QFT, Done Right

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February 26, 2022

Contents

Notations and Conventions

• $\hbar = \mu_0 = \varepsilon_0 = k_B = 4\pi G = 1$.

• The space-time metric has $-+++\cdots$ signature. Your height is a real number and the distance between your house today and your house tomorrow is imaginary!

• The Fourier transforms look as below

$$\tilde{f}(k_1, k_2, \cdots, k_n) \equiv (2\pi)^{-n/2} \int d\mathbf{x}\, f(x_1, x_2, \cdots, x_n) \exp\big[-i(k_1 x_1 + \cdots + k_n x_n)\big]$$

$$f(x_1, x_2, \cdots, x_n) = (2\pi)^{-n/2} \int d\mathbf{k}\, \tilde{f}(k_1, k_2, \cdots, k_n) \exp\big[+i(k_1 x_1 + \cdots + k_n x_n)\big]$$

• Hilbert space operators DO NOT wear hats.

• Space-time dimension is $n+1$ unless otherwise specified.

• Spatial vectors are denoted by boldface characters like $\mathbf{P}$, etc. Whereas the space-time vectors can be recognized from their accompanying Greek indices eg: P^α, etc.

• Complex conjugate of any qunatity Q is Q^* and the Hermitian conjugate will be $Q^\dagger$ (whenever applicable)

• Repeated double indices are always summed over unless there are 3 (or more) of them or one of them is protected by parentheses:

$$A_i B_i, \quad A_\mu B^\mu, \quad A_{ii} B_i, \quad A_i B_{(i)}$$

• Tuples that are not spatial or space-time vectors are denoted by a sub-tilde; eg. $\underset{\sim}{p}$ for $\{p_i\}$ the set of momenta.

• To distinguish between different pictures when dealing with observables, look at the arguments. If there is no time dependence, we are working in the Schroedinger picture, if there is a time dependence, we are working in the Heisenberg picture and finally, if there is a superscript 0 and a time dependence, we are working in the interaction picture.

• An asterisk (*) in the title of a section means that it is still under construction/correction.

Part I

Classical Mechanics

Chapter 1

A Review of Lagrangian and Hamiltonian Mechanics

1.1 * Dynamical Systems, What to expect

A continuous dynamical system is a tuple $(\mathtt{X}, \Omega)$. The first entry, $\mathtt{X}$ is the set of *observables* which are real random variables. It is assumed that the set of observables satisfies

- $f(X) \in \mathtt{X} \qquad \forall\, f : \mathbb{R} \to \mathbb{R},\ X \in \mathtt{X}$
- $\alpha X_1 + \beta X_2 \in \mathtt{X} \qquad \forall\, \alpha, \beta \in \mathbb{R}\, X_1, X_2 \in \mathtt{X}$

The second entry Ω is a convex set of states denoted by ω. A state $\omega \in \Omega$ is a single parameter map from the set of observables $\mathtt{X}$ to the set of probability measures on $\mathbb{R}$, where the single parameter is called (dynamical) *time*. The practical interpretation is that if the system is in the state ω and one measures the observable X at some time t, the result will be a real random variable with distribution $\omega(X)$. This motivates us to assume that the following consistency conditions hold

- $\omega_t(f(X)) = f * \omega_t(X) \qquad \forall\, f : \mathbb{R} \to \mathbb{R},\ \omega \in \Omega,\ X \in \mathtt{X}, t \in \mathbb{R}$
- $\big[p\omega + (1-p)\omega'\big]_t(X) = p\omega_t(X) + (1-p)\omega'_t(X)$

We will generally limit our discussion to systems where it is possible to come up with maps $\varphi^{\text{Sch.}}$ and $\varphi^{\text{Hei.}}$ such that the following identities hold.

$$\omega_t(X) = \varphi^{\text{Sch.}}_{s\to t}(\omega)_s(X)$$

$$\omega_t(X) = \omega_s\Big(\varphi^{\text{Hei.}}_{s\to t}(X)\Big)$$

1.2 Lagrangian Systems

A *classical* mechanical system consists of a number of degrees of freedom, q_i for i in some index set $\mathcal{I}$, and a Lagrangian function L. The Lagrangian is assumed to be a function of the degrees of freedom q_i, their first order time derivatives and possibly of time itself. The equations of motion are derived by optimizing the action

$$S = \int L(q_i, \dot{q}_i; t)\, dt$$

This leads to the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

The equations are clearly invariant under a transformation

$$L \to L + \frac{dX}{dt}$$

for some $X = X(q_i, \dot{q}_i; t)$. This motivates us to call any transformation $q_i \to q_i'$ a symmetry of the system if it leads to a change in the Lagrangian that has the above form. An infinitesimal such transformation looks like below

$$q_i \to q_i + \varepsilon \tau_i(q_j, \dot{q}_j)$$

which in turn results in a change in the Lagrangian

$$L \to L + \varepsilon\big(\tau_i \frac{\partial L}{\partial q_i} + \dot{\tau}_i \frac{\partial L}{\partial \dot{q}_i}\big)$$

$$= L + \varepsilon\Big[\tau_i\big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\big) + \frac{d}{dt}\big(\tau_i \frac{\partial L}{\partial \dot{q}_i}\big)\Big]$$

$$= L + \varepsilon \frac{d}{dt}\big(\tau_i \frac{\partial L}{\partial \dot{q}_i}\big)$$

On the other hand we assumed that $L \to L + \varepsilon \frac{dX}{dt}$.[1] Comparing the two results, we find a conservation law

$$\boxed{\frac{d}{dt}\Big(\tau_i \frac{\partial L}{\partial \dot{q}_i} - X\Big) = 0}$$

This result is famously known as the Noether's theorem.

As a first example, consider a time shift in the degrees of freedom $q_i(t) \to q_i(t+\varepsilon) \approx q_i(t) + \varepsilon \dot{q}_i(t)$. For a Lagrangian with no explicit time dependence, this results in

$$L \to L + \varepsilon \dot{L}$$

and therefore the time shift will be asymmetry of the system with $X = L$. The corresponding conserved charge will be defined to be the Hamiltonian of the system

$$H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

It is customary to define the conjugate momenta as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

then the Hamiltonian may be regarded as the Legendre transformation of the Lagrangian with respect to velocities $\dot{q}_i$ and therefore a function of p_i and q_i

$$H(q_i, p_i) = \dot{q}_j p_j - L$$

[1] There is a subtle difference betweenequations $\delta L = \varepsilon X$ and $\delta L = \varepsilon \tau_i \partial L / \dot{q}_i$. The former is assumed to hold as an identity resulting from the specific properties of the Lagrangian function and has nothing to do with dynamics; however, the latter holds only when the Euler-Lagrange equations are satisfied. This is sometimes called ”on-shell”.

this new function allows us to write the equations of motion as first order differential equations over the phase space (q_i, p_i)

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

The Hamiltonian equations of motion allow us to write down the time derivatives of different phase space functions in a specific format

$$\begin{aligned} \frac{d}{dt} f(q_i, p_i; t) &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial H}{\partial q_j} \\ &= \frac{\partial f}{\partial t} + \{f, H\} \end{aligned}$$

Where in the last line, we have introduced the Poisson brackets

$$\{A, B\} \equiv \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

The above equations, reveal yet another relation between the Hamiltonian and time shifts since the Hamiltonian is generating a time shift. Does such a relation also exist between any other conserved charge Q and its corresponding symmetry transformation? To check this, let us study the infinitesimal transformation generated by some conserved charge

$$\begin{aligned} q_i \to q_i + \varepsilon\{q_i, Q\} &= q_i + \varepsilon\frac{\partial Q}{\partial p_i} \\ &= q_i + \varepsilon\Big[\tau_i + \frac{\partial^2 H}{\partial p_i \partial p_j}\big(p_k\frac{\partial \tau_k}{\partial \dot{q}_j} - \frac{\partial X}{\partial \dot{q}_j}\big)\Big] \end{aligned}$$

On the other hand, the symmetry condition asks the following to hold as an identity, on and off shell

$$\frac{\partial X}{\partial t} + \frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial \dot{q}_i}\ddot{q}_i = \frac{dX}{dt} = \delta L = \frac{\partial L}{\partial q_i}\tau_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \tau_i}{\partial t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \tau_i}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \tau_j}{\partial \dot{q}_i}\ddot{q}_i$$

Since none of the functions depend on second order time derivatives, we may equate the corresponding coefficients to get

$$\frac{\partial X}{\partial \dot{q}_i} = p_j\frac{\partial \tau_j}{\partial \dot{q}_i}.$$

This in turn implies

$$q_i + \varepsilon\{q_i, Q\} = q_i + \varepsilon\tau_i$$

In other words, not only every symmetry gives rise to a conserved charge (Noether's theorem), but also every conserved charge generates a symmetry transformation via Poisson brackets (Converse of Noether's theorem).

Assuming a system with n degrees of freedom has c independent, continuous conserved charges means that its generic integral curve will cover some $2n - c$ dimensional manifold in the phase space. For instance, the Keplerian system has 3 degrees of freedom and 5 conserved charges (Energy, angular momentum vector and the Laplace-Runge-Lenz vector) therefore all Keplerian orbits are closed on themselves.

As an example for the converse, consider the Lagrangian describing two independent harmonic oscillators.

$$L = \frac{1}{2}\Big(\dot{q}_1^2 + \dot{q}_2^2 - q_1^2 - \lambda^2 q_2^2\Big)$$

Regardless of the specific value of λ, the Lagrangian is covariant under *independent* time shifts and therefore the following are conserved

$$H_1 = \frac{1}{2}\left(p_1^2 + q_1^2\right); \quad H_2 = \frac{1}{2}\left(p_2^2 + \lambda^2 q_2^2\right)$$

The remaining 2-dimensional manifold is a torus; let us parametrize it as

$$p_1 = \sqrt{2H_1}\cos\phi_1; \;\; q_1 = \sqrt{2H_1}\sin\phi_1; \;\; p_2 = \sqrt{2H_2}\cos\phi_2; \;\; q_2 = \lambda^{-1}\sqrt{2H_2}\sin\phi_2$$

The integral curves are described by

$$H_1 = E_1; \;\; H_2 = E_2; \;\; \phi_1 = t + \alpha_1; \;\; \phi_2 = \lambda t + \alpha_2$$

For irrational λ, this densely covers the torus and therefore no other conserved charge *could* exist; however, for $\lambda = n_1/n_2$ the curves are 1 dimensional and therefore another conserved charge *must* exist. Indeed the charge is

$$Z = \exp\left[2\pi i(n_1\phi_1 - n_2\phi_2)\right]$$

More often than not, it is the case that we want to work with a statistical/probabilistic description of a mechanical system. This is achieved by having access to a *consistent* set of expected values for all phase space observables $\langle f(q_i, p_i; t)\rangle_t$. However, it is much more convenient to work with a probability density function ρ that produces all the expected values. It is natural to assign the time evolution to the density ρ. We will call this the Schroedinger picture.

$$\langle f(q_i, p_i; t)\rangle_t = \int d\underset{\sim}{p}\, d\underset{\sim}{q}\, f(q_i, p_i; t)\rho(q_i, p_i, t); \qquad \frac{\partial \rho}{\partial t} = -\{\rho, H\}$$

Note that the minus sign is what distinguishes ρ from an observable. It is also possible to assign the time evolution to the observables and consider ρ as being a constant measure in time. This is called the Heisenberg picture.

$$\langle f(q_i, p_i; t)\rangle_t = \int d\underset{\sim}{p}\, d\underset{\sim}{q}\, f_t(q_i, p_i)\rho(q_i, p_i); \qquad \frac{df_t}{dt} = \frac{\partial f}{\partial t} + \{f_t, H\}$$

1.3 * Constrained Systems

In this section we discuss how peculiarities in the Lagrangian/Hamiltonian function may be used to describe constrained systems. This will be useful when we try to quantise systems with gauge symmetries. At first we will try to do the entire analysis without referring to the Hamiltonian formalism. The Dirac theory concerning the Hamiltonian treatment of constrained systems is then presented.

1.3.1 The Lagrangian Formalism

In the previous parts, we always assumed that the Euler-Lagrange equations of motion, derived via optimizing the action integral, are always enough to determine the accelerations, $\ddot{q}_i$, in terms of the positions, q_i, and the velocities, $\dot{q}_i$. That is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \;\; \Rightarrow \;\; \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\dot{q}_j = \frac{\partial L}{\partial q_i} \;\; \Rightarrow \;\; \ddot{q}_i = A_i(q_j, \dot{q}_j)$$

However, this is only possible if the matrix

$$M_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

is not singular. It is not even a serious problem if the matrix is singular on a measure zero surface in the $2N$ dimensional space of $(q_i, \dot{q}_j)$; the equations of motion may still be used arbitrarily near the surface, then gluing the solutions together before and after passing through the singular surface will usually be considered as the answer to the Lagrangian problem. (Cf. exercises)

The interesting case (in this section) happens when the matrix M is singular over a whole domain. At the first glance it seems that in such cases, the accelerations do not exist unless a number of constraints are satisfied on $(q_i, \dot{q}_i)$ and even then, they are not uniquely determined. To see what the constraints are, let us define

$$F_i(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) \equiv \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} \dot{q}_j$$

Then, M's null eigen-vectors $z_i^a(\underset{\sim}{q}, \underset{\sim}{\dot{q}})$ satisfy $z_i^a M_{ij} = 0$; therefore, Lagrange's equations of motion admit solutions only if the following first order constraints are satisfied

$$z_i^a F_i = 0$$

Note that such constraints need to be satisfied even on the boundary/initial conditions; otherwise the action will have no optimal solution. Out of the constraints above, only a subset of them depend on the velocities in an essential way. If the matrix $\left[\frac{\partial (z_i^a F_i)}{\partial \dot{q}_j}\right]$ is singular, a number of the constraints may be separated to form

$$\text{constraints:} \begin{cases} \gamma^a(\underset{\sim}{q}) = 0 & \text{type-A constraints} \\ \gamma^b(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) = 0 & \text{type-B constraints} \end{cases}$$

The constraints above, may further decrease the functional rank of the M_{ij} matrix, thereby adding even more constraints. On the other hand, the constraints not only need to be satisfied at a single moment in time, but also through the whole path, this means we want to have $\dot{\gamma}^a = 0$ and $\dot{\gamma}^b = 0$. The first set leads to more type-B constraints while the second set, give rise to new equations for the accelerations, $\ddot{q}$. Solving the full problem, involves adding these implications and repeating until one reaches a state with the equations

$$\text{constraints:} \begin{cases} \gamma^a(\underset{\sim}{q}) = 0 & \text{type-A constraints} \\ \gamma^b(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) = 0 & \text{type-B constraints} \end{cases} \quad ; \quad M \underset{\sim}{\ddot{q}} = \underset{\sim}{F} \quad ; \quad N_i^c \ddot{q}_i = G_i$$

where

- The null vectors from the acceleration equations, don't add anything to the constraints.

- The time derivative of type-A constraints don't add anything to the type-B constraints.

- The type-B constraints essentially depend on the velocities; that is the matrix $\frac{\partial \gamma^b}{\partial \dot{q}_i}$ is full-rank.

- The time derivative of the type-B constraints does not add anything to the acceleration equations.

- The constraints do not further decrease the rank (increase the singularity) of the acceleration equations.

Even at this final stage, the acceleration equations may not be enough to provide a unique solution, in that case, the dynamical paths that give rise to the optimal action are not unique. It may also be the case that these equations, lead to inconsistencies, in those cases, there is no optimal solution for the action and the Lagrangian method fails.

1.3.2 The Hamiltonian Formalism: Dirac's Theory

Whenever the Lagrange equations of motion fail to determine the accelerations, a similar problem arises in the Hamiltonian formalism and we can not write all the $\underset{\sim}{\dot{q}}$s in terms of the momenta. In fact the necessary and sufficient condition for this process is given via the implicit function theorem as

$$\det\left(\frac{\partial p_i}{\partial \dot{q}_j}\right) = \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = \det(M_{ij}) \neq 0$$

If this is not satisfied, then there are primary (i.e. off shell) constraints connecting the momenta and coordinates.

$$\phi_\alpha(\underset{\sim}{p}, \underset{\sim}{q}) = 0$$

Let N denote the number of degrees of freedom in our system and R the functional rank of the mass matrix M_{ij}; then $N - R$ will be the number of such primary constraints. Of course at this stage (before writing the equations of motion) the coordinates $\underset{\sim}{q}$ are all independent and therefore it would be impossible to use these primary constraints to find type-A constraints on the coordinates; this means that the matrix

$$\left(\frac{\partial \phi_\alpha}{\partial p_i}\right)$$

is full-rank. This allows us to re-write the constraints as

$$p_i - \psi_i(\underset{\sim}{q}, p_1, \cdots, p_R) = 0\ ;\ \ i \in \{R+1, \cdots, N\}$$

This all means that the canonical variables $(\underset{\sim}{p}, \underset{\sim}{q})$ can not serve as a coordinate system for the $2N$ dimensional space of positions and velocities spanned by $(\underset{\sim}{q}, \underset{\sim}{\dot{q}})$. While the former is traditionally called the phase space, we shall call the latter by the name *Lagrange's space*. The phase space contains non-physical points that do not correspond to any set of positions and velocities; in other words, the physical systems are constrained to a sub-manifold of the phase space described by the primary constraints. On the other hand, if we add $N - R$ of the (carefully chosen) velocities as auxiliary coordinates to the $\underset{\sim}{q}$ and the independent momenta, we may construct a healthy coordinate system for the Lagrange's space. In this coordinate system, we can describe the Hamiltonian as

$$H(\underset{\sim}{p}, \underset{\sim}{q}) \equiv \sum_{i=1}^{R} p_i \dot{q}_i + \sum_{i=R+1}^{N} \psi_i \dot{q}_i - L$$

where some of the $\dot{q}_i$'s are independent coordinates and some are functions of other coordinates. Note that this notation implies that the Hamiltonian is only a function of the canonical variables; to see this, we can differentiate both sides while keeping the canonical variables constant.

$$dH_{\underset{\sim}{p}, \underset{\sim}{q}} = \sum_i d\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \sum_i d\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 0$$

Then we can differentiating the Hamiltonian definition with respect to p_i with $i \leq R$, to get

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \sum_{j=R+1}^{N} \dot{q}_j \frac{\partial \psi_j}{\partial p_i}$$

Interestingly, these relations set the auxiliary (undetermined) velocities to be those that correspond to degenerate momenta; i.e. $\dot{q}_i$ with $i > R$.

Now that we know the coordinate systems and the spaces they span, let us focus on the dynamics of the system. In Hamiltonian formalism, we are interested in finding a Hamiltonian flow on the phase space. We have already found expressions for $\dot{q}_i$ with $i \leq R$. The time derivative of the momenta are given by the Euler-Lagrange equations

$$\begin{aligned}
\dot{p}_i = \frac{\partial L}{\partial q_i} &= \sum_{j\leq R} \frac{\partial p_j}{\partial q_i}\dot{q}_j + \sum_{j>R} \frac{\partial \psi_j}{\partial q_i}\dot{q}_j + \sum_{j>R} \frac{\partial \psi_j}{\partial p_k}\frac{\partial p_k}{\partial q_i}\dot{q}_j - \sum_{j\leq R} \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial q_i} - \frac{\partial H}{\partial q_i} \\
&= \sum_{j\leq R} \frac{\partial p_j}{\partial q_i}\Big(\frac{\partial H}{\partial p_j} - \sum_{k>R} \dot{q}_k \frac{\partial \psi_k}{\partial p_j}\Big) + \sum_{j>R} \frac{\partial \psi_j}{\partial q_i}\dot{q}_j + \sum_{j>R} \frac{\partial \psi_j}{\partial p_k}\frac{\partial p_k}{\partial q_i}\dot{q}_j - \sum_{j\leq R} \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial q_i} - \frac{\partial H}{\partial q_i} \\
&= -\frac{\partial H}{\partial q_i} + \sum_{j>R} \dot{q}_j \frac{\partial \psi_j}{\partial q_i}
\end{aligned}$$

It remains to find the $i > R$ velocities in the phase space or prove them irrelevant; this is the focus of what follows.

Dirac's Theory

Let U denote the physical sub-manifold of the phase space where the following primary constraints are satisfied.

$$\phi_i \equiv p_i - \psi_i(q_1, \cdots, q_N, p_1, \cdots, p_R) = 0 \ ; \ i > R$$

Since the constraints need to be satisfied at all times in order for the system to remain physically meaningful, we need the time derivative of the constraints to vanish as well. For a generic function of the canonical variables, we have

$$\begin{aligned}
\frac{df}{dt} &= \frac{\partial f}{\partial p_i}\Big(-\frac{\partial H}{\partial q_i} + \sum_{j>R} \dot{q}_j \frac{\partial \psi_j}{\partial q_i}\Big) + \sum_{i\leq R} \frac{\partial f}{\partial q_i}\Big(\frac{\partial H}{\partial p_i} - \sum_{j>R} \dot{q}_j \frac{\partial \psi_j}{\partial p_i}\Big) + \sum_{j>R} \dot{q}_j \frac{\partial f}{\partial q_j} \\
&= \{f, H\} + \sum_{i>R} \dot{q}_i \Big(\frac{\partial \psi_i}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial \psi_i}{\partial p_j}\frac{\partial f}{\partial q_j} + \frac{\partial f}{\partial q_i}\Big) \\
&= \{f, H\} + \sum_{i>R} \dot{q}_i \{f, \phi_i\}
\end{aligned}$$

For the constraints then we have

$$\frac{d\phi_i}{dt} = \{\phi_i, H\} + \sum_{j>R} \dot{q}_j \{\phi_i, \phi_j\}$$

which leads to equations of motion

$$\sum_{j>R} \dot{q}_j \{\phi_i, \phi_j\} = -\{\phi_i, H\}$$

If the (anti-symmetric) matrix $\{\phi_i, \phi_j\}$ is full-rank on the physical submanifold, then we are done.

$$\dot{q}_i = -\sum_{j>R} C_{ij} \{\phi_j, H\}$$

and therefore the dynamics is given by Dirac's equations of motion

$$\boxed{\frac{df}{dt} = \{f, H\} - \sum_{i,j>R} \{f, \phi_i\} C_{ij} \{\phi_j, H\}}$$

Otherwise, there are secondary constraints corresponding to any null eigenvector of the $\{\phi_i, \phi_j\}$ matrix

$$\lambda_{ai}\{\phi_i, \phi_j\} = 0 \quad \Rightarrow \quad \chi_a \equiv \lambda_{ai}\{\phi_i, H\} = 0$$

these could lead to further secondary constraints by reducing the functional rank of the constraints matrix. Other than that, it would also lead to new equations of motion. The final state is achieved as

$$\text{Auxiliary Equations of Motion} \begin{cases} \text{Primary:} & \{\phi_i, \phi_j\}\dot{q}_j + \{\phi_i, H\} = 0 \\ \\ \text{Secondary:} & \{\chi_a, \phi_i\}\dot{q}_i + \{\chi_a, H\} = 0 \end{cases}$$

$$\text{Constraints} \begin{cases} \text{Primary:} & \phi_i = 0 \\ \\ \text{Secondary:} & \chi_a = 0 \end{cases}$$

when

- Possible singularities in the auxiliary eom's do not add new constraints.

- Time derivative of the constraints, do not add anything to the eom's.

Once again, if the equations of motion are still not enough to determine the dynamics, then the solution may not exists or not be unique.

1.4 Exercises

1. For the Lagrangian $L = \frac{1}{2}q\dot{q}^2$ and initial values $q(0) = 1$, $\dot{q}(0) = -1$ find the solution $q(t)$ for all $t \in \mathbb{R}$.

2. Discuss the corrections necessary to account for explicit time dependencies in constrained Lagrangians and Hamiltonians.

Chapter 2

Classical Tensor Field Theory

By a ”classical field theory”, we mean a mechanical system with uncountably many degrees of freedom. Therefore, our default letter for degrees of freedom changes from q_i to $\phi_a(x)$ where x is some coordinate system on the index manifold and a runs over a finite (or at least countable) set. This is not relativistic yet; time is only a parameter and space is yet to be born. To make things relativistically covariant, we proceed to make the following changes:

- The index manifold is identified with *some* arbitrary spacelike manifold, Σ, embedded in the space-time.
- The Heisenberg picture is (inevitably) used to make sure that Σ is arbitrary and the fields ϕ_a are defined over the whole space-time.
- The integral over time in $S = \int L\,dt$ is replaced with the covariant integral over space-time: $S = \int dx\,\sqrt{-g}\mathcal{L}$.
- The Lagrangian density $\mathcal{L}$, being a function of the fields and their time derivatives, now *must* depend on the first order space-time derivatives in a covariant way: $\mathcal{L} = \mathcal{L}(\phi_a, \nabla_\mu \phi_a; x^\mu)$.
- The fields ϕ_a are covariant objects. In this chapter we focus on ϕ_a being tensors. It costs no generality to assume that all their indices are contravariant: $\phi_a^{\mu_1 \cdots \mu_{m_a}}$

Finally, we get

$$S = \int dx\,\sqrt{-g}\mathcal{L}(\phi_a, \nabla_\mu \phi_a; x^\alpha)$$

The variational principle leads to the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi_a}$$

The action, S, (and therefore the equations of motion) are invariant under

$$\mathcal{L} \to \mathcal{L} + \varepsilon \nabla_\mu \mathcal{X}^\mu$$

for some $\mathcal{X}^\mu(\phi_a, \nabla_\mu \phi_a, x^\alpha)$. Therefore we expect an infinitesimal symmetry transformation to lead to such changes in the Lagrangian density. The transformation must look like

$$\phi_a \to \phi_a + \varepsilon \tau_a$$

The on-shell change in Lagrangian density is

$$\mathcal{L} \to \mathcal{L} + \varepsilon \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi_a} \tau_a$$

We immediately find out that corresponding to any symmetry transformation, there exists a conserved current

$$\nabla_\mu j^\mu = 0; \quad j^\mu \equiv \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi_a}\tau_a - \mathcal{X}^\mu$$

When discussing classical systems with finite or countable degrees of freedom, our first example of a symmetry was a time *shift* for time-independent Lagrangians. The corresponding symmetries in field theory are isometries; these are symmetries of the space-time that may be respected by the Lagrangian. Under such transformations, the fields are simply shifted along a symmetry direction.

In order to discuss the isometries of a given space-time manfiold, let us first introduce the geometric notion of Lie derivatives. Consider an infinitesimal coordinate transformation

$$x^\mu \to x^\mu - \varepsilon\xi^\mu.$$

Under such transformations, the components of a tensor $T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}$ change into

$$T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} \to T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} - \varepsilon\Big[\sum_{r=1}^m T^{\mu_1\cdots\mu_{r-1}\alpha\mu_{r+1}\cdots\mu_m}_{\nu_1\cdots\nu_n}\partial_\alpha\xi^{\mu_r} - \sum_{r=1}^n T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_{r-1}\alpha\nu_{r+1}\cdots\nu_n}\partial_{\nu_r}\xi^\alpha\Big]$$

On the other hand, the components of the same tensor, evaluated at a point with coordinates $x^\mu - \varepsilon\xi^\mu$ in the original coordinate system, are

$$T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} - \varepsilon\xi^\alpha\partial_\alpha T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}$$

The difference between these two expressions, divided by ε is defined to be the Lie derivative of the tensor field T along the vector field ξ^μ this is denoted by

$$\pounds_\xi T \equiv \xi^\alpha\partial_\alpha T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} - \sum_{r=1}^m T^{\mu_1\cdots\mu_{r-1}\alpha\mu_{r+1}\cdots\mu_m}_{\nu_1\cdots\nu_n}\partial_\alpha\xi^{\mu_r} + \sum_{r=1}^n T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_{r-1}\alpha\nu_{r+1}\cdots\nu_n}\partial_{\nu_r}\xi^\alpha$$

A Killing vector field, ξ^μ is one that satisfies $\pounds_\xi g = 0$ or written more explicitly

$$\nabla_{\{\alpha}\xi_{\beta\}} = 0$$

For such a field, we may write the second order derivatives in terms of the Riemann tensor

$$\begin{aligned}
\nabla_\alpha\nabla_\beta\xi_\mu &= -\nabla_\alpha\nabla_\mu\xi_\beta \\
&= -\nabla_\mu\nabla_\alpha\xi_\beta + [\nabla_\mu,\nabla_\alpha]\xi_\beta \\
&= \nabla_\mu\nabla_\beta\xi_\alpha + [\nabla_\mu,\nabla_\alpha]\xi_\beta \\
&= \nabla_\beta\nabla_\mu\xi_\alpha + [\nabla_\mu,\nabla_\beta]\xi_\alpha + [\nabla_\mu,\nabla_\alpha]\xi_\beta \\
&= -\nabla_\beta\nabla_\alpha\xi_\mu + [\nabla_\mu,\nabla_\beta]\xi_\alpha + [\nabla_\mu,\nabla_\alpha]\xi_\beta \\
&= -\nabla_\alpha\nabla_\beta\xi_\mu + [\nabla_\alpha,\nabla_\beta]\xi_\mu + [\nabla_\mu,\nabla_\beta]\xi_\alpha + [\nabla_\mu,\nabla_\alpha]\xi_\beta
\end{aligned}$$

Which may be rearranged to yield

$$\nabla_\alpha\nabla_\beta\xi_\mu = \frac{1}{2}\xi^\nu(R_{\alpha\beta\mu\nu} + R_{\mu\beta\alpha\nu} + R_{\mu\alpha\beta\nu}) = R_{\mu\beta\alpha\nu}\xi^\nu$$

Using this identity, we can prove the following nice result

$$[\nabla_\alpha, \pounds_\xi]T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} = \xi^\beta[\nabla_\alpha,\nabla_\beta]T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} - \sum_{r=1}^m T^{\mu_1\cdots\mu_{r-1}\beta\mu_{r+1}\cdots\mu_m}_{\nu_1\cdots\nu_n}\nabla_\alpha\nabla_\beta\xi^{\mu_r} + \sum_{r=1}^n T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_{r-1}\beta\nu_{r+1}\cdots\nu_n}\nabla_\alpha\nabla_{\nu_r}\xi^\beta = 0$$

where the last inequality is proved by substituting for the second order derivatives of ξ in terms of the Riemann tensor and cancelling the resulting terms with the anti-symmetric derivative operator acting on T.

Now that we are familiar with the Lie derivatives, let us consider the generalised shift transformation

$$\phi_a \to \phi_a + \varepsilon \pounds_\xi \phi_a$$

for some Killing vector ξ^μ. For a Lagrangian that only depends on the space-time position through the metric[1], we have

$$\begin{aligned}\mathcal{L} \to \mathcal{L} + \varepsilon\Big[\frac{\partial\mathcal{L}}{\partial\phi_a}\pounds_\xi\phi_a + \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi_a}\nabla_\alpha\pounds_\xi\phi_a\Big] \\ = \mathcal{L} + \varepsilon\Big[\frac{\partial\mathcal{L}}{\partial\phi_a}\pounds_\xi\phi_a + \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi_a}\pounds_\xi\nabla_\alpha\phi_a + \frac{\partial\mathcal{L}}{\partial g}\pounds_\xi g\Big] \\ = \mathcal{L} + \varepsilon\pounds_\xi\mathcal{L} \\ = \mathcal{L} + \varepsilon\xi^\mu\nabla_\mu\mathcal{L} = \mathcal{L} + \varepsilon\nabla_\mu\mathcal{L}\xi^\mu\end{aligned}$$

and therefore we are dealing with a symmetry transformation. The conserved current will be

$$j^\mu = \mathcal{L}\xi^\mu - \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi_a}\pounds_\xi\phi_a$$

2.1 The Energy-Momentum tensor(s)

Note that the conserved current from an isometry may be written as $j^\alpha = \mathtt{T}^{\alpha\beta}\xi_\beta$ if we define the tensor operator

$$\mathtt{T}^{\alpha\beta} \equiv T^{\alpha\beta}_{can.} + S^{\alpha\beta\gamma}\nabla_\gamma$$

where $T^{\alpha\beta}_{can.}$ is the canonical energy-momentum tensor

$$T^{\alpha\beta}_{can.} \equiv \mathcal{L}g^{\alpha\beta} - \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi_a}\nabla^\beta\phi_a$$

and

$$S^{\alpha\beta\gamma} \equiv \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi_a}\sum_{r=1}^{m_a}\phi_a^{\mu_1\cdots\mu_{r-1}\beta\mu_{r+1}\cdots\mu_{m_a}}g^{\mu_r\gamma}$$

On shell, the canonical energy-momentum tensor satisfies

$$\begin{aligned}\nabla_\alpha T^{\alpha\beta}_{can.} = \nabla^\beta\mathcal{L} - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi_a}\nabla^\beta\phi_a \\ = \nabla^\beta\mathcal{L} - \frac{\partial\mathcal{L}}{\partial g}\nabla^\beta g - \frac{\partial\mathcal{L}}{\partial\phi_a}\nabla^\beta\phi_a - \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi_a}\nabla^\beta\nabla_\alpha\phi_a + \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi_a}[\nabla^\beta,\nabla_\alpha]\phi_a \\ = S^{\alpha\mu\nu}R^\beta{}_{\alpha\mu\nu}\end{aligned}$$

[1]This condition is sometimes called the minimal coupling condition and is justified by the equivalence principle.

Using this we find that for any Killing vector ξ, the conservation equation gives

$$0 = \nabla_\alpha \mathtt{T}^{\alpha\beta}\xi_\beta = \frac{1}{2}\nabla_\alpha \xi_\beta \Big(T^{[\alpha\beta]}_{can.} - \nabla_\mu S^{\mu[\alpha\beta]} \Big)$$

Writing this on a flat (or symmetric enough) space-time, we find the identity

$$T^{[\alpha\beta]}_{can.} = \nabla_\mu S^{\mu[\alpha\beta]}$$

which is an on-shell identity in every space-time.

Now define the Belinfante tensor as

$$B^{\mu\alpha\beta} \equiv \frac{1}{2}\big(S^{\mu\alpha\beta} - S^{\alpha\mu\beta} + S^{\alpha\beta\mu} - S^{\mu\beta\alpha} + S^{\beta\alpha\mu} - S^{\beta\mu\alpha}\big)$$

$$= \frac{1}{2}S^{[\mu\alpha\beta]} + S^{\beta[\alpha\mu]}$$

This definition is in fact the unique tensor built from $S^{\mu\alpha\beta}$ to satisfy

$$B^{\{\mu\alpha\}\beta} = 0; \quad B^{\mu[\alpha\beta]} = S^{\mu[\alpha\beta]}$$

for a general tensor $S^{\mu\alpha\beta}$. The last property guarantees that the Belinfante energy-momentum tensor is symmetric.

$$T^{\alpha\beta}_{Bel.} \equiv T^{\alpha\beta}_{can.} - \nabla_\mu B^{\mu\alpha\beta}$$

This new tensor, is not only symmetric, but also conserved

$$\nabla_\alpha T^{\alpha\beta}_{Bel.} = \nabla_\alpha T^{\alpha\beta}_{can.} - \frac{1}{2}[\nabla_\alpha, \nabla_\mu]B^{\mu\alpha\beta}$$

$$= S^{\alpha\mu\nu}R^\beta{}_{\alpha\mu\nu} - \frac{1}{2}\big(- R_{\alpha\nu}B^{\nu\alpha\beta} + R_{\mu\nu}B^{\mu\nu\beta} + R_{\alpha\mu\beta\nu}B^{\mu\alpha\nu}\big)$$

$$= R^\beta{}_{\alpha\mu\nu}\big(S^{\alpha\mu\nu} + \frac{1}{4}S^{[\alpha\mu\nu]} - \frac{1}{2}S^{\alpha[\mu\nu]}\big) = 0$$

These two properties then imply that for any Killing field ξ, the current $j'^\mu \equiv T^{\mu\nu}_{Bel.}\xi_\nu$ is conserved. In the exercises, the reader proves that this current is equivalent to j^μ in flat space-times and for translational isometries. In summary the Belinfante energy-momentum tensor is

- symmetric,
- conserved,
- equivalent to the canonical energy-momentum tensor in the sense that the total energy and momentum are the conserved charges corresponding to space-time translations; and
- allows one to write the conserved currents as $j^\mu = T^{\mu\nu}_{Bel.}\xi_\nu$.

2.2 The Hamiltonian Formalism

If we denote the momenta as

$$\pi_a(x) \equiv \frac{\partial \mathcal{L}}{\partial \nabla_0 \phi_a}$$

then the Hamiltonian is given by

$$H = \int_\Sigma d\mathbf{x}\,\sqrt{-g}\mathcal{H}; \quad \mathcal{H} \equiv \pi_a(x)\nabla_0\phi_a(x) - \mathcal{L}$$

This could not be more not covariant!

2.3 Exercises

1. Show that on a flat space time, the conserved quantities computed from the canonical and Belinfante tensors are the same.

$$\int d\mathbf{x}\,T^{0\mu}_{can.} = \int d\mathbf{x}\,T^{0\mu}_{Bel.}$$

2. ⚓ For the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\Phi^{\mu;\alpha}\Phi_{\mu;\alpha} - \frac{1}{2}m_\phi^2\Phi^\mu\Phi_\mu - \frac{1}{2}\Psi^{\mu\nu;\alpha}\Psi_{\mu\nu;\alpha} - \frac{1}{2}m_\psi^2\Psi^{\mu\nu}\Psi_{\mu\nu} + g\Psi^{\mu\nu;\alpha}\Phi_{\mu;\nu}\Phi_\alpha$$

   a) Write down the equations of motion.

   b) Compute the tensors $T^{\alpha\beta}_{can.}$, $S^{\mu\nu\alpha}$, $B^{\mu\nu\alpha}$, and $T^{\alpha\beta}_{Bel.}$.

Chapter 3

Classical Spinor Field Theory

When introducing field theories, we argued that the dynamical fields need to be covariant objects. We are already familiar with tensors and their transformation properties. In this section, we introduce another class of objects called spinors. We will also learn how to use spinor-valued tensors in order to get more complicated representations.

Let us start by finding the Lie algebra of the Lorentz group. An infinitesimal transformation keeps the Minkowski metric invariant

$$\big(\delta^\alpha_\mu + A^\alpha_\mu\big)\big(\delta^\beta_\nu + A^\beta_\nu\big)\eta^{\mu\nu} = \eta^{\alpha\beta}$$

which leads to the first order condition

$$A_{\{\alpha\beta\}} = 0$$

Such anti-symmetric tensors are written as linear combinations of the generators $J(\mu,\nu)$ where

$$J(\mu,\nu)^\alpha_\beta = \eta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\eta^{\nu\alpha}$$

The Lie algebra is given by

$$\big[J(\mu,\nu), J(\alpha,\beta)\big] = \eta^{\alpha\nu}J(\mu,\beta) - \eta^{\beta\nu}J(\mu,\alpha) - \eta^{\alpha\mu}J(\nu,\beta) + \eta^{\beta\mu}J(\nu,\alpha)$$

For our brand new representation, corresponding to spinors, the generators are given by

$$J(\mu,\nu) = \frac{1}{4}\big[\gamma(\mu),\gamma(\nu)\big]$$

where $\gamma(\mu)$ are a set of Dirac matrices satisfying the Clifford algebra

$$\big\{\gamma(\mu),\gamma(\nu)\big\} = 2\eta^{\mu\nu}$$

3.1 Trace Technology From Clifford Algebra

The Clifford algebra alone, allows us to compute expressions of the form

$$\mathrm{Tr}\left[\gamma(\mu_1)\cdots\gamma(\mu_r)\right]$$

without using the explicit representation of the γ matrices. First, we need to define the matrix

$$\Gamma \equiv \frac{i}{(n+1)!} \sum_{\pi} \mathrm{sign}(\pi) \prod_{i=0}^{n} \gamma(\pi(0)) \cdots \gamma(\pi(n)) = i\gamma(0) \cdots \gamma(n)$$

This satisfies

$$\Gamma^2 = -\gamma(0) \cdots \gamma(n)\gamma(0) \cdots \gamma(n) = (-1)^{\frac{n(n+1)}{2}}$$

$$\Big\{\Gamma, \gamma(\mu)\Big\} = \eta(\mu,\mu)(-1)^{\mu}\big(1 + (-1)^{n}\big)\Gamma(\backslash\mu)$$

where

$$\Gamma(\backslash\mu) \equiv i\gamma(0) \cdots \gamma(\mu-1)\gamma(\mu+1) \cdots \gamma(n)$$

Part II

Prerequisites from Quantum Mechanics

Chapter 4

Canonical Quantisation

To quantise a mechanical system, one must find a complex Hilbert space $\mathscr{H}$ and Hermitian operators q_i, p_i that satisfy

$$[q_i, p_j] = i\delta_{ij}; \quad [q_i, q_j] = [p_i, p_j] = 0$$

This already means that the q_i have continuous spectra. It is then straightforward to show that the Hilbert space may be considered to be the linear space of all square-integrable functions $\psi(\underset{\sim}{q})$. The operators act as below

$$q_i\psi = q_i\psi(\underset{\sim}{q}); \quad p_i\psi = -i\partial_{q_i}\psi(\underset{\sim}{q})$$

The "state" of the system is nothing but a statistical description containing the expected values for all Hermitian operators. The expected values must be real, linear functions of the operators, therefore they are given by[1]

$$\langle f \rangle = \operatorname{Tr} \rho f$$

For some Hermitian density operator ρ. Normalization is necessary for consistency

$$1 = \langle 1 \rangle = \operatorname{Tr} \rho$$

[1] We will entirely limit our discussion to phase space functions that have unambiguous Hermitian expressions such as p^3, $\sin(aq + bp)$, etc. and not qp, p^q, etc.

Chapter 5

Perturbation Theory

5.1 Time Dependent Schroedinger Equation

The time dependent Schroedinger equation is

$$i\frac{d}{dt}\left|\psi;t\right\rangle = H(t)\left|\psi;t\right\rangle$$

And the solutions are encoded in the Schroedinger propagator $U(t)$

$$\left|\psi;t\right\rangle = U(t)\left|\psi;0\right\rangle .$$

The propagator satisfies

$$i\frac{d}{dt}U(t) = H(t)U(t); \quad U(0) = \mathbb{I}$$

this implies

$$U(t) = \mathbb{I} - i\int_0^t dt'\, H(t')U(t')$$

The integral equation above, suggests the so called Dyson series as the Schroedinger propagator

$$U(t) = \mathbb{I} + \sum_{n=1}^{\infty}(-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \,\cdots \int_0^{t_{n-1}} dt_n\, H(t_1)\cdots H(t_n)$$

This may also be written in the following, compact form

$$U(t) = \mathscr{T}\Big\{\exp\Big[-i\int_0^t dt'\, H(t')\Big]\Big\}$$

where the symbol $\mathscr{T}$ denotes the time ordering: in a product of operators from different times, the order of multiplication from left to right is chronologically decreasing.

If $U_N(t)$ denotes the truncated series after N terms, it is easy to show that

$$\big|\big|U(t) - U_N(t)\big|\big|_2 \leq \frac{t^{n+1}}{(n+1)!}\Big(\sup_{\tau\in[0,t]}\big|\big|H(t)\big|\big|_2\Big)^{n+1}$$

However, for the case of field theories, were $||H(t)||_2 = \infty$ at all times, this inequality is of no practical use; there we only *hope* that the Dyson series yields proper, relevant results.

5.2 The Interaction Picture

Now consider the case of a quantum mechanical system with a Hamiltonian in the form

$$H = H_0 + H_1$$

Assume that we know the *unperturbed* propagator $U_0(t)$ satisfying

$$i\frac{d}{dt}U_0(t) = H_0(t)U_0(t)$$

and want to write the full propagator as

$$U(t) = U_0(t)V(t)$$

The relevant differential equation is

$$\frac{d}{dt}V(t) = -i\big[U_0^\dagger(t)H_1U_0(t)\big]V(t) = -iH_1^0(t)V(t)$$

In words, the compensation propagator, $V(t)$, satisfies the Schroedinger equation with the unperturbed Heisenberg version of the perturbation Hamiltonian. In general, this is a time dependent Hamiltonian; however, since the perturbation term H_1 is asuumed to be small, we may hope to get proper results using the Dyson series.

5.3 The Adiabatic Theorem

Consider a family of Hamiltonians $H(\lambda^a)$. For now assume that for each λ, the Hamiltonian is non-degenerate.

$$H(\lambda^a)\left|n,\lambda^a\right\rangle = E_n(\lambda^a)\left|n,\lambda^a\right\rangle$$

Now for a smooth path $\lambda^a(\sigma)$ for $\sigma\in[0,1]$ consider the time dependent Hamiltonian

$$H(t) = H\big(\lambda^a(t/T)\big);\quad t\in[0,T]$$

The adiabatic theorem is stated as follows

$$\lim_{T\to\infty} U(T)\left|n,\lambda^a(0)\right\rangle\left\langle n,\lambda^a(0)\right| U^\dagger(T) = \left|n,\lambda^a(1)\right\rangle\left\langle n,\lambda^a(1)\right|$$

To prove this, let us start by writing

$$c_{mn}(t) \equiv \exp\left[i\int_0^t dt'\, E_m\big(\lambda^a(t'/T)\big)\right]\left\langle m,\lambda^a(t/T)\right| U(t)\left|n,\lambda^a(0)\right\rangle$$

Differentiating with respect to $\sigma = t/T$ we get

$$\frac{dc_{mn}}{d\sigma} = \sum_l \exp\left[iT\int_0^\sigma d\sigma'\,\big[E_m\big(\lambda^a(\sigma')\big) - E_l\big(\lambda^a(\sigma')\big)\big]\right]c_{ln}(\sigma)\Big(\frac{d\lambda^a}{d\sigma}\frac{\partial}{\partial\lambda^a}\left\langle m,\lambda^a(t)\right|\Big)\left|l,\lambda^a(\sigma)\right\rangle$$

Now in the $T \to \infty$ limit, all the $l \neq m$ terms will be irrelevant[1] and therefore we get to re-write the evolution as

$$c_{mn}(\sigma) = c_{mn}(0) \exp\Big\{ \int_{\lambda(0)}^{\lambda(\sigma)} d\lambda^a \, \big(\frac{\partial}{\partial \lambda^a} \langle m, \lambda^a | \big) \, | m, \lambda^a \rangle \Big\}$$

This completes the proof for the adiabatic theorem as soon as we substitute $c_{mn}(0) = \delta_{mn}$.

The *non − dynamical* phase that the amplitude c_{mn} takes during the adiabatic evolution is called the Berry phase. In fact if we define the following (real) 1-form

$$B_{m,a} \equiv i \, \langle m, \lambda^a | \, \frac{\partial}{\partial \lambda^a} \, | m, \lambda^a \rangle$$

we get to write the Berry phase as

$$\gamma_m = \int d\lambda^a \, B_{m,a}$$

5.4 The Correlation Functions

Suppose we want to compute the correlation function

$$\langle O_1(t_1) \cdots O_n(t_n) \rangle_{|E\rangle\langle E|}$$

for some eigenstate of the perturbed Hamiltonian

$$H \, |E\rangle = (H_0 + H_1) \, |E\rangle = E \, |E\rangle$$

This would be the same as

$$\langle E| \, e^{iH(t_1 - t_0)} O_1(t_0) e^{iH(t_2 - t_1) O_2(t_0)} \cdots e^{iH(t_n - t_{n-1})} O_n(t_0) e^{iH(t_0 - t_n)} \, |E\rangle$$

Ideally, we would want to deal with the unperturbed state $|E_0\rangle$ instead of the perturbed state $|E\rangle$. To do this, we first introduce the adiabatic function $\sigma(t)$.

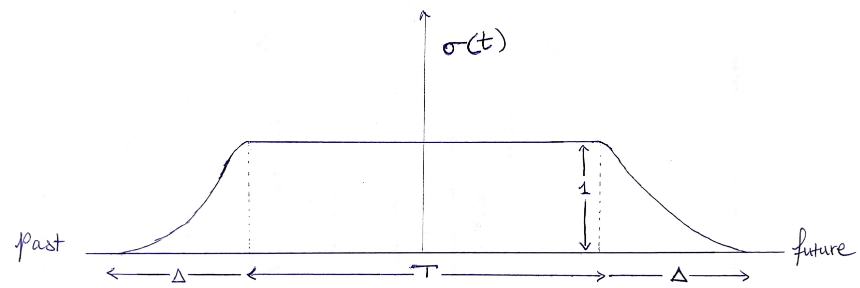
[1]To see this, consider some $\delta\sigma$ such that

$$\frac{1}{|T(E_m - E_l)|} \ll \delta\sigma \ll 1$$

and show that the change

$$\frac{\partial}{\partial c_{ln}} \frac{\delta c_{mn}}{\delta \sigma} \ll 1$$

for $m \neq l$



The adiabatic function $\sigma(t)$. We are dealing with the limit $T, \Delta, T/\Delta \to \infty$

Replacing the perturbed, time independent Hamiltonian with the following

$$H(t) = H_0 + \sigma(t)H_1$$

we see that in the far past and future times, the perturbation is turned off but for all physical, finite times, the Hamiltonian remains the same.

In this section it will be more convenient to work with the double-argument compensator operator

$$V(t_1, t_0) \equiv \mathscr{T}\Big\{\exp\Big[-i\int_{t_0}^{t_1} H_1^0(t)\sigma(t)\,dt\Big]\Big\}$$

Now, using the adiabatic theorem, we know that the state $|E\rangle$ corresponds to some unperturbed state $|E_0\rangle$ in the far past times. Therefore, the correlation function becomes

$$\langle O_1(t_1)\cdots O_n(t_n)\rangle_{|E\rangle\langle E|} = \langle E_0|\, V(\text{past}, t_1)O_1^0(t_1)V(t_1,t_2)O_2^0(t_2)\cdots O_n^0(t_n)V(t_n,\text{past})\,|E_0\rangle$$

The adiabatic theorem also assures us that in the adiabatic limit

$$V(\text{future}, \text{past})\,|E_0\rangle = e^{i\alpha}\,|E_0\rangle$$

and therefore

$$\langle O_1(t_1)\cdots O_n(t_n)\rangle_{|E\rangle\langle E|} = \frac{\langle E_0|\, V(\text{past}, t_1)O_1^0(t_1)V(t_1,t_2)O_2^0(t_2)\cdots O_n^0(t_n)V(t_n,\text{future})\,|E_0\rangle}{\langle E_0|\, V(\text{future},\text{past})\,|E_0\rangle}$$

The operator $S \equiv V(\text{future}, \text{past})$ is called the Scattering matrix or simply the S-matrix for reasons that become clear later on. When computing the S-matrix, we may forget about the small proportions of time duration when the perturbation Hamiltonian is not fully turned on since their contribution to each term in the Dyson formula is infinitesimal.

If, without loss of generality, we assume that $t_1 > t_2 > \cdots > t_n$ are time ordered, then the following formula follows as the final result for this section.

$$\boxed{\langle O_1(t_1)\cdots O_n(t_n)\rangle_{|E\rangle\langle E|} = \frac{\Big\langle E_0\Big|\mathscr{T}\big\{O_1^0(t_1)\cdots O_n^0(t_n)S\big\}\Big|E_0\Big\rangle}{\langle E_0|\, S\,|E_0\rangle}}$$

5.5 Exercises

1. For the perturbed harmonic oscillator

$$H_0 = \frac{p^2}{2} + \frac{x^2}{2}; \quad H_1 = -\lambda x$$

show that the perturbed ground state is centered at

$$\langle x \rangle_g = \lambda$$

Chapter 6

Scattering Theory

In this chapter, we develop the theoretical tools that describe scattering amplitudes, cross sections, and other practical quantities concerning scattering experiments. The Hilbert space corresponds to that of a non relativistic particle living on a Galilean space-time; namely the space of square-integrable wave functions on $\mathbb{E}^n$.

6.1 Gaussian Wave Packets

In this section, we focus on *single-particle* states that best describe a freely moving particle; i.e. Gaussian wave packets. In Fourier space

$$\tilde{\psi}(\mathbf{k}) = A\exp\big[-(\mathbf{k}-\bar{\mathbf{k}})^T\Sigma(\mathbf{k}-\bar{\mathbf{k}}) - i\bar{\mathbf{x}}^T\mathbf{k}\big]$$

We will generally ignore the normalisation constant A in this chapter. The three parameters $(\bar{\mathbf{x}}, \bar{\mathbf{k}}, \Sigma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{C}^{n\times n}$ are defined in a way so that

$$\langle x_i\rangle = \bar{x}_i, \quad \langle p_i\rangle = \bar{k}_i, \quad \mathrm{Cov}(p_i,p_j) = \frac{1}{4}\big(\mathrm{Re}[\Sigma]\big)^{-1}_{ij}, \quad \mathrm{Cov}(x_i,x_j) = \Big(\mathrm{Re}\left[\Sigma^{-1}\right]\Big)^{-1}_{ij}$$

Under the free Hamiltonian $H = p^2/2m$, the wave function remains Gaussian and the parameters evolve as

$$\Sigma(t) = \Sigma(0) + \frac{it}{2m}\mathbb{I}, \quad \bar{\mathbf{k}}(t) = \bar{\mathbf{k}}(0), \quad \bar{\mathbf{x}}(t) = \bar{\mathbf{x}}(0) + \frac{t\bar{\mathbf{k}}}{m}$$

6.2 Spherical Schroedinger Equation

The time-independent Schroedinger equation is

$$\Big(-\frac{\nabla^2}{2m} + V(\mathbf{x})\Big)\psi(\mathbf{x}) = E\psi(\mathbf{x})$$

For a spherically symmetric potential, it is best to work in the spherical coordinates (x,Ω), then the equation is solved as

$$\psi(\mathbf{x}) = Y_{lm}(\Omega)\frac{u(x)}{x^{(n-1)/2}}$$

where

$$(\nabla_\Omega^2 + \lambda_l^2) Y_{lm}(\Omega) = 0; \quad \int d\Omega\, Y_{lm}^*(\Omega) Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$$

and

$$\frac{-1}{2m}\frac{d^2u}{dx^2} + \Big[V(x) + \frac{\lambda_l^2 + (n-1)(n-3)/4}{2mx^2}\Big] u(x) = Eu(x); \quad u(0) = 0; \quad \int |u(x)|^2 dx = 1$$

The main purpose of introducing the spherical equations is to write down a plane wave $\psi = e^{ikx\cos\theta}$ - where θ is the axis angle i.e. $x^1 = x\cos\theta$ -in terms of the spherical *partial waves*. Here the relevant $Y_{\lambda\mu}(\Omega)$ are the generalised Legendre polynomials

$$Y_{lm} \to Q_{\ell n}(\cos\theta)$$

. Here $Q_{\ell n}$ are determined via the equations

$$(1-y^2)\frac{d^2Q}{dy^2} - (n-1)y\frac{dQ}{dy} + \Lambda_{\ell n}^2 Q = 0$$

$$\int_{-1}^{+1} dy\, w_n(y) Q_{\ell n}(y) Q_{\ell' n}(y) = \delta_{\ell\ell'}$$

$$w_n(y) = \frac{2\pi^{(n-1)/2}}{\Gamma\big(\frac{n-1}{2}\big)}(1-y^2)^{(n-3)/2}$$

The first two polynomials are easy to compute

$$Q_{0n}(x) = \sqrt{\frac{\Gamma(n/2)}{2\pi^{n/2}}}; \quad \Lambda_{0n} = 0$$

$$Q_{1n}(x) = \sqrt{\frac{\Gamma(n/2+1)}{\pi^{n/2}}}x; \quad \Lambda_1 = \sqrt{-1}$$

$$\vdots$$

In general it is possible to show that $\Lambda_{\ell n} = \ell(\ell + n - 2)$ (Cf. Exercises).

Since the plane wave does not blow up in the origin, we have the following expansion

$$e^{ikx\cos\theta} = \sum_{\ell=0}^{\infty} A_\ell \frac{Q_{\ell n}(\cos\theta)}{(kx)^{n/2-1}} J_{\ell+\frac{n}{2}-1}(kx)$$

6.3 Stationary Unbounded States

Now let us add a localized potential term to our Hamiltonian

$$H = \frac{p^2}{2m} + V(x), \qquad \lim_{||\mathbf{x}||\to\infty} V(\mathbf{x}) = 0$$

Apart from the $E < 0$ part of its spectrum, where the wave functions decay exponentially for large $\mathbf{x}$, we may also look for stationary states with $E > 0$. These states will not be properly normalisable, however a superposition of them may be normalised properly. For a wave vector $\mathbf{k}$, define

$$\psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}.\mathbf{x}} + f(\Omega)\frac{e^{ikx}}{x^{(n-1)/2}}\big(1 + o(1)\big)$$

where the decaying $o(1)$ term denotes a decaying function of r.

As proved in the exercises, the probability current vector

$$\mathbf{J} \equiv \frac{1}{m} \operatorname{Im}\left[\psi^* \nabla \psi\right]$$

satisfies the continuity equation.

$$\frac{\partial |\psi|^2}{\partial t} + \nabla.\mathbf{J} = 0$$

An evaluation of this current at large r yields a connection between $f(\Omega)$ and the differential cross section as

$$\boxed{D(\Omega) = |f(\Omega)|^2}$$

6.4 Exercises

1. Starting from the separation of variables

$$\psi(\mathbf{x}) = Y_{\lambda\mu}(\Omega) \frac{u(x)}{x^{(n-1)/2}}$$

show that the radial Schroedinger equation is equivalent to a 1D problem with

$$V_{eff}(x) = V(x) + \frac{\lambda^2 + (n-1)(n-3)/4}{2mx^2}$$

Chapter 7

Path Integral Quantisation

In this chapter, we limit ourselves to dynamical systems with a Lagrangian in the form

$$L = \frac{1}{2}\dot{q}_i\dot{q}_j M_{ij}(t) - V(t) + A_i(t)\dot{q}_i$$

and introduce an equivalent quantisation formalism to find the expected values of different observables. This approach will specifically be useful for proving certain theorems in quantum field theory.

7.1 The Fundamental Formula

We know that the canonical quantisation of the Lagrangian

$$L = \frac{1}{2}\dot{q}_i\dot{q}_j M_{ij}(t) - V(t) + A_i(t)\dot{q}_i$$

leads to Schroedinger's wave equation

$$i\frac{\partial}{\partial t}\psi(q,t) = H\psi(q,t)$$

with

$$H = -\frac{1}{2}M^{-1}_{ij}\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j} + iM^{-1}_{ij}A_i\frac{\partial}{\partial q_j} + V.$$

However, the same dynamical equation may be derived using the path integral formalism. In this formalism, the probability amplitude for a transition from q_1 at some time t_1 to q_2 at t_2 is found by *summing* over all *paths* $q_i(t)$ that have proper starting and end points.

$$\text{Amp.}\Big[(q_1,t_1) \longrightarrow (q_2,t_2)\Big] \propto \sum_{\text{paths}} e^{iS[\text{path}]}$$

The constant of proportionality depends on the resolution of the set of paths we are summing over. To fix this constant, let us focus on a very short time interval $[t, t+\delta t]$. In this regime, we may approximate any path with a straight line and write

$$\psi(q,t+\delta t) \propto \int dr\,\psi(q-r,t)\exp\Big[i\Big(r_i r_j\frac{M_{ij}}{2\delta t} + r_i A_i - \delta t V\Big)\Big]$$

$$= \int dr\,\psi(q-r,t)\exp\Big[i\Big((r_i + \delta t M^{-1}_{ik}A_k)(r_j + \delta t M^{-1}_{jl}A_l)\frac{M_{ij}}{2\delta t} - \delta t\big(V + 1/2 A_i A_j M^{-1}_{ij}\big)\Big)\Big]$$

$$\propto \psi(q,t) - i\delta t\Big[\big(V + \frac{1}{2}A_iA_jM^{-1}_{ij}\big) + iM^{-1}_{ij}A_i\frac{\partial\psi(q,t)}{\partial q_j} - \frac{1}{2}M^{-1}_{ij}\frac{\partial^2\psi(q,t)}{\partial q_i\partial q_j}\Big] + o(\delta t)$$

This shows that the only consistent dynamics that we may get from the path integral formalism, is that of Schoroedinger's equation.[1] We invent a special notation for this specific choice of the constant of proportionality and write

$$\boxed{\text{Amp.}\Big[(q_1,t_1)\longrightarrow(q_2,t_2)\Big] = \int_{(q_1,t_1)}^{(q_2,t_2)} Dq\, e^{iS[q]}}$$

Comparing this with the canonical quantisation formalism we get

$$\Big\langle q_2\Big|\mathscr{T}\Big\{\exp\Big[-i\int_{t_1}^{t_2} dt\, H(t)\Big]\Big\}\Big|q_1\Big\rangle = \int_{(q_1,t_1)}^{(q_2,t_2)} Dq\, e^{iS[q]}$$

In general, we may define

$$\boxed{\Big\langle \psi_2\Big|\mathscr{T}\Big\{\exp\Big[-i\int_{t_1}^{t_2} dt\, H(t)\Big]\Big\}\Big|\psi_1\Big\rangle = \int_{(\psi_1,t_1)}^{(\psi_2,t_2)} Dq\, e^{iS[q]} = \int dq_1dq_2\,\psi_2^*(q_2)\psi_1(q_1)\int_{(q_1,t_1)}^{(q_2,t_2)} Dq\, e^{iS[q]}}$$

7.2 The Euclidean Time

For time independent Hamiltonians, it is more often than not that we are interested in thermal states

$$\rho(\beta) \equiv \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$$

Specially the ground state corresponding to $\beta = \infty$ is of great interest in quantum field theories. The operator $e^{-\beta H}$ satisfies the Euclidean Schroedinger equation

$$\frac{d}{d\beta}e^{-\beta H} = -He^{-\beta H}$$

This is similar to Schroedinger's equation if we define the Euclidean time as $t \equiv -i\beta$. In terms of β, the action becomes

$$idS = -\frac{1}{2}M_{ij}(-i\beta)\frac{dq_i}{d\beta}\frac{dq_j}{d\beta} - V(-i\beta)d\beta + iA_i(-i\beta)dq_i$$

here, the expressions depending on $-i\beta$ should be interpreted as the analytic extension of these functions from the real axis to the imaginary axis. From now on, we focus on the special case where the Hamiltonian is time-independent and therefore drop the time/temperature arguments.

7.3 Ground State Expectation Values

The path integral formalism may not be considered as a complete quantisation formalism unless we provide a path integral formula for correlation functions. In general we are interested in

$$\langle\Omega|\,\mathscr{T}\big\{q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\,|\Omega\rangle$$

[1] There is a slight, irrelevant difference here. The potential $V(q)$ now has an extra term that is only a function of time. This won't affect any observation and we need not worry about it.

It is easy to show that the following recipe is consistent with the canonical formalism

$$\boxed{\langle\Omega|\,\mathscr{T}\big\{q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\,|\Omega\rangle = \frac{\int_{\Omega,\text{past}}^{\Omega,\text{future}} Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_n}(t_n)}{\int_{\Omega,\text{past}}^{\Omega,\text{future}} Dq\, e^{iS[q]}}}$$

An advantage of the Euclidean time is that we don't need to specify the initial and final states. From now on, any path integral without limits must be interpreted as a Euclidean time integral over the whole imaginary axis. Moreover, any expectation value $\langle O\rangle$ with no specific states, should be interpreted as the ground state expectation value. Therefore we are allowed to write

$$\boxed{\langle\mathscr{T}\big\{q_{i_1}(-i\beta_1)\cdots q_{i_n}(-i\beta_n)\big\}\rangle = \frac{\int Dq\, e^{iS[q]} q_{i_1}(-i\beta_1)\cdots q_{i_n}(-i\beta_n)}{\int Dq\, e^{iS[q]}}}$$

7.4 Schwinger-Dyson equation and Ward Identities

In classical systems the symmetry of the product rule ($AB = BA$) allows us to freely move differentiation operators inside correlation functions; for example

$$\frac{d}{dt_1}\langle q_{i_1}(t_1)q_{i_2}(t_2)\rangle = \langle\frac{dq_{i_1}}{dt_1}q_{i_2}(t_2)\rangle$$

However, in the quantum case, the time ordered correlation functions may be sensitive specially when two operators have the same time argument. This subtle point, is best observed when the differential operator is such that the classical counter part would correspond to the RHS of the Euler-Lagrange equations of motion which is zero.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

For example and for a simple harmonic oscillator, the classical theory predicts

$$\Big[\big(\frac{d}{dt_1}\big)^2 + \omega^2\Big]\langle X(t_1)X(t_2)\rangle = \Big\langle\Big(\frac{d^2X(t_1)}{dt_1^2} + \omega^2 X(t_1)\Big)X(t_2)\Big\rangle = 0$$

As we will see, the quantum theory provides a different answer.

To find the quantum version of the above equation (in general), let us consider the path integral

$$\int Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_n}(t_n).$$

Using the infinitesimal change of variables

$$q_i' = q_i + \varepsilon f_i(t)$$

we may write

$$\int Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_n}(t_n) = \int Dq'\, e^{iS[q']} q_{i_1}'(t_1)\cdots q_{i_n}'(t_n)$$

$$= \int Dq\, e^{iS[q']} q_{i_1}'(t_1)\cdots q_{i_n}'(t_n)$$

where in the last equality we have used the fact that a *shift* in the integration variables does not change the measure. Expanding up to the first order in ε, this implies

$$\int Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_n}(t_n)\Big[\int dt\, f_i(t)\big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\big)\Big] = i\sum_{r=1}^{n}\int Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_{r-1}}(t_{r-1}) f_{i_r}(t_r) q_{i_{r+1}}(t_{r+1})\cdots q_{i_n}(t_n)$$

By letting $f_i(t) = \delta_{ii_0}\delta(t-t_0)$ for some i_0, t_0 we immediately get to translate these integral equations into correlation function results known as the Schwinger-Dyson equations

$$\boxed{\Big\langle \mathscr{T}\Big\{\big(\frac{\partial L}{\partial q_{i_0}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{i_0}}\big)_{t=t_0} q_{i_1}(t_1)\cdots q_{i_n}(t_n)\Big\}\Big\rangle = i\sum_{r=1}^{n}\delta_{i_r i_0}\delta(t_r - t_0)\Big\langle \mathscr{T}\big\{q_{i_1}(t_1)\cdots q_{i_{r-1}}(t_{r-1}) q_{i_{r+1}}(t_{r+1})\cdots q_{i_n}(t_n)\big\}\Big\rangle}$$

Of course, the operator

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

is zero; what we mean by the left hand side of the Schwinger-Dyson equation is that the differentiation operators are acting on the correlation function. Back to our example of the simple harmonic oscillator, this reads

$$\Big[\big(\frac{d^2}{dt_1^2}\big) + \omega^2\Big]\Big\langle \mathscr{T}\big\{X(t_1)X(t_2)\big\}\Big\rangle = -i\delta(t_1 - t_2)$$

Symmetries of the dynamical system lead to conserved charges in the classical formalism. Since the conservation of a charge X^α is a result of the Euler-Lagrange equations of motion, we may also ask similar questions about the correlation functions involving the observable $\frac{dX^\alpha}{dt}$. Using, the Schwinger-Dyson equation, it is rather straightforward to prove the Ward identities.

$$\boxed{\Big\langle \mathscr{T}\big\{\frac{dX^\alpha}{dt}\big|_{t_0} q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\Big\rangle = -i\sum_{r=1}^{n}\delta(t_0 - t_r)\Big\langle \mathscr{T}\big\{\tau^\alpha_{i_r}(t_0) q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\Big\rangle}$$

The proof for these identities is left as an exercise.

7.5 Exercises

1. Use the Schwinger-Dyson formula and the symmetry properties of the 2-point correlation function for the simple harmoni oscillator to prove

$$\Big\langle \mathscr{T}\big\{X(t_1)X(t_2)\big\}\Big\rangle = \langle X^2(0)\rangle \cos\big(\omega(t_1 - t_2)\big) + \frac{1}{2i\omega}\sin\big(\omega|t|\big)$$

2. Define the ground state generator functional as

$$Z[\sigma] = \int Dq\, e^{iS[q]} e^{-i\int dt\, \sigma_i(t) q_i(t)}$$

show that

$$\Big\langle \mathscr{T}\big\{q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\Big\rangle = \frac{(i\delta)^n}{\delta\sigma_{i_1}(t_1)\cdots\delta\sigma_{i_n}(t_n)}\log(Z[\sigma])\big|_{\sigma=0}$$

3. To prove the Ward identities, using the Schwinger-Dyson equations

   a) Show that
   $$\frac{dX^\alpha}{dt} = -\tau_i^\alpha \Big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \Big)$$

   b) Now use the previous part and the Schwinger-Dyson equations to prove the Ward identity.

Chapter 8

Entropies and Entanglement

In this chapter we review the bare minimum essentials of the quantum information theory. The material is not needed until much further down the road when we discuss holographies and entanglements in quantum field theories.

8.1 Entropies

For a classical probability vector $P(x)$, the Renyi entropies are defined as

$$\mathtt{S}_\alpha \equiv \frac{1}{1-\alpha} \log \Big(\sum_x P^\alpha(x) \Big)$$

For a review of the properties of the Renyi entropies, check out the exercises.

The natural generalisation of these quantities to quantum mechanics is

$$\mathtt{S}_\alpha(\rho) \equiv \frac{1}{1-\alpha} \log \operatorname{Tr} \rho^\alpha$$

Our main motivation for defining these entropies is to have a way of computing the Shannon - Von Neumann entropy. This is done as

$$\begin{aligned} \mathtt{S}_1 = -\langle \log P(x) \rangle &= -\Big\langle \log \Big\{ 1 - \big[1 - P(x) \big] \Big\} \Big\rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \big\langle \big[1 - P(x) \big]^m \big\rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{m} \binom{m}{n} (-1)^n e^{-n \mathtt{S}_{n+1}} \end{aligned}$$

8.2 Exercises

1. By considering the limit $\alpha \to 1$, show that $\mathtt{S}_1$ is the same as the Shannon - Von Neumann entropy.

2. a) One way to measure the correlation between two systems is to compute the covariance for two observables

$$\operatorname{Cov}(O_A, O_B) \equiv \langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle$$

since this quantity depends on the specific choice of the observables O_A and O_B, we define the correlation measure as

$$\mathtt{C}(A:B) \equiv \max_{O_A, O_B} \frac{\mathrm{Cov}(O_A, O_B)}{||O_A||_2 \, ||O_B||_2}$$

Show that the optimal operators are written as $O^* = 2P^* - \mathbb{I}$ where P^* is a projector operator.

b) The total-variation (or trace) distance between two density matrices is defined as

$$\mathrm{TV}(\rho, \sigma) \equiv \frac{1}{2} \mathrm{Tr}(|\rho - \sigma|)$$

Prove

$$\mathtt{C}(A:B) \leq \mathrm{TV}(\rho_{AB}, \rho_A \otimes \rho_B)$$

c) ⚓ The Kullback-Leibler divergence is defined as

$$D_{KL}(\rho||\sigma) \equiv \mathrm{Tr}\left[\rho(\log\rho - \log\sigma)\right]$$

Prove **Pinsker's inequality**

$$\mathrm{TV}(\rho, \sigma) \leq \sqrt{2 D_{KL}(\rho||\sigma)}$$

d) Finally, define the mutual information

$$\mathtt{I}(A:B) \equiv \mathtt{S}_1(A) + \mathtt{S}_1(B) - \mathtt{S}_1(AB)$$

and by proving

$$\mathtt{I}(A:B) = D_{KL}(\rho_{AB}||\rho_A \otimes \rho_B)$$

conclude with

$$\mathtt{C}^2(A:B) \leq 2\mathtt{I}(A:B)$$

Note that this implies the **subadditivity** property for $\mathtt{S}_1$.

$$\mathtt{S}_1(AB) \leq \mathtt{S}_1(A) + \mathtt{S}_1(B)$$

e) 🔔 By proving

$$\mathtt{I}(A:B) \leq D_{KL}(\rho_A \otimes \rho_B||\rho_{AB})$$

show that the result of the previous part is not improved by swapping ρ_{AB} and $\rho_A \otimes \rho_B$.

3. 🔔 Generalise the **Donsker-Varadhan** formula to quantum systems

$$D_{KL}(\rho||\sigma) = \max_{A=A^\dagger} \left\{ \mathrm{Tr}\,\rho A - \log \mathrm{Tr}\,\sigma e^A \right\}$$

4. ⚓ Prove the two versions of **strong subadditivity** property for the Shannon - Von Neumann entropy.

   a)
$$\mathtt{I}(A:B) \leq \mathtt{I}(A:BC)$$

   b)
$$\mathtt{S}_1(A) + \mathtt{S}_1(C) \leq \mathtt{S}_1(AB) + \mathtt{S}_1(BC)$$

5. Prove the **Araki-Lieb** inequality
$$|\mathtt{S}_1(A) - \mathtt{S}_1(B)| \leq \mathtt{S}_1(AB)$$
   Hint: First purify the state ρ_{AB} by adding a third system C and then use the strong subadditivity property.

6. a) Show that the Renyi entropies are decreasing function of α by proving
$$\frac{d\mathtt{S}_\alpha(p)}{d\alpha} = -(1-\alpha)^{-2} D_{KL}(q_\alpha || p)$$
   where $q_\alpha(x) = \frac{p^\alpha(x)}{\sum_x p^\alpha(x)}$.

   b) ⚓ Use the previous result to prove the following inequalities as well
$$\frac{d}{d\alpha}(1-\alpha)\mathtt{S}_\alpha \leq 0$$
$$\frac{d}{d\alpha}(1/\alpha - 1)\mathtt{S}_\alpha \leq 0$$

   c) ⚓ Prove
$$\frac{d^2}{d\alpha^2}(1-\alpha)\mathtt{S}_\alpha \geq 0$$

   d) Show that the Renyi entropy itself is not necessarily convex. Hint: Consider a probability vector with a dominant component.

7. For a thermal state
$$\rho(\beta) = \frac{1}{Z(\beta)} e^{-\beta H}$$
   show that the Renyi entropies are given by
$$\mathtt{S}_\alpha\big(\rho(\beta)\big) = \frac{\alpha\beta}{1-\alpha}\big[F(\beta) - F(\alpha\beta)\big]$$
   where $F(\beta) \equiv -\beta^{-1} \operatorname{Tr} e^{-\beta H}$ is the free energy.

8. ⚓ Consider a random, normalized pure state $|\psi\rangle$ in the N - particle Hilbert space $\mathscr{H} = \mathscr{H}_0^N$ and consider the entanglement entropy between the first xN particles and the rest of the system as a random variable

$$A \equiv \mathtt{S}_1\Big(\operatorname{Tr}_{(1-x)N} |\psi\rangle\langle\psi|\Big)$$

Show that in the large N limit, the following concentration of measure occurs.

$$\frac{A}{N \log \dim \mathscr{H}_0} \to \min(x, 1-x)$$

Part III

Quantum Field Theories on Flat Space-Times

Chapter 9

Quantum Klein-Gordon Fields

Our first quantum field theory, will be that corresponding to the free scalar field theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\sum_a \left[\eta^{\mu\nu}(\partial_\mu \phi_a)(\partial_\nu \phi_a) + m^2\phi_a^2\right]$$

This is a second order function of the fields and their time derivatives, therefore we expect to be able to solve the model exactly by finding the normal modes.

The translation invariance of the model, persuades us to use the Fourier transform

$$\tilde{\phi}_a(\mathbf{k}, t) = (2\pi)^{-n/2}\int d\mathbf{x}\, \phi_a(\mathbf{x}, t)e^{-i\mathbf{k}.\mathbf{x}}$$

$$\tilde{\pi}_a(\mathbf{k}, t) = (2\pi)^{-n/2}\int d\mathbf{x}\, \pi_a(\mathbf{x}, t)e^{-i\mathbf{k}.\mathbf{x}}$$

At first, this looks promising and diagonalizes the Hamiltonian as

$$H = \frac{1}{2}\sum_a \int d\mathbf{x} \left[\pi^2 + ||\boldsymbol{\nabla}\phi||_2^2 + m^2\phi^2\right]$$

$$= \frac{1}{2}\sum_a \int d\mathbf{k} \left[\tilde{\pi}_a\tilde{\pi}_a^\dagger + \omega^2(\mathbf{k})\tilde{\phi}_a\tilde{\phi}_a^\dagger\right]$$

with

$$\omega(\mathbf{k}) \equiv +\sqrt{||\mathbf{k}||_2^2 + m^2}$$

However, some peculiarities arise due to the introduction of complex coefficients $e^{\pm i\mathbf{k}.\mathbf{x}}$. First of all the $\tilde{\phi}_a$ and $\tilde{\pi}_a$ are not Hermitian operators; in fact

$$\tilde{\phi}_a^\dagger(\mathbf{k}, t) = \tilde{\phi}_a(-\mathbf{k}, t); \quad \tilde{\pi}_a^\dagger(\mathbf{k}, t) = \tilde{\pi}_a(-\mathbf{k}, t)$$

Second, $\tilde{\phi}_a$, $\tilde{\pi}_a$ are not conjugate operators, rather

$$[\tilde{\phi}_a(\mathbf{k}, t), \tilde{\pi}_b(\mathbf{k}', t)] = (2\pi)^{-n}\int d\mathbf{x}\, d\mathbf{x}'\, e^{-i(\mathbf{k}.\mathbf{x}+\mathbf{k}'.\mathbf{x}')}[\phi_a(\mathbf{x}, t), \pi_b(\mathbf{x}', t)] = i\delta(\mathbf{k}+\mathbf{k}')\delta_{ab}$$

$$[\tilde{\phi}_a(\mathbf{k}, t), \tilde{\phi}_b(\mathbf{k}', t)] = (2\pi)^{-n}\int d\mathbf{x}d\mathbf{x}'\, e^{-i(\mathbf{k}.\mathbf{x}+\mathbf{k}'.\mathbf{x}')}[\phi_a(\mathbf{x}, t), \phi_b(\mathbf{x}', t)] = 0$$

$$[\tilde{\pi}_a(\mathbf{k},t),\tilde{\pi}_b(\mathbf{k}',t)] = (2\pi)^{-n}\int d\mathbf{x}d\mathbf{x}'\,e^{-i(\mathbf{k}.\mathbf{x}+\mathbf{k}'.\mathbf{x}')}[\pi_a(\mathbf{x},t),\pi_b(\mathbf{x}',t)] = 0$$

In other words, $\tilde{\phi}_a$ are conjugate to $\tilde{\pi}_a^\dagger$.

Despite all this, miracle happens when we define the creation and annihilation operators as

$$a_a(\mathbf{k},t) \equiv \omega(\mathbf{k})\tilde{\phi}_a(\mathbf{k},t) + i\tilde{\pi}_a(\mathbf{k},t)$$

Do they satisfy the commutation relations that we desire? Indeed!

$$[a_a(\mathbf{k},t),a_b(\mathbf{k}',t)] = [a_a^\dagger(\mathbf{k},t),a_b^\dagger(\mathbf{k}',t)] = 0; \quad [a_a(\mathbf{k},t),a_b^\dagger(\mathbf{k}',t)] = 2\omega(\mathbf{k})\delta_{ab}\delta(\mathbf{k}-\mathbf{k}')$$

The only problem here seems to be a bad choice of normalization in defining the creation and annihilation operators. Normally, we would want to have $\delta_{ab}\delta_{\mathbf{kk}'}$ as the RHS. To compensate for this abnormal normalisation choice, we use the following rule of thumb: Any product of $2n$ creation/annihilation operators, must be accompanied by n factors of $\frac{d\mathbf{k}}{2\omega(\mathbf{k})}$, whatever factor that remains is the actual, legitimate weight of the product. The reader now understands the reason behind our specific normalisation choice since this is proportional to the covariant measure over the manifold $k_\mu k^\mu = -m^2$, $k^0 > 0$. In fact

$$\int_{k^0>0}\delta(k_\mu k^\mu + m^2)\,dk = \int \frac{d\mathbf{k}}{2\omega(\mathbf{k})}$$

Now let us write down the Hamiltonian. Using

$$\tilde{\phi}_a(\mathbf{k},t) = \frac{a_a(\mathbf{k},t)+a_a^\dagger(-\mathbf{k},t)}{2\omega(\mathbf{k})}; \quad \tilde{\pi}_a(\mathbf{k},t) = \frac{a_a(\mathbf{k},t)-a_a^\dagger(-\mathbf{k},t)}{2i}$$

the Hamiltonian may be re-written as

$$H = \sum_a\int\frac{d\mathbf{k}}{2\omega(\mathbf{k})}\,\omega(\mathbf{k})a_a^\dagger(\mathbf{k},t)a_a(\mathbf{k},t) + \frac{1}{2}\sum_a\int d\mathbf{k}\,\omega(\mathbf{k})\delta(\mathbf{k}-\mathbf{k})$$

The first term is in the shape we love: this is the $\sum_{\mathrm{i}}\omega_i a_i^\dagger a_i$ part of a harmonic oscillator Hamiltonian written in a covariant manner. This shows us that each mode $\mathbf{k}$ has frequency (surprise:) $\omega(\mathbf{k})$. The second term is among the most evil things one could write down; the integrand is infinite, the bounds are infinite, etc. Although we will eventually drop this term as an irrelevant, constant term in the Hamiltonian that only elevates the ground state, let us take a closer look first. Before anything, note that this may be re-written as

$$\frac{1}{2}\sum_a\sum_{\mathbf{k}}\omega(\mathbf{k}) = \frac{1}{2}\sum_{\text{modes}}\omega_{\text{mode}}$$

which is still as infinite as it was, but now looks like something that we must have foreseen all along. On the other hand, if we limit our space to a box of volume V, the horrible δ term becomes an innocent V. This means, whatever remains is the vacuum energy *density*

$$\rho = \frac{1}{2}\sum_a\int d\mathbf{k}\,\omega(\mathbf{k})$$

This still infinite energy density, may be made finite by some next generation theory that puts a limit on how small the wavelength and therefore how large the $\mathbf{k}$ vector could get. (Say by limiting the number of spatial *points*, etc.)

In order to know the Hilbert space better, we start by diagonalising the positive semi definite number operator

$$N \equiv \sum_a N_a; \quad N_a \equiv \int\frac{d\mathbf{k}}{2\omega(\mathbf{k})}\mathcal{N}_a(\mathbf{k},t); \quad \mathcal{N}_a(\mathbf{k},t) \equiv a_a^\dagger(\mathbf{k},t)a_a(\mathbf{k},t)$$

Because of the ladder structure $[N, a_a(\mathbf{k})] = -a_a(\mathbf{k})$, we know that the whole Hilbert space must end in some *vacuum* subspace $\mathscr{V}$ such that

$$\forall \ |\Omega\rangle \in \mathscr{V}, \ \ a \in [K], \ \ \mathbf{k} \quad a_a(\mathbf{k})\,|\Omega\rangle = 0$$

If we define the Fock space $\mathscr{F}$ to be the Hilbert space with the basis

$$|(b_1, \mathbf{k}_1), \cdots, (b_n, \mathbf{k}_n)\rangle \equiv \Big(\prod_{i=1}^{n} \frac{a^\dagger_{b_i}(\mathbf{k}, t)}{\sqrt{2\omega(\mathbf{k}_i)}}\Big)\,|\Omega\rangle$$

for some $|\Omega\rangle$, then we find that the Hilbert space is (nothing more than)

$$\mathscr{H} = \mathscr{F} \otimes \mathscr{V}$$

The energy-momentum four vector is the operator

$$P^\mu \equiv \int d\mathbf{x}\, T^{0\mu} = (H, \mathbf{P})$$

with

$$\mathbf{P} \equiv -\sum_a \int \pi_a \boldsymbol{\nabla} \phi_a \, d\mathbf{x} = \sum_a \int \frac{d\mathbf{k}}{2\omega(\mathbf{k})}\, \mathbf{k} \mathcal{N}_a(\mathbf{k}, t)$$

So far, we have restricted our discussion of operators to a single space-like surface with constant time. The generalisation to different space sheets is straightforward. We begin with the creation/annihilation operators

$$\frac{d}{dt} a_a(\mathbf{k}, t) = i[H, a_a(\mathbf{k}, t)] = -i\omega(\mathbf{k}) a_a(\mathbf{k}, t)$$

This is a great result: the most interesting operators have an equal number of as and $a^\dagger$s and therefore stay the same through time which is equivalent to commuting with the Hamiltonian.

The fields evolve as

$$\phi_a(x^\mu) = (2\pi)^{-n/2} \int_{k^0 > 0} dk\, \delta(k_\alpha k^\alpha + m^2) a_a(k) e^{i\eta_{\alpha\beta} x^\alpha k^\beta} \quad + \quad h.c.$$

Note that this manifestly satisfies the Klein-Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\phi_a = 0$$

Now Imagine Alice performing some local operation on the fields confined to a space-time region A. This may be described using a unitary operator $U = e^{iO_A}$ where O_A is a local observable.

$$O_A = O_A\Big(\phi_a(A), \pi_a(A)\Big)$$

Meanwhile, Bob is measuring some other local observable at space-time region B which is causally disconnected from A.

$$O_B = O_B\Big(\phi_a(B), \pi_a(B)\Big)$$

For our theory to be causal, we need

$$\langle\psi|U^\dagger O_B U|\psi\rangle = \langle\psi|O_B|\psi\rangle$$

which is equivalent to

$$[\phi_a(x), \phi_b(y)] \overset{!}{=} 0; \;\; [\phi_a(x), \pi_b(y)] \overset{!}{=} 0 \quad \forall\ (x-y)^2 > 0$$

In fact, we only need the first condition since it would immediately imply

$$[\phi_a(x), \pi_b(y)] = \Big[\phi_a(x), \frac{\partial}{\partial y^0}\phi_b(y)\Big] = \frac{\partial}{\partial y^0}[\phi_a(x), \phi_b(y)] = 0; \quad \forall\ (x-y)^2 > 0$$

Now for the field-field commutator we first set $y = 0$ then

$$[\phi_a(x), \phi_b(0)] = 2i\delta_{ab}(2\pi)^{-n}\int_{k^0>0} dk\, \delta(k^2+m^2)\sin(k_\mu x^\mu)$$

which clearly vanishes for $x^2 > 0$ and therefore our theory is a causal one.

9.1 The Correlation Functions and the Wick's Theorem

In order to find all the observable expected values for the Klein-Gordon field theory, it suffices to have access to the vacuum correlation functions.

$$\langle\Omega|\prod_i \phi_{a_i}(x_i)|\Omega\rangle$$

Clearly, we only need to consider the correlation functions for a single type of fields, hence we drop the a_i indices. It also costs no generality to consider only the time ordered correlation functions

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\}\,|\Omega\rangle$$

since we have already computed the field commutators and are therefore able to move from any arbitrary order towards the time order.

Our strategy for evaluating the correlation functions is as follows:
• Write down the field operators in terms of creation/annihilation operators.
• Move the annihilation (creation) operators to the right (left) and get extra terms from the canonical commutation relations.
• Simplify using $a|\Omega\rangle = 0$ to get the final result.

This strategy motivates us to define the *normal* ordering for a product of operators. In a chain of creation/annihilation operators, the normal ordering moves all the annihilation operators to the right and the creation operators to the left. For example

$$\mathscr{N}\big\{a_1 a_2^\dagger a_3 a_4 a_5^\dagger\big\} = a_2^\dagger a_5^\dagger a_1 a_3 a_4$$

To carry out the first step, we define

$$\phi^+(x) \equiv (2\pi)^{-n/2}\int_{k^0>0} dk\, \delta(k^2+m^2)e^{i\eta_{\alpha\beta}k^\alpha x^\beta}a(k); \quad \phi^-(x) = \phi^{+\dagger}(x)$$

Therefore a time ordered product of field operators becomes

$$\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\} = (\phi^+(x_1)+\phi^-(x_1))\cdots(\phi^+(x_n)+\phi^-(x_n))$$

Note that without loss of generality, we have assumed that the labellings are consistent with the time ordering.

The next step would be to move creation operators to the left. While doing this, we would get terms in the form

$$D_F(x-y) \equiv \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 \geq y^0 \\ [\phi^+(y), \phi^-(x)] & y^0 \geq x^0 \end{cases}$$

The space-time function $D_F(x)$ is called the Feynman propagator. We will deal with it at the end of this section.

As an example we may write

$$\mathscr{T}\big\{\phi(x)\phi(y)\big\} = \mathscr{N}\big\{\phi(x)\phi(y)\big\} + D_F(x-y)$$

$$\mathscr{T}\big\{\phi(x)\phi(y)\phi(z)\big\} = \mathscr{N}\big\{\phi(x)\phi(y)\phi(z)\big\} + \mathscr{N}\big\{\phi(x)\big\}D_F(y-z) + \mathscr{N}\big\{\phi(y)\big\}D_F(x-z) + \mathscr{N}\big\{\phi(z)\big\}D_F(x-y)$$

In general, one can see that Wick's theorem holds[1]

$$\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\} = \sum_{\substack{\text{all possible}\\ \text{contractions}}} \Big(\prod_{i\sqcap j} D_F(x_i - x_j)\Big)\mathscr{N}\big\{\text{all uncontracted operators}\big\}$$

Here, by contraction, I mean choosing a pair of field operators and removing them from the operator product chain. You can contract no pairs of operators or contract all the operators (in case there is an even number of them) to get a null product chain. Whatever pair of operators that you contract (a contracted pair is here denoted by $i \sqcap j$) results in a Feynman propagator factor.

Finally, in the last step, when evaluating the expected value in the vacuum state, the normal ordering rids us of all terms with remaining, uncontracted field operators and we get

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\}\,|\Omega\rangle = \sum_{\substack{\text{full}\\ \text{contractions}}} \prod_{i\sqcap j} D_F(x_i - x_j)$$

As a last example

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_4)\big\}\,|\Omega\rangle = D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3)$$

Now let us focus on the Feynman propagator; by definition this is

$$D_F(t,\mathbf{x}) = (2\pi)^{-n}\int \frac{d\mathbf{k}}{2\omega(\mathbf{k})}\, e^{i\,\mathrm{sgn}(t)\big[\mathbf{k}.\mathbf{x}-\omega(\mathbf{k})t\big]}$$

$$= (2\pi)^{-n}\int \frac{d\mathbf{k}}{2\omega(\mathbf{k})}\, \cos(\mathbf{k}.\mathbf{x})e^{-i\omega(\mathbf{k})|t|}$$

In appendix A, we have further discussed the space-time values of the Feynman propagator but for now we are more interested in its Fourier transform.

[1]Here, I assume $\mathscr{N}\{\} = 1$. You could say this is an exception in the definition of time and normal ordering operations, since one could write

$$\mathscr{N}\{1\} = \mathscr{N}\{[a, a^\dagger]\} = 0$$

In fact you shouldn't take $\mathscr{N}$ and $\mathscr{T}$ too seriously; they are only here to make our equations look better ie. shorter.

Using theorems of complex integration it is straightforward to prove the following useful identity

$$-i\pi e^{-i|\lambda|} = \lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty} dz\, \frac{e^{-i\lambda z}}{z^2 - 1 + i\varepsilon}$$

which helps us write

$$\begin{aligned} D_F(t,\mathbf{x}) &= (2\pi)^{-(n+1)} i \int \frac{d\mathbf{k}}{\omega(\mathbf{k})}\, e^{i\mathbf{k}.\mathbf{x}} \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} dz\, \frac{e^{-i\omega(\mathbf{k})tz}}{z^2 - 1 + i\varepsilon} \\ &= (2\pi)^{-(n+1)} \lim_{\varepsilon\to 0}\int dk\, e^{ik_\mu x^\mu}\, \frac{i}{k^2 + m^2 - i\varepsilon} \end{aligned}$$

We encode on this in the symbolic equation

$$\tilde{D}_F(k_\mu) = \frac{i(2\pi)^{-(n+1)/2}}{k^2 + m^2 - i\varepsilon}$$

Exercises

1. The Klein-Gordon Lagrangian for m independent fields $a = 1, 2, \cdots, m$ is invariant under the $SO(m)$ transformations

$$\phi_a \to \phi_a + \varepsilon A_{ab}\phi_b; \quad A_{\{ab\}} = 0$$

These are called the internal symmetries of the field theory. Find the corresponding $\binom{m}{2}$ conserved currents. $\big($*Ans.* $j^\mu(A) = A_{ab}\phi_a\partial^\mu\phi_b$ $\big)$

2. Repeat what we did for the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\left[m^2 A_\alpha A_\beta + \eta^{\mu\nu}(\partial_\mu A_\alpha)(\partial_\nu A_\beta)\right]$$

and find *all* the conserved currents. Compare your results with the previous exercise.

3. ⚓ Repeat the results of this chapter for the Lagrangian

$$\mathcal{L} = -\frac{1}{2}K^{ab}\eta^{\mu\nu}\partial_\mu\phi_a\partial_\nu\phi_b - \frac{1}{2}m^2\phi_a\phi_a$$

where K^{ab} is a constant, positive definite matrix.

4. Let $\mathbf{X}$ be a vector of N centerred random variables (that is $\langle X_i\rangle = 0$). Assume that the covariance matrix is given as

$$C_{ij} = \langle X_i X_j\rangle$$

If the higher order moments satisfy

$$\langle X_{i_1}\cdots X_{i_n}\rangle = \sum_{\substack{\text{full}\\ \text{contractions}\\ \text{of } i_1\cdots i_n}} \prod_{i\sqcap j} C_{ij}$$

then show that the vector $\mathbf{X}$ is normally distributed.

Hint: First, show that without loss of generality, it is possible to take $C_{ij} = \delta_{ij}$ and then show that for the corresponding, *independent* normal distribution, the conditions are satisfied.

Chapter 10

The $\lambda\phi^4$ Theory

We now consider our first interacting quantum field theory called the $\lambda\phi^4$ - theory. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

Assuming $\lambda \ll 1$, it sounds reasonable to use the perturbation theory that we developed in previous chapters. In this regard, the interaction Hamiltonian is

$$H_1 = \frac{\lambda}{4!}\int d\mathbf{x}\,\phi^4(\mathbf{x})$$

Although we are going to use the interaction picture operators for the rest of this chapter (and perhaps more), we drop the superscript: $O^0 \to O$.

Like the $free$ Klein-Gordon theory, we are interested in evaluating the N - point correlation functions.

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\}\,|\Omega\rangle$$

But we already know that this is written as

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\}\,|\Omega\rangle = \frac{\langle\Omega_0|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)S\big\}\,|\Omega_0\rangle}{\langle\Omega_0|\,S\,|\Omega_0\rangle}$$

10.1 The Diagrammatic Notation

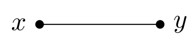
Let us consider the 2-point correlation function. Using Wick's theorem, we may expand this in λ as

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x)\phi(y)\big\}\,|\Omega\rangle = \frac{D_F(x-y) - \frac{i\lambda}{4!}\int dz\,\big[3D_F(x-y)D_F^2(z-z) + 12D_F(x-z)D_F(y-z)D_F(z-z)\big] + \cdots}{1 - 3\frac{i\lambda}{4!}\int dz\,D_F^2(z-z) + \cdots}$$

Our current notation is now clearly getting repetitive, it is definitely easier to describe each term in the expansion with words than to write it down explicitly and since a picture is worth a thousand words we are tempted to use a diagrammatic notation. First of all, to denote a field operator on space-time position x, we simply use a node.

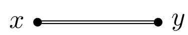
$$x \bullet = \phi(x)$$

The Feynman propagators connect space-time field operators and are therefore denoted by lines such as

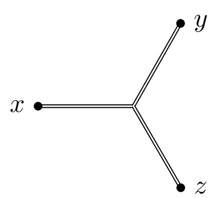


$$= D_F(x-y) = \langle\Omega_0|\, \mathscr{T}\big\{\phi(x)\phi(y)\big\}\, |\Omega_0\rangle$$

The perturbed correlation functions are denoted by a double line connection, for example

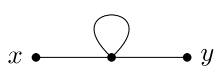


$$= \langle\Omega|\, \mathscr{T}\big\{\phi(x)\phi(y)\big\}\, |\Omega\rangle = \frac{\langle\Omega_0|\, \mathscr{T}\big\{\phi(x)\phi(y)S\big\}\, |\Omega_0\rangle}{\langle\Omega_0|\, S\, |\Omega_0\rangle}$$

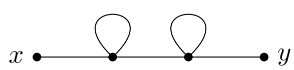


$$= \langle\Omega|\, \mathscr{T}\big\{\phi(x)\phi(y)\phi(z)\big\}\, |\Omega\rangle = \frac{\langle\Omega_0|\, \mathscr{T}\big\{\phi(x)\phi(y)\phi(z)S\big\}\, |\Omega_0\rangle}{\langle\Omega_0|\, S\, |\Omega_0\rangle}$$

Finally, we know that every power of $-i\frac{\lambda}{4!}$ adds 4 field operators at some dummy space-time position z which is to be integrated over. There are also combinatorial factors such as 3 or 12 that correspond to the number of ways that such a contraction of field operators appears. Since the $-i\frac{\lambda}{4!}$, the space-time integral and the combinatorial factor are all determined by the shape of the diagram, we simply do not write them. Therefore and for example we have



$$= \frac{1}{1!}(12)\frac{-i\lambda}{4!}\int dz\, D_F(x-z)D_F(y-z)D_F(z-z)$$



$$= \frac{1}{2!}(288)\Big(\frac{-i\lambda}{4!}\Big)^2\int dz_1 \mathrm{d}z_2\, D_F(x-z_1)D_F(z_1-z_1)D_F(z_1-z_2)D_F(z_2-z_2)D_F(z_2-y)$$

We will denote the combinatorial factor for a diagram D with $\#(D)$. Now consider the diagram D_1D_2 consisting of two disconnected parts D_1 and D_2 with n_1 and n_2 interaction vertices respectively. We have

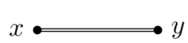
$$\frac{\#(D_1D_2)}{(n_1+n_2)!} = \binom{n_1+n_2}{n_1}\frac{\#(D_1)\#(D_2)}{(n_1+n_2)!} = \frac{\#(D_1)\#(D_2)}{n_1!n_2!}$$

This means that the final, numerical value of a diagram is the product of the numerical value of its connected components.

$$\mathrm{val}(D_1D_2) = \mathrm{val}(D_1)\,\mathrm{val}(D_2)$$

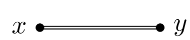
10.2 The 2-Point Correlation Function

Armed with our new, efficient notation it is now straight forward to write down an expression for the perturbed 2-point correlation function

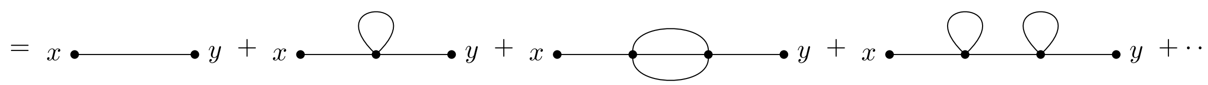


$$= \frac{\sum \text{all possible diagrams}}{1+\cdots}$$

It is possible to factor out the exact same term as in the denominator from the numerator, leaving only the connected diagrams

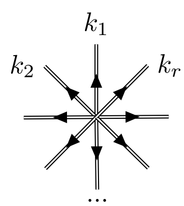


$= \sum$ all connected diagrams

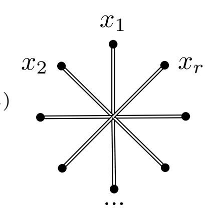


10.3 Fourier Space Diagrams

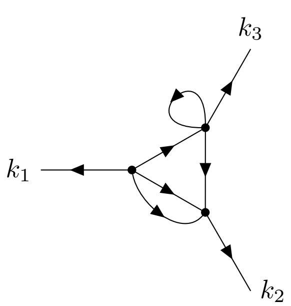
Just like the unperturbed case, we would sometimes prefer working with the Fourier transforms



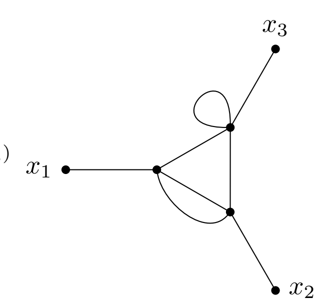
$$\equiv (2\pi)^{-r(n+1)/2} \int dx_1 \cdots dx_r \, e^{-i(k_1.x_1 + \cdots k_r.x_r)}$$



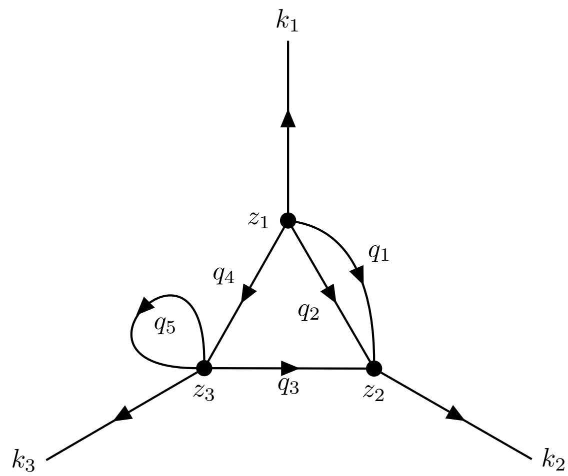
Due to the linearity of Fourier transforms, each diagram is mapped to a corresponding diagram in the Fourier domain. The $\frac{-i\lambda}{4!}$ factors, the exponential factorials and the combinatorial factor $\#(D)$ also remain the same. For example



$$\equiv (2\pi)^{-3(n+1)/2} \int dx_1 dx_2 dx_3 \, e^{-i(k_1.x_1 + k_2.x_2 + k_3.x_3)}$$



It remains to evaluate the *nuclear* value of each diagram. We do this for the example above and generalise the rules based on our observations. The strategy is to replace the propagators with their Fourier transforms. If we label the dummy indices as below



The integral we are dealing with, stripped of the extra factors is written as

$$(2\pi)^{-3(n+1)/2}\int dx_1dx_2dx_3dz_1dz_2dz_3\, e^{-i(k_1.x_1+k_2.x_2+k_3.x_3)}(2\pi)^{-8(n+1)/2}\int dq_1dq_2dq_3dq_4dq_5dk_1'dk_2'dk_3'$$
$$e^{iq_1.(z_2-z_1)}e^{iq_2.(z_2-z_1)}e^{iq_3.(z_2-z_3)}e^{iq_4.(z_3-z_1)}e^{iq_5.(z_3-z_3)}e^{ik_1'.(x_1-z_1)}e^{ik_2'.(x_2-z_2)}e^{ik_3'.(x_3-z_3)}$$
$$\tilde{D}_F(k_1')\tilde{D}_F(k_2')\tilde{D}_F(k_3')\tilde{D}_F(q_1)\tilde{D}_F(q_2)\tilde{D}_F(q_3)\tilde{D}_F(q_4)\tilde{D}_F(q_5)$$
$$=(2\pi)^{3(n+1)/2}(2\pi)^{-8(n+1)/2}\tilde{D}_F(k_1)\tilde{D}_F(k_2)\tilde{D}_F(k_3)\int dz_1dz_2dz_3dq_1dq_2dq_3dq_4dq_5$$
$$e^{iq_1.(z_2-z_1)}e^{iq_2.(z_2-z_1)}e^{iq_3.(z_2-z_3)}e^{iq_4.(z_3-z_1)}e^{iq_5.(z_3-z_3)}e^{-ik_1.z_1}e^{-ik_2.z_2}e^{-ik_3.z_3}\tilde{D}_F(q_1)\tilde{D}_F(q_2)\tilde{D}_F(q_3)\tilde{D}_F(q_4)\tilde{D}_F(q_5)$$
$$=(2\pi)^{3(n+1)/2}(2\pi)^{-8(n+1)/2}(2\pi)^{3(n+1)}\delta(k_1+k_2+k_3)\tilde{D}_F(k_1)\tilde{D}_F(k_2)\tilde{D}_F(k_3)\int dq_1dq_2dq_5$$
$$\tilde{D}_F(q_1)\tilde{D}_F(q_1)\tilde{D}_F(k_2-q_1-q_2)\tilde{D}_F(-k_1-q_1-q_2)\tilde{D}_F(q_5)$$

The same procedure applies to any diagram with translation invariant and even propagators. Since the final term is always proportional to the $\delta(k_1+\cdots+k_r)$, we will label the external k vectors in a manner that they add up to zero and then focus only on the coefficient of the delta function. In general, to evaluate this coefficient, we need to label internal wave vectors in a way that respects conservation of energy-momentum at each vertex, replace each propagator with its Fourier transform and then integrate over the undetermined momenta. The overall $(2\pi)^{(n+1)/2}$ power is given by $2\times\mathcal{V}_{int.}-\mathcal{E}_{int.}$; where $\mathcal{V}_{int.}$ is the number of internal interaction vertices and $\mathcal{E}_{int.}$ is the number of internal propagator edges, sometimes called *virtual particles.*

10.4 The Vacuum Bubbles

10.5 Exercises

1. ⚓ Consider a single simple harmonic oscillator.

$$H=\frac{p^2}{2}+\frac{x^2}{2}$$

a) Calculate the two point correlation function for the ground state

$$D(t) \equiv \langle g_0 | \mathscr{T}\{x(0)x(t)\} | g_0 \rangle$$

b) For the perturbed Hamiltonian

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\beta}{3!}x^3 + \frac{\lambda}{4!}x^4$$

use the diagrammatic method to calculate the width of the perturbed ground state, $\langle x^2 \rangle_g$ up to second order in β and λ

c) Consider a finite number of independent harmonic oscillators and then consider the perturbation

$$H = \sum_a \left(\frac{p_a^2}{2} + \frac{x_a^2}{2} \right) - \sum_a \mu_a x_a + \frac{1}{2} \sum_{ab} \alpha_{ab} x_a x_b$$

use the diagrammatic method to find the exact covariance matrix for the perturbed ground state: $\langle x_a x_b \rangle_g$

Chapter 11

From Theory to Experiment

In this chapter we try to translate our previous, abstract calculations to tangible, experimental predictions.

11.1 The LSZ Reduction Formula

Our first step towards computing experimental quantities is to find transition amplitudes between freely moving particle states

$$|i\rangle \equiv |\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_r\rangle$$

and

$$|f\rangle \equiv \left|\mathbf{k}'_1, \mathbf{k}'_2, \cdots, \mathbf{k}'_s\right\rangle$$

after the initial state is exposed to the interacting Hamiltonian dynamics for a long time. Once again, we assume that the interaction Hamiltonian is adiabatically turned on and off through the adiabatic function, $\sigma(t)$, that we introduced before. The amplitude is then given by

$$\frac{\exp\left(i(\text{future} - \text{past})\sum_{j=1}^{s}\omega(\mathbf{k}'_j)\right)}{\langle\Omega_0|\, V(\text{future}, \text{past})\,|\Omega_0\rangle}\langle\Omega_0|\Big(\prod_{j=1}^{s}\frac{a(\text{future}, \mathbf{k}'_j)}{\sqrt{2\omega(\mathbf{k}'_j)}}\Big)V(\text{future}, \text{past})\Big(\prod_{j=1}^{r}\frac{a^\dagger(\text{past}, \mathbf{k}_j)}{\sqrt{2\omega(\mathbf{k}_j)}}\Big)|\Omega_0\rangle$$

where

$$V(b, a) \equiv \mathscr{T}\Big\{\frac{-i\lambda}{4!}\int_a^b dt \int d\mathbf{x}\,\phi^4(t, \mathbf{x})\Big\}.$$

The phase factor on the left is present to rid us from the irrelevant dynamical phases. Now let us define the modified annihilation operators

$$\alpha(t, \mathbf{k}) \equiv e^{i\omega(\mathbf{k})(t-\text{past})}V^\dagger(t, \text{past})a(\text{past}, \mathbf{k})V(t, \text{past}).$$

Using this, the transition amplitude is written as

$$\mathcal{A}\Big[\,|i\rangle \to |f\rangle\,\Big] = \frac{\langle\Omega_0|\Big(\prod_{j=1}^{s}\frac{\alpha(\text{future}, \mathrm{k}'_j)}{\sqrt{2\omega(\mathbf{k}'_j)}}\Big)\Big(\prod_{j=1}^{r}\frac{\alpha^\dagger(\text{past}, \mathbf{k}_j)}{\sqrt{2\omega(\mathrm{k}_j)}}\Big)|\Omega_0\rangle}{\langle\Omega_0|\, S\,|\Omega_0\rangle}$$

At t = past, the αs are the same as the good old annihilation operators, to find their value in the future, we integrate their time derivative

$$\alpha(\text{future}, \mathbf{k}) = \alpha(\text{past}, \mathbf{k}) + \int_{\text{past}}^{\text{future}} dt\,\dot{\alpha}(t, \mathbf{k})$$

$$= \alpha(\text{past}, \mathbf{k}) + i\int dt\, e^{i\omega(\mathbf{k})(t-\text{past})}V^\dagger(t, \text{past})[H_{int.}, a(t, \mathbf{k})]V(t, \text{past})$$

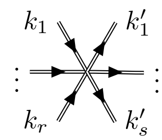
$$
\begin{aligned}
&= \alpha(\text{past}, \mathbf{k}) - i(2\pi)^{-n/2} \int dx \, e^{-ik_\mu x^\mu} \left(\frac{\lambda \phi_H^3}{3!} \right) \\
&= \alpha(0, \mathbf{k}) - i(2\pi)^{-n/2} \int dx \, e^{-ik_\mu x^\mu} (\Box - m^2) \phi_H(x) \\
&= \alpha(0, \mathbf{k}) + i\sqrt{2\pi}(k^2 + m^2) \tilde{\phi}_H(k)
\end{aligned}
$$

where $\phi_H(x) = V^\dagger(t)\phi(x)V(t)$ and the penultimate equality is a direct implication of the equations of motion. Neglecting the case of *unscattered* momenta (ie. $\mathbf{k}_j = \mathbf{k}'_j$), we can unambiguously re-write the transition amplitude as

$$
\mathcal{A}\Big[|i\rangle \to |f\rangle \Big] = \frac{\langle \Omega_0 | \mathscr{T} \Big\{ \Big(\prod_{j=1}^s \frac{\alpha(\text{future}, \mathbf{k}'_j) - \alpha(\text{past}, \mathbf{k}'_j)}{\sqrt{2\omega(\mathbf{k}'_j)}} \Big) \Big(\prod_{j=1}^r \frac{\alpha^\dagger(\text{past}, \mathbf{k}_j) - \alpha^\dagger(\text{future}, \mathbf{k}_j)}{\sqrt{2\omega(\mathbf{k}_j)}} \Big) \Big\} |\Omega_0\rangle}{\langle \Omega_0 | S | \Omega_0 \rangle}
$$

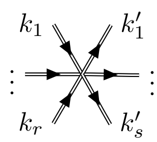
which, given our previous calculations, implies

$$
\boxed{\frac{\langle \mathbf{k}'_1 \cdots \mathbf{k}'_s | S | \mathbf{k}_1 \cdots \mathbf{k}_r \rangle}{\prod_{j=1}^r \Big(\sqrt{\frac{\pi}{\omega(\mathbf{k}_j)}} \Big) \prod_{j=1}^s \Big(\sqrt{\frac{\pi}{\omega(\mathbf{k}'_j)}} \Big)} = i^{r+s} \prod_{j=1}^r (k_j^2 + m^2) \prod_{j=1}^s (k_j'^2 + m^2)}
$$



This is known as the LSZ formula. For translationally invariant theories, the Fourier transforms are proportional to a delta function guaranteeing the conservation of momentum. It will prove convenient to define the matrix element $\mathcal{M}$ as the constant of proportionality

$$
\frac{\langle \mathbf{k}'_1 \cdots \mathbf{k}_s | S | \mathbf{k}_1 \cdots \mathbf{k}_r \rangle}{\prod_{j=1}^r \Big(\sqrt{\frac{\pi}{\omega(\mathbf{k}_j)}} \Big) \prod_{j=1}^s \Big(\sqrt{\frac{\pi}{\omega(\mathbf{k}'_j)}} \Big)} = \mathcal{M} \Big(\quad \Big) \delta(k_1 + \cdots + k_r - k'_1 - \cdots - k'_s)
$$



For the $\lambda\phi^4$ theory, the matrix element corresponding to each Fourier space diagram is given by

$$
\mathcal{M}(D) \equiv (-)^{\mathcal{E}_{ext.}} (2\pi)^{\frac{n+1}{2}(2\mathcal{V} - \mathcal{E})} \int (\text{unconstrained momenta}) \prod_{e \in \mathcal{E}_{int.}} \tilde{D}(e)
$$

11.2 Cross Sections

Now, we are in a situation to compute our first quantity that can be measured conveniently in the lab, namely the two-particle scattering differential cross sections. To do this, consider a two particle state describing a head-on collision with almost definite momenta.

$$
|\psi_A \psi_B\rangle \equiv \int d\mathbf{k}_A d\mathbf{k}_B \, \psi_A(\mathbf{k}_A) \psi_B(\mathbf{k}_B) \, |\mathbf{k}_A \mathbf{k}_B\rangle
$$

to consider a collision with non zero collision parameter $\mathbf{b}$, all we need to do is to multiply the wave function $\psi_B(\mathrm{k}_B)$ by $e^{-i\mathbf{b}.\mathbf{k}_B}$. From all this, we can write the differential cross section for the process

$$
k_A, k_B \longrightarrow k'_1, \cdots, k'_s
$$

as

$$
\frac{d\sigma}{d\mathbf{k}'_1 \cdots d\mathbf{k}'_s} = \int d^{n-1}\mathbf{b} \, \big| \langle \mathbf{k}'_1 \cdots \mathbf{k}'_s | S | \psi_A \psi_B, \mathbf{b} \rangle \big|^2
$$

$$= \int d^{n-1}\mathbf{b} \int d\mathbf{k}_A d\mathbf{k}_B d\mathbf{k}'_A d\mathbf{k}'_B \psi_A(\mathbf{k}_A)\psi_B(\mathbf{k}_B)\psi^*_A(\mathbf{k}'_A)\psi^*_B(\mathbf{k}'_B) e^{i\mathbf{b}.(\mathbf{k}'_B-\mathbf{k}_B)}$$

$$\left\langle \mathbf{k}'_1 \cdots \mathbf{k}'_s \right| S \left| \mathbf{k}_A \mathbf{k}_B \right\rangle \left\langle \mathbf{k}'_A \mathbf{k}'_B \right| S \left| \mathbf{k}'_1 \cdots \mathbf{k}'_s \right\rangle$$

$$= (2\pi)^{n-1} \int d\mathbf{k}_A d\mathbf{k}_B d\mathbf{k}'_A d\mathbf{k}'_B \psi_A(\mathbf{k}_A)\psi_B(\mathbf{k}_B)\psi^*_A(\mathbf{k}'_A)\psi^*_B(\mathbf{k}'_B)\delta_\perp(\mathbf{k}_B - \mathbf{k}'_B)\mathcal{M}\mathcal{M}'^*$$

$$\frac{\pi^s}{\prod_{j=1}^s \omega(\mathbf{k}'_j)} \frac{\pi^2}{\sqrt{\omega(\mathbf{k}_A)\omega(\mathbf{k}_B)\omega(\mathbf{k}'_A)\omega(\mathbf{k}'_B)}} \delta\Big(k_A + k_B - \sum_{j=1}^s k'_j\Big)\delta\Big(k'_A + k'_B - \sum_{j=1}^s k'_j\Big)$$

Where $\mathcal{M}$ and $\mathcal{M}'$ are the relevant matrix elements. Using the delta functions to perform the integrals over $\mathbf{k}'_A$ and $\mathbf{k}'_B$ we get

$$\boxed{\frac{d\sigma}{d\mathbf{k}'_1 \cdots d\mathbf{k}'_s} = (2\pi)^{n-1} \int \frac{d\mathbf{k}_A d\mathbf{k}_B}{|\beta_A - \beta_B|} |\psi_A(\mathbf{k}_A)|^2 |\psi_B(\mathbf{k}_B)|^2 |\mathcal{M}|^2 \delta\Big(k_A + k_B - \sum_{j=1}^s k'_j\Big) \frac{\pi^{2+s}}{\omega(\mathbf{k}_A)\omega(\mathbf{k}_B)\prod_{j=1}^s \omega(\mathbf{k}'_j)}}$$

At this stage, we are almost done in extracting a verifiable result from our theoretic calculations. For example in 3+1 dimensions the formula above may be used to find the two particle to two particle collision cross section in the CM frame as

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{2\pi^6}{E^2_{CM}} |\mathcal{M}|^2}$$

It must be added that the formula above has an extra factor of $4 = 2 \times 2$. This is to convert the Bosonic to Bosonic cross section area to the classical one. Up to the first order in λ, we have only one diagram to evaluate in the matrix element.

$$\mathcal{M}_{(1)} = -i\lambda(2\pi)^{-4}$$

which gives

$$\sigma = \frac{\lambda^2}{32\pi E^2_{CM}}$$

11.3 Decay Rates

The particles in the $\lambda\phi^4$ theory are stable since there are no kinematically allowed decay processes. Therefore, in this section, we use another toy model

$$\mathcal{L} = -\frac{1}{2}\big(\partial\phi\big)^2 - \frac{1}{2}m^2\phi^2 - \frac{Z}{2}\big(\partial\chi\big)^2 - \frac{g}{2}\phi\chi^2$$

Now for an almost still ϕ-particle

$$|\psi\rangle = \int d\mathbf{k}\psi(\mathbf{k}) \left| \mathbf{k}_\phi \right\rangle$$

we want to know the transition probabilities

$$\mathbb{P}\Big[|\psi\rangle \longrightarrow |\mathbf{k}_{1,\chi}, \mathbf{k}_{2,\chi}, \cdots\rangle \Big]$$

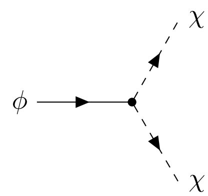
of course such a probability is proportional to the decay rate, Γ, as well as the waiting time: 'future − past'. The full decay rate is therefore given by

$$\frac{1}{\tau} = \Gamma = \lim_{\text{localized } \psi} \sum_{|f\rangle} \frac{|\langle f| S |\psi\rangle|^2}{\text{future} - \text{past}}$$

The first non-zero term for Γ in this theory will be given by

$$\Gamma^{(2)}_{\phi\to\chi+\chi} = \frac{\Omega_{n-1}\pi^2}{4m^2}\left(\frac{m}{2}\right)^{n-3}|\mathcal{M}|^2$$

where $\mathcal{M}$ is the matrix element corresponding to the diagram



The final expression for this quantity is found in the exercises.

The astute reader might have realised by now that higher order diagrams may give infinite results as physical quantities. Alleviating this apparent obstacle in a logically consistent way will be our main task in the next part.

Exercises

1. In $n+1$ space-time dimensions, check that the equations derived in this chapter are dimensionally consistent.

2. Evaluate the first matrix element corresponding to the decay process

$$\phi \to \chi + \chi$$

   and find the first approximation for the half-life of such a particle.

Part IV

Renormalization

Chapter 12

Renormalizing the $\lambda\phi^4$ Theory

While evaluating more complex

Part V

Quantum Electrodynamics

Chapter 13

Free Fermions

Chapter 14

Free Photons

Chapter 15

The QED

Part VI

Conformal Field Theory

Chapter 16

Conformal Transformations

In this chapter we introduce the notion of conformal transformations. This will be essential for our discussion of field theories with corresponding symmetries. A conformal transformation on a manifold is one that does not change the angle between different vectors. From a passive point of view, this is a coordinate transformation $x^\mu \to x'^\mu$ such that

$$g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$$

In this regard, the set of all possible conformal transformations on a manifold, $\mathcal{M}$, forms a group, $\texttt{conf}(\mathcal{M})$, generally known as the conformal group of the manifold.

To find the generators of the conformal group, we seek infinitesimal transformations $x^\mu \to x^\mu + \varepsilon\xi^\mu$ that are conformal. The condition on ξ^μ is called the conformal Killing equation

$$\pounds_\xi g_{\mu\nu} = \nabla_{\{\mu}\xi_{\nu\}} = \kappa g_{\mu\nu}$$

A vector field that satisfies this equation, is called a conformal Killing field and the scalar constant κ is called a conformal Killing factor. It is readily observed that the conformal Killing vector is given by

$$\kappa = \frac{2}{n}\nabla_\mu \xi^\mu$$

where n is the dimension of the manifold.

Our discussion in this chapter will be entirely dedicated to flat manifolds with the constant, flat metric tensor $\eta_{\mu\nu}$. Therefore it is useful to write the alleged conformal Killing field as

$$\xi_\mu = S^{(0)}_\mu + S^{(1)}_{\mu\alpha_1} x^{\alpha_1} + S^{(2)}_{\mu\alpha_1\alpha_2} x^{\alpha_1} x^{\alpha_2} + \cdots$$

$$+\Omega^{(1)}_{\mu\alpha_1} x^{\alpha_1} + \Omega^{(2)}_{\mu\{\alpha_1}\Sigma^{(1)}_{\alpha_2\}} x^{\alpha_1} x^{\alpha_2} + \Omega^{(3)}_{\mu\{\alpha_1}\Sigma^{(2)}_{\alpha_2\alpha_3\}} x^{\alpha_1} x^{\alpha_2} x^{\alpha_3} + \cdots$$

where S^i and Σ^i are completely symmetric tensors and Ω^i are anti-symmetric tensors of rank 2. On the other hand, the second order derivatives of the conformal Killing factor κ are highly restricted by the conformal Killing equation.

$$(n-2)\partial_\alpha\partial_\beta\kappa + g_{\alpha\beta}\partial^\mu\partial_\mu\kappa = g^{\mu\nu}\Big(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta - g_{\mu\alpha}\partial_\nu\partial_\beta - g_{\nu\beta}\partial_\alpha\partial_\mu\Big)\kappa$$

$$= g^{\mu\nu}\Big[\big(\partial_\mu\partial_\nu\partial_\beta - \partial_\nu\partial_\beta\partial_\mu\big)\xi_\alpha + \big(\partial_\mu\partial_\nu\partial_\alpha - \partial_\alpha\partial_\mu\partial_\nu\big)\xi_\beta$$

$$+\big(\partial_\alpha\partial_\beta\partial_\nu - \partial_\nu\partial_\beta\partial_\alpha\big)\xi_\mu + \big(\partial_\alpha\partial_\beta\partial_\mu - \partial_\alpha\partial_\mu\partial_\beta\big)\xi_\nu\Big] = 0$$

Finally, taking a trace, we find the conditions

$$\Box\kappa = 0; \quad (n-2)\partial_\mu\partial_\nu\kappa = 0$$

Clearly, there is a stark difference here between $n = 2$ and $n > 2$ dimensions. We consider each case separately.

16.1 $n > 2$ dimensions

For $n > 2$, we have $\partial_\mu\partial_\nu\kappa = 0$ and therefore

$$\xi_\mu = t_\mu + \omega\eta_{\mu\nu}x^\nu + \Omega_{\mu\nu}x^\nu + \gamma_{\mu\alpha\beta}x^\alpha x^\beta$$

The first three terms correspond to translations, rotations and scale transformations respectively. But what transformation does the tensor γ_{abc} represent? The conformal Killing equation is equivalent to

$$\gamma_{\mu\alpha\beta} + \gamma_{\alpha\mu\beta} = \frac{2}{n}\eta_{\mu\alpha}\gamma^\nu{}_{\sigma\beta}$$

the solution is

$$\gamma_{\mu\alpha\beta} = \eta_{\mu\alpha}\sigma_\beta + \eta_{\mu\beta}\sigma_\alpha - \eta_{\alpha\beta}\sigma_\mu$$

Counting the number of generators, we find that there are $\frac{(n+1)(n+2)}{2}$ independent generators. In fact, it is possible to show that under the commutator operation

$$[A, B]^\mu \equiv B^\nu\partial_\nu A^\mu - A^\nu\partial_\nu B^\mu = \pounds_B A^\mu$$

these generators have the same Lie algebra as the group $SO(p+1, q+1)$ where (p, q) is the signature of the manifold.

16.2 $n = 2$ dimensions

To examine the $n = 2$ case, we start by formally allowing the real coordinates x^μ to take values in the complex plane. Although complex points do not exist on the manifold, we may always analytically the coordinate transformation rules to transformation rules between complex coordinates. As we'll see, this will help us express and classify the conformal transformations in well known terms.

Let us start our discussion with the case of the Euclidean plane; this is the non-compact set of real pairs of numbers (x, y), equipped with the metric

$$ds^2 = dx^2 + dy^2$$

If we define the *independent* complex coordinates

$$z \equiv x + iy; \quad \bar{z} \equiv x - iy$$

then the metric becomes

$$ds^2 = dz\, d\bar{z}.$$

Of course the true manifold points only correspond to coordinates satisfying $\bar{z} = z^*$. Nevertheless, any (extended) coordinate transformation is written as

$$(z, \bar{z}) \to (z', \bar{z}')$$

And therefore the metric transforms as

$$ds^2 = \Big(\frac{\partial z}{\partial z'}dz' + \frac{\partial z}{\partial \bar{z}'}d\bar{z}'\Big)\Big(\frac{\partial \bar{z}}{\partial z'}dz' + \frac{\partial \bar{z}}{\partial \bar{z}'}d\bar{z}'\Big) \overset{!}{=} \Omega^2(z', \bar{z}')dz'\,d\bar{z}'$$

Where the last equality holds, if and only if, the transformation is a conformal one. Clearly, there are only two possibilities here, either this is a holomorphic map, i.e.

$$z' = f(z), \quad \bar{z}' = \bar{f}(\bar{z})$$

or, an anti-holomorphic map, i.e.

$$z' = \bar{f}(\bar{z}), \quad \bar{z}' = f(z)$$

where f and $\bar{f}$ are independent, analytic (otherwise, the partial derivatives would be meaningless) functions. We still need to translate these transformations back into real coordinate systems; there, we need to have both $\bar{z} = z^*$ and $\bar{z}' = z'^*$ to hold. Therefore (dropping the redundant transformation rule for barred coordinates), the two holomorphic and anti-holomorphic cases correspond to

$$z' = f(z); \qquad z' = \Big(f(z)\Big)^*$$

for some analytic function f. We will generally be interested in the connected component of $\texttt{conf}(\mathbb{E}^2)$ that includes the identity, namely the holomorphic transformations.

Using the MacLaurin series for analytic functions, we may introduce the generators L_s that correspond to infinitesimal [1] transformations

$$\big(1 + \varepsilon L_s\big)(z) = z + \varepsilon z^{s+1}, \qquad s \in \mathbb{Z}.$$

Then, it is easy to see that these generators obey the Witt algebra under commutation operations.

$$\text{Witt algebra:} \qquad [L_s, L_r] = (s - r)L_{s+r}$$

Finally, let us add the important remark that although any analytic function is a good local conformal transformation, only the ones with exactly one pole and one zero in $\mathbb{C} \cup \{\infty\}$, namely the Møbius transformations are valid, global conformal transformations.

$$z' = \frac{az + b}{cz + d}; \qquad ad - bc = 1$$

In the exercises, the reader is further familiarized with this *global* subgroup. Note that this subgroup corresponds to translations, rotations, scale transformations and SCTs; just like in $n > 2$ dimensions.

For the 1+1 dimensional Minkowski space time, $\mathbb{E}^{1+1}$ with the metric

$$ds^2 = dx^2 - dt^2,$$

it is more convenient to use the pair of complex coordinates

$$z = x - t; \qquad \bar{z} = x + t.$$

[1] For the moment, we are focusing on a local domain in the complex plane where both $|z|$ and $|1/z|$ are bounded. We will deal with the *global* transformations in a moment.

once again, the metric becomes

$$ds^2 = dz\, d\bar{z}$$

And therefore, a similar piece of reasoning, allows us to write any conformal transformation in one of the following two forms

$$z' = f(z); \qquad \bar{z}' = \bar{f}(\bar{z})$$

$$z' = \bar{f}(\bar{z}); \qquad \bar{z}' = f(z).$$

Here, the actual manifold corresponds to $z, \bar{z} \in \mathbb{R}$ and therefore, a valid transformation corresponds to *real-entire* functions f, $\bar{f}$. Once again, we usually neglect the component that is not connected to the identity. What remains is

$$x - t \to f(x - t); \qquad x + t \to \bar{f}(x + t)$$

with monotonically increasing bijections $f, \bar{f} : \mathbb{R} \to \mathbb{R}$.

16.3 Exercises

1. On flat space-times, we proved that the conformal Killing factors satisfy

$$(n-2)\partial_\mu \partial_\nu \kappa + g_{\mu\nu} \Box \kappa = 0$$

a) First, show that on curved manifolds

$$(n-2)\nabla_\mu \nabla_\nu \kappa + g_{\mu\nu} \Box \kappa$$

$$= 2\nabla_\alpha \big(R^\alpha{}_{\nu\mu\beta}\xi^\beta\big) - 2\nabla_\mu \big(R_{\nu\alpha}\xi^\alpha\big) + R_{\mu\alpha}\big(\nabla^\alpha \xi_\nu - \nabla_\nu \xi^\alpha\big) + R^\alpha{}_{\mu\nu\beta}\big(\nabla_\alpha \xi^\beta - \nabla^\beta \xi_\alpha\big)$$

⚓ b) Simplify the RHS of the previous part as much as possible.

2. For $p + q > 2$, identify the Lie algebra for $\texttt{conf}(\mathbb{E}^{p+q})$ with that of the group $\texttt{SO}(p + 1, q + 1)$.

3. a) For the Euclidean manifold $\mathbb{E}^n$, show that the following discrete transformation is conformal

$$x'^\mu = \frac{x^\mu}{x^\nu x_\nu}$$

b) Find the conformal Killing vector field corresponding to the following, infinitesimal conformal transformation

$$x''^\mu = \frac{x'^\mu + \varepsilon t^\mu}{(x'^\nu + \varepsilon t^\nu)(x_{\prime\nu} + \varepsilon t_\nu)}$$

and show that this is a $\gamma-$transform.

4. a) Show that the Mobius transforms are generated using the Witt generators $\{L_{-1}, L_0, L_{+1}\}$.

   ⚓ b) Show that the Møbius subalgebra is the only non trivial (containing more than 1 and less than all of the generators) subalgebra of the Witt algebra.

   c) Show that under the Møbius transforms, a line is mapped to either a circle or a line and that a circle is mapped to a line or a circle.

   d) Let M_1 and M_2 be two Møbius trnasformations as

$$M_i(z) = \frac{a_i z + b_i}{c_i z + d_i}; \quad i = 1, 2$$

   Show that the composite transform

$$M_3(z) \equiv M_2\big(M_1(z)\big) = \frac{a_3 z + b_3}{c_3 z + d_3}$$

   satisfies

$$\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

Chapter 17

Conformal Fields

From now on and until the end of this part (while talking about CFTs), we call any physical quantity that depends on space-time positions, a field; examples include: the independent fields that appear in the Lagrangian, the Lagrangian itself, the energy-momentum tensor, etc. Of particular interest are the quasi-primary fields. A field Φ is called quasi-primary if, under the infinitesimal conformal transformation

$$x^\mu \to x'^\mu = x^\mu + \varepsilon \xi^\mu$$

it transforms as

$$\Phi \to \Phi' = \Phi - \varepsilon\Big(\pounds_\xi \Phi + \frac{1}{2}\Delta_\Phi \kappa \Phi\Big)$$

where Δ_Φ is a constant, called the scaling dimension of the field Φ. Under this, the Lagrangian changes as

$$\delta\mathcal{L} = -\varepsilon\frac{\kappa}{2}\Big[2g^{\mu\nu}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + \sum_a \Delta_a\big(\frac{\partial\mathcal{L}}{\partial\phi_a} + \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi_a}\nabla_\mu\phi_a\big) - n\mathcal{L}\Big]$$

Considering the Lagrangian as a sum of product terms

$$\mathcal{L} = \sum_i \mathcal{L}_i,$$

we get

$$\delta\mathcal{L} = -\varepsilon\frac{\kappa}{2}\sum_i \mathcal{L}_i\Big(2\#_g^i + \sum_a \Delta_a \#_a^i - n\Big)$$

where $\#_g^i$ and $\#_a^i$ denote the power of inverse metric tensors and ϕ_a in each term. As an example, consider the term

$$\phi^2(\nabla_\alpha\phi)g^{\alpha\beta}(\nabla_\beta\psi_{\mu\nu})g^{\mu\rho}g^{\nu\sigma}\chi_\rho\chi_\sigma$$

this will be conformally invariant, only if

$$6 + 3\Delta_\phi + \Delta_\psi + 2\Delta_\chi = n$$

Most of the time, when we talk about conformal Lagrangians, we are pointing to those that exhibit conformal symmetry with $\Delta_a = 0$. Each term in such a theory must have exactly $n/2$ inverse metric tensors involved. For instance in $3+1$ dimensions, the free electromagnetic theory would be conformal

$$\mathcal{L}_{EM} = -\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}(\nabla_\alpha A_\mu - \nabla_\mu A_\alpha)(\nabla_\beta A_\nu - \nabla_\nu A_\beta)$$

An immediate consequence of such a definition, would be that the Hilbert tensor is traceless. The Hilbert tensor is dfeined as

$$H_{\mu\nu} \equiv -2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}$$

and therefore

$$\begin{aligned} H^{\mu}_{\mu} &= n\mathcal{L} - 2g^{\mu\nu}\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \\ &= (n - 2\#_g)\mathcal{L} = 0 \end{aligned}$$

For many interesting cases, the Hilbert tensor is the same as the Belinfante energy momentum tensor; in those cases, the energy momentum tensor is traceless as well.

$$T^{\mu}_{Bel.\,\mu} = 0$$

17.1 Conformal Correlation Functions

In a conformal theory (more specifically, for a conformal state), we expect the quantum correlation functions to be conformally symmetric as well. This strongly limits the form of such functions.

Part VII

Quantum Field Theories on Curved Space-Times

Chapter 18

Static Space-times

In this chapter, we start by a simple generalisation and assume the space to have arbitrary, but time-invariant geometry. We also assume that the space-time is static, therefore

$$ds^2 = -dt^2 + h_{ij}dx^i dx^j$$

We also neglect the effect of the fields on space-time geometry for now.

To solve the problem of non-interacting fields, we used Fourier transforms. Here, and on a curved space, we can not do that any more and therefore seek alternatives. For two square integrable, complex scalar fields ϕ, ψ, we define the inner product

$$\langle \phi, \psi \rangle \equiv \int_\Sigma dx\, \sqrt{h} \phi^*(x)\psi(x)$$

Integration by parts reveals that for any real vector field V, the operator $V^i\nabla_i$ is anti-hermitian, that is

$$\langle \phi, V^i\nabla_i\psi \rangle = -\langle V^i\nabla_i\phi, \psi \rangle$$

However, the Laplacian operator $\nabla^2 \equiv h^{ij}\nabla_i\nabla_j$ is a negative definite operator. This allows us to define a Fourier-like transform with analysis and synthesis equations as

$$\tilde{\psi}(k, \Pi) \equiv \int dx\, \sqrt{h}\ \chi^*(k, \Pi; x)\psi(x)$$

$$\psi(x) = \sum\!\!\!\!\!\!\!\int_{k,\Pi} \chi(k, \Pi; x)\tilde{\psi}(k, \Pi)$$

where the functions $\chi(k, \Pi; x)$ are the properly normalized eigen-functions of the Laplacian operator with eigenvalue $\lambda = -k^2$ and degeneracy index Π.

18.1 The Klein-Gordon Fields

The Klein-Gordon Lagrangian on a curved space is

$$\mathcal{L} = -\frac{1}{2}\sum_a \left[g^{\mu\nu}\left(\nabla_\mu\phi_a\right)\left(\nabla_\nu\phi_a\right) + m^2\phi_a^2\right]$$

the conjugate momenta become

$$\pi_a(x) = \dot{\phi}_a(x)$$

and the Hamiltonian is

$$H = -\frac{1}{2} \sum_a \int_\Sigma dx \, \sqrt{h} \left[\right]$$

Part VIII

AdS/CFT

Appendix A

The Feynman Propagator in Space-time

For spacelike vectors, we use a Lorentz transformation to set $t = 0$, then the Feynman propagator in a distance r becomes

$$D_F(0, r\hat{\mathbf{z}}) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}(4\pi)^{n/2}} \int_m^\infty d\omega\, (\omega^2 - m^2)^{n/2-1} \int_0^\pi d\theta \cos\left(\sqrt{\omega^2 - m^2}r\cos\theta\right)\sin^{n-2}\theta$$

and for timelike separations, setting $\mathbf{x} = 0$, we get

$$D_F(t, \mathbf{0}) = \frac{1}{\Gamma\left(\frac{n}{2}\right)(4\pi)^{n/2}} \int_m^\infty d\omega\, (\omega^2 - m^2)^{n/2-1} \cos\left(\omega t\right)$$

Although these integrals may diverge in some dimensions, we do not worry too much about it. Such divergences are equivalent to those appearing in a stochastic setting where one considers a white noise; the correlation function may be diverging but at the end of the day, the interesting quantities will turn out to be finite.

Specifically for $n = 3$ these become

$$D_F(0, r\hat{\mathbf{z}}) = \frac{1}{4\pi r} \int_m^\infty d\omega\, \sin\left(r\sqrt{\omega^2 - m^2}\right)$$

$$D_F(t, \mathbf{0}) = \frac{1}{4\pi^2} \int_m^\infty d\omega\, \cos\left(\omega t\right)\sqrt{\omega^2 - m^2}$$

Bibliography

[1] Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* 41 (2019): 190088.

[2] Mark Srednicki...