Arboricity and Degeneracy of Graphs

Equivalence, Bounds, and Intuition

Definitions

Definition 1.1 (Arboricity)**.** *For a graph $G = (V, E)$, the* arboricity $\alpha(G)$ *is the minimum number of edge-disjoint forests whose union is G, equivalently:*

$$\alpha(G) = \max_{H \subseteq G, |V(H)| \geq 2} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil.$$

Definition 1.2 (Degeneracy)**.** *The* degeneracy $\kappa(G)$ *is the smallest k such that every subgraph of G has a vertex of degree at most k. Equivalently:*

$$\kappa(G) = \max_{H \subseteq G} \Big(\min_{v \in V(H)} \deg_H(v) \Big).$$

Relationship Between α and κ

We have the tight bounds:

$$\alpha(G) \;\leq\; \kappa(G) \;\leq\; 2\,\alpha(G) - 1.$$

Combinatorial Proof that $\alpha \leq \kappa$

Let $k = \kappa(G)$, so G is k-degenerate. Then there exists an ordering of the vertices

$$v_n,\, v_{n-1}, \ldots, v_1$$

such that each vertex v_i has at most k neighbors among $\{v_1, \ldots, v_{i-1}\}$.

Every edge of any n-vertex subgraph H is counted exactly once by its later endpoint in this ordering, so

$$|E(H)| \;\leq\; k\,(n-1).$$

By Nash–Williams,

$$\alpha(G) = \max_{H \subseteq G, |V(H)| \geq 2} \left\lceil \tfrac{|E(H)|}{|V(H)|-1} \right\rceil \;\leq\; \max_{H} \lceil k \rceil = k,$$

giving $\alpha(G) \leq \kappa(G)$.

Proof that $\kappa \le 2\alpha - 1$

For any subgraph H with n vertices and m edges, the Nash–Williams bound $m \le (n-1)\alpha$ yields an average degree

$$\bar{d}(H) = \frac{2m}{n} < 2\alpha.$$

Thus

$$\min_{v \in V(H)} \deg_H(v) \le \bar{d}(H) < 2\alpha \quad \Longrightarrow \quad \min_v \deg_H(v) \le 2\alpha - 1.$$

Taking the maximum over H gives $\kappa(G) \le 2\alpha - 1$.

Intuition and Impact

These inequalities show that arboricity and degeneracy both capture graph sparsity, with arboricity measuring forest decompositions and degeneracy capturing peelability. Despite different definitions, they remain within a factor of two of each other.