Introduction to Analysis I

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Abstract

The lecture note of 2025 Fall Introduction to Analysis I by professor 崔茂培. In this note, we will write $\left(X^{(n)}\right)_{n=m}^{\infty}$ and $\left\{X^{(n)}\right\}_{n=m}^{\infty}$ to express a sequence, they are identical, but 崔茂培 use both during lectures, so I follow him.

Contents

Chapter 1

Basic Things

Lecture 1

1.1 Natural Numbers

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The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, \dots\}$. There exists an addition operation

$$1+1=2 \quad 1+1+1=3 \quad \underbrace{1+1+\cdots+1}_{n \text{ times}} = n.$$

1.2 Integers

The set of integers is $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. There is a zero element 0 such that $z + 0 = z$ for any $z \in \mathbb{Z}$. Also, for $n \in \mathbb{N}$, we have $n + (-n) = 0$ and $n - m = n + (-m)$ for all $n, m \in \mathbb{N}$.

$$\mathbb{Z} \xRightarrow[\text{introduce division}]{} \mathbb{Q} \xRightarrow[\text{Completeness axiom}]{} \mathbb{R}$$

1.3 Field

Next, we introduce the concept of field.

Definition 1.3.1 (Fields)**.** A field is a set F together with two binary operations, called addition($+$) and multiplication($*$), such that the following properties hold:

(a) $a + b = b + a$, $a \cdot b = b \cdot a$ for $a, b \in F$.

(b) $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in F$.

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$.

(d) There are distince element 0 and 1 such that $a + 0 = a$, $a \cdot 1 = a$ for $a \in F$.

(e) For each $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$. If $a \neq 0$, there is an element $\frac{1}{a}$ or a^{-1} in F such that $a \cdot \frac{1}{a} = 1$, or $a \cdot a^{-1} = 1$.

Remark 1.3.1. If $a \in F$, then $a + a \in F$. We denote $a + a$ by $2 \cdot a$. Similarly,

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = n \cdot a,$$

and

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$

if $a \in F$ and $n \in \mathbb{N}$.

Remark 1.3.2. In a field, we have subtraction and division $a - b = a + (-b)$ for $a, b \in F$. If $b \neq 0$, then $\frac{a}{b} = a \cdot b^{-1}$ for $a, b \in F$.

In a field F, we have

$$\begin{aligned}(a+b)^2 &= (a+b) \cdot (a+b) \\ &= (a+b) \cdot a + (a+b) \cdot b \\ &= a \cdot a + b \cdot a + a \cdot b + b \cdot b \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2.\end{aligned}$$

Example 1.3.1.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

if $b \neq 0$ and $d \neq 0$.

Proof.

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= a \cdot b^{-1} + c \cdot d^{-1} \\ &= ab^{-1}dd^{-1} + cd^{-1}bb^{-1} \\ &= adb^{-1}d^{-1} + cbd^{-1}b^{-1} \\ &= \frac{ad + bc}{bd}.\end{aligned}$$

Notice that this is true since we have commutativity in multiplication and

$$d^{-1}b^{-1} = (bd)^{-1} = \frac{1}{bd}.$$

∎

Example 1.3.2. The set of rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$ is a field.

Example 1.3.3. The set of real numbers is also a field.

Example 1.3.4. $F_2 = \{0, 1\}$ is also a field since we can define addition and multiplication like $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$, and $0 \cdot 0 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$.

1.4 Order Relation

Next, we introduce the order relation. The real number system is ordered by the relation $<$, which has the following properties.

(f) For each pair of real numbers a and b, exactly one of the follwing is true: $a = b, a < b, b < a$.

(g) If $a < b$ and $b < c$, then $a < c$.

(h) If $a < b$, then $a + c < b + c$ for any c, and if $0 < c$, then $a \cdot c < b \cdot c$.

Definition 1.4.1. A field with an order relation satisfy (f) to (h) is called an ordered field.

Example 1.4.1. The set of rational numbers is an ordered field.

Example 1.4.2. F_2 is not an ordered field.

Proof. If $0 < 1$, then $1 = 0 + 1 < 1 + 1 = 0$, which is a contradiction. If $1 < 0$, then $0 = 1 + 1 < 0 + 1 = 1$, which is also a contradiction. ∎

Notation. In an ordered field, we use $a \leq b$ to denote either $a < b$ or $a = b$.

1.5 Absolute Value and Triangle Inequality

Next, we define the absolute value of a real number

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a \leq 0; \end{cases}$$

Theorem 1.5.1 (Triangle Inequality).

$$|a + b| \leq |a| + |b|$$

for all $a, b \in \mathbb{R}$.

Corollary 1.5.1.

$$||a| - |b|| \leq |a - b| \quad \text{and} \quad ||a| - |b|| \leq |a + b|$$

Proof. We write

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Similarly we have

$$|b| \leq |b - a| + |a|.$$

So

$$-|b - a| \leq |a| - |b| \leq |a - b|.$$

Thus,

$$||a| - |b|| \leq |a - b|.$$

∎

1.6 Supremum and Infimum

Next, we introduce the notion of supremum of a subset of real numbers.

Definition 1.6.1. Let S be a subset of $\mathbb{R}$,

(1) we say b is an upper bound of S if $x \leq b$ for all $x \in S$.

(2) If B is an upper bound of S, and no number smaller than B is an upper bound of S, then B is called the supremum or the least upper bound of S. We write $B = \sup S$.

Corollary 1.6.1. If $B = \sup S$, then

(1) $x \in S$ implies $x \leq B$

(2) If $b < B$, then b is not an upper bound of S, i.e. there exists $x_1 \in S$ such that $b < x_1$.

Definition 1.6.2. Let S be a subset of $\mathbb{R}$,

(1) we say b is an lower bound of S if $x \geq b$ for all $x \in S$.

(2) If α is an lower bound of S, and no number bigger than α is an lower bound of S, then α is called the infimum or the greatest lower bound of S. We write $\alpha = \inf S$.

Corollary 1.6.2. If $\alpha = \inf S$, then

(1) $x \in S$ implies $x \geq \alpha$

(2) If $\alpha < a$, then a is not an lower bound of S, i.e. there exists $x_1 \in S$ such that $x_1 < a$.

Notation (Interval Notation).

$$\begin{aligned}
(a, b) &= \{x \mid a < x < b\} \\
(a, b] &= \{x \mid a < x \leq b\} \\
[a, b) &= \{x \mid a \leq x < b\}
\end{aligned}$$

Example 1.6.1. $S = \{x \mid x < 0\} = (-\infty, 0)$, then $\sup S = 0$ but $\inf S$ does not exists.

Example 1.6.2. $S_1 = \{-1, -2, -3, -4, \dots\} = \{-n \mid n \in \mathbb{N}\}$, then $\sup S = -1$, but $\inf S$ does not exist.

Definition 1.6.3 (Nonempty Sets). A nonempty set is that a set has at least one element. The empty set, written as $\varnothing$, is the set has no elements at all.

Example 1.6.3. $S = \left\{x \mid x \in \mathbb{Q}, x < \sqrt{2}\right\}$

In $\mathbb{Q}$, $\sup S$ does not exist. In $\mathbb{R}$, $\sup S = \sqrt{2}$.

Theorem 1.6.1 (Completeness axiom). If a nonempty set of real numbers (an ordered field) is bounded above, then it has a least upper bound or $\sup S$ exists.

Remark 1.6.1. This is an extra axiom that can't be derived from the properties of ordered field.

Remark 1.6.2. Up to "isomorphism", there is exactly one complete ordered field: the field of real numbers.

Remark 1.6.3. From now, we assume $\mathbb{R}$ satisfies the completeness axiom. Thus, any nonempty subset $S \subseteq \mathbb{R}$ that is bounded above, we have $\sup S$ exists.

We can prove the following property of $\sup S$.

Theorem 1.6.2. If $S \subseteq \mathbb{R}$ is bounded above, then $\sup S$ is the unique real number B such that

(i) $x \leq B$ for all $x \in S$

(ii) for every $\varepsilon > 0$, there exist an $x_0 \in S$ such that $B - \varepsilon < x_0$.

Proof. (i), (ii) follows from the definition. We prove the uniqueness. Suppose $B_1 = \sup S = B_2$. We want to show $B_1 = B_2$. Suppose $B_1 \neq B_2$. Then either $B_1 < B_2$ or $B_2 < B_1$. However, if either one is true, then the other one cannot be $\sup S$. ■

Theorem 1.6.3 (Archimedean Property)**.** If $p > 0$ and $\varepsilon > 0$, then there exists an $n \in \mathbb{N}$ such that $p < n\varepsilon$.

Proof. We prove this contradiction. Suppose it is not true. This implies $n\varepsilon \leq p$ for all $n \in \mathbb{N}$. Consider $S = \{n\varepsilon \mid n \in \mathbb{N}\}$, then p is an upper bound of S, so S is bounded above by p, so we know $B = \sup S$ exists. Hence, $n\varepsilon \leq B$ for all $n \in \mathbb{N}$, so we have $(n+1)\varepsilon \leq B$, which means

$$n\varepsilon \leq B - \varepsilon$$

for all $n \in \mathbb{N}$. This implies $B - \varepsilon$ is also an upper bound of S, which is a contradiction. ■

1.7 Density of other number system

Theorem 1.7.1. Every nonempty subset of the integers that is bounded below has a least element.

Proof. We first introduce an axiom:

Theorem 1.7.2 (Well-Ordering principle)**.** Every non-empty subset of the natural numbers has a least element.

Note 1.7.1. Here, $\mathbb{N}$ can be $\{0, 1, 2, \dots\}$ or $\{1, 2, 3, \dots\}$, which is not that important.

Now we call this subset of integers as S, and suppose we have m as a lower bound of S, then define $S' = \{s - m \mid s \in S\}$, then we know S' is a nonempty subset of $\mathbb{N}$, then by well-ordering principle we know there is a least element in S' and thus there is also a least element in S. ■

Corollary 1.7.1. Every nonempty subset of the integers that is bounded above has a greatest element.

Proof. Suppose M is an upper bound, then define a set $S' = \{M - s \mid s \in S\}$, then by well-ordering principle we know $M - a$ is the least element of S' for some $a \in S$, so we have $M - x \geq M - a$ for all $x \in S$, which means $a \geq x$ for all $x \in S$ and since $a \in S$, so a is the greatest element of S. ■

Theorem 1.7.3. The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{p}{q}$ such that $a < \frac{p}{q} < b$.

Proof. Let $a, b \in \mathbb{R}$, $a < b$. By Archimedean Property, $\exists q \in \mathbb{N}$ such that $q(b-a) > 1$. Let $S = \{m \mid m \text{ is an integer with } m > qa\}$, since we know $S \neq \varnothing$ and S is bounded below. Hence, $p = \inf S$ exists and is an integer by the last theorem. So $qa < p$ and $p - 1 \leq qa$, which means $qa < p \leq qa + 1 < qb$, so we have $a < \frac{p}{q} < b$. ■

Lecture 2

Definition 1.7.1 (Floor Function)**.** For any real number x, the floor function of x is denoted by $\lfloor x \rfloor$, and is defined by the formula $\lfloor n \rfloor$ if $n \leq x < n + 1$ where $n \in \mathbb{Z}$. 4 Sep. 10:20

Corollary 1.7.2.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

Example 1.7.1. $\lfloor 3.7 \rfloor = 3$, $\lfloor -1.2 \rfloor = -2$.

Now by floor function, we can reprove Theorem 1.7.3.

Theorem 1.7.4 (Density of rational number in real number Again). The set of rational numbers is dense in the real number. That is, if a and b are real numbers with $a < b$, then there exists a rational number $\frac{q}{p}$ such that $a < \frac{q}{p} < b$.

Reprove Theorem 1.7.3. Since $a < b$, so we know $b - a > 0$. Now by Archimedean Property, we know there exists $q \in \mathbb{N}$ such that $q(b-a) > 1$. Let $p = \lfloor qa \rfloor + 1$, we have

$$\lfloor qa \rfloor \leq qa < \lfloor qa \rfloor + 1 = p.$$

From our construction, $qb > qa + 1$, so we have

$$p = \lfloor qa \rfloor + 1 \leq qa + 1 < qb,$$

hence we have

$$qa \leq p \leq qb.$$

■

Note 1.7.2. For some reason, p, q in Theorem 1.7.3 and Theorem 1.7.4 are reversed.

Definition 1.7.2 (irrational number). x is called irrational if x is not rational.

Example 1.7.2. $\sqrt{2}$ is irrational.

Theorem 1.7.5. Let $r \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, then

1. $r + x$ is irrational.
2. If $r \neq 0$, then rx is irrational.

sketch of proof.

1. If $r + x = q \in \mathbb{Q}$, then $x = q - r \in \mathbb{Q}$, contradiction.
2. If $rx = q \in \mathbb{Q}$, then $x = \frac{q}{r} \in \mathbb{Q}$ since $r \neq 0$.

■

Theorem 1.7.6 (irrational number dense in real number). The set of irrational number is dense in real number. That is, if $a, b \in \mathbb{R}$ and $a < b$, then there exists a irrational number t such that $a < t < b$.

Proof. By density of rational number, we can find $a < r_1 < r_2 < b$ where $r_1, r_2 \in \mathbb{Q}$, and then let $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$, then we know

$$a < r_1 < t < r_2 < b.$$

Note 1.7.3. We should use Theorem 1.7.5 and the fact that $\sqrt{2}$ is irrational.

■

Definition 1.7.3 (bounded set). A set $S \subseteq \mathbb{R}$ is bounded if there are numbers a, b s.t. $a \leq x \leq b$ for all $x \in S$.

Corollary 1.7.3. A bounded non-empty set in $\mathbb{R}$ has a unique supremum and a unique infimum and $\inf S \leq \sup S$.

1.8 Extended real number system

The real number system, together with ∞ and $-\infty$, then we have the following properties:

(a) If $a \in \mathbb{R}$, then $a + \infty = \infty + a = \infty$ and $a - \infty = -\infty + a = -\infty$, and $\frac{a}{\infty} = \frac{a}{-\infty} = 0$.

(b) If $a > 0$, then $a \cdot \infty = \infty \cdot a = \infty$ and $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$

(c) If $a < 0$, then $a \cdot \infty = \infty \cdot a = -\infty$ and $a \cdot -\infty = -\infty \cdot a = \infty$ and $\infty + \infty = \infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$ and $-\infty - \infty = \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$ and $|-\infty| = |\infty| = \infty$

However, there are some indeterminate form:

Theorem 1.8.1. The following things are not defined:

$$\infty - \infty,\ 0 \cdot \infty,\ \frac{\infty}{\infty},\ \text{and } \frac{0}{0}.$$

1.9 Mathematical Induction

Theorem 1.9.1 (Peano's Postulate). The natural numbers satisfy the following properties

(a) $\mathbb{N}$ is nonempty.

(b) For each natural number n, there exists a unique rational number n called the successor of n.

(c) There exists a natural number $\overline{n}$ that is not the sucessor of any natural number.

(d) Different natural numbers have different sucessors, that is, $n \neq m$ implies $n' \neq m'$.

(e) The only subset of $\mathbb{N}$ that contains $\overline{n}$ and also contains the sucessor of every one of its element is $\mathbb{N}$.

Theorem 1.9.2 (Principle of Mathematical Induction). Let $p_1, p_2, \ldots, p_n$ be propositions, one for each positive integers, such that

(a) p_1 is true.

(b) for each positive integer n, p_n implies p_{n+1}.

then p_n is true for each $n \in \mathbb{N}$.

Proof. Let $M = \{n \mid n \in \mathbb{N} \text{ and } p_n \text{ is true}\}$, then from (a) we know $1 \in M$ and from (b) we know $n \in M$ implies $n + 1 \in M$. Hence, from (e) of Peano's Postulate, we know $M = \mathbb{N}$. ■

Chapter 2

Metric Space

2.1 Definition and examples

Definition 2.1.1. Suppose $x_n \in \mathbb{R}$ for $n \geq m$. We use the notation $(x_n)_{n=m}^{\infty}$ to denote the sequence of numbers

$$x_m, x_{m+1}, \ldots$$

We first recall the definition of a convergent sequence.

Definition 2.1.2 (Convergent Sequence). We say that a sequence $(x_n)_{n=m}^{\infty}$ of real numbers converges to x if for every $\varepsilon > 0$, there exists an $N \geq m$ s.t. $|x_n - x| \leq \varepsilon$ for all $n \geq N$.

Notation. We write $\lim_{n\to\infty} x_n = x$.

On $\mathbb{R}$, we can define the distance function between two points $x, y \in \mathbb{R}$ by $d(x, y) = |x - y|$. We'll discuss this more later.

Lemma 2.1.1. Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number, then $(x_n)_{n=m}^{\infty}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Proof. Assume $(x_n)_{n=m}^{\infty}$ converges to x. Let $\varepsilon > 0$ be arbitrary real number. By definition, there exists an $N \geq m$ such that $|x_n - x| \leq \varepsilon$ for all $n \geq N$. But $d(x_n, x) = |x_n - x|$ by the definition. Hence, $\forall \varepsilon > 0$, $\exists N \geq m$ such that $d(x_n, x) \leq \varepsilon$ fpr all $n \geq N$. This implies that $\forall \varepsilon > 0$, $\exists N \geq m$ such that $|d(x_n, x) - 0| \leq \varepsilon$ for all $n \geq N$. This implies $\lim_{n\to\infty} d(x_n, x) = 0$.

The proof of the other side is the same but writing the above proof from bottom to top again. ∎

Definition 2.1.3 (Metric Space). A metric space (X, d) is the space of X of objects(called points), together with a distance function or metric $d : X \times X \to [0, \infty)$ which associates to each x, y of points in X a nonnegative number $d(x, y) \geq 0$, the following. Furthermore, the metric must satisfy 4 axioms.

(a) For any $x \in X$, $d(x, x) = 0$.

(b) (Positivity) For any distinct $x, y \in X$, we have $d(x, y) > 0$.

(c) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.

(d) (Triangle inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Example 2.1.1. On $\mathbb{R}$, we can define $d(x, y) = |x - y|$.

Proof.

- $d(x, y) = |x - y| \geq 0$.

- $d(x, y) = 0$ iff $|x - y| = 0$ iff $x = y$.
- $|x - y| = |y - x|$, so $d(x, y) = d(y, x)$
- $|x - z| \leq |x - y| + |y - z|$ for all $x, y, z \in \mathbb{R}$.

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Example 2.1.2. Let (X, d) be a metric space and $Y \subseteq X$, then Y inherits a natural distance function

$$d|_{Y \times Y} : Y \times Y \to [0, \infty)$$

defined by $d|_{Y \times Y}(\alpha, \beta) = d(\alpha, \beta)$ for all $\alpha, \beta \in Y$.

Note 2.1.1. $(Y, d|_{Y \times Y})$ is called a metric subspace of (X, d). It is obvious that $d|_{Y \times Y}$ is a metric on Y.

Recall $\mathbb{R}^n$. Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

Definition 2.1.4 (l^2-metric). The l^2-metric is defined by

$$d_2(x, y) = \left(\sum_{i=1}^{n} (x_n - y_n)^2 \right)^{\frac{1}{2}} \text{ (or we called } d_{l_2}(x, y)).$$

Definition 2.1.5 (l^1-metric(taxicab metric)). The l^1-metric is defined by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| (\text{or we called } d_{l_1}(x, y))$$

Definition 2.1.6 (l^∞-metric). The l^∞-metric is defined by

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

Exercise 2.1.1. Verify they are all metrics.

Note 2.1.2. Actually we have to define inner product and norm first and then we can use the triangle inequality of norm to prove d_2 is a metric. (See lecture notes by professor)

Lecture 3

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Definition 2.1.7 (Cartesian Product). Let A, B be sets. The cartesian product of A and B is defined by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Similarly, the cartesian product of $X_1, X_2, \dots, X_n$ is

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \dots, x_n) \mid x_i \in X_i \; \forall 1 \leq i \leq n\}.$$

Definition 2.1.8 (Functions). Let $X_1, X_2, \dots, X_n$ be sets and let Y be another set. A fuction of n variables with codomains is a map $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ which assigns each n-tuple $(x_1, x_2, \dots, x_n)$ with $x_i \in X_i$ a unique element $f(x_1, x_2, \dots, x_n)$.

Definition. We talk about the definition of domain, codomain, and range:

Definition 2.1.9. The domain of f is $X_1 \times X_2 \times \cdots \times X_n$ and Y is the codomain of f.

Definition 2.1.10. The range of f is

$$\{f(x_1, x_2, \dots, x_n) \in Y \mid x_i \in X_i \ \forall i\}.$$

In the definition of metric space, we write (X, d) to emphasize our set X and d is a distance function defined on $X \times X$, i.e.

$$d : X \times X \to [0, \infty) \subseteq \mathbb{R},$$

where

$$d : (x, y) \mapsto d(x, y)$$

for $x, y \in X$. Let (X, d) be a metric space and $Y \subseteq X$. Then $(Y, d|_{Y\times Y})$ is also a metric space with distance function defined by

$$d|_{Y\times Y} \to [0, \infty)$$

and

$$d|_{Y\times Y} : (\alpha, \beta) \mapsto d(\alpha, \beta) \text{ for } \alpha, \beta \in Y.$$

Example 2.1.3. Recall the Taxi-cab metric, it can be used in cryptography. For example, for two binary strings, we know

$$d_1((10010), (10101)) = 3 = \text{the number of mismatched bits.}$$

Example 2.1.4. Recall the l^∞-metric. Suppose two jobs where each consists of 3 tasks, and the time (in hours) to complete each task is represented by a vector

$$x = (2, 4, 6), \ \ y = (3, 7, 5),$$

so

$$d_\infty(x, y) = \max\{|2 - 3|, |4 - 7|, |6 - 5|\} = 3.$$

Definition 2.1.11 (Lipschitz equivalent metrics)**.** Let (X, d_1) and (X, d_2) be two metrics on X. We say d_1 and d_2 are Lipschitz equivalent if $\exists c_1, c_2 > 0$ s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y) \quad \forall x, y \in X$$

Remark 2.1.1. They will have same topology (defined later).

Proposition 2.1.1. For all $x, y \in \mathbb{R}^n$,

$$d_2(x, y) \leq d_1(x, y) \leq \sqrt{n} d_2(x, y) \tag{2.1}$$
$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} d_\infty(x, y) \tag{2.2}$$

Remark 2.1.2.

$$\begin{aligned} d_\infty(x, y) &\geq \frac{1}{\sqrt{n}} d_2(x, y) \\ &\geq \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} d_1(x, y) = \frac{1}{n} d_1(x, y). \end{aligned}$$

Also,

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y).$$

Remark 2.1.3. d_1, d_2, d_∞ are all Lipschitz equivalent.

proof of Proposition 2.1.1 . Recall $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, then

$$d_1(x,y) = \sum_{i=1}^n |x_i - y_i|, \quad d_2(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

By Cauchy-Schurwatz inequality,

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^n |x_i - y_i| \\ &\le \left(\sum_{i=1}^n |x_i - y_i|\right)^{\frac{1}{2}} \left(\sum_{i=1}^n 1^2\right)^{\frac{1}{2}} = \sqrt{n} d_2(x,y). \end{aligned}$$

Now we show that $d_1(x,y) \ge d_2(x,y)$.

$$\begin{aligned} (d_1(x,y))^2 &= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\ &= \sum_{i=1}^n |x_i - y_i|^2 + 2 \sum_{1 \le i < j \le n} |x_i - y_i||x_j - y_j| \\ &\ge \sum_{i=1}^n |x_i - y_i|^2 = d_2(x,y)^2. \end{aligned}$$

Hence, we have $d_1(x,y) \ge d_2(x,y)$.

Now we show that $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$. Note that

$$d_2(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}, \quad d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|.$$

For each i, we know

$$|x_i - y_i| \le d_\infty(x,y),$$

so

$$d_2(x,y)^2 \le \sum_{i=1}^n d_\infty(x,y)^2 = n d_\infty(x,y)^2,$$

so $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$. ■

Definition 2.1.12 (Discrete metric)**.** Let X be any set, define the discrete metric:

$$d_{\text{disc}} : X \times X \to \{0, 1\}$$

where

$$d_{\text{disc}}(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \ne y. \end{cases}$$

Why this is a metric? Because

- $d_{\text{disc}}(x,y) \ge 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$.
- $d_{\text{disc}}(x,y) = d_{\text{disc}}(y,x)$ by definition.
- $d_{\text{disc}}(x,z) \le d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z)$?

proof of triangle inequality in discrete metric. We first consider the case that $x = z$, then

$$d_{\text{disc}}(x, z) = 0,$$

so it is obviously that the triangle inequality is true.

Now if $x \neq z$, then either $y \neq z$ or $y \neq x$ must happen, so the triangle inequality must be true. ■

Example 2.1.5. We can define

$$d(x, x) = 0, \quad d(x, y) = \text{minimal length of a path from } x \text{ to } y,$$

then this is also a metric.

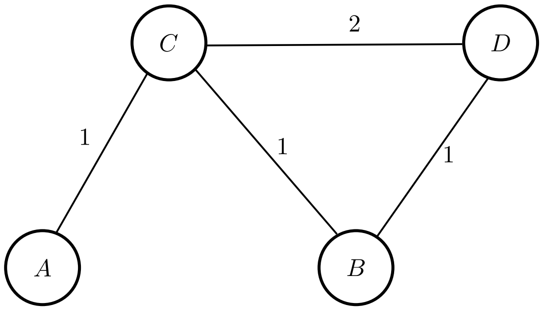

Figure 2.1: Graph metrics

Definition 2.1.13 (Convergence in metric space). Let m be an integer, (X, d) be a metric space, and let $\left(X^{(n)}\right)_{n=m}^{\infty}$ be a sequence of points in X. Let $x \in X$. We say that $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges to x with respect to d iff

$$\lim_{n\to\infty} d\left(X^{(n)}, x\right) = 0,$$

where $\lim_{n\to\infty} d\left(X^{(n)}, x\right) = 0$ iff for every $\varepsilon > 0$, $\exists N \geq m$ s.t. $d\left(X^{(n)}, x\right) \leq \varepsilon$ for all $n \geq N$.

Notation. We also write $\lim_{n\to\infty} X^{(n)} = x$ in (X, d).

Remark 2.1.4. Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges to x in (X, d), then $\left(X^{(n)}\right)_{n=m_1}^{\infty}$ also converges to x in (X, d) if $m_1 \geq m$.

Example 2.1.6. Let $\left(X^{(n)}\right)_{n=1}^{\infty}$ denote the sequence $X^{(n)} = \left(\frac{1}{n}, \frac{1}{n}\right)$ in $\mathbb{R}^2$, then what will this sequence converges to for different metric?

Proof.

- If the metric is d_1, then

$$d_1\left(X^{(n)}, (0,0)\right) = \left|\frac{1}{n} - 0\right| + \left|\frac{1}{n} - 0\right| = \frac{2}{n},$$

so

$$\lim_{n\to\infty} d_1\left(X^{(n)}, (0,0)\right) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

- If the metric is d_2, then

$$d_2\left(X^{(d)},(0,0)\right)=\sqrt{\left(\frac{1}{n}-0\right)^2+\left(\frac{1}{n}-0\right)^2}=\frac{\sqrt{2}}{n}.$$

  Hence, under l_2-metric $\left\{X^{(n)}\right\}$ also converges to 0.

- If the metric is d_∞, then

$$d_\infty\left(X^{(n)},(0,0)\right)=\max\left\{\left|\frac{1}{n}\right|,\left|\frac{1}{n}\right|\right\}=\frac{1}{n},$$

  so it also converges to 0.

- If the metric is discrete metric, then however, it will not converges to $(0,0)$ since

$$\lim_{n\to\infty} d_{\text{disc}}\left(X^{(n)},(0,0)\right)=\lim_{n\to\infty} d_{\text{disc}}\left(\left(\frac{1}{n},\frac{1}{n}\right),(0,0)\right)=1.$$

⊛

Definition. Let $f: X\to Y$ be a function with domain X and codomain Y. The range of $f=\{f(x)\mid x\in X\}\subseteq Y$.

Definition 2.1.14 (injective)**.** We say f is injective or one-to-one if for all $x_1,x_2\in X$, $f(x_1)=f(x_2)$ implies $x_1=x_2$.

Definition 2.1.15 (surjective)**.** We say f is surjective or onto if for every $y\in Y$, $\exists x\in X$ s.t. $f(x)=y$.

Definition 2.1.16 (bijective)**.** We say f is bijective if f is injective and surjective.

Corollary 2.1.1. If f is bijective, then there exists $f^{-1}: Y\to X$ defined by $f^{-1}(y)=x$ if $f(x)=y$. We also have

$$f\left(f^{-1}(y)\right)=y\ \forall y\in Y$$
$$f^{-1}\left(f(x)\right)=x\ \forall x\in X.$$

Example 2.1.7. $\lim_{n\to\infty}\frac{1}{n}=0$ in $(\mathbb{R},d)$, where d is the standard metric in $\mathbb{R}$, which is defined by

$$d(x,y)=|x-y|.$$

But in different metric, $\lim_{n\to\infty}\frac{1}{n}$ may not be 0.

Proof. Define $f:[0,1]\to[0,1]$ defined by

$$f(x)=\begin{cases}x, & \text{if } 0<x<1;\\ 1, & \text{if } x=0;\\ 0, & \text{if } x=1.\end{cases}$$

f is bijective on $[0,1]$ to $[0,1]$

Define another metric d^1 on $[0,1]$ by

$$d^1(x,y) = d(f(x), f(y)).$$

We want to show that d^1 is also a metric on $[0,1]$.

- $d^1(x,y) = d(f(x), f(y)) = |f(x) - f(y)| \geq 0$
- $d^1(x,y) = 0$ iff $f(x) = f(y)$ iff $x = y$ since f is injective.
- The triangle inequality is trivially true since we can just use the triangle inequality in d.

In fact, $\lim_{n\to\infty} \frac{1}{n} = 1$ in $\left([0,1], d^1\right)$ since

$$\lim_{n\to\infty} d^1\left(\frac{1}{n}, 1\right) = \lim_{n\to\infty} d\left(\frac{1}{n}, 0\right) = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0.$$

⊛

2.2 Some point set topology of metric space

Definition 2.2.1 (ball). Let (X,d) be a metric space. let $x_0 \in X$ and $r > 0$. We define the ball $B_{(X,d)}(x_0, r)$ in X, centered at x_0 and with radius r in the metric d, to the set

$$B_{(X_0,d)}(X_0, Y) \coloneqq \{x \in X \mid d(x_0, x) < r\}.$$

Sometimes, we write it as $B_X(x_0, r)$ or $B(x_0, r)$.

Example 2.2.1. In $\mathbb{R}^2$,

$$B_{(\mathbb{R}^2, d_2)}\left((0,0), 1\right) = \left\{(x,y) \mid d_2((x,y),(0,0)) = \sqrt{x^2+y^2} < 1\right\},$$

and

$$B_{(\mathbb{R}^2, d_1)}((0,0), 1) = \{(x,y) \mid d_1((x,y),(0,0)) = |x| + |y| < 1\},$$

and

$$B_{(\mathbb{R}^2, d_\infty)}((0,0), 1) = \{(x,y) \mid d_\infty((x,y),(0,0)) = \max\{|x|, |y|\} < 1\},$$

also we can consider the d_{disc} case but I am too lazy to write it down.

Notation. Let $E \subseteq X$, we will write

$$X \setminus E \coloneqq \{x \in X \mid x \notin E\}.$$

Definition. Let (X,d) be a metric space and $E \subseteq X$. For a point $x_0 \in X$,

Definition 2.2.2 (interior point). x_0 is an interior point of E if $\exists r > 0$ s.t. $B(x_0, r) \subseteq E$.

Definition 2.2.3 (exterior point). x_0 is an exterior point of E if $\exists r > 0$ s.t. $B(x_0, r) \subseteq X \setminus E$.

Definition 2.2.4 (boundary point). x_0 is a boundary point of E if it is neither an interior point nor an exterior point of E.

Proposition 2.2.1. x_0 is a boundary point of E iff for all $r > 0$, $B(x_0, r) \cap E \neq \varnothing$ and $B(x_0, r) \cap (X \setminus E) \neq \varnothing$.

Lecture 4

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Theorem 2.2.1. Let (X, d_1) and (X, d_2) be metrics on X, and suppose d_1 and d_2 are Lipschitz equivalent, then for any sequence $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$, then for any $x \in X$

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_1) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_2).$$

Proof. Since d_1, d_2 are Lipschitz equivalent, so there exists $c_1, c_2 > 0$ s.t.

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y).$$

($\Rightarrow$) Given $\frac{\varepsilon}{c_2} > 0$, since $\lim_{n\to\infty} x^{(n)} = x$ in (X, d_1), so there exists N s.t. $N \geq m$ and

$$d_1(x^{(n)}, x) \leq \frac{\varepsilon}{c_2} \text{ for } n \geq N.$$

This implies $d_2(x^{(n)}, x) \leq c_2 d_1(x^{(n)}, x) \leq \varepsilon$ for $n \geq N$, which means

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_2).$$

($\Leftarrow$) Similar.

■

Remark 2.2.1. On $\mathbb{R}^n$, the metrics d_1, d_2, d_∞ are Lipschitz equivalent, that is,

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_1) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_2) \Leftrightarrow \lim_{n\to\infty} x^{(n)} = x \text{ in } (\mathbb{R}^n, d_\infty)$$

Proposition 2.2.2. Let (X, d_{disc}) be a discrete metric space, and $\{x^{(n)}\}_{n=m}^{\infty} \subseteq X$. Then

$$\lim_{n\to\infty} x^{(n)} = x \text{ in } (X, d_{\text{disc}}) \Leftrightarrow \exists N \geq m \text{ s.t. } x^{(n)} = x \text{ for } n \geq N.$$

Proof. ($\Leftarrow$) Easy.

($\Rightarrow$) Given $\frac{1}{2} > 0$, there exists $N \geq m$ s.t. $d(x_n, x) < \frac{1}{2}$ for $n \geq N$, but $d(x_n, x) < \frac{1}{2}$ implies $d(x_n, x) = 0$, which means $x_n = x$ for all $n \geq N$.

■

Definition. We define the interior, exterior, and boundary point again.

Definition 2.2.5. The set of interior points is denoted by

$$\text{Int}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq E\}.$$

Definition 2.2.6. The set of exterior points is denoted by

$$\text{Ext}(E) = \{x \in X \mid \exists r > 0 \text{ s.t. } B_X(x, r) \subseteq X \setminus E\}.$$

Definition 2.2.7. A point is a boundary points if it is neithe an interior point nor an exterior point, and we define

$$\partial E = \{x \in X \mid x \notin \text{Int}(E) \text{ and } x \notin \text{Ext}(E)\}.$$

Remark 2.2.2.

1. $$x_0 \notin \mathrm{Int}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (X \setminus E) \neq \varnothing.$$

2. $$x_0 \notin \mathrm{Ext}(E) \Leftrightarrow \forall r > 0, B_X(x_0, r) \cap (E) \neq \varnothing.$$

3. $\mathrm{Int}(X \setminus E) = \mathrm{Ext}(E)$.

4. $\partial E = \partial(X \setminus E)$ since

$$x_0 \in \partial E \Leftrightarrow x \notin \mathrm{Int}(E) \text{ and } \mathrm{Ext}(E) \Leftrightarrow x_0 \notin \mathrm{Int}(E) \text{ and } x_0 \notin \mathrm{Int}(X \setminus E).$$

Also,

$$x_0 \in \partial(X \setminus E) \Leftrightarrow x \notin \mathrm{Int}(X \setminus E) \text{ and } \mathrm{Ext}(X \setminus E) \Leftrightarrow x_0 \notin \mathrm{Int}(X \setminus E) \text{ and } x_0 \notin \mathrm{Int}(E).$$

Hence, acutually $\partial E = \partial(X \setminus E)$.

Proposition 2.2.3.

$$x_0 \in \partial E \Leftrightarrow \text{ For any } r > 0, B_X(x_0, r) \cap E \neq \varnothing \text{ and } B_X(x_0, r) \cap (X \setminus E) \neq \varnothing$$

Example 2.2.2. Let $(\mathbb{R}, d)$ be the usual metric on $\mathbb{R}$, where

$$d(x, y) = |x - y|.$$

Then, we know in this space,

$$\begin{aligned} B_{\mathbb{R}}(x_0, r) &= \{x \in \mathbb{R} \mid d(x, x_0) < r\} \\ &= \{x \in \mathbb{R} \mid |x - x_0| < r\} \\ &= \{x \in \mathbb{R} \mid -r + x_0 < x < r + x_0\}. \end{aligned}$$

Hence, suppose $E = [1, 2)$, then $\mathrm{Int}(E) = (1, 2)$ since we know $B(x_0, r) = (x_0 - r, x_0 + r)$, so for all $x \in (1, 2)$, we know there is an open ball $B(x_0, r) \subsetneq [1, 2)$ for some $r > 0$. Also, consider the endpoint $1, 2$, we can verify that these two points are not interior points. Besides, consider the points not in $[1, 2]$, it is trivial that they cannot be interior points.

Example 2.2.3. We consider (X, d_{disc}). Let $E \subseteq X$. If $x \in E$, we know

$$B\left(x, \frac{1}{2}\right) = \left\{y \mid d(y, x) < \frac{1}{2}\right\} = \{x\} \subseteq E.$$

Hence, $E \subseteq \mathrm{Int}(E)$. Besides, for all $x \in \mathrm{Int}(E)$, we know there exists $r > 0$ s.t. $B(x_0, r) \subseteq E$, also we know $x_0 \in B(x_0, r) \subseteq E$, so $x_0 \in E$, and thus $\mathrm{Int}(E) \subseteq E$. Hence, $E = \mathrm{Int}(E)$. Similarly, $\mathrm{Int}(X \setminus E) = X \setminus E$. Suppose there is a $x \in X$ s.t. $x \in \partial E$, then $x \notin \mathrm{Int}(E) = E$ and $x \notin \mathrm{Ext}(E) = \mathrm{Int}(X \setminus E) = X \setminus E$, so such x does not exist.

Definition 2.2.8 (Closure)**.** Let (X, d) be a metric space, and let $E \subseteq X$ and $x_0 \in X$. We say x_0 is a adherent point of E if for every $r > 0$, $B(x_0, r) \cap E \neq \varnothing$. The set of adeherent points is called the closure of E, and denoted by $\overline{E}$.

Proposition 2.2.4 (TFAE)**.**

(a) x_0 is an adherent point of E.

(b) x_0 is either an interior point or a boundary point of E.

(c) $\exists$ a sequence $\left\{X^{(n)}\right\}_{n=1}^{\infty}$ in E which converges to x_0 in (X, d).

proof from (a) to (b). Suppose $x_0 \in \overline{E}$, then $B(x_0, r) \cap E \neq \varnothing$ for all $r > 0$. If $\exists s > 0$ s.t. $B(x_0, s) \subseteq E$, then $x_0 \in \mathrm{Int}(E)$. If such s does not exists, then we know

$$B(x_0, r) \cap E \neq \varnothing \text{ and } B(x_0, r) \cap (X \setminus E) \neq \varnothing \text{ for all } r > 0,$$

so we can use Proposition 2.2.1 to conclude that x_0 must be a boundary point. ∎

proof from (b) to (c). Since either $x_0 \in \mathrm{Int}(E)$ or $x_0 \in \partial E$. If $x_0 \in \mathrm{Int}(E)$, then $x_0 \in E$, then we can choose $X^{(n)} = x_0$ for all $n \geq 1$. If $x_0 \in \partial E$, then given $n \in \mathbb{N}$, $\exists x_n \in B\left(x_0, \frac{1}{n}\right) \cap E \neq \varnothing$. Hence, $x_n \in E$ and $d(x_n, x_0) < \frac{1}{n}$. Pick such x_n to form $\left\{X^{(n)}\right\}_{n=1}^{\infty}$, then we know this sequence converges to x_0. ∎

proof from (c) to (a). Suppose $\left\{X^{(n)}\right\} \subseteq E$ s.t. $\lim_{n\to\infty} d\left(X^{(n)}, x_0\right) = 0$, then we want to show $x_0 \in \overline{E}$. Given any $r > 0$, choose $N \geq 1$ s.t.

$$d\left(X^{(n)}, x_0\right) < r \text{ when } n \geq N.$$

This implies for $n \geq N$, $X^{(n)} \in E$ and $X^{(n)} \in B(x_0, r)$, so we know $E \cap B(x_0, r) \neq \varnothing$ for all $r > 0$, which means $x_0 \in \overline{E}$. ∎

Remark 2.2.3. The equation (a) and (b) implies $\overline{E} = \mathrm{Int}(E) \cup \partial E$.

An alternative proof. Since we know $X = \mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E$ by Theorem 2.2.2, and $\overline{E} \subseteq X$, so

$$\begin{aligned}\overline{E} = \overline{E} \cap X &= \overline{E} \cap (\mathrm{Int}(E) \cup \mathrm{ext}(E) \cup \partial E)\\ &= (\overline{E} \cap \mathrm{Int}(E)) \cup (\overline{E} \cap \mathrm{Ext}(E)) \cup (\overline{E} \cap \partial E).\end{aligned}$$

Also, notice that

$$\overline{E} \cap \mathrm{Int}(E) = \mathrm{Int}(E) \quad \overline{E} \cap \mathrm{Ext}(E) = \varnothing \quad \overline{E} \cap \partial E = \partial E,$$

so $\overline{E} = \mathrm{Int}(E) \cup \partial E$. ∎

Corollary 2.2.1. $\overline{E} = \mathrm{Int}(E) \cup \partial E$.

Theorem 2.2.2. Let (X, d) be a metric space and $E \subseteq X$. Then,

$$X = \mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E$$

Remark 2.2.4. ∂E could be empty. (See previous example.)

Corollary 2.2.2. Let (X, d) be a metric space and $E \subseteq X$. Then

$$\overline{E} = \mathrm{Int}(E) \cup \partial E = X \setminus \mathrm{Ext}(E).$$

Lemma 2.2.1. $\overline{E} = E \cup \partial E$

Proof. We first show that $E \cup \partial E \subseteq \overline{E}$. For every point $x \in E$, we know $x \in B(x, r)$ for all $r > 0$, so $B(x, r) \cap E \neq \varnothing$. Also, by definition, we know $\partial E \subseteq \overline{E}$, so we're done.

Next, we show that $\overline{E} \subseteq E \cup \partial E$. For every $x \in \overline{E}$, if $x \in E$, then $x \in E \cup \partial E$. If not, since $x \in \overline{E}$, so $B(x, r) \cap E \neq \varnothing$ for all $r > 0$. Also, since $x \notin E$, and $x \in B(x, r)$, so $B(x, r) \cap (X \setminus E) \neq \varnothing$, otherwise $x \in B(x,r) \subseteq E$, which is a contradiction. Now we know for every $r > 0$, $B(x,r) \cap E \neq \varnothing$ and $B(x,r) \cap (X \setminus E) \neq \varnothing$, so $x \in \partial E$. ■

Lemma 2.2.2 (Discarded)**.** If $x \in \mathrm{Int}(E)$, then $x \in E$. In other words, $\mathrm{Int}(E) \subseteq E$.

Proof. If $x \in \mathrm{Int}(E)$, then there exists $r > 0$ s.t. $B(x,r) \subseteq E$, and thus $x \in B(x,r) \subseteq E$, which means $x \in E$. ■

Note 2.2.1. I thought we need Lemma 2.2.2 to prove Theorem 2.2.3, but I found it needless. Nevertheless, I still want to keep it since I think it is useful in some elsewhere.

Definition 2.2.9. Let (X,d) be a metric space and $E \subseteq X$. We say E is closed if $\partial E \subseteq E$. We say E is open if it doesn't contain any boundary points i.e. $\partial E \cap E = \varnothing$.

Theorem 2.2.3. E is closed if and only if $\overline{E} = E$.

Proof.

$$\begin{aligned} E \text{ is closed } &\Rightarrow \partial E \subseteq E \Rightarrow \overline{E} = E \cup \partial E = E. \\ E = \overline{E} = E \cup \partial E &\Rightarrow \partial E \subseteq E \Rightarrow E \text{ is closed.} \end{aligned}$$

■

Theorem 2.2.4. E is open. $\Leftrightarrow \mathrm{Int}(E) = E$.

proof of ($\Rightarrow$) **.** E is open means $\partial E \cap E = \varnothing$. Fix $x \in E$, since $x \notin \partial E$, so $\exists r > 0$ s.t. $B(x,r) \cap E = \varnothing$ or $B(x,r) \cap (X \setminus E) = \varnothing$. Since $x \in E$ and $x \in B(x,r)$, so $B(x,r) \cap (X \setminus E) = \varnothing$, which means $B(x,r) \subseteq E$, so $x \in \mathrm{Int}(E)$. Now we know $E \subseteq \mathrm{Int}(E)$. Also, we know $\mathrm{Int}(E) \subseteq E$ by Lemma 2.2.2. Hence, $\mathrm{Int}(E) = E$. ■

proof of ($\Leftarrow$) **.** If $\mathrm{Int}(E) = E$, then given any $x \in E = \mathrm{Int}(E)$, there exists $r > 0$ s.t. $B(x,r) \subseteq E$. Hence, $B(x,r) \cap (X \setminus E) = \varnothing$, so $x \notin \partial E$, and thus $E \cap \partial E = \varnothing$. ■

Theorem 2.2.5. If $E \subseteq X$, then E is open $\Leftrightarrow X \setminus E$ is closed.

proof of ($\Rightarrow$)**.** Since we can write $X = \mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E$, and E is open, so

$$X \setminus E = (\mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E) \setminus E = (\mathrm{Int}(E) \uplus \mathrm{Ext}(E) \uplus \partial E) \setminus \mathrm{Int}(E) = \mathrm{Ext}(E) \uplus \partial E.$$

by Theorem 2.2.4. Now we want to show that $\partial(X \setminus E) \subseteq X \setminus E$, and we know

$$X \setminus E = \mathrm{Ext}(E) \uplus \partial E = \mathrm{Ext}(E) \uplus \partial(X \setminus E)$$

since $\partial E = \partial(X \setminus E)$. Hence, we have $\partial(X \setminus E) \subseteq X \setminus E$. ■

proof of $\Leftarrow$**.** Suppose $X \setminus E$ is closed, then $\partial(X \setminus E) \subseteq X \setminus E$, and since $\partial E = \partial(X \setminus E)$, so $\partial E \subseteq X \setminus E$, and thus $\partial E \cap E = \varnothing$, which means E is open. ■

Lecture 5

Definition 2.2.10. Let (X,d) be a metric space, $E \subseteq X$ and $x_0 \in E$. We say x_0 is an adherent point if for every $r > 0$, $B(x_0,r) \cap E \neq \varnothing$, and we denote $\overline{E}$ to the set of all adherent points. 16 Sep. 10:20

Remark 2.2.5. $E \subseteq \overline{E}$, since given any $x_0 \in E$ and $r > 0$, $x_0 \in B(x_0, r)$, so $B(x_0, r) \cap E \neq \varnothing$, and thus $E \subseteq \overline{E}$.

Remark 2.2.6. $\partial E \subseteq \overline{E}$. Given $x_0 \in \partial E$, we know for any $r > 0$, $B(x_0, r) \cap E \neq \varnothing$, so $x_0 \in \overline{E}$.

Proposition 2.2.5. $x_0 \in \overline{E}$ if and only if there exists $\left(X^{(n)}\right)_{n=1}^{\infty} \subseteq E$ s.t. $\lim_{n\to\infty} X^{(n)}$ exists and $\lim_{n\to\infty} X^{(n)} = x_0$.

proof of (⇒). Given $n \in \mathbb{N}$. Consider $B\left(x_0, \frac{1}{n}\right)$. We know $B\left(x_0, \frac{1}{n}\right) \cap E \neq \varnothing$. Choose $X^{(n)} \in B\left(x_0, \frac{1}{n}\right) \cap E$, then $d\left(x_0, X^{(n)}\right) < \frac{1}{n}$, which means $\lim_{n\to\infty} d\left(x_0, X^{(n)}\right) = 0$. Hence, there exists $\left(X^{(n)}\right) \subseteq E$ s.t. $\lim_{n\to\infty} X^{(n)} = x_0$. ■

proof of (⇐). There exists N s.t. $X^{(n)} \in B(x_0, r)$ when $n \geq N$. Given any $r > 0$, since $\lim_{n\to\infty} X^{(n)} = x_0$, so $\lim_{n\to\infty} d\left(X^{(n)}, x_0\right) = 0$. Hence, there exists N s.t. $d\left(X^{(n)}, x_0\right) < r$ when $n \geq N$. Hence, when $n \geq N$, we have $X^{(n)} \subseteq B(x_0, r)$. Since we know $X^{(n)} \in E$ for all n, so we know $B(x_0, r) \cap E \neq \varnothing$, so $x_0 \in \overline{E}$. ■

Proposition 2.2.6. Let (X, d) be a metric space and $E \subseteq X$, then

$$X = \operatorname{Int}(E) \,\dot\cup\, \operatorname{Ext}(E) \,\dot\cup\, \partial E.$$

Corollary 2.2.3. Let (X, d) be a metric space and $E \subseteq X$. Then,

$$\overline{E} = \operatorname{Int}(E) \cup \partial E = X \setminus \operatorname{Ext}(E) = E \cup \partial E.$$

Proof. Since

$$\begin{aligned} \overline{E} &= \overline{E} \cap X = \overline{E} \cap (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E) \\ &= \left(\overline{E} \cap \operatorname{Int}(E)\right) \cup \left(\overline{E} \cap \operatorname{Ext}(E)\right) \cup \left(\overline{E} \cap \partial E\right) = \operatorname{Int}(E) \,\dot\cup\, \partial E. \end{aligned}$$

Also,

$$X \setminus \operatorname{Ext}(E) = (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E) \setminus \operatorname{Ext}(E) = \operatorname{Int}(E) \cup \partial E = \overline{E}.$$

Besides, we know $\operatorname{Int}(E) \subseteq E \subseteq \overline{E}$, so

$$\overline{E} = \operatorname{Int}(E) \cup \partial E \subseteq E \cup \partial E.$$

Also, by Remark 2.2.5 and Remark 2.2.6, we know $E \cup \partial E \subseteq \overline{E}$, so we know $\overline{E} = E \cup \partial E$. ■

Definition 2.2.11. Let (X, d) be a metric space and $E \subseteq X$. We say E is open iff $\partial E \cap E \neq \varnothing$. We say E is closed iff $\partial E \subseteq E$.

Proposition 2.2.7.

$$E \text{ is open} \Leftrightarrow \operatorname{Int}(E) = E \Leftrightarrow X \setminus E \text{ is closed.}$$

proof of E is open ⇔ $\operatorname{Int}(E) = E$.

(⇒) Since E is open, so $\partial E \cap E = \varnothing$. Hence,

$$\begin{aligned} E &= E \cap X = E \cap (\operatorname{Int}(E) \cup \operatorname{Ext}(E) \cup \partial E) \\ &= (E \cap \operatorname{Int}(E)) \cup (E \cap \operatorname{Ext}(E)) \cup (E \cap \partial E) = \operatorname{Int}(E) \cup (E \cap \partial E) = \operatorname{Int}(E) \end{aligned}$$

since $E \cap \operatorname{Ext}(E) = \varnothing$ and we know $\partial E \cap E = \varnothing$.

(⇐) Since $\operatorname{Int}(E) = E$, and $\operatorname{Int}(E) \cap \partial E = \varnothing$, so $E \cap \partial E = \varnothing$, and thus E is open.

■

proof of E is open $\Leftrightarrow X \setminus E$ is closed.

($\Rightarrow$) $X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E$, so

$$X \setminus E = (\text{Int}(E) \cup \text{Ext}(E) \cup \partial E) \setminus \text{Int}(E) = \text{Ext}(E) \cup \partial E = \text{Int}(X \setminus E) \cup \partial(X \setminus E).$$

Hence, $\partial(X \setminus E) \subseteq X \setminus E$, which means $X \setminus E$ is closed.

($\Leftarrow$) $X \setminus E$ is closed, then $\partial(X \setminus E) \subseteq X \setminus E$, but $\partial E = \partial(X \setminus E)$, so $\partial E \subseteq X \setminus E$, and thus $\partial E \cap E = \varnothing$.

■

Remark 2.2.7. If $\partial E = \varnothing$, then E is open and closed.

Definition 2.2.12 (Clopen). If a set S is closed and open, then S is clopen.

Remark 2.2.8. Let (X, d) be a metric space, then $\varnothing$ is clopen, and we can deduce that X is also clopen since X is the complement of $\varnothing$ and we know S is open iff $X \setminus S$ is closed.

Remark 2.2.9. In $(\mathbb{R}, d)$, where d is the standard metric, then the only clopen set is $\mathbb{R}$ or $\varnothing$.

Remark 2.2.10. Let (X, d_{disc}) be the discrete metric space on X. Let E be any set, then E is open and closed. Given $x_0 \in E$, we know $B_{\text{disc}}\left(x_0, \frac{1}{2}\right) \subseteq E$, so $x_0 \in \text{Int}(E)$, which means $E = \text{Int}(E)$, so E is open. Now since $X \setminus E$ is also open, so E is closed. Thus, E is clopen.

Proposition 2.2.8. The following hold:

(a) E is open iff $E = \text{Int}(E)$.

(b) E is closed iff every convergent sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ in E, then the limit $\lim_{n\to\infty} X^{(n)} \in E$.

(c) Let $r > 0$, then

(i)
$$\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\} \text{ is closed.}$$

(ii)
$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\} \text{ is open.}$$

(d) Any singleton $\{x_0\}$ where $x_0 \in X$ is closed.

(e) E is open iff $X \setminus E$ is closed.

(f) (i) If $E_1, \dots, E_n$ are open sets in X, then $E_1 \cap E_2 \cap \dots \cap E_n$ is open.
 (ii) If $F_1, \dots, F_n$ are closed, then $F_1 \cup \dots \cup F_n$ is closed.

(g) (i) If $\{E_\alpha\}_{\alpha\in I}$ is any collection of open sets in X, then $\bigcup_{\alpha\in I} E_\alpha$ is open.
 (ii) If $\{F_\alpha\}_{\alpha\in I}$ is any collection of closed sets in X, then $\bigcap_{\alpha\in I} F_\alpha$ is closed.

(h) (i) If $E \subseteq X$, then $\text{Int}(E)$ is the largest open set that contained in E i.e. $\text{Int}(E)$ is open and if $V \subseteq E$ and V is open, then $V \subseteq \text{Int}(E)$.
 (ii) If $E \subseteq X$, then $\overline{E}$ is the smallest closed set containing E i.e. $\overline{E}$ is closed and if $E \subseteq K$ and K is closed, then $\overline{E} \subseteq K$.

proof of (b).

($\Rightarrow$) Since E is closed, so $\overline{E} = E$, and we know every convergent sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ converges to x_0 with $x_0 \in \overline{E}$ by Proposition 2.2.4. Thus, we have $x_0 \in E$.

($\Leftarrow$) Assume that every convergent sequence in E has its limit in E. We want to prove that E is closed, i.e. that $X \setminus E$ is open.

Take any point $y \in X \setminus E$. Suppose, for contradiction, that every ball around y meets E. That is, for each $k \in \mathbb{N}$ there exists a point

$$x^{(k)} \in E \cap B\big(y, \tfrac{1}{k}\big)\,.$$

Then, by construction, we have $x^{(k)} \to y$.

By our assumption, the limit of any convergent sequence from E must lie in E. Hence $y \in E$, contradicting the fact that $y \in X \setminus E$.

Therefore, there must exist some radius $r > 0$ such that

$$B(y, r) \cap E = \varnothing,$$

which means $B(y, r) \subseteq X \setminus E$. Thus every point of $X \setminus E$ is an interior point, so $X \setminus E$ is open. Hence E is closed.

$\blacksquare$

proof of (c).

(i) To show that $\overline{B}(x_0, r)$ is closed, it sufficies to show that $X \setminus \overline{B}(x_0, r)$ is open. Note that

$$X \setminus \overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) > r\}\,.$$

Let $y \in X \setminus \overline{B}(x_0, r)$, then define $\varepsilon = d(x_0, y) - r > 0$, then we can similarly prove that $B(y, \varepsilon) \subseteq X \setminus \overline{B}(x_0, r)$. Hence, $X \setminus \overline{B}(x_0, r) = \operatorname{Int}(X \setminus \overline{B}(x_0, r))$, and thus it is open.

(ii) If $y \in B(x_0, r)$, then $d(x_0, y) < r$. Let $\varepsilon = r - d(x_0, y) > 0$, then we claim that $B(y, \varepsilon) \subseteq B(x_0, r)$. Given $z \in B(y, \varepsilon)$, then $d(z, y) < \varepsilon$, then use triangle inequality we know $z \in B(x_0, r)$.

$\blacksquare$

proof of (d). It sufficies to show that $X \setminus \{x_0\}$ is open. Given $y \in X \setminus \{x_0\}$, so we can show that

$$B\left(y, \frac{d(y, x_0)}{2}\right) \subseteq X \setminus \{x_0\}\,.$$

Hence, $y \in \operatorname{Int}(X \setminus \{x_0\})$, and thus $X \setminus \{x_0\}$ is open. $\blacksquare$

proof of (f).

(i) Given $x_0 \in E_1 \cap E_2 \cap \cdots \cap E_n$, then $x_0 \in E_i$ for all $1 \leq i \leq n$. Thus, there exists $r_i > 0$ s.t.

$$B(x_0, r_i) \subseteq E_i \quad \text{for each } 1 \leq i \leq n.$$

Let $r = \min\{r_1, \ldots, r_n\} > 0$, then we know $B(x_0, r) \subseteq B(x_0, r_i) \subseteq E_i$ for all $1 \leq i \leq n$. Hence, $B(x_0, r) \subseteq E_1 \cap E_2 \cap \cdots \cap E_n$, and thus $E_1 \cap \cdots \cap E_n$ is open.

(ii) Now if $F_1, \ldots, F_n$ are closed, then $X \setminus F_1, \ldots, X \setminus F_n$ are open. Since we know $\bigcap_{i=1}^{n}(X \setminus F_i)$ is open, and

$$\bigcap_{i=1}^{n}(X \setminus F_i) = X \setminus \left(\bigcup_{i=1}^{n} F_i\right),$$

so $X \setminus (\bigcup_{i=1}^n F_i)$ is open, which means $\bigcup_{i=1}^n F_i$ is closed.

■

proof of (g).

(i) Suppose $x_0 \in \bigcup_{\alpha\in I} E_\alpha$, then there exists $\mathcal{B} \in I$ s.t. $x_0 \in E_{\mathcal{B}}$. Now since $E_{\mathcal{B}}$ is open, so there exists $r_{x_0} > 0$ s.t.

$$B(x_0, r_{x_0}) \subseteq E_{\mathcal{B}} \subseteq \bigcup_{i\in\alpha} E_\alpha.$$

Hence, $\bigcup_{\alpha\in I} E_\alpha$ is open.

(ii)

$$\left(X \setminus \left(\bigcap_{\alpha\in I} F_\alpha\right)\right) = \bigcup_{\alpha\in I} (X \setminus F_\alpha)$$

is open since $X \setminus F_\alpha$ is open for all $\alpha \in I$, so we have $\bigcap_{\alpha\in I} F_\alpha$ is closed.

Remark 2.2.11.

(1) $\bigcap_{\alpha\in I} E_\alpha$ may NOT be open. For example,

$$\bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

which is closed.

(2) $\bigcup_{\alpha\in I} F_\alpha$ may NOT be closed. For example,

$$\bigcup_{i=1}^{\infty} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] = (-1, 1),$$

which is open.

Note 2.2.2. In the proof of (f), if the index set I is infinite, then we can not pick $\min\{r_1, \ldots, r_n\}$, so we can not deduce that (f) is correct when there infinitely many open sets or closed sets.

■

proof of (h).

(i) We first claim that $\mathrm{Int}(E)$ is open.

Proof. Since for all $x \in \mathrm{Int}(E)$, $\exists r_x > 0$ s.t. $B(x, r_x) \subseteq E$, so

$$\mathrm{Int}(E) = \bigcup_{x\in \mathrm{Int}(E)} B(x, r_x),$$

and by (ii) of (c) and (i) of (g) in Proposition 2.2.8, we know $\mathrm{Int}(E)$ is open. ■

Now if we have $V \subseteq E$ and V is open, then $y \in V$ implies there exists $s > 0$ s.t. $B(y, s) \subseteq V$, and thus $B(y, s) \subseteq E$ since $V \subseteq E$. Hence, we know $y \in \mathrm{Int(E)}$, and thus $V \subseteq \mathrm{Int}(E)$.

(ii) To show $\overline{E}$ is closed, it sufficies to show that $X \setminus \overline{E}$ is open. Note that

$$\overline{E} = X \setminus \mathrm{Ext}(E) = X \setminus \underbrace{\mathrm{Int}(X \setminus E)}_{\text{open}},$$

so $\overline{E}$ is closed. Now if $E \subseteq K$ and K is closed, then if $x \in \overline{E}$, we have $B(x,r) \cap E \neq \varnothing$ for all $r > 0$. Hence, $B(x,r) \cap K \neq \varnothing$ since $E \subseteq K$, so $x \in \overline{K} = K$ (since K is closed). Thus, $\overline{E} \subseteq K$.

■

Lecture 6

2.3 Relative topology

18 Sep. 10:20

Let (X,d) be a metric space and $Y \subseteq X$, then $(Y, d|_{Y\times Y})$ is also a metric space.

Example 2.3.1. Consider $(\mathbb{R}^2, d_2)$ and $X = \{(x,0) \mid x \in \mathbb{R}\}$, then on $(X, d_2|_{X\times X}) = (X,d)$, it is also a metric space.

Proof. Since

$$d((x,0),(y,0)) = \sqrt{(x-y)^2 + 0^2} = |x-y|,$$

so it is obvious that d is a metric.

Note that X is not open in $\mathbb{R}^2$. Also, if $E = \{(x,0) \mid -1 < x < 1\}$, then E is not open in $\mathbb{R}^2$, but E is open in $(X, d_2|_{X\times X})$. ⊛

Example 2.3.2. Suppose $X = (-1,1) \subseteq \mathbb{R}$, then $(X, d|_{X\times X})$ is a metric space. Consider $E = [0,1)$, then we know E is not closed in $(\mathbb{R}, d)$ since $1 \notin \overline{E}$. But E is closed in $(X, d|_{X\times X})$ since $\overline{E} = X$ in $(X, d|_{X\times X})$.

Definition 2.3.1 (relatively open/close). Let (X,d) be a metric space and $Y \subseteq X$. We say E is relatively open (resp. closed) in Y if E is open (resp. closed) in $(Y, d|_{Y\times Y})$.

Note 2.3.1. In the following context, if we say E is open in Y, then we mean E is "relatively" open, and if we say E is closed in Y, then we mean E is relatively closed in Y.

Note 2.3.2. If Y is open/closed in E, then $Y \subseteq E$. Otherwise, we cannot define $d|_{Y\times Y}(a,b)$ for $a, b \in E \setminus Y$.

Remark 2.3.1. If $Y \subseteq X$, and $(X,d), (Y, d|_{Y\times Y})$ are both metric spaces, then

$$B_Y(x,r) = \{y \in Y \mid d(y,x) < r\} = B_X(x,r) \cap Y.$$

Remark 2.3.2. If E is relatively open in Y, then given $x_0 \in E$, $\exists r_0 > 0$ s.t. $B_X(x_0, r_0) \cap Y \subseteq E$. This is because by Remark 2.3.1, we have

$$B_X(x_0,r_0) \cap Y = B_Y(x_0, r_0) \subseteq E.$$

Remark 2.3.3. A set $E \subseteq Y$ is relatively closed in Y if given any $r > 0$ and $x_0 \in Y$,

$$B_Y(x_0, r) \cap E \neq \varnothing,$$

then $x_0 \in E$. This is because "closed" gives $E = \overline{E}_Y$. Note that this statement is equivalent to

$$\text{If } x_0 \in \overline{E}_Y, \text{ then } x_0 \in E = E_Y.$$

Proposition 2.3.1. Let (X, d) be a metric space, and $Y \subseteq X$ and $E \subseteq Y$, then

(1) E is relatively open in Y iff $\exists$ open set V in (X, d) s.t. $E = V \cap Y$.

(2) E is relatively closed in Y iff $\exists$ closed set K in (X, d) s.t. $E = K \cap Y$.

proof of (1).

($\Rightarrow$) Given any $x \in E$, $\exists r_x > 0$ s.t. $B_X(x, r_x) \cap Y \subseteq E$. Let $V = \bigcup_{x \in E} B_X(x, r_x)$. Obviously, $V \cap Y = E$ and V is open.

($\Leftarrow$) Suppose $E = V \cap Y$, then given any $x \in E$, since V is open, so there exists $r > 0$ s.t. $B_X(x, r) \subseteq V$, and then $B_X(x, r) \cap Y \subseteq V \cap Y = E$. Since x is an interior point of E in Y, so $\mathrm{Int}_Y(E) = E$, and thus E is open in Y.

■

proof of (2).

($\Rightarrow$) E is relatively closed in Y, then $Y \setminus E$ is relatively open, so there exists V open in X s.t. $Y \setminus E = V \cap Y$. Hence,

$$\begin{aligned} E = Y \setminus (Y \setminus E) &= (X \setminus (Y \setminus E)) \cap Y = (X \setminus (V \cap Y)) \cap Y \\ &= ((X \setminus V) \cup (X \setminus Y)) \cap Y \\ &= ((X \setminus V) \cap Y) \cup ((X \setminus Y) \cap Y) \\ &= (X \setminus V) \cap Y \end{aligned}$$

Let $E = (X \setminus V) \cap Y = K \cap Y$, then since $K = X \setminus V$ is closed in X, so we're done.

($\Leftarrow$) Suppose $E = K \cap Y$ for some closed K, then $Y \setminus E = (X \setminus K) \cap Y$, which means $Y \setminus E$ is relatively open in Y since $X \setminus K$ is open and by (a), so E is closed in Y.

■

Example 2.3.3. Let $X = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ with $x, y \in X$, then

(i) $[0, 1]$ is open and closed in X.

(ii) $\partial_X [0, 1] = \varnothing$.

Proof.

(i) We want to find V open in $\mathbb{R}$ s.t.

$$[0, 1] = V \cap \overbrace{([0, 1] \cup [2, 3])}^{X},$$

we can choosed $V = \left(-\frac{1}{2}, \frac{3}{2}\right)$, so $[0, 1]$ is open in X.
We want to find K closed in $\mathbb{R}$ and

$$[0, 1] = K \cap ([0, 1] \cup [2, 3]),$$

and we can choosed $K = \left[-\frac{1}{2}, \frac{3}{2}\right]$, so $[0, 1]$ is closed in X.

(ii) If $x \in \partial_X [0, 1]$, then $B_X(x, r) \cap [0, 1]$ and $B_X(x, r) \cap [2, 3]$ are both nonempty for any $r > 0$. However, this is impossible for any x in X, so $\partial_X [0, 1] = \varnothing$.

⊛

2.4 Cauchy sequence and complete metric space

Definition 2.4.1 (subsequence). Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ is a sequence in (X,d). Suppose $m \leq n_1 < n_2 < \ldots$, then $\left(X^{(n_j)}\right)_{j=1}^{\infty}$ is called a subsequence of $\left(X^{(n)}\right)_{n=m}^{\infty}$.

Example 2.4.1. $X^{(n)} = (-1)^n$ for all $n \in \mathbb{N}$.

Proof.

$$\left\{X^{(2n)}\right\}_{n=1}^{\infty}$$

is a subsequence of $\left\{X^{(n)}\right\}_{n=1}^{\infty}$. ⊛

Lemma 2.4.1. Let $\left\{X^{(n)}\right\}_{n=m}^{\infty}$ be a convergent sequence with $\lim_{n\to\infty} X^{(n)} = x$, then every subsequence of $\left\{X^{(n)}\right\}_{n=m}^{\infty}$ also converges to x_0.

Definition 2.4.2 (limit points). Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ is a sequence in (X,d), then we say L is a limit point of $\left(X^{(n)}\right)_{n=m}^{\infty}$ if for every $N \geq m$ and every $\varepsilon > 0$, there exists $n \geq N$ s.t. $d\left(X^{(n)}, L\right) \leq \varepsilon$.

Proposition 2.4.1. L is a limit point of $\left(X^{(n)}\right)_{n=m}^{\infty}$ iff there exists a subsequence

$$\left(X^{(n_j)}\right)_{j=1}^{n}$$

converges to L.

Proof.

($\Rightarrow$) Assume L is a limit point, now we build a subsequence converges to L by an inductive method. Our goal is to build a subsequence $\left\{X^{(n_j)}\right\}_{j=1}^{\infty}$ so that

$$d\left(X^{(n_j)}, L\right) < \frac{1}{j} \quad \forall 1 \leq j.$$

For $j = 1$, pick $N = m$, and pick $\varepsilon < \frac{1}{1}$ to pick $n_1 \geq N$ s.t.

$$d\left(X^{(n_1)}, L\right) \leq \varepsilon < \frac{1}{1}.$$

Now suppose $n_1, n_2, \ldots, n_{k-1}$ are all chosen, then now we can pick $N = n_{k-1} + 1$ and $\varepsilon < \frac{1}{k}$, so that we can pick $n_k \geq N$ s.t. $d\left(X^{(n_k)}, L\right) \leq \varepsilon < \frac{1}{k}$, so we're done. Now we show that this subsequence converges to L. For every $\varepsilon > 0$, we know there exists $0 < \frac{1}{k} < \varepsilon$, so for all $K \geq k$, we have

$$d\left(X^{(K)}, L\right) < \frac{1}{K} \leq \frac{1}{k} < \varepsilon,$$

so we're done.

($\Leftarrow$) Left as exercise to the reader.

■

Proposition 2.4.2. L is a limit point iff $L \in \bigcap_{N=1}^{\infty} \overline{S_N}$ where $S_N = \left\{X^{(K)}\right\}_{K \geq N}$.

Definition 2.4.3 (Cauchy sequence). Let $\left(X^{(n)}\right)_{n=m}^{\infty}$ be a sequence in (X, d). We say this sequence is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \geq m$ s.t. $d\left(X^{(j)}, X^{(k)}\right) < \varepsilon$ for all $j, k \geq N$.

Lemma 2.4.2. Suppose $\left(X^{(n)}\right)_{n=m}^{\infty}$ converges in (X, d), then $\left(X^{(n)}\right)_{n=m}^{\infty}$ is a Cauchy sequence in (X, d).

Proof. Suppose $\lim_{n\to\infty} X^{(n)} = X_0$, then for every $\frac{\varepsilon}{2} > 0$, there exists $N \geq m$ s.t. $d\left(X^{(n)}, X_0\right) < \frac{\varepsilon}{2}$ for all $n \geq N$. If $j, k \geq N$, then

$$d\left(X^{(j)}, X^{(k)}\right) \leq d\left(X^{(j)}, X_0\right) + d\left(X^{(k)}, X_0\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

Example 2.4.2. A sequence in $\mathbb{Q}$ may not converges in $\mathbb{Q}$.

Proof. See teacher's note. ⊛

Definition 2.4.4 (Complete space). A metric space (X, d) is complete iff every Cauchy sequence converges to some points in X.

Remark 2.4.1. $\mathbb{Q} \subseteq \mathbb{R}$, then $(\mathbb{Q}, d)$ is not complete.

Remark 2.4.2. The limit of a convergent sequence in metric space is unique. If

$$\lim_{n\to\infty} x^{(n)} = y \quad \text{and} \quad \lim_{n\to\infty} x^{(n)} = z,$$

then suppose by contradiction, $y \neq z$. Then,

$$0 \leq d(y, z) \leq d\left(y, x^{(n)}\right) + d\left(z, x^{(n)}\right).$$

By squeeze theorem, we know $d(y, z) = 0$ and thus $y = z$.

Proposition 2.4.3. Let (X, d) be a metric space and let $(Y, d|_{Y\times Y})$ be a subspace of (X, d). If $(Y, d|_{Y\times Y})$ is complete, then Y is closed in X.

Proof. We want to show that $Y = \overline{Y}$, so we want to show for all $y \in \overline{Y}$, we have $y \in Y$. Now for every $y \in \overline{Y}$, then by Proposition 2.2.4, we know there exists a convergent sequence $\left\{Y^{(n)}\right\}_{n=1}^{\infty}$ in Y and converges to y. However, every convergent sequence is Cauchy, and since $(Y, d|_{Y\times Y})$ is complete, so $\left\{Y^{(n)}\right\}_{n=1}^{\infty}$ converges in Y, which means $y \in Y$, and we're done. ■

Proposition 2.4.4. If (X, d) is complete and $Y \subseteq X$ is closed, then $(Y, d|_{Y\times Y})$ is complete.

Proof. Given a Cauchy sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ in Y, so this is also a Cauchy sequence in X, so it converges in X. If $\exists x_0 \in X$ s.t. $\lim_{n\to\infty} X^{(n)} = x_0$. Since Y is closed, so $Y = \overline{Y}$, and by Proposition 2.2.4, we know $x_0 \in \overline{Y} = Y$, so $x_0 \in Y$, and thus $\left(X^{(n)}\right)_{n=1}^{\infty}$ also converges in Y. ■

Lecture 7

Completeness of $\mathbb{R}^n$ with d_2, d_1, d_∞ 23 Sep. 09:10

As previously seen. (X, d_1) and (X, d_2) are Lipschitz equivalent if $\exists c_1, c_2 > 0$s.t.

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_2(x,y) \quad \forall x, y \in X.$$

Theorem 2.4.1. Suppose (X, d_1) and (X, d_2) are Lipschitz equivalent, then

$$(X_1, d_1) \text{ is complete} \Leftrightarrow (X, d_2) \text{ is complete.}$$

Proof.

($\Rightarrow$) Given any Cauchy sequence $\left(x^{(n)}\right)_{n=1}^{\infty}$ in (X, d_2), then since $d_1(x,y) \leq \frac{1}{c_1} d_2(x,y)$, so $\left(x^{(n)}\right)_{n=1}^{\infty}$ is Cauchy in (X, d_1). Since (X, d_1) is complete, so there exists $x \in X$ s.t. $\lim_{n\to\infty} x_n = x \in (X, d_1)$. However, $x \in (X, d_2)$, so (X, d_2) is complete.

($\Leftarrow$) Similar.

■

Theorem 2.4.2. $(\mathbb{R}^n, d_2)$ is a complete metric space.

Corollary 2.4.1. ince $(\mathbb{R}^n, d_2), (\mathbb{R}^n, d_1), (\mathbb{R}^n, d_\infty)$ are Lipschitz equivalent, so they are all complete by Theorem 2.4.1 and Theorem 2.4.2.

2.5 Compact metric space

Definition 2.5.1 (Compact space)**.** A metric space (X, d) is compact iff every sequence in (X, d) has at least one convergent subsequence convergeing in X. A subset $Y \subseteq X$ is compact if $(Y, d|_{Y\times Y})$ is compact. That is, $(Y, d|_{Y\times Y})$ is compact if for any sequence $\left(y^{(n)}\right)_{n=1}^{\infty} \subseteq Y$, there exists a subsequence $\left(y^{(n_j)}\right)_{j=1}^{\infty}$ and $y \in Y$ s.t. $\lim_{k\to\infty} y^{(n_k)} = y$.

Definition 2.5.2 (Bounded)**.** Let (X, d) be a metric space and let $Y \subseteq X$. We say Y is bounded iff for any $x \in X$, there exists $r > 0$ s.t. $Y \subseteq B_X(x, r)$.

Theorem 2.5.1.

$$Y \text{ is bounded} \Leftrightarrow \exists x_0 \in X \text{ and } R > 0 \text{ s.t. } Y \subseteq B_X(x_0, R).$$

Proof. The "($\Rightarrow$)" is easy, so we just prove the other direction. Given any $x \in X$, we can choose $r_x = R + d(x, x_0)$.

Claim 2.5.1. $Y \subseteq B_X(x, r_x)$.

Proof. Let $y \in Y$, we know

$$d(y,x) \leq d(y, x_0) + d(x_0, x) < R + d(x_0, x).$$

Hence, $y \in B_X(x, r_x)$. ⊛

■

Proposition 2.5.1. Let (X, d) be a compact metric space. Then (X, d) is complete and bounded.

Proof.

- We want to show that (X,d) is complete. Given any Cauchy sequence $\left(x^{(n)}\right)_{n=1}^{\infty}$ in (X,d), then since (X,d) is compact, so there exists a compact subsequence $\left(x^{(n_k)}\right)_{k=1}^{\infty}$ in X s.t. $\lim_{k\to\infty} x^{(n_k)} = x$. Since $\left(x^{(n)}\right)_{n=1}^{\infty}$ is Cauchy sequence and $\left(x^{(n_k)}\right)_{k=1}^{\infty}$ converges to x, so $\lim_{n\to\infty} x^{(n)} = x$. (See Theorem A.0.1)
- Consider $x_0 \in X$. Suppose X is not bounded, then $B(x_0, n)$ will not contain X for all n. For each $n \in \mathbb{N}$,
$$\exists y^{(n)} \in X \text{ and } y^{(n)} \notin B_X(x_0, n) \text{ i.e. } d\left(y^{(n)}, x_0\right) \geq n.$$
Hence, $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ is a sequence in (X,d) with $d\left(y^{(n)}, x_0\right) \geq n$. Since (X,d) is compact, so there exists a convergent sequence $\left\{y^{(n_k)}\right\}_{k=1}^{\infty}$ and $y \in X$ s.t. $\lim_{k\to\infty} y^{(n_k)} = y$. Hence, there exists $R > 0$ s.t. $d\left(y, y^{(n_k)}\right) < R$ for all k which is big enough, but this means
$$n_k \leq d\left(y^{(n_k)}, x_0\right) \leq d\left(y^{(n_k)}, y\right) + d(y, x_0) < R + d(y, x_0),$$
which is a fixed value, but n_k can be arbitrary large, so this is a contradiction.

■

Corollary 2.5.1. Let (X,d) be a metric space and Y be a compact subset, then Y is closed and bounded.

Proof. Since Y is a compact subset, so $(Y, d|_{Y\times Y})$ is compact. Thus, Y is bounded by Proposition 2.5.1. Hence, $\exists y_0 \in Y$ and $R > 0$ s.t.

$$Y \subseteq B_Y(y_0, R) = B_X(y_0, R) \cap Y \subseteq B_X(y_0, R).$$

Let $y \in \overline{Y}$, then $\exists \left(y^{(n)}\right)_{n=1}^{\infty}$ in Y s.t. $\lim_{n\to\infty} y^{(n)} = y$. Also, since Y is compact, so for the convergent sequence $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, there is a subsequence $\left\{y^{(n_k)}\right\}_{k=1}^{\infty}$ and $y_0 \in Y$ s.t. $\lim_{k\to\infty} y^{(n_k)} = y_0 \in Y$. By uniqueness of limit in metric space, we know $y = y_0$, and thus $y \in \overline{Y}$. Hence, $\overline{Y} = Y$. (Actually, by Lemma 2.4.2, we know $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ is Cauchy, and then by Theorem A.0.1, we know $y = y_0$.) ■

Theorem 2.5.2 (Heine-Borel Theorem). Let $(\mathbb{R}^n, d)$ be $\mathbb{R}^n$ with $d = d_2, d_\infty, d_1$, and let $E \subseteq \mathbb{R}^n$, then

$$E \text{ is compact} \Leftrightarrow E \text{ is closed and bounded.}$$

Proof.

($\Rightarrow$) Trivial by the corollary.

($\Leftarrow$) Suppose E is closed and bounded. Given a sequence $\left(X^{(n)}\right)_{n=1}^{\infty}$ in E. By Bolzano-Weierstrass Theorem , every bounded sequence has a convergent subsequnce. Since E is closed, so $E = \overline{E}$, and thus the convergent subsequence converges in E. Hence, E is compact.

■

Remark 2.5.1. In a metric space, closed and bounded do not imply compact but compact implies closed and bounded.

Example 2.5.1. Consider $(\mathbb{Z}.d_{\text{disc}})$, then $\mathbb{Z}$ is bounded since $\mathbb{Z} \subseteq B_{\text{disc}}(0,2)$ and $\mathbb{Z}$ is closed in $\mathbb{Z}$ but $\mathbb{Z}$ is not compact since any subsequence of $\{n\}_{n\in\mathbb{N}}$ does not converge in (Z, d_{disc}).

Theorem 2.5.3. Let (X, d) be a metric space, let Y be a compact subset of X. Let $(V_\alpha)_{\alpha\in A}$ be a collection of open sets in X, and suppose that $Y \subseteq \bigcup_{\alpha\in A} V_\alpha$ (i.e. $(V_\alpha)_{\alpha\in A}$ covers Y). Then, there exists a finite subset $F \subseteq A$ s.t. $Y \subseteq \bigcup_{\alpha\in F} V_\alpha$.

Proof. We prove by contradiction. Suppose there does not exist a finite subset $F \subseteq A$ s.t. $Y \subseteq \bigcup_{\alpha\in F} V_\alpha$. For each $y \in Y \subseteq \bigcup_{\alpha\in A} V_\alpha$. $\exists \alpha \in A$ s.t. $y \in V_\alpha$. Since V_α is open, so there exists $r > 0$ s.t. $B(y, r) \subseteq V_\alpha$. Define

$$r(y) = \sup\{r > 0 : B_X(y, r) \subseteq V_\alpha \text{ for some } \alpha \in A\}.$$

Note that $r(y) > 0$ for all $y \in Y$. Now if we pick $r_0 = \inf\{r(y) : y \in Y\}$, then $r_0 \geq 0$.

- Case 1: $r_0 = 0$, there exists $y^{(n)} \in Y$ s.t. $0 < r\left(y^{(n)}\right) < \frac{1}{n}$. Thus, $\left(y^{(n)}\right)_{n=1}^{\infty}$ is a sequence in Y, and since Y is compact, so there exists a convergent subsequence $\left(y^{(n_k)}\right)_{k=1}^{\infty}$ converging to $y_0 \in Y$. Also, there exists $\varepsilon > 0$ and $\alpha \in A$ s.t. $B_X(y_0, \varepsilon) \subseteq V_\alpha$. Since $\lim_{k\to\infty} d\left(y^{(n_k)}, y_0\right) = 0$, so there exists $N > 0$ s.t. $j \geq N$ implies

$$y^{(n_j)} \in B_X\left(y_0, \frac{\varepsilon}{2}\right).$$

Claim 2.5.2. For all $j \geq N$, $B\left(y^{(n_j)}, \frac{\varepsilon}{2}\right) \subseteq B(y_0, \varepsilon)$.

Proof. Suppose $z \in B\left(y^{(n_j)}, \frac{\varepsilon}{2}\right)$, then $d\left(z, y^{(n_j)}\right) < \frac{\varepsilon}{2}$, and thus

$$d(z, y_0) \leq d\left(z, y^{(n_j)}\right) + d\left(y^{(n_j)}, y_0\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

⊛

Now since $B_X(y_0, \varepsilon) \subseteq V_\alpha$, so for $j \geq N$, $B\left(y^{(n_j)}, \frac{\varepsilon}{2}\right) \subseteq V_\alpha$, which means

$$r\left(y^{(n_j)}\right) \geq \frac{\varepsilon}{2} > 0.$$

However, this contradicts to the assumption that $r\left(y^{(n_j)}\right) < \frac{1}{n_j}$ for all j. Hence, Case 1 is impossible.

- Case 2: $\infty > r_0 > 0$. We know $r_0 \leq r(y)$ for all $y \in Y$ by definition. Hence, $0 < \frac{r_0}{2} < r(y)$. This means for each $y \in Y$, there exists $\alpha \in A$ s.t. $B_X\left(y, \frac{r_0}{2}\right) \subseteq V_\alpha$. Choose a point $y^{(1)} \in Y$ s.t. $\exists \alpha_1 \in A$ s.t. $B_X\left(y^{(1)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_1}$. Since V_{α_1} cannot cover Y, so there exists $y^{(2)} \in Y$ and $y^{(2)} \notin B_X\left(y^{(1)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_1}$. Hence, $d\left(y^{(2)}, y^{(1)}\right) \geq \frac{r_0}{2}$. Now we set the induction hypothesis: Suppose there exists $y^{(1)}, \ldots, y^{(k)} \in Y$ and $\alpha_1, \ldots, \alpha_k \in A$ s.t.

$$B_X\left(y^{(j)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_j} \text{ and } d\left(y^{(i)}, y^{(j)}\right) \geq \frac{r_0}{2} \quad \forall i \neq j,$$

and $B_X\left(y^{(1)}, \frac{r_0}{2}\right) \cup \cdots \cup B_X\left(y^{(k)}, \frac{r_0}{2}\right)$ cannot cover Y, then we can find

$$y^{(k+1)} \notin B_X\left(y^{(1)}, \frac{r_0}{2}\right) \cup \cdots \cup B_X\left(y^{(k)}, \frac{r_0}{2}\right),$$

and thus $d\left(y^{(k+1)}, y^{(i)}\right) \geq \frac{r_0}{2}$ for $1 \leq i \leq k$. Also, $\exists \alpha_{k+1}$ s.t. $B\left(y^{(k+1)}, \frac{r_0}{2}\right) \subseteq V_{\alpha_{k+1}}$. Now we know $B\left(y^{(1)}, \frac{r_0}{2}\right) \cup \cdots \cup B\left(y^{(k+1)}, \frac{r_0}{2}\right)$ won't cover Y, then $\left\{y^{(k)}\right\}_{k=1}^{\infty}$ is a sequence in Y and $d\left(y^{(j)}, y^{(l)}\right) \geq \frac{r_0}{2}$. Since Y is compact, so there exists a subsequence of $\left\{y^{(k)}\right\}_{k=1}^{\infty}$ which is convergent, but it is impossible, so we have a contradiction.

- Case 3: $r_0 = \infty$. If so, then it means $\inf\{r(y) : y \in Y\} = \infty$, so $r(y) = \infty$ for all $y \in Y$, otherwise if for some $y' \in Y$, $r(y')$ is finite, then $r_0 \leq r(y')$, and will get a contradiction. Now we have $r(y) = \infty$ for all $y \in Y$. This means for all $r > 0$, there exists some $\alpha \in A$ s.t.

$B_X(y,r) \subseteq V_\alpha$. Now since Y is compact, so Y is bounded, which means for all $y \in Y$, there exists r_y s.t. $Y \subseteq B_X(y, r_y)$. However, since $r(y) = \infty$ and by the previous argument, we know $B_X(y, r_y) \subseteq V_{\alpha_y}$ for some $\alpha_y \in A$, and thus $Y \subseteq V_{\alpha_y}$, and thus V_{α_y} covers Y, which is a contradiciton.

■

Lecture 8

25 Sep. 10:20

Theorem 2.5.4 (Review Theorem 2.5.3). Let Y be a compact subset of a metric space (X,d) and let $\{V_\alpha\}_{\alpha\in A}$ be an open cover of Y. Then $\exists$ a finite subcover of $\{V_\alpha\}_{\alpha\in A}$ i.e. $\exists \alpha_1, \ldots, \alpha_n \in A$ s.t. $Y \subseteq \bigcup_{i=1}^n V_{\alpha_i}$.

Remark 2.5.2.

$$Y \text{ is compact } \Leftrightarrow \text{ Any open cover of } Y \text{ has a finite subcover.}$$

Proof. The ($\Rightarrow$) direction is proved. Now we proved the other direction.

Claim 2.5.3. If (X,d) is a metric space and for all open cover of X, there exists finite subcover of X, then X is complete.

Proof. We will prove this by contradiction, using the definition of compactness (the open cover property).

1. Assumption Assume for the sake of contradiction that X is **compact** but **not complete**. Since X is not complete, there exists a **Cauchy sequence** $\{x_n\}_{n=1}^\infty$ in X that **does not converge** to any point $p \in X$.

2. Constructing the Open Cover Since $\{x_n\}$ does not converge to a point $p \in X$, for every $p \in X$, the point p is not the limit of the sequence. This means there exists some $\epsilon_p > 0$ such that the open ball $B_{\epsilon_p}(p)$ contains only a **finite number of terms** of the sequence $\{x_n\}$.

To see why, suppose for a contradiction that there was some $p \in X$ such that for all $\epsilon > 0$, the ball $B_\epsilon(p)$ contains an infinite number of terms of $\{x_n\}$. Let $\{x_{n_k}\}$ be a subsequence with $x_{n_k} \in B_{1/k}(p)$. This subsequence converges to p. Since $\{x_n\}$ is a Cauchy sequence and has a convergent subsequence, the entire sequence $\{x_n\}$ must converge to the same limit p, which contradicts our initial assumption. Therefore, the property holds: for every $p \in X$, there is an $\epsilon_p > 0$ such that $B_{\epsilon_p}(p)$ contains x_n for only finitely many n.

Consider the collection of open balls $\mathcal{U} = \{B_{\epsilon_p}(p) : p \in X\}$. Since the union of these balls covers every point $p \in X$, $\mathcal{U}$ is an **open cover** of X:

$$X \subseteq \bigcup_{p\in X} B_{\epsilon_p}(p).$$

3. Using Compactness to Find a Finite Subcover Since X is **compact**, the open cover $\mathcal{U}$ must have a **finite subcover**. That is, there exist a finite number of points $p_1, p_2, \ldots, p_k \in X$ such that

$$X \subseteq B_{\epsilon_{p_1}}(p_1) \cup B_{\epsilon_{p_2}}(p_2) \cup \cdots \cup B_{\epsilon_{p_k}}(p_k) = \bigcup_{i=1}^k B_{\epsilon_{p_i}}(p_i).$$

4. Reaching the Contradiction By the definition of ϵ_{p_i}, each ball $B_{\epsilon_{p_i}}(p_i)$ contains x_n for only a **finite number** of indices n. The union of a finite number of finite sets is a finite set. Therefore, the finite union $\bigcup_{i=1}^k B_{\epsilon_{p_i}}(p_i)$ can contain x_n for only a finite number of indices n. However, since this finite union covers all of X (step 3), it must contain **all** terms of the sequence $\{x_n\}_{n=1}^\infty$. Since the sequence $\{x_n\}$ is an infinite set of points, this is a **contradiction**.

The initial assumption that X is not complete must be false. Thus, every compact metric space is complete. ⊛

Suppose any open cover of Y has a finite subcover, then given any sequence $\left(y^{(n)}\right)_{n=1}^{\infty}$. Consider

$$\bigcup_{x\in Y} B_Y(x,1),$$

then this is an open cover of Y, and now we know there is a finite subcover

$$\bigcup_{i=1}^{k} B_Y(x_i,1)$$

of Y where $x_i \in Y$ for all i. Now since $\left(y^{(n)}\right)_{n=1}^{\infty}$ has infinitely many terms, so we know for some i, we have infinitely many terms of $\left(y^{(n)}\right)_{n=1}^{\infty} \subseteq B_Y(x_i,1)$ by Pigeonhole principle. Hence, there are infinitely many terms of $\left(y^{(n)}\right)_{n=1}^{\infty}$ are in

$$\left\{y\in Y : 0 \leq d(y,x_i) < \frac{1}{2}\right\} \cup \left\{y\in Y : \frac{1}{2} \leq d(y,x_i) < 1\right\}.$$

Thus, again, by Pigeonhold principle we know there are infinitely many terms of $\left(y^{(n)}\right)_{n=1}^{\infty}$ are in either one of the above two sets. By repeating split the space into half as what we do above, we know for all $k \geq 0$, there are infinitely many terms of $\left(y^{(n)}\right)_{n=1}^{\infty}$ has the following property: Every two terms of these "infinitely many terms" has distance less than $\frac{1}{2^k}$. Note that this means we can pick a subsequence of $\left(y^{(n)}\right)_{n=1}^{\infty}$ so that it is Cauchy, and since every Cauchy sequence converges in Y (Since Claim 2.5.3), so we're done.

■

Corollary 2.5.2. Let (X,d) be a metric space and let $K_1, K_2, \ldots$ be a sequence of nonempty compact subsets of X s.t. $K_{i+1} \subseteq K_i$ for $i \in \mathbb{N}$, that is, $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ for $i \in \mathbb{N}$, then

$$\bigcap_{i=1}^{\infty} K_i \neq \varnothing.$$

Proof. Suppose $\bigcap_{i=1}^{\infty} K_i = \varnothing$. Since K_i's are compact, so they are closed. Also, we have

$$\bigcup_{i=1}^{\infty} (K_1 \setminus K_n) = K_1 \setminus \left(\bigcap_{i=1}^{\infty} K_n\right) = K_1.$$

Let $V_i = K_1 \setminus K_i = K_1 \cap K_i^C$. Note that K_i^C is open in X. Hence, we have V_i is open in K_1, and thus $\{V_i\}_{i=1}^{\infty}$ is an open cover of K_1 in K_1. ($(K_1, d|_{K_1\times K_1})$ is compact.) By Theorem 2.5.3, we know there exists $\alpha_1, \alpha_2, \ldots, \alpha_l$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_l$ s.t.

$$\begin{aligned} K_1 \subseteq \bigcup_{i=1}^{l} V_{\alpha_i} &= \bigcup_{i=1}^{l} (K_1 \setminus K_{\alpha_i}) \\ &= K_1 \setminus \bigcap_{i=1}^{l} K_{\alpha_i} = K \setminus K_{\alpha_l} \end{aligned}$$

since $K_{\alpha_1} \supseteq K_{\alpha_2} \supseteq \cdots \supseteq K_{\alpha_l}$. However, $K_{\alpha_l} \subseteq K_1$ and $K_{\alpha_l} \neq \varnothing$. Thus, we have a contradiction. ■

Example 2.5.2. Consider $I_1 = [0,1]$, and $I_2 = [0,\frac{1}{3}]\cup[\frac{2}{3},1]$, and picking $I_3, I_4, \ldots$ with same method, then $I_{n+1} \subseteq I_n$ for all n and they are compact, so

$$\bigcap_{i=1}^{\infty} I_i \neq \varnothing.$$

Theorem 2.5.5. Let (X, d) be a metric space.

(a) If Y is a compact subset of X, and $Z \subseteq Y$, then Z is compact iff Z is closed.

(b) If $Y_1, \ldots, Y_n$ are a finite collection of compact subsets of X, then $\bigcup_{i=1}^{n} Y_i$ are also compact.

proof of (a). If Z is compact, then by Corollary 2.5.1, we know Z is closed. Now we show that if Z is closed, then Z is comapct. If Z is closed, then $Y \setminus Z$ is open in Y, then we know

$$Y \setminus Z = V \cap Y$$

for some open set $V \subseteq Y$, so note that $(Y \setminus Z) \subseteq V$. Now suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of Z. Hence, we know $\{U_\alpha\}_{\alpha \in A} \cup \{V\}$ is an open cover of Y since the former covers Z and the latter covers $Y \setminus Z$. Now since Y is compact, so we know for some $\alpha_1, \alpha_2, \ldots, \alpha_n$, there is

$$Y \subseteq \left(\bigcup_{i=1}^{n} U_{\alpha_i} \right) \cup V,$$

and thus we can write

$$Z \subseteq Y \subseteq \left(\bigcup_{i=1}^{n} U_{\alpha_i} \right) \cup V.$$

However, note that $Z \cap V = \varnothing$ since

$$Z = Y \setminus (Y \setminus Z) = Y \setminus (V \cap Y) = (Y \setminus V) \cup (Y \setminus Y) = Y \setminus V.$$

Hence, we know

$$Z \subseteq \bigcup_{i=1}^{n} U_{\alpha_i},$$

and thus for any open cover of Z, we know there exists a finite subcover of Z, and we're done. ■

Chapter 3

Continuous functions on metric spaces

Suppose (X, d_x) and (Y, d_y) are metric space. Let $f : X \to Y$ be a function from X to Y. Then we want that if $x \in X$ is close to $y \in X$, then, then $f(x) \in Y$ is close to $f(y) \in Y$.

Definition 3.0.1 (Continuous function)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function. Suppose $x_0 \in X$, we will say f is continous at x_0 iff for every $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{whereas } d_X(x, x_0) < \delta.$$

We say f is continuous if f is continuous at every point $x \in X$.

Definition 3.0.2 (Preimage)**.** Let $f : X \to Y$ be a function from $X \to Y$ and $V \subseteq Y$. The preimage (inverse image) of V is

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}.$$

Example 3.0.1. Suppose $f(x) = x^2$, then what is the preimage of $(1, \infty)$?

Proof.

$$f^{-1}((1, \infty)) = (-\infty, -1) \cup (1, \infty).$$

⊛

Now we build an equivalent definition of continuity. If f is continuous at x_0, then given any $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$f(x) \in B_Y(f(x_0), \varepsilon) \quad \text{whereas } x \in B_X(x_0, \delta).$$

Also, $f(x) \in B_Y(f(x_0), \varepsilon)$ if and only if

$$x \in f^{-1}(B_Y(f(x_0, \varepsilon), \varepsilon)).$$

Hence, we have

Corollary 3.0.1. f is continuous at x_0 if and only if

$$\text{Given any } \varepsilon > 0, \exists \delta > 0, \text{ we have } B_X(x_0, \delta) \subseteq f^{-1}(B_Y(f(x_0), \varepsilon))..$$

Remark 3.0.1. If $f : X \to Y$ is continuous and $K \subseteq X$, then $f|_K : K \to Y$ is continuous.

Proof. Given any point $x_0 \in K \subseteq X$. Since f is continuous at x_0, so $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$d\left(f(x), f(x_0)\right) < \varepsilon \quad \text{if } d_X(x, x_0) < \delta.$$

If $z \in K$ and $d_K(z, x_0) < \delta$, then $d_Y(f(z), f(x)) < \varepsilon$, so f is continuous on K. ⊛

Theorem 3.0.1. Suppose that (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ is a function and let $x_0 \in X$, then TFAE:

(a) f is continuous at x_0.

(b) Whenever $\left(x^{(n)}\right)_{n=1}^{\infty}$ is a sequence in X converges to x_0, then

$$\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0) \text{ in } (Y, d_Y).$$

(c) For every open set $V \subseteq Y$ that contains $f(x_0)$, $\exists$ a open set $U \subseteq X$ containing x_0 s.t. $f(U) \subseteq V$, or equivalently, $U \subseteq f^{-1}(V)$.

proof of (a) ⇒ (b). Given any $\varepsilon > 0$, since f is continuous at x_0, so $\exists\delta > 0$ s.t. if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$. Now if $\lim_{n\to\infty} x^{(n)} = x_0$. Then there exists $N > 0$ s.t. $n \geq N$ implies $d_X\left(x^{(n)}, x_0\right) < \delta$. Hence, we know $d_Y\left(f\left(x^{(n)}\right), f(x_0)\right) < \varepsilon$. Hence, for this ε, we know there exists N s.t. $n \geq N$ implies $d_Y\left(f\left(x^{(n)}\right), f(x_0)\right) < \varepsilon$, and thus $\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0)$. ■

proof of (b) ⇒ (c). Let $f(x_0) \in V \subseteq Y$.

Claim 3.0.1. There exists an open set u s.t. $x_0 \in u \subseteq X$ and $f(u) \subseteq V$.

Proof. If this is not true, then this implies that for every open set u with $x_0 \in u$, consider $B_X\left(x_0, \frac{1}{n}\right)$, $\exists x_u \in u$ and $f(x_u) \notin V$, then pick all of this x_u to be $\left\{x^{(n)}\right\}_{n=1}^{\infty}$, we know $\forall x^{(n)}$ we have $f\left(x^{(n)}\right) \notin V$. Then, $\left\{x^{(n)}\right\}_{n=1}^{\infty}$ is a sequence converges to x_0. By (b), we know $\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0)$. However, by our choice, $f\left(x^{(n)}\right) \notin V$, so $f\left(x^{(n)}\right) \in Y \setminus V$. Since V is open, so $Y \setminus V$ is closed. Hence, we must have $\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x_0) \in Y \setminus V$, which is a contradiction. ⊛

■

proof of (c) ⇒ (a). Suppose (c) holds, then we want to show f is continuous at x_0, which means for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. $B_X(x_0, \delta) \subseteq f^{-1}\left(B_Y(f(x_0), \varepsilon)\right)$. Now consider $V = B_Y(f(x_0), \varepsilon) \subseteq Y$, then by (c) we know there exists open $U \subseteq X$ s.t. $x_0 \in U \subseteq X$ s.t. $U \subseteq f^{-1}(V)$. Now since U is open and $x_0 \in U$, so there exists $B_X(x_0, \delta) \subseteq U$, and thus

$$B_X(x_0, \delta) \subseteq U \subseteq f^{-1}(V) = f^{-1}\left(B_Y(f(x_0), \varepsilon)\right),$$

and we're done. ■

Theorem 3.0.2. Suppose $f : X \to Y$, then TFAE

(a) f is continuous.

(b) If $\lim_{n\to\infty} x^{(n)} = x \in (X, d_X)$, then $\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x)$ in (Y, d_Y).

(c) If V is open in Y, then $f^{-1}(V)$ is open in X.

(d) Whenever F is closed in Y, then $f^{-1}(F)$ is closed in X.

(a) ⇔ (b). By Theorem 3.0.1, we know it is true. ■

(c) is equivalent to (c) in Theorem 3.0.1. For each $x \in f^{-1}(V)$, we have $f(x) \in V$, so there exists an open set u_x s.t. $x \in u_x \subseteq f^{-1}(V)$. Hence,

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} u_x.$$

Since u_x is open, so $f^{-1}(V)$ is open. ■

(c) ⇔ (d). If F is closed in Y, then $Y \setminus F$ is open in Y, and thus $F^{-1}(Y \setminus F)$ is open in X, and since $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is open, so $f^{-1}(F)$ is closed. ■

Remark 3.0.2. If f is continuous, then the image of an open set on f may not be open, and the image of a closed set on f may not be closed.

Example 3.0.2. Consider $f(x) = x^2$, then $f(-1, 1) = [0, 1)$ is not open.

Example 3.0.3. Consider $f(x) = \arctan(x)$, then $f([0, \infty)) = [0, \frac{\pi}{2}]$.

Now if $f : X \to Y$ and $g : X \to Z$, then consider

$$(f, g) : X \to Y \times Z \text{ with } x \mapsto (f(x), g(x)),$$

then $Y \times Z$ has a natural metric. Note that if (y_1, z_1) and (y_2, z_2) are in $Y \times Z$, then

$$d_{Y \times Z}\left((y_1, z_1), (y_2, z_2)\right) = d_Y(y_1, y_2) + d_Z(z_1, z_2).$$

Lemma 3.0.1. Consider $f : X \to Y$ and $g : X \to Z$ and $(f, g) : X \to Y \times Z$, then f and g are both continuous if and only if (f, g) is continuous.

Proof. Given $x \in X$, we know $\lim_{n\to\infty} x^{(n)} = x \in (X, d_X)$ implies

$$\lim_{n\to\infty} f\left(x^{(n)}\right) = f(x) \text{ and } \lim_{n\to\infty} g\left(x^{(n)}\right) = x.$$

Claim 3.0.2.

$$\lim_{n\to\infty} (f, g)(x^n) = (f, g)(x).$$

Proof. Check. ⊛

■

Appendix

Appendix A

Some Extra proof

Theorem A.0.1. For a Cauchy sequence $\{x^{(n)}\}_{n=1}^{\infty}$, if there exists a subsequence $\{x^{(n_j)}\}_{j=1}^{\infty}$ converges to x, then $\{x^{(n)}\}_{n=1}^{\infty}$ also converges to x.

Proof. For all $\varepsilon > 0$, we know there exists $N > 0$ s.t. $j \geq N$ implies

$$d\left(x^{(n_j)}, x\right) < \frac{\varepsilon}{2}.$$

Also, there exists $N' > 0$ s.t. $i, j \geq N'$ implies

$$d\left(x^{(i)}, x^{(j)}\right) < \frac{\varepsilon}{2}.$$

Hence, if we pick some $d \geq N$ and $n_d \geq N'$, then we know for all $n \geq N'$, we have

$$d\left(x^{(n)}, x\right) \leq d\left(x^{(n)}, x^{(n_d)}\right) + d\left(x^{(n_d)}, x\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means $\{x^{(n)}\}_{n=1}^{\infty}$ converges to x. ■

Definition A.0.1. A sequence of intervals I_n $(n \in \mathbb{N})$ is nested if $I_n \neq \varnothing$ and $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. $(I_1 \supseteq I_2 \supseteq \ldots)$.

Now we want to know $\bigcap_{n\in\mathbb{N}}^{\infty} I_n \neq \varnothing$?

Here is some counterexamples. Consider $I_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$. We can show that $\bigcap_{n=1}^{\infty} I_n = \varnothing$ by Archimedean Property. Besides, if $I_n = [n, \infty)$, $n \in \mathbb{N}$, this is trivial that $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

Theorem A.0.2 (Theorem of nested intervals)**.** If I_n $(n \in \mathbb{N})$ is a sequence of bounded closed nested intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

Proof. Write $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$. First, we know I_n is nested iff $a_n \leq b_n$ and a_n is nondecreasing and b_n is nonincreasing. Hence, $\forall n, m \in \mathbb{N}$, we have $a_n \leq a_{\max\{n,m\}} \leq b_{\max\{n,m\}} \leq b_m$. In other words, for every $m \in \mathbb{N}$, b_m is a upper bound of $\{a_n\}$. Hence, we know $c = \lim_{n\to\infty} a_n = \sup\{a_n\}$.exists. Then, $c \leq b_m$ for all $m \in \mathbb{N}$. Also, $c \geq a_n$ for all $n \in \mathbb{N}$. Hence, $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$, and thus we know $c \in I_n$ for all $n \in \mathbb{N}$. Thus, $c \in \bigcap_{n=1}^{\infty} I_n$. ■

Theorem A.0.3 (Bolzano Weierstrass Theorem)**.** Suppose we have a bounded infinite sequence $a_n \in \mathbb{R}^m$, then $\exists$ a subsequence $a_{n(m)}$ such that $a_{n(m)}$ is convergent.

Proof. We just talk about the case $m = 2$, and the higher case is similar. Choose $M > 0$ such that $a_n \in [-M, M] \times [-M, M]$ for all $n \in \mathbb{N}$. Suppose $[-M, M] \times [-M, M]$ is called Q. Divide Q into 4 squares with equal size, and choose one, say Q_1 such that $|\{n \mid a_n \in Q_1\} = \infty|$. Select $n_1 \in \mathbb{N}$ such that $a_{n_1} \in Q_1$. Repeat this step, that is, divide Q_1 into 4 subparts, then says the one subpart with infinite many a_n in it is Q_2 (Q_2 must exists). Select $n_2 \in \mathbb{N}$ such that $a_{n_2} \in Q_2$ and $n_2 > n_1$. Keep repeating this step, then by Theorem A.0.2 we know

$$\bigcap_{n=1}^{\infty} Q_n \neq \varnothing.$$

Note A.0.1. Just think of the nested intervals are in x and y directions.

Actually, $\bigcap_{n=1}^{\infty} Q_n = \{a\}$ for some $a \in \mathbb{R}^2$, otherwise if there are two points in the intersection, then at some moment we will divide them into different subpart, which is a contradiction. It can been seen that $\lim_{k\to\infty} a_{n(k)} = a$. ■

Appendix B

TA Class