Abstract algebra I

Homework 3

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1. For the following pairs of groups, determine whether they are isomorphic. If so, construct an isomorphism, otherwise, explain why they are not.

   (a) $(\mathbb{Z}/15\mathbb{Z})^\times$ and $\mathbb{Z}/8\mathbb{Z}$.

   (b) $\mathbb{Z}/4\mathbb{Z}$ and $\{z \in \mathbb{C} \setminus \{0\} : z^4 = 1\}$

   (c) $\mathbb{Z}$ and $3\mathbb{Z} = \{3n : n \in \mathbb{Z}\}$

   (d) $\mathbb{Z}$ and $\{z \in \mathbb{C} \setminus \{0\} : z^n = 1 \text{ for some } n \geq 1\}$. (The second group is the group of all roots of unity. You may want to verify that this collection is indeed a subgroup of $\mathbb{C} \setminus \{0\}$.)

   (e) S_3 and $D_3 = \langle r, s : r^3 = s^2 = 1, srs = r^{-1} \rangle$. (The second group is described as follows: it is generated by two elements r, s, and they satisfy the given relations. You may easily check that there are a total of six distinct elements. Such a description is called a group presentation.)

   (f) S_4 and $D_4 = \langle r, s : r^4 = s^2 = 1, srs = r^{-1} \rangle$

   (g) $Q = \langle i, j, k : i^2 = j^2 = k^2 = ijk = -1 \rangle$ and $T = \langle a, b : a^6 = 1, b^2 = a^3, ba = a^5 b \rangle$.

   (h) An infinite cyclic group G and one of its non-trivial proper subgroup H, i.e., $H \neq \{e\}$.

Solution: We first give a claim, which will be used in some of the later problems.
Claim: If G, H are groups and they are isomorphic, then

$$\{p : p \text{ is the order of some } g \in G\} = \{q : q \text{ is the order of some } h \in H\}.$$

proof. Suppose $\phi : G \to H$ is an isomorphism, then for every $g \in G$, if $o(g) = m$, then $o(\phi(g)) = m$ since $(\phi(g))^m = \phi(g^m) = \phi(e_G) = e_H$. Also, if there exists some $n < m$ s.t. $(\phi(g))^n = e_H$, then $e_H = (\phi(g))^n = \phi(g^n)$, but since ϕ is injective, so $g^n = e_G$, which is a contradiction since $n < m = o(g)$. Now since we know for all $g \in G$, $o(g) = o(\phi(g))$, and since ϕ is bijective, so our claim is true.

(a) By the claim, since in $\mathbb{Z}/8\mathbb{Z}$, we know $o(1) = 8$, but $(\mathbb{Z}/15\mathbb{Z})^\times$ is not cyclic, and $\left|(\mathbb{Z}/15\mathbb{Z})^\times\right| = 8$, so $\mathbb{Z}/8\mathbb{Z}$ and $(\mathbb{Z}/15\mathbb{Z})^\times$ are not isomorphic.

(b) Note that

$$\left\{z \in \mathbb{C} \setminus \{0\} : z^4 = 1\right\} = \left\{e^{i \cdot \frac{2\pi k}{4}} : k = 0, 1, 2, 3\right\}.$$

Hence, we can define $\phi : \mathbb{Z}/4\mathbb{Z} \to \{z \in \mathbb{C} \setminus \{0\} : z^4 = 1\}$ as

$$\phi\left(\bar{i}\right) = e^{i \cdot \frac{2\pi i}{4}}.$$

Note that this is an isomorphism since it is bijective and is a homomorphism, so $\mathbb{Z}/4\mathbb{Z}$ and $\{z \in \mathbb{C} \setminus \{0\} : z^4 = 1\}$ are isomorphic.

(c) We can construct an isomorphism between them as

$$\phi : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}, \quad \phi(n) = 3n.$$

(d) Note that $\mathbb{Z}$ has no element which has order 2 but in

$$\{z \in \mathbb{C} \setminus \{0\} : z^n = 1 \text{ for some } n \geq 1\}$$

we know $o(-1) = 2$, so by the claim above, they are not isomorphic.

(e) Note that $S_3 = \{(1), (12), (23), (13), (123), (132)\}$ and $D_3 = \{1, s, r, r^2, sr, sr^2\}$, so we can build an isomorphism $\phi : S_3 \to D_3$ as:

$$\begin{aligned}
\phi((1)) &= 1 \\
\phi((12)) &= s \\
\phi((23)) &= sr \\
\phi((13)) &= sr^2 \\
\phi((123)) &= r \\
\phi((132)) &= r^2.
\end{aligned}$$

(f) Note that $|S_4| = 24$ and $|D_4| = 8$, so there does not exist bijective fundtion $\phi : S_4 \to D_4$, which means S_4 and D_4 are not bijective.

(g) Since $|Q| = 8$ and $|T| = 12$, so Q and T are not isomorphic.

(h) Suppose $G = \langle g \rangle$, then since every subgroup of cyclic subgroup is cyclic, so $H = \langle g^k \rangle$ for some $k \geq 1$. Now we can build an isomorphism $\phi : G \to H$ by $\phi\left(g^i\right) = \left(g^k\right)^i$ for all $g^i \in G$. Note that this is an isomorphism.

2. Define the additive group

$$G = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Show that the map $\varphi : G \to \mathbb{C}$ given by $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto a + bi$ is a group isomorphism. You don't have to check for well-definedness.

Solution: We first show that φ is a group homomorphism. Note that

$$\varphi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} ac - bd & -ad - bc \\ bc + ad & -bd + ac \end{pmatrix}\right) = (ac - bd) + (bc + ad)i.$$

$$\varphi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right)\varphi\left(\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\right) = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Hence, we know φ is a group homomorphism. Now we show that φ is bijective. If

$$\varphi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\right),$$

then $a + bi = c + di$, which gives $a = c$ and $b = d$. Hence, φ is injective. Now we show that it is surjective. For all $c \in \mathbb{C}$, we know $c = a + bi$ for unique $a, b \in \mathbb{R}$, and

$$\varphi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right) = a + bi,$$

so we know φ is surjective. Hence, we know φ is bijective and thus an isomorphism.

3. Let H be a normal subgroup of a group G, and N a subgroup of H.

   (a) If H is cyclic, prove that N is a normal subgroup of G.

   (b) If N is a normal subgroup of H, show that N is not necessarily a normal subgroup of G. In other words, give a counterexample.

   Solution:

   (a) If H is cyclic, then suppose $H = \langle h \rangle$, and we want to show that $gNg^{-1} = N$ for all $g \in G$. Since we know H is normal, so $gHg^{-1} = H$ for all $g \in G$. Now fix some $g \in G$, we know $ghg^{-1} = h^k$ for some $k \geq 0$. Now given any $h^m \in N$, we have
   $$gh^m g^{-1} = \left(ghg^{-1}\right)^m = h^{km} = (h^m)^k \in N$$
   since $h^m \in N$. Thus, $gNg^{-1} \subseteq N$. Now we show that $N \subseteq gNg^{-1}$. Given $g^m \in N$, by repeating the above argument but replacing g with g^{-1}, we know there exists some $h^j \in N$ s.t. $g^{-1}h^m g = h^j \in N$. Hence, we have $h^m = gh^j g^{-1} \in gNg^{-1}$, which shows $N \subseteq gNg^{-1}$. Hence, we know $gNg^{-1} = N$, and since this g can be any element in G, so we know N is a normal subgroup of G.

   (b) Consider $G = S_4$, $H = V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$, and $N = \{e, (12)(34)\}$, then H is a normal subgroup of G and N is a normal subgroup of H, but N is not a normal subgroup of S_4 since $(31)(12)(34)(13) = (14)(23) \notin N$.

4. Let $f : G \to H$ be a group homomorphism with H abelian and let N be a subgroup of G containing $\ker f$. Prove that N is normal in G.
   Solution: We want to show that for all $g \in G$, we have $gNg^{-1} = N$. Now fix some $g \in G$ and suppose $n_1 \in N$, then
   $$\begin{aligned} f\left(gn_1g^{-1}n_1^{-1}\right) &= f(g)f\left(n_1\right)f\left(g^{-1}\right)f\left(n_1^{-1}\right) \\ &= f(g)f\left(g^{-1}\right)f(n_1)f\left(n_1^{-1}\right) \\ &= f(g)f(g)^{-1}f(n_1)f(n_1)^{-1} = e_H \end{aligned}$$
   since H is abelian. Hence, $gn_1g^{-1}n_1^{-1} \in \ker f \subseteq N$, and thus
   $$gn_1g^{-1}n_1^{-1} = n_2 \in N$$
   for some $n_2 \in N$ and thus $gn_1g^{-1} = n_2n_1 \in N$. Since g can be arbitrary element of G, so $gNg^{-1} \subseteq N$ for all $g \in G$. Now given $n_2 \in N$, and repeat the same argument but replacing g with g^{-1}, we will get
   $$g^{-1}n_2g = n_3 \in N$$
   for some $n_3 \in N$, and thus $n_2 = gn_3g^{-1}$, which means $N \subseteq gNg^{-1}$ for all $g \in G$. Hence, we have $N = gNg^{-1}$ for all $g \in G$.