Combinatorics I

Kon Yi

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Abstract

The lecture note of Combinatorics I by Shagnik Das, where the NTU cool site is https://cool.ntu.edu.tw/courses/55532/.

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Chapter 1

Chatting

Lecture 1

1.1 Prime Numbers

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Theorem 1.1.1 (Euclid ≈ 300 BCE)**.** There are infinitely many primes.

proof. (Saidak, 2006).

- Every natural number has at least one prime factor.
- No prime divides n and $n+1$, for any $n \in \mathbb{N}$.

Consider a sequence of pronic number

$$p_1 = 2,\ p_{n+1} = p_n(p_n + 1).$$

Then the number of prime factors of p_n is strictly increasing in n: p_{n+1} has all the factors of p_n together with the (disstinct) ones of $p_n + 1$.

Example 1.1.1. $p_1 = 2, p_2 = 6, p_3 = 42, p_4 = 1806$, where the prime factors of them are $\{2\}$, $\{2,3\}, \{2,3,7\}, \{2,3,7,43\}$.

■

1.1.1 How many prime numbers are there?

Definition 1.1.1. We define

$$\pi(n) = |\{p : 1 \leq p \leq n : p \text{ is prime}\}|.$$

Note 1.1.1. By Saidak's proof, we know $\pi(p_n) \geq n$. In fact, $\pi(p_n) \geq \log_2 n$.

Theorem 1.1.2 (Legendre, ≈ 1800 LE)**.**

$$\pi(n) \approx \frac{n}{\ln n} \Leftrightarrow \lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1$$

Note 1.1.2. Proven by Hadamard and independently de la Vallée Poussin(1896).

Theorem 1.1.3 (Better Approximation).
Dirichlet: $\pi(x) \approx Li(x) = \int_2^x \frac{1}{\ln t} dt$.
Known: $\pi(n) = Li(n) + O\left(ne^{-a\sqrt{\ln n}}\right)$
Believed: $\pi(n) = Li(n) + O\left(\sqrt{n}\ln n\right)$

Chapter 2

Elementary Counting Principles

Fundemental problem: Given a set S, and we want to determine $|S|$.

2.1 Sum Rule

Theorem 2.1.1 (Sum Rule). If $S = \dot\bigcup_{i=1}^{k} S_i$, then $|S| = \sum_{i=1}^{k} |S_i|$.

Note 2.1.1. $\dot\bigcup$ means disjoint union.

Example 2.1.1. A drawer contains 8 pairs of yellow socks, 5 pairs of blue socks, and 3 pairs of red socks. How many socks are there in total.

Informal proof. $2 \times (8 + 5 + 3) = 32$. ■

Proof. Let S be the set of socks in the drawer, then $S = \dot\bigcup_{p \in P} S_p$, where P is the set of pairs of socks, and S_p is the set of two socks in the pair where $p \in P$. By the sum rule,

$$|S| = \sum_{p \in P} |S_p| = \sum_{p \in P} 2 = 2|P| = 32.$$

$P = P_{\text{yellow}} \,\dot\cup\, P_{\text{blue}} \,\dot\cup\, P_{\text{red}}$. By the sum rule,

$$|P| = |P_{\text{yellow}}| + |P_{\text{blue}}| + |P_{\text{red}}| = 8 + 5 + 3 = 16.$$

■

Note 2.1.2. Sum rule is the basis for case analysis arguments. It needs two requirements:

- Cover each case.
- Cover each case exactly once.

Example 2.1.2. Counting subset of a general set.

Notation. If X is a set, and $k \in \mathbb{N} \cup \{0\}$, then

$$\binom{X}{k} = \{T : \; T \subseteq X, \; |T| = k\}.$$

We define the binomial coefficient as

$$\binom{|X|}{k} = \left|\binom{X}{k}\right|.$$

i.e. Given $n \geq k \geq 0$, $\binom{n}{k}$ is the number of k-element subsets of a set of size n. ■

Proposition 2.1.1 (Pascal's relation)**.** If $n \geq k \geq 1$, then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof. Proof. Let X be an n-element set (e.g. $X = [n] = \{1, 2, \ldots, n\}$), and let $S = \binom{X}{k} = \{T \subseteq X : |T| = k\}$. Then, by definition, $\binom{n}{k} = |S|$. For each k-element subset, we can ask: "Do you contain n?" Let

$$S_0 = \{T : T \subseteq X, n \notin T, |T| = k\},$$

and

$$S_1 = \{T : T \subseteq X, n \in T, |T| = k\}.$$

Then, $S = S_0 \uplus S_1$. By the sum rule, $|S| = |S_0| + |S_1|$. Observe that

$$\begin{aligned} S_0 &= \{T \subseteq [n], n \notin T, |T| = k\} \\ &= \{T \subseteq [n-1], |T| = k\}, \end{aligned}$$

so by definition,

$$|S_0| = \binom{|[n-1]|}{k} = \binom{n-1}{k}.$$

$$S_1 = \{T \subseteq [n], n \in T, |T| = k\}.$$

Let

$$S_1' = \{T' \subseteq [n-1], |T'| = k-1\},$$

then we know a bijection from S_1 to S_1':

$$T \in S_1 \longleftrightarrow T \setminus \{n\} \in S_1'.$$

Theorem 2.1.2 (bijection rule)**.** Given two sets S and S', if there is a bijection $f : S \to S'$, then $|S| = |S'|$.

By this rule, we know

$$|S_1| = |S_1'| = \binom{|[n-1]|}{k-1} = \binom{n-1}{k-1}.$$

Hence,

$$\binom{n}{k} = |S| = |S_0| + |S_1| = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

■

2.1.1 Pascal's Triangle

We can use Pascal's relation to compute $\binom{n}{k}$.

Note 2.1.3. Boundary case: $\binom{n}{0} = 1$, $\binom{n}{n} = 1$. Also, $\binom{n}{k} = 0$ for $k = -1, n+1$.

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & & & \searrow \quad \swarrow & & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
 & & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}$$

2.2 Product Rule

Theorem 2.2.1. If $S = S_1 \times S_2 \times \cdots \times S_k = \{(x_1, x_2.., x_k), x_i \in S_i\}$, then $|S| = \prod_{i=1}^{k} |S_i|$.

Proof. Induction on k:
Base case: $k = 1$, trivial.
Induction step: seperate into cases bases on choice of $x_{k+1} \in S_{k+1}$. Let

$$S(x) = \{(x_1, \ldots, x_{k+1}) \in S, x_{k+1} = x \in S_{k+1}\},$$

then

$$S = \dot{\bigcup_{x \in S_{k+1}}} S(x) \to |S| = \sum_{x \in S_{k+1}} |S(x)|.$$

But $S(x) = S_1 \times S_2 \times \cdots \times \{x\}$, which is in bijection with $S_1 \times S_2 \times \cdots \times S_k$. By induction rule,

$$|S(x)| = |S_1 \times S_2 \times \cdots \times S_k| \quad \forall x$$

Hence,

$$\begin{aligned}
|S| = \sum_{x \in S_{k+1}} |S(x)| &= \sum_{x \in S_{k+1}} |S_1 \times S_2 \times \ldots S_k| \\
&= |S_1 \times S_2 \times \cdots \times S_k| \times |S_{k+1}| = |S_1| \times |S_2| \times \cdots \times |S_{k+1}|.
\end{aligned}$$

■

Example 2.2.1. Consider binary strings of length n.

Proof.

$$S = \{0,1\}^n \Rightarrow |S| = |\{0,1\}^n| = |\{0,1\}|^n = 2^n.$$

■

Definition 2.2.1 (Power Set)**.** Given a finite set X, let 2^X denote the set of all subsets of X (also denoted $\mathcal{P}(x)$), which is called the power set.

Corollary 2.2.1. $\left|2^X\right| = 2^{|X|}$.

Proof. Without lose of generality, $X = [n]$. We build a bijection between $2^{[n]}$ and the set of binary string of length n. Suppose for every $T \in 2^{[n]}$, we have $\chi_T = (x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} 1, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases}$$

Then,

$$\left|2^{[n]}\right| = |\{0,1\}^n| = 2^n.$$

■

2.3 Double-Counting argument

If we count a set in two different ways, the answer should be equal.

Example 2.3.1. Count $2^{[n]}$.

Proof.

1. Product rule $\to 2^n$.
2. Use the sum rule, split the subsets by size.

$$2^{[n]} = \binom{[n]}{0} \cup \binom{[n]}{1} \cup \ldots \cup \binom{[n]}{n}$$

Hence, we have the following proposition:

Proposition 2.3.1. For all $n \geq 0$,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

⊛

2.4 Permutations

Lecture 2

As previously seen. Instead of choosing the subsets all at once, we could pick one element at a time, then we can try to use product rule. 5 Sep. 13:10

Example 2.4.1. Consider

$$\binom{[10]}{3}.$$

Proof. At the choice of the first element, we have 10 choices, the second one has 9 choisce, while the third one has 8 choice, but we didn't consider the order of each picked elements. ⊛

Definition 2.4.1. Given a set X and $k \in \mathbb{N} \cup \{0\}$, a k-permutation of X is

- an ordered choice of k distinct elements from X.
- a k-tuple $(x_1, x_2, \ldots, x_k)$ with $x_i \in X$ and $x_i \neq x_j$ for each $i \neq j$.
- an injection $f : [k] \to X$.

where these 3 statements are equivalent.

Notation. $X^{\underline{k}} = \{k\text{-permutation of } X\} \subseteq X^k$ where $X^k = X \times X \times \cdots \times X$ allows repitition of the elements but $X^{\underline{k}}$ don't allow repitition.

Note 2.4.1. If $|X| = n$, then

$$n^{\underline{k}} = \left|X^{\underline{k}}\right|.$$

Definition 2.4.2.

- a n-permutation is a n-permutation of $[n]$.
- a X-permutation is a $|X|$-permutation of X.

Theorem 2.4.1 (Generalized Product Rule). Suppose we are enumerating S, and can uniquely determine an element $s \in S$ through a series of k questions, if i-th problem always has n_i possible outcomes, independently to the permutation, then

$$|S| = n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i$$

Proof. Can make a bijection from S to

$$[n_1] \times [n_2] \times \cdots \times [n_k].$$

Map each element in S to the index of its answer in the series of answer.

Our moral is when counting we don't care about what the options are but only how many options. ∎

Proposition 2.4.1.

$$\begin{aligned} n^{\underline{k}} &= n(n-1)\dots(n-(k-1)) \\ &= \prod_{i=0}^{k-1}(n-i) = \frac{n!}{(n-k)!}. \end{aligned}$$

Proof. Use the generalized product rule.

Question i: What is the i-th element in the k-permutation of $[n]$?

We can choose anything except what we're alreafy chosen, so there are $i-1$ forbidden choices and thus there are $n-(i-1)$ possible choices. ∎

Proposition 2.4.2. For all $0 \leq k \leq n$,

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k^{\underline{k}}} = \frac{\left(\frac{n!}{(n-k)!}\right)}{k!} = \frac{n!}{k!(n-k)!}.$$

Proof. Double-count $[n]^{\underline{k}}$ i.e. k-permutation of $[n]$.

- Direct counting $\left|[n]^{\underline{k}}\right| = n^{\underline{k}}$.
- First choose the k elements to appear in the k-permutation, $\binom{n}{k}$ options, then choose the order in which they appear, $k^{\underline{k}}$ options.

  Then, by the generalized product rule, the number of k-permutation of $[n]$ is $\binom{n}{k} \cdot k^{\underline{k}}$.

Hence,

$$n^{\underline{k}} = \left|[n]^{\underline{k}}\right| = \binom{n}{k} \cdot k^{\underline{k}}.$$

∎

Corollary 2.4.1. We can then use this result to reprove Pascal's Property again.

Proof. ∎

Exercise 2.4.1. 6 players at the tennis club want to have three matches involving all the players? How many ways can we arrange the games.

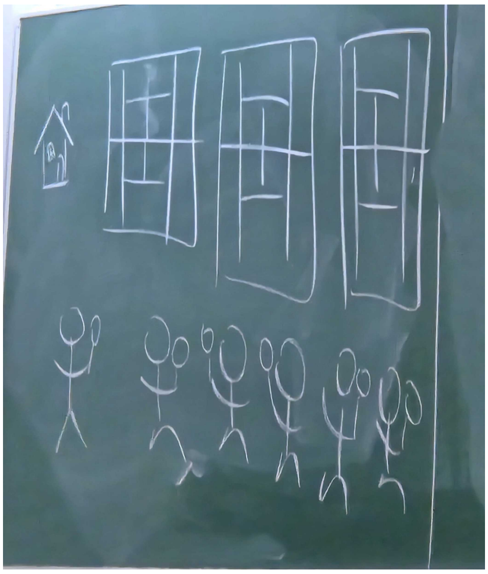

Figure 2.1: Tennis Games

Proof. We only care about who plays against whom, not about which court or who versus first, e.t.c.

The arrangement of games is a set of three disjoint pairs of players.

$$\{\{1,2\},\{3,4\},\{5,6\}\} \neq \{\{1,3\},\{2,4\},\{5,6\}\}.$$

Double-count the arrangements of games where counts do matter.

- Choose a pair of players for Court A: $\binom{6}{2}$
- Choose a pair of players for Court B: $\binom{4}{2}$
- Choose a pair of players for Court C: $\binom{2}{2}$

Generalized product rule tells

$$\text{number of choices} = \binom{6}{2}\binom{4}{2}\binom{2}{2} = 90.$$

Second count: First gets a set of 3 pairs, say there are x possibilities , and assign the three pairs to 3 courts, so there are 3! , so $x \cdot 3! = 90$, and thus $x = \frac{90}{3!} = 15$. ∎

Lecture 3

Actually we have an alternitive prove: 9 Sep. 15:30

proof by direct computation.

- Q1: Who's the opponent for the 1-st player? There are 5 choices.
- Q2: Who plays the next lowest numbered player? There are 3 choices.

The left 2 players are the opponents to each other. Hence, there are $3 \times 5 = 15$possible pairings. ∎

More generally, if we have $n = 2k$ players to pair up, then the first proof gives there are

$$\frac{\binom{n}{2}\binom{n-2}{2}\dots\binom{2}{2}}{\left(\frac{n}{2}\right)!}$$

possible pairings, while the second proof gives that there are

$$(n-1)\cdot(n-3)\cdot(n-5)\dots := (n-1)!! \neq ((n-1)!)!.$$

By this, we know these two numbers must be equal, or more rigorously, we can write

$$\frac{\binom{n}{2}\binom{n-2}{2}\dots\binom{2}{2}}{\left(\frac{n}{2}\right)!} = 2^n \cdot \frac{\frac{n(n-1)}{2}\frac{(n-2)(n-3)}{2}\dots}{n(n-2)(n-4)\dots 2} = (n-1)\cdot(n-3)\cdot\dots$$

Example 2.4.2. How mant shortest routes on the grid are there from $(0,0)$ to (n,m)?

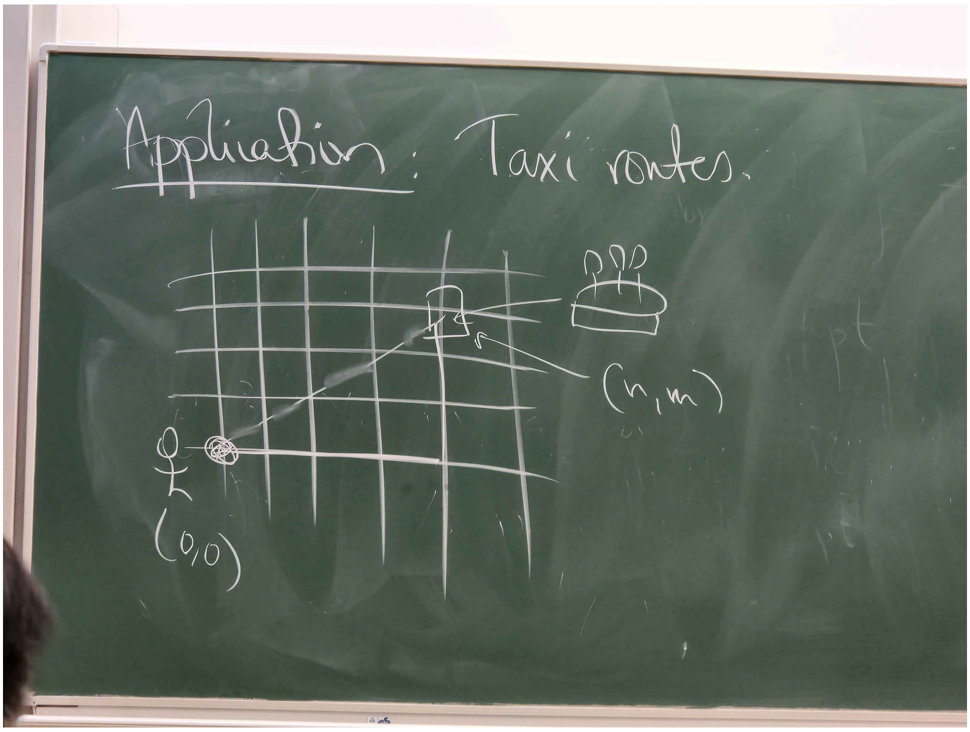

Figure 2.2: Taxi routes

Proof. Shortest route is of length $n+m$,m up-steps and n right-steps. We can think of a shortest route to be a binary string of length $n+m$ with n 1s and m 0s, so we want to count how many such binary strings are there. Choose n of them to be 1s, while the other are 0s. Hence, there are $\binom{n+m}{n}$ possibilities. ⊛

2.5 Binomial Theorem

Theorem 2.5.1 (Binomial Theorem)**.** For any $n \in \mathbb{N}\cup\{0\}$, and $x, y \in \mathbb{R}$, we have

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

Example 2.5.1. $(x+y)^0 = 1 = \sum_{k=0}^{0} x^k y^{0-k}$.

Example 2.5.2. $(x+y)^1 = x+y$, while

$$\sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x.$$

proof of binomial theorem.

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)\dots(x+y)}_{n \text{ factors}}$$

From each factor, we pick a term x or y, multuply chosen factors together. If we choose k x's, then we must choose $n-k$ y's, so the monomial is $x^k y^{n-k}$, where the coefficient of $x^k y^{n-k}$ is the number of ways of choosing k x's. Also, the possible monomials are $x^k y^{n-k}$ for $k = 0, 1, 2, \dots, n$. Hence, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

■

We can use this formula to derive identities for the binomial coefficients, by plugging in values for x and y.

Example 2.5.3. $x = 1, y = 1$.

Proof.

$$2^n = (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

⊛

Example 2.5.4. $y = -1, x = 1$.

Proof.

$$(x+y)^n = (-1+1)^n = 0^n = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \sum_{2|k} \binom{n}{k} - \sum_{2\nmid k} \binom{n}{k}$$

⊛

Corollary 2.5.1.

$$\sum_{2|k} \binom{n}{k} = \sum_{2\nmid k} \binom{n}{k}$$

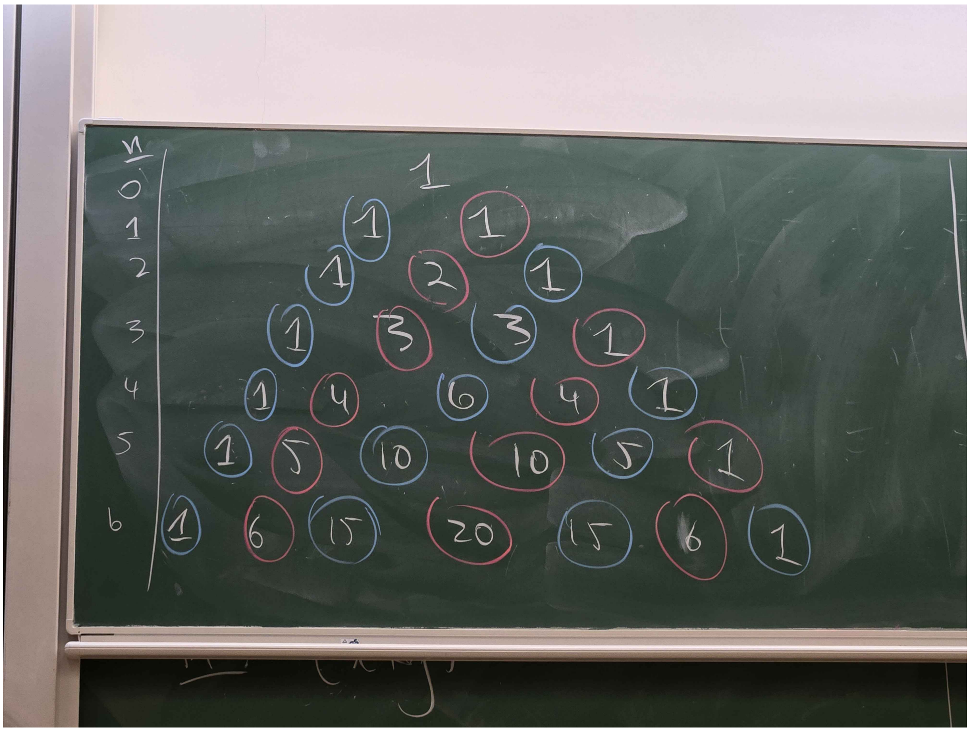

Figure 2.3: The sum of even terms is equal to the sum of odd terms.

Theorem 2.5.2. $\forall n \geq k$, we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-(n-k))!} = \binom{n}{n-k}.$$

Remark 2.5.1. Choosing a subset of k elements from n is equivalent to choose $n-k$ elements to discard, and we can build a bijection between these two methods.

■

For n even.
Consider the bijection

$$S \mapsto S \triangle \{n\} = \begin{cases} S - \{n\}, & \text{if } n \in S; \\ S \cup \{n\}, & \text{if } n \notin S. \end{cases}$$

Hence,

$$|S \triangle \{n\}| \subseteq \{|S| - 1, |S| + 1\},$$

so if $|S|$ is odd, then $S \triangle \{n\}$ is even, and vice versa. We know this is a bijection (self-inverse), so we have odd-sized sets to even-sized set. Hence, $\sum_{2|k} \binom{n}{k} = \sum_{2\nmid k} \binom{n}{k}$.

Example 2.5.5. $x = 2, y = 1$.

Proof.

$$(2+1)^n = 3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Counting partitions $[n] = A \cup B \cup C$, each element has a choice of 3 sets to go into. Hence, the product rule says there are 3^n partitions, while RHS uses sum rule bases on $k = |A \cup B|$. ⊛

2.6 Divisor Function

Definition 2.6.1 (Divisor Functions). Given a natural number $n \in \mathbb{N}$, let $d(n)$ count the number of divisors of n.

Example 2.6.1.

$$\begin{aligned} d(1) &= 1 = |\{1\}| \\ d(2) &= 2 = |\{1,2\}| \\ d(3) &= 2 = |\{1,3\}| \\ d(4) &= 3 = |\{1,2,4\}| \\ d(5) &= 2 = |\{1,5\}|. \end{aligned}$$

Corollary 2.6.1. $d(n) = 2$ if and only if n is a prime.

Now we want to compute the average value of $d(n)$.

Definition 2.6.2.

$$\bar{d}(n) = \frac{\sum_{i=1}^{n} d(i)}{n}.$$

We can use double-counting. First, notice that

$$d(i) = \sum_{\substack{j\in[i]\\ j|i}} 1.$$

Hence,

$$\sum_{i=1}^{n} d(i) = \sum_{i=1}^{n} \sum_{\substack{j\in[i]\\ j|i}} 1.$$

We can exchange the order of summation:

$$n\bar{d}(n) = \sum_{i=1}^{n} d(i) = \sum_{i=1}^{n} \sum_{j:j|i} 1 = \sum_{j=1}^{n} \sum_{\substack{i\in[n]\\ j|i}} 1.$$

For fixed j, we know

$$\sum_{\substack{i\in[n]\\ j|i}} 1 = \left\lfloor \frac{n}{j} \right\rfloor.$$

Hence, we have

$$n\bar{d}(n) = \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor,$$

which is equivalent to

$$\bar{d}(n) = \frac{1}{n} \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor.$$

Observe that

$$\frac{n}{j} - 1 \leq \left\lfloor \frac{n}{j} \right\rfloor \leq \frac{n}{j},$$

so

$$H_n - 1 = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{n}{j} - 1\right) \le \overline{d}(n) \le \frac{1}{n}\sum_{j=1}^{n}\frac{n}{j} = \sum_{j=1}^{n}\frac{1}{j} = H_n \approx \ln n.$$

Hence,

$$H_n - 1 \le \overline{d}(n) \le H_n,$$

which gives $\overline{d}(n) \sim \ln n$.

Chapter 3

Partitions

How many ways can we divide n items into k groups? Need to specify details to get well-posed questions.

1. Items distinguishable or not?
2. Groups distinguishable or not?
3. Can we have empty groups? Can we have group with more than one item?

Example 3.0.1. Professor has 49 students, to distribute 3000% between the students.

Proof. Indistinguishable items: percentage points.
Distinguishable groups: students $k = 49$. No restriction on sizes of groups. Formally, we are enumerating

$$S = \left\{ (x_1, x_2, \ldots, x_{49}) \mid x_i \geq 0, x_i \in \mathbb{Z}, \sum_{i=1}^{49} x_i = 3000 \right\}$$

⊛

Lecture 4

3.1 Number of nonnegative integer solution to $x_1 + \cdots + x_k = n$ 12 Sep. 12:20

We can represent solutions using a "stars and bar" diagaram:

- n stars represent the items
- $k - 1$ bars to divides the groups

Example 3.1.1. $x_1 = 3, x_2 = 1, x_3 = 0, x_4 = 5$. $(k = 4, n = 9)$

Proof.

$$\underbrace{***}_{x_1} \mid \underbrace{*}_{x_2} \mid\mid \underbrace{*****}_{x_3}$$

⊛

Hence, we can use a projection between solution and diagrams with $k - 1$ bars and n stars.
Each diagram consists of $n + k - 1$ symbols. Once we know which are the bars, we know the full diagram.

$$\text{number of diagrams} = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Proposition 3.1.1. The number of non-negative integer solutions to $x_1 + \cdots + x_k = n$ is $\binom{n+k-1}{k-1}$.

Now we have a new problem.

Question. How many solutions are there to $x_1 + \cdots + x_k = n$ with $x_i \geq 1$ for all i?

We can let $y_i = x_i - 1$, then $y_i \geq 0$ and $y_1 + \cdots + y_k = n - k$. Hence, the answer is

$$\binom{(n-k)+(k-1)}{k-1} = \binom{n-1}{k-1}.$$

Definition 3.1.1 (Multisets)**.** An unordered collection of elements with repretition allowed.

$$\{\{1,1,1,2,3\}\} \neq \{\{1,2,3\}\}$$

can represent as an ordered tuple in increasing order.

Example 3.1.2. How many multisets of size n are there from a set of size k?

Proof. Let x_i be the multiplicites of the i-th element in the multiset. Then $x_i \geq 0$ and

$$x_1 + \cdots + x_k = n.$$

Hence, the number of multisets is

$$\binom{n+k-1}{k-1}.$$

⊛

Alternatively, multisets are $(a_1, \ldots, a_n)$ with $1 \leq a_1 \leq \cdots \leq a_n \leq k$. Now if we let $b_i = a_i + i - 1$, then

$$(b_1, \ldots, b_n) = (a_1, a_2 + 1, \ldots, a_n + n - 1) \text{ with } 1 \leq b_1 < b_2 < \cdots < b_n \leq n + k - 1.$$

Note that there is a bijection between $\{(a_1, \ldots, a_n)\}$ and $\{(b_1, \ldots, b_n)\}$. This shows the number of multisets of size n from $[k]$ is the number of subsets of $[n+k-1]$ of size n, which is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Now we add some new setting.

- Distinguishable items
- Indistinguishable groups
- Groups non-empty.

The objects we are counting is

$$\{S_1, S_2, \ldots, S_k\}$$

with $S_1 \cup S_2 \cup \ldots \cup S_k = [n]$ and $S_i \neq \varnothing$ for all i.

Definition 3.1.2 (The Stirling Number of the second kind)**.** $S(n,k)$ is defined to be number of partitions of n distinct items into k indistinguishable non-empty groups.

Example 3.1.3. $S(n,1) = 1$ for all $n \geq 1$. $S(n,n) = 1$ for all n. $S(n,n-1) = \binom{n}{2}$ for all $n \geq 2$. $S(n,2) = 2^{n-1} - 1$.

Proof. We just talk about the $S(n,2)$ one. Since we can choose any subset of $[n]$, so there are 2^n possibilities, but each partition is counted twice, so we have to divide it by 2, and subtract the partition that includes empty group, so it is $2^{n-1}-1$. ⊛

Proposition 3.1.2. For all n, k,

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Proof. Case analysis:

- Case 1: $\{n\}$ is a group.
  This means the remaining $n-1$ elements are partitioned into $k-1$ groups, so there are $S(n-1,k-1)$ possibilities.
- Case 2: $\{n\}$ is not a group.
  $n-1$ left elments is first partitioned into k groups, then we can distribute the n-th element into each group, so there are $kS(n-1,k)$ possibilities.

By sum rule, we know

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

■

Example 3.1.4. Using induction to prove

$$S(n,n-1) = \binom{n}{2}.$$

Proof.

$$\begin{aligned} S(n,n-1) &= S(n-1,n-2) + (n-1)S(n-1,n-1) = S(n-1,n-2) + (n-1) \\ &= \cdots = 1 + 2 + \cdots + n - 1 = \binom{n}{2}. \end{aligned}$$

⊛

Now what if the groups are distinguishable? Also, we have

- items distinguishable
- groups distinguishable
- groups non-empty.

Short answer: $S(n,k)k!$.

Lecture 5

We can observe that the number of ways of partitioning n distinct items into k distinct nonempty groups is $S(n,k)k!$. 16 Sep. 15:30

Question. How many ways can we partition n distinct items into l distinct groups (not necessarily nonempty)?

Answer. l^n: product rule, each element has l choice for which group to go to. ⊛

Alternative method. Count by the number of nonempty groups (k), and then use sum rule. Partition elements into k nonempty indistinguishable groups, which has $S(n,k)$ choices, and then map the k sets to the l groups injectively, so there are $l^{\underline{k}} = l(l-1)\dots(l-k+1)$ choices. Hence, the total number of partition is

$$\sum_{k=0}^{l} S(n,k) l^{\underline{k}}.$$

By double counting, we know

$$l^n = \sum_{k=0}^{l} S(n,k) l^{\underline{k}} = \sum_{k=0}^{n} S(n,k) l^{\underline{k}}.$$

■

Proposition 3.1.3. For any field F, and $x \in F$, $n \in \mathbb{N} \cup \{0\}$, then

$$x^n = \sum_{k=0}^{n} S(n,k) x^{\underline{k}}.$$

(We define $x^{\underline{k}} = x(x-1)\dots(x-(k-1))$.)

Proof. There are polynomials of degree $\leq n$ that agree for all $x \in \mathbb{N}$, so they must agree everywhere. ■

We can observe that $\{x^n \mid n \in \mathbb{N} \cup \{0\}\}$ forms a basis for

$$F[x] = \left\{ \sum_{k=0}^{n} a_k x^k : a_k \in F \right\}.$$

Since x^n is a linear combination of $\{x^{\underline{n}} \mid n \in \mathbb{N} \cup \{0\}\}$, that means this is also a basis for $F[x]$. And the proposition shows that the change of basis matrix is the matrix of Stirling numbers of the second kind:

$$\begin{pmatrix} 1 & & & & 0 & 0 \\ & 1 & & & & 0 \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ S(n,k) & & & & & 1 \end{pmatrix} \begin{pmatrix} x^{\underline{0}} \\ x^{\underline{1}} \\ x^{\underline{2}} \\ \vdots \\ x^{\underline{k}} \\ \vdots \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ \vdots \\ x^k \\ \vdots \end{pmatrix}.$$

3.2 Stirling numbers of the first kind

Recall the permutation π is a bijection from $[n]$ to $[n]$.

Example 3.2.1. $\pi = 32154$, then $\pi(1) = 3, \pi(2) = 2, \pi(3) = 1, \pi(4) = 5, \pi(5) = 4$.

Example 3.2.2. $\pi_1 = 312, \pi_2 = 213$, then $\pi_2 \circ \pi_1 = 321$ and $\pi_1 \circ \pi_2 = 132$.

Claim 3.2.1. $\forall \pi \in S_n$, $\forall x \in [n]$, $\exists i \in [n]$ s.t. $\pi^i(x) = x$.

Proof. Consider $\pi^1(x), \pi^2(x), \dots, \pi^n(x) \in [n]$, if any are equal to x, then we're done. Otherwise, there are only $n-1$ possible values, which are $[n] \setminus \{x\}$. Hence, there are some $j_1, j_2 \in [n]$ with $j_1 > j_2$ and $\pi^{j_1}(x) = \pi^{j_2}(x)$ by Pigeonhole principle. Applying π^{-1} for j_2 times, we get

$$\pi^{j_1 - j_2}(x) = x \quad \text{with } 1 \leq j_1 - j_2 \leq n,$$

which is a contradiction. ■

Definition 3.2.1 (cycle). For the smallest i, $1 \leq i \leq n$ with $\pi^i(x) = x$, we say

$$(x \ \pi(x) \ \pi^2(x) \ \dots \ \pi^{i-1}(x))$$

is the cycle of x.

It follows that every permutation is a union of disjoint cycles. Hence, we have cycle representation of π.

Example 3.2.3. $\pi = 32154$, the cycle form is $(13)(2)(45)$.

Definition 3.2.2 (fixed point and transposition)**.** A fixed point of a permutation is a cycle of length 1 i.e. an element x with $\pi(x) = x$. A transposition is a cycle of length 2. A permutation is cyclic if it has a single cycle (of length n).

Question. How many cyclic permutations of $[n]$ are there?

Answer. $(n-1)!$. We can first fix the head of the cycle to be 1, then for $\pi(1)$, we have $n-1$ choices, and for $\pi^2(1)$, we have $n-2$ choices, and so on, so we have $(n-1)!$ cyclic permutations.

> **Note 3.2.1.** Who is in the head of the cycle is not important.

⊛

Definition 3.2.3 (The Stirling numbers of the first kind)**.** $s_{n,k}$ (or $[s(n,k)]$) enumerate the permutation in S_n with exactly k cycles.

Example 3.2.4. $s_{n,1} = (n-1)!, s_{n,n} = 1, s_{n,n-1} = \binom{n}{2}, s_{n,2} =$not so obvious.

Proof.

$$s_{n,2} = \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(k-1)!(n-k-1)!$$

Note that we multiply it by $\frac{1}{2}$ since we count each cycle-pair twice. Also, we know that a cycle of length n has $(n-1)!$ choices if we fix all n members in the cycle.

Alternatively, say the "first" cycle is the one containing 1 together with $0 \leq k \leq n-2$ other elements. Hence, we have

$$\begin{aligned}
s_{n,2} &= \sum_{k=0}^{n-2}\binom{n-1}{k}(k!)(n-k-2)! \\
&= \sum_{k=0}^{n-2}\frac{(n-1)!}{k!(n-k-1)!}k!(n-k-2)! = (n-1)!\sum_{k=0}^{n-2}\frac{1}{n-1-k} \\
&= (n-1)!\sum_{k=1}^{n-1}\frac{1}{k} \\
&= (n-1)!H_{n-1} \approx (n-1)!\ln n.
\end{aligned}$$

⊛

Proposition 3.2.1. $\forall n, k \geq 1$,

$$s_{n,k} = s_{n-1,k-1} + (n-1)s_{n-1,k}$$

Proof. Case analysis: is n a fixed point?

Case 1: Yes. Removing it, and then the left $n-1$ elements can be permutated with $k-1$ cycles. Hence, there are $s_{n-1,k-1}$ choices.

Case 2: No. We remove n from a cycle to get a permutation of $[n-1]$ with k cycles. Now, we have $n-1$ place to insert n inside. For example, we if $n=7$, and we have $(13)(2)(456)$, then we have $7-1=6$ places to insert 7 inside since (7456) and (4567) are same cycles.

To create a permutation $\pi \in S_n$ with k cycles where n is not a fixed point, we can take a permutation $\pi' \in S_{n-1}$ with k cycles, which has $s_{n-1,k}$ choices, and insert n before any element, so there are $n-1$ ways, so the number of such permutation is $(n-1)s_{n-1,k}$. By sum rule, we have

$$s_{n,k} = s_{n-1,k-1} + (n-1)s_{n-1,k}.$$

■

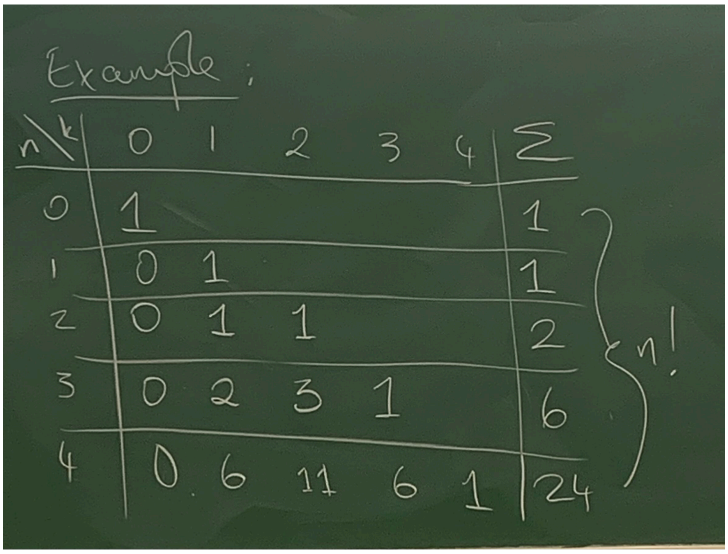

Figure 3.1: table of $s_{n,k}$

Corollary 3.2.1. $\forall n$, we have

$$\sum_{k=0}^{n} s_{n,k} = n!.$$

Proof. The number of permutations are $n!$, and every permutation consists of i cycles where $1 \leq i \leq n$, and then apply the sum rule. ■

Notation. Given $x \in F$, and $k \in \mathbb{N} \cup \{0\}$, we have

- $x^{\underline{k}} = x(x-1)\dots(x-(k-1))$
- $x^{\overline{k}} = x(x+1)\dots(x+(k-1)) = (x+k-1)^{\underline{k}}$.

Proposition 3.2.2. For all $x \in F$, $n \in \mathbb{N} \cup \{0\}$,

$$x^{\overline{n}} = \sum_{k=0}^{n} s_{n,k} x^k.$$

Proof. Induction on n. We know it is true for $n = 0, 1$. Note that

$$\begin{aligned}
x^{\overline{n}} &= x^{\overline{n-1}}(x+n-1) \\
&= (x+n-1)\sum_{k=0}^{n-1} s_{n-1,k}x^k \\
&= x\sum_{k=0}^{n-1} s_{n-1,k}x^k + (n-1)\sum_{k=0}^{n-1} s_{n-1,k}x^k \\
&= \sum_{k=0}^{n-1} s_{n-1,k}x^{k+1} + \sum_{k=0}^{n-1}(n-1)s_{n-1,k}x^k \\
&= \sum_{k=1}^{n} s_{n-1,k-1}x^{k} + \sum_{k=0}^{n-1}(n-1)s_{n-1,k}x^k \\
&= \sum_{k=0}^{n} \left(s_{n-1,k-1} + (n-1)s_{n-1,k}\right)x^k \\
&= \sum_{k=0}^{n} s_{n,k}x^k.
\end{aligned}$$

■

Corollary 3.2.2.

$$x^{\underline{n}} = \sum_{k=0}^{n} \underbrace{(-1)^{n-k}s_{n,k}x^k}_{\substack{\text{signed Stirling numbers}\\ \text{of the first kind}}}.$$

Proof.

$$\begin{aligned}
x^{\underline{n}} &= x(x-1)\dots(x-(n-1)) \\
&= (-1)^n(-x)(-x+1)\dots(-x+(n-1)) \\
&= (-1)^n(-x)^{\overline{n}} \\
&= (-1)^n\sum_{k=0}^{n} s_{n,k}(-x)^k \\
&= \sum_{k=0}^{n}(-1)^{n-k}s_{n,k}x^k.
\end{aligned}$$

■

Lecture 6

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Corollary 3.2.3.

$$\sum_{k=j}^{i}(-1)^{k-j}S(i,k)s_{k,j} = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Proof.

$$
\begin{aligned}
x^i &= \sum_{k=0}^{i} S(i,k)x^{\underline{k}} = \sum_{k=0}^{i} S(i,k)\left[\sum_{j=0}^{k}(-1)^{k-j}s_{k,j}x^k\right] \\
&= \sum_{k=0}^{i}\sum_{j=0}^{k}(-1)^{k-j}S(i,k)s_{k,j}x^j \\
&= \sum_{j=0}^{i}\left(\sum_{k=j}^{i}(-1)^{k-j}S(i,k)s_{k,j}\right)x^j = x^i.
\end{aligned}
$$

Since $\{x^0, x^1, x^2, \dots\}$ is a basis of $F[x]$, the coefficient of x^j is 1 is $i = j$ and is 0 is $i \neq j$. ■

Question. How many ways can we distribute \$100000 of prize money to six players in the tournaments?

- Whole dollars only.
- Nonnegative prices.

It is an arbitrary partition, and there are $k = 6$ distinct groups(players). Hence, there are $\binom{1000005}{5}$ ways of distribution? However, this is not what we want, since in a tournament a better player should get more money. Actually, in this scenario, groups are indistinguishable since largest prize is for first place, and so on. Thus, our goal is to dividing n indistinguishable items into k indistinguishable (non-empty) groups.

Definition 3.2.4 (number partition). A number partition is a decomposition of n and a sum of k unordered natural numbers.

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \text{ s.t. } \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k, \ \sum_{i=1}^{k}\lambda_i = n \text{ with } \lambda_i \in \mathbb{N}.$$

We write $\lambda \vdash n$. We define

$$p(n,k) = \left|\{\lambda = (\lambda_1, \dots, \lambda_k) : \lambda \vdash n\}\right|.$$

We also define

$$
\begin{aligned}
p(n, \leq k) &= \sum_{i=0}^{k} p(n,i) \\
p(n) = p(n, \leq n) &= \sum_{i=0}^{n} p(n,i).
\end{aligned}
$$

Observe that

- $$p(n,0) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n = 1. \end{cases}$$
- $p(n,n) = 1$
- $p(n, n-1) = 1 = |\{2, 1, 1, \dots\}|$
- $p(n,1) = 1.$
- $p(n,2) = \lfloor \frac{n}{2} \rfloor.$

Proposition 3.2.3. $\forall n \geq k \geq 1$,

$$p(n, k) = p(n-1, k-1) + p(n-k, k).$$

Proof. Case analysis based on size of smallest part:

- Case 1: $\lambda_k = 1$.
  Then remove the last part to get a partition of $n-1$ into $k-1$ nonempty parts. (bijective, can add part of size 1 to the end of a partition), so there are $p(n-1, k-1)$ such cases.
- Case 2: $\lambda_k \geq 2$.
  Consider $\lambda' = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$, then $\lambda' \vdash n-k$, and this is a bijection, so there are $p(n-k, k)$ such cases.

■

Appendix