

**Introduction to Mathematical Analysis**  
**Homework 10 Due November 28 (Friday), 2025**  
**Please submit your homework online in PDF format.**

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1. (15 pts) **Exercise 4.7.8** Let  $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$  be the tangent function  $\tan(x) := \sin(x)/\cos(x)$ . Show that  $\tan$  is differentiable and monotone increasing, with

$$\frac{d}{dx} \tan(x) = 1 + \tan(x)^2,$$

and that  $\lim_{x \rightarrow \pi/2} \tan(x) = +\infty$  and  $\lim_{x \rightarrow -\pi/2} \tan(x) = -\infty$ . Conclude that  $\tan$  is in fact a bijection from  $(-\pi/2, \pi/2) \rightarrow \mathbb{R}$ , and thus has an inverse function

$$\tan^{-1} : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$$

(this function is called the *arctangent function*). Show that  $\tan^{-1}$  is differentiable and

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}.$$

2. (15 pts) **Exercise 4.7.9** Recall the arctangent function  $\tan^{-1}$  from Exercise 4.7.8. By modifying the proof of Theorem 4.5.6(e), establish the identity

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for all  $x \in (-1, 1)$ . Using Abel's theorem (Theorem 4.3.1) to extend this identity to the case  $x = 1$ , conclude in particular the identity

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

(Note that the series converges by the alternating series test, Proposition 7.2.11.) Conclude in particular that  $4 - \frac{4}{3} < \pi < 4$ . (One can of course compute  $\pi = 3.1415926 \dots$  to much higher accuracy, though if one wishes to do so it is advisable to use a different formula than the one above, which converges very slowly.)

3. (30 pts) **Exercise 4.7.10** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the function

$$f(x) := \sum_{n=1}^{\infty} 4^{-n} \cos(32^n \pi x).$$

- (a) Show that this series is uniformly convergent, and that  $f$  is continuous.  
(b) Show that for every integer  $j$  and every integer  $m \geq 1$ , we have

$$\left| f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right) \right| \geq 4^{-m}.$$

*Hint: use the identity*

$$\sum_{n=1}^{\infty} a_n = \left( \sum_{n=1}^{m-1} a_n \right) + a_m + \sum_{n=m+1}^{\infty} a_n$$

for certain sequences  $a_n$ . Also, use the fact that the cosine function is periodic with period  $2\pi$ , as well as the geometric series formula  $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$  for any  $|r| < 1$ . Finally, you will need the inequality  $|\cos(x) - \cos(y)| \leq |x - y|$  for any real numbers  $x$  and  $y$ ; this can be proven by using the mean value theorem.

- (c) Using (b), show that for every real number  $x_0$ , the function  $f$  is not differentiable at  $x_0$ . (Hint: for every  $x_0$  and every  $m \geq 1$ , there exists an integer  $j$  such that  $j \leq 32^m x_0 \leq j+1$ , thanks to Exercise 5.4.3.)
- (d) Explain briefly why the result in (c) does not contradict Corollary 3.7.3.

4. (20 pts)

- (a) Prove that

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

or all integers  $n$  and all real  $\theta$ . This is the classical *DeMoivre's theorem*.

- (b) By equating imaginary parts in DeMoivre's formula, prove that

$$\sin n\theta = \sin^n \theta \left\{ \binom{n}{1} \cot^{n-1} \theta - \binom{n}{3} \cot^{n-3} \theta + \binom{n}{5} \cot^{n-5} \theta - \dots \right\}.$$

- (c) If  $0 < \theta < \pi/2$ , prove that

$$\sin(2m+1)\theta = \sin^{2m+1} \theta P_m(\cot^2 \theta)$$

where  $P_m$  is the polynomial of degree  $m$  given by

$$P_m(x) = \binom{2m+1}{1} x^m - \binom{2m+1}{3} x^{m-1} + \binom{2m+1}{5} x^{m-2} - \dots.$$

Use this to show that  $P_m$  has zeros at the  $m$  distinct points

$$x_k = \cot^2 \left( \frac{\pi k}{2m+1} \right), \quad k = 1, 2, \dots, m.$$

- (d) Show that the sum of the zeros of  $P_m$  is given by

$$\sum_{k=1}^m \cot^2 \left( \frac{\pi k}{2m+1} \right) = \frac{m(2m-1)}{3}.$$

5. (20 pts)

This exercise outlines a simple proof of the formula  $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$ . Start with the inequality

$$\sin x < x < \tan x, \quad 0 < x < \frac{\pi}{2},$$

take reciprocals, and square each member to obtain

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x.$$

Now put  $x = \frac{k\pi}{2m+1}$ , where  $k$  and  $m$  are integers with  $1 \leq k \leq m$ , and sum on  $k$  to obtain

$$\sum_{k=1}^m \cot^2 \left( \frac{k\pi}{2m+1} \right) < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^m \frac{1}{k^2} < m + \sum_{k=1}^m \cot^2 \left( \frac{k\pi}{2m+1} \right).$$

Use the formula in problem 4(d) to deduce the inequality

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < \sum_{k=1}^m \frac{1}{k^2} < \frac{2m(m+1)\pi^2}{3(2m+1)^2}.$$

Now let  $m \rightarrow \infty$  to obtain  $\zeta(2) = \pi^2/6$ .