

# Introduction to Analysis I HW 1

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**Problem 0.0.1 (10pts).** Dyadic density via the Archimedean property. Let  $a < b$  be real numbers. Prove that there exists a dyadic rational

$$q = \frac{k}{2^n} \in \mathbb{Q} \quad (k \in \mathbb{Z}, n \in \mathbb{N})$$

such that  $a < q < b$ . Further show that there are infinitely many such dyadic rationals in  $(a, b)$ .

**Proof.** We first need to show a lemma first:

**Lemma 0.0.1.** For any real numbers  $a, b$  such that  $a < b$ , there exists  $n \in \mathbb{N}$  such that  $2^n a > b$ .

**Proof.** By Archimedean Property, we know there exists  $q \in \mathbb{N}$  such that  $qa > b$ , so if we pick  $n = q + 2$ , then we have

$$2^n = 2^{q+2} > q + 2 > q,$$

so we have  $2^n a > qa > b$ , and we're done. ⊗

Now using [Lemma 0.0.1](#), we can get there exists some  $n \in \mathbb{N}$  such that  $2^n(b - a) > 1$ , so if we let  $k = \lfloor 2^n a \rfloor + 1$ , then we have

$$2^n a < \lfloor 2^n a \rfloor + 1 = k \leq 2^n a + 1 < 2^n b.$$

Hence,

$$a < \frac{k}{2^n} < b$$

here. Note that  $n \in \mathbb{N}$  and  $k \in \mathbb{Z}$ , so we can pick  $q = \frac{k}{2^n}$ .

Next we'll show that there are infinitely many such dyadic rationals in  $(a, b)$ . Actually we can just repeat the step above but let  $a$  be  $q^{(0)}$  that  $q^{(0)}$  is the  $q$  we found above and then we know there exists another dyadic rationals  $q^{(1)}$  in  $(q^{(0)}, b)$ , and then doing again this step we know there exists another dyadic rationals  $q^{(2)}$  in  $(q^{(1)}, b)$ . and so on. Then, since  $q^{(i)} \neq q^{(j)}$  if  $i \neq j$ , so we know

$$a < q^{(0)} < q^{(1)} < q^{(2)} < \dots < b,$$

which means there are infinitely many such dyadic rationals in  $(a, b)$ . ■

**Problem 0.0.2 (A tour of the  $p$ -adic world.).** The field  $\mathbb{Q}$  inherits the Euclidean metric from  $\mathbb{R}$ , but it also carries a very different metric: the  $p$ -adic metric.

Given a prime number  $p$  and an integer  $n$ , the  $p$ -adic norm of  $n$  is defined as

$$|n|_p = \frac{1}{p^k},$$

where  $p^k$  is the largest power of  $p$  dividing  $n$ . (We define  $|0|_p := 0$ .) The more factors of  $p$  appear in  $n$ , the smaller the  $p$ -adic norm becomes.

For a rational number  $x = \frac{a}{b}$  with  $a, b \in \mathbb{Z}$ , we may factor  $x$  as

$$x = p^k \cdot \frac{r}{s},$$

where  $k \in \mathbb{Z}$  and  $p$  divides neither  $r$  nor  $s$ . We then define

$$|x|_p = p^{-k}.$$

The  $p$ -adic metric on  $\mathbb{Q}$  is given by

$$d_p(x, y) := |x - y|_p.$$

- (a) To compute the 5-adic norm  $|x|_5$  of a rational number  $x$ , we examine how many factors of 5 occur in  $x$  (in either numerator or denominator).

- If  $x = 5^k \cdot \frac{a}{b}$  with  $a, b$  not divisible by 5 and  $k \in \mathbb{Z}$ , then the 5-adic norm is

$$|x|_5 = 5^{-k}.$$

- **Examples.**

- (a)  $30 = 2 \cdot 3 \cdot 5$ . There is exactly one factor of 5, so

$$|30|_5 = 5^{-1} = \frac{1}{5}.$$

- (b)  $32 = 2^5$ . There is no factor of 5, so

$$|32|_5 = 5^0 = 1.$$

- (c) Compute  $|\frac{1}{250}|_5$ .

$$250 = 2 \cdot 5^3.$$

So

$$\frac{1}{250} = \frac{1}{2 \cdot 5^3} = 5^{-3} \cdot \frac{1}{2},$$

where  $\frac{1}{2}$  has no factor of 5 in numerator or denominator.  
Therefore,

$$|\frac{1}{250}|_5 = 5^{-(-3)} = 5^3 = 125.$$

Hence,

$$\boxed{|\frac{1}{250}|_5 = 125.}$$

Now practice computing the following 5-adic norms: (6 pts)

- (a)  $|75|_5$   
 (b)  $|\frac{10}{9}|_5$   
 (c)  $|\frac{20}{375}|_5$

- (b) (9 pts) Further properties of the 5-adic norm.

- (a) Determine all rational numbers  $x$  satisfying  $|x|_5 \leq 1$ .  
 (b) Which rational numbers  $x$  satisfy  $|x|_5 = 1$ ?  
 (c) What is  $\lim_{n \rightarrow \infty} 5^n$  in  $(\mathbb{Q}, d_5)$  (the 5-adic metric)?  
*Hint: Compute  $d_5(5^n, 0)$ .*

- (c) (15 pts) **Non-Archimedean absolute value and metric.** Prove that  $|\cdot|_p$  satisfies

$$|xy|_p = |x|_p |y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\},$$

and show that  $d_p$  is a metric on  $\mathbb{Q}$ .

**Proof.**

- (a)

- (a) First note that  $75 = 5^2 \cdot 3$ , so  $|75|_5 = 5^{-2} = \frac{1}{25}$ .  
 (b) First note that  $\frac{10}{9} = 5 \cdot \frac{2}{9}$ , so  $|\frac{10}{9}|_5 = 5^{-1} = \frac{1}{5}$ .  
 (c) First note that  $-\frac{4 \cdot 5}{5^3 \cdot 3} = 5^{-2} \cdot \frac{-4}{3}$ , so  $|\frac{20}{375}|_5 = 5^{-(-2)} = 25$ .

(b)

(a) Suppose  $x = 5^k \cdot \frac{r}{s}$  where  $k, r, s \in \mathbb{Z}$  and 5 divides neither  $r$  nor  $s$ , then we know  $|x|_5 = 5^{-k}$ , and we want  $5^{-k} \leq 1$ , which means  $k \geq 0$ . Hence,

$$\{\text{all rational numbers } x \text{ satisfying } |x|_5 \leq 1\} = \left\{5^k \cdot \frac{r}{s} \mid k, r, s \in \mathbb{Z} \text{ and } k \geq 0 \text{ and } 5 \nmid rs\right\}.$$

(b)

$$\{\text{all rational numbers } x \text{ satisfying } |x|_5 = 1\} = \left\{\frac{r}{s} \mid r, s \in \mathbb{Z} \text{ and } 5 \nmid rs\right\}$$

(c) First notice that  $d_5(5^n, 0) = |5^n - 0|_5 = 5^{-n}$ . Also, we know

$$0 = \lim_{n \rightarrow \infty} 5^{-n} = \lim_{n \rightarrow \infty} d_5(5^n, 0),$$

so we know  $\lim_{n \rightarrow \infty} 5^n = 0$  in  $(\mathbb{Q}, d_5)$ .

(c) First we consider the case that  $x, y$  are both not zero. Now suppose  $x = p^{k_1} \frac{r_1}{s_1}$  and  $y = p^{k_2} \frac{r_2}{s_2}$ , where  $p \nmid r_1 s_1 r_2 s_2$ . Hence,  $xy = p^{k_1+k_2} \frac{r_1 r_2}{s_1 s_2}$ , and thus

$$|xy|_p = p^{-(k_1+k_2)}.$$

Also, we know

$$|x|_p = p^{-k_1} \quad |y|_p = p^{-k_2},$$

so

$$|xy|_p = p^{-(k_1+k_2)} = p^{-k_1} p^{-k_2} = |x|_p |y|_p.$$

Now without loss of generality, suppose  $k_1 \geq k_2$ , then we know

$$x + y = p^{k_2} \left( \frac{p^{k_1-k_2} r_1 s_2 + r_2 s_1}{s_1 s_2} \right),$$

and thus

$$|x + y|_p \leq p^{-k_2} = |y|_p = \max\{|x|_p, |y|_p\}.$$

**Note.** When  $k_1 = k_2$ , it may happen that  $|x + y|_p < \max\{|x|_p, |y|_p\}$ .

And the case that  $k_2 \geq k_1$  is similar.

As for the case that either  $x$  or  $y$  is zero, we know that  $|0|_p = 0$ . We first talk about the case that  $x = 0$ , so

$$|xy|_p = |0|_p = 0 = |x|_p |y|_p$$

and

$$|x + y|_p = |y|_p = \max\{|x|_p, |y|_p\}.$$

Similarly, we know the case that  $y = 0$  is also true by repeating the steps above.

Next, we want to show that  $d_p$  is a metric on  $\mathbb{Q}$ . From now on we suppose  $x = p^{k_1} \frac{r_1}{s_1}$ ,  $y = p^{k_2} \frac{r_2}{s_2}$ , and  $z = p^{k_3} \frac{r_3}{s_3}$  for some  $x, y, z \in \mathbb{Q}$  and  $p \nmid r_i s_i$  for  $i = 1, 2, 3$ . Hence,

$$- d_p(x, x) = |0|_p = 0.$$

$$- d_p(x, y) = |x - y|_p = \frac{1}{p^z} \text{ for some } z \in \mathbb{Z}, \text{ so } d_p(x, y) > 0.$$

– Without loss of generality, suppose  $k_1 \geq k_2$ , then

$$x - y = p^{k_2} \left( \frac{p^{k_1 - k_2} r_1 s_2 - r_2 s_1}{s_1 s_2} \right)$$

and

$$y - x = -p^{k_2} \left( \frac{p^{k_1 - k_2} r_1 s_2 - r_2 s_1}{s_1 s_2} \right),$$

so we know

$$d_p(x, y) = |x - y|_p = k_2 = |y - x|_p = d_p(y, x).$$

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$$\begin{aligned} d(x, z) &= |x - z|_p = |(x - y) + (y - z)|_p \\ &\leq \max\{|x - y|_p, |y - z|_p\} \leq |x - y|_p + |y - z|_p = d(x, y) + d(y, z). \end{aligned}$$

By the above four properties of  $d_p$ , we can conclude that  $d_p$  is a metric on  $\mathbb{Q}$ .

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**Problem 0.0.3 (exercise 1.1.3 (20 pts)).** Let  $X$  be a set, and let  $d : X \times X \rightarrow [0, \infty)$  be a function.

- Give an example of a pair  $(X, d)$  which obeys axioms (bcd) of Definition 1.1.2, but not (a). (Hint: modify the discrete metric.)
- Give an example of a pair  $(X, d)$  which obeys axioms (acd) of Definition 1.1.2, but not (b).
- Give an example of a pair  $(X, d)$  which obeys axioms (abd) of Definition 1.1.2, but not (c).
- Give an example of a pair  $(X, d)$  which obeys axioms (abc) of Definition 1.1.2, but not (d). (Hint: try examples where  $X$  is a finite set.)

**Problem 0.0.4 (20 pts).** Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be vectors in  $\mathbb{R}^n$ .

- The  $\ell^1$  metric is defined by

$$d_1(x, y) := \sum_{i=1}^n |x_i - y_i|.$$

Show that  $d_1$  is a metric on  $\mathbb{R}^n$

- The  $\ell^\infty$  metric is defined by

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

Show that  $d_\infty$  is a metric on  $\mathbb{R}^n$

**Problem 0.0.5 (10 pts).** A *vector space*  $V$  over  $\mathbb{R}$  is a set equipped with two operations:

- Vector addition:**  $+: V \times V \rightarrow V$ , written  $(u, v) \mapsto u + v$ .
- Scalar multiplication:**  $\cdot: \mathbb{R} \times V \rightarrow V$ , written  $(\alpha, v) \mapsto \alpha v$ ,

such that the following properties hold for all  $u, v, w \in V$  and  $\alpha, \beta \in \mathbb{R}$ :

- $(u + v) + w = u + (v + w)$  (associativity of addition)
- $u + v = v + u$  (commutativity of addition)
- There exists  $0 \in V$  such that  $u + 0 = u$  (additive identity)

- (VS4) For each  $u \in V$ , there exists  $-u \in V$  such that  $u + (-u) = 0$  (additive inverse)
- (VS5)  $\alpha(u + v) = \alpha u + \alpha v$  (distributivity I)
- (VS6)  $(\alpha + \beta)u = \alpha u + \beta u$  (distributivity II)
- (VS7)  $(\alpha\beta)u = \alpha(\beta u)$  (compatibility of scalar multiplication)
- (VS8)  $1 \cdot u = u$  (identity element of scalar multiplication)

A function  $\|\cdot\| : V \rightarrow [0, \infty)$  is called a *norm* on  $V$  if, for all  $u, v \in V$  and  $\alpha \in \mathbb{R}$ , the following properties hold:

- (N1)  $\|v\| \geq 0$ , and  $\|v\| = 0$  if and only if  $v = 0$ . (positivity)
- (N2)  $\|\alpha v\| = |\alpha| \cdot \|v\|$ . (homogeneity)
- (N3)  $\|u + v\| \leq \|u\| + \|v\|$ . (triangle inequality)

Given a norm  $\|\cdot\|$  on  $V$ , define  $d : V \times V \rightarrow [0, \infty)$  by

$$d(u, v) = \|u - v\|.$$

Prove that  $d$  is a *metric* on  $V$ , that is, for all  $x, y, z \in V$  show that:

1.  $d(x, y) \geq 0$  and  $d(x, y) = 0$  if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$ .
3.  $d(x, z) \leq d(x, y) + d(y, z)$ .

(Thus we conclude that every normed vector space  $(V, \|\cdot\|)$  is also a metric space with metric  $d(u, v) = \|u - v\|$ .)

**Problem 0.0.6 (10 pts).** Let  $S$  be a bounded nonempty set of real numbers, and let  $a$  and  $b$  be fixed nonzero real numbers. Define  $T = \{as + b | s \in S\}$ . Find formulas for  $\sup T$  and  $\inf T$  in terms of  $\sup S$  and  $\inf S$ . Prove your formulas.

**Claim.**  $\sup T = a \sup S + b$ .

**Proof.** First notice that for all  $t \in T$ , we can write  $t = as + b$  for some  $s \in S$ . Hence,

$$t = as + b \leq a \sup S + b,$$

which means  $a \sup S + b$  is an upper bound of  $T$ . Now if  $a \sup S + b \neq \sup T$ , then there exists some  $t' \in T$  such that  $t' > a \sup S + b$ , and we can write  $t' = as' + b$  for some  $s' \in S$ , so we have

$$as' + b = t' > a \sup S + b \Leftrightarrow s' > \sup S,$$

which is a contradiction, so  $\sup T = a \sup S + b$ . ⊗

**Claim.**  $\inf T = a \inf S + b$

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**Proof.** First notice that for all  $t \in T$ , we can write  $t = as + b$  for some  $s \in S$ . Hence,

$$t = as + b \geq a \inf S + b,$$

which means  $a \inf S + b$  is a lower bound of  $T$ . Now if  $a \inf S + b \neq \inf T$ , then there exists some  $t' \in T$  such that  $t' < a \inf S + b$ , and we can write  $t' = as' + b$  for some  $s' \in S$ , so we have

$$as' + b = t' < a \inf S + b \Leftrightarrow s' < \inf S,$$

which is a contradiction, so  $\inf T = a \inf S + b$ . ⊛

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