Linear Algebra I HW6

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Problem 0.0.1. Let W_1, W_2 be subspaces of a finite dimensional vector space V.

(a) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(b) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

Proof.

(a) If $f \in (W_1 + W_2)^0$, then $f(w_1 + w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. Now since $0 \in W_1$ and $0 \in W_2$, so we can pick $w_2 = 0$ so obtain $f(w_1) = 0$ for all $w_1 \in W_1$ and similarly we can obtain $f(w_2) = 0$ for all $w_2 \in W_2$. Hence, $f \in W_1^0 \cap W_2^0$. This means $(W_1 + W_2)^0 \subseteq W_1^0 \cap W_2^0$. Now if $g \in W_1^0 \cap W_2^0$, then since $g(w_1) = 0$ and $g(w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$, so we know $g(w_1 + w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$, and this means $g(w) = 0$ for all $w \in W_1 + W_2$. Hence, $g \in (W_1 + W_2)^0$, which gives $W_1^0 \cap W_2^0 \subseteq (W_1 + W_2)^0$. Hence, $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(b) We first claim that $\dim(W_1 \cap W_2)^0 = \dim\left(W_1^0 + W_2^0\right)$:

$$
\begin{aligned}
\dim\left(W_1 \cap W_2\right)^0 &= \dim V - \dim(W_1 \cap W_2) \\
\dim\left(W_1^0 + W_2^0\right) &= \dim W_1^0 + \dim W_2^0 - \dim\left(W_1^0 \cap W_2^0\right) \\
&= (\dim V - \dim W_1) + (\dim V - \dim W_2) - \dim\left(W_1 + W_2\right)^0 \quad \text{(by (a))} \\
&= 2\dim V - \dim W_1 - \dim W_2 - (\dim V - \dim(W_1 + W_2)) \\
&= \dim V + \dim(W_1 + W_2) - \dim W_1 - \dim W_2 \\
&= \dim V - \dim(W_1 \cap W_2).
\end{aligned}
$$

Hence, we've prove it. Now we prove that $W_1^0 + W_2^0 \subseteq (W_1 \cap W_2)^0$. If $f \in W_1^0 + W_2^0$, then $f = g + h$ for some $g \in W_1^0$ and $h \in W_2^0$. Hence, we know for all $w \in W_1 \cap W_2$, $f(w) = g(w) + h(w) = 0$, which means $f \in (W_1 \cap W_2)^0$, so $W_1^0 + W_2^0 \subseteq (W_1 \cap W_2)^0$.

Now since we know
$$
\begin{cases}
\dim\left(W_1 \cap W_2\right)^0 = \dim\left(W_1^0 + W_2^0\right) \\
W_1^0 + W_2^0 \subseteq \left(W_1 \cap W_2\right)^0,
\end{cases}
$$
so we know $\left(W_1 \cap W_2\right)^0 = W_1^0 + W_2^0$.

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Problem 0.0.2. Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. If f is a linear functional on W, prove that there is a linear functional g on V such that $g(\alpha) = f(\alpha)$ for each α in the subspace W.

Proof. Suppose $B = \{w_1, \ldots, w_n\}$ is a basis of W, and extend it to

$$C = \{w_1, \ldots, w_n, v_{n+1}, \ldots, v_m\},$$

and makes C a basis of V, then if we take dual of C, say

$$C^* = \left\{w_1^*, \ldots, w_n^*, v_{n+1}^*, \ldots, v_m^*\right\},$$

then we know $f = \sum_{i=1}^n \alpha_i w_i^*$ for some α_i's in F since

$$\{w_1^*, w_2^*, \ldots, w_n^*\}$$

is a basis W^* and $f \in W^*$, and thus if we pick $g = \sum_{i=1}^n \alpha_i w_i^* + \sum_{i=n+1}^m v_i^*$, then since we know for all $w \in W$, $v_j^*(w) = 0$ for all $n+1 \le j \le m$, so

$$g(w) = \sum_{i=1}^{n} \alpha_i w_i^*(w) = f(w).$$

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Problem 0.0.3. Let S be a set, F a field, and $V(S; F)$ the space of all functions from S into F:

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x).$$

Let W be any n-dimensional subspace of $V(S; F)$. Show that there exist points $x_1, \ldots, x_n$ in S and functions $f_1, \ldots, f_n$ in W such that $f_i(x_j) = \delta_{ij}$.

Proof. Suppose $B = \{g_1, g_2, \ldots, g_n\}$ is a basis of W, then we define

$$L_x : W \to F, \quad L_x(g) = g(x)$$

where $x \in S$.

Claim 0.0.1. $\exists x_1, x_2, \ldots, x_n \in S$ s.t. $\mathcal{L} = \{L_{x_1}, L_{x_2}, \ldots, L_{x_n}\}$ is linearly independent in W^*.

Proof. Suppose by contradiction, for all $x_1, x_2, \ldots, x_n$, $\{L_{x_1}, L_{x_2}, \ldots, L_{x_n}\}$ is linearly dependent, then we know

$$\dim\left(\operatorname{span}\{L_x : x \in S\}\right) < n,$$

otherwise, we can pick $\left\{\sum_{x\in S} \alpha_{ji} L_x\right\}_{j=1}^{n}$ s.t. this set is linearly independent, but notice that

$$\sum_{x\in S} \alpha_{ji} L_x = L_{\sum_{x\in S} \alpha_{ji} x}$$

by the definition of L_x, and this means we can pick n points $\left\{y_j = \sum_{x\in S} \alpha_{ji} x\right\}_{j=1}^{n}$ s.t. $\left\{L_{y_j}\right\}_{j=1}^{n}$ is linearly independent, which is a contradiction.

Now since $\dim\left(\operatorname{span}\{L_x : x \in S\}\right) < n$, and $\dim W^* = \dim W = n$, so we know

$$\dim\left(\operatorname{span}\{L_x : x \in S\}\right)^0 = \dim W^* - \dim\left(\operatorname{span}\{L_x : x \in S\}\right) \ge 1,$$

so we can pick $T \neq 0$ s.t. $T \in \left(\operatorname{span}\{L_x : x \in S\}\right)^0$. Now since we know

$$\mathcal{J} : W \to W^{**}, \quad \mathcal{J}(w)(\varphi) = \varphi(w) \quad \varphi \in W^*$$

is an isomorphism, so we know there exists $w \in W$ s.t. $\mathcal{J}(w) = T$, and since $T \neq 0$, so $w \neq 0$. Also, since $T \in \left(\operatorname{span}\{L_x : x \in S\}\right)^0$, so for all $x \in S$ we have

$$0 = T(L_x) = \mathcal{J}(w)(L_x) = L_x(w) = w(x),$$

which means w is the zero function in W, which is a contradiction. Hence, there must exists $x_1, x_2, \ldots, x_n \in S$ s.t. $\{L_{x_1}, L_{x_2}, \ldots, L_{x_n}\}$ is linearly independent. ⊛

By the claim above, we can pick $\mathcal{L} = \{L_{x_1}, L_{x_2}, \ldots, L_{x_n}\}$ s.t. $\mathcal{L}$ is linearly independent. Now suppose

$$A = \begin{pmatrix} L_{x_1}(g_1) & L_{x_1}(g_2) & \cdots & L_{x_1}(g_n) \\ L_{x_2}(g_1) & L_{x_2}(g_2) & \cdots & L_{x_2}(g_n) \\ \vdots & \vdots & \ddots & \vdots \\ L_{x_n}(g_1) & L_{x_n}(g_2) & \cdots & L_{x_n}(g_n) \end{pmatrix},$$

and we will show that A is invertible. Suppose

$$v_i = (L_{x_i}(g_1), L_{x_i}(g_2), \ldots, L_{x_i}(g_n)) = (g_1(x_i), g_2(x_i), \ldots, g_n(x_i)), \quad \forall 1 \le i \le n,$$

and suppose $\sum_{i=1}^n v_i = 0$, then we have

$$\alpha_1 g_i(x_1) + \alpha_2 g_i(x_2) + \cdots + \alpha_n g_i(x_n) = 0 \quad \forall 1 \le i \le n.$$

However, since we know $\mathcal{L}$ is linearly independent, so

$$\begin{aligned} \beta_1, \beta_2, \ldots, \beta_n = 0 &\Leftrightarrow \beta_1 L_{x_1} + \beta_2 L_{x_2} + \cdots + \beta_n L_{x_n} = 0 \\ &\Leftrightarrow \beta_1 p(x_1) + \beta_2 p(x_2) + \cdots + \beta_n p(x_n) = 0 \quad \forall p \in W \\ &\Leftrightarrow \beta_1 g_i(x_1) + \beta_2 g_i(x_2) + \cdots + \beta_n g_i(x_n) = 0 \quad \forall 1 \le i \le n. \end{aligned}$$

Hence, we know $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, and thus $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, which shows all rows of A is linearly independent, so A is invertible.

Now since A is invertible, so we can do row operations to make A becomes I_n, and suppose

$$I_n = A' = E_1 E_2 \ldots E_k A,$$

where $E_1, E_2, \ldots, E_k$ are some elementary matrices, then if $A' = \left(a'_{ij}\right)_{n \times n}$, then

$$a'_{ij} = \sum_{k=1}^n \beta_{ki} L_{x_k}(g_j) \quad \text{for some constants } \beta_{ki}\text{'s } \forall 1 \le i \le n.$$

Hence, we know

$$a'_{ij} = L_{\sum_{k=1}^n \beta_{ki} x_k}(g_j),$$

and since $A' = I_n$, so $a'_{ij} = \delta_{ij}$, which means if we pick $y_i = \sum_{k=1}^n \beta_{ki} x_k$ and then we have

$$g_i(y_j) = \delta_{ij}.$$

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Problem 0.0.4. Let n be a positive integer and let V be the space of all polynomial functions over the field of real numbers which have degree at most n, i.e., functions of the form

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

Let D be the differentiation operator on V. Find a basis for the null space of the transpose operator D^t.

Proof. Suppose $B = \left\{1, x, x^2, \ldots, x^n\right\}$, then we know B is a basis of V, and suppose $B^* = \{f_0, f_1, \ldots, f_n\}$ is the dual basis of B, then since

$$\left[D^t\right]_{B^*}^{B^*} = \left([D]_B^B\right)^t,$$

so we can first find out $[D]_B^B$. Note that

$$\begin{aligned} D(1) &= 0 = 0 \cdot 1 + 0 \cdot x + \cdots + 0 \cdot x^n \\ D(x) &= 1 = 1 \cdot 1 + 0 \cdot x + \cdots + 0 \cdot x^n \\ D\left(x^2\right) &= 2x = 0 \cdot 1 + 2 \cdot x + \cdots + 0 \cdot x^n \\ &\vdots \\ D\left(x^n\right) &= nx^{n-1} = 0 \cdot 1 + 0 \cdot x + \cdots + nx^{n-1} + 0 \cdot x^n. \end{aligned}$$

Hence,

$$[D]_B^B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and thus

$$\left[D^t\right]_{B^*}^{B^*} = \left([D]_B^B\right)^t = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n & 0 \end{pmatrix},$$

and if $v = (a_0, a_1, \ldots, a_n)^t \in \ker\,[D^t]_{B^*}^{B^*}$, then we know $[D^t]_{B^*}^{B^*}\, v = 0$, which gives

$$\begin{aligned} 1 \cdot a_0 &= 0 \\ 2 \cdot a_1 &= 0 \\ &\vdots \\ n \cdot a_{n-1} &= 0. \end{aligned}$$

Hence, we know $v = (0, 0, \ldots, 0, a_n)$. Thus,

$$\ker D^t = \operatorname{span}\{f_n\},$$

where

$$f_n\left(b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0\right) = b_n \quad \forall b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \in V.$$

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Problem 0.0.5. Let V be the vector space of $n \times n$ matrices over the field F.

(a) If B is a fixed $n \times n$ matrix, define a function f_B on V by $f_B(A) = \operatorname{Tr}(B^t A)$. Show that f_B is a linear functional on V.

(b) Show that every linear functional f on V is of the above form, i.e., is f_B for some B.

(c) Show that $B \to f_B$ is an isomorphism of V onto V^*.

Proof.

(a) Since we know $f_B : V \to F$, so we just need to show that f_B is linear. Suppose $P, Q \in V$ and $\alpha \in F$, then

$$\begin{aligned} f_B(\alpha P + Q) &= \operatorname{Tr}\left(B^t(\alpha P + Q)\right) \\ &= \operatorname{Tr}\left(B^t(\alpha P) + B^t Q\right) \\ &= \operatorname{Tr}\left(B^t(\alpha P)\right) + \operatorname{Tr}\left(B^t Q\right) \\ &= \alpha \operatorname{Tr}\left(B^t P\right) + \operatorname{Tr}\left(B^t Q\right) \\ &= \alpha f_B(P) + f_B(Q), \end{aligned}$$

so we know f_B is linear, and we're done.

(b) Suppose $E = \{e_{ij}\}_{1 \le i,j \le n}$ is the standard basis of V, then if we take dual basis of E, say it is $E^* = \left\{e_{ij}^*\right\}_{1 \le i,j \le n}$, then if $X = (x_{ij})_{n \times n}$, we have $e_{ij}^*(X) = x_{ij}$. Now if $f \in V^*$, then we know

$$f=\sum_{1\le i,j\le n}\beta_{ij}e_{ij}^*$$

for some constants β_{ij}'s. Hence, we know for all $X=(x_{ij})_{n\times n}$, we have

$$f(X)=\sum_{1\le i,j\le n}\beta_{ij}x_{ij}=\mathrm{Tr}\left(\begin{pmatrix}\beta_{11}&\beta_{21}&\cdots&\beta_{n1}\\\beta_{12}&\beta_{22}&\cdots&\beta_{n2}\\\vdots&\vdots&\ddots&\vdots\\\beta_{1n}&\beta_{2n}&\cdots&\beta_{nn}\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}&\cdots&x_{1n}\\x_{21}&x_{22}&\cdots&x_{2n}\\\vdots&\vdots&\ddots&\vdots\\x_{n1}&x_{n2}&\cdots&x_{nn}\end{pmatrix}\right)=\mathrm{Tr}\big(B^tX\big),$$

where

$$B=\begin{pmatrix}\beta_{11}&\beta_{12}&\cdots&\beta_{1n}\\\beta_{21}&\beta_{22}&\cdots&\beta_{2n}\\\vdots&\vdots&\ddots&\vdots\\\beta_{n1}&\beta_{n2}&\cdots&\beta_{nn}\end{pmatrix}.$$

(c) Suppose T is the map sending B to f_B, then we want to show that T is an isomorphism. We first show that T is linear. Suppose $A,B\in V$ and $\alpha\in F$, then for all $X\in V$, we have

$$\begin{aligned}T\left(\alpha A+B\right)(X)&=f_{\alpha A+B}(X)\\&=\mathrm{Tr}\left((\alpha A+B)^tX\right)\\&=\mathrm{Tr}\left(\alpha A^tX+B^tX\right)\\&=\alpha\,\mathrm{Tr}\left(A^tX\right)+\mathrm{Tr}\left(B^tX\right)\\&=\alpha f_A(X)+f_B(X)=(\alpha T(A)+T(B))(X),\end{aligned}$$

so T is linear. Now we show that T is bijective. Since $\dim V=\dim V^*$, so we just need to show that T is surjective. By (b), we know for all $f\in V^*$, $f=f_B=T(B)$ for some $B\in V$, so T is surjective, and we're done.

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