Thesis

$$\exists\, \alpha, \gamma \in \langle 0, 2\pi\rangle : \begin{bmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{bmatrix} \begin{bmatrix} cos(\frac{\beta}{2}) & -isin(\frac{\beta}{2}) \\ -isin(\frac{\beta}{2}) & cos(\frac{\beta}{2}) \end{bmatrix} \begin{bmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{bmatrix} = \begin{bmatrix} cos(\frac{\beta}{2}) & -sin(\frac{\beta}{2}) \\ sin(\frac{\beta}{2}) & cos(\frac{\beta}{2}) \end{bmatrix}$$

Proof

Firstly, let's multiply the matrices

$$\begin{bmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{bmatrix} \begin{bmatrix} cos(\frac{\beta}{2}) & -isin(\frac{\beta}{2}) \\ -isin(\frac{\beta}{2}) & cos(\frac{\beta}{2}) \end{bmatrix} \begin{bmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{bmatrix} = \begin{bmatrix} cos(\frac{\beta}{2}) & -sin(\frac{\beta}{2}) \\ sin(\frac{\beta}{2}) & cos(\frac{\beta}{2}) \end{bmatrix}$$

$$\begin{bmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{bmatrix} \begin{bmatrix} e^{-i\frac{\gamma}{2}}cos(\frac{\beta}{2}) & -ie^{i\frac{\gamma}{2}}sin(\frac{\beta}{2}) \\ -ie^{-i\frac{\gamma}{2}}sin(\frac{\beta}{2}) & e^{i\frac{\gamma}{2}}cos(\frac{\beta}{2}) \end{bmatrix} = \begin{bmatrix} cos(\frac{\beta}{2}) & -sin(\frac{\beta}{2}) \\ sin(\frac{\beta}{2}) & cos(\frac{\beta}{2}) \end{bmatrix}$$

$$\begin{bmatrix} e^{-i\frac{\alpha+\gamma}{2}}cos(\frac{\beta}{2}) & -ie^{-i\frac{\alpha-\gamma}{2}}sin(\frac{\beta}{2}) \\ -ie^{i\frac{\alpha-\gamma}{2}}sin(\frac{\beta}{2}) & e^{i\frac{\alpha+\gamma}{2}}cos(\frac{\beta}{2}) \end{bmatrix} = \begin{bmatrix} cos(\frac{\beta}{2}) & -sin(\frac{\beta}{2}) \\ sin(\frac{\beta}{2}) & cos(\frac{\beta}{2}) \end{bmatrix}$$

Now we can move to form of 4 equations.

$$\begin{cases} e^{-i\frac{\alpha+\gamma}{2}} & = 1 \\ -ie^{-i\frac{\alpha-\gamma}{2}} & = -1 \\ -ie^{i\frac{\alpha-\gamma}{2}} & = 1 \\ e^{i\frac{\alpha+\gamma}{2}} & = 1 \end{cases}$$

$$\begin{cases} e^{-i\frac{\alpha+\gamma}{2}} & = 1 \\ e^{-i\frac{\alpha-\gamma}{2}} & = -i \\ e^{i\frac{\alpha-\gamma}{2}} & = i \\ e^{i\frac{\alpha+\gamma}{2}} & = 1 \end{cases}$$

$$\begin{cases} cos(\frac{\alpha+\gamma}{2}) - isin(\frac{\alpha+\gamma}{2}) & = 1 \\ cos(\frac{\alpha-\gamma}{2}) - isin(\frac{\alpha-\gamma}{2}) & = -i \\ cos(\frac{\alpha-\gamma}{2}) + isin(\frac{\alpha-\gamma}{2}) & = i \\ cos(\frac{\alpha+\gamma}{2}) + isin(\frac{\alpha+\gamma}{2}) & = 1 \end{cases}$$

Solving the first one:

$$
\begin{aligned}
cos(\frac{\alpha+\gamma}{2}) - isin(\frac{\alpha+\gamma}{2}) = 1 \\
cos(\frac{\alpha+\gamma}{2}) - 1 &= isin(\frac{\alpha+\gamma}{2}) \\
cos^2(\frac{\alpha+\gamma}{2}) - 2cos(\frac{\alpha+\gamma}{2}) + 1 &= -1 \cdot sin^2(\frac{\alpha+\gamma}{2}) \\
cos^2(\frac{\alpha+\gamma}{2}) + sin^2(\frac{\alpha+\gamma}{2}) &= 2cos(\frac{\alpha+\gamma}{2}) - 1 \\
1 &= 2cos(\frac{\alpha+\gamma}{2}) - 1 \\
1 &= cos(\frac{\alpha+\gamma}{2}) \\
\frac{\alpha+\gamma}{2} &= 2\pi + 2\pi n_1, \qquad\qquad n_1 \in Z
\end{aligned}
$$

Solving the 3rd. one:

$$
\begin{aligned}
cos(\frac{\alpha-\gamma}{2}) + isin(\frac{\alpha-\gamma}{2}) = i \\
cos(\frac{\alpha-\gamma}{2}) &= i(1 - sin(\frac{\alpha-\gamma}{2})) \\
cos^2(\frac{\alpha-\gamma}{2}) &= -1(1 - 2sin(\frac{\alpha-\gamma}{2}) + sin^2(\frac{\alpha-\gamma}{2})) \\
cos^2(\frac{\alpha-\gamma}{2}) &= -1 + 2sin(\frac{\alpha-\gamma}{2}) - sin^2(\frac{\alpha-\gamma}{2})) \\
cos^2(\frac{\alpha-\gamma}{2}) + sin^2(\frac{\alpha-\gamma}{2}) &= -1 + 2sin(\frac{\alpha-\gamma}{2}) \\
1 &= -1 + 2sin(\frac{\alpha-\gamma}{2}) \\
1 &= sin(\frac{\alpha-\gamma}{2}) \\
\frac{\alpha-\gamma}{2} &= \frac{\pi}{2} + 2\pi n_2, \qquad\qquad n_2 \in Z
\end{aligned}
$$

Combining the results:

$$
\begin{cases}
\frac{\alpha+\gamma}{2} = 2\pi + 2\pi n_1, & n_1 \in Z \\
\frac{\alpha-\gamma}{2} = \frac{\pi}{2} + 2\pi n_2, & n_2 \in Z
\end{cases}
$$

$$
\begin{cases}
\frac{\alpha+\gamma}{2} + \frac{\alpha-\gamma}{2} = \alpha & = 2\pi + \frac{\pi}{2} + 2\pi(n_1 + n_2),\ n_1 \in Z, n_2 \in Z \\
\frac{\alpha+\gamma}{2} - \frac{\alpha-\gamma}{2} = \gamma & = 2\pi - \frac{\pi}{2} + 2\pi(n_1 - n_2),\ n_1 \in Z, n_2 \in Z
\end{cases}
$$

$$
\begin{cases}
\alpha & = \frac{1}{2}\pi + 2\pi k_1,\ k_1 = (n_1 + n_2 + 1) \\
\gamma & = -\frac{1}{2}\pi + 2\pi k_2,\ k_2 = (n_1 - n_2 + 1)
\end{cases}
$$

Let's choose any angles from the set of solutions and check whether all coefficients have proper values:
$\alpha = \frac{5}{2}\pi, \gamma = \frac{3}{2}\pi$

$$\begin{bmatrix} e^{-i\frac{\alpha+\gamma}{2}}cos(\frac{\beta}{2}) & -ie^{-i\frac{\alpha-\gamma}{2}}sin(\frac{\beta}{2}) \\ -ie^{i\frac{\alpha-\gamma}{2}}sin(\frac{\beta}{2}) & e^{i\frac{\alpha+\gamma}{2}}cos(\frac{\beta}{2}) \end{bmatrix} = \begin{bmatrix} e^{-i2\pi}cos(\frac{\beta}{2}) & -ie^{-i\frac{\pi}{2}}sin(\frac{\beta}{2}) \\ -ie^{i\frac{\pi}{2}}sin(\frac{\beta}{2}) & e^{i2\pi}cos(\frac{\beta}{2}) \end{bmatrix} = \begin{bmatrix} cos(\frac{\beta}{2}) & -sin(\frac{\beta}{2}) \\ sin(\frac{\beta}{2}) & cos(\frac{\beta}{2}) \end{bmatrix}$$