1. (a) There have 14 possible answers for the first letter of the strings, and 13 possible answers for the second letter since we have already used 1 letter.
Similarly, there have 12 possible answers for the third letter, 11 possible answers for the fourth letter, ..., and 8 possible answers for the 7th letter.
So, there have $14 \times 13 \times 12 \times \cdots \times 8 = 17297280$ different strings.

(b) The word "REARRANGEMENTS" has 14 letters with 3 R's 3 E's, 2 A's, 2 N's, 1 G, 1 M, 1 T, and 1 S.
Now, we have 8 groups $R_3, E_3, A_2, N_2, G, M, T, S$.
Let's first consider which 7 letters the strings is composed of. We have 8 cases as belows

$7 = 3+3+1$ (case 1) $= 3+2+2$ (case 2) $= 3+2+1+1$ (case 3)
$= 3+1+1+1+1$ (case 4) $= 2+2+2+1$ (case 5) $= 2+2+1+1+1$ (case 6)
$= 2+1+1+1+1+1$ (case 7) $= 1+1+1+1+1+1+1$ (case 8).

Here, we write $7 = x_1 + \cdots + x_n$ to denote that choosing $x_1, \cdots, x_n$ letters from n different groups. ($n < 8 = |\{R_3, E_3, A_2, N_2, G, M, T, S\}|$)

<u>case1.</u> Obviously, the string should be composed of 3 R's, 3 E's, and one other letter.
$\rightsquigarrow$ There have $\binom{8-2}{1} = 6$ ways to form the string.
Now, to arrange 7 chosen letters, there has $6 \times \frac{7!}{3!\,3!\,1!} = 840$ distinct strings.

<u>case2.</u> We can choose 3 identical letters only from R_3 or E_3.
Also, we can choose 2 identical letters only from R_3, E_3, A_2, or N_2. (But notice that R_3 and E_3 can only be chosen once) $\rightsquigarrow \binom{2}{1} \times \binom{4-1}{2} = 6$ ways to form the string.

Consider the arrangement $\rightsquigarrow 6 \times \frac{7!}{3!2!2!} = 1260$ distinct strings

<u>case3.</u> We can choose 3 identical letters only from R_3 or E_3, choose 2 identical letters from R_3, E_3, A_2, or N_2 and choose 2 different letters from different groups.

(Without choosing from a group more than once)

$\rightsquigarrow \binom{2}{1} \times \binom{4-1}{1} \times \binom{8-2}{2} \times \frac{7!}{3!2!1!1!} = 37800$ distinct strings.

<u>case4.</u> Choose 3 identical letters from R_3 or E_3 and 4 different letters from different groups.

$\rightsquigarrow \binom{2}{1} \times \binom{8-1}{4} \times \frac{7!}{3!1!1!1!1!} = 58800$ distinct strings.

<u>case5.</u> 2 identical letters can be chosen only from R_3, E_3, A_2, or N_2. Then choose 1 letter from other groups.

$\rightsquigarrow \binom{4}{3} \times \binom{8-3}{1} \times \frac{7!}{2!2!2!1!} = 12600$ distinct strings.

<u>case6.</u> 2 identical letters can be chosen only from R_3, E_3, A_2, or N_2. Then we choose 3 different letters from different groups.

$\rightsquigarrow \binom{4}{2} \times \binom{8-2}{3} \times \frac{7!}{2!2!1!1!1!1!} = 151200$ distinct strings.

<u>case7.</u> 2 identical letters can be chosen only from R_3, E_3, A_2, or N_2. Then we choose 5 different letters from different groups.

$\rightsquigarrow \binom{4}{1} \times \binom{8-1}{5} \times \frac{7!}{2!1!1!1!1!1!1!} = 211680$ distinct strings.

<u>case8.</u> Choose 7 different letters from different groups.

$\rightsquigarrow \binom{8}{7} \times \frac{7!}{1!1!1!1!1!1!1!} = 40320$ distinct strings.

$840 + 1260 + 37800 + 58800 + 12600 + 151200 + 211680 + 40320 = 514500$.

In total, there has 514500 distinct strings.

4. As $n=0$, $(x+y)^0 = 1 = \sum_{k=0}^{0}\binom{0}{k}x^k y^{-k}$.

As $n=m$, suppose that $(x+y)^m = \sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k}$ holds.

Then, as $n=m+1$, we have

$$(x+y)^{m+1} = (x+y)\cdot(x+y)^m = (x+y)\cdot\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k}$$

$$= \sum_{k=0}^{m}\binom{m}{k}x^{k+1}y^{m-k} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k+1}$$

$$= \binom{m}{m}x^{m+1} + \sum_{k=0}^{m-1}\binom{m}{k}x^{k+1}y^{m-k} + \sum_{k=1}^{m}\binom{m}{k}x^k y^{m-k+1} + \binom{m}{0}y^{m+1}$$

Consider $\sum_{k=0}^{m-1}\binom{m}{k}x^{k+1}y^{m-k}$. Let $k=k'-1$, we have

$$\sum_{k=0}^{m-1}\binom{m}{k}x^{k+1}y^{m-k} = \sum_{k'=1}^{m}\binom{m}{k'-1}x^{k'}y^{m-k'+1}.$$

Hence

$$(x+y)^{m+1} = x^{m+1} + \sum_{k=1}^{m}\binom{m}{k-1}x^k y^{m-k+1} + \sum_{k=1}^{m}\binom{m}{k}x^k y^{m-k+1} + y^{m+1}$$

$$= \binom{m+1}{m+1}x^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k-1}+\binom{m}{k}\right]x^k y^{m-k+1} + \binom{m+1}{0}y^{m+1}$$

$$= \binom{m+1}{m+1}x^{m+1}y^{m+1-(m+1)} + \sum_{k=1}^{m}\binom{m+1}{k}x^k y^{m+1-k} + \binom{m+1}{0}x^0 y^{m+1-0}$$

$$= \sum_{k=0}^{m+1}\binom{m+1}{k}x^k y^{m+1-k}.$$

By induction, $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ holds for all $n \in \{0\}\cup\mathbb{N}$.