Calculus Note

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1 Differential Rules

1.1 Linear approximations

We think that $y = f(a) + f'(a)(x-a)$ is a good approximation of $y = f(x)$ near $x = a$.

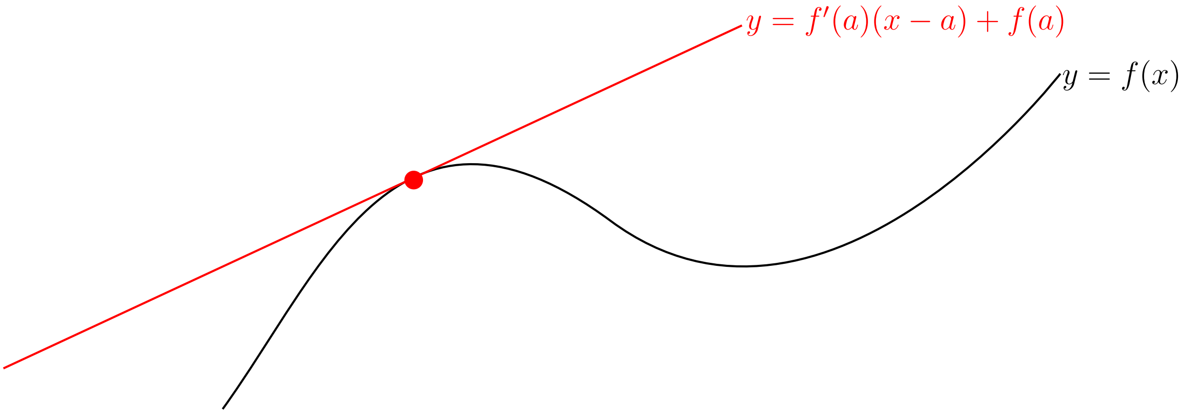


Figure 1.1.1: function f

Definition 1.1.1: Linear Approximation

Let $L(x) := f(a) + f'(a)(x-a)$.

- $L(x)$ *is called the linearization of* f *at* a.
- $f(x) \approx L(x)$ *is called the linear approximation of* f *at* a.

Example 1.1.1. $f(x) = \sqrt{x+3}$, find the linear approximation of f at $x = 1$.

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x+3}} \Rightarrow f'(1) = \frac{1}{4}$$
$$\Rightarrow L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1)$$

Approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

$$\sqrt{3.98} = f(1-0.02) \approx L(1-0.02) = 2 + \frac{1}{4}(-0.02) = 1.995$$
$$\sqrt{4.05} = f(1+0.05) \approx L(1+0.05) = 2 + \frac{1}{4}(0.05) = 2.0125$$

We denote $\Delta y := f(x) - f(a)$, then

$$\Delta y = f(x) - f(a) \approx f'(a)\underbrace{(x-a)}_{\Delta x}$$
$$\Rightarrow \frac{\Delta y}{\Delta x} \approx f'(a) = \frac{dy}{dx}$$

Hence, the idea of linear approximation is to use the slope of the tangent line to approximate the slopes of nearby secant line.(which is opposite to the definition of differentiation)

Definition 1.1.2: dx and dy

If we denote $dx := \Delta x$, define the differential of $y = f(x)$ at a to be

$$dy := f'(a) \cdot dx$$

Using this notation, the linear approximation become

$$\Delta y \approx f'(a)\underset{\Delta x = dx}{\underset{\|}{(x-a)}} = dy.$$

Example 1.1.2. The radius of a sphere is 21cm(measured with a possible error at most 0.05cm). What is the maximal error in computing the volume of the sphere?

The linear approximation of the volume $V(r) = \frac{4}{3}\pi r^3$ at 21 is

$$L(r) = V(21) + 4\pi r_0^2 \cdot (r-21).$$
$$\Rightarrow \Delta V = V(r) - V(21) \approx 4\pi(21)^2 \cdot \underbrace{(r-21)}_{\leq 0.05} \approx 277cm^3$$

Using the notation of differential,

$$\Delta V \approx dv = V'(21) \cdot dr = 4\pi(21)^2 \cdot 0.05$$

If we want the <u>relative error</u> $\frac{\Delta V}{V}$ to be at most 3%, what is the relative error allowed in measuring the radius?

$$\underbrace{\frac{\Delta V}{V}}_{\leq 3\%} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\underbrace{\frac{\Delta r}{r}}_{\leq 1\%}$$

2 Application of differentiation

2.1 Maximum and minimum values

Definition 2.1.1: Absolute Extreme Value

Let $f : U \longrightarrow \mathbb{R}$.

- *If $\exists c \in U$ such that $f(c) \geq f(x) \forall x \in U$, then $f(c)$ is called the* ***absolute maximum value*** *of f on U.*
- *If $\exists c \in U$ such that $f(c) \leq f(x) \forall x \in U$, then $f(c)$ is called the* ***absolute minimum value*** *of f on U.*

The set of absolute maximum and absolute minimum is called the ***extreme value of f on U****.*

- *If there is c such that $f(x) \geq f(x)$ for all x near c, then $f(c)$ is called a* ***local maximum value*** *of f on U.*
- *If there is c such that $f(x) \leq f(x)$ for all x near c, then $f(c)$ is called a* ***local minimum value*** *of f on U.*

Example 2.1.1. By the below figure we can see that global max value is not attained!

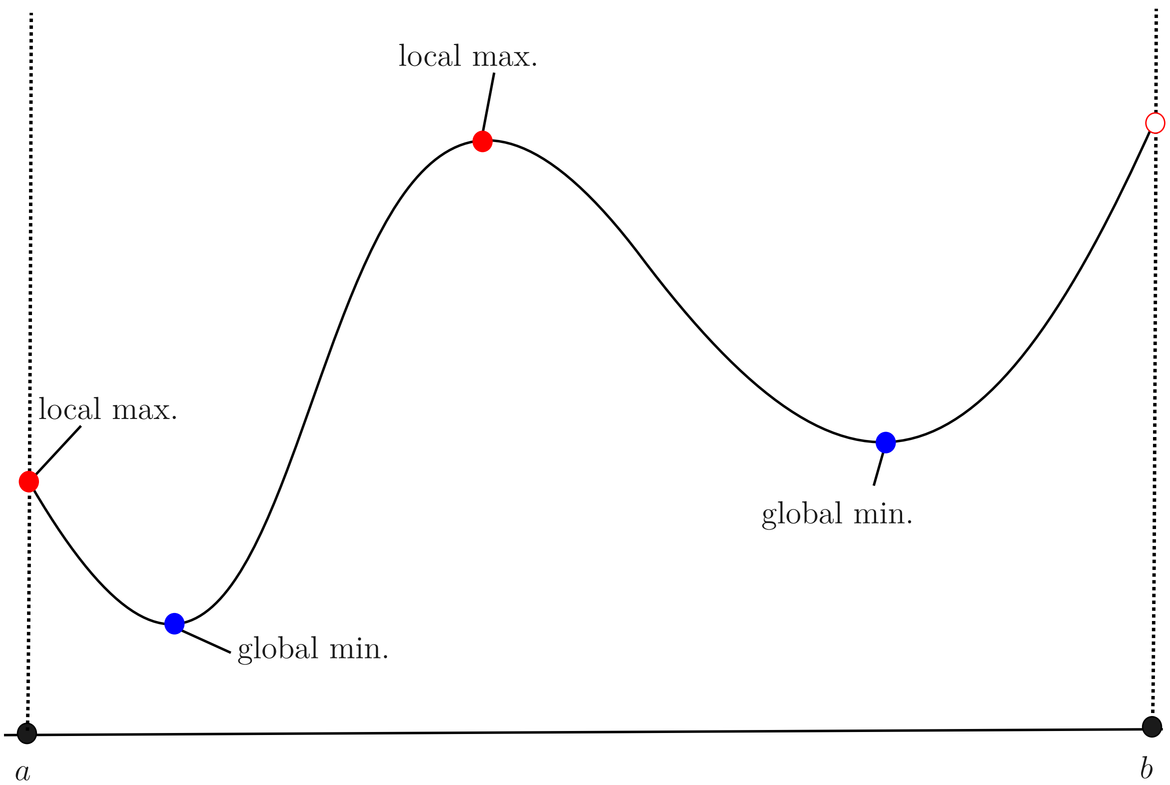
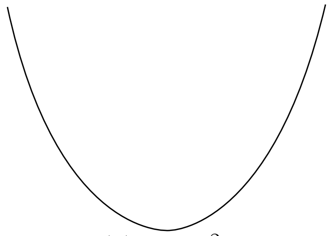
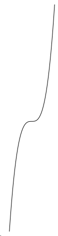


Figure 2.1.1: $f : [a, b) \longrightarrow \mathbb{R}$

Example 2.1.2. (1) $f(x) = x^2$, (2) $g(x) = x^3$



(a) $y = x^2$



(b) $y = x^3$

We can see that the global min value$= 0$, but global value is not attained by any $x \in \mathbb{R}$.

Theorem 2.1.1: Extreme Value Theorem

If f is continuous on $[a, b]$, then f attains an global max value and an global min value on $[a, b]$.

Remark 2.1.1. Being continuous on a close and bounded(compact) interval.

Theorem 2.1.2: Fermat's theorem

Suppose f is differentiable at c and f has a local max./min. at c, then $f'(c) = 0$.

Remark 2.1.2. Local extreme value have "horizontal tangent lines".

Example 2.1.3. $f(x) = x^3$, $x \in \mathbb{R}$.

$f'(c) = 3c^2 = 0 \iff c = 0$. But $f(0) = 0$ is neither a local max nor a local min.
So we know that the converse of Fermat's Theorem does not hold!

Example 2.1.4. $f(x) = |x|$, $x \in \mathbb{R}$.

$f(x) \geq f(0) = 0\ \forall x \in \mathbb{R}$, so f attains a global min at 0. But $f'(0)$ does not exist.

Remark 2.1.3. differentiability is crucial.

Example 2.1.5. $f(x) = \dfrac{1}{x}$, $x \in \mathbb{R}_+$

$f'(x) = -\dfrac{1}{x^2} \neq 0$ on $\mathbb{R}_+$. By Fermat's Theorem, f does not attain any local extereme on $\mathbb{R}_+$.

Proof 2.1.1 (Proof of Fermat's Theorem)**.** Let $f : U \longrightarrow \mathbb{R}$, $c \in U$. Suppose c is a local maximum, then $\exists \delta > 0$ such that if $x \in U$, $|x - c| < \delta$, then $f(x) \leq f(c)$.

Case 1$(x > c)$ For any $c < x < c + \delta$, we have

$$\frac{f(x) - f(c)}{x - c} \leq 0. \Rightarrow \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

Case 2$(x < c)$ For any $c - \delta < x < c$, we have

$$\frac{f(x) - f(c)}{x - c} \geq 0. \Rightarrow \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Since f is differentiable at c,

$$0 \geq \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

$\Rightarrow f'(c) = 0$. Similar argument works if c is a local minimum.

Definition 2.1.2: Critical Number

For $f : U \longrightarrow \mathbb{R}$, define the critical numbers:

$$Crit(f) = \{c \in U : f'(c) = 0 \text{ or } f'(c) \text{ doesn't exist}\}$$

Proposition 2.1.1. Steps to find global max/min of $f : [a, b] \underset{\text{conti.}}{\longrightarrow} \mathbb{R}$:

1) Find $Crit(f)$ in (a, b).
2) Find $f(a)$ and $f(b)$.
3) $\max\{f(x) : x \in Crit(f) \cup \{a, b\}\}$ is the global max.
   $\min\{f(x) : x \in Crit(f) \cup \{a, b\}\}$ is the global min.

Example 2.1.6. Find the global max and global min of $f : [-1, 3] \longrightarrow \mathbb{R}$,

$$f(x) = \begin{cases} -x, & x \in [-1, 0). \\ \sqrt{4 - (x-2)^2}, & x \in [0, 3]. \end{cases}$$

1) $Crit(f) = \{0, 2\}, \ f(2) = 2, \ f(0) = 0.$
2) $f(-1) = 1, \ f(3) = \sqrt{3}.$

So f attains its global max value 2 at $x = 2$, and f attains its global min value 0 at $x = 0$.(You can see the picture of the function in next page.)

Example 2.1.7. $f(x) = x^3 - 3x^2 + 1, \ x \in [\frac{-1}{2}, 4]$.

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

1) $Crit(f) = \{0, 2\}, \ f(0) = 1, \ f(2) = -3.$
2) $f\left(-\frac{1}{2}\right) = \frac{1}{8}, \ f(4) = 17.$

$\Rightarrow$ global max $= 17$ at 4, global min $= 0$ at $x = 0$.

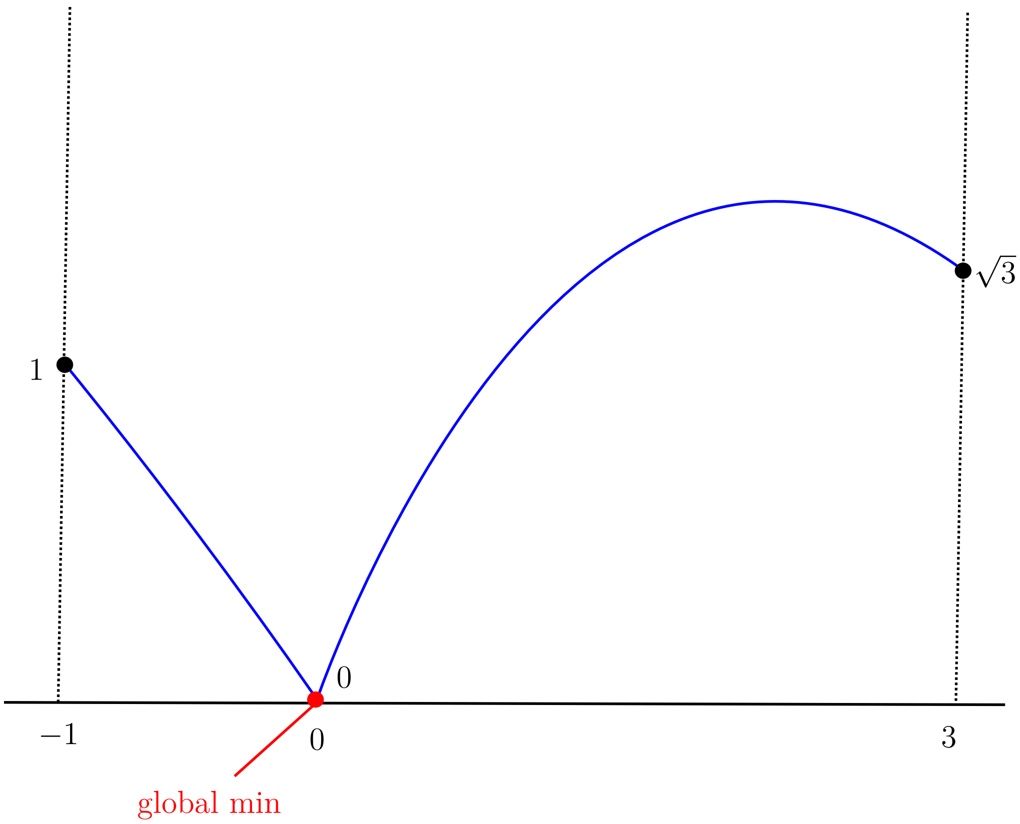


Figure 2.1.2: function f in **Example 2.1.6.**

2.2 TA class week 6

Problem 2.2.1. Find the absolute maximum and absolute minimum values of $f(x) = xe^{x-x^2}$ ont the interval $[-2, 2]$.

Solution 2.2.1. We have

$$f'(x) = e^{x-x^2} + xe^{x-x^2}(1-2x) = e^{x-x^2}(-2x^2+x+1).$$

By this we know the critical points are $x \in \{\frac{-1 \pm \sqrt{9}}{-8},\ 1,\ -\frac{1}{2}\}$. Thus,

$$\begin{cases} f(-2) = 2e^{-2} < 1 \\ f\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-\frac{3}{4}} < -\frac{1}{2}\cdot\frac{1}{8} < \frac{-1}{16} \\ f(1) = 1 \\ f(2) = -2e^{-6} > -\frac{1}{32} \end{cases}$$

$\Rightarrow$ $x = 1$ is absolute maximum, while $x = -2$ is absolute minimum.

Problem 2.2.2. Show for $x > 0$ that

$$x - \frac{x^2}{2} < \log(1+x) < x.$$

Solution 2.2.2. For $x > 0$, then since f is differentiable on $(0, x)$ and thus continuous on $[0, x]$, so by MVT and consider $f(x) = \log(1+x) - (x - \frac{x^2}{2})$:

$$\frac{f(x) - f(0)}{x - 0} = f'(c), \text{ for some } c \in (0, x)$$

Claim 1 $f'(x) > 0, \forall x > 0$

Proof:

$$f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x} > 0$$

So

$$0 < f'(c) = \frac{f(x) - f(0)}{x - 0}$$

and by $f(0) = 0$ and $x > 0$ we can get $f(x) > 0$, which is what we want.
Now consider $g(x) = x - \log(1+x)$, similarly:

$$\exists c' \in (0, x) \text{ such that } g'(c') = \frac{g(x) - g(0)}{x - 0}$$

and also we have:

Claim 2 $g'(x) > 0, \forall x > 0$

Proof:

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$$

and the rest of step is same as $f(x)$, and we're done.

Problem 2.2.3. Show for $x > 0$ that $e^x \geq \sum_{k=0}^{n} \frac{x^k}{k!}$.(Hint: induction)

Solution 2.2.3. We first prove the base case:

$$e^x \geq e^0 \geq 1 = \frac{x^0}{0!}$$

Now suppose for all $x > 0$ we have $e^x \geq \sum_{k=0}^{n} \frac{x^k}{k!}$, for some $n \in \mathbb{N}$ and $0 \leq n \leq n'$. Consider

$$f(x) = e^x - \sum_{k=0}^{n'+1} \frac{x^k}{k!}$$

so by MVT and because

$$\frac{d}{dx}\left(\frac{x^k}{k!}\right) = \frac{x^{k-1}}{(k-1)!} \geq 0$$

so $\exists x' \in (0, x)$ such that

$$f'(x') = \frac{f(x) - f(0)}{x - 0} = e^{x'} - \sum_{k=0}^{n'} \frac{x'^k}{k!} \geq 0 \Rightarrow f(x) > 0 \Leftrightarrow e^x - \sum_{k=0}^{n'+1} \frac{x^k}{k!} \geq 0.$$

Problem 2.2.4. Let $f(x)$ be a twice-differentiable one-to-one function. Let $g(x) = f^{-1}(x)$. Suppose that $f(2) = 1$, $f'(2) = 3$, $f''(2) = e$. Find $g'(1)$, $g''(1)$.

Solution 2.2.4. By the definition of inverse function, we have

$$\begin{aligned}
& g(f(x)) = x \\
\frac{d}{dx} \quad & g'(f(x)) \cdot f'(x) = 1 \\
\frac{d}{dx} \quad & g''(f(x)) \cdot (f'(x))^2 + g'(f(x)) \cdot f''(x) = 0 \\
& \Rightarrow g'(1) \cdot 3 = 1 \Rightarrow g'(1) = \frac{1}{3} \\
& \Rightarrow g''(1) \cdot 9 + \frac{1}{3} \cdot e = 0 \Rightarrow g''(1) = -\frac{e}{27}
\end{aligned}$$

Problem 2.2.5. Suppose $f(x)$ is a continuous function, and that $f(x)$ is differentiable on $(a, x_0) \cup (x_0, b)$. Suppose $f'(x) \to L$ as $x \to x_0$. Show that $f'(x_0)$ exists and is equal to L.

Solution 2.2.5. Suppose $x \in (a, x_0) \Rightarrow \exists c \in (x, x_0)$ such that $f'(c) = \dfrac{f(x) - f(x_0)}{x - x_0}$(By Mean Value Theorem), and take $x \longrightarrow x_0^-$, and then we can obtain

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = f'(c)$$

Note that we can get $c \longrightarrow x_0$ since $c \in (x, x_0)$.
Similarly, suppose $x' \in (x_0, b)$ and take $x' \longrightarrow x_0^+$, so by Mean Value Theorem $\exists c' \in (x_0, x')$ such that

$$f'(c') = \lim_{x' \to x_0^+} \frac{f(x') - f(x_0)}{x' - x_0}$$

Note that we can also get $c' \longrightarrow x_0$ since $c' \in (x_0, x')$.
And since

$$\begin{aligned}
\exists c : f'(c) &= \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \\
\exists c' : f'(c') &= \lim_{x' \to x_0^+} \frac{f(x') - f(x_0)}{x' - x_0} = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}
\end{aligned}$$

and because $\lim_{x \to x_0} f'(x) = L$. Therefore,

$$\begin{aligned}
L &= \lim_{x \to x_0^-} f'(x) = f'(c) \\
L &= \lim_{x' \to x_0^+} f'(x) = f'(c')
\end{aligned}$$

which means

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = L$$

By this, we can get $f'(x_0) = L$.

Problem 2.2.6. Suppose $f(x)$ is differentiable on $\mathbb{R}$, $f(0) = 0$, and $|f'(x)| \leq |f(x)|$ for all x. Show that $f(x) = 0$ identically.

Solution 2.2.6. Suppose $f(t) = 0$ for some t, and define $S = \left[t - \frac{1}{2}, t + \frac{1}{2}\right]$, and by Extreme Value Theorem, we suppose $x = c$ has the absolute maximum in S such that $|f(c)| > |f(x)|$, for all x between c and t. Now by MVT we suppose $\exists\, k$ which is between c and t and have

$$f'(k) = \frac{f(c) - f(t)}{c - t}$$

and we can have:

Claim 1 $|f(k)| > |f(c)|$

Proof by $|c - t| < 1$ and $f(t) = 0$, we can get:

$$|f(k)| = |f'(k)| = \left|\frac{f(c) - f(t)}{c - t}\right| = \left|\frac{f(c)}{c - t}\right| > |f(c)|$$

Claim 2 $|f(k)| \leq |f(c)|$

this is trivial because we suppose $|f(c)| > |f(x)|$ for all x between c and t

So by this we get a contradiction and hence know the maximum of $|f(x)|$ should be 0, which means $f(x) = 0$, and we are done.

2.3 The Mean Value Theorem

The most basic version is Rolle's theorem:

Theorem 2.3.1: Rolle's theorem

Suppose

(1) f *is continuous on* $[a, b]$

(2) f *is differentiable on* (a, b)

(3) $f(a) = f(b)$

Then $\exists\, c \in (a, b)$ *such that* $f'(c) = 0$.
Note: *c is not necessarily unique.*

Proof 2.3.1. We have 3 cases:

Case 1 $f(x) \equiv k$, for all $x \in [a, b]$
$\Rightarrow f'(x) \equiv 0$, for all $x \in (a, b)$. (c is any $x \in (a, b)$)

Case 2 $\exists x \in (a, b)$ such that $f(x) > f(a)$
Claim. $\exists\, c \in (a, b)$ such that $f(c)$ is the global max value of f on $[a, b]$.
Proof of claim: By Extreme Value Theorem, f attains its global max value on $[a, b]$, say at $c \in [a, b]$. If $c = a$, then $f(c) = f(a) < f(x)$, which is a contradiction. Hence, $c \neq a$. Similarly, $c \neq b$, since $f(a) = f(b)$. Therefore, $c \in (a, b)$.
So by Fermat's theorem, $f'(c) = 0$.

Case 3 $\exists x \in (a,b)$ such that $f(x) < f(a)$
Similarly as in Case 2, $\exists c \in (a,b)$ which is the global min of f on $[a,b]$. By Fermat's theorem, $f'(c) = 0$.

Example 2.3.1. Show that $x^3 + x - 1 = 0$ has exactly one root.

$f(1) = 1$, $f(-1) = -3$. By intermediate value theorem, $\exists x_0 \in (-1,1)$ such that $f(x_0) = 0$. Suppose $\exists x \in \mathbb{R}$, $x_1 > x_0$, such that $f(x_1) = 0$. Then since f is continuous on $[-1, x_1+1]$ and differentiable on $(-1, x_1+1)$, by Rolle's Theorem $\exists c \in (-1, x_1+1)$ such that $f'(c) = 0$. But $f'(c) = 3c^2 + 1 \geq 1$, which is a contradiction.

Theorem 2.3.2: Mean Value Theorem

Suppose

(1) f *is continuous on* $[a,b]$*.*

(2) f *is differentiable on* (a,b)

Then $\exists\ c \in (a,b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

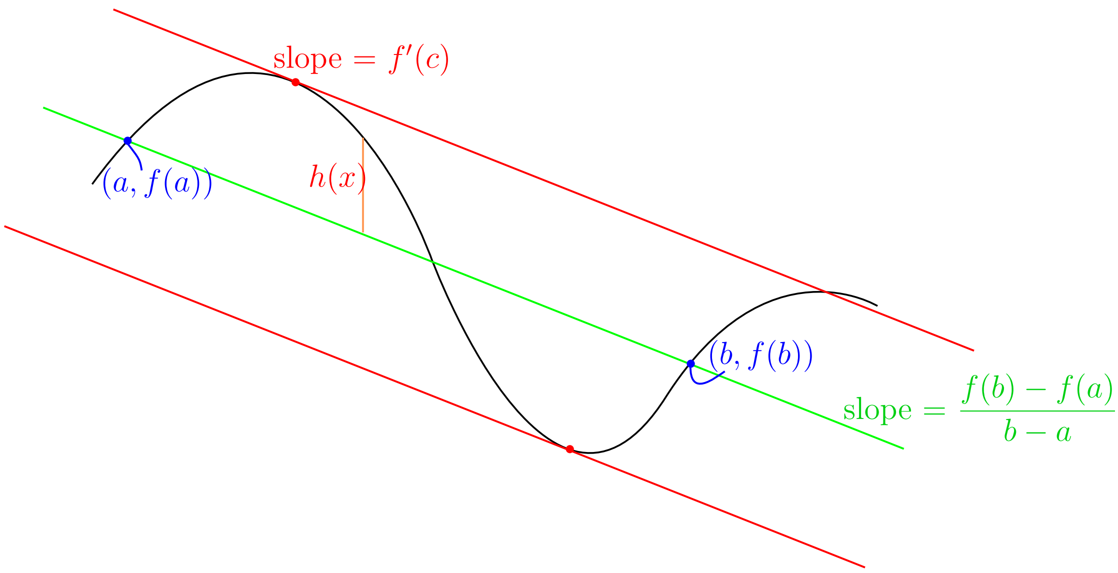


Figure 2.3.1: Mean Value Theorem

Remark 2.3.1. If $f(a) = f(b)$, then MVT reduces to Rolle's theorem.

Proof 2.3.2. Let $a = (a, f(a))$, $B = (b, f(b))$. Then

$$\overleftrightarrow{AB}: \ y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

Let $h(x) = f(x) - \left(f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)\right)$. Then

$$\begin{aligned}
&h(a) = f(a) - f(a) = 0 \\
&h(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0 \\
\Rightarrow &h(a) = h(b)
\end{aligned}$$

Since h is continuous on $[a,b]$(h is the sum of some continuous function) and differentiable on (a,b)(h is the sum of some differentiable function), by Rolles's theorem.

$$\exists\ c \in (a,b) \text{ such that } h'(c) = 0.$$

i.e.

$$0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \Longrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Remark 2.3.2. the Mean Value Theorem is the principle to measure the velocity in a distance interval.

Example 2.3.2. Suppose that f is differentiable on $\mathbb{R}$, $f(0) = -3$ and $f'(x) \leq$, $\forall x \in \mathbb{R}$. How large can $f(2)$ possibly be?

By Mean Value Theorem, $\exists\ c \in (0,2)$ such that

$$\begin{aligned}
&f'(c) = \frac{f(2)-f(0)}{2-0}. \\
&\Rightarrow f(2) - \underbrace{f(0)}_{=-3} = 2\underbrace{f'(c)}_{\leq 5} \leq 10 \\
&\Rightarrow f(2) \leq 10 - 3 = 7
\end{aligned}$$

Now we think that instantaneous information(conditions on the derivative) gives global information(the function itself).

Theorem 2.3.3: Constancy theorem

Suppose f is continuous on $[a,b]$, and $f'(x) = 0,\ \forall x \in (a,b)$. Then f isa constant, i.e. $f(x) = c,\ \forall x \in (a,b)$, for some $c \in \mathbb{R}$.

Proof 2.3.3. Choose $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$. By MVT, $\exists\ d \in (x_1,x_2)$ such that $f'(d) = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$, by the assumption, we say $f'(d) = 0 \ \Rightarrow f(x_2) = f(x_1)$. Since the choice of x_1, x_2 is arbitrary, $f(x) = c,\ \forall x \in (a,b)$.

Corollary 2.3.1. Suppose f, g are continuous on $[a,b]$ and differentiable on (a,b), and $f'(x) = g'(x),\ \forall x \in (a,b)$. Then $f \equiv g + c$ for some constant $c \in \mathbb{R}$ on (a,b).

Proof 2.3.4. Apply Constancy Theorem to $h(x) = f(x) - g(x)$.

Example 2.3.3. Prove that:

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}, \forall x \in \mathbb{R}.$$

let $f(x) = \arctan x + \operatorname{arccot} x$. Then $f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$.

$$f'(x) = \frac{1}{1+x^2} + \frac{-1}{1+x^2} = 0,\ \forall x \in \mathbb{R}.$$

By Constancy Theorem, $f(x) = \frac{\pi}{2},\ \forall x \in \mathbb{R}$.

Theorem 2.3.4: Cauchy's MVT

Suppose f and g are

(1) continuous on $[a, b]$

(2) differentiable on (a, b)

Then $\exists\ c \in (a, b)$ such that

$$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c)$$

Remark 2.3.3. Take $g(x) = x$, then $g'(x) = 1, \forall\ x \in (a, b)$

$$\Rightarrow f(b) - f(a) = (b - a)f'(c)$$

is the original MVT.

Proof 2.3.5. We have 2 cases:

Case 1 $g(a) = g(b)$
By Rolle's theorem, $\exists c \in (a, b)$ such that $g'(c) = 0$. This is as desired.($\because 0 = 0$)

Case 2 $g(a) \neq g(b)$
Consider

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Then $h(a) = h(b) = 0$, and then apply Rolle's theorem, and we are done.

Remark 2.3.4. slope $= \dfrac{\Delta g}{\Delta f} = \dfrac{g(x+\Delta x) - g(x)}{f(x+\Delta x) - f(x)} = \dfrac{\frac{g(x+\Delta x) - g(x)}{\Delta x}}{\frac{f(x+\Delta x) - f(x)}{\Delta x}} \xrightarrow[\Delta x \to 0]{} \dfrac{g'(x)}{f'(x)}$

2.4 L'Hospital's Rule

Theorem 2.4.1: L'Hospital's rule

Suppose $f, g : \underbrace{I}_{open} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *are differentiable except possibly at* $a \in I$. *Then if* $g'(x) \neq 0,\ \forall x \in I,\ \lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists(the limit can be* $\pm\infty$*), and either*

(1) $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

(2) $\lim_{x \to a} f(x) = \pm\infty,\ \lim_{x \to a} g(x) = \pm\infty$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example 2.4.1. (1) $\lim_{x \to 0} \frac{\sin x}{x}$ (2) $\lim_{x \to \infty} \frac{e^x}{x^n}$, $n \in \mathbb{N}$

(1)

$$\lim_{x \to 0} \frac{\sin x}{x} \underset{\text{L'Hospital}}{=} \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = 1.$$

(2)

$$\lim_{x \to \infty} \frac{e^x}{x^n} \underset{\text{H}}{=} \lim_{x \to \infty} \frac{e^x}{nx^{n-1}} \underset{\text{H}}{=} \cdots \underset{\text{H}}{=} \lim_{x \to \infty} \frac{e^x}{n!} = +\infty.$$

This tells us e^x grows faster than polynomials of any order!

Example 2.4.2. $\lim_{x \to \infty} \frac{\ln x}{x^{\frac{1}{n}}}$, $n \in \mathbb{N}$

$$\lim_{x \to \infty} \frac{\ln x}{x^{\frac{1}{n}}} \underset{\text{H}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{n} x^{\frac{1}{n}-1}} = \lim_{x \to \infty} \frac{1}{\frac{1}{n} x^{\frac{1}{n}}} = 0.$$

This tells us $\ln x$ grows slower than $x^{\frac{1}{n}}$, $\forall n \in \mathbb{N}$.

Example 2.4.3. $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$

You should notice that $\lim_{x \to \pi^-} (1 - \cos x) = 2$, so you cannot use L'Hospital rule here.

Example 2.4.4. $\lim_{x \to 0^+} x \ln x$

You should notice that this is the type of $0 \cdot \infty$

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \underset{\text{H}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} (-x) = 0.$$