

Linear Algebra I HW12

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Section 7.1

Problem. 1. Let T be a linear operator on F^2 . Prove that any non-zero vector which is not a characteristic vector for T is a cyclic vector for T . Hence, prove that either T has a cyclic vector or T is a scalar multiple of the identity operator.

Proof. Suppose $v \in F^2$ and $v \neq 0$ and $Tv \notin \text{span}\{v\}$, then $\{v, Tv\}$ is linearly independent and thus a basis of F^2 , which means v is a cyclic vector for T . Now if such v does not exist, then $Tv \in \text{span}\{v\}$ for all $v \in F^2$. Then if $\{p, q\}$ form a basis of F^2 , then $Tp = \lambda p$ and $Tq = \lambda' q$ for some $\lambda, \lambda' \in F$, and we claim that $\lambda = \lambda'$. If $\lambda \neq \lambda'$, then

$$T(p+q) = T(p) + T(q) = \lambda p + \lambda' q \notin \text{span}\{p+q\},$$

which is impossible. Hence, $\lambda = \lambda'$, and thus for all $v \in V$, we know $v = \alpha p + \beta q$ for some $\alpha, \beta \in F$, and thus

$$Tv = \alpha Tp + \beta Tq = \alpha \lambda p + \beta \lambda q = \lambda(\alpha p + \beta q) = \lambda v,$$

which shows T is a scalar multiple of the identity operator. ■

Problem. 7. Let V be an n -dimensional vector space, and let T be a linear operator on V . Suppose that T is diagonalizable.

- (a) If T has a cyclic vector, show that T has n distinct characteristic values.
- (b) If T has n distinct characteristic values, and if $\{\alpha_1, \dots, \alpha_n\}$ is a basis of characteristic vectors for T , show that $\alpha = \alpha_1 + \dots + \alpha_n$ is a cyclic vector for T .

Proof.

- (a) If T has a cyclic vector, then $m_T(x) = \text{ch}_T(x)$, and thus

$$\deg m_T(x) = \deg \text{ch}_T(x) = n,$$

and since T is diagonalizable, so we know $m_T(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_n)$ for some $\lambda_i \in F$. Thus,

$$\text{ch}_T(x) = m_T(x) = (x - \lambda_1) \dots (x - \lambda_n),$$

which means T has n distinct characteristic values.

- (b) Suppose $T\alpha_i = \lambda_i \alpha_i$ for all $i = 1, 2, \dots, n$. Then we know

$$T^i \alpha = T^i \left(\sum_{j=1}^n \alpha_j \right) = \sum_{j=1}^n T^i(\alpha_j) = \sum_{j=1}^n \lambda_j^i \alpha_j.$$

Now we want to show $\{\alpha, T\alpha, \dots, T^{n-1}\alpha\}$ is a basis of V so that we know α is a cyclic vector for T . Note that

$$\begin{aligned} \alpha &= \alpha_1 + \alpha_2 + \dots + \alpha_n \\ T\alpha &= \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \dots + \lambda_n \alpha_n \\ T^2 \alpha &= \lambda_1^2 \alpha_1 + \lambda_2^2 \alpha_2 + \dots + \lambda_n^2 \alpha_n \\ &\vdots \\ T^{n-1} \alpha &= \lambda_1^{n-1} \alpha_1 + \lambda_2^{n-1} \alpha_2 + \dots + \lambda_n^{n-1} \alpha_n, \end{aligned}$$

so we have

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}}_{A \in M_n(F)} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}_{X \in M_n(F)} = \underbrace{\begin{pmatrix} \alpha \\ T\alpha \\ \vdots \\ T^{n-1}\alpha \end{pmatrix}}_{Y \in M_n(F)},$$

and note that $\det(A) \neq 0$ since $\lambda_i \neq \lambda_j$ for all $i \neq j$ and A is the Vandermonde matrix. Hence, A is invertible, and thus

$$\text{rank } Y = \text{rank } AX = \text{rank } X = n$$

since the n rows of X are linearly independent, and thus the n rows of Y are also linearly independent, which shows $\{\alpha, T\alpha, \dots, T^{n-1}\alpha\}$ is a basis of V , and we're done.

■

Section 7.3

Problem. 6. Let A be the complex matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Find the Jordan form for A .

Problem. 10. Let n be a positive integer, $n \geq 2$, and let N be an $n \times n$ matrix over the field F such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that N has no square root, i.e., that there is no $n \times n$ matrix A such that $A^2 = N$.

Problem. 13. If N is a $k \times k$ elementary nilpotent matrix, i.e., $N^k = 0$ but $N^{k-1} \neq 0$, show that N^t is similar to N . Now use the Jordan form to prove that every complex $n \times n$ matrix is similar to its transpose.