Introduction to Analysis I HW5

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Problem 0.0.1 (15pts). (a) Let (X, d_{disc}) be a metric space with the discrete metric. Let E be a subset of X which contains at least two elements. Show that E is disconnected.

(b) Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous if and only if it is constant. *(Hint: use part (a))*

Problem 0.0.2 (15pts). Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha\in I}$ be a collection of connected sets in X with I non-empty. Suppose also that $\bigcap_{\alpha\in I} E_\alpha$ is non-empty. Show that $\bigcup_{\alpha\in I} E_\alpha$ is connected.

Proof. Suppose by contradiction, $\bigcup_{\alpha\in I} E_\alpha$ is disconnected, then there exists non-empty V, W open in $\bigcup_{\alpha\in I} E_\alpha$ s.t. $V \cup W = \bigcup_{\alpha\in I} E_\alpha$ and $V \cap W = \varnothing$. Hence, we know

$$\begin{cases} V = O_1 \cap \left(\bigcup\limits_{\alpha\in I} E_\alpha\right) = \bigcup\limits_{\alpha\in I} (O_1 \cap E_\alpha) \\ W = O_2 \cap \left(\bigcup\limits_{\alpha\in I} E_\alpha\right) = \bigcup\limits_{\alpha\in I} (O_2 \cap E_\alpha) \end{cases},$$

where O_1, O_2 are open in X. Since I is non-empty, so we can suppose $i \in I$ s.t. $O_1 \cap E_i$ and $O_2 \cap E_i$ are both open in E_i. We first claim that there exists $i, j \in I$ s.t. $O_1 \cap E_i$ and $O_2 \cap E_j$ are non-empty. Suppose by contradiction, $O_1 \cap E_i = O_2 \cap E_j = \varnothing$ for all $i, j \in I$, then $V = O_1 \cap \bigcup_{\alpha\in I} E_\alpha = \varnothing$ and $W = O_2 \cap \bigcup_{\alpha\in I} E_\alpha = \varnothing$, which are contradicitons. Now we claim that there exists $i \in I$ s.t. $O_1 \cap V \neq \varnothing$ and $O_2 \cap V \neq \varnothing$. If not, then

$$O_1 \cap O_2 \cap \left(\bigcap_{\alpha\in I} E_\alpha\right) = \varnothing.$$

However, $\exists p \in \bigcap_{\alpha\in I} E_\alpha$ since $\bigcap_{\alpha\in I} E_\alpha$ is non-empty, so either $p \in V$ or $p \in W$, so either $p \in O_1$ or $p \in O_2$. (If $p \in O_1 \cap O_2$, then $p \in O_1 \cap O_2 \cap \bigcup_{\alpha\in I} E_\alpha = V \cap W = \varnothing$.) WLOG, suppose $p \in O_1$, then $O_1 \cap E_k \neq \varnothing$ for all $k \in I$, and since we know there exists $j \in I$ s.t. $O_2 \cap E_j \neq \varnothing$, so we know there exists $i \in I$ s.t. $O_1 \cap E_i$ and $O_2 \cap E_i$ are both non-empty. Now since

$$\bigcup_{\alpha\in I} E_\alpha = V \cup W = \left(O_1 \cap \bigcup_{\alpha\in I} E_\alpha\right) \cup \left(O_2 \cap \bigcup_{\alpha\in I} E_\alpha\right) = (O_1 \cup O_2) \cap \bigcup_{\alpha\in I} E_\alpha.$$

Hence, we have $\bigcup_{\alpha\in I} E_\alpha \subseteq O_1 \cup O_2$, which gives $E_i \subseteq O_1 \cup O_2$. By this, we have

$$(O_1 \cap E_i) \cup (O_2 \cap E_i) = (O_1 \cup O_2) \cap E_i = E_i.$$

Now since

$$\varnothing = V \cap W = \left(O_1 \cap \bigcup_{\alpha\in I} E_\alpha\right) \cap \left(O_2 \cap \bigcup_{\alpha\in I} E_\alpha\right) = O_1 \cap O_2 \cap \bigcup_{\alpha\in I} E_\alpha,$$

so we have

$$(O_1 \cap E_i) \cap (O_2 \cap E_i) = (O_1 \cap O_2) \cap E_i = \varnothing.$$

However, we have shown that for $A = O_1 \cap E_i$ and $B = O_2 \cap E_i$, A, B are open in E_i and $A \cup B = E_i$ and $A \cap B = \varnothing$, which means E_i is disconnected, and this is a contradiction, so $\bigcup_{\alpha\in I} E_\alpha$ must be connected. ■

Problem 0.0.3 (20pts). Let (X, d) be a metric space, and let E be a subset of X. We say that E is *path-connected* iff, for every $x, y \in E$, there exists a continuous function

$$\gamma : [0,1] \to E$$

from the unit interval $[0,1]$ to E such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every non-empty path-connected set is connected. (The converse is false, but is a bit tricky to show and will not be detailed here.)

Problem 0.0.4 (15pts). Let (X, d) be a metric space, and let E be a subset of X. Show that if E is connected, then the closure $\overline{E}$ of E is also connected. Is the converse true?

Proof. If E is connected, and suppose by contradiction, $\overline{E}$ is disconnected, then $\overline{E} = V \cup W$ and $V \cap W = \varnothing$ for some non-empty V, W open in $\overline{E}$. Hence, we can write $V = O_1 \cap \overline{E}$ and $W = O_2 \cap \overline{E}$, where O_1, O_2 are open in X. Now suppose $A = O_1 \cap E$ and $B = O_2 \cap E$, then we know A, B are open in E.

Claim 0.0.1. $E \subseteq O_1 \cup O_2$.

Proof. If $\exists x \in E$ but $x \notin O_1 \cup O_2$, then $x \notin V$ and $x \notin W$ since

$$\begin{cases} V = O_1 \cap \overline{E} \\ W = O_2 \cap \overline{E}. \end{cases}$$

Thus, $x \notin V \cup W = X$, but $x \in E \subseteq X$, so it is a contradiction, and thus $E \subseteq O_1 \cup O_2$. ⊛

Now by this claim, we know

$$A \cup B = (O_1 \cap E) \cup (O_2 \cap E) = (O_1 \cup O_2) \cap E = E.$$

Also,

$$\begin{aligned} A \cap B &= (O_1 \cap E) \cap (O_2 \cap E) = O_1 \cap O_2 \cap E \\ &\subseteq V \cap W = \left(O_1 \cap \overline{E}\right) \cap \left(O_2 \cap \overline{E}\right) = \left(O_1 \cap (E \cup \partial E)\right) \cap \left(O_2 \cap (E \cup \partial E)\right), \end{aligned}$$

and since $V \cap W = \varnothing$, so $A \cap B = \varnothing$. Now we show that A, B are non-empty. If $O_1 \cap E = \varnothing$, then since V is non-empty and

$$V = O_1 \cap \overline{E} = O_1 \cap (E \cup \partial E) = (O_1 \cap E) \cup (O_1 \cap \partial E),$$

so $O_1 \cap \partial E \neq \varnothing$, say $x \in O_1 \cap \partial E$. Then since O_1 is open in X, so there exists $r > 0$ s.t. $B_X(x, r) \subseteq O_1$, and since $x \in \partial E$, so $B_X(x, r) \cap E \neq \varnothing$, say $y \in B_X(x, r) \cap E$. Thus, we have

$$y \in B_X(x, r) \cap E \subseteq O_1 \cap E,$$

but this means $O_1 \cap E$ is non-empty, which is a contradiciton. Hence, $O_1 \cap E$ is non-empty, and we can use similar method to prove $O_2 \cap E$ is non-empty. Now since

$$\begin{cases} A, B \neq \varnothing \\ A \cup B = E \\ A \cap B = \varnothing \\ A, B \text{ are open in } E, \end{cases}$$

so E is connected, which is a contradiction, so $\overline{E}$ must be connected.

Now we show that the converse may not be true. Suppose $X = \mathbb{R}$ and $\overline{E} = [1, 2]$, then we know $\overline{E}$ is connected since in $\mathbb{R}$ connected space is equivalent to an interval. However, we know E may be $(1, 1.5) \cup (1.5, 2)$, and this is disconnected in $\mathbb{R}$, so this is a counterexample.

■

Problem 0.0.5 (20pts). Let (X, d) be a metric space. Let us define a relation $x \sim y$ on X by declaring $x \sim y$ iff there exists a connected subset of X which contains both x and y. Show that this is an equivalence relation (i.e., it obeys the reflexive, symmetric, and transitive axioms). Also, show that the equivalence classes of this relation (i.e., the sets of the form

$$\{y \in X : y \sim x\} \quad \text{for some } x \in X$$

are all closed and connected. *(Hint: use Problem 2 and Problem 4)* These sets are known as the *connected components* of X. You can read a note about equivalence relation in the file at NTU cool.

Proof. We first show that $\sim$ is an equivalence relation.

- reflexive: Note that for all $x \in X$, $\{x\}$ is connected since it cannot be cut into two non-empty part, so $x \sim x$.
- symmetry: This is trivial since if there exists a connected subset of X which contains both x and y, then this connected subset of X contains y and x.
- transitive: If $x \sim y$ and $y \sim z$, then we know there exists connected E_1, E_2 s.t. E_1 contains x, y and E_2 contains y, z. Since $y \in E_1 \cap E_2$, so $E_1 \cap E_2$ is non-empty, and thus by Problem 2 we know $E_1 \cup E_2$ is connected, and since $x, z \in E_1 \cup E_2$, so $x \sim z$.

Now if we fix $x \in X$, and say $[x] = \{y \in X : y \sim x\}$, then now we show that $[x]$ is closed and connected. We first show that $[x]$ is connected. If not, then there exists non-empty V, W s.t. $[x] = V \cup W$ and $V \cap W = \varnothing$ and V, W are open in $[x]$. If $x \in V$, then $x \notin W$ and since W is non-empty, we know there exists $z \neq x$ s.t. $z \in W$. Since $z \in W \subseteq [x]$, so there exists connected E_z s.t. $x, z \in E_z$.

Claim 0.0.2. $E_z \subseteq [x]$.

Proof. For all $p \in E_z$, since we know $x \in E_z$, so E_z is a connected set containing p and x, so $p \sim x$, which means $p \in [x]$. Hence, $E_z \subseteq [x]$. ⊛

Now we know

$$E_z = E_z \cap [x] = E_z \cap (V \cup W) = (E_z \cap V) \cup (E_z \cap W),$$

and since $x \in E_z \cap V$ and $z \in E_z \cap W$, so $E_z \cap V$ and $E_z \cap W$ are non-empty. Also, since $E_z \subseteq [x]$ and V, W are open in $[x]$, so $E_z \cap V$ and $E_z \cap W$ are open in E_z. Besides,

$$(E_z \cap V) \cap (E_z \cap W) = E_z \cap (V \cap W) = E_z \cap \varnothing = \varnothing,$$

so we know E_z is disconnected since

$$\begin{cases} (E_z \cap V), (E_z \cap W) \neq \varnothing \\ E_z = (E_z \cap V) \cup (E_z \cup W) \\ (E_z \cap V) \cap (E_z \cap W) = \varnothing \\ (E_z \cap V), (E_z \cap W) \text{ are open in } E_z. \end{cases}$$

However, E_z is connected, so this is a contradiciton. Hence, $[x]$ is connected.

Now we show that $[x]$ is closed. Since $[x]$ is connected and by Problem 4, we know $\overline{[x]}$ is connected. Note that $x \in \overline{[x]}$ since $x \in [x]$ and thus for all $r > 0$ we have $x \in B_X(x, r) \cap [x]$. However, this means for all $y \in \overline{[x]}$, we have $x, y \in \overline{[x]}$ and $\overline{[x]}$ is connected, so $y \sim x$, which means $y \in [x]$. Hence, $\overline{[x]} \subseteq [x]$, and thus $[x]$ is closed. ■

Problem 0.0.6 (15pts). Let $f : S \to T$ be a function from a metric space S to another metric space T. Assume f is uniformly continuous on a subset A of S and that T is complete. Prove that there is a unique extension of f to $\overline{A}$ which is uniformly continuous on $\overline{A}$.