Geometric Gravitation

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Part I

Newtonian Theory of Gravitation

Chapter 1

Newtonian Mechanics

In this first chapter we look through Newtonian physics. Sir Isaac Newton (1642-1726) needed space-time to describe motion. He knew that only relative positions and times have meaning. It is meaningless to say there is an apple in $\vec{r} = (1.5, 4.1, 7.6)$ meters in the absence of some agreed upon center of coordinates. It is also meaningless to say that Isaac Newton was born in $t = 1642$ years without the letters a.c. In order to describe his laws of motion however, Newton found it useful to assume the existence of an absolute space-time as a mathematical tool.

The absolute space is the three dimensional Euclidean space $\mathbb{E}^3$. That is the set $\mathbb{R}^3$ equipped with the distance function

$$d(\vec{r}_2, \vec{r}_1) \equiv \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The absolute time is just the real line. An absolute space alongside an absolute time is called (by Newton) an inertial reference frame. A motion (of a particle say) is described by giving the position of the mover as a function of time.

$$\vec{r} = \vec{r}(t)$$

The velocity of such a path is defined as

$$\vec{\beta} \equiv \frac{d\vec{r}}{dt}$$

Note that it is perfectly suitable to treat time as a parameter of a particle's (or a more complicated system's) path in space. To define momentum, Newton used his intuition of inertial mass. A car is more *massive* than a piece of chalk intuitively. For now, let's assume that each particle has an inertial mass assigned to it as a mere label m_I. Then the momentum of this particle is defined to be

$$\vec{p} \equiv m_I \vec{\beta}$$

The first law postulates the motion of a $free$ particle to be a motion of constant velocity.

Newton's 1st law: *The motion of a free particle in an inertial frame is one with constant momentum or velocity.*

This is in fact the definition of an inertial frame. The second law $defines$ force.

Newton's 2nd law: $\vec{F} \equiv \frac{d\vec{p}}{dt} = m_I \vec{a}$

The third law defines the quantitative value of inertial mass, or relative inertial mass to be exact.

Newton's 3rd law: *The force that a body $\mathcal{B}_1$ exerts on another body $\mathcal{B}_2$, namely $\vec{F}_{1\to 2}$ is minus the force exerted by $\mathcal{B}_2$ to $\mathcal{B}_1$, namely $\vec{F}_{2\to 1}$.*

Assume you have a particle with known inertial mass m_I and another particle with unknown inertial mass m_I'. Letting them interact, their momenta change through time. We may write

$$\frac{m_I'}{m_I} = \frac{|\dot{\vec{p}}'|}{|\dot{\vec{p}}|}$$

which provides an experimental definition of relative inertial mass. Defining a sample mass as the unit mass, all particles will have well defined inertial mass.

***Exercise:** If all three laws of Newton, are defining new quantities and concepts, how come this justifies as a theory?

Chapter 2

Galilean Relativity

From the definitions and laws presented in the first chapter, the Galilean relativity is achieved. Galilean relativity is the fact that if $\mathcal{S}$ is an inertial frame, then $\mathcal{S}'$ is also an inertial frame assuming the two are related in one of the following ways

a) Space-time translations.

$$(t', \vec{r'}) = (t, \vec{r}) + (t_0, \vec{r_0}) \quad \text{for some } (t_0, \vec{r_0}).$$

b) Spatial rotations/parity

$$(t', \vec{r'}) = (t, O\vec{r}) \quad \text{for any orthogonal matrix } O$$

c) Galilean Boost

$$(t', \vec{r'}) = (t, \vec{r}) + (0, \vec{\beta} t) \quad \text{for some } \vec{\beta}.$$

Principle of Galilean relativity, states that all laws of physics should read the same in all inertial frames.

It is a fairly easy task to check that Newton's laws of motion are consistent with this principle, provided that the force is a vector i.e. it transforms in a similar fashion to the transformation of $\vec{r}$.

*****Exercise:** Find *all* the transformations that keep the Newton's laws of motion unchanged. To do so assume the following

$$(t', \vec{r}') = \mathcal{T}((t, \vec{r}))$$

$$(m', \vec{F}') = (m, \vec{F})$$

Chapter 3

Newtonian Gravitation

The previously discussed laws predict the evolution of physical systems, given the forces acting on the system. In 17^{th} century little was known about electricity and magnetism and it was Newton's natural decision to aim his theory at what was arguably the most observed system of his time, the solar system. His first daring step was to drop the belief that these heavenly bodies in the sky, were different than things on earth in their essence. He assumed his laws to be describing the motion of solar planets as well as bodies on earth. To make predictions he now only needed to come up with a force law since It is was already conjectured that there must be an attraction between sun and planets to make them orbit the sun.

3.1 Kepler

Johannes Kepler, heir to a vast amount of data from Tycho Brahe was trying to find order in the motion of heavenly bodies. He succeeded to come up with three rules that were obeyed by all planets in the solar planet.

i) planets, orbit the sun in elliptical closed orbits. Sun sits at the focus of this ellipse.

ii) The area swept by the line connecting sun to a planet in equal time intervals is the same.

iii) The squared time it takes a planet to make a full orbit, is proportional to the cube of the major semi-axis of it's orbit.

3.2 Principle of Equivalence and Newton's Law of Gravitation

Newton, having access to Kepler's rules tried different distance dependencies for his gravitational force. He finally found that an inverse squared force law is in perfect accordance with all of Kepler's rules.

This was not (and nothing will ever be) a *proof* of an inverse squared force law, Although all the experiments down to a scale of 50 micrometers are in agreement with it.

In 1970, Vera Rubin *et. al.* measured the velocities of bodies orbiting the center of Andromeda galaxy. Newton's law of gravity predicts

$$\frac{\beta^2}{r} \propto \frac{1}{r^2} \Rightarrow \beta \propto r^{-.5}$$

while the observed data went

$$\beta \propto r$$

This problem is still to find a suitable answer and is perhaps the strongest fact against the inverse squared force law and shows that we still lack a deep and complete understanding of gravity.

So far we know $F_g \propto r^{-2}$. Galileo had observed that different bodies with different masses and materials have the same motion when sliding down an inclined plane or swinging as a pendulum. For Newton (knowing that it is earth's gravitation that makes things fall etc.)This meant

$$\frac{F}{m_I} = acceleration = g = const.$$

which (with the help of available symmetries) immediately implies

$$F_g = \frac{GM_I m_I}{r^2}$$

G is called Newton's constant and is a constant of nature. We assume $G = 1$ from now on.

The fact that all bodies experience the same acceleration due to gravity is called (or is a version of) a principle called the *equivalence principle*. This is tested to be true on earth with a precision of a part in 10^{12}. It is believed that this is the best accuracy one may get on earth and to increase the accuracy, Spatial Tests of Equivalence Principle (STEP) need to be performed.

From now on we assume EP to hold and hence drop the emphasis on inertial mass m_I.

Chapter 4

Tidal Forces

Although Newton's theory of gravitation includes action at a distance, a local field theory (still containing immediate interaction) may be developed defining the potential scalar field Φ.

$$\vec{F}_g = -m\vec{\nabla}\Phi$$

$$\nabla^2\Phi = 4\pi\rho$$

In obvious notations.

Now consider two nearby test (i.e. small) masses travelling in space due to gravitational forces. The first path is $\vec{r}(t)$ and the second $\vec{r}(t) + \vec{\delta}(t)$.

$$\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}\Phi(\vec{r})$$

$$\frac{d^2\vec{r}}{dt^2} + \frac{d^2\vec{\delta}}{dt^2} = -\vec{\nabla}\Phi(\vec{r} + \vec{\delta})$$

Subtracting the two equations and keeping terms that are linear in δ, we get

$$\frac{d^2\delta_i}{dt^2} = -\delta_j\partial_j\partial_i\Phi$$

Some times we define

$$K_{ij} \equiv \partial_j\partial_i\Phi$$

This means that the first particle will observe a (fictitious) force on $free$ nearby objects in the form $\vec{F} = K\vec{r}$. This is called the tidal force and the matrix K is called the tidal matrix.

In vacuum, K satisfies

$$\nabla^2 K_{ij} = 0 \qquad \mathrm{Tr}(K) = 0$$

Exercise: Show that the motion described by

$$\frac{d^2\delta_i}{dt^2} = -\delta_j \partial_j \partial_i \Phi$$

in vacuum is not a stable motion. Consider some test masses distributed on the surface of a sphere under tidal forces. Show that they form either a cigar shaped or a pancake shaped ellipsoid in later times.

***Exercise:** Consider earth and another body (moon, sun, etc.) orbiting around each other. Predict the *tides* using the tidal forces on earth. What is the period of the tides you predicted? Using numerical values show that moon makes much stronger tides than that of the sun.

****Exercise:** Consider a universe ($\mathbb{R}^3$), filed with uniform density ρ.

a) Show that the gravitational potential is ill-defined.

b) Correct Newton's law to

$$\nabla^2 \Phi + \Lambda \Phi = 4\pi\rho$$

and show that now there is a constant solution for Φ.

c) Find the force law of this corrected version. Try to find an upper bound on Λ using numerical values available. (hint: try computing the Fourier transform of $e^{-|x|}$ first.)

d) Does this solve the andromeda problem? If not, is there any force law that solves both problems at once?

Part II

Special Relativity

Chapter 5

Light

[1]

From the first days of science the nature of light has been a popular question. The answers and sometimes even what we mean by the question have changed a lot. Newton's picture of light was "very fast particles moving in straight lines", Although he was familiar with other theories that conjectured light to be a wave, propagating through a media.

In 1676, Ole Roemer was observing moons of jupiter when he discovered that they didn't completely follow Kepler's rules. He conjectured that this defective effect might be due to a finite speed of light. Huygens later used his data to calculate the speed of light as c=212000km/s.

Later on, in 1729, English astronomer James Bradley was trying to document the exact location of stars in the sky and come up with a map. He then found out that the coordiantes of stars were changing with seasons. It was known that the change in the location of earth in it's orbit was too small to contribute significantly to this effect. Bradley solved this problem by introducing "aberration of starlight". Imagine it's raining and you are driving in the street. If your are stuck before a red traffic light, you will see rain drops falling vertically say with velocity c_r. But as soon as you start to move with some speed v, the path the drops take will look tilted by some angle $\theta = \arctan \frac{v}{c_r}$ or for slow cars $\theta \simeq \frac{v}{c_r}$. The same

[1] In this chapter we don't assume $G = k_B = \hbar = \varepsilon_0 = \mu_0 = 1$ and use standard notations. Velocities are also denoted by the $u, v, \cdots$ instead of $\beta, \cdots$

calculation, describes the aberration of starlights. Here, the car will be the earth and v will be it's velocity orbiting the sun which was known to be around 30km/s. This yields an aberration angle $\theta \simeq \frac{v}{c}$. Which was in good agreement with observations and led to a more accurate calculation of c.

In 1850, James Clerk Maxwell introduced his theory of electromagnetism.

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t})$$

His equations predicted an electromagnetic wave travelling with speed

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Which was very close to the speed of light. He therefore conjectured that light is an electromagnetic wave. Although in perfect agreement with almost all (c.f. next exercise) observed light related phenomena this was in contradiction with Newtonian mechanics since velocity is a relative quantity and should not be determined for all observers. Maxwell, well aware of this contradiction assumed a preferred reference frame called "the ether" with respect to which the speed of light was determined. Inertial frames moving relative to ether would therefore measure ascribe a different value to the speed of light.

***Exercise:** The aberration of starlight is in contradiction with wave nature of light and Newtonian mechanics.

a) Consider a wave $\propto e^{i(\vec{k}.\vec{r}-\omega t)}$ in a frame S. Show that another frame S' will observe the wave vector$\vec{k}'$ same as $\vec{k}$

b) To describe the aberration of starlight, Lorentz corrected Galilean transformation

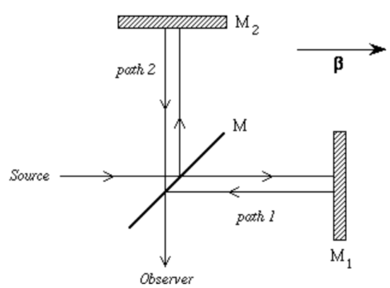
$$t' = t - \frac{\vec{v}.\vec{r}}{c^2}$$

$$\vec{r'} = \vec{r} - \vec{v}t$$

Chapter 6

Lorentz' Transformations

Maxwell's last contribution before his death was to propose a list of experiments aimed to detect ether. Albert Michelson and Max Planck were the first people to perform these experiments. Michelson later became the first american physicist to win a Nobel prize for his famous interferometry experiment. Michelson interferometer was designed to detect β^2 effects with β being the speed of earth in it's orbit around the sun.



Michelson Interferometer. The finite β leads to a time difference of $2L\beta^2 + \mathcal{O}(\beta^4)$. This is a phase difference of $\frac{4\pi L}{\lambda^2}\beta^2$.

Despite his best efforts, Michelson couldn't find any observable interference even when his accuracy was about 10^4 times higher than what was needed. Lorentz' proposed a conjecture to describe this failure. He postulated that physical matter i.e. things are shrunk in the direction of their motion with respect to ether by a Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$ (in other words $L' = L\sqrt{1-\beta^2}$). This rather non intuitive phenomenon later found to be in agreement with relativistic theory and was directly observed in 1932 by Kennedy and Thorndike.

Exercise: Light clock. Consider a cylinder of height L. There is a source of light in the bottom and a mirror on top. The source fires a sharp beam of light, it bounces back from the mirror and comes back to a sensor also placed next to the fire source. This whole process defines a tick of this light clock and measures a time $T = 2L$.

a) Let this clock move along it's axis with speed β with respect to a frame S. Show that S measures the inter-tick times to be $T' = \gamma T$

b) Find the same result for the case where the clock is moving perpendicular to it's axis. This effect was called time dilation.

Later on, Lorentz came up with a set of transformation rules, known as Lorentz' transformations that could explain all the above phenomena.

$$t' = \gamma(t - vec\beta.\vec{r})$$
$$\vec{r'} = \vec{r} + (\gamma - 1)\hat{\beta}\hat{\beta}.\vec{r} - \gamma\vec{\beta}t$$

Nevertheless he insisted on t' being an effect of motion relative to ether which caused clocks to tick slower and not the real time measured by a moving frame. Albert Einstein was the one who leaped the ether barrier and postulated this to be *time*. He no longer needed ether because all previous laws were now consistent in each frame as we'll discuss in the next chapter.

******Exercise:*** Do as Poincare did and using the principle of relativity, show that the only possible transformations that agree to Lorentz' up to order β^2 are the ones stated above. (hint: If S' is related to S through Lorentz' transformation with $\vec{\beta}_1$ and S'' is related to S' through Lorentz' transformation with $\vec{\beta}_2$, then principle of relativity insists (why?) that S'' should also be related to S by Lorentz' transformation of some parameter $\vec{\beta}_3$)

****Exercise:** Addition rule of velocities. Find the addition rule for parallel velocities (c.f. previous exercise)

Answer: $\beta_3 = \tanh[\tanh^{-1}(\beta_1) + \tanh^{-1}(\beta_2)]$

Now generalize your result to non parallel velocities and derive the relativistic relation for aberration of starlight.

****Exercise:** Using Lorentz' transformations (this time including the γ factor) predict the aberration of starlight using wave theory for light.

****Exercise:** Rakhsha's Paradox. A car is moving towards a garage door with velocity β. When it's front lights are at a distance L from the door, the driver presses the remote control button to open the door. The car has a bonnet length of a.

a) Find the conditions under which the door receives the remote control signal before the car hits it.

b) Find the conditions that the driver, observes the signal to hit the door before the car hits it.

c) Find numerical values which satisfy (a) but not (b) or vice versa.

d) What is going on?

Partial Answer: For $\frac{1}{\gamma}(\frac{1}{\beta} - 1) < \frac{a}{L} < \gamma(\frac{1}{\beta} - 1)$, (a) is satisfied but (b) is not.

Chapter 7

Space-time and the debut of greek iundices

To describe the physical phenomena, we need to address every *event*, with it's space-time coordinates, i.e. it's location $\vec{r}$ and it's time t. It will prove useful to pack all the 4 coordinates into a single space-time coordinate as

$$x = x^\mu = (x^0, x^1, x^2, x^3) = (t, r^i)$$

There are two conventions used in this notation. First note that we use upper indices instead of the more convenient lower ones. Unless told otherwise, the quantity x_μ is undefined and has no meaning. Secondly the use of greek index μ is used for x^μ while r^i has latin indices. It is also assumed to hold that greek indices run from 0 to 1 while the latin indices only run in the spatial part 1,2,3. The Lorentz' transformations now read

$$x^\mu = \sum_\nu \Lambda^\mu_\nu x^\nu$$

This is one of the very few times the $\sum$ symbol is used throughout this text. Instead a very simplifying convention is used: "Whenever an index is repeated, once as an upper index and once as a lower one, a sum is assumed to run over the index." Therefore we may write the last equation as

$$x^\mu = \Lambda^\mu_\nu x^\nu$$

Exercise: The inertial frame S' is moving with respect to