Introduction to Analysis I HW2

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Problem 0.0.1 (11pts). If (X, d) is a metric space, define

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that d' is also a metric on X.
Note that $0 \leq d'(x, y) < 1$ for all $x, y \in X$.

Proof. In the first three properties we are going to check, they are all true since we can directly these properties on d to conclude that these properties are also true on d'.

- We know $d'(x, x) = \frac{d(x,x)}{1+d(x,x)} = 0$ for every $x \in X$.
- For every distinct $x, y \in X$, we have

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} > 0.$$

- For any $x, y \in X$, we have $d'(x, y) = d'(y, x)$, which is trivial.
- For any $x, y, z \in X$, suppose

$$a = d(x, z) \quad b = d(x, y) \quad c = d(y, z),$$

we want to show that

$$\frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c},$$

where we know $a, b, c \geq 0$ and $a \leq b + c$. By directly computing, we know it is equivalent to

$$\begin{aligned} & a(1 + b)(1 + c) \leq (1 + a)(1 + c)b + (1 + a)(1 + b)c \\ & \Leftrightarrow a(1 + b + c + bc) \leq (1 + a + c + ac)b + (1 + a + b + ab)c \\ & \Leftrightarrow a \leq b(1 + c) + c(1 + b + ab) = b + c + 2bc + abc. \end{aligned}$$

Hence, we know this inequality holds because we know $a, b, c \geq 0$.

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Problem 0.0.2 ((12 pts) exercise 1.2.4). Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball

$$B := B(x_0, r) = \{x \in X : d(x, x_0) < r\},$$

and let C be the closed ball

$$C := \{x \in X : d(x, x_0) \leq r\}.$$

(a) Show that $\overline{B} \subseteq C$.

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B} \neq C$.

Proof.

(a) For all $b \in \overline{B}$, we know for all $r' > 0$, we have $B(b, r') \cap B(x_0, r) \neq \varnothing$. Now if $d(b, x_0) > r$, say $\varepsilon = d(b, x_0) - r > 0$. Suppose $z \in B(b, \varepsilon)$, we have

$$\begin{aligned} d(z, x_0) &\geq d(b, x_0) - d(z, b) \\ &> d(b, x_0) - \varepsilon = r \end{aligned}$$

by triangle inequality. However, this means $z \notin B(x_0, r)$. Hence, $B(b, \varepsilon) \cap B(x_0, r) = \varnothing$, which is a contradiction. By this, we know $d(b, x_0) \leq r$ for all $b \in \overline{B}$, so $\overline{B} \subseteq C$.

(b) Suppose the metric space is $(\mathbb{R}, d_{\text{disc}})$, where d_{disc} is the discrete metric defined by

$$d_{\text{disc}} = \begin{cases} 1, & \text{if } x \neq y; \\ 0, & \text{if } x = y, \end{cases}$$

and suppose $x_0 = 0$ and $r = 1$ Thus, we know $\overline{B} = B \cup \partial B$, but notice that

$$B = \{x \in X \mid d(x, 0) < 1\} = \{0\},$$

and $\partial B = \varnothing$ since for all $x \neq 0$, we know

$$B\left(x, \frac{1}{2}\right) = \{x\} \subseteq X \setminus B(0, 1),$$

so we know $\text{Ext}(B) = \mathbb{R} \setminus \{0\}$. Also, we know $\text{Int}(B) = \{0\}$ since $B(0, 1) \subseteq B$ and $\text{Ext}(B) \cap \text{Int}(B) = \varnothing$, so $\partial B = \varnothing$. Now we know $\overline{B} = B \cup \partial B = \{0\}$, but

$$C = \{x \in X \mid d(x, 0) \leq 1\} = \mathbb{R},$$

so $\overline{B} \neq C$.

$\blacksquare$

Problem 0.0.3 (21pts). Two metrics d_1 and d_2 on a set X are said to be *Lipschitz equivalent* if there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 d_2(x, y) \leq d_1(x, y) \leq C_2 d_2(x, y) \quad \text{for all } x, y \in X.$$

Let $E \subset X$.

(a) Prove that E is open in (X, d_1) if and only if E is open in (X, d_2).

(b) Prove that E is closed in (X, d_1) if and only if E is closed in (X, d_2).

(c) Two metrics d_1 and d_2 on a set X are said to be *topologically equivalent* if they induce the same topology on X. That is, a set $U \subset X$ is open in (X, d_1) if and only if it is open in (X, d_2). Give examples of topologically equivalent metrics that are not Lipschitz equivalent.

Proof. In the following text, if we write $\text{Int}_1, \text{Int}_2, B_1, B_2$, then the number of the subscript means it is under which metric. For example, $\text{Int}_1(E)$ means the interior points of E in (X, d_1), and the others are similarly defined.

(a) ($\Rightarrow$) If E is open in (X, d_1), then we know $E = \text{Int}_1(E)$. Thus, $\forall x_0 \in E, \exists r > 0$ s.t.

$$B_1(x_0, r) = \{x \in X \mid d_1(x, x_0) < r\} \subseteq E.$$

However, it means for all $x_0 \in E$, we know

$$B_2\left(x_0, \frac{r}{C_2}\right) = \left\{x \in X \mid d_2(x, x_0) < \frac{r}{C_2}\right\} \subseteq B_1(x_0, r) \subseteq E$$

because for all $x \in B_2\left(x_0, \frac{r}{C_2}\right)$, we have $d_2(x, x_0) < \frac{r}{C_2}$, so it must have $d_1(x, x_0) < r$ since

$$d_1(x, x_0) \leq C_2 d_2(x, x_0) < r.$$

Hence, we have $E \subseteq \text{Int}_2(E)$.
Also, for every $x \in \text{Int}_2(E)$, we know there exists $r > 0$ s.t. $B_2(x, r) \subseteq E$, and also $x \in B_2(x, r)$, so $x \in E$, which means $\text{Int}_2(E) \subseteq E$.
Hence, we have $\text{Int}_2(E) = E$, which means E is open in (X, d_2).

($\Leftarrow$) Since we know

$$\frac{1}{C_2}d_1(x,y) \le d_2(x,y) \le \frac{1}{C_1}d_1(x,y) \quad \forall x,y \in X,$$

so we can just use the same method in the ($\Rightarrow$)'s proof to prove ($\Leftarrow$) direction.

(b)

$$\begin{aligned} E \text{ is closed in } (X,d_1) &\Leftrightarrow X \setminus E \text{ is open in } (X,d_1) \\ &\Leftrightarrow X \setminus E \text{ is open in } (X,d_2) \quad \text{(by (a))} \\ &\Leftrightarrow E \text{ is closed in } (X,d_2). \end{aligned}$$

(c) For $X = \mathbb{R}$, $d_1 = |x-y|$, and $d_2 = \frac{d_1}{1+d_1}$, we claim that d_1 and d_2 are not Lipschitz equivalent and are topologically equivalent.

> **Note 0.0.1.** In the course, we have shown that d_1 is a metric, and in Problem 0.0.1 we have shown that d_2 is a metric.

> **Claim 0.0.1.** d_1 and d_2 are not Lipschitz equivalent.
>
> **Proof.** Note that $d_1(x,y)$ can be arbitraty large in $\mathbb{R}$ and $d_2(x,y) < 1$ for any $x,y \in \mathbb{R}$, so there does not exist a constant c s.t. $d_1(x,y) < cd_2(x,y)$, which means d_1 and d_2 are not Lipschitz equivalent. ⊛

Now we show that a set $U \subseteq \mathbb{R}$ is open in $(\mathbb{R}, d_1)$ if and only if U is open in $(\mathbb{R}, d_2)$.

First notice that

$$d_2(x,y) = \frac{d_1(x,y)}{1+d_1(x,y)} \Leftrightarrow d_1(x,y) = \frac{d_2(x,y)}{1-d_2(x,y)}.$$

($\Rightarrow$) If U is open in $(\mathbb{R}, d_1)$, then for all $u \in U$, there exists $r > 0$ s.t.

$$B_1(u,r) = \{x \in X \mid d_1(x,u) < r\} \subseteq X.$$

Also, we know

$$d_1(x,u) < r \Leftrightarrow \frac{d_2(x,u)}{1-d_2(x,u)} < r \Leftrightarrow d_2(x,u) < \frac{r}{1+r}.$$

Thus, we know in $(\mathbb{R}, d_2)$, for all $u \in U$, there exists $\frac{r}{1+r} > 0$ s.t.

$$B_2\left(u, \frac{r}{1+r}\right) = \left\{x \in X \mid d_2(x,u) < \frac{r}{1+r}\right\} \subseteq X,$$

which means $\mathrm{Int}_2(U) = U$ and thus U is open in $(\mathbb{R}, d_2)$.

($\Leftarrow$) If U is open in $(\mathbb{R}, d_2)$, then for all $u \in U$, there exists $r > 0$ s.t.

$$B_2(u,r) = \{x \in X \mid d_2(x,u) < r\} \subseteq X.$$

Besides, we can let $r < 1$. (If $r \ge 1 > r_2$, then $B_2(u,r_2) \subseteq B(u,r) \subseteq X$, and then we can let $r = r_2$.) Also, we know

$$d_2(x,u) < r \Leftrightarrow \frac{d_1(x,u)}{1+d_1(x,u)} < r \Leftrightarrow d_1(x,u) < \frac{r}{1-r}.$$

Notice that since $0 < r < 1$, so $\frac{r}{1-r} > 0$. Thus, we know in $(\mathbb{R}, d_2)$, for all $u \in U$, there exists $\frac{r}{1-r} > 0$ s.t.

$$B_1\left(u, \frac{r}{1-r}\right) = \left\{x \in X \mid d_1(x, u) < \frac{r}{1-r}\right\} \subseteq X,$$

which means $\mathrm{Int}_1(U) = U$ and thus U is open in $(\mathbb{R}, d_1)$.

$\blacksquare$

Problem 0.0.4 (15 pts). Let $\mathcal{M}_n = M_n(\mathbb{R})$ denote the set of all $n \times n$ real matrices. Define a function on $\mathcal{M}_n \times \mathcal{M}_n$ by

$$\rho(A, B) = \mathrm{rank}(A - B).$$

Then ρ is a metric on $\mathcal{M}_n$ and it is topologically equivalent to the discrete metric on $\mathcal{M}_n$.

Proof. We first show that ρ is a metric on $\mathcal{M}_n$.

- For all $A \in \mathcal{M}_n$, we know $\rho(A, A) = \mathrm{rank}(A - A) = \mathrm{rank}\, 0 = 0$.
- For any distinct $A, B \in \mathcal{M}_n$, we know there is a row of $A - B$ not equal to 0-vector, so $\mathrm{rank}(A - B) > 0$.
- For $A, B \in \mathcal{M}_n$, we know $\mathrm{rank}(A - B) = \mathrm{rank}(B - A)$, so $\rho(A, B) = \rho(B, A)$.
- For $A, B, C \in \mathcal{M}_n$, we want to show $\mathrm{rank}(A - C) \le \mathrm{rank}(A - B) + \mathrm{rank}(B - C)$. Suppose $A - B = X, B - C = Y$, then we want to show $\mathrm{rank}(X + Y) \le \mathrm{rank}\, X + \mathrm{rank}\, Y$, which is equivalent to show

  $$\dim \mathrm{Im}(X + Y) \le \dim(\mathrm{Im}\, X) + \dim(\mathrm{Im}\, Y).$$

  Notice that

  $$\mathrm{Im}(X + Y) = \{w \mid (X + Y)v = w \text{ for some } v\} \subseteq \{a + b \mid a \in \mathrm{Im}\, X, b \in \mathrm{Im}\, Y\} = \mathrm{Im}\, X + \mathrm{Im}\, Y.$$

  Hence, we have $\dim \mathrm{Im}(X + Y) \le \dim(\mathrm{Im}\, X + \mathrm{Im}\, Y)$. Also, we know

  $$\dim(\mathrm{Im}\, X + \mathrm{Im}\, Y) = \dim \mathrm{Im}\, X + \dim \mathrm{Im}\, Y - \dim \mathrm{Im}\, X \cap \mathrm{Im}\, Y \le \dim \mathrm{Im}\, X + \dim \mathrm{Im}\, Y.$$

  Hence, we know $\dim \mathrm{Im}(X + Y) \le \dim \mathrm{Im}\, X + \dim \mathrm{Im}\, Y$.

Now we prove that ρ is topologically equivalent to the discrete metric on $\mathcal{M}_n$, called d_{disc}. Now we show that for any set $U \subseteq \mathcal{M}_n$, U is open in $(\mathcal{M}_n, \rho)$ and $(\mathcal{M}, d_{\mathrm{disc}})$. For any $U \subseteq \mathcal{M}_n$, and for all $u \in U$, we know $B_\rho\left(u, \frac{1}{2}\right) = \{u\} \subseteq U$, so $U = \mathrm{Int}_\rho(U)$, which means U is open in $(\mathcal{M}_n, \rho)$. Similarly, for all $u \in U$, $B_{\mathrm{disc}}\left(u, \frac{1}{2}\right) = \{u\} \subseteq U$, so we can similarly conclude that U is open in $(\mathcal{M}_n, d_{\mathrm{disc}})$. Hence, we can say that $U \subseteq X$ is open in $(\mathcal{M}, \rho)$ if and only if U is open in $(\mathcal{M}_n, d_{\mathrm{disc}})$, so these two metrics are topologically equivalent. $\blacksquare$

Problem 0.0.5 (20 pts). Let E be a subset of a metric space (X, d). Prove the following:

(a) The boundary of E is a closed set.

(b) $\partial E = \overline{E} \cap \overline{X \setminus E}$

(c) If E is clopen (closed and open), what is ∂E?

(d) Give an example of $S \subset \mathbb{R}$ such that $\partial(\partial S) \neq \varnothing$, and infer that "the boundary of the boundary $\partial \circ \partial$ is not always zero."

Proof.

(a) We want to show that $\partial(\partial E) \subseteq \partial E$. For all $x \in \partial(\partial E)$, if $x \in \partial E$, then we're done. Now consider the second case: $x \in X \setminus \partial E = \operatorname{Int}(E) \cup \operatorname{Ext}(E)$. Note that for all $r > 0$, we have

$$B(x,r) \cap \partial E \neq \varnothing \quad B(x,r) \cap (X \setminus \partial E) = B(x,r) \cap (\operatorname{Int}(E) \cup \operatorname{Ext}(E)) \neq \varnothing.$$

Case 1: $x \in \operatorname{Int}(E)$.
We know there exists $r' > 0$ s.t. $B(x,r') \subseteq E$. If there exists $c \in B(x,r') \cap \partial E$, then we know $c \in B(x,r') \subseteq E$, so $c \in E$. Also, we know

$$B(c,r'') \cap E \neq \varnothing \quad B(c,r'') \cap (X \setminus E) \neq \varnothing \quad \forall r'' > 0.$$

Now suppose $\varepsilon = d(c,x) < r'$. If we pick some $r'' < r' - \varepsilon$, then for all $p \in B(c,r'')$, we have $d(p,c) < r''$, and by triangle inequality we have

$$d(p,x) \leq d(p,c) + d(c,x) < r'' + \varepsilon < r' - \varepsilon + \varepsilon = r',$$

which means $p \in B(x,r')$. Hence, $B(c,r'') \subseteq B(x,r') \subseteq E$, which means $B(c,r'') \cap (X \setminus E) = \varnothing$, and this is a contradiction, so we know there does not exist $x \in \partial(\partial E)$ s.t. $x \in \operatorname{Int}(E)$.

Case 2: $x \in \operatorname{Ext}(E)$.
We know there exists $r' > 0$ s.t. $B(x,r') \subseteq X \setminus E$. If there exists $c \in B(x,r') \cap \partial E$, then we know $c \in B(x,r') \subseteq X \setminus E$, so $c \in X \setminus E$. Also, we know

$$B(c,r'') \cap E \neq \varnothing \quad B(c,r'') \cap (X \setminus E) \neq \varnothing \quad \forall r'' > 0.$$

Now suppose $\varepsilon = d(c,x) < r'$. If we pick some $r'' < r' - \varepsilon$, then for all $p \in B(c,r'')$, we have $d(p,c) < r''$, and by triangle inequality we have

$$d(p,x) \leq d(p,c) + d(c,x) < r'' + \varepsilon < r' - \varepsilon + \varepsilon = r',$$

which means $p \in B(x,r')$. Hence, $B(c,r'') \subseteq B(x,r') \subseteq X \setminus E$, which means $B(c,r'') \cap E = \varnothing$, and this is a contradiction, so we know there does not exist $x \in \partial(\partial E)$ s.t. $x \in \operatorname{Ext}(E)$.

(b)

$$\begin{aligned}
\text{a point } x \in \partial E &\Leftrightarrow \begin{cases} B(x,r) \cap E \neq \varnothing \\ B(x,r) \cap (X \setminus E) \neq \varnothing \end{cases} \\
&\Leftrightarrow x \in \overline{E} \text{ and } x \in \overline{X \setminus E}. \\
&\Leftrightarrow x \in \overline{E} \cap x \in \overline{X \setminus E}.
\end{aligned}$$

(c) If E is clopen, then we know

$$\begin{cases} \partial E \subseteq E \\ \partial E \cap E = \varnothing. \end{cases}$$

Hence, $\partial E = \varnothing$. Otherwise, if there exists $a \in \partial E$, then $a \in \partial E \subseteq E$, and thus $a \in \partial E \cap E$, which means $\partial E \cap E \neq \varnothing$, and this is a contradiction.

(d) Consdier $S = (-1,1)$, and the metric is defined by $d(x,y) = |x-y|$, then $\{-1,1\} = \partial S$, and for any $r > 0$, we know $-1 \in B(-1,r)$, so $B(-1,r) \cap \partial S \neq \varnothing$. Also, for any $r > 0$, we know $-1 + \min\left\{0.1, \frac{r}{2}\right\} \in B(-1,r)$. Note that $-1 + \min\left\{0.1, \frac{r}{2}\right\} \in X \setminus \partial S$, so we know $B(-1,r) \cap (X \setminus \partial S) \neq \varnothing$. Hence, $-1 \in \partial(\partial S)$, and thus $\partial(\partial S) \neq \varnothing$.

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Problem 0.0.6 (21 pts). In a metric space (X,d), if subsets satisfy $A \subseteq S \subseteq \overline{A}$, where $\overline{A}$ is the closure of A, then A is said to be *dense* in S. For example, the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$.

(a) If A is dense in S and S is dense in T, prove that A is dense in T.

(b) If A is dense in S and if B is open in S, prove that

$$B \subseteq \overline{A \cap B}.$$

(c) If each of A and B is dense in S and if B is open in S, prove that

$$A \cap B \quad \text{is dense in } S.$$

Proof.

(a) We know that

$$\begin{cases} A \subseteq S \subseteq \overline{A} \\ A \subseteq T \subseteq \overline{S}, \end{cases}$$

and we want to show that $A \subseteq T \subseteq \overline{A}$. Since $A \subseteq S \subseteq T$, so we have $A \subseteq T$. Now we want to show $T \subseteq \overline{A}$. If there exists $x \in T$ but $x \notin \overline{A}$, then we know

$$\forall r > 0, \begin{cases} B(x,r) \cap S \neq \varnothing \\ B(x,r) \cap (X \setminus S) \neq \varnothing \end{cases} \quad \text{since } T \subseteq \overline{S}.$$

Also, we know there exists $r' > 0$ s.t. $B(x,r') \cap A = \varnothing$. Now since $A \subseteq S$, so we know

$$B(x,r') \cap (S \setminus A) \neq \varnothing.$$

Besides, note that $S \subseteq \overline{A} = A \cup \partial A$, so $(S \setminus A) \subseteq \partial A$, so we know there exists $y \in \partial A$ s.t. $y \in B(x,r')$. Note that $d(y,x) < r'$, and we define $\varepsilon = r' - d(y,x) > 0$. Also, we have

$$\forall r'' > 0, \begin{cases} B(y,r'') \cap A \neq \varnothing \\ B(y,r'') \cap (X \setminus A) \neq \varnothing. \end{cases}$$

Hence, we know for some $a \in A$, $a \in B(y,r'')$ for all $r'' > 0$, which means $d(y,a) < r''$. If we pick r'' s.t. $0 < r'' < \varepsilon$, then we have

$$\begin{aligned} d(a,x) &\leq d(a,y) + d(y,x) \\ &< \varepsilon + d(y,x) \\ &= r' - d(y,x) + d(y,x) = r'. \end{aligned}$$

Hence, we have $a \in B(x,r')$, but $a \in A$, so $a \in B(x,r') \cap A = \varnothing$, which is a contradiction. Thus, we know for every $x \in T$, we must have $x \in \overline{A}$, which means $T \subseteq \overline{A}$, and we're done.

(b) Since $S \subseteq \overline{A}$, so for all $x \in S$ and $r > 0$, we know $B(x,r) \cap A \neq \varnothing$. We want to show that for all $x \in B$, we have $B(x,r) \cap A \cap B \neq \varnothing$ for all $r > 0$. Now suppose $x \in B \subseteq S$. Since B is open in S, so there exists $O \subseteq X$ s.t. O is open and $B = O \cap S$. Note that for all $x \in B \subseteq S$, there exists $r_1 > 0$ s.t. $B(x,r_1) \subseteq O$. Hence, we have $B(x,r_1) \cap S \subseteq O \cap S = B$. Also, since we know $A \subseteq S$, so

$$B(x,r_1) \cap A \subseteq B(x,r_1) \cap S \subseteq B,$$

which shows that $B(x,r_1) \cap A \cap B \neq \varnothing$. Now if $0 < r_2 < r_1$, then since $B(x,r_2) \subseteq B(x,r_1)$, so we have

$$B(x,r_2) \cap S \subseteq B(x,r_1) \cap S \subseteq B.$$

Also, we still have $B(x,r_2) \cap A \neq \varnothing$, and similarly we have

$$B(x,r_2) \cap A \subseteq B(x,r_1) \cap S \subseteq B,$$

which shows $B(x, r_2) \cap A \cap B \neq \varnothing$. Now if $r_3 > r_1$, then since $B(x, r_1) \subseteq B(x, r_3)$, and we have shown that $B(x, r_1) \cap A \cap B \neq \varnothing$, so we have

$$\varnothing \neq B(x, r_1) \cap A \cap B \subseteq B(x, r_3) \cap A \cap B.$$

Hence, for all $r > 0$, we know $B(x, r) \cap A \cap B \neq \varnothing$, and we're done.

(c) By (b), we know $B \subseteq \overline{A \cap B}$. Also, we always have $A \cap B \subseteq B$, so we have $A \cap B \subseteq B \subseteq \overline{A \cap B}$, which means $A \cap B$ is dense in B. Now since B is dense in S, so by (a) we know $A \cap B$ is dense in S.

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