Math 2213 Introduction to Analysis
Homework 2 Due September 20 (Saturday), 2025
Please submit your homework online in PDF format.

(1) (11 pts) If (X, d) is a metric space, define

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that d' is also a metric on X.
Note that $0 \leq d'(x, y) < 1$ for all $x, y \in X$.

(2) (12 pts) [exercise 1.2.4] Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball

$$B := B(x_0, r) = \{x \in X : d(x, x_0) < r\},$$

and let C be the closed ball

$$C := \{x \in X : d(x, x_0) \leq r\}.$$

(a) Show that $\overline{B} \subseteq C$.
(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B} \neq C$.

(3) (21 pts) Two metrics d_1 and d_2 on a set X are said to be *Lipschitz equivalent* if there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 d_2(x, y) \leq d_1(x, y) \leq C_2 d_2(x, y) \quad \text{for all } x, y \in X.$$

Let $E \subset X$.
(a) Prove that E is open in (X, d_1) if and only if E is open in (X, d_2).
(b) Prove that E is closed in (X, d_1) if and only if E is closed in (X, d_2).
(c) Two metrics d_1 and d_2 on a set X are said to be *topologically equivalent* if they induce the same topology on X. That is, a set $U \subset X$ is open in (X, d_1) if and only if it is open in (X, d_2). Give examples of topologically equivalent metrics that are not Lipschitz equivalent.

(4) (15 pts) Let $\mathcal{M}_n = M_n(\mathbb{R})$ denote the set of all $n \times n$ real matrices. Define a function on $\mathcal{M}_n \times \mathcal{M}_n$ by

$$\rho(A, B) = \operatorname{rank}(A - B).$$

Then ρ is a metric on $\mathcal{M}_n$ and it is topologically equivalent to the discrete metric on $\mathcal{M}_n$.

(5) (20 pts) Let E be a subset of a metric space (X, d). Prove the following:
(a) The boundary of E is a closed set.
(b) $\partial E = \overline{E} \cap \overline{X \setminus E}$
(c) If E is clopen (closed and open), what is ∂E?

(d) Give an example of $S \subset \mathbb{R}$ such that $\partial(\partial S) \neq \varnothing$, and infer that "the boundary of the boundary $\partial \circ \partial$ is not always zero."

(6) (21pts) Let (X, d) be a metric space. If subsets satisfy $A \subseteq S \subseteq \overline{A}^S$, where $\overline{A}^S$ denotes the closure of A with respect to the subspace metric on S, then A is said to be *dense* in S.

Recall that the closure of A in the subspace $(S, d|_{S\times S})$ is defined by

$$\overline{A}^S := \{\, s \in S : \forall r > 0,\ B_S(s,r) \cap A \neq \varnothing \,\},$$

where

$$B_S(s,r) = B_X(s,r) \cap S$$

is the open ball in S relative to X.

Equivalently, A is dense in S if for every $s \in S$ and $r > 0$ one has

$$B_X(s,r) \cap S \cap A \neq \varnothing.$$

Examples. The set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$, and the open interval $(0,1)$ is dense in the closed interval $[0,1]$.

(a) Suppose $A \subseteq S \subseteq T$. If A is dense in S and S is dense in T, prove that A is dense in T. Equivalently,

$$\overline{A}^S = S \quad \text{and} \quad \overline{S}^T = T \implies \overline{A}^T = T,$$

where $\overline{\cdot}^Y$ denotes closure in the subspace Y.

(b) If A is dense in S and B is open in S, prove that

$$B \subseteq \overline{A \cap B}.$$

Note: B is open in S iff $B = V \cap S$ for some open $V \subseteq X$, equivalently, for every $b \in B$ there exists $r > 0$ such that

$$B_S(b,r) = B_X(b,r) \cap S \subseteq B.$$

(c) If A and B are both dense in S and B is open in S, prove that

$$A \cap B \quad \text{is dense in } S.$$