Introduction to Analysis I HW12

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Problem 0.0.1 (Exercise 5.4.1). Show that if $f : \mathbb{R} \to \mathbb{C}$ is both compactly supported and $\mathbb{Z}$–periodic, then it is identically zero.

Hint: A function $f : \mathbb{R} \to \mathbb{C}$ is said to be *compactly supported* if the set

$$\operatorname{supp}(f) := \overline{\{\, x \in \mathbb{R} : f(x) \neq 0 \,\}}$$

is a compact subset of $\mathbb{R}$. Equivalently, f is compactly supported if there exists a bounded closed interval $[a, b] \subset \mathbb{R}$ such that

$$f(x) = 0 \qquad \text{whenever } x \notin [a, b].$$

Problem 0.0.2 (Exercise 5.5.1). Let f be a function in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, and define the *trigonometric Fourier coefficients* a_n, b_n for $n = 0, 1, 2, \ldots$ by

$$a_n := 2 \int_0^1 f(x) \cos(2\pi n x)\, dx, \qquad b_n := 2 \int_0^1 f(x) \sin(2\pi n x)\, dx.$$

(a) Show that the series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\big)$$

converges to f in the L^2-metric.

(b) Show that if $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ are absolutely convergent, then the above series actually converges *uniformly* to f (and not just in L^2).

Problem 0.0.3 (Exercise 5.5.2). Let $f(x)$ be the function defined by $f(x) = (1-2x)^2$ when $x \in [0, 1]$, and extended to be $\mathbf{Z}$-periodic on $\mathbf{R}$.

(a) Using Exercise 5.5.1, show that the series

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x)$$

converges uniformly to f. (You may use the fact that

$$\int_0^1 x\, e^{-2\pi i n x}\, dx = -\frac{1}{2\pi i n}, \qquad (n \neq 0),$$

$$\int_0^1 x^2\, e^{-2\pi i n x}\, dx = -\frac{1}{2\pi i n} + \frac{2}{(2\pi n)^2}, \qquad (n \neq 0).$$

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(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(c) Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(Hint: expand the cosines in terms of exponentials and use Plancherel's theorem.)

Problem 0.0.4 (Exercise 5.5.3). If $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ and P is a trigonometric polynomial, show that

$$\widehat{f * P}(n) = \hat{f}(n)\, c_n = \hat{f}(n)\, \hat{P}(n)$$

for all integers n, where c_n are the Fourier coefficients of P. More generally, if $f, g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, show that

$$\widehat{f * g}(n) = \widehat{f}(n)\,\widehat{g}(n) \quad \text{for all } n \in \mathbf{Z}.$$

Problem 0.0.5 (Exercise 5.5.4). Let $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ be differentiable, and assume its derivative f' is also continuous. Show that

$$\sum_{n=-\infty}^{\infty} |n\,\widehat{f}(n)|^2 < \infty$$

and that the Fourier coefficients of f' satisfy

$$\widehat{f'}(n) = 2\pi i n\,\widehat{f}(n) \qquad \text{for all } n \in \mathbf{Z}.$$