Introduction to Analysis I HW12

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Problem 0.0.1 (Exercise 5.4.1). Show that if $f : \mathbb{R} \to \mathbb{C}$ is both compactly supported and $\mathbb{Z}$–periodic, then it is identically zero.

Hint: A function $f : \mathbb{R} \to \mathbb{C}$ is said to be *compactly supported* if the set

$$\operatorname{supp}(f) := \overline{\{\, x \in \mathbb{R} : f(x) \neq 0 \,\}}$$

is a compact subset of $\mathbb{R}$. Equivalently, f is compactly supported if there exists a bounded closed interval $[a, b] \subset \mathbb{R}$ such that

$$f(x) = 0 \qquad \text{whenever } x \notin [a, b].$$

Problem 0.0.2 (Exercise 5.5.1). Let f be a function in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, and define the *trigonometric Fourier coefficients* a_n, b_n for $n = 0, 1, 2, \ldots$ by

$$a_n := 2\int_0^1 f(x)\cos(2\pi nx)\,dx, \qquad b_n := 2\int_0^1 f(x)\sin(2\pi nx)\,dx.$$

(a) Show that the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(2\pi nx) + b_n\sin(2\pi nx)\big)$$

converges to f in the L^2-metric.

(b) Show that if $\sum_{n=1}^{\infty}|a_n|$ and $\sum_{n=1}^{\infty}|b_n|$ are absolutely convergent, then the above series actually converges *uniformly* to f (and not just in L^2).

(a). Note that

$$\begin{aligned}
\sum_{n=-N}^{N}\hat{f}(n)e_n = \sum_{n=-N}^{N}\langle f, e_n\rangle e_n &= \sum_{n=-N}^{N}\left(\int_0^1 f(x)e^{-2\pi inx}\,\mathrm{d}x\right)e^{2\pi inx} \\
&= \sum_{n=-N}^{N}\left(\int_0^1 f(x)\left(\cos(-2\pi nx) + i\sin(-2\pi nx)\right)\mathrm{d}x\right)\left(\cos(2\pi nx) + i\sin(2\pi nx)\right) \\
&= \sum_{n=-N}^{N}\left(\int_0^1 f(x)\left(\cos(2\pi nx) - i\sin(2\pi nx)\right)\mathrm{d}x\right)\left(\cos(2\pi nx) + i\sin(2\pi nx)\right) \\
&= \sum_{n=-N}^{N}\left(\int_0^1 f(x)\cos(2\pi nx)\,\mathrm{d}x\right)\cos(2\pi nx) + \left(\int_0^1 f(x)\sin(2\pi nx)\,\mathrm{d}x\right)\sin(2\pi nx) \\
&+ i\left(\left(\int_0^1 f(x)\cos(2\pi nx)\,\mathrm{d}x\right)\sin(2\pi nx) - \left(\int_0^1 f(x)\sin(2\pi nx)\,\mathrm{d}x\right)\cos(2\pi nx)\right) \\
&= \sum_{n=-N}^{N}\left(\frac{1}{2}a_n\cos(2\pi nx) + \frac{1}{2}b_n\sin(2\pi nx)\right) + i\left(\frac{1}{2}a_n\sin(2\pi nx) - \frac{1}{2}b_n\cos(2\pi nx)\right).
\end{aligned}$$

Since

$$\begin{aligned}
a_n\cos(2\pi nx) &= a_{-n}\cos(2\pi(-n)x) \\
b_n\sin(2\pi nx) &= b_{-n}\sin(2\pi(-n)x) \\
a_n\sin(2\pi nx) &= -a_{-n}\sin(2\pi(-n)x) \\
b_n\cos(2\pi nx) &= -b_{-n}\cos(2\pi(-n)x),
\end{aligned}$$

we know

$$\sum_{n=-N}^{N}\left(\frac{1}{2}a_n\cos(2\pi nx)+\frac{1}{2}b_n\sin(2\pi nx)\right)+i\left(\frac{1}{2}a_n\sin(2\pi nx)-\frac{1}{2}b_n\cos(2\pi nx)\right)$$
$$=\left(\frac{1}{2}a_0\cos 0+\frac{1}{2}b_0\sin 0\right)+2\sum_{n=1}^{N}\left(\frac{1}{2}a_n\cos(2\pi nx)+\frac{1}{2}b_n\sin(2\pi nx)\right)+i\left(\frac{1}{2}a_0\sin 0-\frac{1}{2}b_0\cos 0\right)$$
$$=\frac{1}{2}a_0+\sum_{n=1}^{N}a_n\cos(2\pi nx)+b_n\sin(2\pi nx).$$

Thus,

$$\sum_{n=-N}^{N}\hat{f}(n)e_n=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(2\pi nx)+b_n\sin(2\pi nx)\right),$$

and we have shown that

$$f\to\sum_{n=-N}^{N}\hat{f}(n)e_n \text{ in the } L^2\text{-metric}$$

in class, so we're done. ■

(b). Note that

$$\sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|=\sum_{n=-\infty}^{\infty}|\langle f,e_n\rangle|=\sum_{n=-\infty}^{\infty}\left|\int_0^1 f(x)e^{-2\pi inx}\,\mathrm{d}x\right|$$
$$=\sum_{n=-\infty}^{\infty}\left|\int_0^1 f(x)\left(\cos(-2\pi nx)+i\sin(-2\pi nx)\right)\,\mathrm{d}x\right|$$
$$=\sum_{n=-\infty}^{\infty}\left|\int_0^1 f(x)\cos(2\pi nx)\,\mathrm{d}x-i\int_0^1 f(x)\sin(2\pi nx)\,\mathrm{d}x\right|$$
$$=\sum_{n=-\infty}^{\infty}\left|\frac{1}{2}a_n-\frac{1}{2}ib_n\right|\leq\sum_{n=-\infty}^{\infty}\frac{1}{2}|a_n|-\frac{1}{2}|b_n|=\sum_{n=1}^{\infty}(|a_n|-|b_n|)+\left(\frac{1}{2}|a_0|-\frac{1}{2}|b_0|\right),$$

which converges since $\sum_{n=1}^{\infty}|a_n|$ and $\sum_{n=1}^{\infty}|b_n|$ converges absolutely and $\left(\frac{1}{2}|a_0|-\frac{1}{2}|b_0|\right)$ is a finite term. Now we know

$$\sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|<\infty,$$

so in class we have shown that

$$f\to\sum_{n=-N}^{N}\hat{f}(n)e_n=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(2\pi nx)+b_n\sin(2\pi nx)\right) \text{ uniformly.}$$

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Problem 0.0.3 (Exercise 5.5.2). Let $f(x)$ be the function defined by $f(x)=(1-2x)^2$ when $x\in[0,1]$, and extended to be $\mathbf{Z}$-periodic on $\mathbf{R}$.

(a) Using Exercise 5.5.1, show that the series

$$\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{\pi^2n^2}\cos(2\pi nx)$$

converges uniformly to f. (You may use the fact that

$$\int_0^1 x\, e^{-2\pi i n x}\, dx = -\frac{1}{2\pi i n}, \qquad (n \neq 0),$$

$$\int_0^1 x^2\, e^{-2\pi i n x}\, dx = -\frac{1}{2\pi i n} + \frac{2}{(2\pi n)^2}, \qquad (n \neq 0).$$

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(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(c) Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(Hint: expand the cosines in terms of exponentials and use Plancherel's theorem.)

(a). Let $F_N = \sum_{n=-N}^{N} \hat{f}(n) e_n$. If $n \neq 0$, then

$$\begin{aligned}
\hat{f}(n) &= \int_0^1 f(x) e^{-2\pi i n x}\, \mathrm{d}x = \int_0^1 \left(1 - 4x + 4x^2\right) e^{-2\pi i n x}\, \mathrm{d}x \\
&= \int_0^1 e^{-2\pi i n x}\, \mathrm{d}x - 4\int_0^1 x e^{-2\pi i n x}\, \mathrm{d}x + 4\int_0^1 x^2 e^{-2\pi i n x}\, \mathrm{d}x \\
&= \frac{1}{-2\pi i n} e^{-2\pi i n x}\Big]_0^1 - 4\left(-\frac{1}{2\pi i n}\right) + 4\left(-\frac{1}{2\pi i n} + \frac{2}{(2\pi n)^2}\right) \\
&= \frac{e^{-2\pi i n}}{-2\pi i n} - \frac{1}{-2\pi i n} + \frac{8}{(2\pi n)^2} = \frac{1}{-2\pi i n} - \frac{1}{-2\pi i n} + \frac{8}{(2\pi n)^2} = \frac{2}{\pi^2 n^2}.
\end{aligned}$$

Besides,

$$\hat{f}(0) = \int_0^1 1 - 4x + 4x^2\, \mathrm{d}x = x - 2x^2 + \frac{4}{3}x^3\Big]_0^1 = \frac{1}{3}.$$

Thus,

$$\begin{aligned}
F_N &= \hat{f}(0) e_0 + \sum_{n=-N}^{-1} \hat{f}(n) e_n + \sum_{n=1}^{N} \hat{f}(n) e_n \\
&= \frac{1}{3} + \sum_{n=-N}^{-1} \frac{2}{\pi^2 n^2} \left(\cos(2\pi n x) + i \sin(2\pi n x)\right) + \sum_{n=1}^{N} \frac{2}{\pi^2 n^2} \left(\cos(2\pi n x) + i \sin(2\pi n x)\right) \\
&= \frac{1}{3} + 2\sum_{n=1}^{N} \frac{2}{\pi^2 n^2} \cos(2\pi n x) = \frac{1}{3} + \sum_{n=1}^{N} \frac{4}{\pi^2 n^2} \cos(2\pi n x).
\end{aligned}$$

Also, since

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right| = \lim_{N \to \infty} \frac{1}{3} + \sum_{n=-N}^{N} \left|\frac{2}{\pi^2 n^2}\right| = \lim_{N \to \infty} \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2} < \infty$$

by p-series test, so we know

$$F_N = \frac{1}{3} + \sum_{n=1}^{N} \frac{4}{\pi^2 n^2} \cos(2\pi n x) \to f \text{ uniformly.}$$

■

(b). By (a), we know

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x) = (1-2x)^2$$

uniformly, so we can plug 0 into it and we have

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} = 1 \Rightarrow \frac{2}{3} = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

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(c). By Plancherel's theorem, we know

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2.$$

Note that

$$\|f\|_2^2 = \int_0^1 |f(x)|^2 \,\mathrm{d}x = \int_0^1 (1-2x)^4 \,\mathrm{d}x = \frac{1}{5}.$$

Thus,

$$\frac{1}{5} = \sum_{n=-\infty}^{-1} \frac{4}{\pi^4 n^4} + \sum_{n=1}^{\infty} \frac{4}{\pi^4 n^4} + \frac{1}{9} \Rightarrow \frac{4}{45} = 2\sum_{n=1}^{\infty} \frac{4}{\pi^4 n^4} \Rightarrow \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

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Problem 0.0.4 (Exercise 5.5.3). If $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ and P is a trigonometric polynomial, show that

$$\widehat{f * P}(n) = \widehat{f}(n)\, c_n = \widehat{f}(n)\, \widehat{P}(n)$$

for all integers n, where c_n are the Fourier coefficients of P. More generally, if $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, show that

$$\widehat{f * g}(n) = \widehat{f}(n)\, \widehat{g}(n) \quad \text{for all } n \in \mathbf{Z}.$$

Problem 0.0.5 (Exercise 5.5.4). Let $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ be differentiable, and assume its derivative f' is also continuous. Show that

$$\sum_{n=-\infty}^{\infty} |n\, \widehat{f}(n)|^2 < \infty$$

and that the Fourier coefficients of f' satisfy

$$\widehat{f'}(n) = 2\pi i n\, \widehat{f}(n) \qquad \text{for all } n \in \mathbf{Z}.$$