Linear Algebra I HW12

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Section 7.1

Problem. 1. Let T be a linear operator on F^2. Prove that any non-zero vector which is not a characteristic vector for T is a cyclic vector for T. Hence, prove that either T has a cyclic vector or T is a scalar multiple of the identity operator.

Proof. Suppose $v \in F^2$ and $v \neq 0$ and $Tv \notin \operatorname{span}\{v\}$, then $\{v, Tv\}$ is linearly independent and thus a basis of F^2, which means v is a cyclic vector for T. Now if such v does not exist, then $Tv \in \operatorname{span}\{v\}$ for all $v \in F^2$. Then if $\{p, q\}$ form a basis of F^2, then $Tp = \lambda p$ and $Tq = \lambda' q$ for some $\lambda, \lambda' \in F$, and we claim that $\lambda = \lambda'$. If $\lambda \neq \lambda'$, then

$$T(p+q) = T(p) + T(q) = \lambda p + \lambda' q \notin \operatorname{span}\{p+q\},$$

which is impossible. Hence, $\lambda = \lambda'$, and thus for all $v \in V$, we know $v = \alpha p + \beta q$ for some $\alpha, \beta \in F$, and thus

$$Tv = \alpha Tp + \beta Tq = \alpha\lambda p + \beta\lambda q = \lambda(\alpha p + \beta q) = \lambda v,$$

which shows T is a scalar multiple of the identity operator. ■

Problem. 7. Let V be an n-dimensional vector space, and let T be a linear operator on V. Suppose that T is diagonalizable.

(a) If T has a cyclic vector, show that T has n distinct characteristic values.

(b) If T has n distinct characteristic values, and if $\{\alpha_1, \dots, \alpha_n\}$ is a basis of characteristic vectors for T, show that $\alpha = \alpha_1 + \cdots + \alpha_n$ is a cyclic vector for T.

Proof.

(a) If T has a cyclic vector, then $m_T(x) = \operatorname{ch}_T(x)$, and thus

$$\deg m_T(x) = \deg \operatorname{ch}_T(x) = n,$$

and since T is diagonalizable, so we know $m_T(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_n)$ for some $\lambda_i \in F$. Thus,

$$\operatorname{ch}_T(x) = \mathrm{m}_T(x) = (x - \lambda_1) \dots (x - \lambda_n),$$

which means T has n distinct characteristic values.

(b) Suppose $T\alpha_i = \lambda_i \alpha_i$ for all $i = 1, 2, \dots, n$. Then we know

$$T^i \alpha = T^i \left(\sum_{j=1}^{n} \alpha_j \right) = \sum_{j=1}^{n} T^i(\alpha_j) = \sum_{j=1}^{n} \lambda_j^i \alpha_j.$$

Now we want to show $\{\alpha, T\alpha, \dots, T^{n-1}\alpha\}$ is a basis of V so that we know α is a cyclic vector for T. Note that

$$\begin{aligned}
\alpha &= \alpha_1 + \alpha_2 + \cdots + \alpha_n \\
T\alpha &= \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \cdots + \lambda_n \alpha_n \\
T^2 \alpha &= \lambda_1^2 \alpha_1 + \lambda_2^2 \lambda_2 + \cdots + \lambda_n \alpha_n \\
&\vdots \\
T^{n-1}\alpha &= \lambda_1^{n-1} \alpha_1 + \lambda_2^{n-1} \alpha_2 + \cdots + \lambda_n^{n-1} \alpha_n,
\end{aligned}$$

so we have

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}}_{A \in M_n(F)} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}_{X \in M_n(F)} = \underbrace{\begin{pmatrix} \alpha \\ T\alpha \\ \vdots \\ T^{n-1}\alpha \end{pmatrix}}_{Y \in M_n(F)},$$

and note that $\det(A) \neq 0$ since $\lambda_i \neq \lambda_j$ for all $i \neq j$ and A is the Vandermonde matrix. Hence, A is invertible, and thus

$$\operatorname{rank} Y = \operatorname{rank} AX = \operatorname{rank} X = n$$

since the n rows of X are linearly independent, and thus the n rows of Y are also linearly independent, which shows $\{\alpha, T\alpha, \dots, T^{n-1}\alpha\}$ is a basis of V, and we're done.

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Section 7.3

Problem. 6. Let A be the complex matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Find the Jordan form for A.

Proof. Note that $\operatorname{ch}_A(x) = (x-2)^5(x+1)$, and since

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix},$$

so we know $\operatorname{rank}(A-2I) = 4$, and thus $\dim\ker(A-2I) = 2$. Also, we have

$$(A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -3 & 9 \end{pmatrix},$$

so $\operatorname{rank}(A-2I)^2 = 3$, and thus $\dim\ker(A-2I)^2 = 3$. Now we know there are two Jordan blocks with characteristic value 2 and there is one Jordan block with characteristic value 2 and of size larger than 2, and since the sum of the size of these two Jordan blocks with characteristic value 2 is 5, so we know the sizes of these two Jordan blocks are 1 and 4, and since the sum of size of the

Jordan block with characteristic value 1 is 1, so we know the Jordan form of A is

$$\left(\begin{array}{cccc|c|c} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

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Problem. 10. Let n be a positive integer, $n \geq 2$, and let N be an $n \times n$ matrix over the field F such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that N has no square root, i.e., that there is no $n \times n$ matrix A such that $A^2 = N$.

Proof. If $A^2 = N$, then $A^{2n} = N^n = 0$ and $A^{2n-2} = N^{n-1} \neq 0$, so

$$m_A(x) \mid x^{2n} \text{ but } m_A(x) \nmid x^{2n-2},$$

so $m_A(x) \in \left\{x^{2n-1}, x^{2n}\right\}$. Hence, $\deg m_A(x) \geq 2n-1$, but $m_A(x) \mid \operatorname{ch}_A(x)$, so

$$n = \deg \operatorname{ch}_A(x) \geq \deg m_A(x) \geq 2n-1,$$

but this gives $1 \geq n$, so this is impossible. ■

Problem. 13. If N is a $k \times k$ elementary nilpotent matrix, i.e., $N^k = 0$ but $N^{k-1} \neq 0$, show that N^t is similar to N. Now use the Jordan form to prove that every complex $n \times n$ matrix is similar to its transpose.

Proof. Note that

$$\left(N^k\right)^t = \left(N^t\right)^k \quad \forall k \geq 0.$$

Hence, $\left(N^t\right)^k = 0$ and $\left(N^t\right)^{k-1} \neq 0$. Note that since $N^k = 0$ and $N^{k-1} \neq 0$, so $m_N(x) = x^k$, and thus if J_N is the Jordan form of N, then J_N's largest Jordan block has size k, which means J_N has exactly one Jordan block. Also, we have similar argument on N^t, so N and N^t has same Jordan form, and thus

$$N \sim J_N \sim N^t.$$

Now for every complex $n \times n$ matrix A, since $\mathbb{C}$ is algebraically closed, so the Jordan form of A exists, say it is J_A, then we know

$$J_A = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{r_i} J_{s_j}(\lambda_i),$$

where k is the number of characteristic value of A, and r_i is the number of Jordan block with characteristic value λ_i, and s_j is the size of the Jordan block. Note that for all i, j, $J_{s_j}(\lambda_i) - \lambda_i I$ is a $s_j \times s_j$ elementary nilpotent matrix, so

$$J_{s_j}(\lambda_i) - \lambda_i I = Q^{-1}(J_{s_j}(\lambda_i) - \lambda_i I)^t Q = Q^{-1} J_{s_j}(\lambda_i)^t Q - \lambda_i Q^{-1} I Q = Q^{-1} J_{s_j}(\lambda_i)^t Q - \lambda_i I,$$

for some Q and thus

$$J_{s_j}(\lambda_i)^t = Q^{-1} J_{s_j}(\lambda_i) Q,$$

which shows

$$J_{s_j}(\lambda_i) \sim J_{s_j}(\lambda_i)^t.$$

Hence, we know

$$J_A = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{r_i} J_{s_j}(\lambda_i) \sim \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{r_i} J_{s_j}(\lambda_i)^t = J_A^t,$$

so

$$J_A^t = R^{-1} J_A R$$

for some R. Now if $A = P^{-1} J_A P$ for some P, then

$$A^t = P^t J_A^t \left(P^{-1}\right)^t = P^t R^{-1} J_A R \left(P^{-1}\right)^t,$$

which shows

$$A^t \sim J_A \sim A,$$

and we're done. ■