

微積分一

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Abstract

This note is the lecture note of the 齊震宇微積分一 on NTU OCW.

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Chapter 1

Some Basics

1.1 Introduction

Lecture 1: Introduction

Definition 1.1.1. Suppose we have a function $f(x)$, then the signed area between $x = a$ and $x = b$ is called the definite integral of the function f on the interval $[a, b]$.

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Notation. $\int_a^b f(x) \, dx$ is the definite integral of f on the interval $[a, b]$.

Let $A(x)$ be the signed area of the $f(x)$ on the interval $[a, x]$ of the below figure,

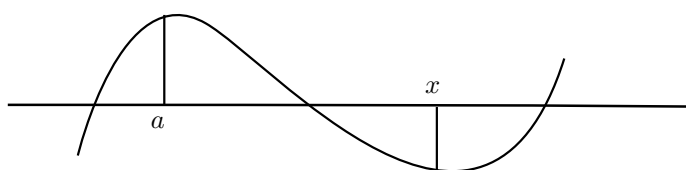


Figure 1.1: $y = f(x)$

then we can draw the figure of $A(x)$.

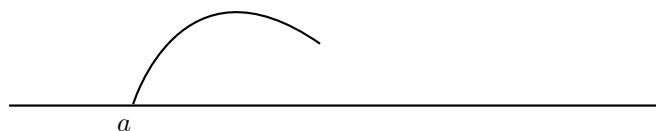


Figure 1.2: A part of $y = A(x)$.

Now we want to show that $y = A'(x)$ is the graph of $y = f(x)$. Compute

$$A'(x) = \frac{dA}{dx}(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \rightarrow 0} \frac{\text{signed area of } f(x) \text{ on } [x, x+h]}{h} \approx f(x)$$

since $h \rightarrow 0$, so the area divided by h is approximately equal to $f(x)$.

Also, we know $\int_a^b f(x) dx = A(b)$, but we only know $A(a) = 0$ and $A'(x) = f(x)$.

Suppose that we have found a function $g(x)$ such that $g'(x) = f(x)$. What is the relation between g and A ?

Consider the function $g(x) - A(x)$, we know its derivative is 0. Hence, we know $g(x) - A(x) = \text{constant}$. Therefore, we have

$$g(b) - g(a) = (A(b) + C) - (A(a) + C) = A(b) - A(a) = A(b).$$

By this, we have $\int_a^b f(x) dx = A(b) = g(b) - g(a)$.

Theorem 1.1.1. If $g' = f$, then $g(b) - g(a) = \int_a^b f(x) dx$.

Remark. Such a function g is called a primitive (function) (原函数) of f .

Remark. We have to believe that if $g'(x) = 0$ for some function g , then g is a constant function to finish the above proof.

Now we talk about continuity. If we give a function f on $[0, \frac{\pi}{2})$. We should notice that not every f has maximum or minimum value. For example, $y = \tan(x)$ has no maximum. We say that a function f defined on an interval I is continuous at $a \in I$ if $\lim_{x \rightarrow a} f(x) = f(a)$. We may think of that if f is a continuous function defined on a closed interval, then it must have maximum and minimum.

Now if we extend the concept of continuity to a parametrized curve $f(t) = (x(t), y(t))$, then we say that the parametrized curve is continuous if both $x(t)$ and $y(t)$ are, and we may think that whether there is a continuous parametrized curve f mapping \mathbb{R} to \mathbb{R}^2 such that $f(\mathbb{R}) = \mathbb{R}^2$. Or, if we think \mathbb{R} is too scary, then we can think that whether there is a continuous f mapping $[0, 1]$ to a triangle on \mathbb{R}^2 . This result is called Peano's curve.

1.2 Upper Bound and Lower Bound

Lecture 2

Definition 1.2.1. Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$. We say that

- r is a upper (resp. lower) bound of S if $\forall s \in S, r \geq$ (resp. \leq) s .
- r is the greatest/largest/highest (resp. least/smallest/lowest) element of S if r is a upper (resp. lower) bound of S and $r \in S$.

Notation. $r = \max S$ (resp. $\min S$).

- r is the least upper (greatest lower) bound of S if $r = \min \{u \in \mathbb{R} \mid u \text{ is a upper bound of } S\}$ (resp. $r = \max \{l \in \mathbb{R} \mid l \text{ is a lower bound of } S\}$).

Notation. $r = \sup S$ (resp. $\inf S$).

Remark. Every $r \in \mathbb{R}$ is a upper bound of \emptyset .

Note. • We write $\sup S = \infty$ (resp. $\inf S = -\infty$) if and only if S has no upper (resp. lower) bound. If this is the case, we say $\sup S$ (resp. $\inf S$) doesn't exist.

- S is bounded above (resp. below) iff S has a upper (resp. lower) bound.

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Definition 1.2.2 (Dedekind cut). Let $A, B \subseteq \mathbb{R}$. We say that (A, B) is a Dedekind cut (of \mathbb{R}) if

1. $A \neq \emptyset \neq B$
2. $A \cup B = \mathbb{R}$
3. $\forall a \in A, b \in B$ we have $a < b$.

We usually call A (resp. B) the lower (resp. upper) part of (A, B) .

Theorem 1.2.1. From now on (until Professor Chi say stop), we assume that \mathbb{R} has the following property (Dedekind's gapless property): If (A, B) is a D-cut of \mathbb{R} , then exactly one of the following happens:

1. $\max A$ exists but $\min B$ doesn't;
2. $\min B$ exists but $\max A$ doesn't.

We call the $\max A$ in 1. (resp. $\min B$ in 2.) the cutting of (A, B) .

Exercise. We may define Dedekind cuts of \mathbb{Q} , \mathbb{Z} similarly. Does the Dedekind gapless property hold for \mathbb{Q} or \mathbb{Z} ?

Hint: Consider $B = \{x \in \mathbb{Q} \mid x > 0, x^2 > 2\}$. We are allowed to use the fact that there does not exist a rational number x such that $x^2 = 2$.

Proof. We first show that for \mathbb{Q} this is incorrect. Consider $B = \{x \in \mathbb{Q} \mid x > 0, x^2 > 2\}$, and let $A = \mathbb{Q} \setminus B$. First, notice that $\min B$ does not exist since if $b = \min B$, then $b^2 > 2$. Now we want to construct a $b - \varepsilon$ such that $(b - \varepsilon)^2 > 2$ with $\varepsilon \in \mathbb{Q}$ and $b - \varepsilon > 0$, which is equivalent to

$$b^2 - 2\varepsilon b + \varepsilon^2 > 2 = b^2 - \delta \Leftrightarrow \delta > 2\varepsilon b - \varepsilon^2 = \varepsilon(2b - \varepsilon).$$

Hence, we can pick some $\varepsilon \in \mathbb{Q}$ such that $\frac{\delta}{2b} > \varepsilon$, then we have

$$\delta > 2b\varepsilon > 2b\varepsilon - \varepsilon^2,$$

then we know $\min B$ does not exist. Notice that by Archimedean Property we must can pick such ε . Also, by same method, we can show that $\max A$ does not exist.

As for \mathbb{Z} , we can let $A = \{z \in \mathbb{Z} \mid z \leq 1\}$ and $B = \{z \in \mathbb{Z} \mid z > 1\}$, then it can be easily seen that $\min B$ and $\max A$ simultaneously exist. ■

Theorem 1.2.2 (Weierstrass). Let $\emptyset \neq S \subseteq \mathbb{R}$. If S has a upper bound, then $\sup S$ exists.

Proof. Let B be the set of every upper bound of S and suppose $A := \mathbb{R} \setminus B$. We need to show that $\min B$ exists. We first show that (A, B) is a Dedekind cut of \mathbb{R} .

Since we know S is not empty, so we know $\forall s \in S$, $s - 1$ is not an upper bound, and thus $s - 1 \in A$, which means A is not empty, and we know B is not empty by the description of the problem.

Also, it is trivial that $A \cup B = \mathbb{R}$.

For $a \in A$ and $b \in B$, we need to show that $a < b$. Were this false, $a \geq b$, and hence a is a upper bound of S since b is a upper bound, so $a \in B$. However, $A \cap B = \emptyset$. Hence, we have shown that (A, B) is a Dedekind cut of \mathbb{R} .

Now we want to use Dedekind's gapless property to say that we must have $\min B$ exists. Were this false, we know $\max A$ exists, denoted by a_0 . Note that $a_0 \in A$, so $a_0 \notin B$, and thus a_0 is not an upper bound of S . Hence, there exists $s_0 \in S$ such that $s_0 > a_0$, but this implies $s_0 \notin A$, so $s_0 \in B$, and thus s_0 is a upper bound of S . Now choose some x such that $a_0 < x < s_0$, so $x \in B$ and thus x is an upper bound of S , but we have $x < s_0 \in S$, which is a contradiction. ■

Theorem 1.2.3 (the Archimedean Property). Prove the following statement: $\forall r \in \mathbb{R}$, then $r > 0$ implies that $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < r$.

Hint: Rephrase this statement in a way linking it to the upper bound of the set $S = \mathbb{N} \subseteq \mathbb{R}$.

Proof. It is equivalent to show that $\forall \frac{1}{r} \in \mathbb{R}^+$, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{r}$. If not, says $\frac{1}{r} \in \mathbb{R}^+$ and for all $n \in \mathbb{N}$ we have $\frac{1}{r} > n$. Hence, $\frac{1}{r}$ is an upper bound of \mathbb{N} . Hence, by Weierstrass we know $\sup \mathbb{N}$ exists. Now we will show that in fact $\sup \mathbb{N}$ does not exist. Suppose $\alpha = \sup \mathbb{N}$, then $\exists m \in \mathbb{N}$ such that $m > \alpha - 1$, otherwise $\alpha - 1 < \alpha = \sup \mathbb{N}$ is also an upper bound of \mathbb{N} which is a contradiction. Thus, we know $m + 1 > \alpha$ and $m + 1 \in \mathbb{N}$, but this implies α is not an upper bound of \mathbb{N} , which is a contradiction. Hence, $\sup \mathbb{N}$ does not exist. ■

1.3 Sequence

Definition. We can define the limit of the sequence as follow.

Definition 1.3.1. Let $a_n (n \in \mathbb{N})$ or $\{a_n\}_{n=1}^{\infty}$ be a sequence in \mathbb{R} and $L \in \mathbb{R}$. We say that a_n converges to L (as $n \rightarrow \infty$) if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \varepsilon$.

Notation. $\lim_{n \rightarrow \infty} a_n = L$.

Note. If such L exists, we call it the limit of a_n and we call $\{a_n\}$ a convergent sequence, otherwise we call it a divergent sequence.

Definition 1.3.2. $\lim_{n \rightarrow \infty} a_n = \infty (-\infty)$ means $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $n > N$ implies $a_n \geq (\leq) M$.

Exercise. 1. $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} a_n = M$ implies $L = M$.

2. $\{a_n\}$ is convergent implies $\{a_n\}$ is bounded.

3. Suppose we have $\{a_n\}$, $\{b_n\}$, and $a_n \leq b_n$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} a_n = L$, $\lim_{n \rightarrow \infty} b_n = M$, prove that $L \leq M$. What if \leq is replaced by $<$, is this statement still correct?

proof of 1. WLOG, suppose $L > M$, then we know $\forall \frac{\varepsilon}{2} > 0$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$$\begin{aligned} n \geq N_1 &\Rightarrow |a_n - L| < \frac{\varepsilon}{2} \\ n \geq N_2 &\Rightarrow |a_n - M| < \frac{\varepsilon}{2}. \end{aligned}$$

Hence, we know $n > \max\{N_1, N_2\}$ implies

$$|L - M| = |(a_n - L) - (a_n - M)| < |a_n - L| + |a_n - M| < \varepsilon.$$

If $|L - M| = \delta > 0$, then we can pick some $\varepsilon' < \delta$ and then we will get a contradiction. Hence, $\delta = 0$, which means $L = M$. ■

proof of 2. First, $\{a_n\}$ converges implies $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N$ implies $|a_n - L| < \varepsilon$ for some L . Say, for $\varepsilon = \varepsilon_1$, the corresponding N is equal to N_1 . Hence, for $n \geq N_1$, we have $a_n < L + \varepsilon_1$. Hence, we know $\max\{a_1, a_2, \dots, a_{N_1-1}, L + \varepsilon_1\}$ is an upper bound of $\{a_n\}$. Similarly, we can know $\min\{a_1, a_2, \dots, a_{N_1-1}, L - \varepsilon_1\}$ is a lower bound of $\{a_n\}$. ■

proof of 3. First, we know $\forall \varepsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 &\Rightarrow |a_n - L| < \varepsilon \\ n \geq N_2 &\Rightarrow |b_n - M| < \varepsilon. \end{aligned}$$

Now if $L > M$, say $L = M + \delta$, where $\delta > 0$, then we can pick $\varepsilon < \frac{\delta}{2}$ so that for $n \geq \max\{N_1, N_2\}$ we have

$$\begin{aligned} L - \varepsilon &< a_n < L + \varepsilon \\ M - \varepsilon &< b_n < M + \varepsilon. \end{aligned}$$

Also, we know

$$b_n < M + \varepsilon < M + \frac{\delta}{2} = L - \frac{\delta}{2} < L - \varepsilon < a_n,$$

which is a contradiction, so $L \leq M$. ■

Remark. Changing or removing finitely many terms in a_n does not affect $\{a_n\}$ is convergent or not (and its limit).

Proposition 1.3.1. If $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = M$, then

1. $\lim_{n \rightarrow \infty} (a_n \pm b_n) = L \pm M$.
2. $\lim_{n \rightarrow \infty} a_n \cdot b_n = LM$.
3. If $M \neq 0$, then $b_n \neq 0$ for all but finitely many n , and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}$.

proof of 1. Consider $|(a_n \pm b_n) - (L \pm M)|$. We can see that

$$|(a_n \pm b_n) - (L \pm M)| = |(a_n - L) \pm (b_n - M)| \leq |a_n - L| + |b_n - M|.$$

Also, we know $\forall \varepsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow |a_n - L| < \varepsilon$ and $n \geq N_2 \Rightarrow |b_n - M| < \varepsilon$. Let $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$|(a_n \pm b_n) - (L \pm M)| \leq |a_n - L| + |b_n - M| < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence, we can pick $\varepsilon_1 = \frac{\varepsilon}{2} = \varepsilon_2$ so that $n \geq N'_1 \Rightarrow |a_n - L| < \varepsilon_1 = \frac{\varepsilon}{2}$ and $n \geq N'_2 \Rightarrow |b_n - M| < \varepsilon_2 = \frac{\varepsilon}{2}$, and thus we know $n \geq \max\{N'_1, N'_2\}$ implies that

$$|(a_n \pm b_n) - (L \pm M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

proof of 2. Consider $|a_n b_n - LM|$. Notice that

$$|a_n b_n - LM| = |a_n b_n - L b_n + L b_n - LM| \leq |a_n - L| |b_n| + |L| |b_n - M|.$$

If we can choose $C > 0$ such that $|b_n| \leq C$ for all $n \in \mathbb{N}$ and $|L| \leq C$, then we have

$$|a_n - L| |b_n| + |L| |b_n - M| \leq C |a_n - L| + C |b_n - M|.$$

Hence, we want to find some N_1, N_2 such that $\forall \varepsilon > 0$, we have $n \geq \max\{N_1, N_2\}$ implies $|a_n - L| \leq \frac{\varepsilon}{2C}$ and $|b_n - M| \leq \frac{\varepsilon}{2C}$, and then we're done.

Now since $\{b_n\}$ is convergent, so it is bounded and thus we can pick such C by a previous exercise. ■

proof of 3. We only need to prove $\lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{1}{M}$, and removing the terms being zeros will not affect the convergence of this sequence. ■

Note. What if $L, M = \pm\infty$?

Exercise. Suppose $a_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}}$, then $\lim_{n \rightarrow \infty} a_n = \frac{1}{1 - \frac{1}{2}}$.

Proof. We want to find an $N \in \mathbb{N}$ so that for all $n \geq N$ we have

$$\left| \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{2}} \right| < \varepsilon.$$

By simplifying this, we have

$$\frac{1}{\varepsilon} < 2^{n-1},$$

so we just need to find an $n \in \mathbb{N}$ such that $n > 1 - \frac{\ln \varepsilon}{\ln 2}$, then we are done. ■

Lecture 3

As previously seen. We can think a sequence as a function mapping from \mathbb{N} to \mathbb{R} , and the sequence converges means when N is big enough, then the value of every term after a_N will be located in a closed interval $[L - \varepsilon, L + \varepsilon]$.

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Example. If $a > 1$, then $\lim_{n \rightarrow \infty} \frac{1}{a^n} = 0$.

Proof. First, we know

$$\frac{1}{a^n} = \frac{1}{(1 + (a - 1))^n} \leq \frac{1}{1 + n(a - 1)} \leq \frac{1}{n(a - 1)}.$$

Then, we can use the deduction that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ to prove that this is correct. (We may need to use the following argument.)

Exercise. Suppose $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = L$, now if $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$, show that $\lim_{n \rightarrow \infty} c_n = L$.

Now we know $0 \leq \frac{1}{a^n} \leq \frac{1}{n(a-1)}$, so we can prove $\lim_{n \rightarrow \infty} \frac{1}{a^n} = 0$. ⊛

Definition 1.3.3. A sequence a_n in \mathbb{R} is

1. nondecreasingly monotone / increasing if $a_n \leq a_{n+1} \forall n \in \mathbb{N}$.
2. nonincreasingly monotone / decreasing if $a_n \geq a_{n+1} \forall n \in \mathbb{N}$.
3. strictly increasing (resp. strictly decreasing) if $\forall n \in \mathbb{N}$, we have $a_n < a_{n+1}$ (resp. $a_n > a_{n+1}$).

Theorem 1.3.1. If a_n is nondecreasing and $\{a_n \mid n \in \mathbb{N}\}$ has a upper bound, then a_n converges to $\sup \{a_n \mid n \in \mathbb{N}\}$.

Proof. $\{a_n \mid n \in \mathbb{N}\}$ has an upper bound, so $L := \sup \{a_n \mid n \in \mathbb{N}\}$ exists. Now we claim that $\lim_{n \rightarrow \infty} a_n = L$.

First, we know $\forall \varepsilon > 0$, $L - \varepsilon < L$, and hence $\exists N \in \mathbb{N}$ such that $L - \varepsilon < a_N$, otherwise $L - \varepsilon$ is an upper bound. Since a_n is nondecreasing, so $\forall n \geq N$, we have

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon.$$

Hence, $|a_n - L| < \varepsilon$. ■

Example. A decimal expression gives a real number, say it is $0.d_1d_2d_3\dots$, and suppose

$$a_n = 0.d_1d_2d_3\dots = \frac{d_1}{10} + \dots + \frac{d_n}{10^n}.$$

Since we know

$$a_n < \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} = \frac{9}{10} \frac{1 - \left(\frac{1}{10}\right)^{n+1}}{1 - \frac{1}{10}} < 1,$$

and since $\{a_n\}$ is nondecreasing, so $\lim_{n \rightarrow \infty} a_n = 0.d_1d_2\dots$

Note. Hence, a real number may have more than one way of representation. For example, 0.1 and 0.09999... are same, this can be seen by the limits of both of the decimal representation.

Example (The natural base e). We first define

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n,$$

but we have to show that this limit exists.

Proof. Suppose

$$a_n = \left(1 + \frac{1}{n}\right)^n = \sum_{i=0}^n \binom{n}{i} \left(\frac{1}{n}\right)^i,$$

we know

$$\begin{aligned} \binom{n}{i} \left(\frac{1}{n}\right)^i &= \frac{n!}{i!(n-i)!} \frac{1}{n^i} = \frac{1}{i!} \left(\frac{n(n-1)\dots(n-i+1)}{n \cdot n \dots n} \right) \\ &= \frac{1}{i!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{i-1}{n}\right). \end{aligned}$$

Hence,

$$a_n = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) \dots + \frac{1}{i!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{i-1}{n}\right) + \dots$$

By this, we can see that $a_n < a_{n+1}$ since we can see that by replacing n with $n+1$ in the above equation then it becomes a_{n+1} . Also, we can see that

$$\begin{aligned} a_n &< 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{i!} + \dots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{i-1}} + \dots + \frac{1}{2^{n-1}} \\ &= 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} < 1 + \frac{1}{\frac{1}{2}} = 3, \end{aligned}$$

so it has upper bound and nondecreasing, and thus it converges and e is well-defined.

Note. In the near future, we will see that $e = \lim_{n \rightarrow \infty} 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}$.

⊛

Definition 1.3.4. A sequence of intervals I_n ($n \in \mathbb{N}$) is nested if $I_n \neq \emptyset$ and $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. ($I_1 \supseteq I_2 \supseteq \dots$).

Now we want to know $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$?

Here is some counterexamples. Consider $I_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$. We can show that $\bigcap_{n=1}^{\infty} I_n = \emptyset$ by Archimedean Property. Besides, if $I_n = [n, \infty)$, $n \in \mathbb{N}$, this is trivial that $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Theorem 1.3.2 (Theorem of nested intervals). If I_n ($n \in \mathbb{N}$) is a sequence of bounded closed nested intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Write $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$. First, we know I_n is nested iff $a_n \leq b_n$ and a_n is nondecreasing and b_n is nonincreasing. Hence, $\forall n, m \in \mathbb{N}$, we have $a_n \leq a_{\max\{n, m\}} \leq b_{\max\{n, m\}} \leq b_m$. In other words, for every $m \in \mathbb{N}$, b_m is an upper bound of $\{a_n\}$. Hence, we know $c = \lim_{n \rightarrow \infty} a_n = \sup \{a_n\}$ exists. Then, $c \leq b_m$ for all $m \in \mathbb{N}$. Also, $c \geq a_n$ for all $n \in \mathbb{N}$. Hence, $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$, and thus we know $c \in I_n$ for all $n \in \mathbb{N}$. Thus, $c \in \bigcap_{n=1}^{\infty} I_n$. ■

Exercise. What if $I_n = (a_n, b_n)$ is nested but a_n is strictly increasing and b_n is strictly decreasing, is theorem of nested interval still correct?

Exercise. $I_n = (a_n, \infty)$ is nested and $\{a_n\}$ bounded from above, is theorem of nested interval still correct?

Exercise. We can use Theorem 1.3.2 and Theorem 1.2.3 to show Theorem 1.2.1.

Now we have a neq question, if we have a sequence $\{a_n\}$ in \mathbb{R} , can we determine whether $\{a_n\}$ converges or not without referring a limit candidate L but concluding according to the mutual behaviour of the terms of $\{a_n\}$.

Definition 1.3.5 (Cauchy Sequence). A sequence $\{a_n\}$ in \mathbb{R} is a Cauchy Sequence if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $n, m \geq N$ implies $|a_n - a_m| < \varepsilon$.

Exercise. a_n is convergent implies a_n is a Cauchy sequence.

Exercise. If a_n is a Cauchy sequence, then a_n is bounded.

Theorem 1.3.3. Let a_n be a sequence in \mathbb{R} , then

$$a_n \text{ is convergent} \Leftrightarrow a_n \text{ is Cauchy.}$$

proof from Cauchy to convergent. We first give a part of the proof as exercise.

Definition. Let a_n be a bounded sequence in \mathbb{R} .

Definition. $u_n := \sup \{a_m \mid m \geq n\}$.

Definition. $l_n := \inf \{a_m \mid m \geq n\}$.

Now we know $l_n \leq a_m \leq u_n$ for all $m, n \in \mathbb{N}$ and $m \geq n$. Also, we know l_n is increasing and u_n is decreasing.

Exercise. a_n converge if and only if $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} l_n$, and if any of both sides holds, then

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} l_n.$$

Let a_n, b_n be two bounded sequences, then

$$\lim_{n \rightarrow \infty} u_{a_n+b_n} \leq \lim_{n \rightarrow \infty} u_{a_n} + \lim_{n \rightarrow \infty} u_{b_n}, \quad \lim_{n \rightarrow \infty} l_{a_n+b_n} \geq \lim_{n \rightarrow \infty} l_{a_n} + \lim_{n \rightarrow \infty} l_{b_n}.$$

(Why?)

Now we start the proof. Assume that a_n is Cauchy. We claim that $\lim_{n \rightarrow \infty} (u_n - l_n) = 0$. For

$\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|a_n - a_m| < \varepsilon$. In particular, $n \geq N$ implies

$$a_N - \varepsilon < a_n < a_N + \varepsilon,$$

which also implies

$$a_N - \varepsilon \leq l_N \leq u_N \leq a_N + \varepsilon,$$

so we have

$$0 \leq u_n - l_n \leq u_N - l_N \leq (a_N + \varepsilon) - (a_N - \varepsilon) = 2\varepsilon.$$

By adjusting the coefficient of ε we can show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $0 \leq u_n - l_n \leq \varepsilon$, which means $\lim_{n \rightarrow \infty} (u_n - l_n) = 0$.

Hence, by the exercise above, we know $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} l_n$, and a_n is convergent. ■

Lecture 4

We first finish the proof above.

11 July. 9:00

Exercise. Let $S \subseteq \mathbb{R}$. Prove that if $|s - s'| \leq 3$ for all $s, s' \in S$, then

1. S is bounded.
2. $\sup S - \inf S \leq 3$.

Exercise. If we change the \leq to $<$ in the above exercise, is it still correct?

1.4 Series

Definition 1.4.1. Let a_n be a sequence in \mathbb{R} . We say that the series $\sum_{i=1}^{\infty} a_i$ converges to / has sum a real number S if $\lim_{n \rightarrow \infty} s_n = S$, where $s_n := \sum_{i=1}^n a_i$, the n -th partial sum of $\sum_i a_i$. If such S exists (resp. does not exist), we say that the series $\sum a_n$ is convergent (resp. divergent).

Example. Suppose $a_n = \frac{1}{2^n}$, then $s_n = 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}}$, so $\lim_{n \rightarrow \infty} s_n = \frac{1}{1 - \frac{1}{2}} = 2$.

Note. However, given a sequence a_n , it is not always possible to write down s_n explicitly.

Example. $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = ?$, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n-1}}{n} = ?$

Notation. For a series $\sum_n a_n$ and $l, m \in \mathbb{N}$, $l < m$, we let

$$s_{l,m} := \sum_{i=l}^m a_i,$$

the (l, m) -tail.

Exercise. $\sum_n a_n = 0$ implies $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. We know $a_n = s_n - s_{n-1}$ and also $\lim_{n \rightarrow \infty} s_n - s_{n-1} = 0$ if $\sum_n a_n$ is convergent. ■

As previously seen. $\sum_n a_n$ converges iff s_n converges iff s_n is Cauchy, i.e. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such

that $n, m \geq N$ implies $|s_n - s_m| \leq \varepsilon$.

Now if $n > m$, then $s_n - s_m = a_{m+1} + \dots + a_n$, so we can rewrite the statement as: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $k > N$ and $l \geq 0$ implies $|a_k + \dots + a_{k+l}| < \varepsilon$. Hence, there is another equivalent condition of converging, which is: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $k > N$ and $l \geq 0$ implies $|a_k| + \dots + |a_{k+l}| < \varepsilon$. If we have this, we can deduce that $\sum_n a_n$ is convergent since by triangle inequality this implies the above Cauchy condition. Also, notice that " $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $k > N$ and $l \geq 0$ implies $|a_k| + \dots + |a_{k+l}| < \varepsilon$ " is equivalent to $\sum_n |a_n|$ is convergent. Besides, notice that

$$\sum_n |a_n| \text{ converges if and only if } \exists M > 0 \forall n \in \mathbb{N} \text{ we have } |a_1| + |a_2| + \dots + |a_n| \leq M,$$

since $\{s_n\}$ is nondecreasing.

This tells us

Corollary 1.4.1. If b_n is a positive sequence, then $\sum_n b_n$ converges if and only if $\exists M > 0$ such that $\forall n \in \mathbb{N}$ we have $b_1 + b_2 + \dots + b_n \leq M$.

and the important result is

Corollary 1.4.2. If $\sum_n |a_n|$ converges, then $\sum_n a_n$ converges.

Notation. If $a_n \geq 0$ for all n , then we write $\sum_n a_n < \infty$ (resp. ∞) to mean that $\sum_n a_n$ converges (resp. diverges).

Example. Is $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent?

Proof. Notice that

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}.$$

Hence, we have $\forall m \in \mathbb{N}$,

$$s_{2^m} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^{m-1} + 1} + \dots + \frac{1}{2^{m-1} + 2^{m-1}} > 1 + \frac{m}{2},$$

so s_n is unbounded, and thus $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. *

Example. Is $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent?

Proof. Since

$$\frac{1}{n^2} < \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n},$$

so we know

$$s_n = 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2} < 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2.$$

Hence, we know s_n is bounded above and nondecreasing, and thus s_n is convergent. *

Example. Is $\sum_{n=1}^{\infty} \frac{\sin n}{n^k}$ convergent for some $k \geq 2$ and $k \in \mathbb{N}$?

Proof. Since we know

$$\left| \frac{\sin 1}{1^k} \right| + \dots + \left| \frac{\sin n}{n^k} \right| < 1 + \dots + \frac{1}{n^k} < 1 + \dots + \frac{1}{n^2} < 2 \quad \forall n \in \mathbb{N}.$$

Hence, we know $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^k} \right| < \infty$, and thus $\sum_{n=1}^{\infty} \frac{\sin n}{n^k}$ converges. *

Exercise. For $a > 1$ and $k \in \mathbb{N}$, show that $\sum_{n=1}^{\infty} \frac{n^k}{a^n} < \infty$.

Definition 1.4.2. Given a sequence a_n , we say that

1. $\sum_n a_n$ converges absolutely if $\sum_n |a_n| < \infty$ (and thus $\sum_n a_n$ converges).
2. $\sum_n a_n$ converges conditionally if $\sum_n |a_n| = \infty$ but $\sum_n a_n$ converges.

1.4.1 Comparison Test

Theorem 1.4.1 (Comparison Test). Given a_n, b_n are two non-negative sequences, then if " $\exists C > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq Cb_n$ ", then " $\sum_n b_n < \infty$ implies $\sum_n a_n < \infty$ ".

Proof. For $n \geq N$, we have

$$\begin{aligned} a_1 + \cdots + a_n &= a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n \leq a_1 + \cdots + a_N + C(b_{N+1} + \cdots + b_n) \\ &\leq a_1 + \cdots + a_N + CM \quad \text{for some } M > 0. \end{aligned}$$

■

Claim. If a_n, b_n are two non-negative sequences, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists implies $\exists C > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq Cb_n$.

Proof. Let $l := \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ and $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $\left| \frac{a_n}{b_n} - l \right| \leq 1$. Hence,

$$\frac{a_n}{b_n} \leq l + 1 \Leftrightarrow a_n \leq (l + 1)b_n.$$

⊛

Corollary 1.4.3. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists, then

$$\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty.$$

Exercise. If $\lim_{n \rightarrow \infty} u_n$ exists, where $u_n = \sup \left\{ \frac{a_m}{b_m} \mid m \geq n \right\}$, then " $\exists C > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq Cb_n$ ".

Exercise (The ratio test). Let a_n be a non-negative sequence. Then show that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \exists N \in \mathbb{N} \text{ and } C < 1 \text{ such that } n \geq N \text{ implies } a_{n+1} \leq Ca_n,$$

and thus $\sum_n a_n$ converges. Besides, prove that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \exists N \in \mathbb{N} \text{ and } C > 1 \text{ such that } n \geq N \text{ implies } a_{n+1} \geq Ca_n,$$

and thus $\sum_n a_n = \infty$. However, if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$, then we cannot conclude anything.

Exercise (The root test). If a_n is a non-negative sequence, then

$$\lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} < 1 \Rightarrow \exists N \in \mathbb{N} \text{ and } C < 1 \text{ such that } n \geq N \text{ implies } a_n \leq C^n,$$

and thus $\sum_n a_n$ converges. Besides, prove that

$$\lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} > 1 \Rightarrow \exists N \in \mathbb{N} \text{ and } C > 1 \text{ such that } n \geq N \text{ implies } a_n \geq C^n,$$

and thus $\sum_n a_n = \infty$.

Now we have a question: Does $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges (conditionally)?

Definition 1.4.3. A series $\sum_n a_n$ is an alternating series if $\exists b_n > 0$ ($n \in \mathbb{N}$) such that

$$a_n = (-1)^{n-1} b_n (n \in \mathbb{N}).$$

Theorem 1.4.2 (Leibniz's criterion). If $\sum_n a_n$ is an alternating series and $b_n = |a_n|$ is decreasing and converging to 0 as $n \rightarrow \infty$, then $\sum_n a_n$ converges.

Proof. Suppose $b_n = (-1)^{n-1} a_n$

$$\begin{aligned} |a_k + \dots + a_{k+l}| &= |(-1)^{k-1} (b_k - b_{k+1} + \dots + (-1)^l b_{k+l})| \\ &= b_k - b_{k+1} + \dots + (-1)^l b_{k+l} \\ &= \begin{cases} b_k - (b_{k+1} - b_{k+2}) - \dots - (b_{k+l-1} - b_{k+l}), & \text{if } 2 \mid l; \\ b_k - (b_{k+1} - b_{k+2}) - \dots - (b_{k+l-2} - b_{k+l-1}) - b_{k+l}, & \text{otherwise.} \end{cases} \\ &\leq b_k = |a_k| \end{aligned}$$

Since $\lim_{n \rightarrow \infty} b_n = 0$, so $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |b_n| < \varepsilon$. Hence, if $k \geq N$, we have

$$|a_k + \dots + a_{k+l}| \leq b_k \leq \varepsilon \quad \forall l > 0.$$

■

Given a sequence a_n , we can separate all terms into two sequences. The first sequence is

$$a_{n_1}, a_{n_2}, \dots,$$

while the second one is

$$a_{n'_1}, a_{n'_2}, \dots$$

with $n_1 < n_2 < \dots$ and $n'_1 < n'_2 < \dots$ and $\{n_1, n_2, \dots\} \cup \{n'_1, n'_2, \dots\} = \mathbb{N}$ such that $a_{n_j} \geq 0$ and $a_{n'_k} \leq 0$ for all $j, k \in \mathbb{N}$. Let $p_j := a_{n_j}$ and $q_k := -a_{n'_k}$ to construct two non-negative sequences.

Exercise. Show that $\sum_n |a_n| < \infty$ iff $\sum_j p_j$ and $\sum_k q_k$ are both convergent. Moreover, if any side holds, then

$$\sum_n |a_n| = \sum_j p_j + \sum_k q_k \text{ and } \sum_n a_n = \sum_j p_j - \sum_k q_k.$$

Exercise. Suppose $\sum_n a_n$ and $\sum_n b_n$ are both convergent, then

$$\sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n.$$

Exercise. Inserting 0s to a series will not affect its convergence / divergence and its sum if the sum exists.

1.4.2 Rearrangement

Definition 1.4.4. We know a sequence is a map from \mathbb{N} to \mathbb{R} , and a subsequence is a double-map from \mathbb{N} to \mathbb{N} to \mathbb{R} , where the first map from \mathbb{N} to \mathbb{N} is increasing. Moreover, a rearrangement is also a double-map from \mathbb{N} to \mathbb{N} to \mathbb{R} , where the first map is a bijection.

Now we have a question. If $\sum_n a_n$ converges, then

1. is $\sum_m a_{n(m)}$ converges where $a_{n(m)}$ is a rearrangement of a_n ?
2. $\sum_m a_{n(m)} = \sum_n a_n$?

Note. We will prove that if the series is absolutely convergent, then rearrangement does not affect the sum of the infinite series.

1. If $a_n \geq 0$ for $n \in \mathbb{N}$, the answers are affirmative.
2. If $\sum_n |a_n| < \infty$, then the answers are affirmative.

In fact, 1. implies 2 (by the 4-th exercise in [subsection 1.4.1](#)). (Why?)

Why does 1. hold? Actually $\sum_n a_n = \sup \{a_{n_1} + a_{n_2} + \cdots + a_{n_k} \mid n_1 < \cdots < n_k, k \in \mathbb{N}\}$ (including the case ∞) (Why?), and hence 1. holds.

Theorem 1.4.3 (Dirichlet's Rearrangement theorem (1829)). If $\sum_n a_n$ converges absolutely, then for every rearrangement $a_{n(m)}$ we have $\sum_{m=1}^{\infty} a_{n(m)} = \sum_{n=1}^{\infty} a_n$.

Theorem 1.4.4 (Riemann's rearrangement theorem (1852)). If $\sum_n a_n$ converges conditionally, then for every $L \in \mathbb{R}$ there exists a rearrangement $a_{n(m)}$ of a_n such that $\sum_{m=1}^{\infty} a_{n(m)} = L$.

Proof. We will only use two known facts given by the conditional convergence of $\sum_n a_n$:

1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$.
2. $\lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} q_n = 0$. (Since p_n, q_n are subsequence of a_n .)

Our thought is to add up $p_1 + p_2 + \cdots + p_{m_1}$ so that

$$p_1 + p_2 + \cdots + p_{m_1-1} < L < p_1 + p_2 + \cdots + p_{m_1}.$$

Now we start to minus $q_1 + q_2 + \cdots + q_{m'_1}$ so that

$$\sum_{i=1}^{m_1} p_i - \sum_{i=1}^{m'_1} q_i < L < \sum_{i=1}^{m_1} p_i - \sum_{i=1}^{m'_1-1} q_i,$$

and we continue to add up some p_i to make the series bigger than L , and jump back by minusing some q_i . This is feasible since L is not a upper bound of $\sum_n p_n$ and not a lower bound of $\sum_n q_n$. Note that this method construct many partial sums of some rearrangement, say the partial sum of the rearrangement is s_n , we want to show $\lim_{n \rightarrow \infty} s_n = L$.

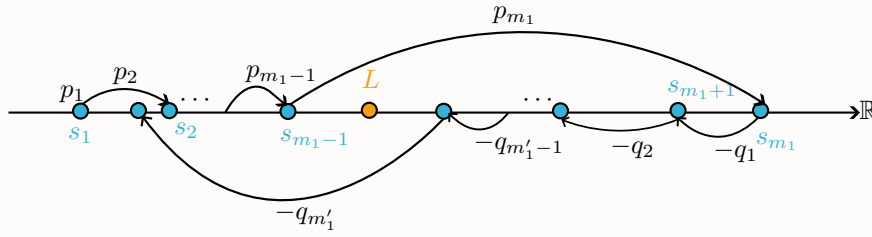


Figure 1.3: Riemann jump

Since \exists natural numbers $m_1 < m_2 < \dots$ and $m'_1 < m'_2 < \dots$ such that

$$\left| s_{m'_{l-1}+m_l+k'} - L \right| < p_{m_l} \quad \text{if } 0 \leq k' < m'_l - m'_{l-1}.$$

Similarly,

$$\left| s_{m_l+m'_l+k} - L \right| < q_{m'_l} \quad \text{if } 0 \leq k < m_{l+1} - m_l$$

if we start jumping from q . Since we know $\lim_{n \rightarrow \infty} p_{m_l} = \lim_{n \rightarrow \infty} q_{m'_l} = 0$, so $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $l \geq N_0$ implies p_{m_l} and $q_{m'_l} < \varepsilon$. Let $N = m'_{N_0-1} + m_{N_0}$. Then $n \geq N$ implies $|s_n - L| < \varepsilon$. ■

Remark. In 1827, Dirichlet made the following observation:

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \\ 2S &= 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \dots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \end{aligned}$$

by combining some terms.

Multiplying absolutely convergent series

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge absolutely.

Theorem 1.4.5. Let $c_n = a_n b_0 + \dots + a_0 b_n = \sum_{i+k=n} a_i b_k$. Then $\sum_n |c_n| < \infty$ and

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{i=0}^{\infty} a_i \right) \left(\sum_{k=0}^{\infty} b_k \right).$$

Proof. First, for all $n \in \mathbb{N}$ we know

$$|c_1| + |c_2| + \dots + |c_n| = \sum_{m=0}^n \left| \sum_{\substack{j+k=m \\ j,k \geq 0}} a_j b_k \right| \leq \sum_{m=0}^n \sum_{\substack{j+k=m \\ j,k \geq 0}} |a_j| |b_k| \leq \left(\sum_{j=0}^n |a_j| \right) \left(\sum_{k=0}^n |b_k| \right) \leq MN$$

for some M, N . Hence, $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

Let $A_n := a_0 + \dots + a_n$, $B_n := b_0 + \dots + b_n$, $C_n := c_0 + \dots + c_n$. Hence, we want to show

$$\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} A_n \cdot \lim_{n \rightarrow \infty} B_n.$$

Claim. $\lim_{n \rightarrow \infty} A_n B_n - C_n = 0$.

Proof. First, we know

$$0 \leq |A_n B_n - C_n| = \sum_{\substack{j+k > n \\ 0 \leq j, k \leq n}} |a_j b_k| \leq \left(\sum_{j=\lfloor \frac{n}{2} \rfloor}^n |a_j| \right) \left(\sum_{k=0}^n |b_k| \right) + \left(\sum_{j=0}^n |a_j| \right) \left(\sum_{k=\lfloor \frac{n}{2} \rfloor}^n |b_k| \right) \\ \rightarrow 0 * M + N * 0$$

when $n \rightarrow \infty$. The last part is derived by convergence implies Cauchy, and the former part can be derived by drawing grids in a square representing the choice of a_j and b_k . \otimes

■

Theorem 1.4.6. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely and $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ is a bijection such that $n \mapsto (j(n), k(n))$, and $c_n := a_{j(n)} b_{k(n)}$ for all $n \in \mathbb{N}$, then $\sum_n |c_n| < \infty$ and

$$\sum_n c_n = \left(\sum_n a_n \right) \left(\sum_n b_n \right).$$

Proof. $\forall n \in \mathbb{N}$, let $l = \max \{j(1), j(2), \dots, j(n), k(1), k(2), \dots, k(n)\}$. Then

$$|c_1| + |c_2| + \dots + |c_n| = |a_{j(1)} b_{k(1)}| + |a_{j(2)} b_{k(2)}| + \dots + |a_{j(n)} b_{k(n)}| \\ \leq \left(\sum_{j=1}^l |a_j| \right) \left(\sum_{k=1}^l |b_k| \right) \leq M \times N.$$

Hence, $\sum_n |c_n|$ converges absolutely. Note that since $\sum_n |c_n|$ converges absolutely, so we can change the order of every term if we want. That is, we can use other bijection to get same value of sum.

Let $A_n = a_1 + \dots + a_n$, $B_n = b_1 + \dots + b_n$, and $C_n = c_1 + \dots + c_n$, then we know

$$A_n B_n = (a_1 + \dots + a_n) (b_1 + \dots + b_n) = \sum_{1 \leq j, k \leq n} a_j b_k.$$

Replace the bijection $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ by:

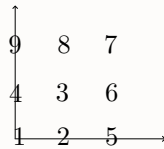


Figure 1.4: New bijection

then we know

$$\sum_{1 \leq j, k \leq n} a_j b_k = C_{n^2}.$$

Hence, we know

$$\lim_{n \rightarrow \infty} A_n B_n = \lim_{n \rightarrow \infty} C_{n^2} = \lim_{n \rightarrow \infty} C_n.$$

■

Lecture 5: Metric Space

1.5 Metric Space

16. July 2025

Recall that we use some concepts of distance, that is, the absolute value, which has the following properties:

1. $|x| \geq 0$ for all $x \in \mathbb{R}$ and the equal sign holds only when $x = 0$.
2. $|x| = |-x|$ for all $x \in \mathbb{R}$.
3. $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

In the definition of limit, we use absolute value, and we want to abstract the concept of absolute value. In \mathbb{R}^m , we know $x = (x_1, x_2, \dots, x_m)$ has $|x| = \sqrt{\sum_{j=1}^m |x_j|^2}$, and $|x|$ also has the properties above. Hence, we want to introduce Distance function / metric space (賦距空間).

Let X be a set.

Definition 1.5.1. A function $X \times X \xrightarrow{d} \mathbb{R}$ is called a distance / metric (function) on X if

1. $\forall x, y \in X$ we have $d(x, y) \geq 0$ and the equal sign holds only when $x = y$.
2. $\forall x, y \in X$, we have $d(x, y) = d(y, x)$.
3. $\forall x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Example. In \mathbb{R}^m , if $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$, and define

$$d_2(x, y) := \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \dots + |x_m - y_m|^2},$$

then d_2 is a metric on \mathbb{R}^m by Cauchy inequality.

Example. Suppose we define

$$d_1(x, y) := |x_1 - y_1| + |x_2 - y_2| + \dots + |x_n - y_n|$$

in \mathbb{R}^n , then d_1 is a metric.

Note. On a set we can define more than one way of defining metric. For example, $d_1(x, y)$ and $d_2(x, y)$.

Example.

$$d_\infty := \max \{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}$$

is also a metric on \mathbb{R}^n .

Example. Suppose X is a set and $x, y \in X$, then let

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

This is called the discrete metric.

Example. Suppose p is a prime number, $x, y \in \mathbb{Q}$, then

$$|x|_{p\text{-adic}} := p^{-m} \text{ if } x = \frac{a}{b} p^m \text{ with } a, b, m \in \mathbb{Z} \text{ and } \gcd(a, p) = \gcd(b, p) = 1$$

and

$$d_{p\text{-adic}}(x, y) := |x - y|_{p\text{-adic}},$$

which is also a metric.

Actually now we have

1. $|x + y|_{p\text{-adic}} \leq \max\{|x|_{p\text{-adic}}, |y|_{p\text{-adic}}\}$
2. $|x + y|_{p\text{-adic}} = |x|_{p\text{-adic}}$ if $|x|_{p\text{-adic}} > |y|_{p\text{-adic}}$.

Intuition. All triangles in \mathbb{Q} are isosceles if we define the distance function as p -adic distance.

Exercise. Suppose (X, d) is a metric space. Show that $\forall x, y, z \in X$,

$$|d(x, y) - d(y, z)| \leq d(x, y).$$

We may generalize the definitions about limits and convergence to metric spaces.

Definition 1.5.2. Let (X, d) be a metric space, a_n ($n \in \mathbb{N}$) be a sequence in X , and L is an element in X , we say that $\lim_{n \rightarrow \infty} a_n = L$ or a_n converges to L or L is the limit of a_n as $n \rightarrow \infty$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow d(a_n, L) < \varepsilon.$$

Note. Notice that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow d(a_n, L) < \varepsilon.$$

is equivalent to

$$\lim_{n \rightarrow \infty} d(a_n, L) = 0.$$

($d(a_n, L)$ is a real number.)

Intuition. Given any open ball centered at L , we can find N such that $n \geq N$ implies a_n is in this open ball.

Exercise. Can we prove that the limit is unique in a metric space?

Exercise. In \mathbb{R}^m , can we prove the basic limit properties hold when the metric is d_2 ? (The basic limit properties are something like $\lim_{n \rightarrow \infty} a_n \pm b_n = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$.)

Definition 1.5.3. Let (X, d) is a metric space and $S \subseteq X$.

1. For $r > 0$ and $x_0 \in X$, we let

$$B_r(x_0) = B(x_0, r) = B_{x_0}(r) := \{x \in X \mid d(x, x_0) < r\},$$

an open ball.

2. S is an open set (of (X, d)) if

$$\forall x_0 \in S, \exists r > 0 \text{ such that } B_r(x_0) \subseteq S.$$

Definition 1.5.4. For $r > 0$ and $x_0 \in X$, we let

$$\overline{B_r(x_0)} = \overline{B(x_0, r)} = \overline{B_{x_0}(r)} := \{x \in X \mid d(x, x_0) \leq r\},$$

a closed ball.

Definition 1.5.5. Let (X, d) be a metric space and $S \subseteq X$.

1. S is a *closed set* (of (X, d)) if $X \setminus S$ is open; that is,

$$S \text{ is closed} \Leftrightarrow X \setminus S \text{ is open.}$$

2. Equivalently, S is closed if for every sequence $\{x_n\} \subseteq S$ that converges to some $x \in X$, we have $x \in S$; that is,

$$\forall \{x_n\} \subseteq S, x_n \rightarrow x \in X \Rightarrow x \in S.$$

Exercise. Suppose (X, d) is a metric space, $x_0 \in X$, and $r > 0$.

1. Show that $B_r(x_0)$ is open.
2. $\{x \in X \mid d(x, x_0) > r\} = X \setminus \overline{B_r(x_0)}$ is open.

Lecture 6: Open Set and Closed Set

As previously seen. $S(\subseteq X)$ is called an open set of X (with respect to d) if $\forall x_0 \in S, \exists r > 0$ such that $B_r(x_0) \subseteq S$.

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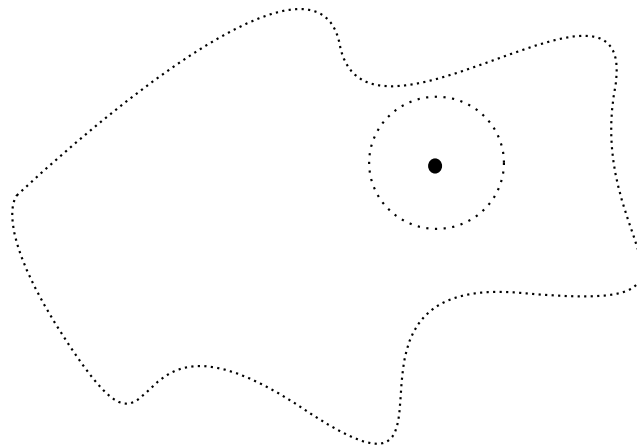


Figure 1.5: open set

As previously seen. $F(\subseteq X)$ is a closed set of X if $X \setminus F \subseteq X$ is open.

Example. A set is either an open set or a closed set?

Proof. No. Consider $(1, 3]$, and the metric is the absolute value.

⊗

Example. A set can not be open and closed at the same time?

Proof. No. Consider \emptyset . By definition, it is open. Also, the universal space is open, so \emptyset is closed.

*

Note. Closed ball is close.

Proof.

*

Exercise. Let (X, d) be a metric space. Show that

1. X and \emptyset are open.
2. O_1 and O_2 are open implies $O_1 \cap O_2$ is open.
3. Suppose we have $O_\alpha \subseteq X$ for all $\alpha \in A$, and they are all open, then $\bigcup_{\alpha \in A} O_\alpha$ is open in X .
4. What if we change "open" to "closed" in the above 3 statements? Should we make some modification to the statements to make them true?

Example. Intersection of infinitely many open set is still open?

Proof. Consider

$$\bigcap_{i=1}^{\infty} \left(-\frac{1}{i}, \frac{1}{i} \right).$$

*

Exercise. Show that

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n = L \\ \Leftrightarrow \lim_{n \rightarrow \infty} d(a_n, L) = 0 \\ \Leftrightarrow \forall \text{ open } U \subseteq X \text{ such that } L \in U, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow a_n \in U \end{aligned}$$

Example. If we have a metric space and a subspace of this space, can we restrict the metric on this subspace, and this space is still a metric space?

Proof. Yes. This does not violate the definition of metric space.

*

Definition 1.5.6. $S \subseteq X$ is bounded with respect to d if $\exists r > 0$ and $x_0 \in X$ such that $S \subseteq B_r(x_0)$.

Theorem 1.5.1 (Bolzano-Weierstrass theorem). Suppose we have a bounded sequence $a_n \in \mathbb{R}^m$, then \exists a subsequence $a_{n(m)}$ such that $a_{n(m)}$ is convergent.

Proof. We just talk about the case $m = 2$, and the higher case is similar. Choose $M > 0$ such that $a_n \in [-M, M] \times [-M, M]$ for all $n \in \mathbb{N}$. Suppose $a_n \in [-M, M] \times [-M, M]$ is called Q . Divide Q into 4 squares with equal size, and choose one, say Q_1 such that $|\{n \mid a_n \in Q_1\}| = \infty$. Select $n_1 \in \mathbb{N}$ such that $a_{n_1} \in Q_1$. Repeat this step, that is, divide Q_1 into 4 subparts, then says the one subpart with infinite many a_n in it is Q_2 (Q_2 must exists). Select $n_2 \in \mathbb{N}$ such that $a_{n_2} \in Q_2$ and $n_2 > n_1$. Keep repeating this step, then by [Theorem 1.3.2](#) we know

$$\bigcap_{n=1}^{\infty} Q_n \neq \emptyset.$$

Note. Just think of the nested intervals are in x and y directions.

Actually, $\bigcap_{n=1}^{\infty} Q_n = \{a\}$ for some $a \in \mathbb{R}^2$, otherwise if there are two points in the intersection, then at some moment we will divide them into different subpart, which is a contradiction. It can be seen that $\lim_{k \rightarrow \infty} a_{n_k} = a$. ■

Exercise. Suppose (X, d) is a metric space, $F \subseteq X$. Show that

$$F \text{ is closed} \Leftrightarrow \text{If } a_n \in F \text{ and } \lim_{n \rightarrow \infty} a_n = a \in X, \text{ then } a \in F.$$

Definition 1.5.7 (Open Cover). Suppose (X, d) is a metric space, and $S \subseteq X$. If we have $O_\alpha \subseteq X$ for $\alpha \in A$ and they are all open. We say that all O_α s form an open cover of S if $S \subseteq \bigcup_{\alpha \in A} O_\alpha$.

Definition 1.5.8 (Compact Set). S is called a compact set if \forall open cover $O_\alpha (\alpha \in A)$ of S , $\exists \alpha_1, \alpha_2, \dots, \alpha_m \in A$ such that

$$S \subseteq \bigcup_{i=1}^m O_{\alpha_i}, \quad \text{where } \bigcup_{i=1}^m O_{\alpha_i} \text{ is a finite subcover.}$$

Example. Is $(0, 1)$ with normal metric a compact set?

Proof. No. Consider

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2 \right),$$

we cannot pick finite many interval to cover $(0, 1)$. *

Example. Is $(1, \infty)$ a compact set?

Proof. No. Consider

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{2}, n \right).$$

*

Theorem 1.5.2 (The Heine Borel theorem). Let $S \subseteq \mathbb{R}^m$, then

$$S \text{ is compact} \Leftrightarrow S \text{ is bounded and closed.}$$

Proof of \Leftarrow . Suppose that S is bounded and closed, and there is an open cover O_α of S , which admits no finite subcover.

First, choose a cube Q containing S . Divide Q into 4 equal-sized cubes and select one of them, say Q_1 , such that $Q_1 \cap S$ cannot be covered by finitely many O_α . Keep repeating this step and get Q_2, Q_3, \dots . Note that we have

$$Q_1 \supseteq Q_2 \supseteq \dots$$

Hence,

$$\bigcap_{n=1}^{\infty} Q_n = \{a\} \quad \text{for some } a.$$

Choose $s_1 \in Q_1 \cap S$, $s_2 \in Q_2 \cap S$ and so on, then we know $\lim_{n \rightarrow \infty} s_n = a$ (think of this is also nested intervals). However, the sequence s_i is in S and S is closed, so by the previous exercise we know $a \in S$. Hence, $\exists \alpha$ such that $a \in O_\alpha$ since $\bigcup_{i=1}^{\infty} O_i$ is a cover of S . Since O_α is open, so there exists an open ball $B_r(a) \subseteq O_\alpha$. However, a is in many subcubes, so $\exists n \in \mathbb{N}$ such that $Q_n \subseteq B_r(a) \subseteq O_\alpha$, and since $O_n \cap S \subseteq Q_n$, so we know $Q_n \cap S \subseteq O_\alpha$, which is a contradiction since we suppose that $Q_i \cap S$ cannot be covered by finitely many O_α .

Note. Note that we need "bounded" and "closed" since we need to use [Theorem 1.3.2](#).



Exercise. How to prove if S is compact, then S is bounded and close?