

**Introduction to Mathematical Analysis**  
**Homework 11 Due December 5 (Friday), 2025**  
**Please submit your homework online in PDF format.**

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1. (20 pts) **Exercise 5.2.6** Let  $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ , and let  $(f_n)_{n=1}^{\infty}$  be a sequence of functions in  $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ .
  - (a) Show that if  $f_n$  converges uniformly to  $f$ , then  $f_n$  also converges to  $f$  in the  $L^2$  metric.
  - (b) Give an example where  $f_n$  converges to  $f$  in the  $L^2$  metric, but does *not* converge to  $f$  uniformly.  
*(Hint: take  $f = 0$ . Try to make the functions  $f_n$  large in sup norm.)*
  - (c) Give an example where  $f_n$  converges to  $f$  in the  $L^2$  metric, but does *not* converge to  $f$  pointwise.  
*(Hint: take  $f = 0$ . Try to make the functions  $f_n$  large at one point.)*
  - (d) Give an example where  $f_n$  converges to  $f$  pointwise, but does *not* converge to  $f$  in the  $L^2$  metric.  
*(Hint: take  $f = 0$ . Try to make the functions  $f_n$  large in  $L^2$  norm.)*

2. (20 pts) Let  $\{\phi_N\} : \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of continuous, periodic functions on  $\mathbb{R}$  (with period 1) which satisfy

$$\int_0^1 \phi_N(t) dt = 1 \quad \text{and} \quad \int_0^1 |\phi_N(t)| dt \leq M < \infty$$

for all  $N \in \mathbb{N}$ , and

$$\lim_{N \rightarrow \infty} \int_{\delta}^{1-\delta} |\phi_N(t)| dt = 0$$

for each  $0 < \delta < 1$ .

Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and periodic with period 1. Prove that

$$\lim_{N \rightarrow \infty} \int_0^1 f(x-t) \phi_N(t) dt = f(x)$$

uniformly for  $x \in \mathbb{R}$ .

3. (15 pts) **Exercise 5.2.3.** If  $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$  is a non-zero function, show that

$$0 < \|f\|_2 \leq \|f\|_{\infty}.$$

Conversely, if  $0 < A \leq B$  are real numbers, show that there exists a non-zero function  $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$  such that

$$\|f\|_2 = A \quad \text{and} \quad \|f\|_{\infty} = B.$$

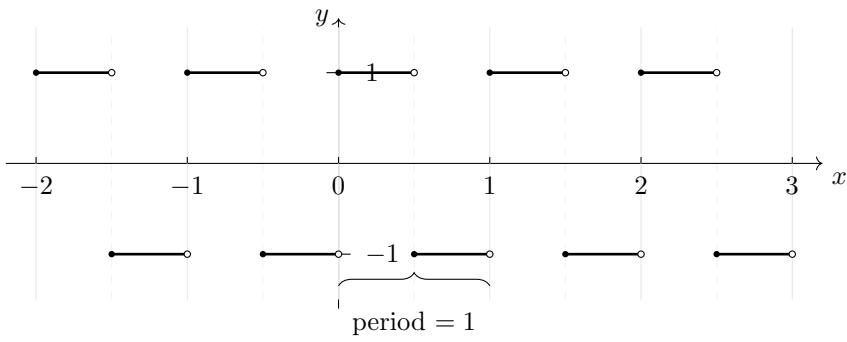
*(Hint: let  $g$  be a non-constant non-negative real-valued function in  $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ , and consider functions of the form  $f = (c + dg)^{1/2}$  for some constant real numbers  $c, d > 0$ .)*

4. (15 pts) A *square wave function* is a  $\mathbb{Z}$ -periodic function defined by

$$f(x) = \begin{cases} 1, & x \in [k, k + \frac{1}{2}), \\ -1, & x \in [k + \frac{1}{2}, k + 1), \end{cases} \quad k \in \mathbb{Z}.$$

Thus  $f$  alternates between 1 and  $-1$  on each half-interval, repeating the same pattern on every interval of length 1.

Find a sequence of continuous periodic functions which converges in  $L^2$  to the square wave function.



5. (15 pts)

(a) Evaluate

$$S_n(\theta) = \sum_{k=1}^n \sin(k\theta).$$

(b) Show that

$$|S_n(\theta)| \leq \pi \varepsilon^{-1} \quad \text{on } [\varepsilon, 2\pi - \varepsilon] \text{ for all } n \geq 1.$$

6. (15 pts) Let  $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{R})$ . We define their *periodic convolution*  $f * g : \mathbb{R} \rightarrow \mathbb{R}$  by

$$(f * g)(x) := \int_0^1 f(y) g(x - y) dy.$$

Prove that  $(f * g)$  is smooth whenever  $f$  is smooth. (Remark: A function is called smooth if it has derivatives of all orders.)