Introduction to Analysis I HW5

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Problem 0.0.1 (15pts). (a) Let (X, d_{disc}) be a metric space with the discrete metric. Let E be a subset of X which contains at least two elements. Show that E is disconnected.

(b) Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous if and only if it is constant. *(Hint: use part (a))*

Problem 0.0.2 (15pts). Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha\in I}$ be a collection of connected sets in X with I non-empty. Suppose also that $\bigcap_{\alpha\in I} E_\alpha$ is non-empty. Show that $\bigcup_{\alpha\in I} E_\alpha$ is connected.

Proof. Suppose by contradiction, $\bigcup_{\alpha\in I} E_\alpha$ is disconnected, then there exists non-empty V, W open in $\bigcup_{\alpha\in I} E_\alpha$ s.t. $V \cup W = \bigcup_{\alpha\in I} E_\alpha$ and $V \cap W = \varnothing$. Hence, we know

$$\begin{cases} V = O_1 \cap \left(\bigcup_{\alpha\in I} E_\alpha\right) = \bigcup_{\alpha\in I} (O_1 \cap E_\alpha) \\ W = O_2 \cap \left(\bigcup_{\alpha\in I} E_\alpha\right) = \bigcup_{\alpha\in I} (O_2 \cap E_\alpha) \end{cases},$$

where O_1, O_2 are open in X. Since I is non-empty, so we can suppose $i \in I$ s.t. $O_1 \cap E_i$ and $O_2 \cap E_i$ are both open in E_i. We first claim that there exists $i, j \in I$ s.t. $O_1 \cap E_i$ and $O_2 \cap E_j$ are non-empty. Suppose by contradiction, $O_1 \cap E_i = O_2 \cap E_j = \varnothing$ for all $i, j \in I$, then $V = O_1 \cap \bigcup_{\alpha\in I} E_\alpha = \varnothing$ and $W = O_2 \cap \bigcup_{\alpha\in I} E_\alpha = \varnothing$, which are contradicitons. Now we claim that there exists $i \in I$ s.t. $O_1 \cap V \neq \varnothing$ and $O_2 \cap V \neq \varnothing$. If not, then

$$O_1 \cap O_2 \cap \left(\bigcap_{\alpha\in I} E_\alpha\right) = \varnothing.$$

However, $\exists p \in \bigcap_{\alpha\in I} E_\alpha$ since $\bigcap_{\alpha\in I} E_\alpha$ is non-empty, so either $p \in V$ or $p \in W$, so either $p \in O_1$ or $p \in O_2$. (If $p \in O_1 \cap O_2$, then $p \in O_1 \cap O_2 \cap \bigcup_{\alpha\in I} E_\alpha = V \cap W = \varnothing$.) WLOG, suppose $p \in O_1$, then $O_1 \cap E_k \neq \varnothing$ for all $k \in I$, and since we know there exists $j \in I$ s.t. $O_2 \cap E_j \neq \varnothing$, so we know there exists $i \in I$ s.t. $O_1 \cap E_i$ and $O_2 \cap E_i$ are both non-empty. Now since

$$\bigcup_{\alpha\in I} E_\alpha = V \cup W = \left(O_1 \cap \bigcup_{\alpha\in I} E_\alpha\right) \cup \left(O_2 \cap \bigcup_{\alpha\in I} E_\alpha\right) = (O_1 \cup O_2) \cap \bigcup_{\alpha\in I} E_\alpha.$$

Hence, we have $\bigcup_{\alpha\in I} E_\alpha \subseteq O_1 \cup O_2$, which gives $E_i \subseteq O_1 \cup O_2$. By this, we have

$$(O_1 \cap E_i) \cup (O_2 \cap E_i) = (O_1 \cup O_2) \cap E_i = E_i.$$

Now since

$$\varnothing = V \cap W = \left(O_1 \cap \bigcup_{\alpha\in I} E_\alpha\right) \cap \left(O_2 \cap \bigcup_{\alpha\in I} E_\alpha\right) = O_1 \cap O_2 \cap \bigcup_{\alpha\in I} E_\alpha,$$

so we have

$$(O_1 \cap E_i) \cap (O_2 \cap E_i) = (O_1 \cap O_2) \cap E_i = \varnothing.$$

However, we have shown that for $A = O_1 \cap E_i$ and $B = O_2 \cap E_i$, A, B are open in E_i and $A \cup B = E_i$ and $A \cap B = \varnothing$, which means E_i is disconnected, and this is a contradiction, so $\bigcup_{\alpha\in I} E_\alpha$ must be connected. ■

Problem 0.0.3 (20pts). Let (X, d) be a metric space, and let E be a subset of X. We say that E is *path-connected* iff, for every $x, y \in E$, there exists a continuous function

$$\gamma : [0, 1] \to E$$

from the unit interval $[0, 1]$ to E such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every non-empty path-connected set is connected. (The converse is false, but is a bit tricky to show and will not be detailed here.)

Problem 0.0.4 (15pts). Let (X, d) be a metric space, and let E be a subset of X. Show that if E is connected, then the closure $\overline{E}$ of E is also connected. Is the converse true?

Problem 0.0.5 (20pts). Let (X, d) be a metric space. Let us define a relation $x \sim y$ on X by declaring $x \sim y$ iff there exists a connected subset of X which contains both x and y. Show that this is an equivalence relation (i.e., it obeys the reflexive, symmetric, and transitive axioms). Also, show that the equivalence classes of this relation (i.e., the sets of the form

$$\{y \in X : y \sim x\} \quad \text{for some } x \in X$$

are all closed and connected. *(Hint: use Problem 2 and Problem 4)* These sets are known as the *connected components* of X. You can read a note about equivalence relation in the file at NTU cool.

Problem 0.0.6 (15pts). Let $f : S \to T$ be a function from a metric space S to another metric space T. Assume f is uniformly continuous on a subset A of S and that T is complete. Prove that there is a unique extension of f to $\overline{A}$ which is uniformly continuous on $\overline{A}$.