

# Introduction to Analysis I

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### Abstract

The lecture note of 2025 Fall Introduction to Analysis I by professor 崔茂培. In this note, we will write  $(X^{(n)})_{n=m}^{\infty}$  and  $\{X^{(n)}\}_{n=m}^{\infty}$  to express a sequence, they are identical, but 崔茂培 use both during lectures, so I follow him.

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# Chapter 1

## Basic Things

### Lecture 1

#### 1.1 Natural Numbers

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The set of natural numbers is denoted by  $\mathbb{N} = \{1, 2, \dots\}$ . There exists an addition operation

$$1 + 1 = 2 \quad 1 + 1 + 1 = 3 \quad \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n.$$

#### 1.2 Integers

The set of integers is  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ . There is a zero element 0 such that  $z + 0 = z$  for any  $z \in \mathbb{Z}$ . Also, for  $n \in \mathbb{N}$ , we have  $n + (-n) = 0$  and  $n - m = n + (-m)$  for all  $n, m \in \mathbb{N}$ .

$$\mathbb{Z} \xrightarrow[\text{introduce division}]{} \mathbb{Q} \xrightarrow[\text{Completeness axiom}]{} \mathbb{R}$$

#### 1.3 Field

Next, we introduce the concept of field.

**Definition 1.3.1 (Fields).** A field is a set  $F$  together with two binary operations, called addition(+) and multiplication(\*), such that the following properties hold:

- (a)  $a + b = b + a$ ,  $a \cdot b = b \cdot a$  for  $a, b \in F$ .
- (b)  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for  $a, b, c \in F$ .
- (c)  $a \cdot (b + c) = a \cdot b + a \cdot c$ .
- (d) There are distinct element 0 and 1 such that  $a + 0 = a$ ,  $a \cdot 1 = a$  for  $a \in F$ .
- (e) For each  $a \in F$ , there exists  $-a \in F$  such that  $a + (-a) = 0$ . If  $a \neq 0$ , there is an element  $\frac{1}{a}$  or  $a^{-1}$  in  $F$  such that  $a \cdot \frac{1}{a} = 1$ , or  $a \cdot a^{-1} = 1$ .

**Remark 1.3.1.** If  $a \in F$ , then  $a + a \in F$ . We denote  $a + a$  by  $2 \cdot a$ . Similarly,

$$\underbrace{a + a + \dots + a}_{n \text{ times}} = n \cdot a,$$

and

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

if  $a \in F$  and  $n \in \mathbb{N}$ .

**Remark 1.3.2.** In a field, we have subtraction and division  $a - b = a + (-b)$  for  $a, b \in F$ . If  $b \neq 0$ , then  $\frac{a}{b} = a \cdot b^{-1}$  for  $a, b \in F$ .

In a field  $F$ , we have

$$\begin{aligned} (a + b)^2 &= (a + b) \cdot (a + b) \\ &= (a + b) \cdot a + (a + b) \cdot b \\ &= a \cdot a + b \cdot a + a \cdot b + b \cdot b \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2. \end{aligned}$$

**Example 1.3.1.**

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

if  $b \neq 0$  and  $d \neq 0$ .

**Proof.**

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= a \cdot b^{-1} + c \cdot d^{-1} \\ &= ab^{-1}dd^{-1} + cd^{-1}bb^{-1} \\ &= adb^{-1}d^{-1} + cbd^{-1}b^{-1} \\ &= \frac{ad + bc}{bd}. \end{aligned}$$

Notice that this is true since we have commutativity in multiplication and

$$d^{-1}b^{-1} = (bd)^{-1} = \frac{1}{bd}.$$

■

**Example 1.3.2.** The set of rational numbers  $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$  is a field.

**Example 1.3.3.** The set of real numbers is also a field.

**Example 1.3.4.**  $F_2 = \{0, 1\}$  is also a field since we can define addition and multiplication like  $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$ , and  $0 \cdot 0 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$ .

## 1.4 Order Relation

Next, we introduce the order relation. The real number system is ordered by the relation  $<$ , which has the following properties.

- (f) For each pair of real numbers  $a$  and  $b$ , exactly one of the following is true:  $a = b, a < b, b < a$ .
- (g) If  $a < b$  and  $b < c$ , then  $a < c$ .
- (h) If  $a < b$ , then  $a + c < b + c$  for any  $c$ , and if  $0 < c$ , then  $a \cdot c < b \cdot c$ .

**Definition 1.4.1.** A field with an order relation satisfy (f) to (h) is called an ordered field.

**Example 1.4.1.** The set of rational numbers is an ordered field.

**Example 1.4.2.**  $F_2$  is not an ordered field.

**Proof.** If  $0 < 1$ , then  $1 = 0 + 1 < 1 + 1 = 0$ , which is a contradiction. If  $1 < 0$ , then  $0 = 1 + 1 < 0 + 1 = 1$ , which is also a contradiction. ■

**Notation.** In an ordered field, we use  $a \leq b$  to denote either  $a < b$  or  $a = b$ .

## 1.5 Absolute Value and Triangle Inequality

Next, we define the absolute value of a real number

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a \leq 0; \end{cases}$$

**Theorem 1.5.1** (Triangle Inequality).

$$|a + b| \leq |a| + |b|$$

for all  $a, b \in \mathbb{R}$ .

**Corollary 1.5.1.**

$$||a| - |b|| \leq |a - b| \quad \text{and} \quad ||a| - |b|| \leq |a + b|$$

**Proof.** We write

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Similarly we have

$$|b| \leq |b - a| + |a|.$$

So

$$-|b - a| \leq |a| - |b| \leq |a - b|.$$

Thus,

$$||a| - |b|| \leq |a - b|. \quad \blacksquare$$

## 1.6 Supremum and Infimum

Next, we introduce the notion of supremum of a subset of real numbers.

**Definition 1.6.1.** Let  $S$  be a subset of  $\mathbb{R}$ ,

- (1) we say  $b$  is an upper bound of  $S$  if  $x \leq b$  for all  $x \in S$ .
- (2) If  $B$  is an upper bound of  $S$ , and no number smaller than  $B$  is an upper bound of  $S$ , then  $B$  is called the supremum or the least upper bound of  $S$ . We write  $B = \sup S$ .

**Corollary 1.6.1.** If  $B = \sup S$ , then

- (1)  $x \in S$  implies  $x \leq B$

(2) If  $b < B$ , then  $b$  is not an upper bound of  $S$ , i.e. there exists  $x_1 \in S$  such that  $b < x_1$ .

**Definition 1.6.2.** Let  $S$  be a subset of  $\mathbb{R}$ ,

- (1) we say  $b$  is a lower bound of  $S$  if  $x \geq b$  for all  $x \in S$ .
- (2) If  $\alpha$  is a lower bound of  $S$ , and no number bigger than  $\alpha$  is a lower bound of  $S$ , then  $\alpha$  is called the infimum or the greatest lower bound of  $S$ . We write  $\alpha = \inf S$ .

**Corollary 1.6.2.** If  $\alpha = \inf S$ , then

- (1)  $x \in S$  implies  $x \geq \alpha$
- (2) If  $\alpha < a$ , then  $a$  is not a lower bound of  $S$ , i.e. there exists  $x_1 \in S$  such that  $x_1 < a$ .

**Notation** (Interval Notation).

$$(a, b) = \{x \mid a < x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

**Example 1.6.1.**  $S = \{x \mid x < 0\} = (-\infty, 0)$ , then  $\sup S = 0$  but  $\inf S$  does not exist.

**Example 1.6.2.**  $S_1 = \{-1, -2, -3, -4, \dots\} = \{-n \mid n \in \mathbb{N}\}$ , then  $\sup S = -1$ , but  $\inf S$  does not exist.

**Definition 1.6.3 (Nonempty Sets).** A nonempty set is that a set has at least one element. The empty set, written as  $\emptyset$ , is the set has no elements at all.

**Example 1.6.3.**  $S = \{x \mid x \in \mathbb{Q}, x < \sqrt{2}\}$

In  $\mathbb{Q}$ ,  $\sup S$  does not exist. In  $\mathbb{R}$ ,  $\sup S = \sqrt{2}$ .

**Theorem 1.6.1 (Completeness axiom).** If a nonempty set of real numbers (an ordered field) is bounded above, then it has a least upper bound or  $\sup S$  exists.

**Remark 1.6.1.** This is an extra axiom that can't be derived from the properties of ordered field.

**Remark 1.6.2.** Up to "isomorphism", there is exactly one complete ordered field: the field of real numbers.

**Remark 1.6.3.** From now, we assume  $\mathbb{R}$  satisfies the completeness axiom. Thus, any nonempty subset  $S \subseteq \mathbb{R}$  that is bounded above, we have  $\sup S$  exists.

We can prove the following property of  $\sup S$ .

**Theorem 1.6.2.** If  $S \subseteq \mathbb{R}$  is bounded above, then  $\sup S$  is the unique real number  $B$  such that

- (i)  $x \leq B$  for all  $x \in S$
- (ii) for every  $\varepsilon > 0$ , there exist an  $x_0 \in S$  such that  $B - \varepsilon < x_0$ .

**Proof.** (i), (ii) follows from the definition. We prove the uniqueness. Suppose  $B_1 = \sup S = B_2$ . We want to show  $B_1 = B_2$ . Suppose  $B_1 \neq B_2$ . Then either  $B_1 < B_2$  or  $B_2 < B_1$ . However, if either one is true, then the other one cannot be  $\sup S$ . ■

**Theorem 1.6.3 (Archimedean Property).** If  $p > 0$  and  $\varepsilon > 0$ , then there exists an  $n \in \mathbb{N}$  such that  $p < n\varepsilon$ .

**Proof.** We prove this contradiction. Suppose it is not true. This implies  $n\varepsilon \leq p$  for all  $n \in \mathbb{N}$ . Consider  $S = \{n\varepsilon \mid n \in \mathbb{N}\}$ , then  $p$  is an upper bound of  $S$ , so  $S$  is bounded above by  $p$ , so we know  $B = \sup S$  exists. Hence,  $n\varepsilon \leq B$  for all  $n \in \mathbb{N}$ , so we have  $(n+1)\varepsilon \leq B$ , which means

$$n\varepsilon \leq B - \varepsilon$$

for all  $n \in \mathbb{N}$ . This implies  $B - \varepsilon$  is also an upper bound of  $S$ , which is a contradiction. ■

## 1.7 Density of other number system

**Theorem 1.7.1.** Every nonempty subset of the integers that is bounded below has a least element.

**Proof.** We first introduce an axiom:

**Theorem 1.7.2 (Well-Ordering principle).** Every non-empty subset of the natural numbers has a least element.

**Note 1.7.1.** Here,  $\mathbb{N}$  can be  $\{0, 1, 2, \dots\}$  or  $\{1, 2, 3, \dots\}$ , which is not that important.

Now we call this subset of integers as  $S$ , and suppose we have  $m$  as a lower bound of  $S$ , then define  $S' = \{s - m \mid s \in S\}$ , then we know  $S'$  is a nonempty subset of  $\mathbb{N}$ , then by well-ordering principle we know there is a least element in  $S'$  and thus there is also a least element in  $S$ . ■

**Corollary 1.7.1.** Every nonempty subset of the integers that is bounded above has a greatest element.

**Proof.** Suppose  $M$  is an upper bound, then define a set  $S' = \{M - s \mid s \in S\}$ , then by well-ordering principle we know  $M - a$  is the least element of  $S'$  for some  $a \in S$ , so we have  $M - x \geq M - a$  for all  $x \in S$ , which means  $a \geq x$  for all  $x \in S$  and since  $a \in S$ , so  $a$  is the greatest element of  $S$ . ■

**Theorem 1.7.3.** The set of rational numbers is dense in the real number. That is, if  $a$  and  $b$  are real numbers with  $a < b$ , then there exists a rational number  $\frac{p}{q}$  such that  $a < \frac{p}{q} < b$ .

**Proof.** Let  $a, b \in \mathbb{R}$ ,  $a < b$ . By [Archimedean Property](#),  $\exists q \in \mathbb{N}$  such that  $q(b - a) > 1$ . Let  $S = \{m \mid m \text{ is an integer with } m > qa\}$ , since we know  $S \neq \emptyset$  and  $S$  is bounded below. Hence,  $p = \inf S$  exists and is an integer by the last theorem. So  $qa < p$  and  $p - 1 \leq qa$ , which means  $qa < p \leq qa + 1 < qb$ , so we have  $a < \frac{p}{q} < b$ . ■

## Lecture 2

**Definition 1.7.1 (Floor Function).** For any real number  $x$ , the floor function of  $x$  is denoted by  $\lfloor x \rfloor$ , and is defined by the formula  $\lfloor n \rfloor$  if  $n \leq x < n + 1$  where  $n \in \mathbb{Z}$ .

**Corollary 1.7.2.**

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

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**Example 1.7.1.**  $\lfloor 3.7 \rfloor = 3$ ,  $\lfloor -1.2 \rfloor = -2$ .

Now by floor function, we can reprove [Theorem 1.7.3](#).

**Theorem 1.7.4** (Density of rational number in real number Again). The set of rational numbers is dense in the real number. That is, if  $a$  and  $b$  are real numbers with  $a < b$ , then there exists a rational number  $\frac{q}{p}$  such that  $a < \frac{q}{p} < b$ .

**Reprove Theorem 1.7.3.** Since  $a < b$ , so we know  $b - a > 0$ . Now by [Archimedean Property](#), we know there exists  $q \in \mathbb{N}$  such that  $q(b - a) > 1$ . Let  $p = \lfloor qa \rfloor + 1$ , we have

$$\lfloor qa \rfloor \leq qa < \lfloor qa \rfloor + 1 = p.$$

From our construction,  $qb > qa + 1$ , so we have

$$p = \lfloor qa \rfloor + 1 \leq qa + 1 < qb,$$

hence we have

$$qa \leq p \leq qb.$$

■

**Note 1.7.2.** For some reason,  $p, q$  in [Theorem 1.7.3](#) and [Theorem 1.7.4](#) are reversed.

**Definition 1.7.2** (irrational number).  $x$  is called irrational if  $x$  is not rational.

**Example 1.7.2.**  $\sqrt{2}$  is irrational.

**Theorem 1.7.5.** Let  $r \in \mathbb{Q}$  and  $x \in \mathbb{R} \setminus \mathbb{Q}$ , then

1.  $r + x$  is irrational.
2. If  $r \neq 0$ , then  $rx$  is irrational.

**sketch of proof.**

1. If  $r + x = q \in \mathbb{Q}$ , then  $x = q - r \in \mathbb{Q}$ , contradiction.
2. If  $rx = q \in \mathbb{Q}$ , then  $x = \frac{q}{r} \in \mathbb{Q}$  since  $r \neq 0$ .

■

**Theorem 1.7.6** (irrational number dense in real number). The set of irrational number is dense in real number. That is, if  $a, b \in \mathbb{R}$  and  $a < b$ , then there exists a irrational number  $t$  such that  $a < t < b$ .

**Proof.** By [density of rational number](#), we can find  $a < r_1 < r_2 < b$  where  $r_1, r_2 \in \mathbb{Q}$ , and then let  $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$ , then we know

$$a < r_1 < t < r_2 < b.$$

**Note 1.7.3.** We should use [Theorem 1.7.5](#) and the fact that  $\sqrt{2}$  is irrational.

■

**Definition 1.7.3 (bounded set).** A set  $S \subseteq \mathbb{R}$  is bounded if there are numbers  $a, b$  s.t.  $a \leq x \leq b$  for all  $x \in S$ .

**Corollary 1.7.3.** A bounded non-empty set in  $\mathbb{R}$  has a unique supremum and a unique infimum and  $\inf S \leq \sup S$ .

## 1.8 Extended real number system

The real number system, together with  $\infty$  and  $-\infty$ , then we have the following properties:

- (a) If  $a \in \mathbb{R}$ , then  $a + \infty = \infty + a = \infty$  and  $a - \infty = -\infty + a = -\infty$ , and  $\frac{a}{\infty} = \frac{a}{-\infty} = 0$ .
- (b) If  $a > 0$ , then  $a \cdot \infty = \infty \cdot a = \infty$  and  $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$
- (c) If  $a < 0$ , then  $a \cdot \infty = \infty \cdot a = -\infty$  and  $a \cdot (-\infty) = -\infty \cdot a = \infty$  and  $\infty + \infty = \infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$  and  $-\infty - \infty = \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$  and  $|\infty| = |-\infty| = \infty$

However, there are some indeterminate form:

**Theorem 1.8.1.** The following things are not defined:

$$\infty - \infty, 0 \cdot \infty, \frac{\infty}{\infty}, \text{ and } \frac{0}{0}.$$

## 1.9 Mathematical Induction

**Theorem 1.9.1 (Peano's Postulate).** The natural numbers satisfy the following properties

- (a)  $\mathbb{N}$  is nonempty.
- (b) For each natural number  $n$ , there exists a unique rational number  $n$  called the successor of  $n$ .
- (c) There exists a natural number  $\bar{n}$  that is not the successor of any natural number.
- (d) Different natural numbers have different successors, that is,  $n \neq m$  implies  $n' \neq m'$ .
- (e) The only subset of  $\mathbb{N}$  that contains  $\bar{n}$  and also contains the successor of every one of its element is  $\mathbb{N}$ .

**Theorem 1.9.2 (Principle of Mathematical Induction).** Let  $p_1, p_2, \dots, p_n$  be propositions, one for each positive integers, such that

- (a)  $p_1$  is true.
- (b) for each positive integer  $n$ ,  $p_n$  implies  $p_{n+1}$ .

then  $p_n$  is true for each  $n \in \mathbb{N}$ .

**Proof.** Let  $M = \{n \mid n \in \mathbb{N} \text{ and } p_n \text{ is true}\}$ , then from (a) we know  $1 \in M$  and from (b) we know  $n \in M$  implies  $n + 1 \in M$ . Hence, from (e) of [Peano's Postulate](#), we know  $M = \mathbb{N}$ . ■

# Chapter 2

## Metric Space

### 2.1 Definition and examples

**Definition 2.1.1.** Suppose  $x_n \in \mathbb{R}$  for  $n \geq m$ . We use the notation  $(x_n)_{n=m}^{\infty}$  to denote the sequence of numbers

$$x_m, x_{m+1}, \dots$$

We first recall the definition of a convergent sequence.

**Definition 2.1.2 (Convergent Sequence).** We say that a sequence  $(x_n)_{n=m}^{\infty}$  of real numbers converges to  $x$  if for every  $\varepsilon > 0$ , there exists an  $N \geq m$  s.t.  $|x_n - x| \leq \varepsilon$  for all  $n \geq N$ .

**Notation.** We write  $\lim_{n \rightarrow \infty} x_n = x$ .

On  $\mathbb{R}$ , we can define the distance function between two points  $x, y \in \mathbb{R}$  by  $d(x, y) = |x - y|$ . We'll discuss this more later.

**Lemma 2.1.1.** Let  $(x_n)_{n=m}^{\infty}$  be a sequence of real numbers, and let  $x$  be another real number, then  $(x_n)_{n=m}^{\infty}$  converges to  $x$  if and only if  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ .

**Proof.** Assume  $(x_n)_{n=m}^{\infty}$  converges to  $x$ . Let  $\varepsilon > 0$  be arbitrary real number. By definition, there exists an  $N \geq m$  such that  $|x_n - x| \leq \varepsilon$  for all  $n \geq N$ . But  $d(x_n, x) = |x_n - x|$  by the definition. Hence,  $\forall \varepsilon > 0, \exists N \geq m$  such that  $d(x_n, x) \leq \varepsilon$  for all  $n \geq N$ . This implies that  $\forall \varepsilon > 0, \exists N \geq m$  such that  $|d(x_n, x) - 0| \leq \varepsilon$  for all  $n \geq N$ . This implies  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ .

The proof of the other side is the same but writing the above proof from bottom to top again.

**Definition 2.1.3 (Metric Space).** A metric space  $(X, d)$  is the space of  $X$  of objects (called points), together with a distance function or metric  $d : X \times X \rightarrow [0, \infty)$  which associates to each  $x, y$  of points in  $X$  a nonnegative number  $d(x, y) \geq 0$ . Furthermore, the metric must satisfy 4 axioms.

- (a) For any  $x \in X$ ,  $d(x, x) = 0$ .
- (b) (Positivity) For any distinct  $x, y \in X$ , we have  $d(x, y) > 0$ .
- (c) (Symmetry) For any  $x, y \in X$ , we have  $d(x, y) = d(y, x)$ .
- (d) (Triangle inequality) For any  $x, y, z \in X$ , we have  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Example 2.1.1.** On  $\mathbb{R}$ , we can define  $d(x, y) = |x - y|$ .

**Proof.** •  $d(x, y) = |x - y| \geq 0$ .

- $d(x, y) = 0$  iff  $|x - y| = 0$  iff  $x = y$ .
- $|x - y| = |y - x|$ , so  $d(x, y) = d(y, x)$
- $|x - z| \leq |x - y| + |y - z|$  for all  $x, y, z \in \mathbb{R}$ .

⊗

**Example 2.1.2.** Let  $(X, d)$  be a metric space and  $Y \subseteq X$ , then  $Y$  inherits a natural distance function

$$d|_{Y \times Y} : Y \times Y \rightarrow [0, \infty)$$

defined by  $d|_{Y \times Y}(\alpha, \beta) = d(\alpha, \beta)$  for all  $\alpha, \beta \in Y$ .

**Note 2.1.1.**  $(Y, d|_{Y \times Y})$  is called a metric subspace of  $(X, d)$ . It is obvious that  $d|_{Y \times Y}$  is a metric on  $Y$ .

Recall  $\mathbb{R}^n$ . Let  $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ .

**Definition 2.1.4** ( $l^2$ -metric). The  $l^2$ -metric is defined by

$$d_2(x, y) = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}} \quad (\text{or we called } d_{l_2}(x, y)).$$

**Definition 2.1.5** ( $l^1$ -metric (taxicab metric)). The  $l^1$ -metric is defined by

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i| \quad (\text{or we called } d_{l_1}(x, y))$$

**Definition 2.1.6** ( $l^\infty$ -metric). The  $l^\infty$ -metric is defined by

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

**Exercise 2.1.1.** Verify they are all metrics.

**Note 2.1.2.** Actually we have to define inner product and norm first and then we can use the triangle inequality of norm to prove  $d_2$  is a metric. (See lecture notes by professor)

## Lecture 3

**Definition 2.1.7** (Cartesian Product). Let  $A, B$  be sets. The cartesian product of  $A$  and  $B$  is defined by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Similarly, the cartesian product of  $X_1, X_2, \dots, X_n$  is

$$X_1 \times X_2 \times \dots \times X_n = \{(x_1, x_2, \dots, x_n) \mid x_i \in X_i \forall 1 \leq i \leq n\}.$$

**Definition 2.1.8** (Functions). Let  $X_1, X_2, \dots, X_n$  be sets and let  $Y$  be another set. A function of  $n$  variables with codomains is a map  $f : X_1 \times X_2 \times \dots \times X_n \rightarrow Y$  which assigns each  $n$ -tuple  $(x_1, x_2, \dots, x_n)$  with  $x_i \in X_i$  a unique element  $f(x_1, x_2, \dots, x_n)$ .

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**Definition.** We talk about the definition of domain, codomain, and range:

**Definition 2.1.9.** The domain of  $f$  is  $X_1 \times X_2 \times \cdots \times X_n$  and  $Y$  is the codomain of  $f$ .

**Definition 2.1.10.** The range of  $f$  is

$$\{f(x_1, x_2, \dots, x_n) \in Y \mid x_i \in X_i \forall i\}.$$

In the definition of metric space, we write  $(X, d)$  to emphasize our set  $X$  and  $d$  is a distance function defined on  $X \times X$ , i.e.

$$d : X \times X \rightarrow [0, \infty) \subseteq \mathbb{R},$$

where

$$d : (x, y) \mapsto d(x, y)$$

for  $x, y \in X$ . Let  $(X, d)$  be a metric space and  $Y \subseteq X$ . Then  $(Y, d|_{Y \times Y})$  is also a metric space with distance function defined by

$$d|_{Y \times Y} \rightarrow [0, \infty)$$

and

$$d|_{Y \times Y} : (\alpha, \beta) \mapsto d(\alpha, \beta) \text{ for } \alpha, \beta \in Y.$$

**Example 2.1.3.** Recall the **Taxi-cab metric**, it can be used in cryptography. For example, for two binary strings, we know

$$d_1((10010), (10101)) = 3 = \text{the number of mismatched bits.}$$

**Example 2.1.4.** Recall the  **$l^\infty$ -metric**. Suppose two jobs where each consists of 3 tasks, and the time (in hours) to complete each task is represented by a vector

$$x = (2, 4, 6), \quad y = (3, 7, 5),$$

so

$$d_\infty(x, y) = \max\{|2 - 3|, |4 - 7|, |6 - 5|\} = 3.$$

**Definition 2.1.11 (Lipschitz equivalent metrics).** Let  $(X, d_1)$  and  $(X, d_2)$  be two metrics on  $X$ . We say  $d_1$  and  $d_2$  are Lipschitz equivalent if  $\exists c_1, c_2 > 0$  s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y) \quad \forall x, y \in X$$

**Remark 2.1.1.** They will have same topology (defined later).

**Proposition 2.1.1.** For all  $x, y \in \mathbb{R}^n$ ,

$$d_2(x, y) \leq d_1(x, y) \leq \sqrt{n} d_2(x, y) \tag{2.1}$$

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} d_\infty(x, y) \tag{2.2}$$

**Remark 2.1.2.**

$$\begin{aligned} d_\infty(x, y) &\geq \frac{1}{\sqrt{n}} d_2(x, y) \\ &\geq \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} d_1(x, y) = \frac{1}{n} d_1(x, y). \end{aligned}$$

Also,

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y).$$

**Remark 2.1.3.**  $d_1, d_2, d_\infty$  are all Lipschitz equivalent.

**proof of Proposition 2.1.1** . Recall  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ , then

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|, \quad d_2(x, y) = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

By Cauchy-Schurwatz inequality,

$$\begin{aligned} d_1(x, y) &= \sum_{i=1}^n |x_i - y_i| \\ &\leq \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n 1^2 \right)^{\frac{1}{2}} = \sqrt{n} d_2(x, y). \end{aligned}$$

Now we show that  $d_1(x, y) \geq d_2(x, y)$ .

$$\begin{aligned} (d_1(x, y))^2 &= \left( \sum_{i=1}^n |x_i - y_i| \right)^2 \\ &= \sum_{i=1}^n |x_i - y_i|^2 + 2 \sum_{1 \leq i < j \leq n} |x_i - y_i| |x_j - y_j| \\ &\geq \sum_{i=1}^n |x_i - y_i|^2 = d_2(x, y)^2. \end{aligned}$$

Hence, we have  $d_1(x, y) \geq d_2(x, y)$ .

Now we show that  $d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$ . Note that

$$d_2(x, y) = \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}, \quad d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

For each  $i$ , we know

$$|x_i - y_i| \leq d_\infty(x, y),$$

so

$$d_2(x, y)^2 \leq \sum_{i=1}^n d_\infty(x, y)^2 = n d_\infty(x, y)^2,$$

so  $d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$ . ■

**Definition 2.1.12 (Discrete metric).** Let  $X$  be any set, define the discrete metric:

$$d_{\text{disc}} : X \times X \rightarrow \{0, 1\}$$

where

$$d_{\text{disc}}(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

Why this is a metric? Because

- $d_{\text{disc}}(x, y) \geq 0$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ .
- $d_{\text{disc}}(x, y) = d_{\text{disc}}(y, x)$  by definition.
- $d_{\text{disc}}(x, z) \leq d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z)$ ?

**proof of triangle inequality in discrete metric.** We first consider the case that  $x = z$ , then

$$d_{\text{disc}}(x, z) = 0,$$

so it is obviously that the triangle inequality is true.

Now if  $x \neq z$ , then either  $y \neq z$  or  $y \neq x$  must happen, so the triangle inequality must be true. ■

**Example 2.1.5.** We can define

$$d(x, x) = 0, \quad d(x, y) = \text{minimal length of a path from } x \text{ to } y,$$

then this is also a metric.

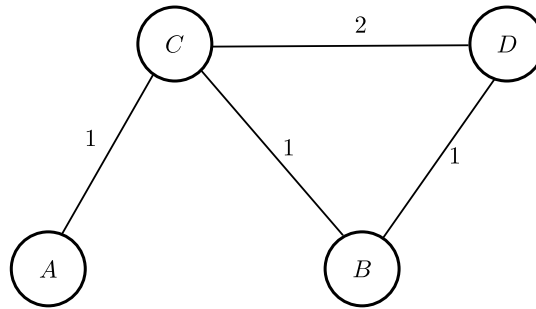


Figure 2.1: Graph metrics

**Definition 2.1.13 (Convergence in metric space).** Let  $m$  be an integer,  $(X, d)$  be a metric space, and let  $(X^{(n)})_{n=m}^{\infty}$  be a sequence of points in  $X$ . Let  $x \in X$ . We say that  $(X^{(n)})_{n=m}^{\infty}$  converges to  $x$  with respect to  $d$  iff

$$\lim_{n \rightarrow \infty} d(X^{(n)}, x) = 0,$$

where  $\lim_{n \rightarrow \infty} d(X^{(n)}, x) = 0$  iff for every  $\varepsilon > 0$ ,  $\exists N \geq m$  s.t.  $d(X^{(n)}, x) \leq \varepsilon$  for all  $n \geq N$ .

**Notation.** We also write  $\lim_{n \rightarrow \infty} X^{(n)} = x$  in  $(X, d)$ .

**Remark 2.1.4.** Suppose  $(X^{(n)})_{n=m}^{\infty}$  converges to  $x$  in  $(X, d)$ , then  $(X^{(n)})_{n=m_1}^{\infty}$  also converges to  $x$  in  $(X, d)$  if  $m_1 \geq m$ .

**Example 2.1.6.** Let  $(X^{(n)})_{n=1}^{\infty}$  denote the sequence  $X^{(n)} = (\frac{1}{n}, \frac{1}{n})$  in  $\mathbb{R}^2$ , then what will this sequence converges to for different metric?

**Proof.**

- If the metric is  $d_1$ , then

$$d_1(X^{(n)}, (0, 0)) = \left| \frac{1}{n} - 0 \right| + \left| \frac{1}{n} - 0 \right| = \frac{2}{n},$$

so

$$\lim_{n \rightarrow \infty} d_1(X^{(n)}, (0, 0)) = \lim_{n \rightarrow \infty} \frac{2}{n} = 0.$$

- If the metric is  $d_2$ , then

$$d_2 \left( X^{(d)}, (0, 0) \right) = \sqrt{\left( \frac{1}{n} - 0 \right)^2 + \left( \frac{1}{n} - 0 \right)^2} = \frac{\sqrt{2}}{n}.$$

Hence, under  $l_2$ -metric  $\{X^{(n)}\}$  also converges to 0.

- If the metric is  $d_\infty$ , then

$$d_\infty \left( X^{(n)}, (0, 0) \right) = \max \left\{ \left| \frac{1}{n} \right|, \left| \frac{1}{n} \right| \right\} = \frac{1}{n},$$

so it also converges to 0.

- If the metric is discrete metric, then however, it will not converges to  $(0, 0)$  since

$$\lim_{n \rightarrow \infty} d_{\text{disc}} \left( X^{(n)}, (0, 0) \right) = \lim_{n \rightarrow \infty} d_{\text{disc}} \left( \left( \frac{1}{n}, \frac{1}{n} \right), (0, 0) \right) = 1.$$

⊗

**Definition.** Let  $f : X \rightarrow Y$  be a function with domain  $X$  and codomain  $Y$ . The range of  $f = \{f(x) \mid x \in X\} \subseteq Y$ .

**Definition 2.1.14 (injective).** We say  $f$  is injective or one-to-one if for all  $x_1, x_2 \in X$ ,  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ .

**Definition 2.1.15 (surjective).** We say  $f$  is surjective or onto if for every  $y \in Y$ ,  $\exists x \in X$  s.t.  $f(x) = y$ .

**Definition 2.1.16 (bijective).** We say  $f$  is bijective if  $f$  is injective and surjective.

**Corollary 2.1.1.** If  $f$  is bijective, then there exists  $f^{-1} : Y \rightarrow X$  defined by  $f^{-1}(y) = x$  if  $f(x) = y$ . We also have

$$\begin{aligned} f(f^{-1}(y)) &= y \quad \forall y \in Y \\ f^{-1}(f(x)) &= x \quad \forall x \in X. \end{aligned}$$

**Example 2.1.7.**  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  in  $(\mathbb{R}, d)$ , where  $d$  is the standard metric in  $\mathbb{R}$ , which is defined by

$$d(x, y) = |x - y|.$$

But in different metric,  $\lim_{n \rightarrow \infty} \frac{1}{n}$  may not be 0.

**Proof.** Define  $f : [0, 1] \rightarrow [0, 1]$  defined by

$$f(x) = \begin{cases} x, & \text{if } 0 < x < 1; \\ 1, & \text{if } x = 0; \\ 0, & \text{if } x = 1. \end{cases}$$

$f$  is bijective on  $[0, 1]$  to  $[0, 1]$

Define another metric  $d^1$  on  $[0, 1]$  by

$$d^1(x, y) = d(f(x), f(y)).$$

We want to show that  $d^1$  is also a metric on  $[0, 1]$ .

- $d^1(x, y) = d(f(x), f(y)) = |f(x) - f(y)| \geq 0$
- $d^1(x, y) = 0$  iff  $f(x) = f(y)$  iff  $x = y$  since  $f$  is injective.
- The triangle inequality is trivially true since we can just use the triangle inequality in  $d$ .

In fact,  $\lim_{n \rightarrow \infty} \frac{1}{n} = 1$  in  $([0, 1], d^1)$  since

$$\lim_{n \rightarrow \infty} d^1\left(\frac{1}{n}, 1\right) = \lim_{n \rightarrow \infty} d\left(\frac{1}{n}, 0\right) = \lim_{n \rightarrow \infty} \left|\frac{1}{n}\right| = 0.$$

⊛

## 2.2 Some point set topology of metric space

**Definition 2.2.1 (ball).** Let  $(X, d)$  be a metric space. Let  $x_0 \in X$  and  $r > 0$ . We define the ball  $B_{(X, d)}(x_0, r)$  in  $X$ , centered at  $x_0$  and with radius  $r$  in the metric  $d$ , to the set

$$B_{(X, d)}(x_0, r) := \{x \in X \mid d(x_0, x) < r\}.$$

Sometimes, we write it as  $B_X(x_0, r)$  or  $B(x_0, r)$ .

**Example 2.2.1.** In  $\mathbb{R}^2$ ,

$$B_{(\mathbb{R}^2, d_2)}((0, 0), 1) = \{(x, y) \mid d_2((x, y), (0, 0)) = \sqrt{x^2 + y^2} < 1\},$$

and

$$B_{(\mathbb{R}^2, d_1)}((0, 0), 1) = \{(x, y) \mid d_1((x, y), (0, 0)) = |x| + |y| < 1\},$$

and

$$B_{(\mathbb{R}^2, d_\infty)}((0, 0), 1) = \{(x, y) \mid d_\infty((x, y), (0, 0)) = \max\{|x|, |y|\} < 1\},$$

also we can consider the  $d_{\text{disc}}$  case but I am too lazy to write it down.

**Notation.** Let  $E \subseteq X$ , we will write

$$X \setminus E := \{x \in X \mid x \notin E\}.$$

**Definition.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . For a point  $x_0 \in X$ ,

**Definition 2.2.2 (interior point).**  $x_0$  is an interior point of  $E$  if  $\exists r > 0$  s.t.  $B(x_0, r) \subseteq E$ .

**Definition 2.2.3 (exterior point).**  $x_0$  is an exterior point of  $E$  if  $\exists r > 0$  s.t.  $B(x_0, r) \subseteq X \setminus E$ .

**Definition 2.2.4 (boundary point).**  $x_0$  is a boundary point of  $E$  if it is neither an interior point nor an exterior point of  $E$ .

**Proposition 2.2.1.**  $x_0$  is a boundary point of  $E$  iff for all  $r > 0$ ,  $B(x_0, r) \cap E \neq \emptyset$  and  $B(x_0, r) \cap (X \setminus E) \neq \emptyset$ .

## Lecture 4

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**Theorem 2.2.1.** Let  $(X, d_1)$  and  $(X, d_2)$  be metrics on  $X$ , and suppose  $d_1$  and  $d_2$  are Lipschitz equivalent, then for any sequence  $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$  and any  $x \in X$ , we have

$$\lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (X, d_1) \Leftrightarrow \lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (X, d_2).$$

**Proof.** Since  $d_1, d_2$  are Lipschitz equivalent, so there exists  $c_1, c_2 > 0$  s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y).$$

( $\Rightarrow$ ) Given  $\frac{\varepsilon}{c_2} > 0$ , since  $\lim_{n \rightarrow \infty} x^{(n)} = x$  in  $(X, d_1)$ , so there exists  $N$  s.t.  $N \geq m$  and

$$d_1(x^{(n)}, x) \leq \frac{\varepsilon}{c_2} \text{ for } n \geq N.$$

This implies  $d_2(x^{(n)}, x) \leq c_2 d_1(x^{(n)}, x) \leq \varepsilon$  for  $n \geq N$ , which means

$$\lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (X, d_2).$$

( $\Leftarrow$ ) Similar. ■

**Remark 2.2.1.** On  $\mathbb{R}^n$ , the metrics  $d_1, d_2, d_{\infty}$  are Lipschitz equivalent, that is,

$$\lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_1) \Leftrightarrow \lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_2) \Leftrightarrow \lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_{\infty})$$

**Proposition 2.2.2.** Let  $(X, d_{\text{disc}})$  be a discrete metric space, and  $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$ . Then

$$\lim_{n \rightarrow \infty} x^{(n)} = x \text{ in } (X, d_{\text{disc}}) \Leftrightarrow \exists N \geq m \text{ s.t. } x^{(n)} = x \text{ for } n \geq N.$$

**Proof.** ( $\Leftarrow$ ) Easy.

( $\Rightarrow$ ) Given  $\frac{1}{2} > 0$ , there exists  $N \geq m$  s.t.  $d(x_n, x) < \frac{1}{2}$  for  $n \geq N$ , but  $d(x_n, x) < \frac{1}{2}$  implies  $d(x_n, x) = 0$ , which means  $x_n = x$  for all  $n \geq N$ . ■

**Definition.** We define the interior, exterior, and boundary point again.

**Definition 2.2.5.** The set of interior points is denoted by

$$\text{Int}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq E\}.$$

**Definition 2.2.6.** The set of exterior points is denoted by

$$\text{Ext}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq X \setminus E\}.$$

**Definition 2.2.7.** A point is a boundary points if it is neither an interior point nor an exterior point, and we define

$$\partial E = \{x \in X \mid x \notin \text{Int}(E) \text{ and } x \notin \text{Ext}(E)\}.$$

**Remark 2.2.2.**

1.

$$x_0 \notin \text{Int}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (X \setminus E) \neq \emptyset.$$

2.

$$x_0 \notin \text{Ext}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (E) \neq \emptyset.$$

3.  $\text{Int}(X \setminus E) = \text{Ext}(E)$ .4.  $\partial E = \partial(X \setminus E)$  since

$$x_0 \in \partial E \Leftrightarrow x \notin \text{Int}(E) \text{ and } \text{Ext}(E) \Leftrightarrow x_0 \notin \text{Ext}(X \setminus E) \text{ and } x_0 \notin \text{Int}(X \setminus E).$$

Also,

$$x_0 \in \partial(X \setminus E) \Leftrightarrow x \notin \text{Int}(X \setminus E) \text{ and } \text{Ext}(X \setminus E) \Leftrightarrow x_0 \notin \text{Ext}(E) \text{ and } x_0 \notin \text{Int}(E).$$

Hence, acutually  $\partial E = \partial(X \setminus E)$ .**Proposition 2.2.3.**

$$x_0 \in \partial E \Leftrightarrow \text{For any } r > 0, B_X(x_0, r) \cap E \neq \emptyset \text{ and } B_X(x_0, r) \cap (X \setminus E) \neq \emptyset$$

**Example 2.2.2.** Let  $(\mathbb{R}, d)$  be the usual metric on  $\mathbb{R}$ , where

$$d(x, y) = |x - y|.$$

Then, we know in this space,

$$\begin{aligned} B_{\mathbb{R}}(x_0, r) &= \{x \in \mathbb{R} \mid d(x, x_0) < r\} \\ &= \{x \in \mathbb{R} \mid |x - x_0| < r\} \\ &= \{x \in \mathbb{R} \mid -r + x_0 < x < r + x_0\}. \end{aligned}$$

Hence, suppose  $E = [1, 2)$ , then  $\text{Int}(E) = (1, 2)$  since we know  $B(x_0, r) = (x_0 - r, x_0 + r)$ , so for all  $x \in (1, 2)$ , we know there is an open ball  $B(x_0, r) \subseteq [1, 2)$  for some  $r > 0$ . Also, consider the endpoint 1, 2, we can verify that these two points are not interior points. Besides, consider the points not in  $[1, 2]$ , it is trivial that they cannot be interior points.

**Example 2.2.3.** We consider  $(X, d_{\text{disc}})$ . Let  $E \subseteq X$ . If  $x \in E$ , we know

$$B\left(x, \frac{1}{2}\right) = \left\{y \mid d(y, x) < \frac{1}{2}\right\} = \{x\} \subseteq E.$$

Hence,  $E \subseteq \text{Int}(E)$ . Besides, for all  $x \in \text{Int}(E)$ , we know there exists  $r > 0$  s.t.  $B(x_0, r) \subseteq E$ , also we know  $x_0 \in B(x_0, r) \subseteq E$ , so  $x_0 \in E$ , and thus  $\text{Int}(E) \subseteq E$ . Hence,  $E = \text{Int}(E)$ . Similarly,  $\text{Int}(X \setminus E) = X \setminus E$ . Suppose there is a  $x \in X$  s.t.  $x \in \partial E$ , then  $x \notin \text{Int}(E) = E$  and  $x \notin \text{Ext}(E) = \text{Int}(X \setminus E) = X \setminus E$ , so such  $x$  does not exist.

**Definition 2.2.8 (Closure).** Let  $(X, d)$  be a metric space, and let  $E \subseteq X$  and  $x_0 \in X$ . We say  $x_0$  is an adherent point of  $E$  if for every  $r > 0$ ,  $B(x_0, r) \cap E \neq \emptyset$ . The set of adherent points is called the closure of  $E$ , and denoted by  $\overline{E}$ .

**Proposition 2.2.4 (TFAE).**(a)  $x_0$  is an adherent point of  $E$ .

- (b)  $x_0$  is either an interior point or a boundary point of  $E$ .  
 (c)  $\exists$  a sequence  $\{X^{(n)}\}_{n=1}^{\infty}$  in  $E$  which converges to  $x_0$  in  $(X, d)$ .

**proof from (a) to (b).** Suppose  $x_0 \in \overline{E}$ , then  $B(x_0, r) \cap E \neq \emptyset$  for all  $r > 0$ . If  $\exists s > 0$  s.t.  $B(x_0, s) \subseteq E$ , then  $x_0 \in \text{Int}(E)$ . If such  $s$  does not exist, then we know

$$B(x_0, r) \cap E \neq \emptyset \text{ and } B(x_0, r) \cap (X \setminus E) \neq \emptyset \text{ for all } r > 0,$$

so we can use [Proposition 2.2.1](#) to conclude that  $x_0$  must be a boundary point. ■

**proof from (b) to (c).** Since either  $x_0 \in \text{Int}(E)$  or  $x_0 \in \partial E$ . If  $x_0 \in \text{Int}(E)$ , then  $x_0 \in E$ , then we can choose  $X^{(n)} = x_0$  for all  $n \geq 1$ . If  $x_0 \in \partial E$ , then given  $n \in \mathbb{N}$ ,  $\exists x_n \in B(x_0, \frac{1}{n}) \cap E \neq \emptyset$ . Hence,  $x_n \in E$  and  $d(x_n, x_0) < \frac{1}{n}$ . Pick such  $x_n$  to form  $\{X^{(n)}\}_{n=1}^{\infty}$ , then we know this sequence converges to  $x_0$ . ■

**proof from (c) to (a).** Suppose  $\{X^{(n)}\} \subseteq E$  s.t.  $\lim_{n \rightarrow \infty} d(X^{(n)}, x_0) = 0$ , then we want to show  $x_0 \in \overline{E}$ . Given any  $r > 0$ , choose  $N \geq 1$  s.t.

$$d(X^{(n)}, x_0) < r \text{ when } n \geq N.$$

This implies for  $n \geq N$ ,  $X^{(n)} \in E$  and  $X^{(n)} \in B(x_0, r)$ , so we know  $E \cap B(x_0, r) \neq \emptyset$  for all  $r > 0$ , which means  $x_0 \in \overline{E}$ . ■

**Remark 2.2.3.** The equation (a) and (b) implies  $\overline{E} = \text{Int}(E) \cup \partial E$ .

**An alternative proof.** Since we know  $X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$  by [Theorem 2.2.2](#), and  $\overline{E} \subseteq X$ , so

$$\begin{aligned} \overline{E} &= \overline{E} \cap X = \overline{E} \cap (\text{Int}(E) \cup \text{ext}(E) \cup \partial E) \\ &= (\overline{E} \cap \text{Int}(E)) \cup (\overline{E} \cap \text{Ext}(E)) \cup (\overline{E} \cap \partial E). \end{aligned}$$

Also, notice that

$$\overline{E} \cap \text{Int}(E) = \text{Int}(E) \quad \overline{E} \cap \text{Ext}(E) = \emptyset \quad \overline{E} \cap \partial E = \partial E,$$

so  $\overline{E} = \text{Int}(E) \cup \partial E$ . ■

**Corollary 2.2.1.**  $\overline{E} = \text{Int}(E) \cup \partial E$ .

**Theorem 2.2.2.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . Then,

$$X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$$

**Remark 2.2.4.**  $\partial E$  could be empty. (See previous example.)

**Corollary 2.2.2.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . Then

$$\overline{E} = \text{Int}(E) \cup \partial E = X \setminus \text{Ext}(E).$$

**Lemma 2.2.1.**  $\overline{E} = E \cup \partial E$

**Proof.** We first show that  $E \cup \partial E \subseteq \overline{E}$ . For every point  $x \in E$ , we know  $x \in B(x, r)$  for all  $r > 0$ , so  $B(x, r) \cap E \neq \emptyset$ . Also, by definition, we know  $\partial E \subseteq \overline{E}$ , so we're done.

Next, we show that  $\overline{E} \subseteq E \cup \partial E$ . For every  $x \in \overline{E}$ , if  $x \in E$ , then  $x \in E \cup \partial E$ . If not, since  $x \in \overline{E}$ , so  $B(x, r) \cap E \neq \emptyset$  for all  $r > 0$ . Also, since  $x \notin E$ , and  $x \in B(x, r)$ , so  $B(x, r) \cap (X \setminus E) \neq \emptyset$ ,

otherwise  $x \in B(x, r) \subseteq E$ , which is a contradiction. Now we know for every  $r > 0$ ,  $B(x, r) \cap E \neq \emptyset$  and  $B(x, r) \cap (X \setminus E) \neq \emptyset$ , so  $x \in \partial E$ . ■

**Lemma 2.2.2 (Discarded).** If  $x \in \text{Int}(E)$ , then  $x \in E$ . In other words,  $\text{Int}(E) \subseteq E$ .

**Proof.** If  $x \in \text{Int}(E)$ , then there exists  $r > 0$  s.t.  $B(x, r) \subseteq E$ , and thus  $x \in B(x, r) \subseteq E$ , which means  $x \in E$ . ■

**Note 2.2.1.** I thought we need [Lemma 2.2.2](#) to prove [Theorem 2.2.3](#), but I found it needless. Nevertheless, I still want to keep it since I think it is useful in some elsewhere.

**Definition 2.2.9.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . We say  $E$  is closed if  $\partial E \subseteq E$ . We say  $E$  is open if it doesn't contain any boundary points i.e.  $\partial E \cap E = \emptyset$ .

**Theorem 2.2.3.**  $E$  is closed if and only if  $\overline{E} = E$ .

**Proof.**

$$\begin{aligned} E \text{ is closed} &\Rightarrow \partial E \subseteq E \Rightarrow \overline{E} = E \cup \partial E = E. \\ E = \overline{E} = E \cup \partial E &\Rightarrow \partial E \subseteq E \Rightarrow E \text{ is closed.} \end{aligned}$$

■

**Theorem 2.2.4.**  $E$  is open.  $\Leftrightarrow \text{Int}(E) = E$ .

**proof of  $(\Rightarrow)$ .**  $E$  is open means  $\partial E \cap E = \emptyset$ . Fix  $x \in E$ , since  $x \notin \partial E$ , so  $\exists r > 0$  s.t.  $B(x, r) \cap E = \emptyset$  or  $B(x, r) \cap (X \setminus E) = \emptyset$ . Since  $x \in E$  and  $x \in B(x, r)$ , so  $B(x, r) \cap (X \setminus E) = \emptyset$ , which means  $B(x, r) \subseteq E$ , so  $x \in \text{Int}(E)$ . Now we know  $E \subseteq \text{Int}(E)$ . Also, we know  $\text{Int}(E) \subseteq E$  by [Lemma 2.2.2](#). Hence,  $\text{Int}(E) = E$ . ■

**proof of  $(\Leftarrow)$ .** If  $\text{Int}(E) = E$ , then given any  $x \in E = \text{Int}(E)$ , there exists  $r > 0$  s.t.  $B(x, r) \subseteq E$ . Hence,  $B(x, r) \cap (X \setminus E) = \emptyset$ , so  $x \notin \partial E$ , and thus  $E \cap \partial E = \emptyset$ . ■

**Theorem 2.2.5.** If  $E \subseteq X$ , then  $E$  is open  $\Leftrightarrow X \setminus E$  is closed.

**proof of  $(\Rightarrow)$ .** Since we can write  $X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$ , and  $E$  is open, so

$$X \setminus E = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus E = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus \text{Int}(E) = \text{Ext}(E) \cup \partial E.$$

by [Theorem 2.2.4](#). Now we want to show that  $\partial(X \setminus E) \subseteq X \setminus E$ , and we know

$$X \setminus E = \text{Ext}(E) \cup \partial E = \text{Ext}(E) \cup \partial(X \setminus E)$$

since  $\partial E = \partial(X \setminus E)$ . Hence, we have  $\partial(X \setminus E) \subseteq X \setminus E$ . ■

**proof of  $(\Leftarrow)$ .** Suppose  $X \setminus E$  is closed, then  $\partial(X \setminus E) \subseteq X \setminus E$ , and since  $\partial E = \partial(X \setminus E)$ , so  $\partial E \subseteq X \setminus E$ , and thus  $\partial E \cap E = \emptyset$ , which means  $E$  is open. ■

## Lecture 5

**Definition 2.2.10.** Let  $(X, d)$  be a metric space,  $E \subseteq X$  and  $x_0 \in E$ . We say  $x_0$  is an adherent point if for every  $r > 0$ ,  $B(x_0, r) \cap E \neq \emptyset$ , and we denote  $\overline{E}$  to the set of all adherent points.

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**Remark 2.2.5.**  $E \subseteq \overline{E}$ , since given any  $x_0 \in E$  and  $r > 0$ ,  $x_0 \in B(x_0, r)$ , so  $B(x_0, r) \cap E \neq \emptyset$ , and thus  $E \subseteq \overline{E}$ .

**Remark 2.2.6.**  $\partial E \subseteq \overline{E}$ . Given  $x_0 \in \partial E$ , we know for any  $r > 0$ ,  $B(x_0, r) \cap E \neq \emptyset$ , so  $x_0 \in \overline{E}$ .

**Proposition 2.2.5.**  $x_0 \in \overline{E}$  if and only if there exists  $(X^{(n)})_{n=1}^{\infty} \subseteq E$  s.t.  $\lim_{n \rightarrow \infty} X^{(n)}$  exists and  $\lim_{n \rightarrow \infty} X^{(n)} = x_0$ .

**proof of  $(\Rightarrow)$ .** Given  $n \in \mathbb{N}$ . Consider  $B(x_0, \frac{1}{n})$ . We know  $B(x_0, \frac{1}{n}) \cap E \neq \emptyset$ . Choose  $X^{(n)} \in B(x_0, \frac{1}{n}) \cap E$ , then  $d(x_0, X^{(n)}) < \frac{1}{n}$ , which means  $\lim_{n \rightarrow \infty} d(x_0, X^{(n)}) = 0$ . Hence, there exists  $(X^{(n)}) \subseteq E$  s.t.  $\lim_{n \rightarrow \infty} X^{(n)} = x_0$ . ■

**proof of  $(\Leftarrow)$ .** There exists  $N$  s.t.  $X^{(n)} \in B(x_0, r)$  when  $n \geq N$ . Given any  $r > 0$ , since  $\lim_{n \rightarrow \infty} X^{(n)} = x_0$ , so  $\lim_{n \rightarrow \infty} d(X^{(n)}, x_0) = 0$ . Hence, there exists  $N$  s.t.  $d(X^{(n)}, x_0) < r$  when  $n \geq N$ . Hence, when  $n \geq N$ , we have  $X^{(n)} \subseteq B(x_0, r)$ . Since we know  $X^{(n)} \in E$  for all  $n$ , so we know  $B(x_0, r) \cap E \neq \emptyset$ , so  $x_0 \in \overline{E}$ . ■

**Proposition 2.2.6.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ , then

$$X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E.$$

**Corollary 2.2.3.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . Then,

$$\overline{E} = \text{Int}(E) \cup \partial E = X \setminus \text{Ext}(E) = E \cup \partial E.$$

**Proof.** Since

$$\begin{aligned} \overline{E} &= \overline{E} \cap X = \overline{E} \cap (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \\ &= (\overline{E} \cap \text{Int}(E)) \cup (\overline{E} \cap \text{Ext}(E)) \cup (\overline{E} \cap \partial E) = \text{Int}(E) \cup \partial E. \end{aligned}$$

Also,

$$X \setminus \text{Ext}(E) = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus \text{Ext}(E) = \text{Int}(E) \cup \partial E = \overline{E}.$$

Besides, we know  $\text{Int}(E) \subseteq E \subseteq \overline{E}$ , so

$$\overline{E} = \text{Int}(E) \cup \partial E \subseteq E \cup \partial E.$$

Also, by Remark 2.2.5 and Remark 2.2.6, we know  $E \cup \partial E \subseteq \overline{E}$ , so we know  $\overline{E} = E \cup \partial E$ . ■

**Definition 2.2.11.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . We say  $E$  is open iff  $\partial E \cap E = \emptyset$ . We say  $E$  is closed iff  $\partial E \subseteq E$ .

**Proposition 2.2.7.**

$$E \text{ is open} \Leftrightarrow \text{Int}(E) = E \Leftrightarrow X \setminus E \text{ is closed.}$$

**proof of  $E \text{ is open} \Leftrightarrow \text{Int}(E) = E$ .**

$(\Rightarrow)$  Since  $E$  is open, so  $\partial E \cap E = \emptyset$ . Hence,

$$\begin{aligned} E &= E \cap X = E \cap (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \\ &= (E \cap \text{Int}(E)) \cup (E \cap \text{Ext}(E)) \cup (E \cap \partial E) = \text{Int}(E) \cup (E \cap \partial E) = \text{Int}(E) \end{aligned}$$

since  $E \cap \text{Ext}(E) = \emptyset$  and we know  $\partial E \cap E = \emptyset$ .

$(\Leftarrow)$  Since  $\text{Int}(E) = E$ , and  $\text{Int}(E) \cap \partial E = \emptyset$ , so  $E \cap \partial E = \emptyset$ , and thus  $E$  is open.

**proof of  $E$  is open  $\Leftrightarrow X \setminus E$  is closed.**

( $\Rightarrow$ )  $X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$ , so

$$X \setminus E = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus \text{Int}(E) = \text{Ext}(E) \cup \partial E = \text{Int}(X \setminus E) \cup \partial(X \setminus E).$$

Hence,  $\partial(X \setminus E) \subseteq X \setminus E$ , which means  $X \setminus E$  is closed.

( $\Leftarrow$ )  $X \setminus E$  is closed, then  $\partial(X \setminus E) \subseteq X \setminus E$ , but  $\partial E = \partial(X \setminus E)$ , so  $\partial E \subseteq X \setminus E$ , and thus  $\partial E \cap E = \emptyset$ .

**Remark 2.2.7.** If  $\partial E = \emptyset$ , then  $E$  is open and closed.

**Definition 2.2.12 (Clopen).** If a set  $S$  is closed and open, then  $S$  is clopen.

**Remark 2.2.8.** Let  $(X, d)$  be a metric space, then  $\emptyset$  is clopen, and we can deduce that  $X$  is also clopen since  $X$  is the complement of  $\emptyset$  and we know  $S$  is open iff  $X \setminus S$  is closed.

**Remark 2.2.9.** In  $(\mathbb{R}, d)$ , where  $d$  is the standard metric, then the only clopen set is  $\mathbb{R}$  or  $\emptyset$ .

**Remark 2.2.10.** Let  $(X, d_{\text{disc}})$  be the discrete metric space on  $X$ . Let  $E$  be any set, then  $E$  is open and closed. Given  $x_0 \in E$ , we know  $B_{\text{disc}}(x_0, \frac{1}{2}) \subseteq E$ , so  $x_0 \in \text{Int}(E)$ , which means  $E = \text{Int}(E)$ , so  $E$  is open. Now since  $X \setminus E$  is also open, so  $E$  is closed. Thus,  $E$  is clopen.

**Proposition 2.2.8.** The following hold:

- (a)  $E$  is open iff  $E = \text{Int}(E)$ .
- (b)  $E$  is closed iff for every convergent sequence  $(X^{(n)})_{n=1}^{\infty}$  in  $E$ , then the limit  $\lim_{n \rightarrow \infty} X^{(n)} \in E$ .
- (c) Let  $r > 0$ , then
  - (i)  $\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$  is closed.
  - (ii)  $B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$  is open.
- (d) Any singleton  $\{x_0\}$  where  $x_0 \in X$  is closed.
- (e)  $E$  is open iff  $X \setminus E$  is closed.
- (f)
  - (i) If  $E_1, \dots, E_n$  are open sets in  $X$ , then  $E_1 \cap E_2 \cap \dots \cap E_n$  is open.
  - (ii) If  $F_1, \dots, F_n$  are closed, then  $F_1 \cup \dots \cup F_n$  is closed.
- (g)
  - (i) If  $\{E_\alpha\}_{\alpha \in I}$  is any collection of open sets in  $X$ , then  $\bigcup_{\alpha \in I} E_\alpha$  is open.
  - (ii) If  $\{F_\alpha\}_{\alpha \in I}$  is any collection of closed sets in  $X$ , then  $\bigcap_{\alpha \in I} F_\alpha$  is closed.
- (h)
  - (i) If  $E \subseteq X$ , then  $\text{Int}(E)$  is the largest open set that contained in  $E$  i.e.  $\text{Int}(E)$  is open and if  $V \subseteq E$  and  $V$  is open, then  $V \subseteq \text{Int}(E)$ .
  - (ii) If  $E \subseteq X$ , then  $\overline{E}$  is the smallest closed set containing  $E$  i.e.  $\overline{E}$  is closed and if  $E \subseteq K$  and  $K$  is closed, then  $\overline{E} \subseteq K$ .

**proof of (b).**

( $\Rightarrow$ ) Since  $E$  is closed, so  $\overline{E} = E$ , and we know every convergent sequence  $(X^{(n)})_{n=1}^{\infty}$  converges to  $x_0$  with  $x_0 \in \overline{E}$  by [Proposition 2.2.4](#). Thus, we have  $x_0 \in E$ .

( $\Leftarrow$ ) Assume that every convergent sequence in  $E$  has its limit in  $E$ . We want to prove that  $E$  is closed, i.e. that  $X \setminus E$  is open.

Take any point  $y \in X \setminus E$ . Suppose, for contradiction, that every ball around  $y$  meets  $E$ . That is, for each  $k \in \mathbb{N}$  there exists a point

$$x^{(k)} \in E \cap B(y, \frac{1}{k}).$$

Then, by construction, we have  $x^{(k)} \rightarrow y$ .

By our assumption, the limit of any convergent sequence from  $E$  must lie in  $E$ . Hence  $y \in E$ , contradicting the fact that  $y \in X \setminus E$ .

Therefore, there must exist some radius  $r > 0$  such that

$$B(y, r) \cap E = \emptyset,$$

which means  $B(y, r) \subseteq X \setminus E$ . Thus every point of  $X \setminus E$  is an interior point, so  $X \setminus E$  is open. Hence  $E$  is closed. ■

**proof of (c).**

(i) To show that  $\overline{B}(x_0, r)$  is closed, it suffices to show that  $X \setminus \overline{B}(x_0, r)$  is open. Note that

$$X \setminus \overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) > r\}.$$

Let  $y \in X \setminus \overline{B}(x_0, r)$ , then define  $\varepsilon = d(x_0, y) - r > 0$ , then we can similarly prove that  $B(y, \varepsilon) \subseteq X \setminus \overline{B}(x_0, r)$ . Hence,  $X \setminus \overline{B}(x_0, r) = \text{Int}(X \setminus \overline{B}(x_0, r))$ , and thus it is open.

(ii) If  $y \in B(x_0, r)$ , then  $d(x_0, y) < r$ . Let  $\varepsilon = r - d(x_0, y) > 0$ , then we claim that  $B(y, \varepsilon) \subseteq B(x_0, r)$ . Given  $z \in B(y, \varepsilon)$ , then  $d(z, y) < \varepsilon$ , then use triangle inequality we know  $z \in B(x_0, r)$ . ■

**proof of (d).** It suffices to show that  $X \setminus \{x_0\}$  is open. Given  $y \in X \setminus \{x_0\}$ , so we can show that

$$B\left(y, \frac{d(y, x_0)}{2}\right) \subseteq X \setminus \{x_0\}.$$

Hence,  $y \in \text{Int}(X \setminus \{x_0\})$ , and thus  $X \setminus \{x_0\}$  is open. ■

**proof of (f).**

(i) Given  $x_0 \in E_1 \cap E_2 \cap \cdots \cap E_n$ , then  $x_0 \in E_i$  for all  $1 \leq i \leq n$ . Thus, there exists  $r_i > 0$  s.t.

$$B(x_0, r_i) \subseteq E_i \quad \text{for each } 1 \leq i \leq n.$$

Let  $r = \min\{r_1, \dots, r_n\} > 0$ , then we know  $B(x_0, r) \subseteq B(x_0, r_i) \subseteq E_i$  for all  $1 \leq i \leq n$ . Hence,  $B(x_0, r) \subseteq E_1 \cap E_2 \cap \cdots \cap E_n$ , and thus  $E_1 \cap \cdots \cap E_n$  is open.

(ii) Now if  $F_1, \dots, F_n$  are closed, then  $X \setminus F_1, \dots, X \setminus F_n$  are open. Since we know  $\bigcap_{i=1}^n (X \setminus F_i)$  is open, and

$$\bigcap_{i=1}^n (X \setminus F_i) = X \setminus \left( \bigcup_{i=1}^n F_i \right),$$

so  $X \setminus (\bigcup_{i=1}^n F_i)$  is open, which means  $\bigcup_{i=1}^n F_i$  is closed. ■

**proof of (g).**

- (i) Suppose  $x_0 \in \bigcup_{\alpha \in I} E_\alpha$ , then there exists  $\mathcal{B} \in I$  s.t.  $x_0 \in E_{\mathcal{B}}$ . Now since  $E_{\mathcal{B}}$  is open, so there exists  $r_{x_0} > 0$  s.t.

$$B(x_0, r_{x_0}) \subseteq E_{\mathcal{B}} \subseteq \bigcup_{i \in \alpha} E_\alpha.$$

Hence,  $\bigcup_{\alpha \in I} E_\alpha$  is open.

- (ii)

$$\left( X \setminus \left( \bigcap_{\alpha \in I} F_\alpha \right) \right) = \bigcup_{\alpha \in I} (X \setminus F_\alpha)$$

is open since  $X \setminus F_\alpha$  is open for all  $\alpha \in I$ , so we have  $\bigcap_{\alpha \in I} F_\alpha$  is closed.

**Remark 2.2.11.**

- (1)  $\bigcap_{\alpha \in I} E_\alpha$  may NOT be open. For example,

$$\bigcap_{i=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = \{0\},$$

which is closed.

- (2)  $\bigcup_{\alpha \in I} F_\alpha$  may NOT be closed. For example,

$$\bigcup_{i=1}^{\infty} \left[ -1 + \frac{1}{n}, 1 - \frac{1}{n} \right] = (-1, 1),$$

which is open.

**Note 2.2.2.** In the proof of (f), if the index set  $I$  is infinite, then we can not pick  $\min \{r_1, \dots, r_n\}$ , so we can not deduce that (f) is correct when there infinitely many open sets or closed sets. ■

**proof of (h).**

- (i) We first claim that  $\text{Int}(E)$  is open.

**Proof.** Since for all  $x \in \text{Int}(E)$ ,  $\exists r_x > 0$  s.t.  $B(x, r_x) \subseteq E$ , so

$$\text{Int}(E) = \bigcup_{x \in \text{Int}(E)} B(x, r_x),$$

and by (ii) of (c) and (i) of (g) in [Proposition 2.2.8](#), we know  $\text{Int}(E)$  is open. ■

Now if we have  $V \subseteq E$  and  $V$  is open, then  $y \in V$  implies there exists  $s > 0$  s.t.  $B(y, s) \subseteq V$ , and thus  $B(y, s) \subseteq E$  since  $V \subseteq E$ . Hence, we know  $y \in \text{Int}(E)$ , and thus  $V \subseteq \text{Int}(E)$ .

- (ii) To show  $\overline{E}$  is closed, it suffices to show that  $X \setminus \overline{E}$  is open. Note that

$$\overline{E} = X \setminus \text{Ext}(E) = X \setminus \underbrace{\text{Int}(X \setminus E)}_{\text{open}},$$

so  $\overline{E}$  is closed. Now if  $E \subseteq K$  and  $K$  is closed, then if  $x \in \overline{E}$ , we have  $B(x, r) \cap E \neq \emptyset$  for all  $r > 0$ . Hence,  $B(x, r) \cap K \neq \emptyset$  since  $E \subseteq K$ , so  $x \in \overline{K} = K$  (since  $K$  is closed). Thus,  $\overline{E} \subseteq K$ . ■

## Lecture 6

### 2.3 Relative topology

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Let  $(X, d)$  be a metric space and  $Y \subseteq X$ , then  $(Y, d|_{Y \times Y})$  is also a metric space.

**Example 2.3.1.** Consider  $(\mathbb{R}^2, d_2)$  and  $X = \{(x, 0) \mid x \in \mathbb{R}\}$ , then on  $(X, d_2|_{X \times X}) = (X, d)$ , it is also a metric space.

**Proof.** Since

$$d((x, 0), (y, 0)) = \sqrt{(x - y)^2 + 0^2} = |x - y|,$$

so it is obvious that  $d$  is a metric.

Note that  $X$  is not open in  $\mathbb{R}^2$ . Also, if  $E = \{(x, 0) \mid -1 < x < 1\}$ , then  $E$  is not open in  $\mathbb{R}^2$ , but  $E$  is open in  $(X, d_2|_{X \times X})$ . \*

**Example 2.3.2.** Suppose  $X = (-1, 1) \subseteq \mathbb{R}$ , then  $(X, d|_{X \times X})$  is a metric space. Consider  $E = [0, 1)$ , then we know  $E$  is not closed in  $(\mathbb{R}, d)$  since  $1 \in \overline{E}$  but  $1 \notin E$ . But  $E$  is closed in  $(X, d|_{X \times X})$  since  $\overline{E} = E$  in  $(X, d|_{X \times X})$ .

**Definition 2.3.1 (relatively open/close).** Let  $(X, d)$  be a metric space and  $Y \subseteq X$ . We say  $E$  is relatively open (resp. closed) in  $Y$  if  $E$  is open (resp. closed) in  $(Y, d|_{Y \times Y})$ .

**Note 2.3.1.** In the following context, if we say  $E$  is open in  $Y$ , then we mean  $E$  is "relatively" open, and if we say  $E$  is closed in  $Y$ , then we mean  $E$  is relatively closed in  $Y$ .

**Note 2.3.2.** If  $E$  is open/closed in  $Y$ , then  $E \subseteq Y$ . Otherwise, we cannot define  $d|_{Y \times Y}(a, b)$  for  $a, b \in E \setminus Y$ .

**Remark 2.3.1.** If  $Y \subseteq X$ , and  $(X, d), (Y, d|_{Y \times Y})$  are both metric spaces, then

$$B_Y(x, r) = \{y \in Y \mid d(y, x) < r\} = B_X(x, r) \cap Y.$$

**Remark 2.3.2.** If  $E$  is relatively open in  $Y$ , then given  $x_0 \in E$ ,  $\exists r_0 > 0$  s.t.  $B_X(x_0, r_0) \cap Y \subseteq E$ . This is because by [Remark 2.3.1](#), we have

$$B_X(x_0, r_0) \cap Y = B_Y(x_0, r_0) \subseteq E.$$

**Remark 2.3.3.** A set  $E \subseteq Y$  is relatively closed in  $Y$  if given any  $r > 0$  and  $x_0 \in Y$ ,

$$B_Y(x_0, r) \cap E \neq \emptyset,$$

then  $x_0 \in E$ . This is because "closed" gives  $E = \overline{E}_Y$ . Note that this statement is equivalent to

$$\text{If } x_0 \in \overline{E}_Y, \text{ then } x_0 \in E = E_Y.$$

**Proposition 2.3.1.** Let  $(X, d)$  be a metric space, and  $Y \subseteq X$  and  $E \subseteq Y$ , then

- (1)  $E$  is relatively open in  $Y$  iff  $\exists$  open set  $V$  in  $(X, d)$  s.t.  $E = V \cap Y$ .
- (2)  $E$  is relatively closed in  $Y$  iff  $\exists$  closed set  $K$  in  $(X, d)$  s.t.  $E = K \cap Y$ .

**proof of (1).**

- ( $\Rightarrow$ ) Given any  $x \in E$ ,  $\exists r_x > 0$  s.t.  $B_X(x, r_x) \cap Y \subseteq E$ . Let  $V = \bigcup_{x \in E} B_X(x, r_x)$ . Obviously,  $V \cap Y = E$  and  $V$  is open.
- ( $\Leftarrow$ ) Suppose  $E = V \cap Y$ , then given any  $x \in E$ , since  $V$  is open, so there exists  $r > 0$  s.t.  $B_X(x, r) \subseteq V$ , and then  $B_X(x, r) \cap Y \subseteq V \cap Y = E$ . Since  $x$  is an interior point of  $E$  in  $Y$ , so  $\text{Int}_Y(E) = E$ , and thus  $E$  is open in  $Y$ . ■

**proof of (2).**

- ( $\Rightarrow$ )  $E$  is relatively closed in  $Y$ , then  $Y \setminus E$  is relatively open, so there exists  $V$  open in  $X$  s.t.  $Y \setminus E = V \cap Y$ . Hence,

$$\begin{aligned} E &= Y \setminus (Y \setminus E) = (X \setminus (Y \setminus E)) \cap Y = (X \setminus (V \cap Y)) \cap Y \\ &= ((X \setminus V) \cup (X \setminus Y)) \cap Y \\ &= ((X \setminus V) \cap Y) \cup ((X \setminus Y) \cap Y) \\ &= (X \setminus V) \cap Y \end{aligned}$$

Let  $E = (X \setminus V) \cap Y = K \cap Y$ , then since  $K = X \setminus V$  is closed in  $X$ , so we're done.

- ( $\Leftarrow$ ) Suppose  $E = K \cap Y$  for some closed  $K$ , then  $Y \setminus E = (X \setminus K) \cap Y$ , which means  $Y \setminus E$  is relatively open in  $Y$  since  $X \setminus K$  is open and by (a), so  $E$  is closed in  $Y$ . ■

**Example 2.3.3.** Let  $X = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$  with the standard metric  $d(x, y) = |x - y|$  with  $x, y \in X$ , then

- (i)  $[0, 1]$  is open and closed in  $X$ .
- (ii)  $\partial_X[0, 1] = \emptyset$ .

**Proof.**

- (i) We want to find  $V$  open in  $\mathbb{R}$  s.t.

$$[0, 1] = V \cap \overbrace{([0, 1] \cup [2, 3])}^X,$$

we can choose  $V = (-\frac{1}{2}, \frac{3}{2})$ , so  $[0, 1]$  is open in  $X$ .

We want to find  $K$  closed in  $\mathbb{R}$  and

$$[0, 1] = K \cap ([0, 1] \cup [2, 3]),$$

and we can choose  $K = [-\frac{1}{2}, \frac{3}{2}]$ , so  $[0, 1]$  is closed in  $X$ .

- (ii) If  $x \in \partial_X[0, 1]$ , then  $B_X(x, r) \cap [0, 1]$  and  $B_X(x, r) \cap [2, 3]$  are both nonempty for any  $r > 0$ . However, this is impossible for any  $x$  in  $X$ , so  $\partial_X[0, 1] = \emptyset$ . ⊛

## 2.4 Cauchy sequence and complete metric space

**Definition 2.4.1 (subsequence).** Suppose  $(X^{(n)})_{n=m}^{\infty}$  is a sequence in  $(X, d)$ . Suppose  $m \leq n_1 < n_2 < \dots$ , then  $(X^{(n_j)})_{j=1}^{\infty}$  is called a subsequence of  $(X^{(n)})_{n=m}^{\infty}$ .

**Example 2.4.1.**  $X^{(n)} = (-1)^n$  for all  $n \in \mathbb{N}$ .

**Proof.**

$$\{X^{(2n)}\}_{n=1}^{\infty}$$

is a subsequence of  $\{X^{(n)}\}_{n=1}^{\infty}$ . ⊛

**Lemma 2.4.1.** Let  $\{X^{(n)}\}_{n=m}^{\infty}$  be a convergent sequence with  $\lim_{n \rightarrow \infty} X^{(n)} = x$ , then every subsequence of  $\{X^{(n)}\}_{n=m}^{\infty}$  also converges to  $x_0$ .

**Definition 2.4.2 (limit points).** Suppose  $(X^{(n)})_{n=m}^{\infty}$  is a sequence in  $(X, d)$ , then we say  $L$  is a limit point of  $(X^{(n)})_{n=m}^{\infty}$  if for every  $N \geq m$  and every  $\varepsilon > 0$ , there exists  $n \geq N$  s.t.  $d(X^{(n)}, L) \leq \varepsilon$ .

**Proposition 2.4.1.**  $L$  is a limit point of  $(X^{(n)})_{n=m}^{\infty}$  iff there exists a subsequence

$$(X^{(n_j)})_{j=1}^{\infty}$$

converges to  $L$ .

**Proof.**

( $\Rightarrow$ ) Assume  $L$  is a limit point, now we build a subsequence converges to  $L$  by an inductive method. Our goal is to build a subsequence  $\{X^{(n_j)}\}_{j=1}^{\infty}$  so that

$$d(X^{(n_j)}, L) < \frac{1}{j} \quad \forall 1 \leq j.$$

For  $j = 1$ , pick  $N = m$ , and pick  $\varepsilon < \frac{1}{1}$  to pick  $n_1 \geq N$  s.t.

$$d(X^{(n_1)}, L) \leq \varepsilon < \frac{1}{1}.$$

Now suppose  $n_1, n_2, \dots, n_{k-1}$  are all chosen, then now we can pick  $N = n_{k-1} + 1$  and  $\varepsilon < \frac{1}{k}$ , so that we can pick  $n_k \geq N$  s.t.  $d(X^{(n_k)}, L) \leq \varepsilon < \frac{1}{k}$ , so we're done. Now we show that this subsequence converges to  $L$ . For every  $\varepsilon > 0$ , we know there exists  $0 < \frac{1}{k} < \varepsilon$ , so for all  $K \geq k$ , we have

$$d(X^{(K)}, L) < \frac{1}{K} \leq \frac{1}{k} < \varepsilon,$$

so we're done.

( $\Leftarrow$ ) Left as exercise to the reader. ■

**Proposition 2.4.2.**  $L$  is a limit point iff  $L \in \bigcap_{N=1}^{\infty} \overline{S_N}$  where  $S_N = \{X^{(K)}\}_{K \geq N}$ .

**Definition 2.4.3 (Cauchy sequence).** Let  $(X^{(n)})_{n=m}^{\infty}$  be a sequence in  $(X, d)$ . We say this sequence is a Cauchy sequence if for every  $\varepsilon > 0$ , there exists  $N \geq m$  s.t.  $d(X^{(j)}, X^{(k)}) < \varepsilon$  for all  $j, k \geq N$ .

**Lemma 2.4.2.** Suppose  $(X^{(n)})_{n=m}^{\infty}$  converges in  $(X, d)$ , then  $(X^{(n)})_{n=m}^{\infty}$  is a Cauchy sequence in  $(X, d)$ .

**Proof.** Suppose  $\lim_{n \rightarrow \infty} X^{(n)} = X_0$ , then for every  $\frac{\varepsilon}{2} > 0$ , there exists  $N \geq m$  s.t.  $d(X^{(n)}, X_0) < \frac{\varepsilon}{2}$  for all  $n \geq N$ . If  $j, k \geq N$ , then

$$d(X^{(j)}, X^{(k)}) \leq d(X^{(j)}, X_0) + d(X^{(k)}, X_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

**Example 2.4.2.** A sequence in  $\mathbb{Q}$  may not converges in  $\mathbb{Q}$ .

**Proof.** See teacher's note.

⊛

**Definition 2.4.4 (Complete space).** A metric space  $(X, d)$  is complete iff every Cauchy sequence converges to some points in  $X$ .

**Remark 2.4.1.**  $\mathbb{Q} \subseteq \mathbb{R}$ , and  $(\mathbb{Q}, d)$  is not complete.

**Remark 2.4.2.** The limit of a convergent sequence in metric space is unique. If

$$\lim_{n \rightarrow \infty} x^{(n)} = y \quad \text{and} \quad \lim_{n \rightarrow \infty} x^{(n)} = z,$$

then suppose by contradiction,  $y \neq z$ . Then,

$$0 \leq d(y, z) \leq d(y, x^{(n)}) + d(x^{(n)}, z).$$

By squeeze theorem, we know  $d(y, z) = 0$  and thus  $y = z$ .

**Proposition 2.4.3.** Let  $(X, d)$  be a metric space and let  $(Y, d|_{Y \times Y})$  be a subspace of  $(X, d)$ . If  $(Y, d|_{Y \times Y})$  is complete, then  $Y$  is closed in  $X$ .

**Proof.** We want to show that  $Y = \overline{Y}$ , so we want to show for all  $y \in \overline{Y}$ , we have  $y \in Y$ . Now for every  $y \in \overline{Y}$ , then by [Proposition 2.2.4](#), we know there exists a convergent sequence  $\{Y^{(n)}\}_{n=1}^{\infty}$  in  $Y$  and converges to  $y$ . However, every convergent sequence is Cauchy, and since  $(Y, d|_{Y \times Y})$  is complete, so  $\{Y^{(n)}\}_{n=1}^{\infty}$  converges in  $Y$ , which means  $y \in Y$ , and we're done. ■

**Proposition 2.4.4.** If  $(X, d)$  is complete and  $Y \subseteq X$  is closed, then  $(Y, d|_{Y \times Y})$  is complete.

**Proof.** Given a Cauchy sequence  $(X^{(n)})_{n=1}^{\infty}$  in  $Y$ , so this is also a Cauchy sequence in  $X$ , so it converges in  $X$ . If  $\exists x_0 \in X$  s.t.  $\lim_{n \rightarrow \infty} X^{(n)} = x_0$ . Since  $Y$  is closed, so  $Y = \overline{Y}$ , and by [Proposition 2.2.4](#), we know  $x_0 \in \overline{Y} = Y$ , so  $x_0 \in Y$ , and thus  $(X^{(n)})_{n=1}^{\infty}$  also converges in  $Y$ . ■

## Lecture 7

Completeness of  $\mathbb{R}^n$  with  $d_2, d_1, d_{\infty}$

23 Sep. 09:10

As previously seen.  $(X, d_1)$  and  $(X, d_2)$  are Lipschitz equivalent if  $\exists c_1, c_2 > 0$  s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y) \quad \forall x, y \in X.$$

**Theorem 2.4.1.** Suppose  $(X, d_1)$  and  $(X, d_2)$  are Lipschitz equivalent, then

$$(X, d_1) \text{ is complete} \Leftrightarrow (X, d_2) \text{ is complete.}$$

**Proof.**

( $\Rightarrow$ ) Given any Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$  in  $(X, d_2)$ , then since  $d_1(x, y) \leq \frac{1}{c_1} d_2(x, y)$ , so  $(x^{(n)})_{n=1}^{\infty}$  is Cauchy in  $(X, d_1)$ . Since  $(X, d_1)$  is complete, so there exists  $x \in X$  s.t.  $\lim_{n \rightarrow \infty} x_n = x \in (X, d_1)$ . However,  $x \in (X, d_2)$ , so  $(X, d_2)$  is complete.

( $\Leftarrow$ ) Similar. ■

**Theorem 2.4.2.**  $(\mathbb{R}^n, d_2)$  is a complete metric space.

**Corollary 2.4.1.** Since  $(\mathbb{R}^n, d_2)$ ,  $(\mathbb{R}^n, d_1)$ ,  $(\mathbb{R}^n, d_{\infty})$  are Lipschitz equivalent, so they are all complete by [Theorem 2.4.1](#) and [Theorem 2.4.2](#).

## 2.5 Compact metric space

**Definition 2.5.1 (Compact space).** A metric space  $(X, d)$  is compact iff every sequence in  $(X, d)$  has at least one convergent subsequence converging in  $X$ . A subset  $Y \subseteq X$  is compact if  $(Y, d|_{Y \times Y})$  is compact. That is,  $(Y, d|_{Y \times Y})$  is compact if for any sequence  $(y^{(n)})_{n=1}^{\infty} \subseteq Y$ , there exists a subsequence  $(y^{(n_j)})_{j=1}^{\infty}$  and  $y \in Y$  s.t.  $\lim_{k \rightarrow \infty} y^{(n_k)} = y$ .

**Definition 2.5.2 (Bounded).** Let  $(X, d)$  be a metric space and let  $Y \subseteq X$ . We say  $Y$  is bounded iff for any  $x \in X$ , there exists  $r > 0$  s.t.  $Y \subseteq B_X(x, r)$ .

**Theorem 2.5.1.**

$$Y \text{ is bounded} \Leftrightarrow \exists x_0 \in X \text{ and } R > 0 \text{ s.t. } Y \subseteq B_X(x_0, R).$$

**Proof.** The " $\Rightarrow$ " is easy, so we just prove the other direction. Given any  $x \in X$ , we can choose  $r_x = R + d(x, x_0)$ .

**Claim 2.5.1.**  $Y \subseteq B_X(x, r_x)$ .

**Proof.** Let  $y \in Y$ , we know

$$d(y, x) \leq d(y, x_0) + d(x_0, x) < R + d(x_0, x).$$

Hence,  $y \in B_X(x, r_x)$ . ⊗

**Proposition 2.5.1.** Let  $(X, d)$  be a compact metric space. Then  $(X, d)$  is complete and bounded. ■

**Proof.**

- We want to show that  $(X, d)$  is complete. Given any Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$  in  $(X, d)$ , then since  $(X, d)$  is compact, so there exists a compact subsequence  $(x^{(n_k)})_{k=1}^{\infty}$  in  $X$  s.t.  $\lim_{k \rightarrow \infty} x^{(n_k)} = x$ . Since  $(x^{(n)})_{n=1}^{\infty}$  is Cauchy sequence and  $(x^{(n_k)})_{k=1}^{\infty}$  converges to  $x$ , so  $\lim_{n \rightarrow \infty} x^{(n)} = x$ . (See [Theorem A.1.1](#))
- Consider  $x_0 \in X$ . Suppose  $X$  is not bounded, then  $B(x_0, n)$  will not contain  $X$  for all  $n$ . For each  $n \in \mathbb{N}$ ,

$$\exists y^{(n)} \in X \text{ and } y^{(n)} \notin B_X(x_0, n) \text{ i.e. } d(y^{(n)}, x_0) \geq n.$$

Hence,  $\{y^{(n)}\}_{n=1}^{\infty}$  is a sequence in  $(X, d)$  with  $d(y^{(n)}, x_0) \geq n$ . Since  $(X, d)$  is compact, so there exists a convergent sequence  $\{y^{(n_k)}\}_{k=1}^{\infty}$  and  $y \in X$  s.t.  $\lim_{k \rightarrow \infty} y^{(n_k)} = y$ . Hence, there exists  $R > 0$  s.t.  $d(y, y^{(n_k)}) < R$  for all  $k$  which is big enough, but this means

$$n_k \leq d(y^{(n_k)}, x_0) \leq d(y^{(n_k)}, y) + d(y, x_0) < R + d(y, x_0),$$

which is a fixed value, but  $n_k$  can be arbitrary large, so this is a contradiction. ■

**Corollary 2.5.1.** Let  $(X, d)$  be a metric space and  $Y$  be a compact subset, then  $Y$  is closed and bounded.

**Proof.** Since  $Y$  is a compact subset, so  $(Y, d|_{Y \times Y})$  is compact. Thus,  $Y$  is bounded by [Proposition 2.5.1](#). Hence,  $\exists y_0 \in Y$  and  $R > 0$  s.t.

$$Y \subseteq B_Y(y_0, R) = B_X(y_0, R) \cap Y \subseteq B_X(y_0, R).$$

Let  $y \in \overline{Y}$ , then  $\exists (y^{(n)})_{n=1}^{\infty}$  in  $Y$  s.t.  $\lim_{n \rightarrow \infty} y^{(n)} = y$ . Also, since  $Y$  is compact, so for the convergent sequence  $\{y^{(n)}\}_{n=1}^{\infty}$ , there is a subsequence  $\{y^{(n_k)}\}_{k=1}^{\infty}$  and  $y_0 \in Y$  s.t.  $\lim_{k \rightarrow \infty} y^{(n_k)} = y_0 \in Y$ . By uniqueness of limit in metric space, we know  $y = y_0$ , and thus  $y \in \overline{Y}$ . Hence,  $\overline{Y} = Y$ . (Actually, by [Lemma 2.4.2](#), we know  $\{y^{(n)}\}_{n=1}^{\infty}$  is Cauchy, and then by [Theorem A.1.1](#), we know  $y = y_0$ .) ■

**Theorem 2.5.2 (Heine-Borel Theorem).** Let  $(\mathbb{R}^n, d)$  be  $\mathbb{R}^n$  with  $d = d_2, d_{\infty}, d_1$ , and let  $E \subseteq \mathbb{R}^n$ , then

$$E \text{ is compact} \Leftrightarrow E \text{ is closed and bounded.}$$

**Proof.**

( $\Rightarrow$ ) Trivial by the corollary.

( $\Leftarrow$ ) Suppose  $E$  is closed and bounded. Given a sequence  $(x^{(n)})_{n=1}^{\infty}$  in  $E$ . By [Bolzano-Weierstrass Theorem](#), every bounded sequence has a convergent subsequence. Since  $E$  is closed, so  $E = \overline{E}$ , and thus the convergent subsequence converges in  $E$ . Hence,  $E$  is compact. ■

**Remark 2.5.1.** In a metric space, closed and bounded do not imply compact but compact implies closed and bounded.

**Example 2.5.1.** Consider  $(\mathbb{Z}, d_{\text{disc}})$ , then  $\mathbb{Z}$  is bounded since  $\mathbb{Z} \subseteq B_{\text{disc}}(0, 2)$  and  $\mathbb{Z}$  is closed in  $\mathbb{Z}$  but  $\mathbb{Z}$  is not compact since any subsequence of  $\{n\}_{n \in \mathbb{N}}$  does not converge in  $(\mathbb{Z}, d_{\text{disc}})$ .

**Theorem 2.5.3.** Let  $(X, d)$  be a metric space, let  $Y$  be a compact subset of  $X$ . Let  $(V_\alpha)_{\alpha \in A}$  be a collection of open sets in  $X$ , and suppose that  $Y \subseteq \bigcup_{\alpha \in A} V_\alpha$  (i.e.  $(V_\alpha)_{\alpha \in A}$  covers  $Y$ ). Then, there exists a finite subset  $F \subseteq A$  s.t.  $Y \subseteq \bigcup_{\alpha \in F} V_\alpha$ .

**Proof.** We prove by contradiction. Suppose there does not exist a finite subset  $F \subseteq A$  s.t.  $Y \subseteq \bigcup_{\alpha \in F} V_\alpha$ . For each  $y \in Y \subseteq \bigcup_{\alpha \in A} V_\alpha$ ,  $\exists \alpha \in A$  s.t.  $y \in V_\alpha$ . Since  $V_\alpha$  is open, so there exists  $r > 0$  s.t.  $B(y, r) \subseteq V_\alpha$ . Define

$$r(y) = \sup \{r > 0 : B_X(y, r) \subseteq V_\alpha \text{ for some } \alpha \in A\}.$$

Note that  $r(y) > 0$  for all  $y \in Y$ . Now if we pick  $r_0 = \inf \{r(y) : y \in Y\}$ , then  $r_0 \geq 0$ .

- Case 1:  $r_0 = 0$ , there exists  $y^{(n)} \in Y$  s.t.  $0 < r(y^{(n)}) < \frac{1}{n}$ . Thus,  $(y^{(n)})_{n=1}^\infty$  is a sequence in  $Y$ , and since  $Y$  is compact, so there exists a convergent subsequence  $(y^{(n_k)})_{k=1}^\infty$  converging to  $y_0 \in Y$ . Also, there exists  $\varepsilon > 0$  and  $\alpha \in A$  s.t.  $B_X(y_0, \varepsilon) \subseteq V_\alpha$ . Since  $\lim_{k \rightarrow \infty} d(y^{(n_k)}, y_0) = 0$ , so there exists  $N > 0$  s.t.  $j \geq N$  implies

$$y^{(n_j)} \in B_X\left(y_0, \frac{\varepsilon}{2}\right).$$

**Claim 2.5.2.** For all  $j \geq N$ ,  $B(y^{(n_j)}, \frac{\varepsilon}{2}) \subseteq B(y_0, \varepsilon)$ .

**Proof.** Suppose  $z \in B(y^{(n_j)}, \frac{\varepsilon}{2})$ , then  $d(z, y^{(n_j)}) < \frac{\varepsilon}{2}$ , and thus

$$d(z, y_0) \leq d(z, y^{(n_j)}) + d(y^{(n_j)}, y_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

⊛

Now since  $B_X(y_0, \varepsilon) \subseteq V_\alpha$ , so for  $j \geq N$ ,  $B(y^{(n_j)}, \frac{\varepsilon}{2}) \subseteq V_\alpha$ , which means

$$r(y^{(n_j)}) \geq \frac{\varepsilon}{2} > 0.$$

However, this contradicts to the assumption that  $r(y^{(n_j)}) < \frac{1}{n_j}$  for all  $j$ . Hence, Case 1 is impossible.

- Case 2:  $\infty > r_0 > 0$ . We know  $r_0 \leq r(y)$  for all  $y \in Y$  by definition. Hence,  $0 < \frac{r_0}{2} < r(y)$ . This means for each  $y \in Y$ , there exists  $\alpha \in A$  s.t.  $B_X(y, \frac{r_0}{2}) \subseteq V_\alpha$ . Choose a point  $y^{(1)} \in Y$  s.t.  $\exists \alpha_1 \in A$  s.t.  $B_X(y^{(1)}, \frac{r_0}{2}) \subseteq V_{\alpha_1}$ . Since  $V_{\alpha_1}$  cannot cover  $Y$ , so there exists  $y^{(2)} \in Y$  and  $y^{(2)} \notin B_X(y^{(1)}, \frac{r_0}{2}) \subseteq V_{\alpha_1}$ . Hence,  $d(y^{(2)}, y^{(1)}) \geq \frac{r_0}{2}$ . Now we set the induction hypothesis: Suppose there exists  $y^{(1)}, \dots, y^{(k)} \in Y$  and  $\alpha_1, \dots, \alpha_k \in A$  s.t.

$$B_X\left(y^{(j)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_j} \text{ and } d(y^{(i)}, y^{(j)}) \geq \frac{r_0}{2} \quad \forall i \neq j,$$

and  $B_X(y^{(1)}, \frac{r_0}{2}) \cup \dots \cup B_X(y^{(k)}, \frac{r_0}{2})$  cannot cover  $Y$ , then we can find

$$y^{(k+1)} \notin B_X(y^{(1)}, \frac{r_0}{2}) \cup \dots \cup B_X(y^{(k)}, \frac{r_0}{2}),$$

and thus  $d(y^{(k+1)}, y^{(i)}) \geq \frac{r_0}{2}$  for  $1 \leq i \leq k$ . Also,  $\exists \alpha_{k+1}$  s.t.  $B(y^{(k+1)}, \frac{r_0}{2}) \subseteq V_{\alpha_{k+1}}$ . Now we know  $B(y^{(1)}, \frac{r_0}{2}) \cup \dots \cup B(y^{(k+1)}, \frac{r_0}{2})$  won't cover  $Y$ , then  $\{y^{(k)}\}_{k=1}^\infty$  is a sequence in  $Y$  and  $d(y^{(j)}, y^{(l)}) \geq \frac{r_0}{2}$ . Since  $Y$  is compact, so there exists a subsequence of  $\{y^{(k)}\}_{k=1}^\infty$  which is convergent, but it is impossible, so we have a contradiction.

- Case 3:  $r_0 = \infty$ . If so, then it means  $\inf \{r(y) : y \in Y\} = \infty$ , so  $r(y) = \infty$  for all  $y \in Y$ , otherwise if for some  $y' \in Y$ ,  $r(y')$  is finite, then  $r_0 \leq r(y')$ , and will get a contradiction. Now we have  $r(y) = \infty$  for all  $y \in Y$ . This means for all  $r > 0$ , there exists some  $\alpha \in A$  s.t.

$B_X(y, r) \subseteq V_\alpha$ . Now since  $Y$  is compact, so  $Y$  is bounded, which means for all  $y \in Y$ , there exists  $r_y$  s.t.  $Y \subseteq B_X(y, r_y)$ . However, since  $r(y) = \infty$  and by the previous argument, we know  $B_X(y, r_y) \subseteq V_{\alpha_y}$  for some  $\alpha_y \in A$ , and thus  $Y \subseteq V_{\alpha_y}$ , and thus  $V_{\alpha_y}$  covers  $Y$ , which is a contradiction. ■

## Lecture 8

25 Sep. 10:20

**Theorem 2.5.4** (Review [Theorem 2.5.3](#)). Let  $Y$  be a compact subset of a metric space  $(X, d)$  and let  $\{V_\alpha\}_{\alpha \in A}$  be an open cover of  $Y$ . Then  $\exists$  a finite subcover of  $\{V_\alpha\}_{\alpha \in A}$  i.e.  $\exists \alpha_1, \dots, \alpha_n \in A$  s.t.  $Y \subseteq \bigcup_{i=1}^n V_{\alpha_i}$ .

### Remark 2.5.2.

$Y$  is compact  $\Leftrightarrow$  Any open cover of  $Y$  has a finite subcover.

**Proof.** The  $(\Rightarrow)$  direction is proved. Now we prove the other direction.

**Claim 2.5.3.** If  $(X, d)$  is a metric space and for all open cover of  $X$ , there exists finite subcover of  $X$ , then  $X$  is complete.

**Proof.** We will prove this by contradiction, using the definition of compactness (the open cover property).

**1. Assumption** Assume for the sake of contradiction that  $X$  is **compact** but **not complete**. Since  $X$  is not complete, there exists a **Cauchy sequence**  $\{x_n\}_{n=1}^\infty$  in  $X$  that **does not converge** to any point  $p \in X$ .

**2. Constructing the Open Cover** Since  $\{x_n\}$  does not converge to a point  $p \in X$ , for every  $p \in X$ , the point  $p$  is not the limit of the sequence. This means there exists some  $\epsilon_p > 0$  such that the open ball  $B_{\epsilon_p}(p)$  contains only a **finite number of terms** of the sequence  $\{x_n\}$ .

To see why, suppose for a contradiction that there was some  $p \in X$  such that for all  $\epsilon > 0$ , the ball  $B_\epsilon(p)$  contains an infinite number of terms of  $\{x_n\}$ . Let  $\{x_{n_k}\}$  be a subsequence with  $x_{n_k} \in B_{1/k}(p)$ . This subsequence converges to  $p$ . Since  $\{x_n\}$  is a Cauchy sequence and has a convergent subsequence, the entire sequence  $\{x_n\}$  must converge to the same limit  $p$ , which contradicts our initial assumption. Therefore, the property holds: for every  $p \in X$ , there is an  $\epsilon_p > 0$  such that  $B_{\epsilon_p}(p)$  contains  $x_n$  for only finitely many  $n$ .

Consider the collection of open balls  $\mathcal{U} = \{B_{\epsilon_p}(p) : p \in X\}$ . Since the union of these balls covers every point  $p \in X$ ,  $\mathcal{U}$  is an **open cover** of  $X$ :

$$X \subseteq \bigcup_{p \in X} B_{\epsilon_p}(p).$$

**3. Using Compactness to Find a Finite Subcover** Since  $X$  is **compact**, the open cover  $\mathcal{U}$  must have a **finite subcover**. That is, there exist a finite number of points  $p_1, p_2, \dots, p_k \in X$  such that

$$X \subseteq B_{\epsilon_{p_1}}(p_1) \cup B_{\epsilon_{p_2}}(p_2) \cup \dots \cup B_{\epsilon_{p_k}}(p_k) = \bigcup_{i=1}^k B_{\epsilon_{p_i}}(p_i).$$

**4. Reaching the Contradiction** By the definition of  $\epsilon_{p_i}$ , each ball  $B_{\epsilon_{p_i}}(p_i)$  contains  $x_n$  for only a **finite number** of indices  $n$ . The union of a finite number of finite sets is a finite set. Therefore, the finite union  $\bigcup_{i=1}^k B_{\epsilon_{p_i}}(p_i)$  can contain  $x_n$  for only a finite number of indices  $n$ . However, since this finite union covers all of  $X$  (step 3), it must contain **all** terms of the sequence  $\{x_n\}_{n=1}^\infty$ . Since the sequence  $\{x_n\}$  is an infinite set of points, this is a **contradiction**.

The initial assumption that  $X$  is not complete must be false. Thus, every compact metric space is complete. ⊛

Suppose any open cover of  $Y$  has a finite subcover, then given any sequence  $(y^{(n)})_{n=1}^{\infty}$ . Consider

$$\bigcup_{x \in Y} B_Y(x, 1),$$

then this is an open cover of  $Y$ , and now we know there is a finite subcover

$$\bigcup_{i=1}^k B_Y(x_i, 1)$$

of  $Y$  where  $x_i \in Y$  for all  $i$ . Now since  $(y^{(n)})_{n=1}^{\infty}$  has infinitely many terms, so we know for some  $i$ , we have infinitely many terms of  $(y^{(n)})_{n=1}^{\infty} \subseteq B_Y(x_i, 1)$  by Pigeonhole principle. Hence, there are infinitely many terms of  $(y^{(n)})_{n=1}^{\infty}$  are in

$$\left\{ y \in Y : 0 \leq d(y, x_i) < \frac{1}{2} \right\} \cup \left\{ y \in Y : \frac{1}{2} \leq d(y, x_i) < 1 \right\}.$$

Thus, again, by Pigeonhold principle we know there are infinitely many terms of  $(y^{(n)})_{n=1}^{\infty}$  are in either one of the above two sets. By repeating split the space into half as what we do above, we know for all  $k \geq 0$ , there are infinitely many terms of  $(y^{(n)})_{n=1}^{\infty}$  has the following property: Every two terms of these "infinitely many terms" has distance less than  $\frac{1}{2^k}$ . Note that this means we can pick a subsequence of  $(y^{(n)})_{n=1}^{\infty}$  so that it is Cauchy, and since every Cauchy sequence converges in  $Y$  (Since [Claim 2.5.3](#)), so we're done. ■

**Corollary 2.5.2.** Let  $(X, d)$  be a metric space and let  $K_1, K_2, \dots$  be a sequence of nonempty compact subsets of  $X$  s.t.  $K_{i+1} \subseteq K_i$  for  $i \in \mathbb{N}$ , that is,  $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$  for  $i \in \mathbb{N}$ , then

$$\bigcap_{i=1}^{\infty} K_i \neq \emptyset.$$

**Proof.** Suppose  $\bigcap_{i=1}^{\infty} K_i = \emptyset$ . Since  $K_i$ 's are compact, so they are closed. Also, we have

$$\bigcup_{i=1}^{\infty} (K_1 \setminus K_n) = K_1 \setminus \left( \bigcap_{i=1}^{\infty} K_n \right) = K_1.$$

Let  $V_i = K_1 \setminus K_i = K_1 \cap K_i^C$ . Note that  $K_i^C$  is open in  $X$ . Hence, we have  $V_i$  is open in  $K_1$ , and thus  $\{V_i\}_{i=1}^{\infty}$  is an open cover of  $K_1$  in  $K_1$ . ( $(K_1, d|_{K_1 \times K_1})$  is compact.) By [Theorem 2.5.3](#), we know there exists  $\alpha_1, \alpha_2, \dots, \alpha_l$  with  $\alpha_1 < \alpha_2 < \dots < \alpha_l$  s.t.

$$\begin{aligned} K_1 &\subseteq \bigcup_{i=1}^l V_{\alpha_i} = \bigcup_{i=1}^l (K_1 \setminus K_{\alpha_i}) \\ &= K_1 \setminus \bigcap_{i=1}^l K_{\alpha_i} = K_1 \setminus K_{\alpha_l} \end{aligned}$$

since  $K_{\alpha_1} \supseteq K_{\alpha_2} \supseteq \dots \supseteq K_{\alpha_l}$ . However,  $K_{\alpha_l} \subseteq K_1$  and  $K_{\alpha_l} \neq \emptyset$ . Thus, we have a contradiction. ■

**Example 2.5.2.** Consider  $I_1 = [0, 1]$ , and  $I_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ , and picking  $I_3, I_4, \dots$  with same method, then  $I_{n+1} \subseteq I_n$  for all  $n$  and they are compact, so

$$\bigcap_{i=1}^{\infty} I_i \neq \emptyset.$$

**Theorem 2.5.5.** Let  $(X, d)$  be a metric space.

- (a) If  $Y$  is a compact subset of  $X$ , and  $Z \subseteq Y$ , then  $Z$  is compact iff  $Z$  is closed.
- (b) If  $Y_1, \dots, Y_n$  are a finite collection of compact subsets of  $X$ , then  $\bigcup_{i=1}^n Y_i$  are also compact.

**proof of (a).** If  $Z$  is compact, then by [Corollary 2.5.1](#), we know  $Z$  is closed. Now we show that if  $Z$  is closed, then  $Z$  is compact. If  $Z$  is closed, then  $Y \setminus Z$  is open in  $Y$ , then we know

$$Y \setminus Z = V \cap Y$$

for some open set  $V \subseteq Y$ , so note that  $(Y \setminus Z) \subseteq V$ . Now suppose  $\{U_\alpha\}_{\alpha \in A}$  is an open cover of  $Z$ . Hence, we know  $\{U_\alpha\}_{\alpha \in A} \cup \{V\}$  is an open cover of  $Y$  since the former covers  $Z$  and the latter covers  $Y \setminus Z$ . Now since  $Y$  is compact, so we know for some  $\alpha_1, \alpha_2, \dots, \alpha_n$ , there is

$$Y \subseteq \left( \bigcup_{i=1}^n U_{\alpha_i} \right) \cup V,$$

and thus we can write

$$Z \subseteq Y \subseteq \left( \bigcup_{i=1}^n U_{\alpha_i} \right) \cup V.$$

However, note that  $Z \cap V = \emptyset$  since

$$Z = Y \setminus (Y \setminus Z) = Y \setminus (V \cap Y) = (Y \setminus V) \cup (Y \setminus Y) = Y \setminus V.$$

Hence, we know

$$Z \subseteq \bigcup_{i=1}^n U_{\alpha_i},$$

and thus for any open cover of  $Z$ , we know there exists a finite subcover of  $Z$ , and we're done. ■

## Chapter 3

# Continuous functions on metric spaces

Suppose  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces. Let  $f : X \rightarrow Y$  be a function from  $X$  to  $Y$ . Then we want that if  $x \in X$  is close to  $y \in X$ , then, then  $f(x) \in Y$  is close to  $f(y) \in Y$ .

**Definition 3.0.1 (Continuous function).** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and let  $f : X \rightarrow Y$  be a function. Suppose  $x_0 \in X$ , we will say  $f$  is continuous at  $x_0$  iff for every  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{whereas } d_X(x, x_0) < \delta.$$

We say  $f$  is continuous if  $f$  is continuous at every point  $x \in X$ .

**Definition 3.0.2 (Preimage).** Let  $f : X \rightarrow Y$  be a function from  $X \rightarrow Y$  and  $V \subseteq Y$ . The preimage (inverse image) of  $V$  is

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \subseteq X.$$

**Example 3.0.1.** Suppose  $f(x) = x^2$ , then what is the preimage of  $(1, \infty)$ ?

**Answer.**

$$f^{-1}((1, \infty)) = (-\infty, -1) \cup (1, \infty).$$

⊛

Now we build an equivalent definition of continuity. If  $f$  is continuous at  $x_0$ , then given any  $\varepsilon > 0$ ,  $\exists \delta > 0$  s.t.

$$f(x) \in B_Y(f(x_0), \varepsilon) \quad \text{whereas } x \in B_X(x_0, \delta).$$

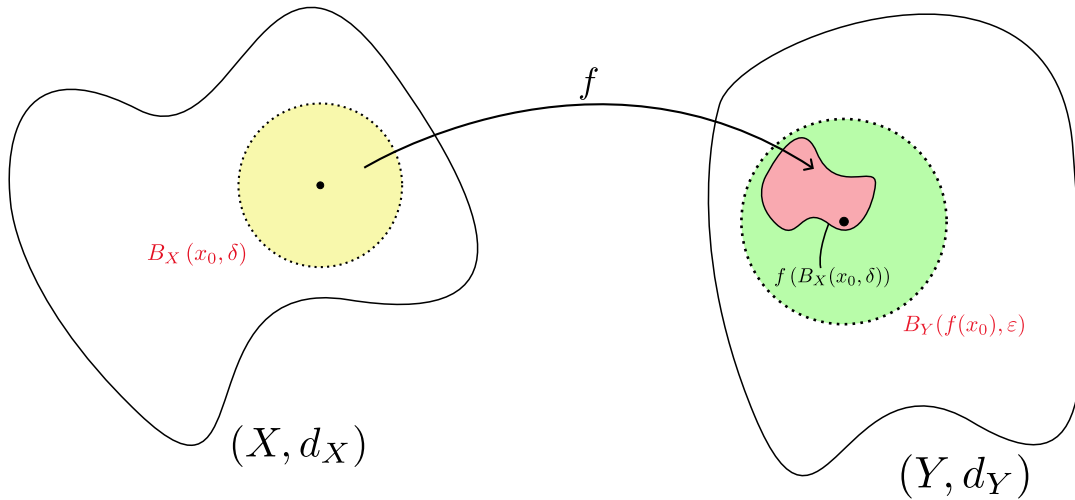
Also,  $f(x) \in B_Y(f(x_0), \varepsilon)$  if and only if

$$x \in f^{-1}(B_Y(f(x_0), \varepsilon)).$$

Hence, we have

**Corollary 3.0.1.**  $f$  is continuous at  $x_0$  if and only if

$$\text{Given any } \varepsilon > 0, \exists \delta > 0 \text{ s.t. } B_X(x_0, \delta) \subseteq f^{-1}(B_Y(f(x_0), \varepsilon)).$$

Figure 3.1: Continuous function from  $(X, d_X)$  to  $(Y, d_Y)$ 

**Remark 3.0.1.** If  $f : X \rightarrow Y$  is continuous and  $K \subseteq X$ , then  $f|_K : K \rightarrow Y$  is continuous.

**Proof.** Given any point  $x_0 \in K \subseteq X$ . Since  $f$  is continuous at  $x_0$ , so  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{if } d_X(x, x_0) < \delta.$$

If  $z \in K$  and  $d_K(z, x_0) < \delta$ , then  $d_Y(f(z), f(x_0)) < \varepsilon$  since  $d_K$  is actually  $d_X$  but intersected to  $K$ , so  $f$  is continuous on  $K$ .  $\circledast$

**Theorem 3.0.1.** Suppose that  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and  $f : X \rightarrow Y$  is a function and let  $x_0 \in X$ , then TFAE:

- (a)  $f$  is continuous at  $x_0$ .
- (b) Whenever  $(x^{(n)})_{n=1}^{\infty}$  is a sequence in  $X$  converges to  $x_0$ , then

$$\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x_0) \text{ in } (Y, d_Y).$$

- (c) For every open set  $V \subseteq Y$  that contains  $f(x_0)$ ,  $\exists$  an open set  $U \subseteq X$  containing  $x_0$  s.t.  $f(U) \subseteq V$ , or equivalently,  $U \subseteq f^{-1}(V)$ .

**proof of (a)  $\Rightarrow$  (b).** Given any  $\varepsilon > 0$ , since  $f$  is continuous at  $x_0$ , so  $\exists \delta > 0$  s.t. if  $d_X(x, x_0) < \delta$ , then  $d_Y(f(x), f(x_0)) < \varepsilon$ . Now if  $\lim_{n \rightarrow \infty} x^{(n)} = x_0$ . Then there exists  $N > 0$  s.t.  $n \geq N$  implies  $d_X(x^{(n)}, x_0) < \delta$ . Hence, we know  $d_Y(f(x^{(n)}), f(x_0)) < \varepsilon$ . Hence, for this  $\varepsilon$ , we know there exists  $N$  s.t.  $n \geq N$  implies  $d_Y(f(x^{(n)}), f(x_0)) < \varepsilon$ , and thus  $\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x_0)$ .  $\blacksquare$

**proof of (b)  $\Rightarrow$  (c).** Let  $f(x_0) \in V \subseteq Y$  for some open  $V$ .

**Claim 3.0.1.** There exists an open set  $U$  s.t.  $x_0 \in U \subseteq X$  and  $f(U) \subseteq V$ .

**Proof.** If this is not true, then this implies that for every open set  $U$  with  $x_0 \in U$ , consider  $U_n = B_X(x_0, \frac{1}{n})$  for all  $n$ , then  $\exists x_n \in U_n$  s.t.  $f(x_n) \notin V$ , then pick all of this  $x_n$  to be  $\{x^{(n)}\}_{n=1}^{\infty}$  with  $x^{(n)} \in U_n$  for all  $n$ , we know  $\forall x^{(n)}$  we have  $f(x^{(n)}) \notin V$ . Then,  $\{x^{(n)}\}_{n=1}^{\infty}$  is a sequence converges to  $x_0$ . By (b), we know  $\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x_0) \in V$ . However, by our choice,  $f(x^{(n)}) \notin V$ , so  $f(x^{(n)}) \in Y \setminus V$ . Since  $V$  is open, so  $Y \setminus V$  is closed. Hence, we must have  $\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x_0) \in Y \setminus V$ , which is a contradiction.  $\otimes$

**proof of (c)  $\Rightarrow$  (a).** Suppose (c) holds, then we want to show  $f$  is continuous at  $x_0$ , which means for all  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $B_X(x_0, \delta) \subseteq f^{-1}(B_Y(f(x_0), \varepsilon))$ . Now consider  $V = B_Y(f(x_0), \varepsilon) \subseteq Y$ , then by (c) we know there exists open  $U \subseteq X$  s.t.  $x_0 \in U \subseteq f^{-1}(V)$ . Now since  $U$  is open and  $x_0 \in U$ , so there exists  $B_X(x_0, \delta) \subseteq U$ , and thus

$$B_X(x_0, \delta) \subseteq U \subseteq f^{-1}(V) = f^{-1}(B_Y(f(x_0), \varepsilon)),$$

and we're done.  $\blacksquare$

**Theorem 3.0.2.** Suppose  $f : X \rightarrow Y$ , then TFAE

- (a)  $f$  is continuous.
- (b) If  $\lim_{n \rightarrow \infty} x^{(n)} = x \in (X, d_X)$ , then  $\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x)$  in  $(Y, d_Y)$ .
- (c) If  $V$  is open in  $Y$ , then  $f^{-1}(V)$  is open in  $X$ .
- (d) If  $F$  is closed in  $Y$ , then  $f^{-1}(F)$  is closed in  $X$ .

**Note 3.0.1.** (a) says  $f$  is continuous, so it is a statement for every point in  $X$  and [Theorem 3.0.1](#) is a theorem for a single  $x_0 \in X$ .

**(a)  $\Leftrightarrow$  (b).** By [Theorem 3.0.1](#), we know it is true.  $\blacksquare$

**(c) holds iff for every  $x \in X$ , (c) in Theorem 3.0.1 holds.** If we have (c) in [Theorem 3.0.1](#) holding for all  $x \in X$  and we know  $V$  is open in  $Y$ , then for each  $x \in f^{-1}(V)$ , we have  $f(x) \in V$ , so there exists an open set  $u_x$  s.t.  $x \in u_x \subseteq f^{-1}(V)$ . Hence,

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} u_x.$$

Since  $u_x$  is open, so  $f^{-1}(V)$  is open.

Now if we have (c), then

If  $V$  is open in  $Y$ , then  $f^{-1}(V)$  is open in  $X$ .

Now for all open  $V \subseteq Y$  that contains  $f(x_0)$  for some  $x_0 \in X$ , we know  $f(x_0) \in V$ , so  $x_0 \in f^{-1}(V)$ , and since  $f^{-1}(V)$  is open in  $X$ , so we can just pick  $U = f^{-1}(V)$ , and we're done since this proof is valid for all  $x_0 \in X$ .  $\blacksquare$

**(c)  $\Leftrightarrow$  (d).** If  $F$  is closed in  $Y$ , then  $Y \setminus F$  is open in  $Y$ , and thus  $f^{-1}(Y \setminus F)$  is open in  $X$ , and since  $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$  is open, so  $f^{-1}(F)$  is closed. This is the proof from (c) to (d), and the proof from (d) to (c) is similar.  $\blacksquare$

**Note 3.0.2.** [Theorem 3.0.2](#) is a stronger version of [Theorem 3.0.1](#) since it states the version holding for all  $x$ .

**Remark 3.0.2.** [Theorem 3.0.2](#) tells us that continuity tells us if the image is open/closed, then the preimage is open/closed. However, if  $f$  is continuous, then the image of an open set on  $f$  may not be open, and the image of a closed set on  $f$  may not be closed.

**Example 3.0.2.** Consider  $f(x) = x^2$ , then  $f(-1, 1) = [0, 1)$  is not open.

**Example 3.0.3.** Consider  $f(x) = \arctan(x)$ , then  $f([0, \infty)) = [0, \frac{\pi}{2}]$ .

Now if  $f : X \rightarrow Y$  and  $g : X \rightarrow Z$ , then consider

$$(f, g) : X \rightarrow Y \times Z \text{ with } x \mapsto (f(x), g(x)),$$

we know  $Y \times Z$  has a natural metric, which is defined as: If  $(y_1, z_1)$  and  $(y_2, z_2)$  are in  $Y \times Z$ , then

$$d_{Y \times Z}((y_1, z_1), (y_2, z_2)) = d_Y(y_1, y_2) + d_Z(z_1, z_2).$$

**Lemma 3.0.1.** Consider  $f : X \rightarrow Y$  and  $g : X \rightarrow Z$  and  $(f, g) : X \rightarrow Y \times Z$ , then  $f$  and  $g$  are both continuous if and only if  $(f, g)$  is continuous.

**Proof.** Suppose  $f, g$  are both continuous, then given  $x \in X$ , we know  $\lim_{n \rightarrow \infty} x^{(n)} = x \in (X, d_X)$  implies

$$\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x) \text{ and } \lim_{n \rightarrow \infty} g(x^{(n)}) = g(x).$$

**Claim 3.0.2.**

$$\lim_{n \rightarrow \infty} (f, g)(x^{(n)}) = (f, g)(x).$$

**Proof.** Check. ⊗

The other direction is also easy to prove. ■

## Lecture 9

**Corollary 3.0.2.** Let  $(X, d_X)$ ,  $(Y, d_Y)$ , and  $(Z, d_Z)$  be metric spaces.

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- (a) If  $f : X \rightarrow Y$  is continuous at  $x_0 \in X$ ,  $g : Y \rightarrow Z$  is continuous at  $f(x_0)$ , then  $g \circ f : X \rightarrow Z$  is continuous at  $x_0$ .
- (b) If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then  $g \circ f : X \rightarrow Z$  is continuous.

**proof of (a).** Fix  $x_0 \in X$ . Since  $f$  is continuous at  $x_0$ , we have

$$\lim_{n \rightarrow \infty} x^{(n)} = x_0 \Rightarrow \lim_{n \rightarrow \infty} f(x^{(n)}) = f(x_0) \in Y.$$

Recall  $(g \circ f)(x) = g(f(x))$ . Since  $g$  is continuous at  $f(x_0) \in Y$ . It follows that

$$\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x_0) \Rightarrow \lim_{n \rightarrow \infty} g(f(x^{(n)})) = g(f(x_0)).$$

Note that this means

$$\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x_0) \Rightarrow \lim_{n \rightarrow \infty} (g \circ f)(x^{(n)}) = (g \circ f)(x_0).$$

Hence,

$$\lim_{n \rightarrow \infty} x^{(n)} = x_0 \Rightarrow \lim_{n \rightarrow \infty} (g \circ f)(x^{(n)}) = (g \circ f)(x_0),$$

which means  $g \circ f$  is continuous at  $x_0$ . ■

### 3.1 Continuity and Product Spaces

Given two functions  $f : X \rightarrow Y$  and  $g : X \rightarrow Z$ , we can define the pairing  $(f, g) : X \rightarrow Y \times Z$  by

$$(f, g)(x) = (f(x), g(x)).$$

**Example 3.1.1.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = x^2 + 3$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $g(x) = 4x$ , then we can define  $(f, g) : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  by

$$(f, g)(x) = (x^2 + 3, 4x).$$

**Definition 3.1.1 (Product metric).** Let  $(Y, d_Y)$  and  $(Z, d_Z)$  be metric spaces. Define a metric  $d_{Y \times Z}^1$  on  $Y \times Z$  by

$$d_{Y \times Z}^1((y_1, z_1), (y_2, z_2)) = d_Y(y_1, y_2) + d_Z(z_1, z_2).$$

Also, we can define  $d_{Y \times Z}^\infty$  by

$$d_{Y \times Z}^\infty((y_1, z_1), (y_2, z_2)) = \max\{d_Y(y_1, y_2), d_Z(z_1, z_2)\}.$$

Finally, we can define

$$d_{Y \times Z}^2((y_1, z_1), (y_2, z_2)) = \sqrt{(d_Y(y_1, y_2))^2 + (d_Z(z_1, z_2))^2}.$$

**Proposition 3.1.1.** Let  $(Y, d_Y)$  and  $(Z, d_Z)$  be metric spaces, then  $d_{Y \times Z}^1, d_{Y \times Z}^2, d_{Y \times Z}^\infty$  are metrics on  $Y \times Z$ .

**Proof.** Here we only prove  $d_{Y \times Z}^1$  is a metric. For  $(y_1, z_1), (y_2, z_2), (y_3, z_3)$  in  $Y \times Z$ .

- $d_{Y \times Z}^1((y_1, z_1), (y_2, z_2)) = d_Y(y_1, y_2) + d_Z(z_1, z_2) \geq 0$ .
- If  $d_{Y \times Z}^1((y_1, z_1), (y_2, z_2)) = 0$ , then  $y_1 = y_2 = z_1 = z_2 = 0$ .
- $d_{Y \times Z}^1((y_1, z_1), (y_2, z_2)) = d_{Y \times Z}^1((y_2, z_2), (y_1, z_1))$ .
- 

$$\begin{aligned} d_{Y \times Z}^1((y_1, z_1), (y_3, z_3)) &= d_Y(y_1, y_3) + d_Z(z_1, z_3) \\ &\leq d_Y(y_1, y_2) + d_Y(y_2, y_3) + d_Z(z_1, z_2) + d_Z(z_2, z_3) \\ &= d_{Y \times Z}^1((y_1, z_1), (y_2, z_2)) + d_{Y \times Z}^1((y_2, z_2), (y_3, z_3)). \end{aligned}$$

■

**Exercise 3.1.1.** Show that  $d_{Y \times Z}^2$  is a metric.

**Proof.** We only show the triangle inequality holds here. We use  $d$  instead of  $d_{Y \times Z}^2$  here. For  $(y_1, z_1), (y_2, z_2), (y_3, z_3)$  here, we have

$$\begin{aligned} d((y_1, z_1), (y_3, z_3))^2 &= d_Y(y_1, y_3)^2 + d_Z(z_1, z_3)^2 \leq (d_Y(y_1, y_2) + d_Y(y_2, y_3))^2 + (d_Z(z_1, z_2) + d_Z(z_2, z_3))^2 \\ &= d_Y(y_1, y_2)^2 + d_Y(y_2, y_3)^2 + d_Z(z_1, z_2)^2 + d_Z(z_2, z_3)^2 + 2(d_Y(y_1, y_2)d_Y(y_2, y_3) + d_Z(z_1, z_2)d_Z(z_2, z_3)) \\ &\leq d_Y(y_1, y_2)^2 + d_Y(y_2, y_3)^2 + d_Z(z_1, z_2)^2 + d_Z(z_2, z_3)^2 + 2\sqrt{d_Y(y_1, y_2)^2 + d_Z(z_1, z_2)^2}\sqrt{d_Y(y_2, y_3)^2 + d_Z(z_2, z_3)^2} \\ &= \left(\sqrt{d_Y(y_1, y_2)^2 + d_Z(z_1, z_2)^2} + \sqrt{d_Y(y_2, y_3)^2 + d_Z(z_2, z_3)^2}\right)^2 \\ &= (d((y_1, z_1), (y_2, z_2)) + d((y_2, z_2), (y_3, z_3)))^2. \end{aligned}$$

■

**Proposition 3.1.2.**  $\lim_{n \rightarrow \infty} (y_n, z_n) = (y, z)$  in  $Y \times Z$  w.r.t.  $d^1, d^2, d^\infty$  iff

$$\lim_{n \rightarrow \infty} y_n = y \text{ in } (Y, d_Y) \text{ and } \lim_{n \rightarrow \infty} z_n = z \text{ in } (Z, d_Z),$$

**Proof.** We prove the case w.r.t.  $d_{Y \times Z}^\infty$  metric. Since

$$\lim_{n \rightarrow \infty} (y_n, z_n) = (y, z) \Leftrightarrow \lim_{n \rightarrow \infty} d_{Y \times Z}^\infty((y_n, z_n), (y, z)) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \max\{d_Y(y_n, y), d_Z(z_n, z)\} = 0.$$

Also,

$$\begin{aligned} 0 &\leq d_Y(y_n, y) \leq \max\{d_Y(y_n, y), d_Z(z_n, z)\} \\ 0 &\leq d_Z(z_n, z) \leq \max\{d_Y(y_n, y), d_Z(z_n, z)\}. \end{aligned}$$

Hence, by squeeze theorem, we must have  $\lim_{n \rightarrow \infty} d_Y(y_n, y) = \lim_{n \rightarrow \infty} d_Z(z_n, z) = 0$ .

If  $\lim_{n \rightarrow \infty} d_Y(y_n, y) = 0$  and  $\lim_{n \rightarrow \infty} d_Z(z_n, z) = 0$ , then

$$\lim_{n \rightarrow \infty} d_{Y \times Z}^\infty((y_n, z_n), (y, z)) = \lim_{n \rightarrow \infty} \max\{d_Y(y_n, y), d_Z(z_n, z)\} = 0.$$

Hence,  $\lim_{n \rightarrow \infty} (y_n, z_n) = (y, z)$  in  $d_{Y \times Z}^\infty$  metrics. ■

**Theorem 3.1.1.** Let  $(Y, d_Y)$  and  $(Z, d_Z)$  be metric spaces. On  $Y \times Z$ , we have the metric  $d^1, d^2, d^\infty$ . The map  $(f, g) : X \rightarrow Y \times Z$  is continuous iff  $f : X \rightarrow Y$  and  $g : X \rightarrow Z$  are both continuous.

**Proof.** If  $(f, g)$  is continuous. Take any sequence  $x^{(n)}$  with  $\lim_{n \rightarrow \infty} x^{(n)} = x$  in  $(X, d_X)$ . Then,

$$\lim_{n \rightarrow \infty} (f, g)(x^{(n)}) = (f, g)(x) \in Y \times Z.$$

Recall that  $(f, g)(x^{(n)}) = (f(x^{(n)}), g(x^{(n)}))$ . Hence, we have

$$\lim_{n \rightarrow \infty} f(x^{(n)}) = f(x) \in (Y, d_Y) \text{ and } \lim_{n \rightarrow \infty} g(x^{(n)}) = g(x) \in (Z, d_Z)$$

by [Proposition 3.1.2](#). Thus,  $f, g$  are both continuous at  $x$ . Since  $x$  can be arbitrary point in  $(X, d_X)$ , so  $f, g$  are both continuous. ■

**Lemma 3.1.1.** Let  $f : X \rightarrow \mathbb{R}$  and  $g : X \rightarrow \mathbb{R}$  be functions, and define  $(f, g) : X \rightarrow \mathbb{R}^2$ . We give  $\mathbb{R}^2$  the Euclidean metric. Then,  $f, g$  are both continuous iff  $(f, g)$  is continuous.

**Proof.** By [Theorem 3.1.1](#). Choose  $d^2$  on  $\mathbb{R} \times \mathbb{R}$ , then  $d^2 = d_2 =$  Euclidean metric. ■

**Lemma 3.1.2.** The following functions are continuous.

- $(x, y) \mapsto x + y$  on  $\mathbb{R}^2$
- $(x, y) \mapsto x - y$  on  $\mathbb{R}^2$
- $(x, y) \mapsto x \cdot y$  on  $\mathbb{R}^2$
- $(x, y) \mapsto \max\{x, y\}$  on  $\mathbb{R}^2$
- $(x, y) \mapsto \frac{x}{y}$  on  $\mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}$ .
- $x \mapsto cx$  on  $\mathbb{R}$  for any  $c \in \mathbb{R}$ .

**Proof.** We prove the  $(x, y) \mapsto x \cdot y$  case. Define  $M(x, y) = x \cdot y$ . Choose  $(x_0, y_0) \in \mathbb{R}^2$ . Given  $\varepsilon > 0$ , we want to find  $\delta > 0$  s.t.

$$d_2((x, y), (x_0, y_0)) = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |M(x, y) - M(x_0, y_0)| < \varepsilon.$$

We will choose  $\delta$  later. Now suppose we have chosen some appropriate  $\delta$ , then we have  $|x - x_0|, |y - y_0| < \delta$ . Then,

$$|M(x, y) - M(x_0, y_0)| = |xy - x_0y_0| = |x(y - y_0) + y_0(x - x_0)|.$$

Now we choose some  $\delta \leq 1$ . Then since  $|x - x_0| < \delta \leq 1$ , we have

$$|x| = |x - x_0 + x_0| < |x - x_0| + |x_0| < 1 + |x_0|.$$

Thus, we have

$$\begin{aligned} |M(x, y) - M(x_0, y_0)| &\leq |x||y - y_0| + |y_0||x - x_0| \\ &< (1 + |x_0|)|y - y_0| + |y_0||x - x_0| \\ &< (1 + |x_0|)\delta + |y_0|\delta \\ &= (1 + |x_0| + |y_0|)\delta. \end{aligned}$$

Hence, we can choose

$$\delta = \min \left\{ 1, \frac{\varepsilon}{1 + |x_0| + |y_0|} \right\},$$

and we will have

$$|M(x, y) - M(x_0, y_0)| < \varepsilon \text{ whenever } d_2((x, y), (x_0, y_0)) < \delta.$$

■

**Proof.** Here we prove the  $(x, y) \mapsto \max\{x, y\}$  case. Note that

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}.$$

Then, we have

$$\begin{aligned} |\max\{x, y\} - \max\{a, b\}| &= \left| \frac{x + y + |x - y|}{2} - \frac{a + b + |a - b|}{2} \right| \\ &= \frac{1}{2} |(x - a) + (y - b) + |x - y| - |a - b|| \\ &\leq \frac{1}{2} (|x - a| + |y - b| + ||x - y| - |a - b||) \\ &\leq \frac{1}{2} (|x - a| + |y - b| + |(x - y) - (a - b)|) \\ &= \frac{1}{2} (|x - a| + |y - b| + |(x - a) + (b - y)|) \\ &\leq |x - a| + |b - y|. \end{aligned}$$

Note that if  $d_2((x, y), (a, b)) < \delta$ , then  $|x - a| < \delta$  and  $|y - b| < \delta$ . Hence, for every  $\varepsilon > 0$ , we can just pick  $\delta = \frac{\varepsilon}{2}$ , and then we can show that

$$|\max\{x, y\} - \max\{a, b\}| < \varepsilon.$$

■

**Corollary 3.1.1.** Let  $(X, d)$  be a metric space, and let  $f, g : X \rightarrow \mathbb{R}$  be functions. Let  $c \in \mathbb{R}$ . If  $f, g$  are continuous on  $X$ , then  $f + g, f - g, f \cdot g, \max\{f, g\}, \min\{f, g\}, cf$  are continuous. Also,  $\frac{f}{g}$  is also continuous at  $x_0$  if  $g(x_0) \neq 0$ .

**Proof.** For example, we know

$$(f + g)(x) = \text{Add} \circ (f, g)(x) = \text{Add}(f(x), g(x)) = f(x) + g(x).$$

Since  $(f, g)$  is continuous and Add is continuous, so the composition function  $\text{Add} \circ (f, g)$  is also

continuous. ■

## 3.2 Continuity and Compactness

**Theorem 3.2.1.** Let  $f : X \rightarrow Y$  be a continuous function from  $(X, d_X)$  to  $(Y, d_Y)$ . Let  $K \subseteq X$  be a compact subset of  $X$ . Then

$$f(K) = \{f(x) \mid x \in K\}$$

is also compact in  $Y$ .

**Proof.** Given any sequence  $(y^{(n)})_{n=1}^\infty$  in  $f(K)$ , then  $\exists x^{(n)} \in K$  s.t.  $f(x^{(n)}) = y^{(n)}$ . Since  $K$  is compact, then there exists a convergent subsequence  $(x^{(n_k)})_{k=1}^\infty$  s.t.  $\lim_{k \rightarrow \infty} x^{(n_k)} = x_* \in K$ . Since  $f$  is continuous, so  $\lim_{n \rightarrow \infty} f(x^{(n_k)}) = f(x_*)$  in  $Y$ . Hence,  $\lim_{n \rightarrow \infty} y^{(n_k)} = f(x_*) \in f(K)$ . Thus,  $f(K)$  is compact. ■

**Another method.** Let  $\{V_\alpha : \alpha \in A\}$  be any open cover of  $f(K)$  i.e.  $f(K) \subseteq \bigcup_{\alpha \in A} V_\alpha$  and  $V_\alpha$  is open in  $Y$ . Hence,

$$K \subseteq \bigcup_{\alpha \in A} f^{-1}(V_\alpha).$$

Note that since  $f$  is continuous and  $V_\alpha$  is open, so  $f^{-1}(V_\alpha)$  is open by [Theorem 3.0.2](#). Hence,  $\{f^{-1}(V_\alpha) : \alpha \in A\}$  is an open cover of  $K$ . Since  $K$  is compact, then

$$K \subseteq \bigcup_{i=1}^n f^{-1}(V_{\alpha_i})$$

Hence,  $f(K) \subseteq \bigcup_{i=1}^n V_{\alpha_i}$ , which shows  $f(K)$  is compact. ■

## Lecture 10

**Proposition 3.2.1.** Let  $(X, d)$  be a compact metric space and let  $f : X \rightarrow \mathbb{R}$  be a continuous map. Then

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(1)  $f$  is bounded on  $X$  i.e.  $f(X)$  is bounded in  $\mathbb{R}$ .

(2)  $\exists x_{\max}, x_{\min} \in X$  s.t.

$$f(x_{\max}) = \max_{x \in X} f(x) \quad f(x_{\min}) = \min_{x \in X} f(x).$$

**Proof.** Since  $X$  is compact and  $f$  is continuous, so  $f(X)$  is compact in  $\mathbb{R}$  and thus  $f(X)$  is bounded by [Corollary 2.5.1](#). Now since  $\{f(x) \mid x \in X\}$  is bounded in  $\mathbb{R}$ , so  $\exists p_1, p_2$  s.t.  $p_1 \leq f(x) \leq p_2$  for all  $x \in X$ . Let  $M = \sup_{x \in X} f(x)$  and  $P = \inf_{x \in X} f(x)$ . Thus, there exists  $\{y_n\}_{n=1}^\infty \subseteq f(X)$  s.t.  $y_n \leq M$  and  $\lim_{n \rightarrow \infty} y_n = M$  for all  $n$ . Hence,  $\exists \{x^{(n)}\}_{n=1}^\infty$  s.t.  $y_n = f(x^{(n)})$  for all  $n$ . Since  $X$  is compact, so there exists a subsequence  $\{x^{(n_k)}\}_{k=1}^\infty$  s.t.

$$\lim_{k \rightarrow \infty} x^{(n_k)} = x_* \in X.$$

Since  $f$  is continuous, so  $\lim_{k \rightarrow \infty} f(x^{(n_k)}) = f(x_*)$ , which means  $\lim_{k \rightarrow \infty} y_{n_k} = f(x_*)$ . Since  $\lim_{n \rightarrow \infty} y_n = M$ , so  $f(x_*) = M$ , and thus  $f(x_*) = \max_{x \in X} f(x)$ .

**Question.** Why we can always find  $\{y_n\}_{n=1}^\infty \subseteq f(X)$  converges to  $M = \sup \{f(X)\}$ ?

**Answer.** Recall the definition of sup, we know  $\forall \varepsilon > 0$ ,  $M - \varepsilon < y_\varepsilon \leq M$  for some  $y_\varepsilon \in f(X)$ , so for all  $\varepsilon = \frac{1}{N}$ , we can pick all  $y_N$  to form a sequence converge to  $M$ . ⊗

**Question.** Why  $\max_{x \in X} f(x) = \sup_{x \in X} f(x)$  here?

**Answer.** Since  $X$  is compact, so  $x_* \in X$ , which has been proved, and thus  $M = f(x_*) \in f(X)$ , which means  $\sup_{x \in X} f(x) \in f(X)$ . This is equivalent to  $\sup_{x \in X} f(X) = \max_{x \in X} f(x)$ .  $\circledast$

**Example 3.2.1.** This proposition is false if  $X$  is not compact.

**Proof.** Consider  $f : (0, 1) \rightarrow \mathbb{R}$  and  $f(x) = x$ , then  $f$  can't achieve its sup on  $(0, 1)$ .

**Note 3.2.1.**  $(0, 1)$  is not compact since if it is compact, then it is closed, and thus all convergent sequence in  $(0, 1)$  converging in  $(0, 1)$ , but consider the sequence  $\{\frac{1}{n}\}_{n=2}^{\infty}$ , and this sequence converges to  $0 \notin (0, 1)$ .

$\circledast$

### 3.3 Uniformly Continuous

**Definition 3.3.1 (uniformly continuous).** Let  $f : X \rightarrow Y$  be a map between metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , we say  $f$  is uniformly continuous if  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.

$$d_Y(f(x), f(x')) < \varepsilon \text{ whenever } x, x' \in X \text{ and } d_X(x, x') < \delta.$$

**Remark 3.3.1.** This  $\delta$  is independent of  $x'$ .

**Example 3.3.1.**  $f(x) = \frac{1}{x}$  is continuous on  $(0, 1]$ , and  $f$  is not uniformly continuous on  $(0, 1]$ .

**Proof.** Let  $\varepsilon = 10$ . Suppose  $\exists \delta > 0$  s.t.

$$|f(x) - f(y)| < \varepsilon \text{ if } |x - y| < \delta.$$

We may assume  $\delta < 1$ . Choose  $x = \delta$  and  $y = \frac{\delta}{11}$ , then  $|x - y| = \frac{10}{11}\delta < \delta$ , and

$$|f(x) - f(y)| = \left| \frac{1}{\delta} - \frac{11}{\delta} \right| = \frac{10}{\delta} > 10.$$

$\circledast$

**Theorem 3.3.1.** Suppose  $X$  is compact, then  $f : X \rightarrow Y$  is continuous iff  $f$  is uniformly continuous.

**Proof.** If  $f$  is uniformly continuous, then  $f$  is continuous. Now we show the other direction. If  $f$  is continuous, then by contradiction, if it is not uniformly continuous, then  $\exists \varepsilon > 0$  s.t. no matter how small  $\delta$  is, then  $\exists p, q$  s.t.  $d_X(p, q) < \delta$  and  $d_Y(f(p), f(q)) \geq \varepsilon$ . Choose  $\delta = \frac{1}{n}$ , then exists  $p^{(n)}, q^{(n)} \in X$  s.t.

$$d_X(p^{(n)}, q^{(n)}) < \frac{1}{n} \text{ and } d_Y(f(p^{(n)}), f(q^{(n)})) \geq \varepsilon.$$

Since  $X$  is compact, so  $\{p^{(n)}\}$  has a convergent subsequence, say

$$\lim_{k \rightarrow \infty} p^{(n_k)} = p \in X.$$

Also, there exists  $\{q^{(n_k)}\}$  s.t.  $\lim_{k \rightarrow \infty} q^{(n_k)} = p$  since  $\lim_{k \rightarrow \infty} d_X(p^{(n_k)}, q^{(n_k)}) = 0$ . (By [Theorem A.1.4](#)). Thus, we have

$$\lim_{k \rightarrow \infty} d_Y(f(p^{(n_k)}), f(q^{(n_k)})) = 0$$

since  $f$  is continuous. Hence, it is a contradiction, since

$$\lim_{n \rightarrow \infty} d_Y \left( f \left( p^{(n)} \right), f \left( q^{(n)} \right) \right) \geq \varepsilon.$$

■

### 3.4 Connectedness

**Definition 3.4.1 (disconnected/connected).** Let  $(X, d)$  be a metric space. We say  $X$  is disconnected iff  $\exists$  non-empty open  $V, W$  in  $X$  s.t.  $V \cup W = X$  and  $V \cap W = \emptyset$ . Also, we called  $X$  is connected if it is non-empty and not disconnected.

**Remark 3.4.1.**  $X$  is disconnected iff  $X$  has a nonempty proper subset which is both open and closed.

**Proof.**

- ( $\Rightarrow$ )  $V \cup W = X$  and  $V \cap W = \emptyset$ , so  $X \setminus V = W$  is open, and thus  $V$  is closed. Since we already know  $V$  is open, so  $V$  is a proper subset of  $X$  that is clopen.
- ( $\Leftarrow$ ) Suppose  $V$  is clopen in  $X$  and  $V \neq X$ . Let  $W = X \setminus V \neq \emptyset$ , then we know  $W$  is open since  $V$  is closed and thus  $V \cup W = X$  and  $V \cap W = \emptyset$  and  $V, W$  are both non-empty open in  $X$ , so  $X$  is disconnected.

■

**Example 3.4.1.** Suppose  $X = [1, 2] \cup [3, 4]$ , then  $X$  is disconnected.

**Proof.** Since

$$[1, 2] = (-\infty, 2.5) \cap X,$$

so  $[1, 2]$  is open in  $X$ , and similarly we can show  $[3, 4]$  is open in  $X$ , and  $[1, 2] \cup [3, 4] = X$ , and  $[1, 2] \cap [3, 4] = \emptyset$ , so  $X$  is disconnected. \*

**Definition 3.4.2.** Let  $(X, d)$  be a metric space and  $Y \subseteq X$ . We say  $Y$  is connected (resp. disconnected) iff the metric space  $(Y, d|_{Y \times Y})$  is connected (resp. connected).

**Remark 3.4.2.**  $Y \subseteq X$  is disconnected iff  $\exists U, V$  open in  $X$  s.t.  $Y \subseteq U \cup V$  with  $U \cap Y \neq \emptyset$  and  $V \cap Y \neq \emptyset$  and  $U \cap V \cap Y = \emptyset$ .

**Proof.**

- ( $\Rightarrow$ ) If  $Y$  is disconnected, then  $\exists$  open sets  $O_1, O_2$  in  $Y$  s.t.  $Y \subseteq O_1 \cup O_2$  and  $O_1 \neq \emptyset$  and  $O_2 \neq \emptyset$  and  $O_1 \cap O_2 = \emptyset$ . Since  $O_1$  is open in  $Y$ , so there exists open set  $U_1$  in  $X$  s.t.  $O_1 = U_1 \cap Y \neq \emptyset$ . Similarly, we know there exists  $U_2$  open in  $X$  s.t.  $O_2 = U_2 \cap Y \neq \emptyset$ . Since

$$Y \subseteq O_1 \cup O_2 = (U_1 \cap Y) \cup (U_2 \cap Y) = (U_1 \cup U_2) \cap Y \subseteq U_1 \cup U_2,$$

and we know

$$\emptyset = O_1 \cap O_2 = (U_1 \cap Y) \cap (U_2 \cap Y) = U_1 \cap U_2 \cap Y.$$

- ( $\Leftarrow$ ) Choose  $O_1 = U \cap Y$  and  $O_2 = V \cap Y$ . Note that  $O_1, O_2$  are non-empty and open in  $Y$ , and we can easily check that  $O_1 \cup O_2 = Y$  and  $O_1 \cap O_2 = \emptyset$ , so  $Y$  is disconnected.

■

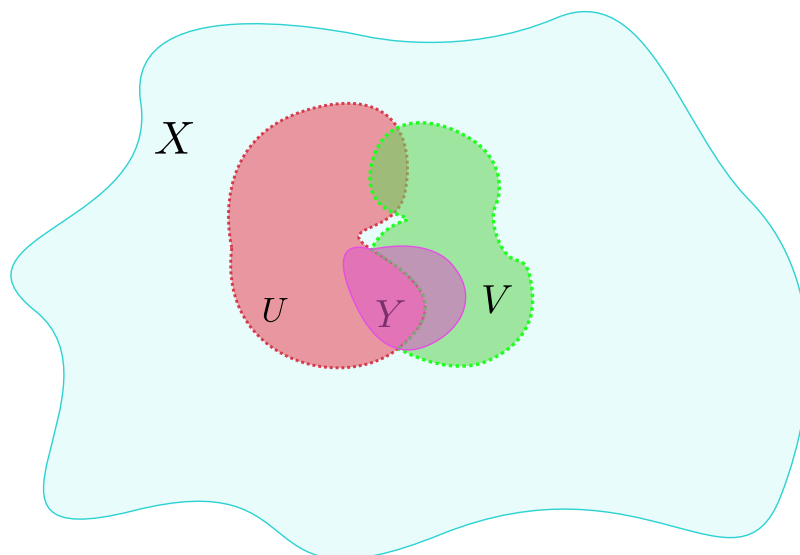


Figure 3.2: Disconnected set condition in Remark 3.4.2

**Theorem 3.4.1.** Let  $f : X \rightarrow Y$  be a continuous map between metric spaces. Let  $E$  be any connected subset of  $X$ . Then  $f(E)$  is connected in  $Y$ .

**Proof.** Suppose not,  $f(E)$  is disconnected in  $Y$ , then  $\exists$  open set  $U, V$  in  $Y$  s.t.  $f(E) \subseteq U \cup V$  and  $U \cap f(E) \neq \emptyset$  and  $V \cap f(E) \neq \emptyset$  and  $U \cap V \cap f(E) = \emptyset$ . Since  $f$  is continuous, so  $f^{-1}(U)$  and  $f^{-1}(V)$  are both open and non-empty and  $(E \cap f^{-1}(U)) \cap (E \cap f^{-1}(V)) = \emptyset$ , and note that  $E \subseteq (E \cap f^{-1}(U)) \cup (E \cap f^{-1}(V))$ , so  $E$  is disconnected.

**Remark 3.4.3.**  $E \cap f^{-1}(U)$  and  $E \cap f^{-1}(V)$  are open in  $E$ .

**Theorem 3.4.2.** Let  $X$  be a nonempty subset of  $\mathbb{R}$ , then TFAE:

- (a)  $X$  is connected.
- (b) Whenever  $x, y \in X$  and  $x < y$ , we have  $[x, y] \subseteq X$ .
- (c)  $X$  is an interval.

**proof from (a) to (b).** Suppose not, then there exists  $z \notin X$  s.t.  $x < z < y$ . Hence, we can pick  $U = (-\infty, z) \cap X$  and  $V = (z, \infty) \cap X$ , then  $U \neq \emptyset$  and  $V \neq \emptyset$  and  $U, V$  both open in  $X$  and  $U \cap V = \emptyset$  and  $X \subseteq U \cup V$ , so  $X$  is disconnected, which is a contradiction. ■

**proof from (b) to (a).** Suppose not, then  $X$  is disconnected, and thus there exists  $V, W$  open in  $X$  s.t.  $X = V \cup W$  and  $V \cap W = \emptyset$ . Now fix  $x \in V$  and  $y \in W$ . WLOG, suppose  $x < y$ , then  $[x, y] \subseteq X = V \cup W$  by the hypothesis. Now suppose  $S = [x, y] \cap V \subseteq X \subseteq \mathbb{R}$ , then since  $y \geq s$  for all  $s \in S$ , so  $S$  is a subset of  $\mathbb{R}$  which is bounded above, and thus  $z = \sup S$  exists. Note that  $z \leq y$ , so  $z \in [x, y] \subseteq X = V \cup W$ .

- Case 1:  $z \in V$ , then  $z < y$  since  $y \in W$  and  $V \cap W = \emptyset$ . Now since  $V$  is open in  $X$ , so there exists  $\varepsilon > 0$  s.t.  $B_X(z, \varepsilon) \subseteq V$ , which means  $(z - \varepsilon, z + \varepsilon) \cap X \subseteq V$ . In particular, we have  $(z - \varepsilon, z + \varepsilon) \cap [x, y] \subseteq V$ , and thus

$$(z, z + \varepsilon) \cap [x, y] \subseteq V \cap [x, y] = S.$$

Now since  $z < y$ , so  $p \in (z, z + \varepsilon) \cap [x, y]$  for some  $p$ , which means  $p \in S$ . However,  $p \in (z, z + \varepsilon)$  gives  $p > z$ , so  $S$  contains a  $p > z = \sup S$ , which is a contradiction.

- Case 2:  $z \in W$ , then there exists  $\varepsilon > 0$  s.t.  $B_X(z, \varepsilon) \subseteq W$  since  $W$  is open in  $X$ . Hence,

$$(z - \varepsilon, z + \varepsilon) \cap [x, y] \subseteq (z - \varepsilon, z + \varepsilon) \cap X \subseteq W.$$

Hence, we have  $(z - \varepsilon, z + \varepsilon) \cap [x, y] \cap V = \emptyset$  since  $V \cap W = \emptyset$ . Note that this means  $(z - \varepsilon, z + \varepsilon) \cap S = \emptyset$ . However, we can construct a sequence of  $S$  converges to  $\sup S$  (See [Theorem A.1.5](#)), so there exists  $y \in S$  s.t.  $y \in (z - \frac{\varepsilon}{2}, z + \frac{\varepsilon}{2}) \subseteq (z - \varepsilon, z + \varepsilon)$ , which means  $y \in (z - \varepsilon, z + \varepsilon) \cap S = \emptyset$ , so it is a contradiction. ■

**Remark 3.4.4.** The fact that (b) is equivalent to (c) is trivial, so we don't give a proof.

## Lecture 11

**Theorem 3.4.3** (Review of [Theorem 3.4.1](#)). Let  $f : X \rightarrow Y$  be a continuous map and let  $E$  be a connected subset of  $X$ . Then  $f(E)$  is connected in  $Y$ .

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**Corollary 3.4.1** (Intermediate value theorem). Let  $f : X \rightarrow \mathbb{R}$  be a continuous map from  $(X, d_X)$  to  $\mathbb{R}$ . Let  $E \subseteq X$  be any connected subset, and let  $a, b \in E$ . Suppose  $y$  is a real number between  $f(a)$  and  $f(b)$  i.e.

$$\min \{f(a), f(b)\} \leq y \leq \max \{f(a), f(b)\},$$

then  $\exists c \in E$  s.t.  $f(c) = y$ .

**Proof.** There are 3 cases:

- Case 1:  $f(a) = f(b)$ , then trivial.
- Case 2:  $f(a) < f(b)$ , Since  $E$  is connected and  $f$  is continuous, so  $f(E)$  is connected in  $\mathbb{R}$ . Hence, for  $f(a), f(b) \in f(E)$ , we know  $(f(a), f(b)) \subseteq f(E)$  by [Theorem 3.4.2](#), so if  $f(a) < y < f(b)$ , then  $\exists c \in E$  s.t.  $f(c) = y$ .
- Case 3:  $f(a) > f(b)$ , then let  $a' = b$  and  $b' = a$ , then  $f(a') < f(b')$  and use the result of Case 2.

## 3.5 Topological space

In metric space  $(X, d_X)$ , we define open ball

$$B_X(x, r) = \{y \mid d_X(y, x) < r\},$$

and a set  $u$  is open if for any  $x \in u$ ,  $\exists r_x > 0$  s.t.  $B_X(x, r_x) \subseteq u$ , so  $u = \bigcup_{x \in u} B_X(x, r_x)$ . Hence, in metric space open sets are in fact union of open balls. We also proved that

- $\emptyset$  and  $X$  are open.
- If  $u_1, \dots, u_n$  are open in  $X$ , then  $\bigcap_{i=1}^n u_i$  is open in  $X$ .
- If  $\{u_i\}_{i \in A}$  are open in  $X$ , then  $\bigcup_{i \in A} u_i$  is also open.

Now we want to extend this concept.

**Definition 3.5.1 (Power sets).** For a given set  $X$ , we define  $2^X$  the power set of  $X$  i.e.

$$2^X := \{A : A \subseteq X\}$$

the collection of all subsets of  $X$ .

**Example 3.5.1.**  $X = \{a, b\}$  for  $a \neq b$ , then

$$2^X = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

**Definition 3.5.2 (Topological space).** A topological space is a pair  $(X, \mathcal{F})$ , where  $X$  is a set and  $\mathcal{F} \subseteq 2^X$  is a collection of subsets of  $X$ , called the open sets. The collection  $\mathcal{F}$  must satisfy

- $\emptyset$  and  $X$  are all in  $\mathcal{F}$ .
- If  $u_1, \dots, u_n$  are in  $\mathcal{F}$ , then  $\bigcap_{i=1}^n u_i$  is in  $\mathcal{F}$ .
- If  $\{u_i\}_{i \in A}$  are in  $\mathcal{F}$ , then  $\bigcup_{i \in A} u_i$  is in  $\mathcal{F}$ .

**Remark 3.5.1.** In a metric space, let

$$\mathcal{F} = \text{the set of open sets in } (X, d_X) = \{u \mid \forall x \in u, \exists r_x > 0 \text{ s.t. } B_X(x, r_x) \subseteq u\},$$

then  $(X, \mathcal{F})$  is a topological space.

**Example 3.5.2.** On any set  $X \neq \emptyset$ , we have a trivial topology on  $X$  i.e.  $\mathcal{F} = \{\emptyset, X\}$ , which means  $(X, \mathcal{F})$  is a topological space in  $X$ .

**Example 3.5.3.** Consider  $\mathcal{F} = 2^X$ , then  $(X, 2^X)$  is also a topological space on  $X$ .

**Definition 3.5.3 (Neighborhood).** Let  $(X, \mathcal{F})$  be a topological space, and let  $x \in X$ . A neighborhood of  $x$  is any open set  $u \in \mathcal{F}$  s.t.  $x \in u$ .

**Example 3.5.4.**  $X = \{a, b\}$  and  $a \neq b$ ,  $\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ , then  $\{a\}, \{a, b\}$  are neighborhoods of  $a$  and  $\{b\}, \{a, b\}$  are neighborhoods of  $b$ .

**Definition 3.5.4 (Interior/Exterior/Boundary point).** Let  $(X, \mathcal{F})$  be a topological space, and  $E \subseteq X$  be a subset. We say that

- $x_0$  is an interior point of  $E$  if  $\exists$  a neighborhood  $V$  of  $x_0$  s.t.  $V \subseteq E$ .
- $x_0$  is an exterior point of  $E$  if  $\exists$  a neighborhood  $V$  of  $x_0$  s.t.  $V \subseteq X \setminus E$ .
- $x_0$  is a boundary point if it is neither interior or exterior.

**Corollary 3.5.1.**

$$X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E.$$

**Proof.** ■

DIY

**Definition 3.5.5 (Adherent point).** We say  $x_0$  is an adherent point of  $E$  if every neighborhood  $V$  of  $x_0$  has a nonempty intersection with  $E$ , and we called  $\overline{E}$  the set of all adherent points.

**Corollary 3.5.2.**  $\overline{E} = \text{Int}(E) \cup \partial E$ .

**Proof.** We first show that  $\text{Int}(E) \cup \partial E \subseteq \overline{E}$ . Suppose  $x_0 \in \text{Int}(E)$ , then  $x_0 \in E$ , so  $x_0 \in \overline{E}$  since  $E \subseteq \overline{E}$ . Now if  $x_0 \in \partial E$ , then any neighborhood  $V$  of  $x_0$  contains points in  $E$ , so  $x_0 \in \overline{E}$ . Now we show that  $\overline{E} \subseteq \text{Int}(E) \cup \partial E$ . Since  $\overline{E} \subseteq X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$ , so we want to show for all  $x_0 \in \overline{E}$ , we have  $x_0 \notin \text{Ext}(E)$ . By the definition of exterior point, we can easily show this. ■

**Definition (Open and Closed Sets).** Suppose  $(X, \mathcal{F})$  is a topological space. Then:

- A set  $O \subseteq X$  is called *open* if  $O \in \mathcal{F}$ .
- A set  $F \subseteq X$  is called *closed* if its complement  $X \setminus F$  is open, i.e.,  $X \setminus F \in \mathcal{F}$ .

**Corollary 3.5.3.** A set  $E \subseteq X$  is open if and only if  $E = \text{Int}(E)$ .

**Proof.**

( $\Rightarrow$ ) If  $E$  is open, then since

$$\text{Int}(E) = \bigcup \{O \subseteq E : O \in \mathcal{F}\}, \quad (3.1)$$

we know  $E \subseteq \text{Int}(E)$ .

( $\Leftarrow$ ) By the definition of topological space and Equation 3.1, we know  $\text{Int}(E)$  is open, so  $E = \text{Int}(E)$  implies  $E$  is open. ■

**Corollary 3.5.4.** A set  $F \subseteq X$  is closed if and only if  $\overline{F} = F$ .

**Proof.** If  $F$  is closed, then  $X \setminus F$  is open, so  $X \setminus F = \text{Int}(X \setminus F)$ , so

$$F = X \setminus (X \setminus F) = X \setminus \text{Int}(X \setminus F) = X \setminus \text{Ext}(F) = \text{Int}(F) \cup \partial(F) = \overline{F}.$$

The other direction is similar. ■

**Definition 3.5.6 (Topological subspace).** Let  $(X, \mathcal{F})$  be a topological space and  $Y \subseteq X$ . We define  $\mathcal{F}_Y = \{V \cap Y \mid V \in \mathcal{F}\}$  and call  $(Y, \mathcal{F}_Y)$  the topological subspace of  $(X, \mathcal{F})$  induced by  $Y$ . We can show that  $\mathcal{F}_Y$  is a topology on  $Y$ .

**Definition 3.5.7 (Continuous map).** Let  $(X, \mathcal{F})$  and  $(Y, g)$  be topological spaces and let  $f : X \rightarrow Y$  be a function. We say  $f$  is continuous at  $x_0 \in X$  if for every neighborhood  $V \in g$  of  $f(x_0)$ , there exists a neighborhood  $u \in \mathcal{F}$  of  $x_0$  s.t.  $u \subseteq f^{-1}(V)$ .

**Definition 3.5.8 (Convergence).** Let  $m$  be an integer,  $(X, \mathcal{F})$  be a topological space, and let  $(x^{(n)})_{n=m}^{\infty}$  be a sequence of points in  $X$ . We say  $x^{(n)} \rightarrow x$  iff for every neighborhood  $V$  of  $x$ ,  $\exists N \geq m$  s.t.  $x^{(n)} \in V$  for all  $n \geq N$ .

**Remark 3.5.2.** In a topological space, a sequence may converge to more than one points.

**Example 3.5.5.** Let  $X = \{a, b\}$  and  $a \neq b$ . Consider the trivial topology  $\mathcal{F} = \{\emptyset, X\}$ . If we have the constant sequence  $\{x^{(n)}\}_{n=m}^{\infty}$  with  $x^{(n)} = a$ , then  $\lim_{n \rightarrow \infty} x^{(n)} = a$  and  $\lim_{n \rightarrow \infty} x^{(n)} = b$ .

**Proof.** Any neighborhood of  $a$  must be  $\{a, b\}$ , and any neighborhood of  $b$  must also be  $\{a, b\}$ , so  $\lim_{n \rightarrow \infty} x^{(n)} = a$  and  $\lim_{n \rightarrow \infty} x^{(n)} = b$ . (⊛)

**Example 3.5.6.** On  $\mathbb{R}$ , we consider

$$\mathcal{F} = \{\emptyset\} \cup \{u : \mathbb{R} \setminus u \text{ is finite points}\},$$

then  $\mathcal{F}$  is a topology on  $\mathbb{R}$ .

**Proof.** Since  $\emptyset, \mathbb{R} \in \mathcal{F}$ , so it satisfies the first rule of topology. Now if  $u_1 \in \mathcal{F}$  and  $u_2 \in \mathcal{F}$ , then we want to show  $u_1 \cap u_2 \in \mathcal{F}$ . We assume  $u_1, u_2$  are non-empty, otherwise it is trivial. Consider  $\mathbb{R} \setminus (u_1 \cap u_2) = (\mathbb{R} \setminus u_1) \cup (\mathbb{R} \setminus u_2)$ , since  $\mathbb{R} \setminus u_1$  and  $\mathbb{R} \setminus u_2$  are both finite points, so  $u_1 \cap u_2$  is finite points. Hence, we know for finitely many  $u_1, \dots, u_n$ ,  $\bigcap_{i=1}^n u_i$  is in  $\mathcal{F}$ . Now we know for  $\mathcal{F}_\alpha \in \mathcal{F}$ ,

$$\mathbb{R} \setminus \left( \bigcup_{\alpha} \mathcal{F}_{\alpha} \right) = \bigcap_{\alpha} (\mathbb{R} \setminus \mathcal{F}_{\alpha}) \subseteq \mathbb{R} \setminus \mathcal{F}_{\alpha_i}$$

for some  $\alpha_i$  in the index set, so  $\mathbb{R} \setminus (\bigcup_{\alpha} \mathcal{F}_{\alpha})$  is also finite points, and we're done.  $\circledast$

**Remark 3.5.3.** In the topological space induced by the topology in [Example 3.5.6](#). If we consider  $\{x^{(n)}\}_{n=1}^{\infty}$  with  $x^{(n)} = n$ , then  $\lim_{n \rightarrow \infty} x^{(n)} = p$  for any  $p \in \mathbb{R}$ .

**Proof.** Since any neighborhood  $u$  of  $p$  has

$$\mathbb{R} \setminus u = \{p_1, \dots, p_k\}$$

with  $p_1 < p_2 < \dots < p_k$ , so we have  $x^{(n)} \in u$  for  $n > p_k$ .  $\blacksquare$

## Lecture 12

Last time, we show that the sequence  $\{x_n\}_{n=1}^{\infty}$  with  $x_n = n$  converges to any point  $p$  in  $\mathbb{R}$  in cofinite topology i.e. 9 Oct. 10:20

$$\mathcal{F} = \{\emptyset\} \cup \{\mathbb{R} \setminus \{\text{finite points}\}\}$$

because each non-empty neighborhood of  $p$  is very big. In general, a sequence in a topological space may converge to more than one points.

**Definition 3.5.9 (Hausdorff).** A topological space  $(X, \mathcal{F})$  is called Hausdorff if given any two distinct points  $x, y \in X$ , there exists open sets  $U, V \in \mathcal{F}$  s.t.  $x \in U$  and  $y \in V$  and  $U \cap V = \emptyset$ .

**Example 3.5.7.** A metric space is Hausdorff since given  $x \neq y$ ,  $B_X(x, \frac{r}{2})$  and  $B_X(y, \frac{r}{2})$  are open and they separate  $x$  and  $y$  where  $r = \frac{d(x, y)}{2}$ .

**Theorem 3.5.1.** Suppose  $(X, \mathcal{F})$  is a Hausdorff topological space, then the limit of a convergent sequence is unique.

**Proof.** If  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} x_n = y$  for some  $x \neq y$ , then since  $(X, \mathcal{F})$  is Hausdorff, so there exists neighborhood  $U$  of  $x$  and  $V$  of  $y$  and  $U \cap V = \emptyset$ . Also, there exists  $N_1 > 0$  s.t.  $x_n \in U$  if  $n \geq N_1$ , and there exists  $N_2 > 0$  s.t.  $x_n \in V$  if  $n \geq N_2$ . Hence, for all  $n \geq \max\{N_1, N_2\}$ , we know  $x_n \in U \cap V = \emptyset$ , which is a contradiction. Hence, the limit of a convergence sequence is unique.  $\blacksquare$

**Definition 3.5.10 (Compact).** Let  $(X, \mathcal{F})$  be topological space, we say  $X$  is compact if for every open cover

$$\{U_{\alpha} : \alpha \in A\} \subseteq \mathcal{F} \text{ with } X \subseteq \bigcup_{\alpha \in A} U_{\alpha},$$

there exists a finite subcover  $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$  s.t.

$$X \subseteq \bigcup_{i=1}^n U_{\alpha_i}.$$

**Theorem 3.5.2.** Let  $f : (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$  be a continuous map between topological spaces. If  $K \subseteq X$  is compact, then  $f(K)$  is also compact in  $(Y, \mathcal{G})$ .

**Proof.** Let  $\{V_\alpha\}_{\alpha \in A}$  be an open cover of  $f(K)$  i.e.  $\{V_\alpha\}_{\alpha \in A} \subseteq \mathcal{G}$  and  $f(K) \subseteq \bigcup_{\alpha \in A} V_\alpha$ . Since  $f$  is continuous, so  $f^{-1}(V_\alpha) \in \mathcal{F}$  and

$$K \subseteq \bigcup_{\alpha \in A} f^{-1}(V_\alpha).$$

Now since  $K$  is compact, so there exists finite subcover, which means  $K \subseteq \bigcup_{i=1}^n f^{-1}(V_{\alpha_i})$ , so  $f(K) \subseteq \bigcup_{i=1}^n V_{\alpha_i}$ , and thus  $f(K)$  has a finite subcover of  $\{V_\alpha\}_{\alpha \in A}$ , which means  $f(K)$  is compact.

**Remark 3.5.4.** In topological space, we also have:  $f : X \rightarrow Y$  is continuous if and only if whenever  $V \subseteq Y$  is open (resp. closed), we have  $f^{-1}(V)$  is open (resp. closed) in  $X$ .

**Proof.** ■

DIY

**Proposition 3.5.1.** Let  $(X, \mathcal{F})$  be a compact topological space and  $f : X \rightarrow \mathbb{R}$  is continuous, then

- (1)  $f$  is bounded on  $X$ .
- (2) If  $X \neq \emptyset$ , then  $\exists x_{\min}, x_{\max} \in X$  s.t.  $f(x_{\max}) = \max_{x \in X} f(x)$  and  $f(x_{\min}) = \min_{x \in X} f(x)$ .

**Proof.**

- (1) Since  $X$  is compact, so  $f(X)$  is compact in  $\mathbb{R}$  by [Theorem 3.5.2](#), and since  $\mathbb{R}$  is a metric space, so  $f(X)$  is closed and bounded in  $\mathbb{R}$ , which means  $f$  is bounded on  $X$ .
- (2) Now since  $f(X)$  is bounded, so  $\sup_{x \in X} f(x)$  and  $\inf_{x \in X} f(x)$  exists. Thus, we can pick  $(y_n) \in f(X)$  and  $(z_n) \in f(X)$  s.t.  $y_n \rightarrow \sup_{x \in X} f(x)$  and  $z_n \rightarrow \inf_{x \in X} f(x)$  by [Theorem A.1.5](#). Now since  $f(X)$  is closed, so

$$\sup_{x \in X} f(x) \in \overline{f(X)} = f(X),$$

so there exists  $x^*$  s.t.  $f(x^*) = \sup_{x \in X} f(x)$  and similarly the "min" case can be proved. ■

## Chapter 4

# Uniform Convergence

In a metric space  $(X, d)$ , we define the convergence of a sequence  $\{x^{(n)}\}_{n=m}^{\infty}$ ,  $\lim_{n \rightarrow \infty} x^{(n)} = x$ , by "Given any  $\varepsilon > 0$ ,  $\exists N \geq m$  s.t.  $d(x^{(n)}, x) < \varepsilon$  for all  $n \geq N$ ". Now suppose

$$f^{(n)} : X \rightarrow Y \quad \forall n \in \mathbb{N},$$

where  $X, Y$  are metric spaces, then if we define the convergence of these functions at some point  $x$  to be:

$$f(x) = \lim_{n \rightarrow \infty} f^{(n)}(x).$$

Do we have

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f^{(n)}(x) = \lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} f^{(n)}(x)?$$

Short answer: Not always true.

In this chapter, we will discuss the concept of limiting function, that is,

$$\lim_{n \rightarrow \infty} f^{(n)} = f.$$

### 4.1 Limiting values of functions

**Definition 4.1.1.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and let  $E \subseteq X$  and  $f : E \rightarrow Y$  be a function. If  $x_0 \in X$  is an adherent point of  $E$ , and  $L \in Y$ , we say that

$$\lim_{x \rightarrow x_0, x \in E} f(x) = L$$

if for every  $\varepsilon > 0$ ,  $\exists \delta > 0$  s.t.  $d_X(x, x_0) < \delta$  and  $x \in E$  implies  $d_Y(f(x), L) < \varepsilon$ .

**Remark 4.1.1.** In this definition, we need not  $x_0 \in E$ , we just need  $x_0 \in \overline{E}$ .

**Remark 4.1.2.** In other textbook, if

$$f(x) = \begin{cases} |x|, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0, \end{cases}$$

then  $\lim_{x \rightarrow 0} f(x) = 0$  because it does not consider  $x = 0$ . More precisely, the definition of

$$\lim_{x \rightarrow x_0, x \in E} f(x) = L$$

in other textbook is " $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  s.t.  $d_Y(f(x), L) < \varepsilon$  for all  $x \in E$  and  $0 < d_X(x, x_0) < \delta$ ". Note that it exclude the case  $x = x_0$ .

However, if  $x_0 \in E$ , then by Terrence Tao's definition,  $f(x_0) = L$  if  $d_Y(f(x), L) < \varepsilon$  for all  $\varepsilon > 0$  and for  $d_X(x, x_0) < \delta$  for the corresponding  $\delta$ . Also, if  $x_0 \notin E$ , then since  $x_0 \in \overline{E}$ , so  $\exists x \in E$  s.t.

$d(x, x_0) < \delta$ , so the definition of  $\lim_{x \rightarrow x_0, x \in E} f(x)$  is well-defined. In our notation, other textbooks' definition is like

$$\lim_{x \rightarrow x_0, x \in E \setminus \{x_0\}} f(x) = L.$$

**Lemma 4.1.1.** If  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces, then  $f : X \rightarrow Y$  is continuous at  $x_0 \in X$  is in fact

$$\lim_{x \rightarrow x_0, x \in X} f(x) = f(x_0).$$

**Proof.** Since  $f$  is continuous at  $x_0$  means for all  $(x_n) \rightarrow x_0$ , we have  $f(x_n) \rightarrow f(x_0)$ , so this is true. ■

**Proposition 4.1.1.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and  $f : X \rightarrow Y$  be a function. Let  $x_0 \in X$  be an adherent point of  $E \subseteq X$  and  $L \in Y$ , then TFAE:

- (a)  $\lim_{x \rightarrow x_0, x \in E} f(x) = L$ .
- (b) For every sequence  $\{x^{(n)}\}_{n=1}^{\infty}$  in  $E$  converges to  $x_0$ , the sequence  $\lim_{n \rightarrow \infty} f(x^{(n)}) = L$  in  $Y$ .
- (c) For every open set  $V \subseteq Y$  containing  $L$ , there exists an open set  $U \subseteq X$  containing  $x_0$  s.t.  $U \cap E \subseteq f^{-1}(V)$ .
- (d) If one define  $g : E \cup \{x_0\} \rightarrow Y$  by

$$g(x) = \begin{cases} f(x), & \text{if } x \in E \setminus \{x_0\}; \\ L, & \text{if } x = x_0, \end{cases}$$

then  $g$  is continuous at  $x_0$  on  $E$ .

**proof from (a) to (b).** We know for all  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $x \in E$  and  $d_X(x, x_0) < \delta$  implies  $d_Y(f(x), L) < \varepsilon$ . Also, if we have a sequence  $\{x^{(n)}\}_{n=1}^{\infty} \subseteq E$  converges to  $x_0$ , then there exists  $N > 0$  s.t.  $n \geq N$  implies  $d_X(x^{(n)}, x_0) < \delta$ , so for all  $n \geq N$ , we know  $d_Y(f(x^{(n)}), L) < \varepsilon$ . ■

**proof from (b) to (c).** Suppose by contradiction,  $V \subseteq Y$  is an open neighborhood of  $L$ , and there does not exist an open neighborhood  $U \subseteq X$  of  $x_0$  has

$$U \cap E \subseteq f^{-1}(V),$$

then for all  $n \in \mathbb{N}$ , we know  $\exists y_n \in B_X(x_0, \frac{1}{n}) \cap E$  and  $y_n \notin f^{-1}(V)$ , and note that  $(y_n)_{n=1}^{\infty}$  is a sequence converges to  $x_0$  in  $E$ , so by (b) we know  $(f(y_n))_{n=1}^{\infty}$  must converges to  $L$ , which means for the neighborhood  $V$  of  $L$ , there exists  $N > 0$  s.t.  $n \geq N$  implies  $f(y_n) \in V$ . ■

**proof from (c) to (d).** ■

**proof from (d) to (a).** ■

DIY

DIY

## 4.2 Pointwise and Uniform Convergence

**Definition 4.2.1 (Pointwise convergence).** Let  $(f^{(n)})_{n=1}^{\infty}$  be a sequence of functions from  $(X, d_X)$  to  $(Y, d_Y)$ , and let  $f : X \rightarrow Y$  be another function. We say that  $f^{(n)}$  converges pointwise to  $f$  on  $X$  if for every  $x \in X$  and every  $\varepsilon > 0$ ,  $\exists N_x > 0$  s.t.

$$d_Y(f^{(n)}(x), f(x)) < \varepsilon \quad \forall n \geq N_x.$$

**Definition 4.2.2 (Uniformly convergence).** We say  $f^{(n)}$  converges uniformly to  $f$  on  $X$  if for every  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.

$$d_Y \left( f^{(n)}(x), f(x) \right) < \varepsilon$$

for all  $n \geq N$  and all  $x \in X$ . ( $N$  is independent of  $x$ )

**Example 4.2.1.** Suppose  $f_n(x) = \frac{x}{n}$ , then  $f_n \rightarrow 0$  pointwise (0 is the zero function here). However,  $\{f^{(n)}\}$  is not uniformly convergent since given  $\varepsilon = 1$ , if  $\exists N > 0$  s.t.

$$|f_n(x) - 0| < 1$$

for all  $n \geq N$  and  $x \in \mathbb{R}$ , then

$$\left| \frac{x}{n} \right| < 1$$

for all  $n \geq N$  and  $x \in X$ , but if we pick  $n = N$  and  $x = 2N$ , then it gives a contradiction.

**Example 4.2.2.**  $f_n(x) = x^n$  on  $[0, 1]$ , then we know

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ 1, & \text{if } x = 1, \end{cases}$$

so  $f_n$  continuous and pointwise convergent but not uniformly convergent.

**Example 4.2.3.** Suppose  $f_n(x) = \frac{x}{n}$  on  $[0, 1]$ , then  $\lim_{n \rightarrow \infty} f_n(x) = 0$  uniformly.

**Proof.** ■

DIY

**Example 4.2.4.** Consider

$$f_n(x) = \begin{cases} 2n, & \text{if } x \in \left[ \frac{1}{2n}, \frac{1}{n} \right]; \\ 0, & \text{if } x \in \mathbb{R} \setminus \left[ \frac{1}{2n}, \frac{1}{n} \right], \end{cases}$$

then  $\lim_{n \rightarrow \infty} f_n(x) = 0$  pointwisely, but if we integrate on both side, then

$$\int_0^1 f^{(n)}(x) dx = 2n \left( \frac{1}{n} - \frac{1}{2n} \right) = 2n \cdot \frac{1}{2n} = 1,$$

but  $\int_0^1 0 dx = 0$ , so we know pointwise convergence will not implies they will be equal after integration.

**Remark 4.2.1.** We will learn that uniform convergence can ensure integration after taking limit takes same value of taking limit after integration.

## Lecture 13

**As previously seen.**  $\lim_{n \rightarrow \infty} f_n = f$  uniformly where  $f_n, f : X \rightarrow Y$  iff given any  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.

$$d_Y(f_n(x), f(x)) < \varepsilon \quad \forall x \in X \text{ and } n \geq N.$$

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**Theorem 4.2.1 (考試會考).** Suppose  $(f^{(n)})_{n=1}^{\infty}$  is a sequence of functions from one metric space  $(X, d_X)$  to  $(Y, d_Y)$  and suppose this sequence of functions converge uniformly to another function

$f : X \rightarrow Y$ . Let  $x_0 \in X$ . If each  $f^{(n)}$  is continuous at  $x_0$ , then  $f$  is continuous at  $x_0$ .

**Proof.** Since  $f_n \rightarrow f$  uniformly. Given  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.

$$d_Y \left( f^{(n)}(x), f(x) \right) < \frac{\varepsilon}{3} \quad \text{for all } x \in X, n \geq N.$$

Since  $f^{(N)}$  is continuous at  $x_0$ , so there exists  $\exists \delta > 0$  s.t. if  $d_X(x, x_0) < \delta$ , then

$$d_Y \left( f^{(N)}(x), f^{(N)}(x_0) \right) < \frac{\varepsilon}{3}.$$

Now if  $d_X(x, x_0) < \delta$ , then

$$\begin{aligned} d_Y(f(x), f(x_0)) &\leq d_Y \left( f(x), f^{(N)}(x) \right) + d_Y \left( f^{(N)}(x), f^{(N)}(x_0) \right) + d_Y \left( f^{(N)}(x_0), f(x_0) \right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence,  $f$  is continuous at  $x_0$ . ■

**Corollary 4.2.1.** Let  $f^{(n)}, f : X \rightarrow Y$ . Suppose  $f^{(n)} : X \rightarrow Y$  are continuous for all  $n$ . If  $\lim_{n \rightarrow \infty} f^{(n)} = f$  uniformly, then  $f$  is also continuous.

**Example 4.2.5.** Suppose  $f_n(x) = x^n$  define on  $[0, 1]$  for all  $n \in \mathbb{N}$ , then we know

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ 1, & \text{if } x = 1. \end{cases}$$

Now suppose

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ 1, & \text{if } x = 1. \end{cases},$$

then  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  pointwise. Note that  $f_n = x^n$  is continuous on  $[0, 1]$  but  $f$  is not continuous. Hence,  $f_n$  does not converge uniformly to  $f$ .

**Remark 4.2.2.** This example tells us if we change converge uniformly to converge pointwise in [Theorem 4.2.1](#), then it may not be true.

Now we want to know is

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0, x \in E} f^{(n)}(x) = \lim_{x \rightarrow x_0, x \in E} \lim_{n \rightarrow \infty} f^{(n)}(x)?$$

We will later show that the equality holds if  $f^{(n)} \rightarrow f$  uniformly and  $Y$  complete, where we define  $f^{(n)}, f : X \rightarrow Y$ .

**Proposition 4.2.1.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces with  $Y$  complete, and let  $E \subseteq X$ . Suppose  $(f^{(n)})_{n=1}^{\infty}$  is a sequence of functions from  $E$  to  $Y$  that converges uniformly to some function  $f : E \rightarrow Y$ . Let  $x_0 \in X$  be adherent point of  $E$ , and suppose that for each  $n$ ,  $\lim_{x \rightarrow x_0, x \in E} f^{(n)}(x)$  exists. Then the limit  $\lim_{x \rightarrow x_0, x \in E} f(x)$  also exists, and moreover,

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0, x \in E} f^{(n)}(x) = \lim_{x \rightarrow x_0, x \in E} \lim_{n \rightarrow \infty} f^{(n)}(x).$$

**Proof.** Since  $\lim_{x \rightarrow x_0, x \in E} f^{(n)}(x)$  exists, so we can let  $L_n := \lim_{x \rightarrow x_0, x \in E} f^{(n)}(x) \in Y$ . First, we show that  $\{L_n\}_{n=1}^{\infty}$  is Cauchy. Since  $\lim_{n \rightarrow \infty} f^{(n)} = f$  uniformly on  $E$ . Given  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.

$$d_Y \left( f^{(n)}(x), f(x) \right) < \frac{\varepsilon}{6} \quad \text{for all } x \in E, n \geq N.$$

Hence,  $n, m \geq N$  implies

$$d_Y \left( f^{(n)}(x), f^{(m)}(x) \right) \leq d_Y \left( f^{(n)}(x), f(x) \right) + d_Y \left( f(x), f^{(m)}(x) \right) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

Now since  $\lim_{x \rightarrow x_0, x \in E} f^{(n)}(x) = L_n$  and  $\lim_{x \rightarrow x_0, x \in E} f^{(m)}(x) = L_m$ , so there exists  $\delta_n, \delta_m > 0$  s.t. for all  $x \in E$ ,

$$\begin{aligned} d(x, x_0) < \delta_n &\Rightarrow d_Y \left( f^{(n)}(x), L_n \right) < \frac{\varepsilon}{3} \\ d(x, x_0) < \delta_m &\Rightarrow d_Y \left( f^{(m)}(x), L_m \right) < \frac{\varepsilon}{3}. \end{aligned}$$

Choose  $\delta = \min \{\delta_n, \delta_m\}$  and fix  $x \in E$  with  $d_X(x, x_0) < \delta$  (since  $x_0 \in \overline{E}$  so this is possible), then

$$d_Y(L_n, L_m) \leq d_Y(L_n, f^{(n)}(x)) + d_Y(f^{(n)}(x), f^{(m)}(x)) + d_Y(f^{(m)}(x), L_m) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence,  $\{L_n\}_{n=1}^\infty$  is Cauchy. Since  $Y$  is complete, so  $\lim_{n \rightarrow \infty} L_n = L$  for some  $L \in Y$ .

**Claim 4.2.1.**  $\lim_{x \rightarrow x_0, x \in E} f(x) = L$ .

**Proof.** Fix  $\varepsilon > 0$ , then  $\lim_{n \rightarrow \infty} L_n = L$  and  $\lim_{n \rightarrow \infty} f^{(n)}(x) = f(x)$  uniformly on  $E$ , so there exists  $N > 0$  s.t.

$$d_Y(L_n, L) < \frac{\varepsilon}{3} \text{ and } d_Y(f^{(n)}(x), f(x)) < \frac{\varepsilon}{3} \text{ for all } x \in E, n \geq N.$$

Now since  $L_n = \lim_{x \rightarrow x_0, x \in E} f^{(n)}(x)$ , so there exists  $\delta > 0$  s.t. for all  $x \in E$  and for all  $n$ ,

$$d_X(x, x_0) < \delta \Rightarrow d_Y(f^{(n)}(x), L_n) < \frac{\varepsilon}{3}.$$

For this  $\delta$ , if  $d_X(x, x_0) < \delta$  and  $x \in E$ , then we know

$$d_Y(f(x), L) \leq d_Y(f(x), f^{(N)}(x)) + d_Y(f^{(N)}(x), L_N) + d_Y(L_N, L) < \varepsilon.$$

⊛

Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0, x \in E} f^{(n)}(x) &= \lim_{n \rightarrow \infty} L_n = L \\ \lim_{x \rightarrow x_0, x \in E} \lim_{n \rightarrow \infty} f^{(n)}(x) &= \lim_{x \rightarrow x_0, x \in E} f(x) = L. \end{aligned}$$

This means  $\lim_{x \rightarrow x_0, x \in E}$  and  $\lim_{n \rightarrow \infty}$  is exchangeable here. ■

**Proposition 4.2.2.** Let  $f^{(n)} : X \rightarrow Y$  be a sequence of continuous functions. If  $\lim_{n \rightarrow \infty} f^{(n)} = f$  uniformly. Let  $\lim_{n \rightarrow \infty} x^{(n)} = x$  in  $X$ , then  $\lim_{n \rightarrow \infty} f^{(n)}(x^{(n)}) = f(x)$ .

**Proof.** Since  $f^{(n)} \rightarrow f$  uniformly, so given  $\varepsilon > 0$ , there exists  $N_1 > 0$  s.t.

$$d_Y(f^{(n)}(x), f(x)) < \frac{\varepsilon}{2} \text{ for all } x \in X, n \geq N_1.$$

Since  $f^{(n)} \rightarrow f$  uniformly, so  $f$  is also continuous by [Corollary 4.2.1](#), so there exists  $\delta_x > 0$  s.t.

$$d_X(y, x) < \delta_x \Rightarrow d_Y(f(y), f(x)) < \frac{\varepsilon}{2}.$$

Since  $\lim_{n \rightarrow \infty} x^{(n)} = x$ , so there exists  $N_2 > 0$  s.t.  $d_X(x^{(n)}, x) < \delta_x$  for all  $n \geq N_2$ . Let

$N = \max \{N_1, N_2\}$ , then  $n \geq N$  implies

$$d_Y \left( f^{(n)} \left( x^{(n)} \right), f(x) \right) \leq d_Y \left( f^{(n)} \left( x^{(n)} \right), f \left( x^{(n)} \right) \right) + d_Y \left( f \left( x^{(n)} \right), f(x) \right) < \varepsilon.$$

■

**Definition 4.2.3 (bounded function).** A function  $f : X \rightarrow Y$  from  $(X, d_X)$  to  $(Y, d_Y)$  is called bounded if its image  $f(X)$  is bounded in  $Y$  i.e. there exists  $y_0 \in Y$  and  $R > 0$  s.t.

$$d_Y (f(x), y_0) < R \text{ for all } x \in X.$$

**Proposition 4.2.3.** Let  $(f^{(n)})_{n=1}^{\infty}$  be a sequence of functions from  $(X, d_X)$  to  $(Y, d_Y)$ . Suppose  $\lim_{n \rightarrow \infty} f^{(n)} = f$  uniformly where  $f : X \rightarrow Y$ . If each  $f^{(n)}$  is bounded on  $X$ , then  $f$  is also bounded.

**Proof.** Since  $\lim_{n \rightarrow \infty} f^{(n)} = f$  uniformly, so given  $\varepsilon = 1$ , there exists  $N > 0$  s.t.

$$d_Y \left( f^{(n)}(x), f(x) \right) < 1 \text{ for all } x \in X, n \geq N.$$

Since  $f^{(N)}$  is bounded, so there exists  $y_0$  and  $R_N > 0$  s.t.

$$d_Y \left( f^{(N)}(x), y_0 \right) < R_N \text{ for all } x \in X.$$

Hence,

$$\begin{aligned} d_Y (f(x), y_0) &\leq d_Y \left( f(x), f^{(N)}(x) \right) + d_Y \left( f^{(N)}(x), y_0 \right) \\ &< 1 + R_N \end{aligned}$$

for all  $x \in X$ , so  $f$  is bounded. ■

### 4.3 The Metric of Uniform Convergence

**Definition 4.3.1.** Suppose  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. We let  $B(X \rightarrow Y)$  denotes the set of all bounded functions from  $X$  to  $Y$  i.e.

$$B(X \rightarrow Y) = \{f \mid f : X \rightarrow Y \text{ is bounded}\}.$$

If  $X \neq \emptyset$ , we define a metric  $d_{\infty}$  on  $B(X \rightarrow Y)$  by

$$d_{\infty} (f, g) = \sup_{x \in X} d_Y (f(x), g(x))$$

for all  $f, g \in B(X \rightarrow Y)$ .

**Proposition 4.3.1.** If  $X$  is non-empty, then  $d_{\infty}$  is a metric on  $B(X \rightarrow Y)$ .

**Proof.** Given  $f, g \in B(X \rightarrow Y)$ , then there exists  $y_f, y_g \in Y$  and  $M_f, M_g > 0$  s.t.

$$d_Y (f(x), y_f) < M_f \text{ and } d_Y (g(x), y_g) < M_g \text{ for all } x \in X.$$

Hence,

$$d_Y (f(x), g(x)) \leq d_Y (f(x), y_f) + d_Y (y_f, y_g) + d_Y (y_g, g(x)) < M_f + M_g + d_Y (y_f, y_g),$$

which means  $\sup_{x \in X} d_Y (f(x), g(x))$  exists and  $\geq 0$ , and thus  $d_{\infty}$  is well-defined.

Now since

$$(1) d_\infty(f, g) = \sup_{x \in X} d_Y(f(x), g(x)) \geq 0.$$

$$(2) d_\infty(f, g) = d_\infty(g, f).$$

$$(3) \text{ For } f, g, h \in B(X \rightarrow Y),$$

$$d_Y(f(x), h(x)) \leq d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \leq d_\infty(f, g) + d_\infty(g, h),$$

$$\text{so } \sup_{x \in X} d_Y(f(x), h(x)) \leq d_\infty(f, g) + d_\infty(g, h).$$

$$(4) d_\infty(f, g) = 0 \text{ iff } \sup_{x \in X} d_Y(f(x), g(x)) = 0 \text{ iff } d_Y(f(x), g(x)) = 0 \text{ for all } x \in X \text{ iff } f(x) = g(x) \text{ for all } x \in X.$$

So  $d_\infty$  is a metric. ■

**Proposition 4.3.2.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $(f^{(n)})_{n=1}^\infty$  be a sequence of functions in  $B(X \rightarrow Y)$ , and let  $f$  be another function in  $B(X \rightarrow Y)$ . Then

$$f^{(n)} \rightarrow f \text{ in } d_\infty (= d_{B(X \rightarrow Y)}) \Leftrightarrow f^{(n)} \rightarrow f \text{ uniformly.}$$

**Proof.**

( $\Rightarrow$ ) Since  $\lim_{n \rightarrow \infty} d_\infty(f^{(n)}, f) = 0$ . Given  $\varepsilon > 0$ , there exists  $N > 0$  s.t.  $n \geq N$  implies

$$\sup_{x \in X} d_Y(f^{(n)}(x), f(x)) = d_\infty(f^{(n)}, f) < \varepsilon,$$

so

$$d_Y(f^{(n)}(x), f(x)) \leq \sup_{x \in X} d_Y(f^{(n)}(x), f) < \varepsilon$$

whenever  $n \geq N$ , so  $f^{(n)} \rightarrow f$  uniformly.

( $\Leftarrow$ ) DIY

Now let  $C(X \rightarrow Y)$  be the set of bounded and continuous function, so  $C(X \rightarrow Y) \subseteq B(X \rightarrow Y)$ .

**Theorem 4.3.1.** If  $Y$  is complete, then  $C(X \rightarrow Y)$  is a complete metric space. (The metric is  $d_\infty$  intersected to  $C(X \rightarrow Y)$ ).

**Proof.** Given any Cauchy sequence  $\{f^{(n)}\}_{n=1}^\infty$  in  $(C(X \rightarrow Y), d_\infty)$ , then  $f^{(n)} : X \rightarrow Y$  is continuous and bounded. Given  $\varepsilon > 0$ , there exists  $N > 0$  s.t.

$$\sup_{x \in X} d_Y(f^{(n)}(x), f^{(m)}(x)) = d_\infty(f^{(n)}, f^{(m)}) < \frac{\varepsilon}{2} \text{ for all } n, m \geq N.$$

Now fix  $x \in X$ . Consider the sequence  $\{f^{(n)}(x)\}_{n=1}^\infty$  in  $Y$ , so  $\{f^{(n)}(x)\}_{n=1}^\infty$  is Cauchy in  $Y$ . Now since  $Y$  is complete, so  $\lim_{n \rightarrow \infty} f^{(n)}(x)$  exists. Let  $f(x) = \lim_{n \rightarrow \infty} f^{(n)}(x)$ . If we set up  $f(x)$  similarly for all  $x \in X$  to construct  $f$ , then we give a claim.

**Claim 4.3.1.**  $\lim_{n \rightarrow \infty} d_\infty(f^{(n)}, f) = 0$ .

**Proof.** Fix  $\varepsilon > 0$ , choose  $N > 0$  s.t.  $d_\infty(f^{(n)}, f^{(m)}) < \frac{\varepsilon}{2}$  for all  $n, m \geq N$ . Then for all  $n \geq N$  we know for all  $x \in X$

$$\begin{aligned} d_Y(f^{(n)}(x), f(x)) &= \lim_{m \rightarrow \infty} d(f^{(n)}(x), f^{(m)}(x)) \leq \sup_{m \geq N, x \in X} d(f^{(n)}(x), f^{(m)}(x)) \\ &= \sup_{m \geq N} d_\infty(f^{(n)}, f^{(m)}) \leq \frac{\varepsilon}{2}. \end{aligned}$$

Hence,  $\lim_{n \rightarrow \infty} d_\infty(f^{(n)}, f) = 0$ . ⊗

By this claim, we know  $f^{(n)} \rightarrow f$ , so  $Y$  is complete.

Now since  $f^{(n)} \rightarrow f$  on  $d_\infty$ , so  $f^{(n)} \rightarrow f$  uniformly by [Proposition 4.3.2](#), and since  $\{f^{(n)}\}_{n=1}^\infty$  is continuous, so  $f$  is continuous by [Corollary 4.2.1](#) we know  $f$  is continuous, and by [Proposition 4.2.3](#), we know  $f$  is bounded, so  $f \in C(X \rightarrow Y)$ . Thus, we know  $\{f^{(n)}\}_{n=1}^\infty$  converges in  $C(X \rightarrow Y)$ , so  $C(X \rightarrow Y)$  is complete. ■

## Lecture 14

**Lemma 4.3.1 (HW6 P4).** Let  $(X, \mathcal{F})$  be a topological space, and  $K \subseteq X$  and  $K$  is compact, then if  $X$  is Hausdorff, then  $K$  is closed.

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**Lemma 4.3.2.** Let  $(Y, d_Y)$  be a metric space and for any  $a \in Y$ , and we have a map  $\varphi : Y \rightarrow \mathbb{R}$  defined by  $\varphi(y) = d_Y(a, y)$ , then  $\varphi$  is continuous. In fact,

$$|\varphi(y) - \varphi(y')| \leq d_Y(y, y').$$

**Proof.** Since

$$\varphi(y) = d_Y(a, y) \leq d_Y(a, y') + d_Y(y', y) = \varphi(y') + d_Y(y', y),$$

so

$$|\varphi(y) - \varphi(y')| \leq d_Y(y, y').$$

Thus, if given  $\varepsilon > 0$ , then choose  $\delta = \varepsilon$ , then we know if  $d_Y(y, y') < \delta = \varepsilon$ , then

$$|\varphi(y) - \varphi(y')| \leq d_Y(y, y') < \delta = \varepsilon,$$

so  $\varphi$  is continuous. ■

**Remark 4.3.1.** Suppose  $\lim_{n \rightarrow \infty} y_n = y$  in  $Y$ , then  $\lim_{n \rightarrow \infty} d_Y(y_n, a) = d_Y(y, a)$ .

**Lemma 4.3.3.** If  $\lim_{n \rightarrow \infty} a_n = a$  in  $(Y, d_Y)$ . Given any  $N \in \mathbb{N}$ , then

$$\inf_{m \geq N} a_m \leq \lim_{n \rightarrow \infty} a_n \leq \sup_{m \geq N} a_m.$$

**Proof.** For any  $K \geq N$ , we have

$$\inf_{m \geq N} a_m \leq a_K \leq \sup_{m \geq N} a_m,$$

We know that  $\lim_{K \rightarrow \infty} a_K$  exists by Squeeze Theorem, so we have

$$\inf_{m \geq N} a_m \leq \lim_{k \rightarrow \infty} a_k \leq \sup_{m \geq N} a_m.$$

**Theorem 4.3.2 (Reprove Theorem 4.3.1).** Let  $(X, d_X)$  be a metric space, and  $(Y, d_Y)$  be a com-

plete metric space. Let  $C(X \rightarrow Y)$  be the space of continuous and bounded function. Then  $(C(X \rightarrow Y), d_\infty)$  is a complete metric space, where

$$d_\infty(f, g) = \sup_{x \in X} d_Y(f(x), g(x)) \text{ for } f, g \in C(X \rightarrow Y).$$

**Proof.** Given a Cauchy sequence  $\{f^{(n)}\}_{n=1}^\infty$  in  $(C(X \rightarrow Y), d_\infty)$ , then we want to show there exists  $f \in C(X \rightarrow Y)$  s.t.  $\lim_{n \rightarrow \infty} d_\infty(f_n, f) = 0$ . i.e.  $f_n \rightarrow f$  uniformly by [Proposition 4.3.2](#). Given  $\varepsilon > 0$ , then there exists  $N > 0$  s.t.  $\sup_{x \in X} d_Y(f_n(x), f_m(x)) = d_\infty(f_n, f_m) < \frac{\varepsilon}{2}$  for all  $n, m \geq N$ . This implies that  $\{f^{(n)}(x)\}_{n=1}^\infty$  is Cauchy in  $Y$  for all  $x \in X$ . Since  $Y$  is complete, so  $\lim_{n \rightarrow \infty} f^{(n)}(x)$  exists in  $Y$  for all  $x \in X$ . Now we define  $f(x) := \lim_{n \rightarrow \infty} f^{(n)}(x)$  for all  $x \in X$ , then for any  $x \in X$ , consider  $n \geq N$ , we know

$$d_Y(f_n(x), f(x)) = \lim_{m \rightarrow \infty} d_Y(f_n(x), f_m(x)) \text{ by Lemma 4.3.2 and } \lim_{m \rightarrow \infty} f_m(x) = f(x) \in Y.$$

By [Lemma 4.3.3](#), we know

$$d_Y(f_n(x), f(x)) = \lim_{m \rightarrow \infty} d_Y(f_n(x), f_m(x)) \leq \sup_{m \geq N} d_Y(f_n(x), f_m(x)) \leq \sup_{m \geq N} d_\infty(f_n, f_m) \leq \frac{\varepsilon}{2} < \varepsilon,$$

so  $f_n$  converges uniformly to  $f$ , and by [Corollary 4.2.1](#) and [Proposition 4.2.3](#), we know that  $f \in C(X \rightarrow Y)$ , and we're done. ■

**Example 4.3.1.** If  $Y$  is not complete, then  $(C(X \rightarrow Y), d_\infty)$  may not be complete.

**Proof.** Let  $X = [0, 1]$  with standard metric, and let  $Y = \mathbb{Q}$ , then note that  $Y$  is not complete. Let  $r_n \in \mathbb{Q}$  and  $\lim_{n \rightarrow \infty} r_n = \sqrt{2}$ . If we define

$$f_n : [0, 1] \rightarrow \mathbb{Q}, \quad f_n(x) = r_n \quad \forall x \in [0, 1],$$

then

$$d_\infty(f_n, f_m) = \sup_{x \in [0, 1]} |f_n(x) - f_m(x)| = \sup |r_n - r_m|.$$

Since  $\lim_{n \rightarrow \infty} r_n = \sqrt{2}$ , so  $\{r_n\}_{n=1}^\infty$  is Cauchy, so  $\{f_n\}_{n=1}^\infty$  is Cauchy in  $(C(X \rightarrow Y), d_\infty)$ . Note that  $\lim_{n \rightarrow \infty} f_n = f$  but  $f$  is not a function from  $[0, 1]$  to  $\mathbb{Q}$  since  $f(x) = \sqrt{2}$  for all  $x \in [0, 1]$ . \*

**Note 4.3.1.** I think uniformly convergent is unique, i.e. if  $(f_n)$  converges to  $f$  uniformly, then  $f$  is unique, but i am not sure.

End of  
midterm

## 4.4 Series of functions

Suppose  $(X, d_X)$  is a metric space and  $f^{(n)} : X \rightarrow \mathbb{R}$  and  $f : X \rightarrow \mathbb{R}$  for all  $n \geq 1$ . We define the partial sum of  $\{f^{(n)}\}_{n=1}^\infty$  by

$$S_N(x) = \sum_{i=1}^N f^{(i)}(x).$$

**Definition 4.4.1.** Let  $(X, d_X)$  be a metric space, and let  $\{f^{(n)}\}_{n=1}^\infty$  be a sequence of functions from  $X$  to  $\mathbb{R}$  and let  $f : X \rightarrow \mathbb{R}$  be another function. We say that the infinite series  $\sum_{i=1}^\infty f^{(i)}$  converges pointwise to  $f$  if

$$\lim_{n \rightarrow \infty} S_N(x) = f(x) \text{ pointwise where } S_N(x) = \sum_{i=1}^N f^{(i)}(x),$$

and we say  $\sum_{i=1}^{\infty} f^{(i)}$  converges to  $f$  uniformly if

$$\lim_{N \rightarrow \infty} S_N(x) = f(x) \text{ uniformly.}$$

**Definition 4.4.2 (Sup norm).** Let  $f : X \rightarrow \mathbb{R}$  be a bounded valued function, then we can define

$$\|f\|_{\infty} := \sup_{x \in X} |f(x)| = d_{\infty}(f, 0).$$

**Theorem 4.4.1 (Weierstrass  $M$ -test).** Let  $(X, d_X)$  be a metric space, and  $(f^{(n)})_{n=1}^{\infty}$  be a sequence of bounded, real-valued, and continuous functions on  $X$ . Suppose  $\sum_{n=1}^{\infty} \|f^{(n)}\|_{\infty}$  converges, then  $\sum_{n=1}^{\infty} f^{(n)}$  converges uniformly on  $X$  to a bounded and continuous, real-valued functions on  $X$ .

**Proof.** Let  $S_N(x) = \sum_{i=1}^N f^{(i)}(x)$ , then recall that  $(C(X \rightarrow \mathbb{R}), d_{\infty})$  is a complete metric space. (Since  $\mathbb{R}$  is complete with standard metric). Let  $M_n := \|f^{(n)}\|_{\infty} \in \mathbb{R}$ . We know that  $\sum_{n=1}^{\infty} M_n$  converges, then we know  $\left(\sum_{n=1}^k M_n\right)_{k=1}^{\infty}$  converges, so we can define  $t_N = \sum_{i=1}^N M_i$ . Thus, we know  $\{t_n\}_{n=1}^{\infty}$  is Cauchy, so given  $\varepsilon > 0$ , we know there exists  $N > 0$  s.t.  $m > n \geq N$  implies

$$\left| \sum_{i=n+1}^m M_i \right| = |t_m - t_n| < \varepsilon.$$

Hence, for all  $m > n \geq N$ ,

$$\begin{aligned} d_{\infty}(S_m, S_n) &= \sup_{x \in X} \left| \sum_{i=1}^m f^{(i)}(x) - \sum_{i=1}^n f^{(i)}(x) \right| \\ &= \sup_{x \in X} \left| \sum_{i=n+1}^m f^{(i)}(x) \right| \leq \sup_{x \in X} \sum_{i=n+1}^n |f^{(i)}(x)| \leq \sum_{i=n+1}^n M_i < \varepsilon. \end{aligned}$$

Thus,  $\{S_n\}_{n=1}^{\infty}$  is Cauchy in  $(C(X \rightarrow \mathbb{R}), d_{\infty})$ , and since  $\mathbb{R}$  is complete, so we know  $S_n \rightarrow f$  in  $(C(X \rightarrow \mathbb{R}), d_{\infty})$ , which means  $S_n \rightarrow f$  uniformly in  $C(X \rightarrow \mathbb{R})$  by [Theorem 4.3.1](#) and [Proposition 4.3.2](#).

**Note 4.4.1.**  $S_N$  is bounded and continuous since  $f^{(i)}$  is bounded and continuous for all  $i \in \mathbb{N}$  and the sum of bounded and continuous function is still bounded and continuous. ■

**Example 4.4.1.**

- $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$  pointwise when  $x \in (-1, 1)$ .
- $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$  uniformly when  $x \in [-r, r]$  where  $0 < r < 1$ .

**Proof.** Consider  $S_N(x) = \sum_{i=1}^N x^n$ , then

$$S_N(x) - xS_N(x) = x - x^{N+1} \Rightarrow S_N(x) = \frac{x - x^{N+1}}{1 - x}.$$

When  $|x| < 1$ , then  $\lim_{N \rightarrow \infty} x^{N+1} = 0$ , so

$$\lim_{N \rightarrow \infty} S_N(x) = \lim_{N \rightarrow \infty} \frac{x - x^{N+1}}{1 - x} = \frac{x}{1 - x}.$$

Thus,  $\sum_{i=1}^{\infty} x^i$  converges to  $\frac{x}{1-x}$  pointwise.

Now if  $x \in [-r, r]$  with  $0 < r < 1$ , then  $|x^i| \leq r^i$  for  $x \in [-r, r]$ . Thus,  $\|x^i\|_{\infty} = r^i$ , so  $\sum_{i=1}^{\infty} \|x^i\|_{\infty} = \sum_{i=1}^{\infty} r^i$  converges to  $\frac{r}{1-r}$ , and thus  $\sum_{i=1}^{\infty} x^i$  converges uniformly to  $\frac{x}{1-x}$  by Theorem 4.4.1.  $\circledast$

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**Example 4.4.2.** Suppose

$$f_n(x) = \begin{cases} 0, & \text{if } x \leq 0 \text{ or } x \geq \frac{1}{n}; \\ 4n^2x, & \text{if } 0 \leq x \leq \frac{1}{2n}; \\ -4n^2x + 4n, & \text{otherwise } \frac{1}{2n} \leq x \leq \frac{1}{n}, \end{cases}$$

then  $\lim_{n \rightarrow \infty} f_n(x) = 0$  pointwise. Also,  $f_n(x)$  is continuous and  $\int_{-\infty}^{\infty} f_n(x) dx = 1$ . However,  $\lim_{n \rightarrow \infty} f_n(x) = f(x) = 0$  for all  $x \in \mathbb{R}$ , so

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx \neq \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx,$$

and we will talk about when will exchanging lim and integration get same result.

**Note 4.4.2.** The problem is that  $f_n \rightarrow f$  only pointwise.

Now we quickly review Riemann integral. Assume  $f$  is continuous on  $[a, b]$ , then we partition the interval  $[a, b]$  into subintervals  $I_k = [x_{k-1}, x_k]$  for all  $1 \leq k \leq n$ . On  $I_k$ , find the maximum and the minimum of  $f$  on  $I_k$ , and let  $M_k = \sup_{x \in I_k} f(x)$  and  $m_k = \inf_{x \in I_k} f(x)$ , and use the area of the rectangle with height  $M_k$  and base  $x_k - x_{k-1} = \Delta X_k$ , and similarly use the area of the rectangle with height  $m_k$  and base  $x_k - x_{k-1} = \Delta X_k$ , then we know

$$m_k \cdot \Delta X_k \leq \text{The real area of } f \text{ on } I_k \leq M_k \cdot \Delta X_k,$$

and thus we can define the upper Riemann sum to be  $\sum_{k=1}^n M_k \cdot \Delta X_k$  and the lower Riemann sum to be  $\sum_{k=1}^n m_k \cdot \Delta X_k$ . Hence, we know

$$\text{Lower Riemann Sum} \leq \text{Area} \leq \text{Upper Riemann Sum}.$$

Also, since upper Riemann sum decreases if one has more partition and lower Riemann sum increases if one has more partition, so we can approach the area by having more partition.

**Definition 4.4.3.** Let  $f : I \rightarrow \mathbb{R}$  be a bounded function on a bounded interval, and let  $p$  be a partition of  $I$ , we define the upper Riemann integral  $U(f, p)$  and the lower Riemann sum  $L(f, p)$  by

$$U(f, p) = \sum_{J \in p, J \neq \emptyset} \left( \sup_{x \in J} f(x) \right) |J|, \quad L(f, p) = \sum_{J \in p, J \neq \emptyset} \left( \inf_{x \in J} f(x) \right) |J|$$

**Proposition 4.4.1** (From Analysis I). Let  $f : I \rightarrow \mathbb{R}$  be a bounded function on a bounded interval  $I$ , then we define

$$\begin{aligned} \overline{\int_I} f &= \inf \{ U(f, p) : p \text{ is a partition of } I \} \quad (\text{upper integral}) \\ \underline{\int_I} f &= \sup \{ L(f, p) : p \text{ is a partition of } I \} \quad (\text{lower integral}), \end{aligned}$$

and we have

$$(1) \underline{\int_I} f \leq \overline{\int_I} f$$

(2) Suppose  $f \leq g$  on  $I$ , then we have

$$\underline{\int_I} f \leq \underline{\int_I} g, \text{ and } \overline{\int_I} f \leq \overline{\int_I} g.$$

**Proof.**

(1) Since  $\underline{\int_I} f \leq \text{real area} \leq \overline{\int_I} f$ , so this is true. (Not rigorous)

(2) Since we fix any partition  $p$  on  $I$ , we have

$$L(f, p) \leq L(g, p) \quad U(f, p) \leq U(g, p),$$

so this is true.

**Remark 4.4.1.** More intuitively, since lower Riemann integral and upper Riemann integral are both approaching the real area (if the function is not weird), so this can be intuitively accepted.

■

**Definition 4.4.4.** Let  $f : I \rightarrow \mathbb{R}$  be a bounded function. We say  $f$  is Riemann integrable on  $I$  if  $\underline{\int_I} f = \overline{\int_I} f$ , and we denote it by  $\int_I f$ .

**Remark 4.4.2.** To prove  $f$  is (Riemann) integrable, we need to prove  $\underline{\int_I} f = \overline{\int_I} f$ .

**Example 4.4.3.** Suppose  $f : I \rightarrow \mathbb{R}$  with  $I = [0, 1]$  and

$$f(x) = \begin{cases} 1, & \text{if } x \in [0, 1], x \in \mathbb{Q}; \\ 0, & \text{if } x \in [0, 1], x \notin \mathbb{Q}, \end{cases}$$

then  $\overline{\int_I} f = 1$  and  $\underline{\int_I} f = 0$ , so  $f$  is not integrable.

**Note 4.4.3.**  $f(x) = 0$  almost everywhere on  $[0, 1]$ , and we say  $f$  is Lebesgue integrable and its Lebesgue integral is 0. We'll discuss this next semester.

**Theorem 4.4.2.** Let  $f^{(n)} : [a, b] \rightarrow \mathbb{R}$  be a sequence of Riemann integrable functions. Suppose  $\lim_{n \rightarrow \infty} f^{(n)} = f$  uniformly where  $f : [a, b] \rightarrow \mathbb{R}$ , then  $f$  is also Riemann integrable and  $\lim_{n \rightarrow \infty} \int_I f^{(n)} = \int_I f$ , or equivalently

$$\lim_{n \rightarrow \infty} \int_I f^{(n)} = \int_I \lim_{n \rightarrow \infty} f^{(n)}.$$

**Proof.** First, we want to show that  $\overline{\int_I} f = \underline{\int_I} f$ , then since  $f_n \rightarrow f$  uniformly on  $[a, b]$ , so given  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.  $|f_n(x) - f(x)| < \varepsilon$  for all  $n \geq N$  and  $x \in [a, b]$ . Thus, we have

$$f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon \quad \forall n \geq N \text{ and } x \in [a, b].$$

Hence, we have

$$\underline{\int_I} (f_n - \varepsilon) \leq \underline{\int_I} f, \quad \overline{\int_I} f \leq \overline{\int_I} (f_n + \varepsilon) \quad \forall n \geq N,$$

which gives

$$\int_I f_n - \varepsilon(b-a) \leq \int_I f, \quad \overline{\int_I f} \leq \overline{\int_I f_n} + \varepsilon(b-a) \quad \forall n \geq N.$$

Now since  $f_n$  is Riemann integrable, so  $\int_I f_n = \overline{\int_I f_n} = \int_I f_n$ . Hence, we have

$$\int_I f_n - \varepsilon(b-a) \leq \int_I f \leq \int_I f_n + \varepsilon(b-a) \quad \forall n \geq N.$$

Hence,

$$\overline{\int_I f} - \int_I f \leq 2\varepsilon(b-a) \quad \forall \varepsilon > 0,$$

which gives  $\overline{\int_I f} = \int_I f$ . Hence,  $f$  is Riemann integrable.

Also, we have

$$\int_I f_n - \varepsilon(b-a) \leq \int_I f \leq \int_I f_n + \varepsilon(b-a) \quad \forall n \geq N,$$

so we have

$$\left| \int_I f - \int_I f_n \right| \leq \varepsilon(b-a) \quad \forall n \geq N,$$

which gives  $\lim_{n \rightarrow \infty} \int_I f^{(n)} = \int_I f$ . ■

**Theorem 4.4.3.** Let  $f^{(n)} : [a, b] \rightarrow \mathbb{R}$  be a sequence of Riemann integrable function. Suppose  $\sum_{n=1}^{\infty} f^{(n)}(x)$  converges uniformly, and let  $f(x) := \sum_{n=1}^{\infty} f^{(n)}(x)$ , then  $f$  is Riemann integrable and

$$\sum_{n=1}^{\infty} \int_I f^{(n)} = \int_I f = \int_I \sum_{n=1}^{\infty} f^{(n)}.$$

**Proof.** Let  $S_k(x) = \sum_{i=1}^k f^{(i)}(x)$ , then since  $\sum_{i=1}^{\infty} f^{(i)}(x)$  converges uniformly, so  $\lim_{k \rightarrow \infty} S_k(x) = f(x)$  where  $f(x) = \sum_{i=1}^{\infty} f^{(i)}(x)$ . Now  $S_k = \sum_{i=1}^k f^{(i)}$  is a sum of Riemann integrable functions, so  $S_k$  is also Riemann integrable. By Theorem 4.4.2, we know  $f$  is Riemann integrable, and  $\lim_{k \rightarrow \infty} \int_I S_k = \int_I f$ , which means  $\lim_{k \rightarrow \infty} \sum_{i=1}^k \int_I f^{(i)} = \int_I f$  by the linearity of Riemann integral, and thus

$$\sum_{n=1}^{\infty} \int_I f^{(n)} = \int_I f.$$

**Note 4.4.4.**

$$\int_I S_k = \int_I \sum_{i=1}^k f^{(i)} = \sum_{i=1}^k \int_I f^{(i)}$$

for finite  $k$ . ■

**Example 4.4.4.**  $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$  pointwise for  $x \in (-1, 1)$ , and  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  pointwise for  $x \in (-1, 1)$ , but if we fix  $r \in (-1, 1)$ , then  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  uniformly on  $[-r, r]$ . Now since  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  uniformly on  $[0, r]$  for  $-1 < r < 1$ , then by Theorem 4.4.3, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \int_0^r x^n dx &= \int_0^r \frac{1}{1-x} dx \\ \Rightarrow \sum_{n=0}^{\infty} \frac{r^{n+1}}{n+1} &= -\ln(1-r) + \ln 1 = -\ln(1-r). \end{aligned}$$

## 4.5 Uniform Convergence & Derivatives

We talk about two examples to show that uniform convergence does not preserve the value of derivatives at some point.

**Example 4.5.1.** Suppose  $f_n : [0, 2\pi] \rightarrow \mathbb{R}$  and

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}},$$

then  $\lim_{n \rightarrow \infty} f_n = 0$  uniformly since

$$|f_n(x)| = \left| \frac{\sin(nx)}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{n}},$$

but its derivative is

$$f'_n(x) = \frac{\cos(nx) \cdot n}{\sqrt{n}},$$

so  $f'_n(0) = \sqrt{n}$ , and thus

$$\lim_{n \rightarrow \infty} f'_n(0) = \infty \neq f'(0) = 0.$$

**Example 4.5.2.** Suppose  $f_n(x) = \sqrt{\frac{1}{n^2} + x^2}$ , then  $\lim_{n \rightarrow \infty} f_n(x) = |x|$  uniformly since

$$\begin{aligned} 0 \leq f_n(x) - |x| &= \sqrt{\frac{1}{n^2} + x^2} - |x| = \frac{\left(\sqrt{\frac{1}{n^2} + x^2} - |x|\right) \left(\sqrt{\frac{1}{n^2} + x^2} + |x|\right)}{\sqrt{\frac{1}{n^2} + x^2} + |x|} \\ &= \frac{\frac{1}{n^2}}{\sqrt{\frac{1}{n^2} + x^2}} \leq \frac{\frac{1}{n^2}}{\sqrt{\frac{1}{n^2}}} = \frac{1}{n}. \end{aligned}$$

Let  $f(x) = |x|$ . We have  $\lim_{n \rightarrow \infty} f_n = f$  uniformly, and note that  $f_n(x) = \sqrt{\frac{1}{n^2} + x^2}$  and  $f'_n(x) = \frac{x}{\sqrt{\frac{1}{n^2} + x^2}}$ . Note that  $f'(0)$  does not exist and  $\lim_{n \rightarrow \infty} f'_n(0) = 0$ , so in this case  $f_n \rightarrow f$  uniformly and  $f'_n$  exists, but  $f'(0)$  doesn't exist.

**Theorem 4.5.1.** Let  $[a, b]$  be an interval and for  $n \geq 1$ , let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a differentiable function whose derivative  $f'_n : [a, b] \rightarrow \mathbb{R}$  is continuous. Suppose  $f'_n \rightarrow g$  uniformly where  $g : [a, b] \rightarrow \mathbb{R}$ . Suppose  $\exists x_0 \in [a, b]$  s.t.  $\lim_{n \rightarrow \infty} f_n(x_0)$  exists, then  $\exists$  a differentiable  $f$  s.t.  $\lim_{n \rightarrow \infty} f_n = f$  uniformly and  $f' = g$ .

**Proof.** Since  $f'_n$  is continuous and  $f'_n \rightarrow g$  uniformly, then  $g$  is Riemann integrable since  $f'_n$  is Riemann integrable for all  $n \in \mathbb{N}$  and by [Theorem 4.4.2](#). Also, we know

$$\lim_{n \rightarrow \infty} \int_{x_0}^x f'_n(s) ds = \int_{x_0}^x g(s) ds \text{ uniformly for } x_0, x \in [a, b]$$

since

$$\left| \int_{x_0}^x (f'_n - g)(s) ds \right| \leq \int_{x_0}^x |f'_n(s) - g(s)| ds \leq \frac{\varepsilon}{b-a} \cdot |x - x_0| \leq \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon \quad \forall \varepsilon > 0, n \geq N$$

for some  $N \in \mathbb{N}$ , and Fundamental theorem of Calculus tells us  $\lim_{n \rightarrow \infty} (f_n(x) - f_n(x_0)) = \int_{x_0}^x g(s) ds$  uniformly, which means  $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f_n(x_0) + \int_{x_0}^x g(s) ds$  uniformly, and we can let  $f$  to be R.H.S. ■

**Remark 4.5.1.** Informally, the theorem states that if  $f'_n$  is continuous and converges uniformly and  $f_n(x_0)$  converges for some  $x_0$ , then  $f_n$  itself converges uniformly, and moreover

$$\frac{d}{dx} \left( \lim_{n \rightarrow \infty} f_n(x) \right) = \lim_{n \rightarrow \infty} f'_n(x).$$

**Remark 4.5.2.** We need  $f'_n$  to be continuous to ensure that  $f'_n$  is Riemann integrable (Note that we have  $f'_n([a, b])$  is bounded and continuous, so  $f'_n$  is Riemann integrable). Also, we need continuity to use Fundamental Theorem of Calculus.

**Corollary 4.5.1.** Let  $[a, b]$  be an interval and for  $n \geq 1$ , let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a differentiable function whose derivatives  $f'_n : [a, b] \rightarrow \mathbb{R}$  is continuous. Suppose  $\sum_{n=1}^{\infty} \|f'_n\|_{\infty}$  converges. Suppose also that  $\sum_{n=1}^{\infty} f^{(n)}(x_0)$  converges for some  $x_0 \in [a, b]$ , then  $\sum_{n=1}^{\infty} f^{(n)}$  converges uniformly to a differentiable function and

$$\frac{d}{dx} \left( \sum_{n=1}^{\infty} f_n \right) = \sum_{n=1}^{\infty} f'_n \text{ for } x \in [a, b].$$

**Proof.** Suppose  $S_k(x) = \sum_{i=1}^k f_i(x)$  and  $S'_k(x) = \sum_{i=1}^k f'_i(x)$ , then since  $\sum_{i=1}^{\infty} \|f'_i\|_{\infty}$  converges, then by [Weierstrass M-test](#), we know  $\lim_{k \rightarrow \infty} S'_k = \sum_{i=1}^{\infty} f'_i$  uniformly. We also know that  $S_k(x_0) = \sum_{i=1}^k f_i(x_0)$  converges to  $\sum_{i=1}^{\infty} f_i(x_0)$  by our conditions. Now by [Theorem 4.5.1](#) (Suppose  $S_k$  here is  $f_n$  in [Theorem 4.5.1](#)), then we know  $\lim_{k \rightarrow \infty} S_k = S$  uniformly for some function  $S : [a, b] \rightarrow \mathbb{R}$  and  $S' = \sum_{i=1}^{\infty} f'_i$ , which means

$$\sum_{n=1}^{\infty} f'_n = \frac{d}{dx} S = \frac{d}{dx} \left( \lim_{k \rightarrow \infty} S_k \right) = \frac{d}{dx} \left( \sum_{n=1}^{\infty} f_n \right).$$

**Note 4.5.1.**  $\sum_{n=1}^{\infty} \|f'_n\|_{\infty}$  converges implies  $f'_n$  is bounded for all  $n$ , so we can use Weierstrass M-test. In fact,  $f'_n$  is continuous gives  $f'_n$  is bounded since its domain is an bounded interval. ■

**Example 4.5.3.**  $f(x) = \sum_{n=1}^{\infty} 4^{-n} \cos(32^n \pi x)$ , and  $f(x)$  is not differentiable anywhere.

**Proof.** Since

$$|4^{-n} \cos(32^{-n} \pi x)| \leq 4^{-n},$$

and  $\sum_{n=1}^{\infty} 4^{-n}$  converges, so  $\sum_{n=1}^{\infty} 4^{-n} \cos(32^{-n} \pi x)$  converges uniformly by Weierstrass M-test. Also, we know

$$(4^{-n} \cos(32^{-n} \pi x))' = -8^n \sin(32^n \pi x),$$

and we will learn that  $f'(x)$  does not exist at any point from the exercise. (See Exercise 4.7.10 in the textbook) \*

# Chapter 5

## Formal Power Series

### Lecture 16

#### 5.1 Review of series

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**Definition 5.1.1.** Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers,

- (a) The limit superior or (lim sup) of a sequence  $(a_n)$  is defined by

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup_{k \geq n} a_k.$$

Let  $S_n = \sup_{k \geq n} a_k$ . Note that the index set  $\{k \geq n\}$  is larger than  $\{k \geq n+1\}$ , so  $S_{n+1} \leq S_n$ . Equivalently, we can define  $\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} S_n$ , and since  $S_n$  is decreasing, so  $\lim_{n \rightarrow \infty} S_n$  exists, but it could be  $\infty$  or  $-\infty$ .

- (b) Similarly we can define

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf_{k \geq n} a_k.$$

Let  $I_n = \inf_{k \geq n} a_k$ , then we know  $I_n \leq I_{n+1}$ , so  $I_n$  is increasing, so  $\lim_{n \rightarrow \infty} I_n = \liminf_{n \rightarrow \infty} a_n$  exists, but it could be  $\infty$  or  $-\infty$ .

**Example 5.1.1.**

$$\limsup_{n \rightarrow \infty} n = \lim_{n \rightarrow \infty} \sup_{k \geq n} k = \infty.$$

$$\liminf_{n \rightarrow \infty} n = \lim_{n \rightarrow \infty} \inf_{k \geq n} k = \lim_{n \rightarrow \infty} n = \infty.$$

**Definition 5.1.2.** Let  $\sum_{n=1}^{\infty} a_n$  be a series of real numbers, and let  $S_N = \sum_{n=1}^N a_n$ , then we say  $\sum_{n=1}^{\infty} a_n$  converges if  $\lim_{N \rightarrow \infty} S_N$  exists and

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} S_N.$$

**Theorem 5.1.1.** If  $(a_n)$  is a sequence of real numbers, and if  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Proof.** Suppose  $S_N = \sum_{n=1}^N a_n$ , then we know  $S_{n+1} - S_n = a_{n+1}$ , then we know

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} S_{n+1} - S_n = \lim_{n \rightarrow \infty} S_{n+1} - \lim_{n \rightarrow \infty} S_n = 0.$$

■

**Corollary 5.1.1.** If  $(a_n)$  is a sequence of real numbers, and if  $\lim_{n \rightarrow \infty} a_n$  doesn't exist or  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

**Definition 5.1.3.** We say a real series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent if  $\sum_{n=1}^{\infty} |a_n|$  converges.

**Theorem 5.1.2.** If  $(a_n)$  is a sequence of real numbers, and if  $\sum_{n=1}^{\infty} a_n$  converges absolutely, then  $\sum_{n=1}^{\infty} a_n$  converges.

**Proof.** Let  $S_n = \sum_{i=1}^n a_i$ , then suppose  $n \geq m$ , then we have

$$|S_n - S_m| = \left| \sum_{i=m+1}^n a_i \right| \leq \sum_{i=m+1}^n |a_i|.$$

Let  $T_n = \sum_{i=1}^n |a_i|$ , then we know  $|T_n - T_m| = \sum_{i=m+1}^n |a_i|$ . Since  $\sum_{n=1}^{\infty} a_n$  converges absolutely, so  $\lim_{n \rightarrow \infty} T_n$  exists, and thus  $\{T_n\}$  is Cauchy. Since

$$|S_n - S_m| \leq |T_n - T_m|,$$

so  $\{S_n\}$  is also Cauchy, which means it is convergent. ■

## 5.2 Formal Power Series

**Definition 5.2.1.** Let  $a \in \mathbb{R}$ . A formal power series centered at  $a$  is any series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n$$

where  $\{c_n\}_{n=0}^{\infty}$  is a sequence of real numbers. We refer  $c_n$  to the  $n$ -th coefficient of the series. Each term  $c_n (x - a)^n$  is a function of  $x$ .

**Example 5.2.1.**  $\sum_{n=0}^{\infty} n!(x - 2)^n$  is a formal power series centered at 2 but  $\sum_{n=0}^{\infty} 2^x (x - 3)^n$  is not a formal power series since  $2^x$  depends on  $x$  not on  $n$ .

**Definition 5.2.2.** Let  $\sum_{n=0}^{\infty} c_n (x - a)^n$  be a formal power series. We define the radius of convergence  $R$  of this series to be the quantity  $R = \frac{1}{\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}}$  with the convention  $\frac{1}{0} = +\infty$  and  $\frac{1}{+\infty} = 0$ .

**Remark 5.2.1.** The radius of convergence must be non-negative.

**Lemma 5.2.1.** Let  $(a_n)$  be a sequence of real number. Suppose  $L = \limsup_{n \rightarrow \infty} a_n \in [0, \infty)$ , then for any  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.  $a_n < L + \varepsilon$  for all  $n \geq N$ .

**Proof.** If  $L = \infty$ , then this is true. Now if  $L$  is a real number, then  $L = \lim_{n \rightarrow \infty} S_n$  where  $S_n = \sup_{k \geq n} a_k$ . Given any  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.  $|S_n - L| < \varepsilon$  for all  $n \geq N$ , so  $S_n < L + \varepsilon$  for all  $n \geq N$ . Hence,

$$a_n \leq \sup_{k \geq n} a_k = S_n < L + \varepsilon \quad \forall n \geq N.$$

■

**Lemma 5.2.2.** Let  $(a_n)$  be a sequence of real numbers, and let  $p := \limsup_{n \rightarrow \infty} a_n$ . Then for every  $\varepsilon > 0$ ,  $\exists$  infinitely many indices  $n$  s.t.  $a_n > p - \varepsilon$ .

**Proof.** We know  $p = \lim_{n \rightarrow \infty} S_n$  where  $S_n = \sup_{k \geq n} a_k$ , so given  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.

$$|S_n - p| < \frac{\varepsilon}{2} \quad \forall n \geq N,$$

so  $p - \frac{\varepsilon}{2} < S_n = \sup_{k \geq n} a_k$ , which means  $\exists k_1 \geq n$  s.t.  $p - \frac{\varepsilon}{2} < a_{k_1}$ , and let  $n = k_1 + 1$  and repeat this step to get  $k_2$  and so on, then we know  $\{k_n\}$  is an infinite set. ■

**Theorem 5.2.1 (Ratio Test).** Suppose  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \in [0, \infty)$ . If  $L < 1$ , then  $\sum_{n=0}^{\infty} a_n$  converges absolutely. If  $L > 1$ , then  $\sum_{n=0}^{\infty} a_n$  diverges. If  $L = 1$ , then no conclusion.

**Proof.** If  $L < 1$ , then we know there exists  $r$  s.t.  $L < r < 1$ , so there exists  $N > 0$  s.t. for all  $n \geq N$  we have

$$\left| \frac{a_{n+1}}{a_n} \right| \leq r,$$

so we know  $|a_{n+1}| \leq r|a_n|$  for all  $n \geq N$ , which means  $|a_{n+k}| \leq r^k|a_n|$ , so

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{N-1} |a_n| + \sum_{k=0}^{\infty} |a_{N+k}| = C + \sum_{k=0}^{\infty} r^k |a_N| = C + |a_N| \sum_{k=0}^{\infty} r^k,$$

which means  $\sum_{n=0}^{\infty} |a_n|$  is convergent since  $r < 1$ .

If  $L > 1$  (including  $L = \infty$ ), then by the definition of limit, we know for some  $\varepsilon > 0$ , there exists  $N > 0$  s.t.

$$\left| \frac{a_{n+1}}{a_n} \right| \geq 1 + \varepsilon$$

for all  $n \geq N$ , and thus  $|a_{n+1}| \geq (1 + \varepsilon)|a_n|$  for all  $n \geq N$ . In particular, there must exist some  $k > 0$  with  $|a_k| > 0$ , otherwise  $L$  is not well-defined, and thus we know  $\lim_{n \rightarrow \infty} a_n \neq 0$  since  $|a_n|$  is strictly increasing, so by [Theorem 5.1.1](#), we know  $\sum_{n=0}^{\infty} a_n$  diverges.

If  $L = 1$ , then for example, we know  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges but  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges. ■

**Theorem 5.2.2.** Let  $\sum_{n=0}^{\infty} c_n(x-a)^n$  be a formal power series, and let  $R$  be its radius of convergence  $R = \frac{1}{\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}}$ , then

- (a)  $\sum_{n=0}^{\infty} c_n(x-a)^n$  diverges if  $|x-a| > R$ .
- (b)  $\sum_{n=0}^{\infty} c_n(x-a)^n$  converges absolutely if  $|x-a| < R$ .
- (c)  $\sum_{n=0}^{\infty} c_n(x-a)^n$  converges uniformly on  $[a-r, a+r]$  when  $0 < r < R$ .
- (d) Let  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ , and if  $|x-a| < R$ , then  $f$  is differentiable and for any  $0 < r < R$ ,  $\sum_{n=1}^{\infty} n c_n(x-a)^{n-1}$  converges uniformly to  $f'$  on  $[a-r, a+r]$ .
- (e) For any  $[y, z] \subset (a-R, a+R)$ ,

$$\int_y^z f(x) dx = \sum_{n=0}^{\infty} \frac{c_n(z-a)^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{c_n(y-a)^{n+1}}{n+1}.$$

**proof from (c) to (e).** Trivial by [Theorem 4.4.3](#). We may have to show the radius of convergence remains the same to show that we can write

$$\sum_{n=0}^{\infty} \frac{c_n(z-a)^{n+1}}{n+1} - \frac{c_n(y-a)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{c_n(z-a)^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{c_n(y-a)^{n+1}}{n+1}.$$

■

**proof of (a).** Write  $L := \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$ , so  $L = \frac{1}{R}$ . Now if  $|x-a| > R$ , then let  $S = |x-a| > R$ ,

and we have

$$\limsup_{n \rightarrow \infty} |c_n(x-a)^n|^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} |x-a| = |x-a| \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} = SL = \frac{S}{R} > 1.$$

Choose  $\varepsilon > 0$  s.t.  $\frac{S}{R} - \varepsilon > 1$ . From [Lemma 5.2.2](#), there exists infinitely many  $n$  s.t.

$$|c_n|^{\frac{1}{n}} |x-a| > \frac{S}{R} - \varepsilon > 1,$$

so there are infinitely many  $n$  has  $|c_n(x-a)^n| > 1$ , so  $\lim_{n \rightarrow \infty} c_n(x-a)^n \neq 0$ , and thus we know  $\sum_{n=0}^{\infty} c_n(x-a)^n$  diverges. ■

**proof of (b).** If  $|x-a| < R$ , then  $\frac{|x-a|}{R} < 1$  i.e.  $|x-a|L < 1$  ( $L$  is same as it is in proof of (a)). Now we have  $\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} = L$ , so we can choose  $\varepsilon > 0$  s.t.  $|x-a|(L+\varepsilon) < 1$ , so there exists  $N > 0$  s.t.

$$|c_n|^{\frac{1}{n}} < L + \varepsilon \quad \forall n \geq N \Leftrightarrow |c_n| < (L + \varepsilon)^n \Leftrightarrow |c_n(x-a)^n| \leq (L + \varepsilon)^n |x-a|^n$$

by [Lemma 5.2.1](#), so

$$\sum_{n=0}^{\infty} |c_n(x-a)^n| \leq \sum_{n=0}^{N-1} |c_n(x-a)^n| + \sum_{n=N}^{\infty} |c_n(x-a)^n| \leq \sum_{n=0}^{N-1} |c_n| |x-a|^n + \sum_{n=N}^{\infty} |(L + \varepsilon)(x-a)|^n$$

and R.H.S. converges since the left term is finite and for the right term we have  $|(L + \varepsilon)(x-a)| < 1$ . ■

**proof of (c).** Now if  $|x-a| \leq r < R$ , then we can choose  $\varepsilon > 0$  s.t.  $q = (L + \varepsilon)r < 1$  since  $Lr = \frac{r}{R} < 1$ . Note that

$$|(L + \varepsilon)(x-a)|^n \leq ((L + \varepsilon)r)^n = q^n,$$

and this is independent of  $x$ . (Note that  $L > 0$  so we can get rid of the absolute value) Note that this means there exists  $N > 0$  s.t.

$$\begin{aligned} \sum_{n=0}^{\infty} \|c_n(x-a)^n\|_{\infty} &= \sum_{n=0}^{\infty} |c_n r^n| = \sum_{n=0}^{N-1} |c_n r^n| + \sum_{n=N}^{\infty} |c_n r^n| \\ &= C + \sum_{n=N}^{\infty} \left| c_n^{\frac{1}{n}} r \right|^n \leq C + \sum_{n=N}^{\infty} |(L + \varepsilon)r|^n \leq C + \sum_{n=N}^{\infty} q^n, \end{aligned}$$

by [Lemma 5.2.1](#), and we know this means  $\sum_{n=0}^{\infty} \|c_n(x-a)^n\|_{\infty}$  is convergent since  $q < 1$ , so by [Weierstrass M-test](#), we know  $\sum_{n=0}^{\infty} c_n(x-a)^n$  converges uniformly. ■

**proof of (d).** Suppose  $S_N(x) = \sum_{n=0}^N c_n(x-a)^n$  and

$$S'_N(x) = \sum_{n=1}^N n c_n(x-a)^{n-1} = \sum_{n=0}^{N-1} (n+1) c_{n+1}(x-a)^n.$$

Then, we know  $S'_N$  is continuous for all  $N > 0$ . Now we claim that the radius of convergence of  $\sum_{n=0}^{\infty} (n+1) c_{n+1}(x-a)^n$  is  $R$ . Note that

$$\limsup_{n \rightarrow \infty} |(n+1) c_{n+1}|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \sup_{k \geq n} (k+1)^{\frac{1}{k}} |c_{k+1}|^{\frac{1}{k}} = \lim_{n \rightarrow \infty} (n+1)^{\frac{1}{n}} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}}.$$

Note that we can show that  $\lim_{n \rightarrow \infty} (n+1)^{\frac{1}{n}} = 1$ , which is easy, so we skip here, and we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} |(n+1) c_{n+1}|^{\frac{1}{n}} &= \lim_{n \rightarrow \infty} (n+1)^{\frac{1}{n}} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} = \lim_{n \rightarrow \infty} (n+1)^{\frac{1}{n}} \lim_{n \rightarrow \infty} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} \\ &= \lim_{n \rightarrow \infty} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} = \limsup_{n \rightarrow \infty} |c_{n+1}|^{\frac{1}{n}}. \end{aligned}$$

Now we show that  $\limsup_{n \rightarrow \infty} |c_{n+1}|^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} = L$ . For all  $\varepsilon > 0$ , there exists  $N > 0$  s.t. for all  $n \geq N$  we have

$$|c_n|^{\frac{1}{n}} < L + \varepsilon \Leftrightarrow |c_{n+1}|^{\frac{1}{n+1}} < L + \varepsilon \Leftrightarrow |c_{n+1}|^{\frac{1}{n}} < (L + \varepsilon)^{\frac{n+1}{n}} = (L + \varepsilon)^{1 + \frac{1}{n}}.$$

Hence, we have

$$\sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} \leq (L + \varepsilon)^{1 + \frac{1}{n}} \quad \forall n \geq N$$

since  $(L + \varepsilon)^{1 + \frac{1}{n}} \geq (L + \varepsilon)^{1 + \frac{1}{k}} > |c_{k+1}|^{\frac{1}{k}}$  for all  $k \geq n$ . Hence, we know

$$\lim_{n \rightarrow \infty} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} \leq \lim_{n \rightarrow \infty} (L + \varepsilon)^{1 + \frac{1}{n}} = L + \varepsilon.$$

Since for every  $\varepsilon > 0$  this is true, so  $\lim_{n \rightarrow \infty} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} \leq L$ . Now we show

$$\lim_{n \rightarrow \infty} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} \geq L,$$

and then we can conclude that

$$\lim_{n \rightarrow \infty} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} = L = \lim_{n \rightarrow \infty} \sup_{k \geq N} |c_n|^{\frac{1}{n}}.$$

For all  $\varepsilon > 0$ , we know there exists infinitely many  $n$  has

$$|c_n|^{\frac{1}{n}} > L - \varepsilon.$$

Hence, for every  $n \in \mathbb{N}$ , there exists  $s_n > n$  has  $|c_{s_n}|^{\frac{1}{s_n}} > L - \varepsilon$ , and we collect  $\{s_n\}_{n=1}^{\infty}$ . Thus, we have

$$|c_{s_n}|^{\frac{1}{s_n-1}} > (L - \varepsilon)^{\frac{s_n}{s_n-1}} = (L - \varepsilon)^{1 + \frac{1}{s_n-1}}$$

for all  $s_n$ . This means

$$\sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} \geq |c_{(s_{n+1}-1)+1}|^{\frac{1}{s_{n+1}-1}} = |c_{s_{n+1}}|^{\frac{1}{s_{n+1}-1}} > (L - \varepsilon)^{1 + \frac{1}{s_{n+1}-1}},$$

and thus

$$\lim_{n \rightarrow \infty} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} > \lim_{n \rightarrow \infty} (L - \varepsilon)^{1 + \frac{1}{s_{n+1}-1}} = (L - \varepsilon) \lim_{n \rightarrow \infty} (L - \varepsilon)^{\frac{1}{s_{n+1}-1}} = L - \varepsilon$$

since  $s_{n+1} - 1 > n$ . Note that this is true for all  $\varepsilon > 0$ , so we know  $\lim_{n \rightarrow \infty} \sup_{k \geq n} |c_{k+1}|^{\frac{1}{k}} \geq L$ , and we're done.

Now since we know the radius of convergence of  $\lim_{n \rightarrow \infty} S'_n$  is  $R$ , so by (c) we know  $S'_n$  converges uniformly to  $\sum_{n=0}^{\infty} (n+1)c_{n+1}(x-a)^n$  since from the problem condition we have  $|x-a| < R$ . Also, we know  $\lim_{n \rightarrow \infty} S_n(a) = 0$ , which means the limit exists, so by [Theorem 4.5.1](#), we know there exists differentiable  $f$  s.t.  $S_N \rightarrow f$  uniformly and  $f' = g$  if we let  $S'_n \rightarrow g$  uniformly. This means  $S'_n \rightarrow f'$  uniformly, where  $f = \sum_{n=0}^{\infty} c_n(x-a)^n$ , and thus

$$S'_n = \sum_{n=0}^{\infty} (n+1)c_{n+1}(x-a)^n \rightarrow f'$$

uniformly. ■

**Remark 5.2.2.** If  $|x-a| = R$ , then  $\sum_{n=0}^{\infty} c_n(x-a)^n$  may converge or diverge, and there is no conclusion.

**Remark 5.2.3.** If  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$  is a formal power series, and  $R$  is the radius of convergence

of  $f$ , then the radius of convergence of

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \quad \int_y^x f(x) dx = \sum_{n=0}^{\infty} \frac{c_n (z-a)^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{c_n (y-a)^{n+1}}{n+1}$$

are both  $R$  on  $[a-r, a+r]$  for some  $r < R$  and  $[y, z] \subseteq (a-R, a+R)$ .

## Lecture 17

### 5.3 Real Analytic Functions

4 Nov. 9:10

A function  $f(x)$  that is representable as a power series near a point is called real analytic at that point.

- Intuitively:  $f$  can be written as  $\sum_{n=0}^{\infty} c_n (x-a)^n$  in some neighborhood of  $a$ .
- Consequences (later): such an  $f$  is continuous and  $C^\infty$  in that neighborhood since no matter how many times you differentiate this  $f$ , it remains a power series.

**Definition 5.3.1 (Real analytic functions).** Let  $E \subseteq \mathbb{R}$  and  $f : E \rightarrow \mathbb{R}$ . If  $a$  is an interior point of  $E$ , we say that  $f$  is real analytic at  $a$  if there exists  $r > 0$  and a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

centered at  $a$  with radius of convergence  $\geq r$ , which converges to  $f(x)$  for all  $x \in (a-r, a+r) \subseteq E$ . If  $E$  is open and  $f$  is real analytic at every  $a \in E$ , then  $f$  is real analytic on  $E$ .

**Remark 5.3.1.** This power series converges to uniformly automatically by (c) of [Theorem 5.2.2](#).

**Remark 5.3.2.** Informally, a function is real analytic at  $a$  implies it can be represented as a power series whose radius of convergence is bigger than 0.

**Corollary 5.3.1.** If  $f$  is real analytic at  $a$ , and say

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

converges to  $f(x)$  for all  $x \in (a-r, a+r)$ , then  $f$  is real analytic for all  $x \in (a-r, a+r)$  i.e.  $f$  is real analytic on  $(a-r, a+r)$ .

**Proof.** Let  $x_0 \in (a-r, a+r)$ . We will show that  $f$  is analytic at  $x_0$ .

Since  $|x_0 - a| < r$ , for any  $x$  we can write

$$x - a = (x - x_0) + (x_0 - a).$$

Substituting this into the given series and applying the binomial theorem gives

$$(x-a)^n = ((x-x_0) + (x_0-a))^n = \sum_{m=0}^n \binom{n}{m} (x-x_0)^m (x_0-a)^{n-m}.$$

Hence

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n \sum_{m=0}^n \binom{n}{m} (x-x_0)^m (x_0-a)^{n-m}.$$

We now interchange the order of summation (which will be justified below) to obtain

$$f(x) = \sum_{m=0}^{\infty} \left( \sum_{n=m}^{\infty} c_n \binom{n}{m} (x_0 - a)^{n-m} \right) (x - x_0)^m.$$

Define

$$d_m = \sum_{n=m}^{\infty} c_n \binom{n}{m} (x_0 - a)^{n-m}.$$

Then

$$f(x) = \sum_{m=0}^{\infty} d_m (x - x_0)^m.$$

To justify the interchange of summation, fix a number  $r'$  such that  $|x_0 - a| < r' < r$ . Since the original series converges for  $|x - a| < r$ , there exists  $M > 0$  such that  $|c_n| r'^n \leq M$  for all  $n$ . Hence

$$|c_n| \leq \frac{M}{r'^n}.$$

For each  $m$ ,

$$|c_n \binom{n}{m} (x_0 - a)^{n-m}| \leq \frac{M}{r'^n} \binom{n}{m} |x_0 - a|^{n-m}.$$

Since  $\binom{n}{m} \leq \frac{n^m}{m!}$  and  $|x_0 - a|/r' < 1$ , we obtain

$$|c_n \binom{n}{m} (x_0 - a)^{n-m}| \leq \frac{M}{m!} n^m \left( \frac{|x_0 - a|}{r'} \right)^n,$$

and the right-hand side defines a convergent series in  $n$  because  $0 < |x_0 - a|/r' < 1$ . Therefore the inner sum defining  $d_m$  converges absolutely, and interchanging the order of summation is justified.

Furthermore, from the last inequality it follows that the new series  $\sum_{m=0}^{\infty} d_m (x - x_0)^m$  converges at least for  $|x - x_0| < r - |x_0 - a|$ .

Thus  $f$  can be represented by a convergent power series around  $x_0$ , so  $f$  is analytic at  $x_0$ . Since  $x_0$  was arbitrary in  $(a - r, a + r)$ ,  $f$  is analytic on the entire interval. ■

**Example 5.3.1.** Consider the function  $f : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$  defined by

$$f(x) := \frac{1}{1-x}.$$

Show that  $\frac{1}{1-x}$  is real analytic on  $\mathbb{R} \setminus \{1\}$ .

**Proof.** Let  $f(x) = \frac{1}{1-x}$  and fix any  $a \in \mathbb{R} \setminus \{1\}$ . Set  $y = x - a$ , so that  $x = a + y$ . Then,

$$f(x) = \frac{1}{1 - (a + y)} = \frac{1}{(1 - a) - y} = \frac{1}{1 - a} \cdot \frac{1}{1 - \frac{y}{1-a}}.$$

Since  $|1 - a| > 0$ , we may apply the geometric series expansion:

$$\frac{1}{1 - \frac{y}{1-a}} = \sum_{n=0}^{\infty} \left( \frac{y}{1-a} \right)^n \quad \text{for } \frac{|y|}{|1-a|} < 1 \text{ i.e. } |x - a| < |1 - a|.$$

Hence,

$$\frac{1}{1-x} = \frac{1}{1-a} \sum_{n=0}^{\infty} \left( \frac{y}{1-a} \right)^n = \sum_{n=0}^{\infty} \frac{y^n}{(1-a)^{n+1}} = \sum_{n=0}^{\infty} \frac{(x-a)^n}{(1-a)^{n+1}},$$

for  $|x - a| < |1 - a|$ . Hence,  $f$  is real analytic on  $\mathbb{R} \setminus \{1\}$ . ⊛

**Remark 5.3.3.** Real analyticity parallels complex analyticity (a topic of complex analysis). In this section we focus on real functions, their power-series convergence, and differentiability properties.

From (c), (d) of [Theorem 5.2.2](#), we know if  $f$  is real analytic at  $a$ , then  $f$  is continuous and infinitely differentiable in some neighborhood (inside the interval of convergence i.e.  $x \in (a - R, a + R)$ ) of  $a$ . We now formalize higher-order differentiability.

**Definition 5.3.2** ( *$k$ -times differentiability*). Let  $E \subseteq \mathbb{R}$  with every point a limit point. A function  $f : E \rightarrow \mathbb{R}$  is once differentiable on  $E$  if  $f'$  exists on  $E$ . For  $k \geq 2$ ,  $f$  is  $k$ -time differentiable on  $E$  if  $f'$  is  $(k - 1)$ -times differentiable. Recursively define

$$f^{(1)} := f', \quad f^{(k)} := \left(f^{(k-1)}\right)' \quad (k \geq 2), \quad f^{(0)} := f.$$

We say  $f$  is infinitely differentiable or smooth or  $C^\infty$  if  $f$  is  $k$ -times differentiable for every  $k \geq 0$ .

**Example 5.3.2.** The function  $f(x) := |x|^3$  is twice differentiable on  $\mathbb{R}$ , but not three times differentiable.

**Proof.** We can write

$$f(x) = \begin{cases} x^3, & x \geq 0, \\ (-x)^3 = -x^3, & x < 0. \end{cases}$$

Hence,

$$f'(x) = \begin{cases} 3x^2, & x > 0, \\ -3x^2, & x < 0. \end{cases}$$

At  $x = 0$ , we check

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{x^3 - 0}{x - 0} = 0, \quad \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-x^3 - 0}{x - 0} = 0,$$

so  $f'(0) = 0$ . Therefore,

$$f'(x) = \begin{cases} 3x^2, & x > 0, \\ 0, & x = 0, \\ -3x^2, & x < 0. \end{cases}$$

Similarly, we can check

$$f''(x) = \begin{cases} 6x, & x > 0, \\ 0, & x = 0, \\ -6x, & x < 0. \end{cases}$$

Hence, we know  $f''(x) = 6|x|$ , and it can easily shown that  $f''(x)$  is not differentiable at  $x = 0$ .  $\circledast$

This section will be extended with further theorems, proofs, and examples as more material is provided.

- (a) Uniform convergence and termwise differentiation of power series.
- (b) Uniqueness of power-series expansions and radii of convergence.

**Proposition 5.3.1** (*Differentiability of real analytic functions*). Let  $E \subseteq \mathbb{R}$ , and  $a$  be an interior point of  $E$ , and  $f : E \rightarrow \mathbb{R}$  be real analytic at  $a$ . Then, there exists  $r > 0$  s.t.

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n, \quad x \in (a - r, a + r).$$

For every integer  $k \geq 0$ ,

$$f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!} (x-a)^n,$$

so  $f$  is  $k$ -times differentiable on  $(a-r, a+r)$ . In particular, evaluating at  $x = a$  gives the coefficient identity

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

**Proof.** Fix  $r > 0$  so that the power series  $\sum_{n=0}^{\infty} c_n (x-a)^n$  converges to  $f(x)$  for all  $x \in (a-r, a+r)$  (We can pick  $r$  to be the radius of convergence). For  $0 < \rho < r$ , the series converges uniformly on  $[a-\rho, a+\rho]$  by [Weierstrass M-test](#), hence  $f$  is continuous there. Now we do induction on  $k$  to obtain the  $k$ -th derivative formula.

- Base case  $k = 0$ . Trivial:

$$f^{(0)}(x) = f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

- Step  $k = 1$ . Fix  $0 < \rho < r$ . On  $[a-\rho, a+\rho]$ ,

$$\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

converges uniformly (again by the M-test), since for  $|x-a| \leq \rho < r$ ,

$$|n c_n (x-a)^{n-1}| \leq n |c_n| \rho^{n-1},$$

and  $\sum_{n=1}^{\infty} n |c_n| \rho^{n-1}$  converges whenever  $\rho < r$  (The radius of convergence remains the same). Therefore we may differentiate termwise to get

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=0}^{\infty} c_{n+1} (n+1) (x-a)^n.$$

- Induction step. Assume for some  $k \geq 1$  that

$$f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!} (x-a)^n \text{ for } x \in (a-r, a+r),$$

and that the series on the right converges uniformly on each  $[a-\rho, a+\rho]$  with  $0 < \rho < r$ . Then, for  $|x-a| < \rho$ ,

$$\left| \frac{(n+k)!}{n!} c_{n+k} (x-a)^n \right| \leq \frac{(n+k)!}{n!} |c_{n+k}| \rho^n.$$

Since a power series and all its termwise derivatives share the same radius of convergence  $r$ , the series  $\sum_{n=0}^{\infty} \frac{(n+k+1)!}{n!} |c_{n+k+1}| \rho^n$  converges for  $\rho < r$ , hence by the  $M$ -test the differentiated series is uniformly convergent on  $[a-\rho, a+\rho]$ . Therefore,

$$\begin{aligned} f^{(k+1)}(x) &= \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!} n (x-a)^{n-1} \\ &= \sum_{n=1}^{\infty} c_{n+k} \frac{(n+k)!}{(n-1)!} (x-a)^{n-1} \\ &= \sum_{m=0}^{\infty} c_{m+k+1} \frac{(m+k+1)!}{m!} (x-a)^m \end{aligned}$$

which is exactly the desired formula with  $k$  replaced by  $k + 1$ . This completes the induction. Finally, setting  $x = a$  in the  $k$ -th derivative formula gives

$$f^{(k)}(a) = \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!} (a-a)^n = c_k k!,$$

and hence

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

**Remark 5.3.4.** Why we can deduce  $\sum_{n=0}^{\infty} \frac{(n+k+1)!}{n!} |c_{n+k+1}| \rho^n$  converges for  $\rho < r$ ? The answer is since we know the radius of convergence of  $\sum_{n=0}^{\infty} \frac{(n+k+1)!}{n!} |c_{n+k+1}| (x-a)^n$  is equal to that of the radius of convergence of  $\sum_{n=0}^{\infty} \frac{(n+k+1)!}{n!} c_{n+k+1} (x-a)^n$  by definition, and we know the radius of convergence of latter is equal to the radius of convergence of  $f^{(k)}$ .

■

**Remark 5.3.5.** If  $f$  is real analytic on  $(a-r, a+r)$ , then it admits a convergent Taylor series expansion about  $a$ . In particular,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k, \quad x \in (a-r, a+r).$$

Thus, the sequence of derivatives  $\{f^{(k)}(a)\}_{k=0}^{\infty}$  completely determines the values of  $f$  on the interval  $(a-r, a+r)$ .

**Corollary 5.3.2** (Analytic to  $C^\infty$ ). Let  $E$  be open and  $f : E \rightarrow \mathbb{R}$  real analytic on  $E$ . Then  $f$  is infinitely differentiable on  $E$ , and each  $f^{(k)}$  is itself real analytic.

**Proof sketch.**

- (a) For any  $a \in E$ ,  $f$  has an convergent power series on  $(a-r, a+r)$ .
- (b) [Proposition 5.3.1](#) gives  $f^{(k)}$  via a power series, so  $f^{(k)}$  is analytic at  $a$ .
- (c) Since  $a$  can be arbitrary, so  $f$  is  $C^\infty$  and every derivative is analytic.

■

**Example 5.3.3.** Let  $f(x) = |x|$ . Then  $f$  is defined on  $\mathbb{R}$  but it is not analytic on  $\mathbb{R}$ .

**Proof.** Fix  $a \neq 0$ . If  $a > 0$ , then on  $(0, \infty)$  we have  $f(x) = x$ , so we may write

$$f(x) = x = x + (x-a),$$

which holds for all  $x > 0$ . Hence, if  $f$  is real analytic at  $a$ , then the domain of convergence of the Taylor series about  $a$  (the largest open interval centered at  $a$  contained in  $(0, \infty)$ ) is

$$(0, 2a) = \{x : |x-a| < a\}.$$

(Since this formula holds only when  $x > 0$ ) If  $a < 0$ , then on  $(-\infty, 0)$  we have  $f(x) = -x$ , so we may write

$$f(x) = -x = -a - (x-a),$$

which holds for all  $x < 0$ . Hence,  $f$  is real analytic at  $a$ . Hence, the domain of convergence of the

Taylor series about  $a$  (the largest open interval centered at  $a$  contained in  $(-\infty, \infty)$ ) is

$$(2a, 0) = \{x : |x - a| < |a|\}.$$

If  $a = 0$ , then if  $f(x)$  is real analytic at  $a$ , then  $f$  is differentiable at  $a$ , which is false, so  $f(x)$  is not real analytic at 0. Therefore,  $f$  is real analytic on  $\mathbb{R} \setminus \{0\}$ .

- (a) Real analyticity is a local property.
- (b) Failure at one point does not preclude analyticity elsewhere.

⊛

**Remark 5.3.6.**  $C^\infty$  does not imply analytic, one counterexample is in the next example.

**Example 5.3.4** (Bump function).

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

**Proof.** Since all derivatives exist and vanish at  $x = 0$  i.e.  $f^{(k)}(0) = 0$ . Hence, the Taylor series at 0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0.$$

But  $f(x) > 0$  for  $x \neq 0$ , so the series does not represent  $f$  near 0. This means  $f$  can not be analytic at 0.

In the following, we explain why  $f^{(k)}(0) = 0$  for all  $k \in \mathbb{N}$ .

**Claim 5.3.1.** For every  $k \in \mathbb{N}$ , there exists a polynomial  $P_k(t)$  s.t.

$$f^{(k)}(x) = P_k\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} \quad \text{for } x \neq 0.$$

**Proof.** Let  $t = \frac{1}{x}$  and set  $g(t) = e^{-t^2}$ , so that  $f(x) = g(t)$  with  $t = \frac{1}{x}$ . Since  $g'(t) = -2te^{-t^2}$  and  $\frac{dt}{dx} = -\frac{1}{x^2} = -t^2$ , we obtain

$$f'(x) = g'(t) \frac{dt}{dx} = (-2te^{-t^2})(-t^2) = 2t^3e^{-t^2} = P_1(t)e^{-t^2} = P_1(t)g(t),$$

where  $P_1(t) = 2t^3$ . Thus, the formula holds for  $k = 1$ . Now assume the formula holds for some  $k \in \mathbb{N}$ , i.e.

$$f^{(k)}(x) = P_k(t)g(t), \quad t = \frac{1}{x}.$$

Differentiating and using the chain rule, together with  $g'(t) = -2tg(t)$  and  $\frac{dt}{dx} = -t^2$ , we have

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx} (P_k(t)g(t)) = (P'_k(t)g(t) + P_k(t)g'(t)) \frac{dt}{dx} \\ &= (P'_k(t)g(t) - 2tP_k(t)g(t))(-t^2) = (-t^2P'_k(t) + 2t^3P_k(t))g(t). \end{aligned}$$

Hence,

$$f^{(k+1)}(x) = P_{k+1}(t)g(t), \quad \text{where } P_{k+1}(t) := -t^2P'_k(t) + 2t^3P_k(t),$$

and  $P_{k+1}(t)$  is a polynomial whenever  $P_k(t)$  is. This completes the induction. ⊛

Now we can show  $f^{(k)}(0) = 0$  for all  $k \in \mathbb{N}$ . First, note that

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x}.$$

We first compute the right-hand limit. Let  $t = \frac{1}{x}$ , so that  $x \rightarrow 0^+$  corresponds to  $t \rightarrow +\infty$ . Then

$$\lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{t \rightarrow \infty} \frac{t}{e^{t^2}} = \lim_{t \rightarrow \infty} \frac{1}{2te^{t^2}} = 0$$

by the L'Hopital's rule. Similarly, for the left-hand limit,

$$\lim_{x \rightarrow 0^-} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{t \rightarrow -\infty} \frac{t}{e^{t^2}} = \lim_{t \rightarrow -\infty} \frac{1}{2te^{t^2}} = 0$$

again by the L'Hopital's rule. Since both one-sided limits are equal to 0, we conclude that  $f'(0) = 0$ .

We now show by induction that  $f^{(k)}(0) = 0$  for all  $k \in \mathbb{N}$ . For  $k = 1$ , we have shown that  $f'(0) = 0$ . Assume  $f^{(n)}(0) = 0$  for some  $n \geq 1$ . By [Claim 5.3.1](#), we know

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}},$$

and by induction we have  $f^{(n)}(0) = 0$ . Then,

$$f^{(n+1)}(0) = \lim_{x \rightarrow 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x \rightarrow 0} \frac{P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}}{x}.$$

Similarly suppose  $t = \frac{1}{x}$ , and then we have

$$\begin{cases} \lim_{x \rightarrow 0^+} \frac{P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}}{x} = \lim_{t \rightarrow \infty} \frac{tP_n(t)}{e^{t^2}} = 0, \\ \lim_{x \rightarrow 0^-} \frac{P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}}{x} = \lim_{t \rightarrow -\infty} \frac{tP_n(t)}{e^{t^2}} = 0 \end{cases}$$

since  $P_n(t)$  is polynomial and  $e^{t^2}$  is an exponential function. By this, we can conclude that  $f^{(n+1)}(0) = 0$ , and we're done.  $\circledast$

**Corollary 5.3.3** (Taylor's formula). If  $f$  is real analytic at  $a$  with

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad x \in (a-r, a+r),$$

then for all  $k \geq 0$ ,

$$f^{(k)}(a) = k!c_k,$$

and hence

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

**Remark 5.3.7.** Taylor's formula applies only to real analytic functions. There exist functions that are  $C^\infty$  but not real analytic, whose Taylor series fails to represent the function. For example,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

**Corollary 5.3.4.** If  $f$  is real analytic at  $a$  and admits two power series expansions

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ and } f(x) = \sum_{n=0}^{\infty} d_n(x-a)^n,$$

then  $c_n = d_n$  for all  $n \geq 0$ .

**Proof.** By Taylor's formula,  $f^{(k)}(a) = k!c_k = k!d_k$ , and since  $k! \neq 0$ , so we can conclude  $c_k = d_k$  for all  $k$ . ■

**Remark 5.3.8.** This means the power series expansion about a given center is unique.

**Remark 5.3.9.** Although the expansion at a fixed point  $a$  is unique, different centers yield distinct series.

**Example 5.3.5.** We have shown that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(x-a)^n}{(1-a)^{n+1}}, \quad \text{for } |x-a| < |1-a|.$$

Hence, we have

$$\text{At } a = 0 : \quad f(x) = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

$$\text{At } a = \frac{1}{2} : \quad f(x) = \sum_{n=0}^{\infty} 2^{n+1} \left(x - \frac{1}{2}\right)^n, \quad \left|x - \frac{1}{2}\right| < \frac{1}{2}.$$

Note that this means the radius of convergence is related to the choice of  $a$  since the choice of  $a$  affects the coefficient of the power series.

**Theorem 5.3.1.** Let  $f$  be real-analytic on the open interval  $(a-r, a+r)$ . Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

has radius of convergence  $R \geq r$  and converges to  $f(x)$  for all  $x \in (a-r, a+r)$ .

**Proof.** Since  $f$  is real-analytic on  $(a-r, a+r)$ , by definition, for every point  $x_0 \in (a-r, a+r)$ , there exists some  $\rho > 0$  and a power series

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n$$

that converges to  $f(x)$  for all  $x$  with  $|x-x_0| < \rho$ . In particular,  $f$  is infinitely differentiable on  $(a-r, a+r)$ .

**Step 1: Cauchy's estimates for derivatives.**

Fix  $0 < r' < r$  and consider the closed interval  $[a-r', a+r'] \subset (a-r, a+r)$ . By analyticity, there exists a constant  $M > 0$  such that for all  $n \geq 0$ ,

$$\sup_{x \in [a-r', a+r']} |f^{(n)}(x)| \leq M \frac{n!}{(r')^n}.$$

This inequality is a real-analytic analogue of *Cauchy's estimates*, which can be derived, for example, by extending  $f$  to a complex-analytic function in a strip of width  $r'$  around the real interval and applying the standard Cauchy derivative formula.

**Step 2: Bound on Taylor coefficients.**

Let  $c_n = \frac{f^{(n)}(a)}{n!}$  be the Taylor coefficients at  $a$ . Then, using the above estimate,

$$|c_n| = \frac{|f^{(n)}(a)|}{n!} \leq \frac{M n! / (r')^n}{n!} = M \left( \frac{1}{r'} \right)^n.$$

**Step 3: Ratio (or root) test for radius of convergence.**

Consider the Taylor series  $\sum_{n=0}^{\infty} c_n(x-a)^n$ . By the root test,

$$\limsup_{n \rightarrow \infty} |c_n|^{1/n} \leq \limsup_{n \rightarrow \infty} \left( M \left( \frac{1}{r'} \right)^n \right)^{1/n} = \frac{1}{r'}.$$

The root test then implies that the series converges absolutely for all  $x$  such that

$$|x-a| < \frac{1}{\limsup_{n \rightarrow \infty} |c_n|^{1/n}} \geq r'.$$

Since  $r' \in (0, r)$  is arbitrary, we can take  $r' \rightarrow r$  and conclude that the radius of convergence  $R$  of the Taylor series satisfies

$$R \geq r.$$

**Step 4: Conclusion.**

Therefore, the Taylor series at  $a$  converges for all  $x \in (a-r, a+r)$  and has radius of convergence at least  $r$ . ■

**Remark 5.3.10.** By [Corollary 5.3.1](#) and [Theorem 5.3.1](#), we know if  $f$  is the function, then the power series of  $f$  centered at  $a$  converges on  $(a-r, a+r)$  iff  $f$  is real analytic on  $(a-r, a+r)$ .

## Lecture 18

### 5.4 Abel's Theorem: Boundary behaviour of power series

6 Nov. 10:20

Let  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$  have radius of convergence  $0 < R < \infty$ . Then we know the series converges absolutely for  $|x-a| < R$  and diverges for  $|x-a| > R$ . However, on the boundary  $|x-a| = R$ , convergence is delicate: the series may converge or diverge.

**Key message:** If the series does converge at a boundary point, Abel's theorem guarantees continuity there and identifies the boundary value.

**Theorem 5.4.1 (Abel's theorem).** Let  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$  be a power series with radius of convergence  $0 < R < \infty$ .

(i) If the series converges at  $x = a + R$ , then

$$\lim_{x \rightarrow a+R: x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n R^n.$$

(ii) If the series converges at  $x = a - R$ , then

$$\lim_{x \rightarrow a-R: x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n (-R)^n.$$

In particular,  $f$  is continuous at any boundary point where the series converges.

**Remark 5.4.1.** This theorem is particularly important because in general, for an arbitrary series of

functions, convergence at a boundary point does not guarantee continuity there. Abel's theorem ensures that power series behave much more regularly at the endpoints of the interval of convergence.

**Remark 5.4.2.** Abel's theorem tells us if the boundary series  $\sum_{n=0}^{\infty} c_n R^n$  or  $\sum_{n=0}^{\infty} c_n (-R)^n$  converges, then the series is continuous at the boundary points, and the limiting value is exactly substituting  $a + R$  or  $a - R$  into the series. (note the definition of continuity).

**Example 5.4.1** (Convergence at endpoint does not imply continuity).

$$f_n(x) := \begin{cases} x^n, & 0 \leq x < 1, \\ 0, & x = 1. \end{cases} \quad S(x) := \sum_{n=1}^{\infty} f_n(x).$$

**Proof.** For  $x \in [0, 1)$ , we know  $f_n(x) = x^n$  and  $\sum f_n(x) = \frac{x}{1-x}$ . At  $x = 1$ , each  $f_n(1) = 0$ , so  $S(1) = 0$ . However,

$$\lim_{x \rightarrow 1^-} S(x) = \lim_{x \rightarrow 1^-} \frac{x}{1-x} = +\infty.$$

Hence, we know  $S(x)$  converges at  $x = 1$  but is discontinuous at  $x = 1$ . \*

**Lemma 5.4.1** (Summation by parts formula). Let  $(a_n)_{n=0}^{\infty}$  and  $(b_n)_{n=0}^{\infty}$  be real sequences with limits  $A = \lim_{n \rightarrow \infty} a_n$  and  $B = \lim_{n \rightarrow \infty} b_n$ . Assume  $\sum_{n=0}^{\infty} (a_{n+1} - a_n) b_n$  converges. Then

$$\sum_{n=0}^{\infty} a_{n+1} (b_{n+1} - b_n)$$

also converges and

$$\sum_{n=0}^{\infty} (a_{n+1} - a_n) b_n = AB - a_0 b_0 - \sum_{n=0}^{\infty} a_{n+1} (b_{n+1} - b_n).$$

**Proof.** Define the partial sums  $S_N := \sum_{n=0}^N (a_{n+1} - a_n) b_n$ . Then,

$$\begin{aligned} S_N &= \sum_{n=0}^N a_{n+1} b_n - \sum_{n=0}^N a_n b_n = \sum_{n=1}^{N+1} a_n b_{n-1} - \sum_{n=0}^N a_n b_n \\ &= -a_0 b_0 + \sum_{n=1}^N a_n (b_{n-1} - b_n) + a_{N+1} b_N \\ &= -a_0 b_0 - \sum_{n=1}^N a_n (b_n - b_{n-1}) + a_{N+1} b_N \\ &= -a_0 b_0 - \sum_{n=1}^N a_n (b_n - b_{n-1}) + a_{N+1} b_N \\ &= -a_0 b_0 - \sum_{n=0}^{N-1} a_{n+1} (b_{n+1} - b_n) + a_{N+1} b_N. \end{aligned}$$

Taking  $N \rightarrow \infty$  and using  $a_{N+1} \rightarrow A$  and  $b_N \rightarrow B$  gives

$$\sum_{n=0}^{\infty} (a_{n+1} - a_n) b_n = AB - a_0 b_0 - \sum_{n=0}^{\infty} a_{n+1} (b_{n+1} - b_n),$$

■

**Remark 5.4.3** (Analogy with integration by parts). This identity above is the discrete analogue of

integration by parts:

$$\int_0^\infty f'(x)g(x) \, dx = f(x)g(x)|_0^\infty - \int_0^\infty f(x)g'(x) \, dx.$$

Summation by parts transfers difference from  $(a_n)$  to  $(b_n)$  just as integration by parts transfers derivatives from  $f$  to  $g$ .

Now we prove Abel's theorem.

**Proof of Abel's theorem.** It suffices to prove the first assertion, namely that

$$\lim_{\substack{x \rightarrow a+R^- \\ x \in (a-R, a+R)}} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n R^n,$$

whenever the series  $\sum_{n=0}^{\infty} c_n R^n$  converges. The corresponding result at  $x = a - R$  follows immediately by replacing  $c_n$  with  $(-1)^n c_n$ .

We first simplify our notations. Set  $d_n := c_n R^n$  and let  $y := \frac{x-a}{R}$ . Then  $(x-a) = Ry$ , and

$$c_n(x-a)^n = c_n R^n y^n = d_n y^n.$$

Thus, the assertion above is equivalent to showing that

$$\lim_{\substack{y \rightarrow 1^- \\ y \in (-1, 1)}} \sum_{n=0}^{\infty} d_n y^n = \sum_{n=0}^{\infty} d_n,$$

whenever the series  $\sum_{n=0}^{\infty} d_n$  converges.

Now let  $D := \sum_{n=0}^{\infty} d_n$  and define

$$S_N := \left( \sum_{n=0}^{N-1} d_n \right) - D, \quad \text{such that } S_{n+1} - S_n = d_n \text{ for } n \geq 0.$$

Choosing  $S_0$  so that  $S_1 - S_0 = d_0$  gives  $S_0 = -D$ . Then  $S_N \rightarrow 0$  as  $N \rightarrow \infty$ , and

$$d_n = S_{n+1} - S_n.$$

Hence, for any  $y \in (-1, 1)$ ,

$$\sum_{n=0}^{\infty} d_n y^n = \sum_{n=0}^{\infty} (S_{n+1} - S_n) y^n.$$

Now by [Summation by parts formula](#). If  $A = \lim_{n \rightarrow \infty} a_n$  and  $B = \lim_{n \rightarrow \infty} b_n$ , and if  $\sum_{n=0}^{\infty} (a_{n+1} - a_n) b_n$  converges, then

$$\sum_{n=0}^{\infty} (a_{n+1} - a_n) b_n = AB - a_0 b_0 - \sum_{n=0}^{\infty} a_{n+1} (b_{n+1} - b_n).$$

Applying this formula with  $a_n = S_n$  and  $b_n = y^n$ , and noting that  $\lim_{n \rightarrow \infty} y^n = 0$  for  $y \in (-1, 1)$ , we obtain

$$\sum_{n=0}^{\infty} d_n y^n = -S_0 y^0 - \sum_{n=0}^{\infty} S_{n+1} (y^{n+1} - y^n).$$

Since  $S_0 = -D$ , this becomes

$$\sum_{n=0}^{\infty} d_n y^n = D - \sum_{n=0}^{\infty} S_{n+1} (y^{n+1} - y^n).$$

Thus, to complete the proof, it remains to show that

$$\lim_{\substack{y \rightarrow 1^- \\ y \in (-1, 1)}} \sum_{n=0}^{\infty} S_{n+1} (y^{n+1} - y^n) = 0.$$

- **Step 1:** Restrict to  $y \in [0, 1)$  since the limit  $y \rightarrow 1^-$  is taken from below.
- **Step 2:** Estimate by absolute values:

$$\left| \sum_{n=0}^{\infty} S_{n+1} (y^{n+1} - y^n) \right| \leq \sum_{n=0}^{\infty} |S_{n+1}| |y^{n+1} - y^n| = \sum_{n=0}^{\infty} |S_{n+1}| (y^n - y^{n+1}).$$

- **Step 3:** Use the squeeze principle: It suffices to show that

$$\lim_{y \rightarrow 1^-} \sum_{n=0}^{\infty} |S_{n+1}| (y^n - y^{n+1}) = 0.$$

- **Step 4:** Since  $S_n \rightarrow 0$ , for any  $\varepsilon > 0$  there exists  $N$  s.t.  $|S_n| \leq \varepsilon$  for all  $n > N$ . Then

$$\sum_{n=0}^{\infty} |S_{n+1}| (y^n - y^{n+1}) \leq \sum_{n=0}^N |S_{n+1}| (y^n - y^{n+1}) + \sum_{n=N+1}^{\infty} \varepsilon (y^n - y^{n+1}).$$

- **Step 5:** The tail sum telescopes:  $\sum_{n=N+1}^{\infty} (y^n - y^{n+1}) = y^{N+1}$ . Thus

$$\sum_{n=0}^{\infty} |S_{n+1}| (y^n - y^{n+1}) \leq \sum_{n=0}^N |S_{n+1}| (y^n - y^{n+1}) + \varepsilon y^{N+1}.$$

- **Step 6:** Let  $y \rightarrow 1^-$ . The finite sum  $\sum_{n=0}^N |S_{n+1}| (y^n - y^{n+1}) \rightarrow 0$ , and  $y^{N+1} \rightarrow 1$ , which gives

$$\limsup_{y \rightarrow 1^-} \sum_{n=0}^{\infty} |S_{n+1}| (y^n - y^{n+1}) \leq \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, the limit is 0.

Hence,

$$\lim_{y \rightarrow 1^-} \sum_{n=0}^{\infty} S_{n+1} (y^{n+1} - y^n) = 0, \quad \text{and therefore} \quad \lim_{y \rightarrow 1^-} \sum_{n=0}^{\infty} d_n y^n = D.$$

Recalling that  $d_n = c_n R^n$ , this gives

$$\lim_{x \rightarrow a+R^-} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n R^n.$$

**Remark 5.4.4.**  $\limsup_{y \rightarrow 1^-}$  means taking the limit of  $y$  to  $1^-$  and then take sup, or equivalently

$$\limsup_{y \rightarrow 1^-} f(y) = \lim_{\delta \rightarrow 0^+} \sup \{f(y) \mid 0 < 1 - y < \delta\}.$$

■

**Proposition 5.4.1** (The  $p$ -series test). The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

(conventionally called a  $p$ -series) converges iff  $p > 1$ , and diverges for  $0 < p \leq 1$ .

**Proposition 5.4.2** (Alternating harmonic series).

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges (conditionally) by the Leibniz criterion.}$$

Now we talk about some examples.

**Remark 5.4.5.** Fix  $a \in \mathbb{R}$  and  $R > 0$ . For a power series written as

$$\sum_{n=0}^{\infty} c_n (x - a)^n,$$

its radius of convergence is

$$\rho = \frac{1}{\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}}.$$

When  $c_n$  contains a factor  $\frac{1}{n}$  or  $\frac{1}{n^2}$ , we interpret the series starting at  $n = 1$ ; the  $n = 0$  term is arbitrary and does not affect  $\rho$  or endpoint behaviour.

**Example 5.4.2.** Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 R^n} (x - a)^n.$$

Here  $c_n = \frac{1}{n^2 R^n}$ . Then

$$\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{R} n^{-\frac{2}{n}} = \frac{1}{R}, \quad \Rightarrow \rho = R.$$

At endpoints:

- At  $x = a + R$ :  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges.
- At  $x = a - R$ :  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  converges absolutely.

Hence, this series converges at both  $x = a \pm R$ . Hence, by Abel's theorem, we know this series is continuous at two endpoints.

**Example 5.4.3.** Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{R^n} (x - a)^n = \sum_{n=0}^{\infty} \left( \frac{x - a}{R} \right)^n \text{ (geometric).}$$

Here  $c_n = \frac{1}{R^n}$ . Then

$$\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} = \frac{1}{R}, \quad \Rightarrow \rho = R.$$

At endpoints:

- At  $x = a + R$ :  $\sum_{n=0}^{\infty} 1$  diverges.
- At  $x = a - R$ :  $\sum_{n=0}^{\infty} (-1)^n$  oscillates, hence diverges.

Hence, it diverges at both  $x = a \pm R$ .

## Lecture 19

### 5.5 Multiplication of Power series

11 Nov. 10:20

**Theorem 5.5.1 (Analysis I).** Let  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$  be a function such that the double series

$$\sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(n, m)$$

is absolutely convergent, i.e.

$$\sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} |f(n, m)|$$

converges. Then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(n, m) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(n, m).$$

In particular, if  $\sum_{n,m} a_n a_m$  absolutely convergent, then

$$\sum_m \sum_n a_n a_m = \sum_n \sum_m a_m a_n.$$

**Theorem 5.5.2.** Let  $f : (a - r, a + r) \rightarrow \mathbb{R}$  and  $g : (a - r, a + r) \rightarrow \mathbb{R}$  be functions analytic on  $(a - r, a + r)$ , with power series expansions

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n, \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} d_n (x - a)^n,$$

then  $f \cdot g : (a - r, a + r) \rightarrow \mathbb{R}$  is also analytic on  $(a - r, a + r)$ , with power series expansion

$$f(x)g(x) = \sum_{n=0}^{\infty} e_n (x - a)^n,$$

where

$$e_n = \sum_{m=0}^n c_m d_{n-m}.$$

**Proof.** We need to show that the series

$$\sum_{n=0}^{\infty} e_n (x - a)^n$$

converges to  $f(x)g(x)$  for all  $x \in (a - r, a + r)$ .

- **Step 1:** Absolute convergence of  $f$  and  $g$ . By our assumption  $f$  and  $g$  are analytic on  $(a - r, a + r)$ , so  $f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$  and  $g(x) = \sum_{n=0}^{\infty} d_n (x - a)^n$  converges absolutely when  $|x - a| < r$ . In particular, for each fixed  $x \in (a - r, a + r)$ , the sum

$$C := \sum_{n=0}^{\infty} |c_n (x - a)^n| \quad \text{and} \quad D := \sum_{n=0}^{\infty} |d_n (x - a)^n|$$

are finite.

- **Step 2:** Bounding the double series. Consider any  $N \geq 0$ , the sum

$$\begin{aligned} \sum_{n=0}^N \sum_{m=0}^{\infty} |c_m (x - a)^m d_n (x - a)^n| &= |d_0| \left( \sum_{m=0}^{\infty} |c_m (x - a)^m| \right) + \cdots + |d_N (x - a)^N| \left( \sum_{m=0}^{\infty} |c_m (x - a)^m| \right) \\ &= \left( \sum_{n=0}^N |d_n (x - a)^n| \right) \cdot \left( \sum_{m=0}^{\infty} |c_m (x - a)^m| \right) \leq D \cdot C, \end{aligned}$$

where  $D, C$  are fixed constants independent of  $N$ . This implies

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |c_m(x-a)^m d_n(x-a)^n|$$

converges absolutely. By Fubini's theorem,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m d_n (x-a)^m (x-a)^n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_m d_n (x-a)^m (x-a)^n.$$

- **Step 3:** First computations of the double series. We compute the sum in two different ways. Note that

$$\sum_{n=0}^{\infty} d_n (x-a)^n \sum_{m=0}^{\infty} c_m (x-a)^m = \left( \sum_{n=0}^{\infty} d_n (x-a)^n \right) f(x) = g(x)f(x).$$

Also, we have

$$\begin{aligned} f(x)g(x) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_m (x-a)^m d_n (x-a)^n \\ &= \sum_{n=0}^{\infty} d_n (x-a)^n \sum_{m=0}^{\infty} c_m (x-a)^m \end{aligned}$$

- **Step 4:** Second computation of the double series.

$$f(x)g(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_m d_n (x-a)^{m+n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m d_n (x-a)^{m+n}$$

by Fubini's theorem.

- **Step 5:** Change of variables. Let  $n' = m + n$ , so

$$f(x)g(x) = \sum_{m=0}^{\infty} \sum_{n'=m}^{\infty} c_m d_{n'-m} (x-a)^{n'},$$

where we adopt the convention  $d_j = 0$  if  $j < 0$ . Hence,

$$\begin{aligned} f(x)g(x) &= \sum_{m=0}^{\infty} \sum_{n'=m}^{\infty} c_m d_{n'-m} (x-a)^{n'} + \sum_{m=0}^{\infty} \sum_{n'=0}^{m-1} c_m d_{n'-m} (x-a)^{n'} \\ &= \sum_{m=0}^{\infty} \sum_{n'=0}^{\infty} c_m d_{n'-m} (x-a)^{n'} \end{aligned}$$

- **Step 6:** Interchanging the order of summation.

$$f(x)g(x) = \sum_{n'=0}^{\infty} \left( \sum_{m=0}^{\infty} c_m d_{n'-m} \right) (x-a)^{n'} = \sum_{n'=0}^{\infty} \left( \sum_{m=0}^{n'} c_m d_{n'-m} \right) (x-a)^{n'}$$

■

## 5.6 The exponential and Logarithm functions

**Definition 5.6.1.** For every real number  $x$ , we define

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

**Theorem 5.6.1.** The exponential function satisfies the following properties:

(a) For every  $x \in \mathbb{R}$ ,  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  is absolutely convergent. The radius of convergence of  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  is  $\infty$ , so  $\exp(x)$  is analytic on  $(-\infty, \infty)$ .

(b)  $\exp(x)$  is differentiable on  $\mathbb{R}$  and  $(\exp(x))' = \exp(x)$ .

(c)  $\exp$  is continuous on  $\mathbb{R}$  and for every interval  $[a, b]$ , we have

$$\int_a^b \exp(x) \, dx = \exp(b) - \exp(a).$$

(d) For  $x, y \in \mathbb{R}$ , we have  $\exp(x + y) = \exp(x) \exp(y)$ .

(e) We have  $\exp(0) = 1$ ,  $\exp(x) > 0$  for all  $x$  and  $\exp(-x) = \frac{1}{\exp(x)}$ .

(f)  $\exp(x) > \exp(y)$  if  $x > y$ .

**proof of (a).** By ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n!x}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0$$

for every  $x \in \mathbb{R}$ , so it is absolutely convergent for every  $x \in \mathbb{R}$ , and the radius of convergence is  $\infty$  by definition, so  $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all  $x \in \mathbb{R}$  and  $\exp(x)$  is analytic on  $\mathbb{R}$ . ■

**proof of (b).**

$$(\exp(x))' = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} \left( \frac{d}{dx} \frac{x^n}{n!} \right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \exp(x).$$

**proof of (c).** Since  $\exp$  is differentiable, so it is continuous. Since

$$\frac{d}{dx} \exp(x) = \exp(x),$$

so by Fundamental Theorem of Calculus, we know

$$\int_a^b \exp(x) \, dx = \exp(b) - \exp(a).$$

**proof of (d).**

$$\begin{aligned} \exp(x) \exp(y) &= \left( \sum_{m=0}^{\infty} \frac{x^m}{m!} \right) \left( \sum_{n=0}^{\infty} \frac{y^n}{n!} \right) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{1}{m!(k-m)!} x^m y^{k-m} = \exp(x + y) \end{aligned}$$

by binomial theorem. ■

**proof of (e).**  $\exp(0)$  is trivial. If  $x > 0$ , then

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} > 1,$$

and since  $\exp(x + (-x)) = \exp(0) = 1$ , and thus by (d) we know

$$\exp(x) \exp(-x) = \exp(x + (-x)) = \exp(0) = 1.$$

■

**proof of (f).** Trivial. ■

**Definition 5.6.2** (Euler number).

$$e := \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots$$

**Proposition 5.6.1.**

$$\exp(x) = e^x \quad \forall x \in \mathbb{R}.$$

**Proof.** If  $n \in \mathbb{Z}$ , then  $\exp(n) = e^n$ , which can be shown by induction. When  $n = 0$ , this is trivial. If  $n > 0$ , then

$$\exp(n) = \exp(1 + 1 + \dots + 1) = \exp(1) \exp(1) \dots \exp(1) = (\exp(1))^n = e^n,$$

so this is true. Now if  $n < 0$ , then  $\exp(-n) = e^{-n}$ , and thus

$$\exp(n) = \frac{1}{\exp(-n)} = \frac{1}{e^{-n}} = e^n.$$

Next, we show  $\exp\left(\frac{p}{q}\right) = e^{\frac{p}{q}}$  for  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$ . Note that we have

$$\left(\exp\left(\frac{p}{q}\right)\right)^q = \exp\left(\frac{p}{q} + \dots + \frac{p}{q}\right) = \exp(p) = e^p,$$

so

$$\exp\left(\frac{p}{q}\right) = (\exp(p))^{\frac{1}{q}} = (e^p)^{\frac{1}{q}} = e^{\frac{p}{q}}.$$

Hence,  $\exp(x) = e^x$  for  $x \in \mathbb{Q}$ . Now if  $x \in \mathbb{R} \setminus \mathbb{Q}$ , then there exists  $(x_n) \in \mathbb{Q}$  s.t.  $\lim_{n \rightarrow \infty} x_n = x$ . Since  $\exp(x)$  is continuous, so

$$\lim_{n \rightarrow \infty} \exp(x_n) = \exp(x),$$

and since  $x_n \in \mathbb{Q}$ , so  $\lim_{n \rightarrow \infty} e^{x_n} = \exp(x)$ . Also, since  $e^x$  is continuous, so

$$\lim_{n \rightarrow \infty} e^{x_n} = e^x,$$

so combining these two results we have  $\exp(x) = e^x$ . ■

# Appendix

# Appendix A

## Some Extra proof

### A.1 Uncategorized

**Theorem A.1.1.** For a Cauchy sequence  $\{x^{(n)}\}_{n=1}^{\infty}$ , if there exists a subsequence  $\{x^{(n_j)}\}_{j=1}^{\infty}$  converges to  $x$ , then  $\{x^{(n)}\}_{n=1}^{\infty}$  also converges to  $x$ .

**Proof.** For all  $\varepsilon > 0$ , we know there exists  $N > 0$  s.t.  $j \geq N$  implies

$$d(x^{(n_j)}, x) < \frac{\varepsilon}{2}.$$

Also, there exists  $N' > 0$  s.t.  $i, j \geq N'$  implies

$$d(x^{(i)}, x^{(j)}) < \frac{\varepsilon}{2}.$$

Hence, if we pick some  $d \geq N$  and  $n_d \geq N'$ , then we know for all  $n \geq N'$ , we have

$$d(x^{(n)}, x) \leq d(x^{(n)}, x^{(n_d)}) + d(x^{(n_d)}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means  $\{x^{(n)}\}_{n=1}^{\infty}$  converges to  $x$ . ■

**Definition A.1.1.** A sequence of intervals  $I_n$  ( $n \in \mathbb{N}$ ) is nested if  $I_n \neq \emptyset$  and  $I_{n+1} \subseteq I_n$  for all  $n \in \mathbb{N}$ . ( $I_1 \supseteq I_2 \supseteq \dots$ ).

Now we want to know  $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$ ?

Here is some counterexamples. Consider  $I_n = (0, \frac{1}{n})$ ,  $n \in \mathbb{N}$ . We can show that  $\bigcap_{n=1}^{\infty} I_n = \emptyset$  by Archimedean Property. Besides, if  $I_n = [n, \infty)$ ,  $n \in \mathbb{N}$ , this is trivial that  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ .

**Theorem A.1.2 (Theorem of nested intervals).** If  $I_n$  ( $n \in \mathbb{N}$ ) is a sequence of bounded closed nested intervals, then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**Proof.** Write  $I_n = [a_n, b_n]$  for all  $n \in \mathbb{N}$ . First, we know  $I_n$  is nested iff  $a_n \leq b_n$  and  $a_n$  is nondecreasing and  $b_n$  is nonincreasing. Hence,  $\forall n, m \in \mathbb{N}$ , we have  $a_n \leq a_{\max\{n, m\}} \leq b_{\max\{n, m\}} \leq b_m$ . In other words, for every  $m \in \mathbb{N}$ ,  $b_m$  is an upper bound of  $\{a_n\}$ . Hence, we know  $c = \lim_{n \rightarrow \infty} a_n = \sup \{a_n\}$  exists. Then,  $c \leq b_m$  for all  $m \in \mathbb{N}$ . Also,  $c \geq a_n$  for all  $n \in \mathbb{N}$ . Hence,  $a_n \leq c \leq b_n$  for all  $n \in \mathbb{N}$ , and thus we know  $c \in I_n$  for all  $n \in \mathbb{N}$ . Thus,  $c \in \bigcap_{n=1}^{\infty} I_n$ . ■

**Theorem A.1.3 (Bolzano Weierstrass Theorem).** Suppose we have a bounded infinite sequence  $a_n \in \mathbb{R}^m$ , then  $\exists$  a subsequence  $a_{n(m)}$  such that  $a_{n(m)}$  is convergent.

**Proof.** We just talk about the case  $m = 2$ , and the higher case is similar. Choose  $M > 0$  such that  $a_n \in [-M, M] \times [-M, M]$  for all  $n \in \mathbb{N}$ . Suppose  $[-M, M] \times [-M, M]$  is called  $Q$ . Divide  $Q$  into 4

squares with equal size, and choose one, say  $Q_1$  such that  $|\{n \mid a_n \in Q_1\}| = \infty$ . Select  $n_1 \in \mathbb{N}$  such that  $a_{n_1} \in Q_1$ . Repeat this step, that is, divide  $Q_1$  into 4 subparts, then says the one subpart with infinite many  $a_n$  in it is  $Q_2$  ( $Q_2$  must exists). Select  $n_2 \in \mathbb{N}$  such that  $a_{n_2} \in Q_2$  and  $n_2 > n_1$ . Keep repeating this step, then by [Theorem A.1.2](#) we know

$$\bigcap_{n=1}^{\infty} Q_n \neq \emptyset.$$

**Note A.1.1.** Just think of the nested intervals are in  $x$  and  $y$  directions.

Actually,  $\bigcap_{n=1}^{\infty} Q_n = \{a\}$  for some  $a \in \mathbb{R}^2$ , otherwise if there are two points in the intersection, then at some moment we will divide them into different subpart, which is a contradiction. It can be seen that  $\lim_{k \rightarrow \infty} a_{n(k)} = a$ . ■

**Theorem A.1.4.** If  $(X, d)$  is a metric space and  $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subseteq X$ . Now if  $\lim_{n \rightarrow \infty} d(a_n, b_n) = 0$  and  $\lim_{n \rightarrow \infty} a_n = p$  for some  $p \in X$ , then  $\lim_{n \rightarrow \infty} b_n = p$ .

**Proof.** Since we know for all  $\varepsilon > 0$ ,  $\exists N > 0$  s.t.  $n \geq N$  implies  $d(a_n, p) < \varepsilon$ , and there exists  $N_1, N_2 > 0$  s.t.  $n \geq N_1$  implies  $d(b_n, a_n) < \frac{\varepsilon}{2}$  and  $n \geq N_2$  implies  $d(a_n, p) < \frac{\varepsilon}{2}$ , so now for  $n \geq \max\{N_1, N_2\}$ , we know

$$d(b_n, p) = d(b_n, a_n) + d(a_n, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

**Theorem A.1.5.** If  $S \subseteq \mathbb{R}$ , and  $\sup S$  exists for some set  $S$ , then there exists a sequence of  $S$  converging to  $\sup S$ .

**Proof.** By the definition of  $\sup$ , we know for all  $\varepsilon > 0$ ,  $\exists s \in S$  s.t.  $\sup S \geq s > \sup S - \varepsilon$ , so pick  $\varepsilon = \frac{1}{n}$  for all  $n \in \mathbb{N}$ , we can form  $(s^{(n)})_{n=1}^{\infty}$  converging to  $\sup S$ . ■

## A.2 The uniqueness of the convergence of function

**Theorem A.2.1.** If  $(f^{(n)})_{n=1}^{\infty}$  converges pointwise to  $f$  and converges pointwise to  $g$ , then  $f = g$ .

**Proof.** Write down the definition and use triangle inequality. ■

**Theorem A.2.2.** If  $(f^{(n)})_{n=1}^{\infty}$  converges uniformly to  $f$  and converges uniformly to  $g$ , then  $f = g$ .

**Proof.** Since uniform convergence implies pointwise convergence, so  $(f^{(n)})_{n=1}^{\infty}$  converges uniformly to  $f$  and  $g$ , so  $f = g$  by [Theorem A.2.1](#). ■

**Theorem A.2.3.** If  $(f^{(n)})_{n=1}^{\infty}$  converges uniformly to  $f$  and converges pointwise to  $g$ , then  $f = g$ .

**Proof.** Since uniform convergence implies pointwise convergence, so  $(f^{(n)})_{n=1}^{\infty}$  converges pointwise to  $f$ , and by [Theorem A.2.1](#), we know  $f = g$ . ■

## Appendix B

### TA Class