Linear Algebra I HW5

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Secion 1.4

Problem 0.0.1. Suppose R and R' are 2×3 row-reduced echelon matrices and that the systems $RX = 0$ and $R'X = 0$ have exactly the same solutions. Prove that $R = R'$.

Section 3.2

Problem 0.0.2. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

Problem 0.0.3. Let p, m, and n be positive integers and F a field. Let V be the space of $m \times n$ matrices over F and let W be the space of $p \times n$ matrices over F. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

Section 3.5

Problem 0.0.4. If A and B are $n \times n$ matrices over the field F, show that $\text{trace}(AB) = \text{trace}(BA)$. Now show that similar matrices have the same trace.

Problem 0.0.5. Let V be the vector space of all polynomial functions p from $\mathbb{R}$ into $\mathbb{R}$ which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)\,dx, \qquad f_2(p) = \int_0^2 p(x)\,dx, \qquad f_3(p) = \int_0^{-1} p(x)\,dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is the dual.