Introduction to Analysis I HW6

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Problem 0.0.1 (20pts).

Definition 0.0.1 (Totally ordered set). A *totally ordered set* (or *linearly ordered set*) is a pair $(X, \leq)$ consisting of a nonempty set X together with a binary relation $\leq$ on X satisfying the following properties:

1. **Reflexivity:** For all $x \in X$, $x \leq x$.
2. **Antisymmetry:** For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.
3. **Transitivity:** For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
4. **Totality (or Comparability):** For all $x, y \in X$, either $x \leq y$ or $y \leq x$.

A relation $\leq$ satisfying only (1)–(3) is called a *partial order*. If, in addition, (4) holds, the order is said to be *total*, meaning that any two elements of X can be compared.

Definition 0.0.2 (Hausdorff space). A topological space $(X, \mathcal{F})$ is called a *Hausdorff space* (or T_2 space) if for every pair of distinct points $x, y \in X$ there exist neighborhoods $U, V \in \mathcal{F}$ such that

$$x \in U, \quad y \in V, \quad \text{and } U \cap V = \varnothing.$$

(a) Given any totally ordered set X with order relation $\leq$, declare a set $V \subseteq X$ to be open if for every $x \in V$ there exists a set I, which is an interval $\{y \in X : a < y < b\}$ for some $a, b \in X$, or $\{y \in X : a < y\}$ for some $a \in X$, or $\{y \in X : y < b\}$ for some $b \in X$, or the whole space X, which contains x and is contained in V. Let $\mathcal{F}$ be the set of all open subsets of X. Show that $(X, \mathcal{F})$ is a topology (this is the *order topology* on the totally ordered set $(X, \leq)$ which is Hausdorff in the sense of Definition 2.5.4-2 or the definition above).

(b) Show that on the real line $\mathbb{R}$ (with the standard ordering $\leq$), the order topology matches the standard topology (i.e., the topology arising from the standard metric).

(c) If instead one defines V to be open if the extended real line $\mathbb{R} \cup \{\pm\infty\}$ has an open set with boundary $\{\pm\infty\}$, then $(X, \mathcal{F})$ is a sequence of numbers in $\mathbb{R}$ (and hence in $\mathbb{R}$), show that $x_n \to +\infty$ if and only if $\inf_{n \geq N} x_n \to +\infty$, and $x_n \to -\infty$ if and only if $\sup_{n \geq N} x_n \to -\infty$.

(a). First note that $\varnothing, X \subseteq \mathcal{F}$, which is trivial by the definition of $\mathcal{F}$. Next, we give a claim:

Claim 0.0.1. If $V_1, V_2 \in \mathcal{F}$, then $V_1 \cap V_2 \in \mathcal{F}$.

Proof. For all $x \in V_1 \cap V_2$, there exists I_1, I_2 s.t. $x \in I_1 \subseteq V_1$ and $x \in I_2 \subseteq V_2$ and $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$ (a_1, b_1, a_2, b_2 may be $\pm\infty$ or some element in X, please see following remark).

Remark 0.0.1. To be convenient, if I_1 or I_2 is $\{y \in X : a < y < b\}$, then we use (a, b) to denote them, and if it is $\{y \in X : a < y\}$, then we use (a, ∞) to denote them, and if it is $\{y \in X : y < b\}$, then we use $(-\infty, b)$ to denote them. Also, if it is the whole X, then we use $(-\infty, \infty)$ to denote. Also, we suppose $-\infty < c$ and $\infty > c$ for all $c \in X$. This is notation may be not formal, but it is useful.

Now we can pick $I_3 = I_1 \cap I_2 = (\max\{a_1, a_2\}, \min\{b_1, b_2\})$ (min, max is similarly defined as when $\leq$ is defined in $\mathbb{R}$.) Hence, we know $x \in I_3$ and $I_3 \subseteq V_1 \cap V_2$, so $V_1 \cap V_2 \subseteq \mathcal{F}$.

Note that I_3 is well-defined since $x \in I_1 \cap I_2$, so I_3 is not empty, and it will not happen that $\min\{b_1, b_2\} \leq \max\{a_1, a_2\}$. ⊛

Now if we have $V_1, V_2, \ldots, V_n \in \mathcal{F}$, then by Claim 0.0.1, we know $V_1 \cap V_2 \cap \mathcal{F}$, and applying

Claim 0.0.1 again, then we know $V_1 \cap V_2 \cap V_3 \in \mathcal{F}$, then repeating this we have

$$\bigcap_{i=1}^{n} V_i \in \mathcal{F}.$$

Now if we have $\{V_\alpha\}_{\alpha \in A}$, then for all $x \in \bigcup_{\alpha \in A} V_\alpha$, we can pick some $\alpha_0 \in A$ s.t. $x \in V_{\alpha_0}$, and we know there exists I_{α_0} s.t. $x \in I_{\alpha_0} \subseteq V_{\alpha_0} \subseteq \bigcup_{\alpha \in A} V_\alpha$ and I_{α_0} is an interval, so we know $\bigcup_{\alpha \in A} V_\alpha \in \mathcal{F}$.

By above arguments, we know $(X, \mathcal{F})$ is a topology. ■

(b). Suppose $\mathcal{F}'$ is the order topology on $\mathbb{R}$ and $\mathcal{F}$ is the standard topology on $\mathbb{R}$, then if $V \in \mathcal{F}'$, then for all $x \in V$, we know there exists intevral I s.t. $x \in I \subseteq V$, then similarly we use the notation in Remark 0.0.1, which means $I = (a, b)$, and this time, if $I \neq X$, then we know

$$x \in B_{\mathbb{R}}(x, \min\{x - a, b - x\}),$$

where we define $\infty - x$ is still ∞ and $x - \infty$ is $-\infty$ and $-\infty - x$ is $-\infty$ and $x - (-\infty)$ is ∞, then since $I \neq X$, so we know $\min\{x - a, b - x\}$ must be some $r \in \mathbb{R}$, and thus $B_{\mathbb{R}}(x, \min\{x - a, b - x\})$ is well-defined. In this case, $V \in \mathcal{F}$. If $I = X = \mathbb{R}$, then $\mathbb{R} = I \subseteq V \subseteq \mathbb{R}$, so $V = \mathbb{R}$ and thus $V \in \mathcal{F}$. Thus, we know $\mathcal{F}' \subseteq \mathcal{F}$.

Now if $V \in \mathcal{F}$, then for all $x \in V$, we know there exists $r_x > 0$ s.t. $B_{\mathbb{R}}(x, r_x) \subseteq V$, so

$$x \in (x - r_x, x + r_x) \subseteq V,$$

and this means $V \in \mathcal{F}'$ by definition. Thus, $\mathcal{F} \subseteq \mathcal{F}'$.

Thus, we can conclude $\mathcal{F} = \mathcal{F}'$. ■

(c). We first show the $x_n \to +\infty$ if and only if $\inf_{n \geq N} x_n \to +\infty$ part:

(⇒) Now if $x_n \to +\infty$, then for all $(a, +\infty)$, there exists $N > 0$ s.t. $n \geq N$ implies $x_n \in (a, +\infty)$. Hence, we know $a < x_n$ for all $n \geq N$ and thus $a \leq \inf_{n \geq N} x_n$, so we know

$$\inf_{n \geq N} x_n \in (a - 1, +\infty),$$

which means $\inf_{n \geq N} x_n \to +\infty$.

(⇐) Now if $\inf_{n \geq N} x_n \to +\infty$, then for all $(a, +\infty)$, we know there exists $N_1 > 0$ s.t. $N \geq N_1$ implies

$$\inf_{n \geq N} x_n \in (a, +\infty),$$

so for all $n \geq N_1$, we have $x_n \in (a, +\infty)$, which means $x_n \to +\infty$.

Next, we show that $x_n \to -\infty$ if and only if $\sup_{n \geq N} \to -\infty$:

(⇒) Suppose $x_n \to -\infty$. Then for all intervals $(-\infty, a)$, there exists $N > 0$ such that $n \geq N$ implies $x_n \in (-\infty, a)$. Hence, we know $x_n < a$ for all $n \geq N$, and thus $\sup_{n \geq N} x_n \leq a$. Therefore,

$$\sup_{n \geq N} x_n \in (-\infty, a + 1),$$

which means $\sup_{n \geq N} x_n \to -\infty$.

(⇐) Now suppose $\sup_{n \geq N} x_n \to -\infty$. Then for all intervals $(-\infty, a)$, there exists $N_1 > 0$ such that $N \geq N_1$ implies

$$\sup_{n \geq N} x_n \in (-\infty, a).$$

Hence, for all $n \geq N_1$, we have $x_n \in (-\infty, a)$, which means $x_n \to -\infty$.

■

Problem 0.0.2 (15pts).

Definition 0.0.3 (Metrizable space). A topological space $(X, \mathcal{F})$ is said to be *metrizable* if there exists a metric $d : X \times X \to [0, \infty)$ such that the topology $\mathcal{F}$ coincides with the topology $\mathcal{F}_d$ induced by d. That is,

$$\mathcal{F} = \mathcal{F}_d := \{ U \subseteq X : \forall x \in U, \exists\, \varepsilon > 0 \text{ such that } B_d(x, \varepsilon) \subseteq U \},$$

where $B_d(x, \varepsilon) := \{ y \in X : d(x, y) < \varepsilon \}$ denotes the open ball centered at x with radius ε.

If no such metric d exists, then $(X, \mathcal{F})$ is said to be *not metrizable*. In other words, its topology cannot arise from any metric on X.

(a) Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or cofinite (which means that $X \setminus E$ is finite). Show that $(X, \mathcal{F})$ is a topology (this is called the *cofinite topology* on X) which is not Hausdorff and is compact.

(b) Show that if $\{V_i : i \in I\}$ is any countable collection of open sets containing x, then $\bigcap_i V_i \neq \varnothing$. Use this to show that the cofinite topology cannot be derived from any metric (i.e., $(X, \mathcal{F})$ is not metrizable). (Hint: what is the set $\bigcap_{n=1}^{\infty} B(x, 1/n)$ equal to in a metric space?)

(a).

Claim 0.0.2. $(X, \mathcal{F})$ is topology.

Proof.

- It is obvious that $\varnothing \in \mathcal{F}$ and $X = X \setminus \varnothing \in \mathcal{F}$
- If $u_1, u_2, ...u_n \in \mathcal{F}$, $\bigcap_{i=1}^{n} u_i = X \setminus \bigcup_{i=1}^{n}(X \setminus u_i)$, since for all $i \in [n]$, $(X \setminus u_i)$ is finite, $\bigcup_{i=1}^{n}(X \setminus u_i)$ is also finite, so $\bigcap_{i=1}^{n} u_i = X \setminus \bigcup_{i=1}^{n}(X \setminus u_i) \in \mathcal{F}$.
- Given $\{u_i\}_{i \in I} \in F$, $X \setminus \bigcup_{i \in I} u_i = \bigcap_{i \in I}(X \setminus u_i)$, since for all $i \in I$, $(X \setminus u_i)$ is finite, so $\bigcap_{i \in I}(X \setminus u_i)$ is also finite, and hence $\bigcup_{i \in I} u_i \in \mathcal{F}$.

⊛

Then we prove that $(X, \mathcal{F})$ is not Hausdorff and it is compact.

- not Hausdorff: Proof by contradiction.
  Suppose it is, then given $x, y \in X, \exists U, V \in \mathcal{F}$ such that $x \in U, y \in V$ and $U \cap V = \varnothing$.
  For such U and V, $U \neq \varnothing$ and $V \neq \varnothing$.
  Since $U, V \in \mathcal{F}$, $X \setminus U$ and $X \setminus V$ are finite, and hence $(X \setminus U) \cup (X \setminus V)$ is also finite. Since $U \cap V = \varnothing$, $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) = X$. However, we know $(X \setminus U) \cup (X \setminus V)$ is finite but X is uncountable so it have infinitely many elements. So we get a contradiction.
- Compact:
  Given $\{U_\alpha : \alpha \in A\} \subseteq \mathcal{F}$ with $X \subseteq \bigcup_{\alpha \in A} U_\alpha$, we need to construct a finite set $F \subseteq A$ such that $X \subseteq \bigcup_{\alpha \in F} U_\alpha$.
  We can do the following construction:
  We first find the non-empty member in $\{U_\alpha\}_A$, let it U_1 and $X \setminus U_1$ is finite set since $U_1 \in \mathcal{F}$, then $\forall x \in X \setminus U_1$, we pick one member U_x in $\{U_\alpha\}_A$ such that $x \in U_x$ (since $\{U_\alpha\}_A$ is cover so U_x must exist), and collect those U_x to get $V = \{\text{collection of those } U_x\}$, and V is finite large, then we can pick $F = \{U_1\} \cup V$ and it satisfy $X \subseteq \bigcup_{\alpha \in F} U_\alpha$, and hence we done.

■

(b).
$\{V_i : i \in I\}$ is countable collection of open sets containing x, so $x \in \bigcap_{i \in I} V_i$. And we know

$X \setminus \bigcap_{i\in I} V_i = \bigcup_{i\in I}(X \setminus V_i)$.
Since we know V_i is cofinite, so $(X \setminus V_i)$ is finite, and $\bigcup_{i\in I}(X \setminus V_i)$ is the union of countable finite set, so $\bigcup_{i\in I}(X \setminus V_i)$ is also countable. So we know that $\bigcap_{i\in I} V_i$ is the complement of countable set, so it is uncountable, this means $\{x\} \subsetneq \bigcap_{i\in I} V_i$.
If the cofinite topology are metrizable for some d, then for each x, $\{B(x, \frac{1}{n})\}_{n=1}^{\infty}$ is a countable set of open sets all containing x, and from previous proof, we know $\bigcap_{n=1}^{\infty} B(x, \frac{1}{n})$ should be uncountable. However, in metric space, $\bigcap_{n=1}^{\infty} B(x, \frac{1}{n}) = \{x\}$, so we get a contradiction, and hence the cofinite topology are not metrizable for any d. ■

Problem 0.0.3 (15pts)**.** Let $(X, \mathcal{F})$ be a compact topological space. Assume that this space is first countable, which means that for every $x \in X$ there exist countable collections of open sets $V_1, V_2, \dots$ of neighborhoods of x, such that every neighborhood of x contains one of the V_n. Show that every sequence in X has a convergent subsequence (see Exercise 1.5.11).

Proof. Suppose$\{x_n\}_{n=1}^{\infty}$ is a sequence in X, then we can define

$$C_m = \overline{\{x_n : n \geq m\}},$$

and we give some claims.

Claim 0.0.3. C_m is closed for all $m \in \mathbb{N}$.

Proof. Since

$$C_m = \overline{\{x_n : n \geq m\}} = X \setminus \operatorname{Ext}(\{x_n : n \geq m\}) = X \setminus \operatorname{Int}(X \setminus \{x_n : n \geq m\}),$$

and

$$\operatorname{Int}(X \setminus \{x_n : n \geq m\}) = \bigcup_{x \in \operatorname{Int}(X \setminus \{x_n : n \geq m\})} V_x,$$

where V_x is a neighborhood of x s.t. $V_x \subseteq X \setminus \{x_n : n \geq m\}$. Thus, $\operatorname{Int}(X \setminus \{x_n : n \geq m\})$ is open since it is the union of a collection of open sets, and thus C_m is closed since it is the complement of an open set. ⊛

Claim 0.0.4. $\bigcap_{i=1}^{\infty} C_i \neq \varnothing$.

Proof. Suppose by contradiction, $\bigcap_{i=1}^{\infty} C_i = \varnothing$, then

$$X = X \setminus \bigcap_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty}(X \setminus C_i),$$

and note that $X \setminus C_i$ is open for all $i \in \mathbb{N}$ since C_i is closed for all $i \in \mathbb{N}$ by Claim 0.0.3. Now since X is compact, and thus we know there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{N}$ s.t.

$$X \subseteq \bigcup_{i=1}^{n}(X \setminus C_{\alpha_i}) = X \setminus \left(\bigcap_{i=1}^{n} C_{\alpha_i}\right).$$

Thus, $\bigcap_{i=1}^{n} C_{\alpha_i} = \varnothing$. However, note that $C_{i+1} \subseteq C_i$ for all $i \in \mathbb{N}$, which is trivial by the definition of C_i. Hence, we know $\varnothing = \bigcap_{i=1}^{n} C_{\alpha_i} = C_{\alpha_n}$. However,

$$x_{\alpha_n} \in \{x_k : k \geq \alpha_n\} \subseteq C_{\alpha_n},$$

so C_{α_n} is non-empty, which is a contradiction. ⊛

Hence, by Claim 0.0.4, we can pick some $x' \in \bigcap_{i=1}^{\infty} C_i$. Now suppose $A = \{x_n : n \geq \mathbb{N}\}$, then we have two cases:

- Case 1: A is a finite set. Then by pigeonhole principle, we know there exists $e \in A$ s.t. e

appears in $\{x_n\}_{n=1}^{\infty}$ for infinitely many times. Hence, we can pick a subsequence of $\{x_n\}_{n=1}^{\infty}$, say $\{x_{n_i}\}_{i=1}^{\infty}$, and $x_{n_i} = e$ for all $i \in \mathbb{N}$. By this pick, we know $\{x_{n_i}\}_{i=1}^{\infty}$ converges to e, and we're done.

- Case 2: A is an infinite set. We first give a claim:

Claim 0.0.5. For any neighborhood of x', say $V_{x'}$, there are infinitely many x_n's are contained in $V_{x'}$ i.e. there exists $\{k_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ with $k_i < k_{i+1}$ for all $i \in \mathbb{N}$ s.t. $x_{k_i} \in V_{x'}$ for all $i \in \mathbb{N}$.

Proof. Suppose by contradiction, only for all $p \in \{p_i\}_{i=1}^{n}$ we have $x_p \in V_{x'}$ and we have $p_i < p_{i+1}$ for all $1 \leq i \leq n$, then we have

$$V_{x'} \cap \{x_k : k \geq p_n + 1\} = \varnothing.$$

However, $x' \in \bigcap_{i=1}^{\infty} C_i \subseteq C_{p_n+1}$, so we know $V_{x'} \cap \{x_k : k \geq p_n + 1\} \neq \varnothing$ by the definition of C_{p_n+1}. Hence, we have a contradiction, and we're done. ⊛

Now since X is first countable, so there exists a countable collection of open sets $V_1, V_2, \ldots$ of neighborhoods of x' s.t. every neighborhood of x' contains one of V_n. Suppose $U_n := \bigcap_{i=1}^{n} V_i$ for all $n \in \mathbb{N}$, then note that $\{U_n\}_{n=1}^{\infty}$ is a collection of neighborhood of x', so by Claim 0.0.5, so we know there are infinitely many terms of $\{x_n\}_{n=1}^{\infty}$ are contained in U_k for all $k \in \mathbb{N}$. Now we can construct a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ by the following method:

1. Choose $x_{n_1} \in U_1$ for some $n_1 \in \mathbb{N}$.
2. Choose some $n_2 > n_1$ s.t. $x_{n_2} \in U_2$.
3. Keep choosing $n_i > n_{i-1}$ s.t. $x_{n_i} \in U_i$ for all $i \in \mathbb{N}$ and $i > 2$.

We know this method is well-defined since for any $i \in \mathbb{N}$ and $i \geq 2$, there must exists $n_i > n_{i-1}$ s.t. $x_{n_i} \in U_i$, otherwise U_i contains only finitely many terms of $\{x_n\}_{n=1}^{\infty}$, which is a contradiction. By this method, we can show that $\{x_{n_i}\}_{i=1}^{\infty}$ converges to x'. For all neighborhood of x', say W, then by first countable property, we know there exists $i \in \mathbb{N}$ s.t. $V_i \subseteq W$, so we know $U_i \subseteq V_i \subseteq W$. Hence, for all $k \geq i$, we know

$$x_{n_k} \in U_k \subseteq U_i \subseteq V_i \subseteq W,$$

and we're done.

■

Problem 0.0.4 (15pts). Let $(X, \mathcal{F})$ be a compact topological space and $(Y, \mathcal{G})$ be a Hausdorff topological space. If $f : X \to Y$ is continuous, then f is a *closed map*; i.e., for every closed subset $F \subseteq X$, the image $f(F)$ is closed in Y.

Claim 0.0.6. For all closed $F \subseteq X$, F is compact.

Proof. Suppose $\{V_\alpha\}_{\alpha\in A}$ is an open cover of F, and since $X \setminus F$ is open, so for all $x \in X \setminus F$, there exists a neighborhood of x, U_x s.t. $U_x \subseteq X \setminus F$. Thus, we know

$$X \setminus F = \bigcup_{x\in X\setminus F} U_x,$$

and thus

$$X = F \cup (X \setminus F) = \left(\bigcup_{\alpha\in A} V_\alpha\right) \cup \left(\bigcup_{x\in X\setminus F} U_x\right),$$

so $\{V_\alpha\}_{\alpha\in A} \cup \{U_x\}_{x\in X\setminus F}$ is an open cover of X. Now since X is compact, so there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ and $x_1, x_2, \dots, x_m \in X \setminus F$ s.t.

$$X \subseteq \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cup \left(\bigcup_{i=1}^{m} U_{x_i}\right).$$

Now since $\bigcup_{i=1}^m U_{x_i} \subseteq X \setminus F$, so we know $F \subseteq \bigcup_{i=1}^n V_{\alpha_i}$, which shows F is compact. ⊛

Hence, for all $F \subseteq X$, since F is compact by Claim 0.0.6 and f is continuous, so $f(F)$ is comapct in Y. Now we show that $f(F)$ is closed in Y. Thus, we want to show $Y \setminus f(F)$ is open. Now suppose $y \in Y \setminus f(F)$, then for all $x \in f(F)$, there exists a neighborhood of x, $V_x \in \mathcal{G}$, and a neighborhood of y, $U_x \in \mathcal{G}$ s.t. $U_x \cap V_x = \varnothing$ since $(Y, \mathcal{G})$ is Hausdorff. Note that $U_x \subseteq Y \setminus V_x$ for all $x \in X$. Also, we know

$$f(F) \subseteq \bigcup_{x\in f(F)} V_x,$$

and since $f(F)$ is compact, so there exists $x_1, x_2, \dots, x_n \in f(F)$ s.t.

$$f(F) \subseteq \bigcup_{i=1}^{n} V_{x_i}.$$

Hence, we have

$$\bigcap_{i=1}^{n} (Y \setminus V_{x_i}) = Y \setminus \bigcup_{i=1}^{n} V_{x_i} \subseteq Y \setminus f(F),$$

and since $U_x \subseteq Y \setminus V_x$ for all $x \in f(F)$, so we know

$$\bigcap_{i=1}^{n} U_{x_i} \subseteq \bigcap_{i=1}^{n} (Y \setminus V_{x_i}) \subseteq Y \setminus f(F),$$

and by the definition of topology, we know $\bigcap_{i=1}^n U_{x_i} \in \mathcal{G}$ and it is a neighborhood of y, so $Y \setminus f(F)$ is open, and we're done.

■

Problem 0.0.5 (20pts). Let $\{f_n\}$ be a sequence of continuous functions real-valued defined on a compact metric space S and assume that $\{f_n\}$ converges pointwise on S to a limit function f. Prove that $f_n \to f$ uniformly on S if, and only if, the following two conditions hold:

(i) The limit function f is continuous on S.

(ii) For every $\varepsilon > 0$, there exist $m > 0$ and $\delta > 0$ such that $n > m$ and

$$|f_k(x) - f(x)| < \delta \Rightarrow |f_{k+n}(x) - f(x)| < \varepsilon$$

for all $x \in S$ and all $k = 1, 2, \dots$.

Hint. To prove the sufficiency of (i) and (ii), show that for each $x_0 \in S$ there is a neighborhood $B(x_0, R)$ and an integer k (depending on x_0) such that

$$|f_k(x) - f(x)| < \delta \quad \text{if } x \in B(x_0, R).$$

By compactness, a finite set of integers, say $A = \{k_1, \ldots, k_r\}$, has the property that for each $x \in S$, some $k \in A$ satisfies $|f_k(x) - f(x)| < \delta$. Uniform convergence is an easy consequence of this fact.

Proof.

($\Rightarrow$)
- (i). Since f_n is continuous on $x \in X$ and $f_n \to f$ uniformly, so f is continuous on $x \in X$.
- (ii). Since f is uniformly continuous,

$$\forall \varepsilon > 0, \exists N > 0 \text{ such that } \forall x \in X, \forall n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

Then $\forall \varepsilon > 0$, we can take $m = N$, $\delta = 114514$. $\forall k, k + n > k + m > m = N$.
So $\forall f_{k+n}, |f_{k+n}(x) - f(x)| < \varepsilon$ for all $x \in S$. Note that δ is actually not important here.

($\Leftarrow$) $\forall \varepsilon > 0, \exists m_\varepsilon > 0, \delta_\varepsilon > 0$ such that $n > m_\varepsilon$,

$$|f_k(x) - f(x)| < \delta_\varepsilon \Rightarrow |f_{k+n}(x) - f(x)| < \varepsilon, \forall x \in S, k = 1, 2, 3, \ldots$$

$\forall x_0 \in S$, since $f_n(x_0) \to f(x_0)$ pointwise, $\exists k_{x_0}$ such that $|f_{k_{x_0}}(x_0) - f(x_0)| < \delta_\varepsilon$. Since $f_{k_{x_0}}$ and f are both continuous, so $f_{k_{x_0}} - f$ is also continuous on x_0, and hence $\exists R_{x_0} > 0$ such that if $B(x_0, x) < R_{x_0}$, then $|f_{k_{x_0}}(x) - f(x)| < \delta_\varepsilon$.
$\forall x_0$, we can get such ball, then we collect those balls to get $I = \{B(x_0, R_{x_0}) : x_0 \in S\}$, by compactness of S, exist a finite subset $F = \{x_1, x_2, \ldots, x_r\}$ such that $\{B(x_i, R_{x_i})\}_{i=1}^r \subseteq I$ and $S \subseteq \bigcup_{i=1}^r B(x_i, R_{x_i})$.
For each x_i, $\exists k_{x_i}$ such that $|f_{k_{x_i}}(x) - f(x)| < \delta_\varepsilon$. Let $A = \{k_{x_1}, k_{x_2}, \ldots, k_{x_r}\}$, and define $N = \max A + m_\varepsilon + 1$.

Claim 0.0.7. $\forall t \geq N$, $|f_t(x) - f(x)| < \varepsilon$ $\forall x \in S$.

Proof. $\forall x \in S$, $x \in B(x_j, R_{x_j})$ for some $j \in [r]$.
So $\forall x \in S, |f_{k_{x_j}}(x) - f(x)| < \delta_\varepsilon$ for some j, by (ii) we know $\forall n > m_\varepsilon, |f_{k_{x_j}+n}(x) - f(x)| < \varepsilon$.
Since

$$t \geq N = \max A + m_\varepsilon + 1 > k_{x_j} + m_\varepsilon, \forall j \in [r], t - k_{x_j} > m_\varepsilon$$

, and hence $|f_t(x) - f(x)| < \varepsilon, \forall t \geq N$. ⊛

Since $\forall t \geq N$, $|f_t(x) - f(x)| < \varepsilon$ $\forall x \in S$, so f_n is uniformly converge to f on S by definition.

■

Problem 0.0.6 (15pts). The purpose of this exercise is to demonstrate a concrete relationship between continuity and pointwise convergence, and between uniform continuity and uniform convergence.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For any $a \in \mathbb{R}$, let $f_a : \mathbb{R} \to \mathbb{R}$ be the shifted function defined by

$$f_a(x) := f(x - a).$$

(a) Show that f is continuous if and only if, whenever $(a_n)_{n=0}^\infty$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge pointwise to f.

(b) Show that f is uniformly continuous if and only if, whenever $(a_n)_{n=0}^\infty$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge uniformly to f.

proof of (a).

($\Rightarrow$) If f is continuous and suppose $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to 0, then given any $x \in \mathbb{R}$ and $\varepsilon > 0$, we know there exists $\delta > 0$ s.t. $|a_n| = |(x - a_n) - x| < \delta$ implies

$$|f(x - a_n) - f(x)| < \varepsilon,$$

and since $(a_n)_{n=0}^{\infty}$ converges to 0, so there exists $N > 0$ s.t. $n \geq N$ implies $|a_n| < \delta$. Thus, for all $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f_{a_n}(x) - f(x)| = |f(x - a_n) - f(x)| < \varepsilon,$$

which means f_{a_n} converge pointwise to f.

($\Leftarrow$) Now if we have a sequence in $\mathbb{R}$, $\{b_n\}_{n=0}^{\infty}$, converges to $b \in \mathbb{R}$, then we know $\{c_n = b - b_n\}_{n=0}^{\infty}$ is a sequence converges to 0, so f_{c_n} converge pointwise to f. This means for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f(x - b + b_n) - f(x)| = |f(x - c_n) - f(x)| = |f_{c_n}(x) - f(x)| < \varepsilon,$$

so if we pick $x = b \in \mathbb{R}$, we know for all $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f(b_n) - f(b)| < \varepsilon,$$

which means $\lim_{n\to\infty} f(b_n) = f(b)$, so f is continuous.

■

proof of (b).

($\Rightarrow$) If f is uniformly continuous and $(a_n)_{n=0}^{\infty} \to 0$, then for all $\varepsilon > 0$, we know there exists $\delta > 0$ s.t. if $|a_n| = |(x - a_n) - x| < \delta$, then $|f(x - a_n) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$, and since $(a_n)_{n=0}^{\infty}$ converges to 0, so there exists $N > 0$ s.t. $n \geq N$ implies $|a_n| < \delta$. Thus, for all $\varepsilon > 0$, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f_{a_n}(x) - f(x)| = |f(x - a_n) - f(x)| < \varepsilon \text{ for all } x \in \mathbb{R},$$

so f_{a_n} converges uniformly to f.

($\Leftarrow$) We want to show that for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Now suppose by contradiction, there exists $\varepsilon_1 > 0$ s.t. for all $\delta > 0$, there exists $x, y \in \mathbb{R}$ s.t. $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_1$. Then for all $\delta = \frac{1}{k}$ with $k \in \mathbb{N}$, we can pick x_k, y_k s.t. $|x_k - y_k| < \frac{1}{k}$ and $|f(x_k) - f(y_k)| \geq \varepsilon_1$. Hence, we know $\{c_n = y_n - x_n\}_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ which converges to 0. Thus, f_{c_n} converges uniformly to f. Thus, there exists $N > 0$ s.t. $n \geq N$ implies

$$|f(x - y_n + x_n) - f(x)| = |f(x - c_n) - f(x)| = |f_{c_n}(x) - f(x)| < \varepsilon_1$$

for all $x \in \mathbb{R}$, so if we pick $x = y_N$, then for $n = N$ we know

$$|f(x_N) - f(y_N)| < \varepsilon_1$$

but this is impossible, so this is a contradiction. Hence, f is uniformly continuous.

■