Abstract algebra I

Homework 3

Due: 1st October 2025

1) For the following pairs of groups, determine whether they are isomorphic. If so, construct an isomorphism, otherwise, explain why they are not.

   (a) $(\mathbb{Z}/15\mathbb{Z})^\times$ and $\mathbb{Z}/8\mathbb{Z}$.

   (b) $\mathbb{Z}/4\mathbb{Z}$ and $\{z \in \mathbb{C} \setminus \{0\} : z^4 = 1\}$.

   (c) $\mathbb{Z}$ and $3\mathbb{Z} = \{3n : n \in \mathbb{Z}\}$.

   (d) $\mathbb{Z}$ and $\{z \in \mathbb{C} \setminus \{0\} : z^n = 1 \text{ for some } n \geq 1\}$. *(The second group is the group of all roots of unity. You may want to verify that this collection is indeed a subgroup of $\mathbb{C} \setminus \{0\}$.)*

   (e) S_3 and $D_3 = \langle r, s : r^3 = s^2 = 1, srs = r^{-1} \rangle$. *(The second group is described as follows: it is generated by two elements r, s, and they satisfy the given relations. You may easily check that there are a total of six distinct elements. Such a description is called a group presentation.)*

   (f) S_4 and $D_4 = \langle r, s : r^4 = s^2 = 1, srs = r^{-1} \rangle$.

   (g) $Q = \langle i, j, k : i^2 = j^2 = k^2 = ijk = -1 \rangle$ and $T = \langle a, b : a^4 = 1, b^2 = a^2, ba = a^3 b \rangle$.

   (h) An infinite cyclic group G and one of its non-trivial proper subgroup H, i.e., $H \neq \{e\}$.

2) Define the *additive* group

$$G = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

   Show that the map $\varphi : G \to \mathbb{C}$ given by $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto a + bi$ is a group isomorphism. You don't have to check for well-definedness.

3) Let H be a normal subgroup of a group G, and N a subgroup of H.

   (a) If H is cyclic, prove that N is a normal subgroup of G.

   (b) If N is a normal subgroup of H, show that N is not necessarily a normal subgroup of G. In other words, give a counterexample.

4) Let $f : G \to H$ be a group homomorphism with H abelian and let N be a subgroup of G containing $\ker f$. Prove that N is normal in G.