Introduction to Analysis I HW3

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Problem 0.0.1. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set

$$S = \{x^{(n)} : n \geq m\}.$$

Is the converse true?

Problem 0.0.2. The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

(a) Given any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, we introduce the *formal limit*

$$\mathrm{LIM}_{n\to\infty} x_n.$$

We say that two formal limits $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Show that this equality relation obeys the reflexive, symmetry, and transitive axioms, i.e. that it is an equivalence relation.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, modulo the above equivalence relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to [0, \infty)$ by

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) := \lim_{n\to\infty} d(x_n, y_n).$$

Show that this function is well-defined (the limit exists and does not depend on the choice of representatives) and that it satisfies the axioms of a metric. Thus $(\overline{X}, d_{\overline{X}})$ is a metric space.

(c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.

(d) We identify an element $x \in X$ with the corresponding constant Cauchy sequence $(x, x, x, \dots)$, i.e. with the formal limit $\mathrm{LIM}_{n\to\infty} x$. Show that this is legitimate: for $x, y \in X$,

$$x = y \quad \iff \quad \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y.$$

With this identification, show that

$$d(x, y) = d_{\overline{X}}(x, y),$$

and thus (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) Show that the closure of X in $\overline{X}$ is $\overline{X}$ itself. (This explains the choice of notation.)

(f) Finally, show that the formal limit agrees with the actual limit: if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X that converges in X, then

$$\lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n \quad \text{in } \overline{X}.$$

a. We verify the following properties:

- Reflexive: $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} x_n$ are equal since d is metric, so $\forall n, d(x_n, x_n) = 0$.
- Symmetry: If $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal, this mean $\lim_{n\to\infty} d(x_n, y_n) = 0$. And since d is metric, so $\lim_{n\to\infty} d(y_n, x_n) = 0$, hence $\mathrm{LIM}_{n\to\infty} y_n$ and $\mathrm{LIM}_{n\to\infty} x_n$ are equal.
- Transitive: If $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal and $\mathrm{LIM}_{n\to\infty} y_n$ and $\mathrm{LIM}_{n\to\infty} z_n$ are equal, then we have $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, z_n) = 0$. By definition, there exists $N_1, N_2 > 0$ s.t. for all $n \geq N_1$, we have $d(x_n, y_n) < \frac{\varepsilon}{2}$ and for all $n \geq N_2$ we have $d(y_n, z_n) < \frac{\varepsilon}{2}$.

Thus, for all $n \geq \max\{N_1, N_2\}$, we have

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means $\lim_{n\to\infty} d(x_n, z_n) = 0$, and thus $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} z_n$.

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b. We first show that the limit exists. Note that $\lim_{n\to\infty} d(x_n, y_n) \in \mathbb{R}_{\geq 0}$ for all Cauchy sequence $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ in X. We already know $(\mathbb{R}, |\cdot|)$ is complete, so we know $(\mathbb{R}_{\geq 0}, |\cdot|)$ is also complete as it is a closed subspace of $(\mathbb{R}, |\cdot|)$. Now we define $u_n \coloneqq d(x_n, y_n)$ for all $n \geq 1$, then we want to show that $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}_{\geq 0}$, and then we can conclude that $\{u_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}_{\geq 0}$, and thus $\lim_{n\to\infty} d(x_n, y_n)$ exists.

Claim 0.0.1. Suppose (X, d) is a metric space, then for all $a, b, c, d \in X$ we have

$$|d(a, b) - d(c, d)| \leq d(a, c) + d(b, d)$$

Proof. Since

$$\begin{cases} d(a, b) \leq d(a, c) + d(c, b) \leq d(a, c) + d(c, d) + d(d, b) \\ d(c, d) \leq d(c, a) + d(a, d) \leq d(c, a) + d(a, b) + d(b, d), \end{cases}$$

so we have

$$\begin{cases} d(a, b) - d(c, d) \geq d(a, c) + d(d, b) \\ -d(c, a) - d(b, d) \leq d(a, b) - d(c, d), \end{cases}$$

so we can conbine these two equations and get the result. ⊛

By Claim 0.0.1, we know for all $p, q \geq 1$, we have

$$|u_p - u_q| = |d(x_p, y_p) - d(x_q, y_q)| \leq d(x_p, x_q) + d(y_p, y_q).$$

Now since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy, so for every $\varepsilon > 0$, there exists $N_1, N_2 > 0$ s.t.

$$\begin{cases} d(x_p, x_q) < \dfrac{\varepsilon}{2} & \forall p, q \geq N_1 \\ d(y_p, y_q) < \dfrac{\varepsilon}{2} & \forall p, q \geq N_2. \end{cases}$$

Thus, for all $p, q \geq \max\{N_1, N_2\}$, we know

$$|u_p - u_q| \leq d(x_p, x_q) + d(y_p, y_q) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we know $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}_{\geq 0}, |\cdot|$.

Now we show that $d_{\overline{X}}$ is well-defined. In other words, if $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} z_n$, then we want to show

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) = d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} z_n, \mathrm{LIM}_{n\to\infty} y_n) \quad \forall \text{ Cauchy } \{y_n\}_{n=1}^{\infty} \text{ in } (X, d).$$

Equivalently, we want to show $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(z_n, y_n)$. Note that we have

$$\lim_{n\to\infty} d(x_n, z_n) = 0 \text{ and } d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n),$$

so we know

$$\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(z_n, y_n).$$

Also, we have $d(z_n, y_n) \leq d(z_n, x_n) + d(x_n, y_n)$, so we know

$$\lim_{n\to\infty} d(z_n, y_n) \leq \lim_{n\to\infty} d(z_n, x_n) + \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n, y_n),$$

and thus we can conclude that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(z_n, y_n)$.

Finally, we want to show that $(\overline{X}, d_{\overline{X}})$ is a metric space.

- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty} \in X$, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} x_n) = \lim_{n\to\infty} d(x_n, x_n) = 0$.
- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \in X$,

$$\begin{aligned} d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) = \lim_{n\to\infty} d(x_n, y_n) &= \lim_{n\to\infty} d(y_n, x_n) \\ &= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} y_n, \mathrm{LIM}_{n\to\infty} x_n) \end{aligned}$$

- $\forall$ Cauchy $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \in X$,

$$\begin{aligned} d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} z_n) &= \lim_{n\to\infty} d(x_n, z_n) \\ &\le \lim_{n\to\infty} (d(x_n, y_n) + d(y_n, z_n)) = \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\ &= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) + d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} y_n, \mathrm{LIM}_{n\to\infty} z_n). \end{aligned}$$

Hence, we know $(\overline{X}, d_{\overline{X}})$ is a metric space. ■

c. ■

Problem 0.0.3. In the following, all the sets are subsets of a metric space (X, d).

(a) If $\overline{A} \cap \overline{B} = \varnothing$, then

$$\partial(A \cup B) = \partial A \cup \partial B.$$

(b) For a finite family $\{A_i\}_{i=1}^{n} \subseteq X$, show that

$$\mathrm{int}\Big(\bigcap_{i=1}^{n} A_i\Big) = \bigcap_{i=1}^{n} \mathrm{int}(A_i).$$

(c) For an arbitrary (possibly infinite) family $\{A_\alpha\}_{\alpha\in F} \subseteq X$, prove that

$$\mathrm{int}\Big(\bigcap_{\alpha\in F} A_\alpha\Big) \subseteq \bigcap_{\alpha\in F} \mathrm{int}(A_\alpha).$$

(d) Give an example where the inclusion in part (c) is strict (i.e., equality fails).

(e) For any family $\{A_\alpha\}_{\alpha\in F} \subseteq M$, prove that

$$\bigcup_{\alpha\in F} \mathrm{int}(A_\alpha) \subseteq \mathrm{int}\Big(\bigcup_{\alpha\in F} A_\alpha\Big).$$

(f) Give an example of a finite collection F in which equality does not hold in part (e).

Problem 0.0.4. Let (X, d) be a metric space and $Y \subset X$ be an open subset. For any subset $A \subset Y$, show that A is open in Y if and only if it is open in X.

Problem 0.0.5. On the space $(0, 1]$, we may consider the topology induced by the metric space $(\mathbb{R}, d)$ defined by $d(x, y) = |x - y|$. Alternatively, we may also define a distance d' on $(0, 1]$, given by

$$d'(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \qquad \forall x, y \in (0, 1].$$

(a) Show that d' is a metric on $(0, 1]$

(b) Let $x \in (0,1]$ and $\varepsilon > 0$. Let $B = B_d(x,\varepsilon) = \{y||y-x| < \varepsilon\} \cap (0,1]$ be the open ball centered at x of radius ε for the metric d in $(0,1]$. Show that for any $y \in B$, we may find $\varepsilon' > 0$ such that

$$B_{d'}(y,\varepsilon') \subseteq B = B_d(x,\varepsilon).$$

(c) Show that an open ball in $((0,1], d')$ is also an open ball in $((0,1], d)$.

(d) Conclude that the metric spaces $((0,1], d)$ and $((0,1], d')$ are topologically equivalent, that is, a set A is open in one space if and only if it is also open in the other one.

(e) Is $((0,1], d')$ a complete metric space? How about $((0,1], d)$?

Problem 0.0.6. (a) We say that a family of closed balls

$$\left(\overline{B}(x_n, r_n)\right)_{n\geq 1}$$

is a *decreasing sequence of closed balls* if the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \ \subseteq \ \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Give an example of a decreasing sequence of closed balls in a complete metric space with empty intersection.

(b) We say that a family of closed balls

$$\left(\overline{B}(x_n, r_n)\right)_{n\geq 1}$$

is a *decreasing sequence of closed balls with radii tending to zero* if

$$r_n \ \to \ 0 \quad \text{as } n \to \infty,$$

and the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \ \subseteq \ \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Show that a metric space (M, d) is complete if and only if every decreasing sequence of closed balls with radii going to zero has a nonempty intersection.