

# Combinatorics I

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## **Abstract**

The lecture note of Combinatorics I by Shagnik Das, where the NTU cool site is <https://cool.ntu.edu.tw/courses/55532/>.

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# Chapter 1

## Chatting

### Lecture 1

#### 1.1 Prime Numbers

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**Theorem 1.1.1** (Euclid  $\approx 300$  BCE). There are infinitely many primes.

**proof.** (Saidak, 2006).

- Every natural number has at least one prime factor.
- No prime divides  $n$  and  $n + 1$ , for any  $n \in \mathbb{N}$ .

Consider a sequence of pronic number

$$p_1 = 2, p_{n+1} = p_n(p_n + 1).$$

Then the number of prime factors of  $p_n$  is strictly increasing in  $n$ :  $p_{n+1}$  has all the factors of  $p_n$  together with the (disstinct) ones of  $p_n + 1$ .

**Example 1.1.1.**  $p_1 = 2, p_2 = 6, p_3 = 42, p_4 = 1806$ , where the prime factors of them are  $\{2\}, \{2, 3\}, \{2, 3, 7\}, \{2, 3, 7, 43\}$ .

■

##### 1.1.1 How many prime numbers are there?

**Definition 1.1.1.** We define

$$\pi(n) = |\{p : 1 \leq p \leq n : p \text{ is prime}\}|.$$

**Note 1.1.1.** By Saidak's proof, we know  $\pi(p_n) \geq n$ . In fact,  $\pi(p_n) \geq \log_2 n$ .

**Theorem 1.1.2** (Legendre,  $\approx 1800$  LE ).

$$\pi(n) \approx \frac{n}{\ln n} \Leftrightarrow \lim_{n \rightarrow \infty} \frac{\pi(n) \ln n}{n} = 1$$

**Note 1.1.2.** Proven by Hadamard and independently de la Vallée Poussin(1896).

**Theorem 1.1.3** (Better Approximation).

Dirichlet:  $\pi(x) \approx Li(x) = \int_2^x \frac{1}{\ln t} dt$ .

Known:  $\pi(n) = Li(n) + O\left(ne^{-a\sqrt{\ln n}}\right)$

Believed:  $\pi(n) = Li(n) + O(\sqrt{n} \ln n)$

# Chapter 2

## Elementary Counting Principles

Fundemental problem: Given a set  $S$ , and we want to determine  $|S|$ .

### 2.1 Sum Rule

**Theorem 2.1.1** (Sum Rule). If  $S = \bigcup_{i=1}^k S_i$ , then  $|S| = \sum_{i=1}^k |S_i|$ .

**Note 2.1.1.**  $\bigcup$  means disjoint union.

**Example 2.1.1.** A drawer contains 8 pairs of yellow socks, 5 pairs of blue socks, and 3 pairs of red socks. How many socks are there in total.

**Informal proof.**  $2 \times (8 + 5 + 3) = 32$ . ■

**Proof.** Let  $S$  be the set of socks in the drawer, then  $S = \bigcup_{p \in P} S_p$ , where  $P$  is the set of pairs of socks, and  $S_p$  is the set of two socks in the pair where  $p \in P$ . By the sum rule,

$$|S| = \sum_{p \in P} |S_p| = \sum_{p \in P} 2 = 2|P| = 32.$$

$P = P_{\text{yellow}} \cup P_{\text{blue}} \cup P_{\text{red}}$ . By the sum rule,

$$|P| = |P_{\text{yellow}}| + |P_{\text{blue}}| + |P_{\text{red}}| = 8 + 5 + 3 = 16. ■$$

**Note 2.1.2.** Sum rule is the basis for case analysis arguments. It needs two requirements:

- Cover each case.
- Cover each case exactly once.

**Example 2.1.2.** Counting subset of a general set.

**Notation.** If  $X$  is a set, and  $k \in \mathbb{N} \cup \{0\}$ , then

$$\binom{X}{k} = \{T : T \subseteq X, |T| = k\}.$$

We define the binomial coefficient as

$$\binom{|X|}{k} = \left| \binom{X}{k} \right|.$$

i.e. Given  $n \geq k \geq 0$ ,  $\binom{n}{k}$  is the number of  $k$ -element subsets of a set of size  $n$ . ■

**Proposition 2.1.1** (Pascal's relation). If  $n \geq k \geq 1$ , then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

**Proof.** Let  $X$  be an  $n$ -element set (e.g.  $X = [n] = \{1, 2, \dots, n\}$ ), and let  $S = \binom{X}{k} = \{T \subseteq X : |T| = k\}$ . Then, by definition,  $\binom{n}{k} = |S|$ . For each  $k$ -element subset, we can ask: "Do you contain  $n$ ?" Let

$$S_0 = \{T : T \subseteq X, n \notin T, |T| = k\},$$

and

$$S_1 = \{T : T \subseteq X, n \in T, |T| = k\}.$$

Then,  $S = S_0 \cup S_1$ . By the sum rule,  $|S| = |S_0| + |S_1|$ . Observe that

$$\begin{aligned} S_0 &= \{T \subseteq [n], n \notin T, |T| = k\} \\ &= \{T \subseteq [n-1], |T| = k\}, \end{aligned}$$

so by definition,

$$|S_0| = \binom{|[n-1]|}{k} = \binom{n-1}{k}.$$

$$S_1 = \{T \subseteq [n], n \in T, |T| = k\}.$$

Let

$$S'_1 = \{T' \subseteq [n-1], |T'| = k-1\},$$

then we know a bijection from  $S_1$  to  $S'_1$ :

$$T \in S_1 \longleftrightarrow T \setminus \{n\} \in S'_1.$$

**Theorem 2.1.2** (bijection rule). Given two sets  $S$  and  $S'$ , if there is a bijection  $f : S \rightarrow S'$ , then  $|S| = |S'|$ .

By this rule, we know

$$|S_1| = |S'_1| = \binom{|[n-1]|}{k-1} = \binom{n-1}{k-1}.$$

Hence,

$$\binom{n}{k} = |S| = |S_0| + |S_1| = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

■

### 2.1.1 Pascal's Triangle

We can use Pascal's relation to compute  $\binom{n}{k}$ .

**Note 2.1.3.** Boundary case:  $\binom{n}{0} = 1$ ,  $\binom{n}{n} = 1$ . Also,  $\binom{n}{k} = 0$  for  $k = -1, n+1$ .



## 2.2 Product Rule

**Theorem 2.2.1.** If  $S = S_1 \times S_2 \times \cdots \times S_k = \{(x_1, x_2, \dots, x_k), x_i \in S_i\}$ , then  $|S| = \prod_{i=1}^k |S_i|$ .

**Proof.** Induction on  $k$ :

Base case:  $k = 1$ , trivial.

Induction step: separate into cases based on choice of  $x_{k+1} \in S_{k+1}$ . Let

$$S(x) = \{(x_1, \dots, x_{k+1}) \in S, x_{k+1} = x \in S_{k+1}\},$$

then

$$S = \bigcup_{x \in S_{k+1}} S(x) \rightarrow |S| = \sum_{x \in S_{k+1}} |S(x)|.$$

But  $S(x) = S_1 \times S_2 \times \cdots \times \{x\}$ , which is in bijection with  $S_1 \times S_2 \times \cdots \times S_k$ . By induction rule,

$$|S(x)| = |S_1 \times S_2 \times \cdots \times S_k| \quad \forall x$$

Hence,

$$\begin{aligned}
 |S| &= \sum_{x \in S_{k+1}} |S(x)| = \sum_{x \in S_{k+1}} |S_1 \times S_2 \times \cdots \times S_k| \\
 &= |S_1 \times S_2 \times \cdots \times S_k| \times |S_{k+1}| = |S_1| \times |S_2| \times \cdots \times |S_{k+1}|.
 \end{aligned}$$

■

**Example 2.2.1.** Consider binary strings of length  $n$ .

**Proof.**

$$S = \{0, 1\}^n \Rightarrow |S| = |\{0, 1\}^n| = |\{0, 1\}|^n = 2^n.$$

■

**Definition 2.2.1 (Power Set).** Given a finite set  $X$ , let  $2^X$  denote the set of all subsets of  $X$  (also denoted  $\mathcal{P}(X)$ ), which is called the power set.

**Corollary 2.2.1.**  $|2^X| = 2^{|X|}$ .

**Proof.** Without loss of generality,  $X = [n]$ . We build a bijection between  $2^{[n]}$  and the set of binary string of length  $n$ . Suppose for every  $T \in 2^{[n]}$ , we have  $\chi_T = (x_1, x_2, \dots, x_n)$ , where

$$x_i = \begin{cases} 1, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases}$$

Then,

$$|2^{[n]}| = |\{0, 1\}^n| = 2^n.$$

■

## 2.3 Double-Counting argument

If we count a set in two different ways, the answer should be equal.

**Example 2.3.1.** Count  $2^{[n]}$ .

**Proof.**

1. Product rule  $\rightarrow 2^n$ .
2. Use the sum rule, split the subsets by size.

$$2^{[n]} = \binom{[n]}{0} \cup \binom{[n]}{1} \cup \dots \cup \binom{[n]}{n}$$

Hence, we have the following proposition:

**Proposition 2.3.1.** For all  $n \geq 0$ ,

$$2^n = \sum_{k=0}^n \binom{n}{k}.$$

(\*)

## 2.4 Permutations

### Lecture 2

**As previously seen.** Instead of choosing the subsets all at once, we could pick one element at a time, then we can try to use product rule.

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**Example 2.4.1.** Consider

$$\binom{[10]}{3}.$$

**Proof.** At the choice of the first element, we have 10 choices, the second one has 9 choices, while the third one has 8 choices, but we didn't consider the order of each picked elements. (\*)

**Definition 2.4.1.** Given a set  $X$  and  $k \in \mathbb{N} \cup \{0\}$ , a  $k$ -permutation of  $X$  is

- an ordered choice of  $k$  distinct elements from  $X$ .
- a  $k$ -tuple  $(x_1, x_2, \dots, x_k)$  with  $x_i \in X$  and  $x_i \neq x_j$  for each  $i \neq j$ .
- an injection  $f : [k] \rightarrow X$ .

where these 3 statements are equivalent.

**Notation.**  $X^k = \{k\text{-permutation of } X\} \subseteq X^k$  where  $X^k = X \times X \times \dots \times X$  allows repetition of the elements but  $X^k$  don't allow repetition.

**Note 2.4.1.** If  $|X| = n$ , then

$$n^k = |X^k|.$$

**Definition 2.4.2.**

- a  $n$ -permutation is a  $n$ -permutation of  $[n]$ .
- a  $X$ -permutation is a  $|X|$ -permutation of  $X$ .

**Theorem 2.4.1 (Generalized Product Rule).** Suppose we are enumerating  $S$ , and can uniquely determine an element  $s \in S$  through a series of  $k$  questions, if  $i$ -th problem always has  $n_i$  possible outcomes, independently to the permutation, then

$$|S| = n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^k n_i$$

**Proof.** Can make a bijection from  $S$  to

$$[n_1] \times [n_2] \times \cdots \times [n_k].$$

Map each element in  $S$  to the index of its answer in the series of answer.

Our moral is when counting we don't care about what the options are but only how many options. ■

**Proposition 2.4.1.**

$$\begin{aligned} n^k &= n(n-1)\dots(n-(k-1)) \\ &= \prod_{i=0}^{k-1} (n-i) = \frac{n!}{(n-k)!}. \end{aligned}$$

**Proof.** Use the generalized product rule.

Question  $i$ : What is the  $i$ -th element in the  $k$ -permutation of  $[n]$ ?

We can choose anything except what we're already chosen, so there are  $i-1$  forbidden choices and thus there are  $n-(i-1)$  possible choices. ■

**Proposition 2.4.2.** For all  $0 \leq k \leq n$ ,

$$\binom{n}{k} = \frac{n^k}{k^k} = \frac{\left(\frac{n!}{(n-k)!}\right)}{k!} = \frac{n!}{k!(n-k)!}.$$

**Proof.** Double-count  $[n]^k$  i.e.  $k$ -permutation of  $[n]$ .

- Direct counting  $\left|[n]^k\right| = n^k$ .
- First choose the  $k$  elements to appear in the  $k$ -permutation,  $\binom{n}{k}$  options, then choose the order in which they appear,  $k^k$  options.

Then, by the generalized product rule, the number of  $k$ -permutation of  $[n]$  is  $\binom{n}{k} \cdot k^k$ .

Hence,

$$n^k = \left|[n]^k\right| = \binom{n}{k} \cdot k^k.$$

■

**Corollary 2.4.1.** We can then use this result to reprove Pascal's Property again.

**Proof.** ■

**Exercise 2.4.1.** 6 players at the tennis club want to have three matches involving all the players? How many ways can we arrange the games.



Figure 2.1: Tennis Games

**Proof.** We only care about who plays against whom, not about which court or who versus first, e.t.c.

The arrangement of games is a set of three disjoint pairs of players.

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \neq \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}.$$

Double-count the arrangements of games where counts do matter.

- Choose a pair of players for Court A:  $\binom{6}{2}$
- Choose a pair of players for Court B:  $\binom{4}{2}$
- Choose a pair of players for Court C:  $\binom{2}{2}$

Generalized product rule tells

$$\text{number of choices} = \binom{6}{2} \binom{4}{2} \binom{2}{2} = 90.$$

Second count: First gets a set of 3 pairs, say there are  $x$  possibilities , and assign the three pairs to 3 courts, so there are  $3!$  , so  $x \cdot 3! = 90$ , and thus  $x = \frac{90}{3!} = 15$ . ■

## Lecture 3

Actually we have an alternative prove:

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**proof by direct computation.**

- Q1: Who's the opponent for the 1-st player? There are 5 choices.
- Q2: Who plays the next lowest numbered player? There are 3 choices.

The left 2 players are the opponents to each other. Hence, there are  $3 \times 5 = 15$  possible pairings. ■

More generally, if we have  $n = 2k$  players to pair up, then the first proof gives there are

$$\frac{\binom{n}{2} \binom{n-2}{2} \cdots \binom{2}{2}}{\left(\frac{n}{2}\right)!}$$

possible pairings, while the second proof gives that there are

$$(n-1) \cdot (n-3) \cdot (n-5) \cdots := (n-1)!! \neq ((n-1)!)!.$$

By this, we know these two numbers must be equal, or more rigorously, we can write

$$\frac{\binom{n}{2} \binom{n-2}{2} \cdots \binom{2}{2}}{\left(\frac{n}{2}\right)!} = 2^n \cdot \frac{\frac{n(n-1)}{2} \frac{(n-2)(n-3)}{2} \cdots}{n(n-2)(n-4) \cdots 2} = (n-1) \cdot (n-3) \cdots$$

**Example 2.4.2.** How many shortest routes on the grid are there from  $(0,0)$  to  $(n,m)$ ?



Figure 2.2: Taxi routes

**Proof.** Shortest route is of length  $n+m$ ,  $m$  up-steps and  $n$  right-steps. We can think of a shortest route to be a binary string of length  $n+m$  with  $n$  1s and  $m$  0s, so we want to count how many such binary strings are there. Choose  $n$  of them to be 1s, while the other are 0s. Hence, there are  $\binom{n+m}{n}$  possibilities. ⊗

## 2.5 Binomial Theorem

**Theorem 2.5.1** (Binomial Theorem). For any  $n \in \mathbb{N} \cup \{0\}$ , and  $x, y \in \mathbb{R}$ , we have

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

**Example 2.5.1.**  $(x + y)^0 = 1 = \sum_{k=0}^0 x^k y^{0-k}$ .

**Example 2.5.2.**  $(x + y)^1 = x + y$ , while

$$\sum_{k=0}^1 \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x.$$

**proof of binomial theorem.**

$$(x + y)^n = \underbrace{(x + y)(x + y)(x + y) \dots (x + y)}_{n \text{ factors}}$$

From each factor, we pick a term  $x$  or  $y$ , multiply chosen factors together. If we choose  $k$   $x$ 's, then we must choose  $n - k$   $y$ 's, so the monomial is  $x^k y^{n-k}$ , where the coefficient of  $x^k y^{n-k}$  is the number of ways of choosing  $k$   $x$ 's. Also, the possible monomials are  $x^k y^{n-k}$  for  $k = 0, 1, 2, \dots, n$ . Hence, we have

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

■

We can use this formula to derive identities for the binomial coefficients, by plugging in values for  $x$  and  $y$ .

**Example 2.5.3.**  $x = 1, y = 1$ .

**Proof.**

$$2^n = (x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^n \binom{n}{k}.$$

(\*)

**Example 2.5.4.**  $y = -1, x = 1$ .

**Proof.**

$$(x + y)^n = (-1 + 1)^n = 0^n = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^n \binom{n}{k} (-1)^k = \sum_{2|k} \binom{n}{k} - \sum_{2\nmid k} \binom{n}{k}$$

(\*)

**Corollary 2.5.1.**

$$\sum_{2|k} \binom{n}{k} = \sum_{2\nmid k} \binom{n}{k}$$



Figure 2.3: The sum of even terms is equal to the sum of odd terms.

**Theorem 2.5.2.**  $\forall n \geq k$ , we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

**Proof.**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-(n-k))!} = \binom{n}{n-k}.$$

**Remark 2.5.1.** Choosing a subset of  $k$  elements from  $n$  is equivalent to choose  $n - k$  elements to discard, and we can build a bijection between these two methods. ■

For  $n$  even.

Consider the bijection

$$S \mapsto S \Delta \{n\} = \begin{cases} S - \{n\}, & \text{if } n \in S; \\ S \cup \{n\}, & \text{if } n \notin S. \end{cases}$$

Hence,

$$|S \Delta \{n\}| \in \{|S| - 1, |S| + 1\},$$

so if  $|S|$  is odd, then  $S \Delta \{n\}$  is even, and vice versa. We know this is a bijection (self-inverse), so we have odd-sized sets to even-sized set. Hence,  $\sum_{2|k} \binom{n}{k} = \sum_{2\nmid k} \binom{n}{k}$ .

**Example 2.5.5.**  $x = 2, y = 1$ .

**Proof.**

$$(2+1)^n = 3^n = \sum_{k=0}^n \binom{n}{k} 2^k$$

Counting partitions  $[n] = A \cup B \cup C$ , each element has a choice of 3 sets to go into. Hence, the product rule says there are  $3^n$  partitions, while RHS uses sum rule bases on  $k = |A \cup B|$ .  $\circledast$

## 2.6 Divisor Function

**Definition 2.6.1** (Divisor Functions). Given a natural number  $n \in \mathbb{N}$ , let  $d(n)$  count the number of divisors of  $n$ .

**Example 2.6.1.**

$$\begin{aligned} d(1) &= 1 = |\{1\}| \\ d(2) &= 2 = |\{1, 2\}| \\ d(3) &= 2 = |\{1, 3\}| \\ d(4) &= 3 = |\{1, 2, 4\}| \\ d(5) &= 2 = |\{1, 5\}|. \end{aligned}$$

**Corollary 2.6.1.**  $d(n) = 2$  if and only if  $n$  is a prime.

Now we want to compute the average value of  $d(n)$ .

**Definition 2.6.2.**

$$\bar{d}(n) = \frac{\sum_{i=1}^n d(i)}{n}.$$

We can use double-counting. First, notice that

$$d(i) = \sum_{\substack{j \in [i] \\ j|i}} 1.$$

Hence,

$$\sum_{i=1}^n d(i) = \sum_{i=1}^n \sum_{\substack{j \in [i] \\ j|i}} 1.$$

We can exchange the order of summation:

$$n\bar{d}(n) = \sum_{i=1}^n d(i) = \sum_{i=1}^n \sum_{j:j|i} 1 = \sum_{j=1}^n \sum_{\substack{i \in [n] \\ j|i}} 1.$$

For fixed  $j$ , we know

$$\sum_{\substack{i \in [n] \\ j|i}} 1 = \left\lfloor \frac{n}{j} \right\rfloor.$$

Hence, we have

$$n\bar{d}(n) = \sum_{j=1}^n \left\lfloor \frac{n}{j} \right\rfloor,$$

which is equivalent to

$$\bar{d}(n) = \frac{1}{n} \sum_{j=1}^n \left\lfloor \frac{n}{j} \right\rfloor.$$

Observe that

$$\frac{n}{j} - 1 \leq \left\lfloor \frac{n}{j} \right\rfloor \leq \frac{n}{j},$$

so

$$H_n - 1 = \frac{1}{n} \sum_{j=1}^n \left( \frac{n}{j} - 1 \right) \leq \bar{d}(n) \leq \frac{1}{n} \sum_{j=1}^n \frac{n}{j} = \sum_{j=1}^n \frac{1}{j} = H_n \approx \ln n.$$

Hence,

$$H_n - 1 \leq \bar{d}(n) \leq H_n,$$

which gives  $\bar{d}(n) \sim \ln n$ .

# Chapter 3

## Partitions

How many ways can we divide  $n$  items into  $k$  groups? Need to specify details to get well-posed questions.

1. Items distinguishable or not?
2. Groups distinguishable or not?
3. Can we have empty groups? Can we have group with more than one item?

**Example 3.0.1.** Professor has 49 students, to distribute 3000% between the students.

**Proof.** Indistinguishable items: percentage points.

Distinguishable groups: students  $k = 49$ . No restriction on sizes of groups. Formally, we are enumerating

$$S = \left\{ (x_1, x_2, \dots, x_{49}) \mid x_i \geq 0, x_i \in \mathbb{Z}, \sum_{i=1}^{49} x_i = 3000 \right\}$$

(\*)

## Lecture 4

### 3.1 Number of nonnegative integer solution to $x_1 + \dots + x_k = n$

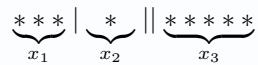
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We can represent solutions using a "stars and bar" diagaram:

- $n$  stars represent the items
- $k - 1$  bars to divides the groups

**Example 3.1.1.**  $x_1 = 3, x_2 = 1, x_3 = 0, x_4 = 5$ . ( $k = 4, n = 9$ )

**Proof.**



(\*)

Hence, we can use a projection between solution and diagrams with  $k - 1$  bars and  $n$  stars.

Each diagram consists of  $n + k - 1$  symbols. Once we know which are the bars, we know the full diagram.

$$\text{number of diagrams} = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

**Proposition 3.1.1.** The number of non-negative integer solutions to  $x_1 + \dots + x_k = n$  is  $\binom{n+k-1}{k-1}$ .

Now we have a new problem.

**Question.** How many solutions are there to  $x_1 + \dots + x_k = n$  with  $x_i \geq 1$  for all  $i$ ?

We can let  $y_i = x_i - 1$ , then  $y_i \geq 0$  and  $y_1 + \dots + y_k = n - k$ . Hence, the answer is

$$\binom{(n-k)+(k-1)}{k-1} = \binom{n-1}{k-1}.$$

**Definition 3.1.1 (Multisets).** An unordered collection of elements with repetition allowed.

$$\{\{1, 1, 1, 2, 3\}\} \neq \{\{1, 2, 3\}\}$$

can be represented as an ordered tuple in increasing order.

**Example 3.1.2.** How many multisets of size  $n$  are there from a set of size  $k$ ?

**Proof.** Let  $x_i$  be the multiplicities of the  $i$ -th element in the multiset. Then  $x_i \geq 0$  and

$$x_1 + \dots + x_k = n.$$

Hence, the number of multisets is

$$\binom{n+k-1}{k-1}.$$

(\*)

Alternatively, multisets are  $(a_1, \dots, a_n)$  with  $1 \leq a_1 \leq \dots \leq a_n \leq k$ . Now if we let  $b_i = a_i + i - 1$ , then

$$(b_1, \dots, b_n) = (a_1, a_2 + 1, \dots, a_n + n - 1) \text{ with } 1 \leq b_1 < b_2 < \dots < b_n \leq n + k - 1.$$

Note that there is a bijection between  $\{(a_1, \dots, a_n)\}$  and  $\{(b_1, \dots, b_n)\}$ . This shows the number of multisets of size  $n$  from  $[k]$  is the number of subsets of  $[n+k-1]$  of size  $n$ , which is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Now we add some new setting.

- Distinguishable items
- Indistinguishable groups
- Groups non-empty.

The objects we are counting is

$$\{S_1, S_2, \dots, S_k\}$$

with  $S_1 \cup S_2 \cup \dots \cup S_k = [n]$  and  $S_i \neq \emptyset$  for all  $i$ .

**Definition 3.1.2 (The Stirling Number of the second kind).**  $S(n, k)$  is defined to be number of partitions of  $n$  distinct items into  $k$  indistinguishable non-empty groups.

**Example 3.1.3.**  $S(n, 1) = 1$  for all  $n \geq 1$ .  $S(n, n) = 1$  for all  $n$ .  $S(n, n-1) = \binom{n}{2}$  for all  $n \geq 2$ .  $S(n, 2) = 2^{n-1} - 1$ .

**Proof.** We just talk about the  $S(n, 2)$  one. Since we can choose any subset of  $[n]$ , so there are  $2^n$  possibilities, but each partition is counted twice, so we have to divide it by 2, and subtract the

partition that includes empty group, so it is  $2^{n-1} - 1$ . (\*)

**Proposition 3.1.2.** For all  $n, k$ ,

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k).$$

**Proof.** Case analysis:

- Case 1:  $\{n\}$  is a group.

This means the remaining  $n - 1$  elements are partitioned into  $k - 1$  groups, so there are  $S(n - 1, k - 1)$  possibilities.

- Case 2:  $\{n\}$  is not a group.

$n - 1$  left elements is first partitioned into  $k$  groups, then we can distribute the  $n$ -th element into each group, so there are  $kS(n - 1, k)$  possibilities.

By sum rule, we know

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k).$$

■

**Example 3.1.4.** Using induction to prove

$$S(n, n - 1) = \binom{n}{2}.$$

**Proof.**

$$\begin{aligned} S(n, n - 1) &= S(n - 1, n - 2) + (n - 1)S(n - 1, n - 1) = S(n - 1, n - 2) + (n - 1) \\ &= \dots = 1 + 2 + \dots + n - 1 = \binom{n}{2}. \end{aligned}$$

\*)

Now what if the groups are distinguishable? Also, we have

- items distinguishable
- groups distinguishable
- groups non-empty.

Short answer:  $S(n, k)k!$ .

## Lecture 5

We can observe that the number of ways of partitioning  $n$  distinct items into  $k$  distinct nonempty groups is  $S(n, k)k!$ . 16 Sep. 15:30

**Question.** How many ways can we partition  $n$  distinct items into  $l$  distinct groups (not necessarily nonempty)?

**Answer.**  $l^n$ : product rule, each element has  $l$  choice for which group to go to. (\*)

**Alternative method.** Count by the number of nonempty groups ( $k$ ), and then use sum rule. Partition elements into  $k$  nonempty indistinguishable groups, which has  $S(n, k)$  choices, and then map the  $k$  sets to the  $l$  groups injectively, so there are  $l^k = l(l - 1)\dots(l - k + 1)$  choices. Hence, the total number of partition is

$$\sum_{k=0}^l S(n, k)l^k.$$

By double counting, we know

$$l^n = \sum_{k=0}^n S(n, k) l^k = \sum_{k=0}^n S(n, k) l^k.$$

■

**Proposition 3.1.3.** For any field  $F$ , and  $x \in F$ ,  $n \in \mathbb{N} \cup \{0\}$ , then

$$x^n = \sum_{k=0}^n S(n, k) x^k.$$

(We define  $x^k = x(x-1)\dots(x-(k-1))$ .)

**Proof.** There are polynomials of degree  $\leq n$  that agree for all  $x \in \mathbb{N}$ , so they must agree everywhere. ■

We can observe that  $\{x^n \mid n \in \mathbb{N} \cup \{0\}\}$  forms a basis for

$$F[x] = \left\{ \sum_{k=0}^n a_k x^k : a_k \in F \right\}.$$

Since  $x^n$  is a linear combination of  $\{x^n \mid n \in \mathbb{N} \cup \{0\}\}$ , that means this is also a basis for  $F[x]$ . And the proposition shows that the change of basis matrix is the matrix of Stirling numbers of the second kind:

$$\begin{pmatrix} 1 & & 0 & 0 \\ & 1 & & 0 \\ & & 1 & \\ & & & \ddots \\ S(n, k) & & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix}.$$

## 3.2 Stirling numbers of the first kind

Recall the permutation  $\pi$  is a bijection from  $[n]$  to  $[n]$ .

**Example 3.2.1.**  $\pi = 32154$ , then  $\pi(1) = 3, \pi(2) = 2, \pi(3) = 1, \pi(4) = 5, \pi(5) = 4$ .

**Example 3.2.2.**  $\pi_1 = 312, \pi_2 = 213$ , then  $\pi_2 \circ \pi_1 = 321$  and  $\pi_1 \circ \pi_2 = 132$ .

**Claim 3.2.1.**  $\forall \pi \in S_n, \forall x \in [n], \exists i \in [n]$  s.t.  $\pi^i(x) = x$ .

**Proof.** Consider  $\pi^1(x), \pi^2(x), \dots, \pi^n(x) \in [n]$ , if any are equal to  $x$ , then we're done. Otherwise, there are only  $n - 1$  possible values, which are  $[n] \setminus \{x\}$ . Hence, there are some  $j_1, j_2 \in [n]$  with  $j_1 > j_2$  and  $\pi^{j_1}(x) = \pi^{j_2}(x)$  by Pigeonhole principle. Applying  $\pi^{-1}$  for  $j_2$  times, we get

$$\pi^{j_1-j_2}(x) = x \quad \text{with } 1 \leq j_1 - j_2 \leq n,$$

which is a contradiction. ■

**Definition 3.2.1 (cycle).** For the smallest  $i$ ,  $1 \leq i \leq n$  with  $\pi^i(x) = x$ , we say

$$(x \ \pi(x) \ \pi^2(x) \ \dots \ \pi^{i-1}(x))$$

is the cycle of  $x$ .

It follows that every permutation is a union of disjoint cycles. Hence, we have cycle representation of  $\pi$ .

**Example 3.2.3.**  $\pi = 32154$ , the cycle form is  $(13)(2)(45)$ .

**Definition 3.2.2 (fixed point and transposition).** A fixed point of a permutation is a cycle of length 1 i.e. an element  $x$  with  $\pi(x) = x$ . A transposition is a cycle of length 2. A permutation is cyclic if it has a single cycle (of length  $n$ ).

**Question.** How many cyclic permutations of  $[n]$  are there?

**Answer.**  $(n - 1)!$ . We can first fix the head of the cycle to be 1, then for  $\pi(1)$ , we have  $n - 1$  choices, and for  $\pi^2(1)$ , we have  $n - 2$  choices, and so on, so we have  $(n - 1)!$  cyclic permutations.

**Note 3.2.1.** Who is in the head of the cycle is not important.

(\*)

**Definition 3.2.3 (The Stirling numbers of the first kind).**  $s_{n,k}$  (or  $[s(n, k)]$ ) enumerate the permutation in  $S_n$  with exactly  $k$  cycles.

**Example 3.2.4.**  $s_{n,1} = (n - 1)!$ ,  $s_{n,n} = 1$ ,  $s_{n,n-1} = \binom{n}{2}$ ,  $s_{n,2}$  =not so obvious.

**Proof.**

$$s_{n,2} = \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} (k-1)! (n-k-1)!$$

Note that we multiply it by  $\frac{1}{2}$  since we count each cycle-pair twice. Also, we know that a cycle of length  $n$  has  $(n - 1)!$  choices if we fix all  $n$  members in the cycle.

Alternatively, say the "first" cycle is the one containing 1 together with  $0 \leq k \leq n - 2$  other elements. Hence, we have

$$\begin{aligned} s_{n,2} &= \sum_{k=0}^{n-2} \binom{n-1}{k} (k!) (n-k-2)! \\ &= \sum_{k=0}^{n-2} \frac{(n-1)!}{k!(n-k-1)!} k! (n-k-2)! = (n-1)! \sum_{k=0}^{n-2} \frac{1}{n-1-k} \\ &= (n-1)! \sum_{k=1}^{n-1} \frac{1}{k} \\ &= (n-1)! H_{n-1} \approx (n-1)! \ln n. \end{aligned}$$

(\*)

**Proposition 3.2.1.**  $\forall n, k \geq 1$ ,

$$s_{n,k} = s_{n-1,k-1} + (n-1)s_{n-1,k}$$

**Proof.** Case analysis: is  $n$  a fixed point?

- Case 1: Yes. Removing it, and then the left  $n - 1$  elements can be permuted with  $k - 1$  cycles. Hence, there are  $s_{n-1,k-1}$  choices.

- Case 2: No. We remove  $n$  from a cycle to get a permutation of  $[n - 1]$  with  $k$  cycles. Now, we have  $n - 1$  place to insert  $n$  inside. For example, we if  $n = 7$ , and we have  $(13)(2)(456)$ , then we have  $7 - 1 = 6$  places to insert 7 inside since  $(7456)$  and  $(4567)$  are same cycles.

To create a permutation  $\pi \in S_n$  with  $k$  cycles where  $n$  is not a fixed point, we can take a permutation  $\pi' \in S_{n-1}$  with  $k$  cycles, which has  $s_{n-1,k}$  choices, and insert  $n$  before any element, so there are  $n - 1$  ways, so the number of such permutation is  $(n - 1)s_{n-1,k}$ . By sum rule, we have

$$s_{n,k} = s_{n-1,k-1} + (n - 1)s_{n-1,k}.$$

■

Example :

$n \setminus k$	0	1	2	3	4	$\sum$
0	1					1
1	0	1				1
2	0	1	1			2
3	0	2	3	1		6
4	0	6	11	6	1	24

$n!$

Figure 3.1: table of  $s_{n,k}$

**Corollary 3.2.1.**  $\forall n$ , we have

$$\sum_{k=0}^n s_{n,k} = n!.$$

**Proof.** The number of permutations are  $n!$ , and every permutation consists of  $i$  cycles where  $1 \leq i \leq n$ , and then apply the sum rule. ■

**Notation.** Given  $x \in F$ , and  $k \in \mathbb{N} \cup \{0\}$ , we have

- $x^k = x(x - 1) \dots (x - (k - 1))$
- $x^{\bar{k}} = x(x + 1) \dots (x + (k - 1)) = (x + k - 1)^k$ .

**Proposition 3.2.2.** For all  $x \in F$ ,  $n \in \mathbb{N} \cup \{0\}$ ,

$$x^{\bar{n}} = \sum_{k=0}^n s_{n,k} x^k.$$

**Proof.** Induction on  $n$ . We know it is true for  $n = 0, 1$ . Note that

$$\begin{aligned}
 x^{\bar{n}} &= x^{\overline{n-1}}(x + n - 1) \\
 &= (x + n - 1) \sum_{k=0}^{n-1} s_{n-1,k} x^k \\
 &= x \sum_{k=0}^{n-1} s_{n-1,k} x^k + (n-1) \sum_{k=0}^{n-1} s_{n-1,k} x^k \\
 &= \sum_{k=0}^{n-1} s_{n-1,k} x^{k+1} + \sum_{k=0}^{n-1} (n-1) s_{n-1,k} x^k \\
 &= \sum_{k=1}^n s_{n-1,k-1} x^k + \sum_{k=0}^{n-1} (n-1) s_{n-1,k} x^k \\
 &= \sum_{k=0}^n (s_{n-1,k-1} + (n-1) s_{n-1,k}) x^k \\
 &= \sum_{k=0}^n s_{n,k} x^k.
 \end{aligned}$$

■

### Corollary 3.2.2.

$$x^n = \sum_{k=0}^n \underbrace{(-1)^{n-k} s_{n,k}}_{\substack{\text{signed Stirling numbers} \\ \text{of the first kind}}} x^k.$$

**Proof.**

$$\begin{aligned}
 x^n &= x(x-1)\dots(x-(n-1)) \\
 &= (-1)^n (-x)(-x+1)\dots(-x+(n-1)) \\
 &= (-1)^n (-x)^{\bar{n}} \\
 &= (-1)^n \sum_{k=0}^n s_{n,k} (-x)^k \\
 &= \sum_{k=0}^n (-1)^{n-k} s_{n,k} x^k.
 \end{aligned}$$

■

## Lecture 6

### Corollary 3.2.3.

$$\sum_{k=j}^i (-1)^{k-j} S(i, k) s_{k,j} = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

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**Proof.** By Proposition 3.1.3, we have

$$\begin{aligned} x^i &= \sum_{k=0}^i S(i, k) x^k = \sum_{k=0}^i S(i, k) \left[ \sum_{j=0}^k (-1)^{k-j} s_{k,j} x^j \right] \\ &= \sum_{k=0}^i \sum_{j=0}^k (-1)^{k-j} S(i, k) s_{k,j} x^j \\ &= \sum_{j=0}^i \left( \sum_{k=j}^i (-1)^{k-j} S(i, k) s_{k,j} \right) x^j = x^i. \end{aligned}$$

Since  $\{x^0, x^1, x^2, \dots\}$  is a basis of  $F[x]$ , the coefficient of  $x^j$  is 1 if  $i = j$  and is 0 if  $i \neq j$ . ■

**Question.** How many ways can we distribute \$100000 of prize money to six players in the tournaments?

- Whole dollars only.
- Nonnegative prices.

It is an arbitrary partition, and there are  $k = 6$  distinct groups(players). Hence, there are  $\binom{100000}{5}$  ways of distribution? However, this is not what we want, since in a tournament a better player should get more money. Actually, in this scenario, groups are indistinguishable since largest prize is for first place, and so on. Thus, our goal is to dividing  $n$  indistinguishable items into  $k$  indistinguishable (non-empty) groups.

**Definition 3.2.4 (number partition).** A number partition is a decomposition of  $n$  and a sum of  $k$  unordered natural numbers.

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \text{ s.t. } \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k, \quad \sum_{i=1}^k \lambda_i = n \text{ with } \lambda_i \in \mathbb{N}.$$

We write  $\lambda \vdash n$ . We define

$$p(n, k) = |\{\lambda = (\lambda_1, \dots, \lambda_k) : \lambda \vdash n\}|.$$

We also define

$$\begin{aligned} p(n, \leq k) &= \sum_{i=0}^k p(n, i) \\ p(n) &= p(n, \leq n) = \sum_{i=0}^n p(n, i). \end{aligned}$$

Observe that

- $p(n, 0) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$
- $p(n, n) = 1$
- $p(n, n-1) = 1 = |\{2, 1, 1, \dots\}|$
- $p(n, 1) = 1$ .
- $p(n, 2) = \lfloor \frac{n}{2} \rfloor$ .

**Proposition 3.2.3.**  $\forall n \geq k \geq 1$ ,

$$p(n, k) = p(n - 1, k - 1) + p(n - k, k).$$

**Proof.** Case analysis based on size of smallest part:

- Case 1:  $\lambda_k = 1$ .  
Then remove the last part to get a partition of  $n - 1$  into  $k - 1$  nonempty parts. (bijective, can add part of size 1 to the end of a partition), so there are  $p(n - 1, k - 1)$  such cases.
- Case 2:  $\lambda_k \geq 2$ .  
Consider  $\lambda' = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_k - 1)$ , then  $\lambda' \vdash n - k$ , and this is a bijection, so there are  $p(n - k, k)$  such cases.

■

## Lecture 7

**Definition 3.2.5 (Ferrers diagram).** Visual representation of  $\lambda \vdash n$ . Each  $\lambda_i$  pictured as a row of  $\lambda_i$  dots.

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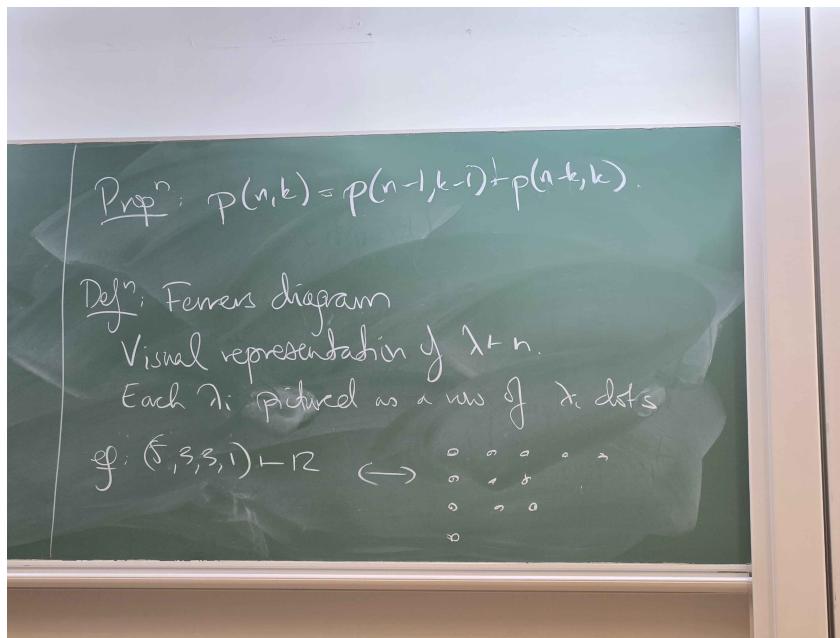


Figure 3.2: Ferrers diagram

**Note 3.2.2.** If we see the Ferrers diagram from the columns, then note that the number of dots in the columns is decreasing.

**Definition 3.2.6.** Given a partition  $\lambda \vdash n$ , the conjugate partition  $\lambda^* \vdash n$  is given by

$$\lambda_j^* = |\{i : \lambda_i \geq j\}|.$$

Visually,  $\lambda^*$  is the partition obtained by reflecting  $\lambda$  in the diagonal  $y = -x$ .

Observe that  $\lambda^*$  is indeed a partition of  $n$ :

$$\lambda_1^* \geq \lambda_2^* \geq \dots$$

is obvious from the definition, and

$$\sum_j \lambda_j^* = \sum_j |\{i : \lambda_i \geq j\}| = \sum_i \lambda_i = n.$$

Also, note that  $(\lambda^*)^* = \lambda$ .

**Proposition 3.2.4.** The number of partition of  $n$  into at most  $k$  parts = The number of partitions of  $n$  into parts of size  $\leq k$ .

**Proof.** The largest part of  $\lambda$  is the number of parts in  $\lambda^*$ . And so conjugation gives a bijection between these two choices of partition of  $n$ . ■

**Definition 3.2.7.** A partition  $\lambda \vdash n$  is called self-conjugate if  $\lambda^* = \lambda$ .

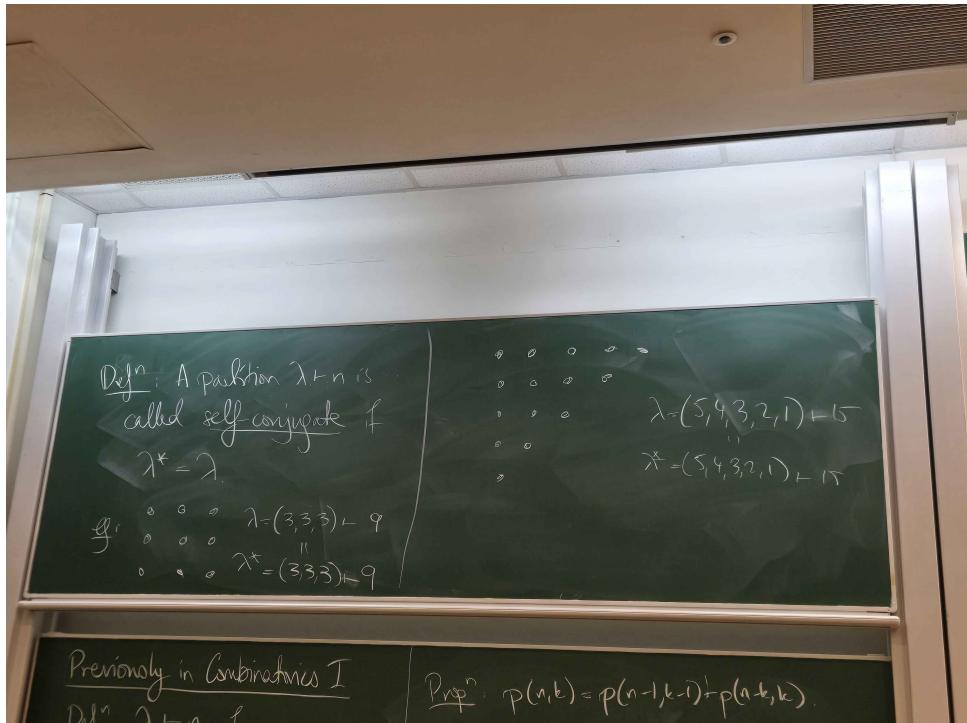


Figure 3.3: Self-conjugate

**Proposition 3.2.5.** The number of self-conjugate partition of  $n$  is the number of partition of  $n$  into distinct odd parts, which means

$$(\lambda_1, \lambda_2, \dots, \lambda_k) : \lambda_1 > \lambda_2 > \dots > \lambda_k \geq 1, \quad \forall 1 \leq i \leq k, \quad \lambda_i \equiv 1 \pmod{2}.$$

**Proof.** Let  $\lambda$  be a self-conjugate partition. (See Figure 3.4) If we consider the dots in the first row or column (we called it a hook), since  $\lambda = \lambda^*$ , we have  $2\lambda_1 - 1$  dots, which is an odd part. If we take the  $i$ -th part of the new partition to be the points in the  $i$ -th row or  $i$ -th column not-yet counted, then we get

$$(\lambda_i - (i-1)) + (\lambda_i - (i-1)) - 1,$$

say  $\mu_i = 2(\lambda_i - (i-1)) - 1$ , then  $\mu \vdash n$  and

$$\begin{aligned} \mu_{i+1} &= 2\lambda_{i+1} - 2(i+1) + 1 \\ &\leq 2\lambda_i - 2(i+1) + 1 \\ &< 2\lambda_i - 2i + 1 = \mu_i, \end{aligned}$$

so  $\mu$  has distinct parts and clearly  $\mu_i$  is odd for all  $i$ . Hence, we have mapped our self-conjugate  $\lambda$  into a partition  $\mu$  with distinct odd parts. This is indeed a bijection.

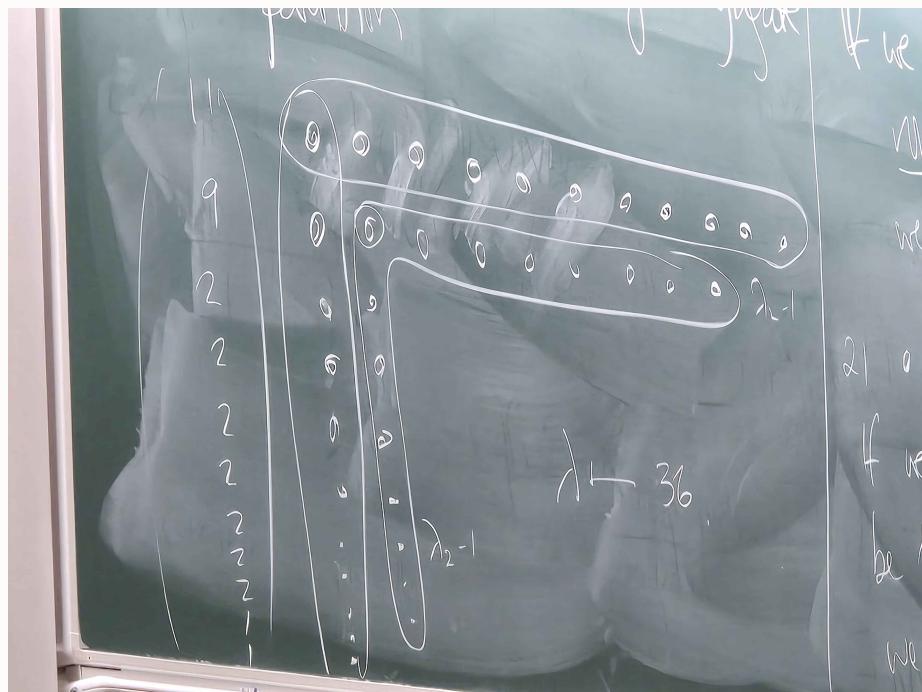


Figure 3.4: Use hook to obtain bijection

Examples:		Self-conjugate	Distinct odd parts	#
$n = 1$		✓		
$n = 2$		X, 0 X	X, 0 X	1
$n = 3$		X, 1, 0 X	X, 1, 0 X	0
$n = 4$		X, 1, X, 1, X	X, 1, X, 1, X	1

$\equiv 1 \pmod{2}$

Figure 3.5: Some cases of small  $n$ .

**Example 3.2.5.** Square partition  $\lambda = \underbrace{(k, k, \dots, k)}_{k \text{ parts}} \vdash k^2$  are self conjugate.

**Corollary 3.2.4.** The sum of the first  $k$  odd numbers is  $k^2$ .

**Proof.** By drawing hooks, it is trivial.



Figure 3.6: Drawing hooks to get the first  $k$  odd numbers from a square

(\*)

### 3.3 The twelvefold way of Counting

**Question.** How many ways can we partition  $n$  items into  $k$  groups?

Items	Groups	Partition
numbered indistinguishable	numbered indistinguishable	injective(group of size $\leq 1$ ) surjective(group of size $\geq 1$ ) arbitrary

Table 3.1: All types of partition problem.

	Injective	Surjective	Arbitrary
Items, groups numbered	$k^n$	$S(n, k) \cdot k!$	$k^n$
Items numbered, groups not	$\begin{cases} 1, & \text{if } k \geq n; \\ 0, & \text{if } k < n. \end{cases}$	$S(n, k)$	$\sum_{j=0}^k S(n, j)$
Items not, groups numbered	$\binom{k}{n}$	$\binom{n-1}{k-1}$	$\binom{n+k-1}{k-1}$
Items, groups not numbered	$\begin{cases} 1, & \text{if } k \geq n; \\ 0, & \text{if } k < n. \end{cases}$	$p(n, k)$	$\sum_{j=0}^k p(n, j)$

Table 3.2: All solution to all kinds of partition problem

# Chapter 4

## Generating Functions



Figure 4.1: Din Tai Fung branches number

We have a recurrence relation:  $\forall n \geq 2$

$$F_n = F_{n-1} + F_{n-2}$$

**Example 4.0.1.** If

$$F'_n = F'_{n-1} + F'_{n-2},$$

then  $F'_n = 2^n F'_0$ .

Suppose  $\{F_n\}_{n=0}^{\infty}$  is a recurring sequence with  $F_0 = 0, F_1 = 1$ , then we can define a power series as

$$F(x) = F_0 + F_1x + F_2x^2 + \cdots = \sum_{n=0}^{\infty} F_n x^n.$$

Thus, we have

$$xF(x) = F_0x + F_1x^2 + \cdots = \sum_{n=0}^{\infty} F_n x^{n+1} = \sum_{n=1}^{\infty} F_{n-1} x^n.$$

If we do it again, then we can get

$$x^2 F(x) = F_0 x^2 + F_1 x^3 + \cdots = \sum_{n=0}^{\infty} F_n x^{n+2} = \sum_{n=2}^{\infty} F_{n-2} x^n.$$

Now we have

$$F(x) - xF(x) - x^2 F(x) = F_0 x^0 + F_1 x^1 - F_0 x^1 + \sum_{n=2}^{\infty} \underbrace{(F_n - F_{n-1} - F_{n-2})}_{=0} x^n = x.$$

Hence,  $(1 - x - x^2)F(x) = x$ , and thus

$$F(x) = \frac{x}{1 - x - x^2} = \frac{A}{1 - \alpha_1 x} + \frac{B}{1 - \alpha_2 x}.$$

Now we solve the  $A, B, \alpha_1, \alpha_2$ .

$$\begin{aligned} \frac{A}{1 - \alpha_1} + \frac{B}{1 - \alpha_2} &= \frac{A(1 - \alpha_2 x) + B(1 - \alpha_1 x)}{(1 - \alpha_1 x)(1 - \alpha_2 x)} \\ &= \frac{(A + B) - (A\alpha_2 + B\alpha_1)x}{1 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2 x^2} = \frac{x}{1 - x - x^2}. \end{aligned}$$

Hence, we want

$$\begin{cases} A + B = 0 \\ A\alpha_2 + B\alpha_1 = -1 \\ \alpha_1 + \alpha_2 = 1 \\ \alpha_1 \alpha_2 = -1 \end{cases},$$

by solving  $\alpha_1, \alpha_2$  first, we can get  $\alpha_1 = \frac{1+\sqrt{5}}{2}$  and  $\alpha_2 = \frac{1-\sqrt{5}}{2}$ , and thus we can solve  $A = \frac{1}{\sqrt{5}}$  and  $B = -\frac{1}{\sqrt{5}}$ . Hence, we have

$$F(x) = \frac{x}{1 - x - x^2} = \frac{\frac{1}{\sqrt{5}}}{1 - \left(\frac{1+\sqrt{5}}{2}\right)x} - \frac{\frac{1}{\sqrt{5}}}{1 - \left(\frac{1-\sqrt{5}}{2}\right)x}.$$

Now since we know

$$\frac{1}{1 - \alpha} = 1 + \alpha + \alpha^2 + \dots,$$

so we can get

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}} \left( \left( 1 + \left( \frac{1+\sqrt{5}}{2} \right)x + \left( \left( \frac{1+\sqrt{5}}{2} \right)x \right)^2 + \dots \right) - \left( 1 + \left( \frac{1-\sqrt{5}}{2}x + \left( \left( \frac{1-\sqrt{5}}{2} \right)x \right)^2 + \dots \right) \right) \right) \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right) x^n = \sum_{n=0}^{\infty} F_n x^n. \end{aligned}$$

Hence, we have

$$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right).$$

## Lecture 8

Observe that

$$\left| \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n \right| < \frac{1}{2}.$$

Hence,  $F_n$  is the integer closed to

$$\frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n.$$

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The idea is to encode a sequence of numbers

$$a_0, a_1, a_2, \dots$$

as coefficients in a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

**Proposition 4.0.1.** Let  $(a_0, a_1, \dots)$  be a sequence of real numbers. If  $|a_n| < K^n$  for all  $n \in \mathbb{N}$ , then

$$\forall x \in \left(-\frac{1}{K}, \frac{1}{K}\right), \text{ we have } A(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges absolutely.

**Proof.** Suppose  $x \in \left(-\frac{1}{K}, \frac{1}{K}\right)$ , then

$$A(x) = \sum_{n=0}^{\infty} |a_n x^n| \leq \sum_{n=0}^{\infty} |K^n x^n| = \sum_{n=0}^{\infty} (|Kx|)^n,$$

which is a geometric series, and since  $|Kx| < 1$ , so it converges. ■

$A(x)$  has derivatives of all orders at  $x = 0$ , and for all  $n \geq 0$ ,

$$A^{(n)}(0) = a_n n!.$$

In particular, the values of  $A(x)$  around the origin determine this sequence  $(a_n)$  uniquely. We treat  $A(x)$  as a formal power series. Thus, we can usually easily verify results using induction.

**Definition 4.0.1.** Given a sequence  $(a_0, a_1, \dots)$  of real numbers, the generating function of the sequence is the (formal) power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

**Example 4.0.2.** Suppose we have a sequence  $(1, 1, 1, \dots)$ , then

$$A(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges for  $|x| < 1$ .

**Example 4.0.3.** Suppose we have a sequence  $(0, 1, \frac{1}{2}, \dots)$ , then

$$A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

converges for  $|x| < 1$ .

**Example 4.0.4.** Suppose we have a sequence  $(1, 1, \frac{1}{2}, \dots, \frac{1}{n!}, \dots)$ , then

$$A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

converges for all  $x \in \mathbb{R}$ .

**Example 4.0.5.** Suppose  $r$  is a fixed number and we have a sequence

$$\left( \binom{r}{0}, \binom{r}{1}, \dots \right),$$

then

$$A(x) = \sum_{n=0}^{\infty} \binom{r}{n} x^n = (1+x)^r.$$

converges for  $|x| < 1$ .

**Remark 4.0.1.** The special case:

$$\begin{aligned} \frac{1}{(1-x)^t} &= (1-x)^{-t} = \sum_{n=0}^{\infty} \binom{-t}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{-t}{n} (-1)^n x^n \\ &= \sum_{n=0}^{\infty} \binom{t+n-1}{n} x^n \end{aligned}$$

since

$$(-1)^n \binom{-t}{n} = (-1)^n \frac{(-t)(-t-1)\dots(-t-n+1)}{n!} = \frac{t(t+1)\dots(t+n-1)}{n!} = \binom{t+n-1}{n}.$$

## 4.1 Dictionary for operations

- Sum:

$$\begin{aligned} A(x) &\sim (a_0, a_1, \dots) \\ B(x) &\sim (b_0, b_1, \dots) \\ A(x) + B(x) &\sim (a_0 + b_0, a_1 + b_1, \dots) \end{aligned}$$

- Scalar multiplication:

$$\begin{aligned} A(x) &\sim (a_0, a_1, \dots) \\ \lambda A(x) &\sim (\lambda a_0, \lambda a_1, \dots) \quad \forall \lambda > 0. \end{aligned}$$

- Shifting to the right:

$$\begin{aligned} (a_0, a_1, \dots) &\sim \sum_{n=0}^{\infty} a_n x^n \\ (0, a_0, a_1, \dots) &\sim \sum_{n=1}^{\infty} a_{n-1} x^n = x \sum_{n=0}^{\infty} a_n x^n \\ A(x) &\rightarrow xA(x) \end{aligned}$$

**Note 4.1.1.** By repeating shifting to the right, we can get

$$x^k A(x) \sim (\underbrace{0, 0, \dots, 0}_k, a_0, a_1, \dots).$$

- Shifting to the left:

$$(a_0, a_1, \dots) \sim \sum_{n=0}^{\infty} a_n x^n$$

$$(a_1, a_2, \dots) \sim \sum_{n=1}^{\infty} a_n x^{n-1} = \frac{A(x) - a_0}{x}.$$

**Note 4.1.2.** By repeating

$$\frac{A(x) - a_0 - a_1 x - \cdots - a_{k-1} x^{k-1}}{x^k},$$

we can shift to the left by  $k$  terms.

- Substituting  $\lambda x$  for  $x$  with some  $\lambda \in \mathbb{R}$ .

$$A(\lambda x) = \sum_{n=0}^{\infty} a_n (\lambda x)^n = \sum_{n=0}^{\infty} (a_n \lambda^n) x^n$$

and it corresponds to  $(a_0, \lambda a_1, \lambda^2 a_2, \dots)$ .

**Example 4.1.1.** Suppose we want  $(1, \lambda, \lambda^2, \dots)$ , then taking  $(1, 1, \dots)$  and substituting  $x$  by  $\lambda x$ , so we will change  $\frac{1}{1-x}$  to  $\frac{1}{1-\lambda x}$ , and this means change  $(1, 1, \dots)$  to  $(1, \lambda, \lambda^2, \dots)$ .

## Lecture 9

### 4.2 Recurrence relation

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#### 4.2.1 Linear homogeneous constant-coefficient recurrence relations

Suppose

$$a_n = \alpha_{k-1} a_{n-1} + \alpha_{k-2} a_{n-2} + \cdots + \alpha_1 a_{n-k+1} + \alpha_0 a_{n-k} \quad (4.1)$$

holds for all  $n \geq k$  and we have initial conditions  $a_0, a_1, \dots, a_{k-1}$ . Then if we define the generating function:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then we have

$$\begin{aligned} \alpha_{k-1} x A(x) &= \sum_{n=1}^{\infty} \alpha_{k-1} a_{n-1} x^n \\ \alpha_{k-2} x^2 A(x) &= \sum_{n=2}^{\infty} \alpha_{k-2} a_{n-2} x^n \\ &\vdots \\ \alpha_0 x^k A(x) &= \sum_{n=k}^{\infty} \alpha_0 a_{n-k} x^n, \end{aligned}$$

so we have

$$\begin{aligned} A(x) [1 - \alpha_{k-1} x - \alpha_{k-2} x^2 - \cdots - \alpha_0 x^k] &= \sum_{n=k}^{\infty} (a_n - \alpha_{k-1} a_{n-1} - \cdots - \alpha_0 a_{n-k}) x^n + R(x) \\ &= R(x), \end{aligned}$$

where  $R(x)$  is a polynomial of degree  $k-1$  depending on coefficient  $\alpha_i$  and the initial terms  $a_0, a_1, \dots, a_{k-1}$ . Hence, we have

$$A(x) = \frac{R(x)}{1 - \alpha_{k-1}x - \alpha_{k-2}x^2 - \cdots - \alpha_0x^k}.$$

If

$$1 - \alpha_{k-1}x - \alpha_{k-2}x^2 - \cdots - \alpha_0x^k = (1 - \lambda_1x)(1 - \lambda_2x) \cdots (1 - \lambda_kx),$$

then we have

$$A(x) = \frac{A_1}{1 - \lambda_1x} + \frac{A_2}{1 - \lambda_2x} + \cdots + \frac{A_k}{1 - \lambda_kx}.$$

for some constants  $A_1, A_2, \dots, A_k$ , which means

$$a_n = A_1\lambda_1^n + A_2\lambda_2^n + \cdots + A_k\lambda_k^n$$

by comparing the  $n$ -th coefficient of  $A(x)$  and R.H.S.

**Definition 4.2.1.** Given the recurrence relation [Equation 4.1](#), then the characteristic polynomial is

$$p(z) = z^k - \alpha_{k-1}z^{k-1} - \alpha_{k-2}z^{k-2} - \cdots - \alpha_1z - \alpha_0.$$

If we let  $z = \frac{1}{x}$ , then multiplying

$$1 - \alpha_{k-1}x - \alpha_{k-2}x^2 - \cdots - \alpha_{k-1}x^{k-1} - \alpha_0x^k$$

by  $z^k$ , we have

$$z^k - \alpha_{k-1}z^{k-1} - \alpha_{k-2}z^{k-2} - \cdots - \alpha_1z - \alpha_0.$$

Hence,  $(1 - \lambda_1x)(1 - \lambda_2x) \cdots (1 - \lambda_kx)$  becomes  $(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_k)$  and thus

$$\{\lambda_i : 1 \leq i \leq k\}$$

are the roots of  $p(z)$ .

**Note 4.2.1.** This method is only true when  $\lambda_i \neq \lambda_j$  for all  $i \neq j$ .

**Question.** What if there is repeated root?

For example, if

$$p(z) = (z - \lambda_1)(z - \lambda_2)^2,$$

then

$$A(x) = \frac{A_1}{1 - \lambda_1x} + \frac{A_2 + A_3x}{(1 - \lambda_2x)^2}.$$

**Theorem 4.2.1.** Suppose a sequence is defined by

$$a_n = \alpha_{k-1}a_{n-1} + \cdots + \alpha_0a_{n-k} \quad \forall n \geq k$$

with initial conditions  $a_0, a_1, \dots, a_{k-1}$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be the roots of the characteristic polynomial  $p(z)$ .

(1) If the roots are distinct, then

$$a_n = \sum_{i=1}^k A_i \lambda_i^n$$

for constants  $A_1, A_2, \dots, A_k$  determined by  $a_0, \dots, a_{k-1}$ .

(2) If we have repeated roots, say

$$p(z) = (z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - x_q)^{k_q},$$

then

$$a_n = \sum_{i=1}^q \left( \sum_{j=0}^{k_i-1} C_{ij} n^j \right) \lambda_i^n.$$

## Lecture 10

### 4.3 Generating function operation

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- Substituting  $x^k$  for  $x$ , then

$$(a_0, a_1, \dots) \rightarrow \left( a_0, \underbrace{0, 0, \dots, 0}_k, \dots, \begin{cases} 0, & \text{if } k \nmid n; \\ a_{\frac{n}{k}}, & \text{if } k \mid n. \end{cases} \right)$$

since for  $A(x) = \sum_{i=0}^{\infty} a_i x^i$ , we have

$$A(x^k) = \sum_{i=0}^{\infty} a_i (x^k)^i = \sum_{i=0}^{\infty} a_i x^{ki}.$$

- Differentiation:

$$(a_0, a_1, \dots) \rightarrow (a_1, 2a_2, 3a_3, \dots).$$

- Integration:

$$(a_0, a_1, \dots) \rightarrow \left( 0, \frac{a_1}{1}, \frac{a_2}{2}, \dots \right)$$

since

$$\int_0^x A(t) dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} a_n \left[ \frac{t^{n+1}}{n+1} \right]_0^x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

**Example 4.3.1.** Find the generating functions for the sequences:

- (i)  $a_n = 2^{\lfloor \frac{n}{2} \rfloor}$
- (ii)  $a_n = (n+1)^2$ .

**Proof.**

- (i) Note that  $(a_i)_{i=0}^{\infty} = (1, 1, 2, 2, 4, 4, 8, 8, 16, 16, \dots)$ , and we can write it as

$$(1, 0, 2, 0, 4, 0, 8, 0, 16, 0, \dots) + (0, 1, 0, 2, 0, 4, 0, 8, 0, 16, \dots),$$

where the first term is  $B(x) = (1, 2, 4, 8, 16, \dots)$  spread by 2, which is  $B(x^2)$ , and the second term is  $xB(x^2)$ . Note that  $B(x) = \frac{1}{1-2x}$ .

- (ii) If  $b_n = n$  corresponds to  $B(x)$ , then

$$A(x) = \frac{dB(x)}{dx}$$

has  $a_n = (n+1)b_{n+1} = (n+1)^2$ . Also, if the sequence  $c_n = 1$  has generating function  $C(x) = \frac{1}{1-x}$ , then

$$\frac{d}{dx} C(x) \sim (1, 2, 3, \dots),$$

so

$$B(x) = x \frac{d}{dx} C(x),$$

and we have

$$A(x) = \left[ x \left( \frac{1}{1-x} \right)' \right]' = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}.$$

(\*)

## 4.4 Products of Generating Functions

Suppose  $A(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $B(x) = \sum_{n=0}^{\infty} b_n x^n$ ,

**Question.** What can we say about  $C(x) = A(x)B(x)$ ?

$$\begin{aligned} A(x)B(x) &= \left( \sum_{n=0}^{\infty} a_n x^n \right) \left( \sum_{n=0}^{\infty} b_n x^n \right) \\ &= \sum_{n,m \geq 0} a_n b_m x^{n+m} \\ &= \sum_{r \geq 0} \left( \sum_{\substack{n,m \geq 0 \\ n+m=r}} a_n b_m \right) x^r = \sum_{r \geq 0} \left( \sum_{n=0}^r a_n b_{r-n} \right) x^r \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) x^n \end{aligned}$$

i.e.  $C(x)$  corresponds to the sequence  $c_n = \sum_{k=0}^n a_k b_{n-k}$ , and we call it a convolution.

### Combinatorial interpretation

Suppose  $(a_n)$  represents the number of ways of completing task 1 with a budget of \$ $n$ , and  $(b_n)$  represents the number of ways of completing task 2 with a budget of \$ $n$ . Then the convolution  $c_n = \sum_{k=0}^n a_k b_{n-k}$  represents the number of ways of completing both task 1 and 2 with a combinatorial budget of \$ $n$ .

**Example 4.4.1.** Designing a physics course for  $n$  days, which has theoretical part including one midterm exam and it is followed by a practical part, which includes two experiments. This is a convolution of the sequences

$(a_n) = \#$  of ways of planning theory and  $(b_n) = \#$  of ways of planning practical.

Note that  $a_n = n$  and  $b_n = \binom{n}{2}$ , so

$$A(x) = x \left( \frac{1}{1-x} \right)' = \frac{x}{(1-x)^2} \quad B(x) = \frac{x^2}{(1-x)^3}$$

since  $b_n = \binom{n}{2} = \frac{n(n-1)}{2}$ . Hence,  $C(x) = \frac{x^3}{(1-x)^5}$  is the generating function for the sequence  $(c_n)$  where  $c_n$  is the number of ways of designing an  $n$ -day course. Now since

$$C(x) = \frac{x^3}{(1-x)^5} = x^3(1-x)^{-5}$$

and

$$(1-x)^{-5} = \sum_{n=0}^{\infty} \binom{-5}{n} (-x)^n,$$

and

$$\binom{-5}{n} = \frac{(-5)(-5-1)(-5-2)\dots(-5-(n-1))}{n!},$$

so we have

$$\binom{-5}{n}(-x)^n = \frac{5(5+1)\dots(5+(n-1))}{n!}x^n = \binom{n+4}{n}x^n,$$

and we have to shift it by 3, so

$$c_n = \binom{n+1}{n-3} = \binom{n+1}{4}.$$

**Remark 4.4.1.** If we think of putting 4 bars in  $n+1$  space, then it can be easily thought that  $c_n = \binom{n+1}{4}$ .

**Note 4.4.1.** We can think of there are  $n+1$  spaces, and the boundary between the theoretical part and the practical part is  $|$ , and the midterm of theoretical part is  $a$ , while the experiments of practical part are  $b_1, b_2$ , and the other symbols are  $*$ , which are some normal course days, then we can first choose 4 spaces out of  $n+1$  spaces, and put on  $a, |, b_1, b_2$  in order, then use  $*$  to fill all the other spaces. Note that this method corresponds to a way of designing the courses, so  $c_n = \binom{n+1}{4}$ .

## 4.5 Catalan Numbers

**Example 4.5.1.** Bank balance goes up by \$1 or down by \$1, and bank balance should always be  $\geq 0$ .

**Question.** Start with \$0. How many ways can we have \$0 after  $2n$  days?

**Answer.** Define  $c_n$  = number of ways of having \$0 after  $2n$  days. Consider the first time we have \$0. Suppose it happens on Day  $2k$ ,  $k \geq 1$ , then  $c_{n-k}$  ways proceeding from Day  $2k$  to Day  $2n$ , where  $c_0 = 1$ . For the initial period: Since we know during Day 1 to Day  $2k-1$ , we've never been to \$0, and in Day 1, we should have +1, and in Day  $2k$ , we must have -1, so in between, we must have at least \$1, so it is Dyck path from Day 1 to Day  $2k-1$  with axis  $y=1$ . Hence, by the sum rule,

$$c_n = \sum_{k=1}^n c_{k-1}c_{n-k} \quad \forall n \geq 1 \text{ with } c_0 = 1.$$



(\*)

$$\begin{aligned}
 C_0 &= 1 \\
 C_1 &= \sum_{k=1}^1 C_{k-1} C_{1-k} = C_0 C_0 = 1 \\
 C_2 &= C_0 C_1 + C_1 C_0 = 1 \cdot 1 + 1 \cdot 1 = 2 \\
 C_3 &= C_0 C_2 + C_1 C_1 + C_2 C_0 = 5
 \end{aligned}$$

Diagrams illustrating the paths for \$C\_1, C\_2, C\_3\$:

Figure 4.2: Example of  $c_i$ .

## Lecture 11

As previously seen. Suppose

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$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad B(x) = \sum_{n=0}^{\infty} b_n x^n,$$

and

$$C(x) = A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n,$$

then  $c_n = \sum_{k=0}^n a_k b_{n-k}$ . We call this a convolution.

**As previously seen.** Catalan numbers

$$c_n = \left| \left\{ (b_1, b_2, \dots, b_{2n}) \in \{-1, 1\}^{2n} : \sum_{i=1}^{2n} b_i = 0, \sum_{j=1}^i b_j \geq 0 \quad \forall j \right\} \right|.$$

If we count recursively, based on first returned to 0 (On Day  $2k$ ). Then, from Day  $2k$  to Day  $2n$ , we start at \$0 and end at \$0, and never drop below to \$0, so there are  $c_{n-k}$  ways. Also, in the first  $2k$  days, it starts at \$0 and ends at \$0. Hence, from Day 1 to Day  $2k-1$ , it always has at least \$1, and thus there is a bijection, which means there are  $c_{k-1}$  probabilities. Therefore, by the product and sum rules,

$$c_n = \sum_{k=1}^n c_{k-1} c_{n-k},$$

where  $c_0 = 1$ .

Define  $v_n$  to be the number of very Catalan sequences, which is the sequence of length  $2n$  that start with +1 and end at \$0, and never drop below \$1 in between. Then,

$$v_n = \begin{cases} 0, & \text{if } n = 0; \\ c_{n-1}, & \text{if } n \geq 1. \end{cases}$$

Then,

$$c_n = \sum_{k=1}^n c_{k-1} c_{n-k} = \sum_{k=1}^n v_k c_{n-k} = \sum_{k=0}^n v_k c_{n-k},$$

so  $c_n$  is the convolution of  $(v_n)$  and  $(c_n)$ . Thus, if

$$C(x) = \sum_{n=0}^{\infty} c_n x^n, \quad V(x) = \sum_{n=0}^{\infty} v_n x^n,$$

then

$$C(x)V(x) = \sum_{n=0}^{\infty} \sum_{k=0}^n v_k c_{n-k} x^n = \sum_{n=0}^{\infty} c_n x^n - 1 = C(x) - 1.$$

since

$$\sum_{k=0}^n v_k c_{n-k} = \begin{cases} c_0 - 1, & \text{if } n = 0; \\ c_n, & \text{if } n \geq 1. \end{cases}$$

Since

$$v_n = \begin{cases} 0, & \text{if } n = 0; \\ c_{n-1}, & \text{if } n \geq 1, \end{cases}$$

so  $V(x) = xC(x)$ . Hence,

$$(xC(x))C(x) = C(x) - 1 \Rightarrow xC(x)^2 - C(x) + 1 = 0.$$

Hence, we have

$$C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

However, which one is correct? Note that

$$\lim_{x \rightarrow 0} C(x) = C(0) = 1,$$

and

$$\lim_{x \rightarrow 0} \frac{1 + \sqrt{1 - 4x}}{2x} = \text{D.N.E.} \quad \lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - 4x}}{2x} = 1.$$

Besides,

$$\sqrt{1 - 4x} = (1 - 4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n = \underbrace{1}_{n=0} + \underbrace{\frac{1}{2}(-4)x}_{n=1} + \underbrace{\left(-\frac{1}{8}\right)(16)x^2}_{n=2} + \dots$$

Thus, the coefficient of  $x^n$  in  $(1 - 4x)^{\frac{1}{2}}$  is

$$\begin{aligned} \binom{\frac{1}{2}}{n}(-4)^n &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-(n-1))}{n!}(-4)^n \\ &= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2n-3}{2})}{n!}(-4)^n \\ &= \frac{-(2n-3)!!}{n!}2^n \\ &= -\frac{(2n-2)!}{n!(n-1)!2^{n-1}}2^n \\ &= (-2)\frac{(2n-2)!}{n!(n-1)!}. \end{aligned}$$

Thus, coefficient of  $x^n$  in  $C(x)$  ( $n \geq 1$ ) is

$$-\frac{1}{2} \times \left\{ \text{coefficient of } x^{n+1} \text{ in } (1 - 4x)^{\frac{1}{2}} \right\},$$

which means

$$c_n = -\frac{1}{2}(-2)\frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}.$$

**Question.** How many ways are there of starting from \$0 and going  $+/- \$1$  each day and ending at \$0 after  $2n$  days if we can go below \$0 in between?

**Answer.**  $\binom{2n}{n}$  since we just have to make sure  $+\$1$  is as many as  $-\$1$ . ⊗

### *k*-wise products of generating functions

Suppose we have sequences

$$a_0^{(j)}, a_1^{(j)}, \dots, a_n^{(j)}, \dots$$

enumerating the number of ways we can carry out Task  $j$  with a budget of  $\$n$  with  $1 \leq j \leq k$ . Let

$$A^{(j)}(x) = \sum_{n=0}^{\infty} a_n^{(j)} x^n$$

be the generating functions. Then, what does

$$A(x) = \prod_{j=1}^k A^{(j)}(x)$$

represent?

The answer is: If  $A(x) = \sum_{n=0}^{\infty} a_n x^n$ , then  $a_n$  is the number of ways of taking a budget of  $\$n$  splitting it between the  $k$  tasks, and carrying out each task with its budget since

$$a_n = \sum_{\substack{l_1, l_2, \dots, l_k \\ l_j \geq 0, \sum_{j=1}^k l_j = n}} \prod_{j=1}^k a_{l_j}^{(j)}.$$

### Composition of generating functions

If  $A(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $B(x) = \sum_{n=0}^{\infty} b_n x^n$ , then what does  $B(A(x))$  represent? Suppose

$$B(A(x)) = \sum_{n=0}^{\infty} b_n (A(x))^n = \sum_{n=0}^{\infty} c_n x^n,$$

then

$$c_0 = \underbrace{b_0}_{n=0} + \underbrace{b_1 a_0}_{n=1} + \underbrace{b_2 a_0^2}_{n=2} + \cdots = \sum_{n=0}^{\infty} b_n a_0^n.$$

In order for this to be finite, we require  $a_0 = 0$ . If  $a_0 = 0$ , then computing any coefficient  $c_n$  in  $C(x) = B(A(x))$  is a finite computation, so it is a well-defined power series.

**Example 4.5.2.**  $A(x) = \lambda x$  or  $A(x) = x^k$ .

A special case is that if  $B(x) = \frac{1}{1-x}$ , then  $b_n = 1$  for all  $n$ , and

$$B(A(x)) = \sum_{n=0}^{\infty} A(x)^n.$$

**Claim 4.5.1.** If  $a_n$  is the number of ways of carrying out a task with a budget of \$n with  $a_0 = 0$ , then  $\frac{1}{1-A(x)}$  is the generating function for the number of ways of carrying out the task any number of times with a total budget of \$n.

**Example 4.5.3.** Suppose we have an army with  $n$  (identical) soldiers. How many ways can we divide the soldiers into units, and pick a captain of each unit?

**Proof.** Suppose  $a_n$  is the number of ways of picking a captain from a unit of  $n$  soldiers, then we know  $a_n = n$ , so we know  $A(x)$  can be obtained by differentiating  $\frac{1}{1-x}$  and shifting it to the right by 1 unit, so  $A(x) = \frac{x}{(1-x)^2}$ . Now if  $F(x)$  is the generating function for forming units and then picking a captain for each, then

$$F(x) = \frac{1}{1-A(x)} = \frac{1-2x+x^2}{1-3x+x^2} = 1 + \frac{x}{1-3x+x^2} = 1 + \frac{\alpha}{1-\lambda_1 x} + \frac{\beta}{1-\lambda_2 x},$$

where  $(1-\lambda_1 x)(1-\lambda_2 x) = 1-3x+x^2$ , and  $\alpha, \beta$  are chosen appropriately. By solving it, we know

$$F(x) = 1 + \frac{\frac{1}{\sqrt{5}}}{1 - \left(\frac{\sqrt{5}+3}{2}\right)x} + \frac{-\frac{1}{\sqrt{5}}}{1 - \left(\frac{-\sqrt{5}+3}{2}\right)x},$$

so

$$f_n = \frac{1}{\sqrt{5}} \left( \left( \frac{3+\sqrt{5}}{2} \right)^n - \left( \frac{3-\sqrt{5}}{2} \right)^n \right)$$

for all  $n \geq 1$  and  $f_0 = 1$ . (\*)

## Lecture 12

**As previously seen.** If  $A_i(x)$  is the generating function for the number of ways of completing Task  $i$  with a budget of size  $n$ , then  $\prod_{i=1}^k A_i(x)$  is the generating function for the number of ways of taking a budget of size  $n$  and

- (1) splitting it arbitrary into  $k$  parts.
- (2) using the  $i$ -th part to carry out Task  $i$ .

There is one special case  $A(x)^k$ , which is the number of ways that carrying a task  $k$  times with total budget of size  $n$ .

Note that

$$B(A(x)) = \sum_{k=0}^{\infty} b_k A(x)^k = \sum_{k=0}^{\infty} b_k \left( \sum_{n=0}^{\infty} a_n x^n \right)^k,$$

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and we need  $a_0 = 0$  for this to be well-defined. There is a special case:  $b_n = 1$ , which means  $B(x) = \frac{1}{1-x}$ , so

$$B(A(x)) = \frac{1}{1 - A(x)} = \sum_{k=0}^{\infty} A(x)^k,$$

where the coefficient of  $x^n$  is the number of ways of carrying out a task on arbitrary number of times with a budget of  $n$ .

The general case:  $B(A(x)) = \sum_{k=0}^{\infty} b_k A(x)^k$ , where  $a_n$  is the number of ways carrying out a task with a budget of  $n$ , while  $b_k$  is the number of ways carrying out a second task on a set of  $k$  items. Thus, the coefficient of  $x^n$  in  $B(A(x))$  counts the number of ways to take a budget of  $n$ , and

- (1) carry out the first task an arbitrary number of times, and
- (2) carry out the second task on the outcomes from (1).

**Example 4.5.4 (Designing Labubus).** We have  $n$  minutes to design new Labubus, and then decide which new models to discard, sell, or sell as premium models.

**Proof.** We have two types of tasks.

- (1)  $A(x)$ : Designing of new models in  $n$  minutes.
- (2)  $B(x)$ : Classification of designs as goodbye/okay/premium.

Thus,

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where  $a_n$  is the number of ways of designing a new Labubus in  $n$  minutes. Every minute we can change a different feature or not. Hence, there are  $2^n$  options in  $n$  minutes but cannot have no changes, so  $a_n = 2^n - 1$ , where  $a_0 = 0$ , so it is well-defined. Note that  $2^n \leftrightarrow \frac{1}{1-2x}$  and  $1 \leftrightarrow \frac{1}{1-x}$ , so

$$2^n - 1 \leftrightarrow \frac{1}{1-2x} - \frac{1}{1-x} = A(x) \Rightarrow A(x) = \frac{x}{1-3x+2x^2}.$$

Also, we know  $B(x) = \sum_{k=0}^{\infty} b_k x^k$ , so  $b_k = 3^k$  and thus  $B(x) = \frac{1}{1-3x}$ , so

$$B(A(x)) = \frac{1}{1-3A(x)} = 1 + \frac{3x}{1-6x+2x^2} = 1 + \frac{\alpha}{1-\lambda_1 x} + \frac{\beta}{1-\lambda_2 x},$$

we can solve that  $\lambda_{1,2} = 3 \pm \sqrt{7}$  since  $\lambda_1, \lambda_2$  are the roots of the characteristic polynomial  $z^2 - 6z + 2$ . By solving  $\alpha, \beta$ , we can get

$$B(A(x)) = 1 + \frac{3}{2\sqrt{7}} \left( \frac{1}{1-(3+\sqrt{7})x} - \frac{1}{1-(3-\sqrt{7})x} \right),$$

so the coefficient of  $x^n$  is

$$f_n := \begin{cases} 1, & \text{if } n = 0; \\ \frac{3}{2\sqrt{7}} \left( (3+\sqrt{7})^n - (3-\sqrt{7})^n \right), & \text{if } n \geq 1. \end{cases}$$

(\*)

# Appendix