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• فرایند میانگین متحرک : $x_t = z_t + \theta_1 z_{t-1} + \cdots + \theta_q z_{t-q}$

این قبیل سری‌ها را میانگین متحرک مرتبه q نامیده و به صورت مختصر $MA(q)$ گویند.

$$MA(1) \Rightarrow x_t = z_t + \theta_1 z_{t-1}$$

• $\operatorname{cov}(x_t, x_{t-1}) = E[(x_t-\mu)(x_{t-1}-\mu)] = E[x_t x_{t-1}]$

$$= E[(z_t + \theta_1 z_{t-1})(z_{t-1} + \theta_1 z_{t-2})]$$

$$= E[z_t z_{t-1} + \theta_1 z_t z_{t-2} + \theta_1 z_{t-1}^2 + \theta_1^2 z_{t-1} z_{t-2}]$$

$$= E[z_t z_{t-1}] + \theta_1 E[z_t z_{t-2}] + \theta_1 E[z_{t-1}^2] + \theta_1 E[z_{t-1} z_{t-2}]$$

$$z_i \overset{iid}{\sim} N(0,\sigma^2) \Longrightarrow \theta_1 E[z_{t-1}^2] = \boxed{\theta_1 \sigma^2 = \operatorname{cov}(x_t, x_{t-1}) = \gamma_1}$$

بنابراین به طور خلاصه برای سری $MA(1)$ خواهیم داشت:

$$E[x_t] = 0$$

$$\operatorname{var}(x_t) = E[x_t^2] - E^2[x_t] = E[x_t^2] = E[(z_t + \theta_1 z_{t-1})^2]$$

$$E[z_t^2 + \theta_1^2 z_{t-1}^2 + 2 z_t \theta_1 z_{t-1}]$$

$$= E[z_t^2] + \theta_1^2 E[z_{t-1}^2] + 2\theta_1 E[z_t z_{t-1}]$$

$$= \sigma^2 + \theta_1^2 \sigma^2 = \boxed{\sigma^2(1+\theta_1^2) = \operatorname{var}(x_t)}$$

$$\gamma_1 = +\theta\sigma^2$$

$$\rho_1 = \frac{\theta\sigma^2}{\sigma^2(1+\theta_1^2)} = \frac{\theta}{1+\theta_1^2} \quad , \quad \boxed{\rho_1 = \frac{\theta}{1+\theta_1^2}}$$

Θz_{t-2} $\Theta_2 z_{t-1}$

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$x_t = z_t + \Theta_1 z_{t-1} + \Theta_2 z_{t-2}$ فرایند میانگین متحرک مرتبه دوم:

$\gamma_0 = Var(x_t) = E[x_t^2] - 0 = E[x_t^2] =$

$E[z_t^2] + \Theta_1 E[z_t z_{t-1}] + \Theta_2 E[z_t z_{t-2}] +$

$\Theta_1 E[z_t z_{t-1}] + \Theta_1^2 E[z_{t-1}^2] + \Theta_1\Theta_2 E[z_{t-1} z_{t-2}] +$

$\Theta_2 E[z_t z_{t-2}] + \Theta_2\Theta_1 E[z_{t-1} z_{t-2}] + \Theta_2^2 E[z_{t-2}^2]$

$= \sigma^2 + \Theta_1^2\sigma^2 + \Theta_2^2\sigma^2 = \boxed{\sigma^2(1+\Theta_1^2+\Theta_2^2) = \gamma_0}$

$\gamma_1 = Cov(x_t, x_{t-1}) = E[x_t x_{t-1}] =$

$E[z_t z_{t-1} + \Theta_1 z_{t-1}^2 + \Theta_2 z_{t-2} z_{t-1} +$

$\Theta_1 z_t z_{t-2} + \Theta_1^2 z_{t-1} z_{t-2} + \Theta_1\Theta_2 \quad +$

$\Theta_2 z_t z_{t-3} + \Theta_2\Theta_1 z_{t-1} z_{t-3} + \Theta_2^2 z_{t-2} z_{t-3}]$

$= E[\Theta_1 z_{t-1}^2 + \Theta_1\Theta_2 z_{t-2}^2] = \Theta_1 E[z_{t-1}^2] + \Theta_1\Theta_2 E[z_{t-2}^2]$

$= \Theta_1\sigma^2 + \Theta_1\Theta_2\sigma^2 = \boxed{\Theta_1(1+\Theta_2)\sigma^2 = \gamma_1}$

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بنابراین به طور خلاصه برای فرایند میانگین متحرک مرتبه دوم داریم:

$$\gamma_0 = (1+\theta_1^2+\theta_2^2)\sigma^2 \qquad \rho_0 = \frac{1}{1+\theta_1^2+\theta_2^2}$$

$$\gamma_1 = \theta_1(\theta_2+1)\sigma^2 \qquad \rho_1 = \frac{\theta_1(\theta_2+1)}{1+\theta_1^2+\theta_2^2}$$

$$\gamma_2 = \theta_2\sigma^2 \qquad \rho_2 = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$\gamma_2 = \operatorname{cov}(x_t, x_{t-2}) = E\left[(z_t+\theta_1 z_{t-1}+\theta_2 z_{t-2})(z_{t-2}+\theta_1 z_{t-3}+\theta_2 z_{t-4})\right]$$

$$= E\left[z_t z_{t-2}+\theta_1 z_t z_{t-3}+\theta_2 z_t z_{t-4}+\theta_1 z_{t-1} z_{t-2}+\theta_1^2 z_{t-1} z_{t-3}+\theta_1 z_{t-1} z_{t-4}+\theta_2 z_{t-2}^2+\theta_2\theta_1 z_{t-2} z_{t-3}+\theta_2^2 z_{t-2} z_{t-4}\right]$$

$$= E\left[\theta_2 z_{t-2}^2\right] = \theta_2 E\left[z_{t-1}^2\right] = \boxed{\theta_2\sigma^2 = \gamma_2}$$

به طور کلی:

$$\rho_k = \begin{cases} \dfrac{+\theta_k+\theta_1\theta_{k+1}+\cdots+\theta_{q-k}\theta_q}{1+\theta_1^2+\cdots+\theta_q^2} & 1 \le k \le q \\ 0 & k > q \end{cases}$$

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$$x_t = \phi_1 x_{t-1} + \text{—} + \phi_p x_{t-p} + z_t$$ • فرایندهای اتورگرسیو:

$$x_t = \phi_1 x_{t-1} + z_t$$ • فرایند اتورگرسیو مرتبه اول:

• $\gamma_0 = var(x_t) = var(\phi_1 x_{t-1} + z_t) = \phi_1^2 \gamma_0 + \sigma^2$

$$\Rightarrow \gamma_0 - \phi_1^2 = \sigma^2 \Rightarrow \gamma_0 = \frac{\sigma^2}{1-\phi_1^2}$$

$$\gamma_1 = cov(x_t, x_{t-1}) = E[(x_t - \mu)(x_{t-1} - \mu)]$$

• important: if we assume stationary process; this means

$E[x_t] = \phi_1 E[x_{t-1}] + E[z_t] = \mu = \phi_1 \mu + 0 \Rightarrow$

since $|\phi_1| < 1$ (because of stationary) $\Rightarrow \mu = E[x_t] = 0$

• important: why is $|\phi_1| < 1$:

$$x_t = \phi_1 x_{t-1} + z_t = \phi_1(\phi_1 x_{t-2} + z_{t-1}) + z_t$$

$$= \phi_1^2 x_{t-2} + \phi_1 z_{t-1} +$$

$$= \phi_1^3 x_{t-3} + \phi_1^2 z_{t-2} + \phi_1 z_{t-1} + z_t$$

$$= \phi_1^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j z_{t-j} \overset{k\to\infty}{\approx} \phi_1^k x_{t-k}$$

for being statitionary $\lim_{k\to\infty} \phi_1^k x_{t-k} = 0 \Rightarrow \begin{cases} |\phi_1| < 1 \\ |\phi_1| = 1 \\ |\phi_1| > 1 \end{cases}$

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- if $\phi_1 = 1$: $x_t = \phi_1 x_{t-1} + z_t = x_{t-1} + z_t$

This is a random walk, variance increases forever

Therefore, it is not stationary

- if $\phi_1 = -1$: $x_t = -x_{t-1} + z_t$

This means the process oscilates forever, mean exists

but variance does not converge $\Rightarrow$ not stationary

- if $|\phi_1| > 1$: Then $\phi_1^k x_{t-k}$ explodes
- if $|\phi_1| < 1$: Then the process is stationary $\Rightarrow E[x_t] = 0$

$\gamma_1 = E[x_t x_{t-1}] = \quad x_{t-1}^2 + z_t x_{t-1}] =$

$\phi_1 E[x_{t-1}^2] + E[x_{t-1} z_t] = \phi_1 \dfrac{\sigma^2}{1-\phi_1^2} = \boxed{\gamma_1 = \dfrac{\phi_1}{1-\phi_1^2}\sigma^2}$

$E[x_{t-1} z_t]$ $\boxed{\text{$\varepsilon_t$ is independent from past values}}$ $\boxed{\text{$x_{t-1}$ depends only on ε_{t-k}}}$ $\boxed{\text{$\varepsilon_t$ is independent from ε_{t-k}}}$

$\Rightarrow E[x_{t-1} z_t] = 0$

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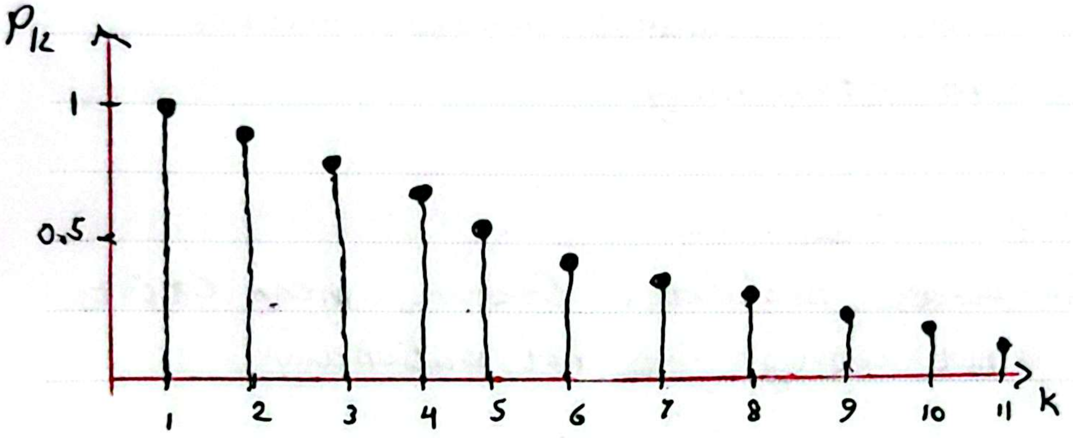
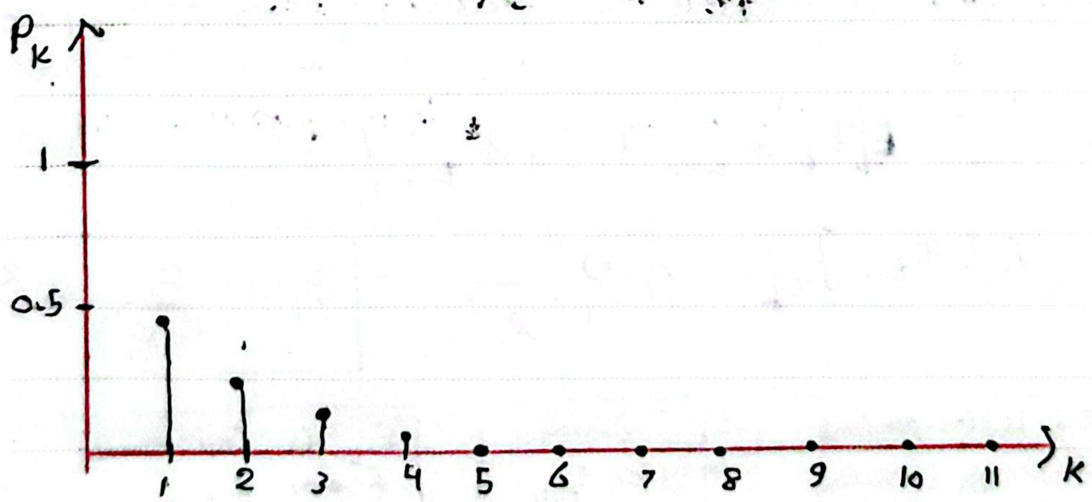
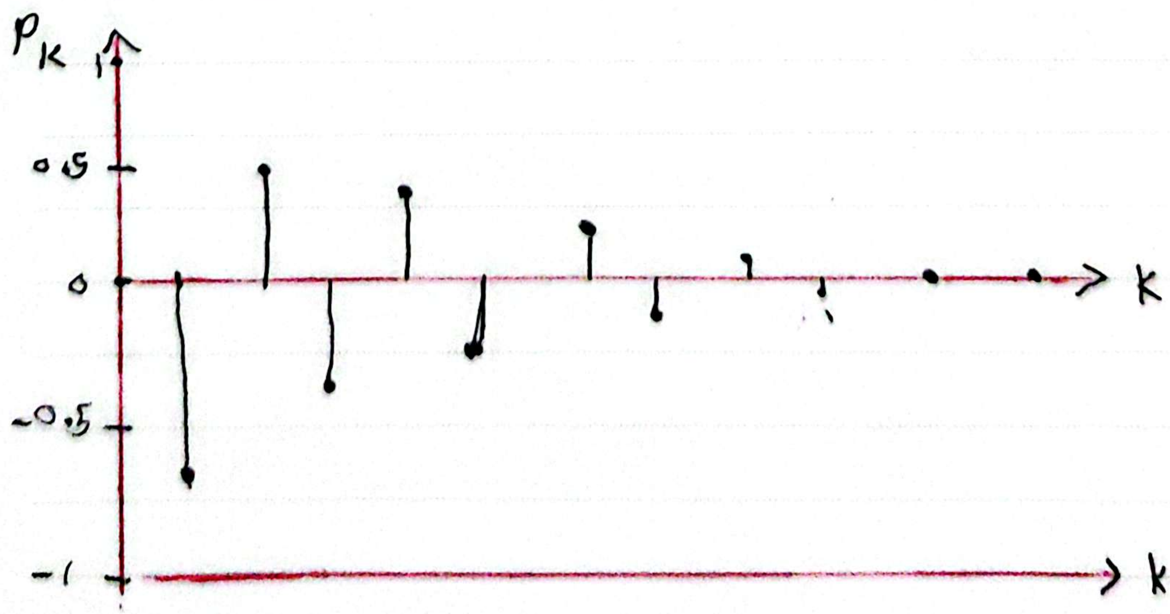
• توابع خودهمبستگی AR(1)



$\phi_1 = 0.9$



$\phi_1 = 0.4$



$\phi_1 = -0.7$

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$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + z_t$$ • اتورگرسیو مرتبه دوم:

$$\gamma_0 = var(x_t) = E[x_t^2] =$$

$$E[\phi_1 x_t x_{t-1} + \phi_2 x_t x_{t-2} + x_t z_t] =$$

$$\gamma_0 = \phi_1 E[x_t x_{t-1}] + \phi_2 E[x_t x_{t-2}] + E[x_t z_t]$$

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + E[\phi_1 x_{t-1} z_t + \phi_2 x_{t-2} z_t + z_t^2]$$

= note that z_t is independent from all past values x_{t-k}, becaus x_{t-k} is only dependendent on z_{t-k}

= and z_t iid $WN(0, \sigma^2)$, hence z_t is independent of z_{t-k}

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + E[z_t^2] = \boxed{\phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2 = \gamma_0}$$

$$\gamma_1 = E[x_t x_{t-1}] = \phi_1 E[x_{t-1}^2] + \phi_2 E[x_{t-2} x_{t-1}] + E[z_t x_{t-1}]$$

$$\Rightarrow \boxed{\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1 + 0} \Longrightarrow \boxed{\gamma_1 = \frac{\phi_1}{1-\phi_2} \gamma_0}$$

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$$\gamma_2 = E[(\phi_1 x_{t-1} + \phi_2 x_{t-2} + z_t)(x_{t-2})]$$

$$\gamma_2 = \phi_1 E[x_{t-1} x_{t-2}] + \phi_2 E[x_{t-2}^2] + E[z_t x_{t-2}]$$

$$\boxed{\gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0}$$

$$\Rightarrow \gamma_0 = \gamma_0 \left(\frac{\phi_1^2}{1-\phi_2} + \frac{\phi_2 \phi_1^2}{1-\phi_2} + \phi_2^2 \right) + \sigma^2$$

$$\rightarrow \boxed{\gamma_0 = \frac{\sigma^2 (1-\phi_2)}{(1+\phi_2)((1-\phi_2^2)-\phi_1^2)}}$$

$$\boxed{\gamma_1 = \frac{\phi_1}{1-\phi_2} \gamma_0} \quad , \quad \boxed{\gamma_2 = \left(\frac{\phi_1^2}{1-\phi_2} + \phi_2 \right) \gamma_0}$$

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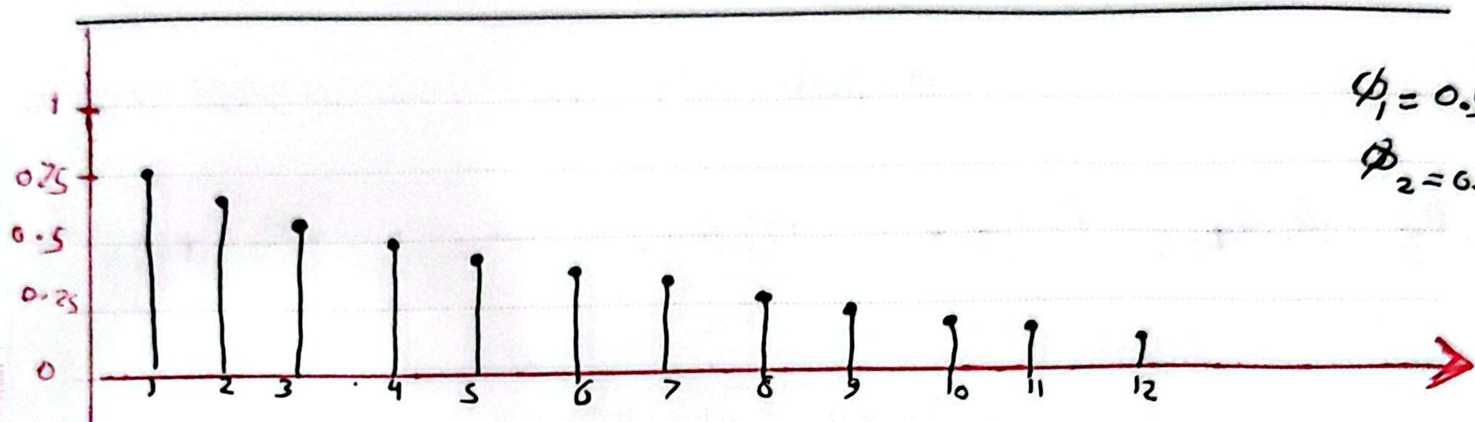
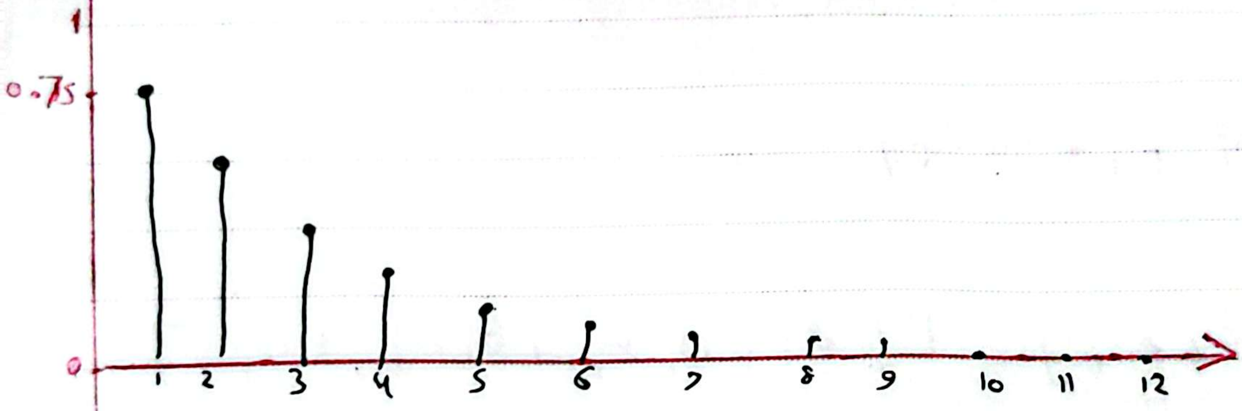
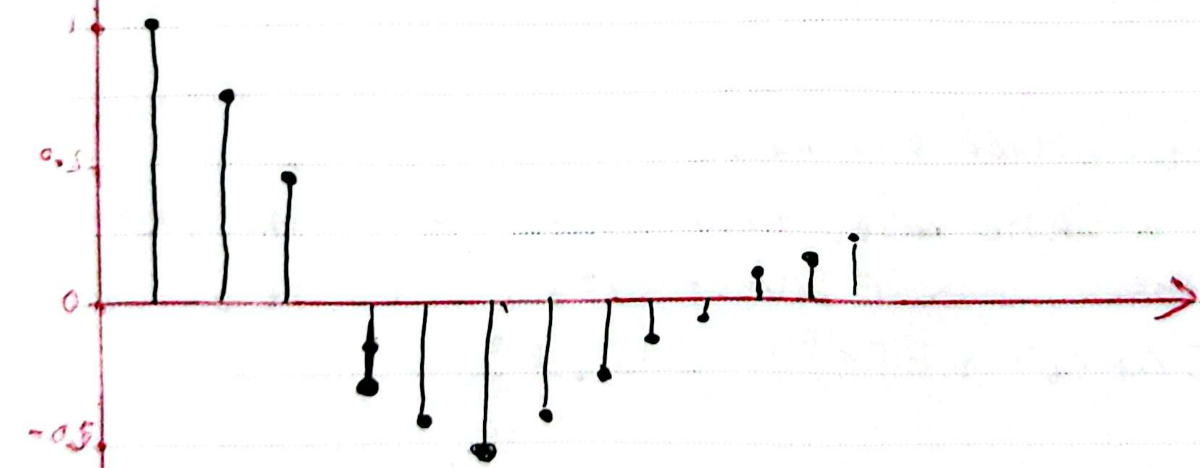


$\phi_1 = 0.5$
$\phi_2 = 0.25$

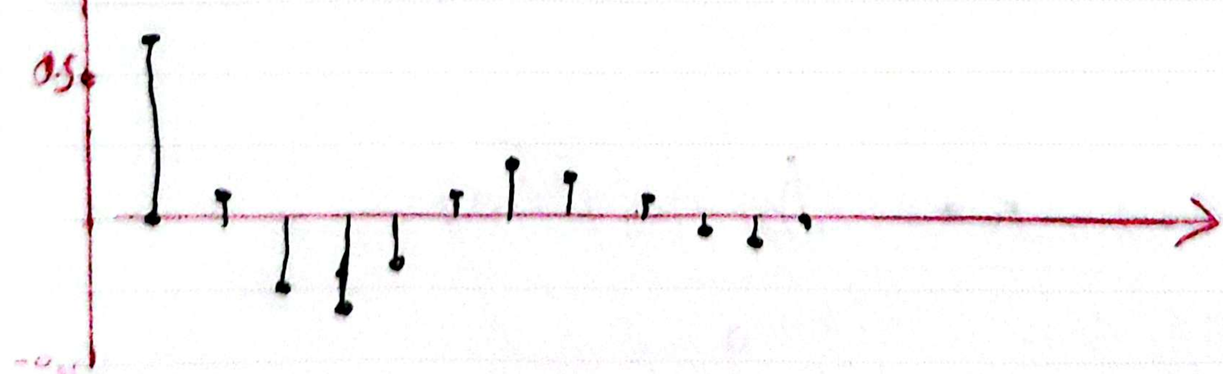


$\phi_1 = 1$
$\phi_2 = -0.25$



$\phi_1 = 1.5$
$\phi_2 = -0.75$

$\phi_1 -1$
$\phi_2 = -0.6$



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• فرایند اتورگرسیو میانگین متحرک $ARMA(p,q)$

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \theta_1 z_{t-1} + \cdots + \theta_q z_{t-q} + z_t$$

• اتورگرسیو میانگین متحرک مرتبه اول : $ARMA(1,1)$:

$$x_t = \phi_1 x_{t-1} + \theta_1 z_{t-1} + z_t$$

$$\Rightarrow E[x_t z_t] = E[z_t(\phi_1 x_{t-1} + \theta_1 z_{t-1} + z_t)]$$

$$E[\phi_1 x_{t-1} z_t + \theta_1 x_{t-1} z_{t-1} + z_t^2]$$

note: $x_t = ((\text{stuff from past})) + z_t = A_t + z_t$

z_t is white noise, uncorrelated to all the past values, $\Longrightarrow E[(A_t + z_t) z_t] = E[A_t z_t + z_t]$

$$= E[A_t z_t] + E[z_t^2] = E[z_t^2]$$

$$\Rightarrow E[x_t z_t] = E[z_t^2] = var(z_t) = \boxed{\sigma^2 = E[x_t z_t]}$$

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- $E[x_t z_{t-1}] = E[(A_t + z_t)(z_{t-1})] = E[A_t z_{t-1} + z_t z_{t-1}] =$

$E[A_t z_{t-1}] + E[z_t z_{t-1}] =$

$E[(\phi_1 x_{t-1} + \theta_1 z_{t-1}) z_{t-1}] + 0 = E[\phi_1 x_{t-1} + \theta_1 z_{t-1}^2] + 0$

$= E[\phi_1 x_{t-1} z_{t-1}] + \theta E[z_{t-1}^2] = \phi_1 \sigma^2 + \theta_1 \sigma^2$

$$\boxed{E[x_t z_{t-1}] = \sigma^2(\phi_1 + \theta_1)}$$

not: $var(A+B+C) = var(A) + var(B) + var(C)$
$+ 2cov(A,B) + 2cov(B,C) + 2cov(A,C)$

- $\gamma_0 = var(x_t) = var(\phi_1 x_{t-1} + \theta_1 z_{t-1} + z_t) =$

$\phi_1^2 var(x_{t-1}) + \theta_1^2 var(z_{t-1}) + var(z_t) + \underbrace{2cov(\phi_1 x_{t-1}, z_t)}_{0}$

$+ 2cov(\theta_1 z_{t-1}, z_t) + 2cov(\phi_1 x_{t-1}, \theta_1 z_{t-1})$

$\phi_1^2 \sigma^2 + \theta_1^2 \sigma^2 + \sigma^2 + 2 \times 0 + 2 \times 0 + 2\theta_1 \phi_1 \sigma^2$

$\Rightarrow \gamma_0 = \phi_1^2 \gamma_0 + (\theta_1^2 + 1 + 2\theta_1 \phi_1) \sigma^2$

$$\Rightarrow \boxed{\gamma_0 = \frac{\theta_1^2 + 2\theta_1 \phi_1 + 1}{1 - \phi_1^2} \sigma^2}$$

note: $cov(x_{t-1}, z_t) = E[x_{t-1} z_t] - \cancel{E[x_{t-1}] E[z_t]} = E[x_{t-1} z_t]$

$E[x_{t-1} z_t] = \underbrace{\phi_1 E[x_{t-2} z_t]}_{0} + \underbrace{\cancel{E[z_{t-1} z_t]}}_{0} + \underbrace{\theta_1 E[z_{t-2} z_t]}_{0}$

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$$\boxed{E[x_{t-1} z_t] = 0}$$

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• بنابراین برای (۱،۱) داریم:

$$\boxed{\gamma_0 = \frac{1+2\theta_1\phi_1+\theta_1^2}{1-\phi_1^2}\sigma^2}$$

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$$\gamma_1 = cov(x_t, x_{t-1}) = E[x_t\, x_{t-1}] =$$

$$E[(\phi_1 x_{t-1} + z_t + \theta_1 z_{t-1})(x_{t-1})] =$$

$$E[\phi_1 x_{t-1}^2] + \theta_1 E[x_{t-1} z_{t-1}] + E[x_{t-1} z_t] =$$

$$= \phi_1\, var(x_{t-1}) + \theta_1\sigma^2 + 0 = \phi_1\gamma_0 + \theta_1\sigma^2$$

$$\gamma_1 = \frac{\phi_1 + 2\theta_1\phi_1^2 + \theta_1^2\phi_1}{1-\phi_1^2}\sigma^2 + \theta_1\sigma^2$$

$$\gamma_1 = \left(\frac{\phi_1 + 2\theta_1\phi_1^2 + \theta_1^2\phi_1 + \theta - \theta_1\phi_1^2}{1-\phi_1^2}\right)\sigma^2$$

$$\boxed{\gamma_1 = \frac{\phi_1 + \theta_1\phi_1^2 + \theta_1^2\phi_1 + \theta}{1-\phi_1^2}\sigma^2}$$

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$$\gamma_2 = cov(x_t, x_{t-2}) = E[x_t x_{t-2}] =$$

$$E[(\phi_1 x_{t-1} + \theta_1 z_{t-1} + z_t) x_{t-2}] =$$

$$\phi_1 E[x_{t-1} x_{t-2}] + \theta_1 E[z_{t-1} x_{t-2}] + E[z_t x_{t-2}]$$

$$= \phi_1 \gamma_1 + 0 + 0 = \phi_1 \gamma_1$$

$\Longrightarrow$

برای ARMA(1,1) داریم:

$$\boxed{\gamma_0 = \frac{1 + 2\theta_1 \phi_1 + \theta_1^2}{1 - \phi_1^2} \sigma^2} \Longrightarrow$$

$$\boxed{\gamma_1 = \frac{\phi_1 + \theta_1 \phi_1^2 + \theta_1^2 \phi_1 + \theta}{1 - \phi_1^2} \sigma^2}$$

$$\boxed{\gamma_2 = \phi_1 \gamma_1}$$

$$\gamma_k = \phi_1 \gamma_{k-1} \qquad \forall k \geq 2, k \in \mathbb{N}$$

$$\Longrightarrow \boxed{\rho_0 = 1, \quad \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{(\phi_1 + \theta_1)(1 + \phi_1 \theta_1)}{1 + 2\theta_1 \phi_1 + \theta_1^2}},$$

$$\boxed{\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\phi_1 (\phi_1 + \theta_1)(1 + \phi_1 \theta_1)}{1 + 2\theta_1 \phi_1 + \theta_1^2}}$$

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$z_t \overset{iid}{\sim} WN(0,\sigma^2)$ $E[z_t]=0$, $cov(z_i,z_j)\neq 0$

- MA(q) behaviour in ACF:
  - $x_t = z_t + \theta_1 z_{t-1} + \cdots + \theta_q z_{t-q}$

  - each x_t is a finite linear combination of white-noise terms $z_t, z_{t-1}, \ldots, z_{t-q}$. When you compute $cov(x_t, x_{t+k})$, for $k>q$, there is no common z-index in the sums. So, every expectation vanishes. $E[z_i z_j]=0, \forall i\neq j$

    hence:
$$\gamma_k = cov(x, x_{t-k}) = 0, \quad k>q$$

- AR(p) general behaviour in ACF:
  - $x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + z_t$

  - The ACF of a non-trivial AR(p) does not become exactly zero at a finite lag. it decays, typicaly exponentialy, or as damped sinusoid

- for AR(1): $\rho(k) = \phi^k$
  - if $\phi>0$: ACF is always positive monotone decrease
  - if $\phi<0$: ACF alternates sign and damping sinusoid

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- AR(2): ACF obeys Yule walker relation.

$$\rho_1 = \frac{\phi_1}{1-\phi_2} \quad , \quad \rho_2 = \phi_1 \rho_1 + \phi_2$$

solve for ϕ_1 , ϕ_2: $\quad \phi_1 = \rho_1 (1 - \phi_2) \qquad \phi_2 = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}$

- AR(p,q):

- short-run lag structure: controlled by MA(q). the MA part fixes exact values of γ_k for $k \leq q$.

- long-run tail structure: controlled by AR(p). It determines whether the tail decays exponentially, oscillates, and how slowly/fast it decays.