November 6, 2025

Introduction to Mathematical Analysis
Homework 8 Due November 14 (Friday), 2025
Please submit your homework online in PDF format.

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1. (25 pts) Give examples of a formal power series

$$\sum_{n=0}^{\infty} c_n x^n$$

centered at 0 with radius of convergence 1, which

(a) diverges at both $x = 1$ and $x = -1$;

(b) diverges at $x = 1$ but converges at $x = -1$;

(c) converges at $x = 1$ but diverges at $x = -1$;

(d) converges at both $x = 1$ and $x = -1$;

(e) converges pointwise on $(-1, 1)$, but does not converge uniformly on $(-1, 1)$.

2. (25 pts) **Exercise 4.2.7.** Let $m \geq 0$ be a positive integer, and let $0 < r$ be real numbers. Prove the identity

$$\frac{r}{r-x} = \sum_{n=0}^{\infty} x^n r^{-n}$$

for all $x \in (-r, r)$.

Using Proposition 4.2.6, conclude the identity

$$\frac{r}{(r-x)^{m+1}} = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}$$

for all integers $m \geq 0$ and all $x \in (-r, r)$. Also explain why the series on the right-hand side is absolutely convergent.

3. (25 pts) Let E be a subset of $\mathbb{R}$, let a be an interior point of E, and let $f : E \to \mathbb{R}$ be a function which is real analytic at a and has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

at a which converges on the interval $(a-r,\, a+r)$. Let $(b-s,\, b+s)$ be any subinterval of $(a-r,\, a+r)$ for some $s > 0$.

(a) Prove that $|a - b| \leq r - s$, so in particular $|a - b| < r$.

(b) Show that for every $0 < \varepsilon < r$, there exists a $C > 0$ such that $|c_n| \leq C(r-\varepsilon)^{-n}$ for all integers $n \geq 0$. *(Hint: what do we know about the radius of convergence of the series $\sum_{n=0}^{\infty} c_n (x-a)^n$?)*

(c) Show that the numbers $d_0, d_1, \ldots$, given by the formula

$$d_m := \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} (b-a)^{n-m} c_n \qquad \text{for all integers } m \geq 0,$$

are well-defined, in the sense that the above series is absolutely convergent. *(Hint: use (b) and the comparison test, Corollary 7.3.2, followed by Exercise 4.2.7.)*

(d) Show that for every $0 < \varepsilon < s$ there exists a $C > 0$ such that

$$|d_m| \leq C(s-\varepsilon)^{-m}$$

for all integers $m \geq 0$. *(Hint: use the comparison test, and Exercise 4.2.7.)*

(e) Show that the power series $\sum_{m=0}^{\infty} d_m(x-b)^m$ is absolutely convergent for $x \in (b-s,\, b+s)$ and converges to $f(x)$. (You may need Fubini's theorem for infinite series, Theorem 8.2.2 of *Analysis I*, as well as Exercise 4.2.5. One may also need to use a variant of the d_m in which the c_n are replaced by $|c_n|$.)

Note. You can use Exercise 4.2.5. Let a, b be real numbers, and let $n \geq 0$ be an integer. Prove the identity

$$(x-a)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m$$

for any real number x.

(f) Conclude that f is real analytic at b, and thus analytic at every point in $(a-r,\, a+r)$.

4. (25 pts)

(a) If each $a_n \geq 0$ and if $\sum a_n$ diverges, show that $\sum a_n x^n \to +\infty$ as $x \to 1^-$. (Assume $\sum a_n x^n$ converges for $|x| < 1$.)

(b) If each $a_n \geq 0$ and if $\lim_{x\to 1^-} \sum a_n x^n$ exists and equals A, prove that $\sum a_n$ converges and has sum A.

You can do the following problems to practice. You don't have to submit the following problems.

1. Let the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converge for $-1 < x < 1$. For each n, define the partial sum

$$s_n = \sum_{k=0}^{n} a_k, \qquad \sigma_n = \sum_{k=0}^{n} k|a_k|.$$

Suppose that $\lim\limits_{x\to 1^-} f(x) = S$ and $\lim\limits_{n\to\infty} na_n = 0$.

In this problem, you will show that the series $\sum_{n=0}^{\infty} a_n$ converges and that its sum is S.

(a) **Preliminary Identity.** Show that for any $x \in (0,1)$,

$$s_n - f(x) = \sum_{k=0}^{n} a_k(1-x^k) - \sum_{k=n+1}^{\infty} a_k x^k.$$

(b) **Bounding the First Sum.** Show that for all $m \geq 1$ and $x \in (0,1)$,

$$1 + x + \cdots + x^{m-1} \leq \frac{1}{1-x},$$

and deduce that

$$|1-x^k| = (1-x)(1+x+\cdots+x^{k-1}) \leq k(1-x).$$

(c) **Application of the Bound.** Use part (b) to prove that for $x \in (0,1)$,

$$\left|\sum_{k=0}^{n} a_k(1-x^k)\right| \leq (1-x)\sigma_n.$$

(d) **Estimate of the Tail.** Use the assumption that $\lim_{n\to\infty} n|a_n| = 0$ to show that for any $\varepsilon > 0$, there exists N such that for all $n \geq N$,

$$n|a_n| < \frac{\varepsilon}{3}.$$

Then prove that for such n and all $x \in (0,1)$,

$$\left|\sum_{k=n+1}^{\infty} a_k x^k\right| \leq \frac{\varepsilon}{3(1-x)}.$$

(e) **Putting the Estimates Together.** Combine parts (a)–(d) to show that for all $n \geq N$ and $x \in (0,1)$,

$$|s_n - S| \leq |f(x) - S| + (1-x)\sigma_n + \frac{\varepsilon}{3(1-x)}.$$

(f) **Strategic Choice of x.** Let $x = x_n = 1 - \frac{1}{n}$. Use part (e) to show that when n is sufficiently large,

$$|s_n - S| < \varepsilon.$$

Conclude that $s_n \to S$, and therefore

$$\sum_{n=0}^{\infty} a_n = S.$$

2. (1) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Show that the radius of convergence of $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ is $+\infty$.

   (2) Suppose that the power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ has radius of convergence $R < +\infty$. What can we say about the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$?

3. Let $(a_n)_{n\geq 1}$ be a sequence of nonzero real numbers such that

$$\frac{|a_{n+2}|}{|a_n|} \xrightarrow[n\to\infty]{} 2.$$

Show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{\sqrt{2}}$.