, u\}$ forms a basis of $\mathbb{R}^2$ since it is an linearly independent set of size 2. Suppose $\mathcal{B} = \{u, v\}$, then
$$[S]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
since $S(u) = u = 1 \cdot u + 0 \cdot v$ and $S(v) = 0 = 0 \cdot u + 0 \cdot v$.
- Case 3: $\dim \operatorname{Im} S = 2$, then S is bijective and thus S^{-1} exists, so
$$S = (S^2)\left(S^{-1}\right) = SS^{-1} = I,$$
which shows $S = I$.

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Problem 0.0.5. Let V be an n-dimensional vector space over the field F, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V.

(a) According to Theorem 1, there is a unique linear operator T on V such that
$$T\alpha_i = \alpha_{i+1}, \quad j = 1, \ldots, n-1, \qquad T\alpha_n = 0.$$
What is the matrix A of T in the ordered basis $\mathcal{B}$?

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

(c) Let S be any linear operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis $\mathcal{B}'$ for V such that the matrix of S in the ordered basis $\mathcal{B}'$ is the matrix A of part (a).

(d) Prove that if M and N are $n \times n$ matrices over F such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$, then M and N are similar.

Proof.

(a) By definition, we know $A = (a_{ij})_{n \times n}$ is
$$a_{ij} = \begin{cases} 1, & \text{if } i = j+1; \\ 0, & \text{otherwise.} \end{cases}$$

(b) For all $\sum_{i=1}^n c_i\alpha_i \in V$, we have
$$T^n\left(\sum_{i=1}^n c_i\alpha_i\right) = \sum_{i=1}^n c_i T^n(\alpha_i) = \sum_{i=1}^n c_i \cdot 0 = 0.$$
However, we have
$$T^{n-1}\left(\sum_{i=1}^n c_i\alpha_i\right) = \sum_{i=1}^n c_i T^{n-1}(\alpha_i) = c_1\alpha_n.$$

Hence, if we pick some vector with $c_1 \neq 0$, then T^{n-1} will not map it to 0, and thus $T^{n-1} \neq 0$.

(c) If we pick some $v \in V$ with $S^{n-1}(v) \neq 0$, then we claim that

$$\mathcal{B}' = \{v, S(v), S^2(v), \ldots, S^{n-1}(v)\}$$

is a basis of V. Suppose $\sum_{i=0}^{n-1} c_i S^i(v) = 0$, then applying S on both sides, we will get

$$c_0 S(v) + c_1 S^2(v) + \cdots + c_{n-2} S^{n-1}(v) = 0.$$

Keep doing this, we will have $c_0 S^{n-1}(v) \neq 0$. Now since $S^{n-1}(v) \neq 0$, so $c_0 = 0$, and go back to the last equation, which is

$$c_0 S^{n-2}(v) + c_1 S^{n-1}(v) = 0,$$

we have $c_1 = 0$, and keep going back, we will get $c_i = 0$ for all $0 \leq i \leq n-1$, which means $\mathcal{B}'$ is linearly independent, and since it is a set of size $n = \dim V$, so $\mathcal{B}'$ is a basis, and thus we know $[S]_{\mathcal{B}'}$ is the matrix A of part (a).

(d) By (c), M, N are both similar to the matrix A of part (a), so M, n are similar since "similar" is an equivalence relation.

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