Understanding ”Subsumes” and ”Dominates” in Asymptotic Notation

Definitions

- **Big-O Notation (O)**: A function $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.
- **Soft-O ($\widetilde{O}$)**: Ignores logarithmic factors. Formally, $\widetilde{O}(g(n)) = O(g(n) \cdot \log^k n)$ for some k.

Key Terms

1. **”Subsumes”**: $O(X)$ subsumes $O(Y)$ if $Y = O(X)$. This means X grows at least as fast as Y, so $O(Y)$ is a subset of $O(X)$.

   Example: $O(n^2)$ subsumes $O(n)$ because $n = O(n^2)$.

2. **”Dominates”**: $O(X)$ dominates $O(Y)$ if X grows faster than Y, so X determines the asymptotic behavior when both terms are present.

   Example: In $O(n^3 + n^2)$, n^3 dominates n^2.

Application to the Triangle Counting Problem

Theorem 1.2

The theorem states that the space complexity is $\widetilde{O}\left(\frac{m\kappa}{T}\right)$. This bound:

- **Subsumes $\widetilde{O}\left(\frac{m^{3/2}}{T}\right)$**:

$$\text{Since } \kappa \leq \sqrt{2m}, \quad \begin{aligned} \frac{m\kappa}{T} &\leq \frac{m\sqrt{2m}}{T} \\ &= O\left(\frac{m^{3/2}}{T}\right). \end{aligned}$$

  Thus, $\widetilde{O}\left(\frac{m\kappa}{T}\right)$ includes $\widetilde{O}\left(\frac{m^{3/2}}{T}\right)$ as a special case.

- **Dominates $\widetilde{O}\left(\frac{m}{\sqrt{T}}\right)$ when $T = \Omega(\kappa^2)$**:

$$\text{If } T \geq c\kappa^2 \text{ (for some } c > 0\text{), then } \begin{aligned} \frac{m\kappa}{T} &\leq \frac{m\kappa}{c\kappa^2} \\ &= \frac{m}{c\kappa} \\ &\leq \frac{m}{c\sqrt{T}} \quad (\text{since } \kappa \leq \sqrt{T} \text{ when } T \geq \kappa^2) \\ &= O\left(\frac{m}{\sqrt{T}}\right). \end{aligned}$$

  Here, $\frac{m\kappa}{T}$ becomes the smaller (dominant) term.

Concrete Example: Wheel Graph

For a wheel graph with n vertices:

- $m = T = \Theta(n)$,
- $\kappa = O(1)$ (planar graph).

The space complexity becomes:

$$\widetilde{O}\left(\frac{m\kappa}{T}\right) = \widetilde{O}(1),$$

which is exponentially better than existing bounds like $\Omega(\sqrt{n})$.

Why This Matters for Real-World Graphs

Real-world graphs often have:

- High triangle counts ($T = \Omega(\kappa^2)$),
- Bounded degeneracy ($\kappa = O(1)$ for social/media networks).

The new bound $\widetilde{O}\left(\frac{m\kappa}{T}\right)$ adapts to these properties, providing significantly better space efficiency than previous results.

- **Our algorithm uses**

$$\widetilde{O}\Big(\frac{m\,\kappa}{T}\Big)$$

  words of space (that's the **upper bound**).

- **No algorithm can do better than**

$$\Omega\Big(\min(\tfrac{m^{3/2}}{T},\ \tfrac{m}{\sqrt{T}})\Big)$$

  words (that's the **lower bound**).

Then, by checking the two regimes

1. When $\frac{m^{3/2}}{T} \le \frac{m}{\sqrt{T}}$, we show
   $\frac{m\,\kappa}{T} \le O(m^{3/2}/T)$.
2. When $\frac{m}{\sqrt{T}} < \frac{m^{3/2}}{T}$, we show (for $T = \Omega(\kappa^2)$)
   $\frac{m\,\kappa}{T} \le O(m/\sqrt{T})$.

This proves

$$\widetilde{O}\Big(\frac{m\,\kappa}{T}\Big) \;=\; \widetilde{\Theta}\Big(\min(\tfrac{m^{3/2}}{T},\ \tfrac{m}{\sqrt{T}})\Big).$$

In other words, **the algorithm's space usage is tight up to poly-logarithmic factors**—it is both

- **at most** $\widetilde{O}(m\kappa/T)$, and
- **at least** $\Omega(\min(m^{3/2}/T,\ m/\sqrt{T}))$,

and we've shown those two quantities coincide (up to logs). So yes: the algorithm runs in $\widetilde{\Theta}(m\,\kappa/T)$ space, matching the information-theoretic lower bound.