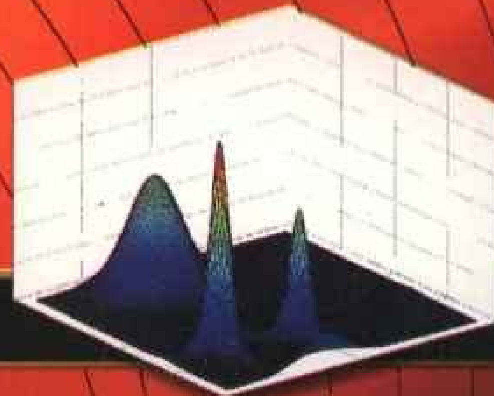
42nd Edition



Higher Engineering Mathematics

Dr. B.S. Grewal

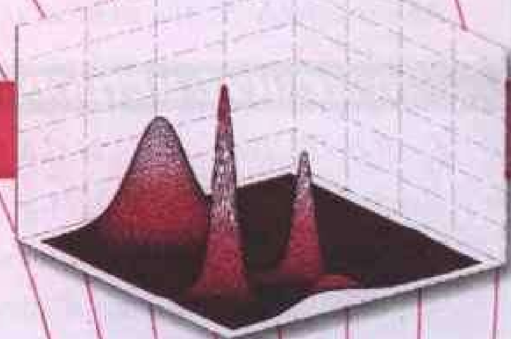
KHANNA PUBLISHERS

42nd Edition

Higher Engineering Mathematics

Dr. B.S. Grewal

42nd Edition



Higher Engineering Mathematics

42nd Edition

Higher Engineering Mathematics

B.S. GREWAL, *Ph.D.*

Professor of Applied Mathematics

Principal Scientific Officer (Ex.)

Defence Research & Development Organisation,

New Delhi

Formerly of

College of Military Engineering, Poona,

Delhi College of Engineering, Delhi

J.S. GREWAL, *M.I.E., I. Engr. (U.K.), M.I. Mar. Engg.*

KHANNA PUBLISHERS

4575/15, Onkar House, Opp. Happy School

Daryaganj, New Delhi-110002

Phone : 011-2324 30 42, 9811541460 ; *Fax* : 011-2324 30 43

e-mail : khannapublishers@yahoo.in

Website : www.khannapublishers.in

Published by :
R.C. Khanna
&
Vineet Khanna
for KHANNA PUBLISHERS
Nai Sarak, Delhi-110006 (India).

ISBN No. : 978-81-7409-195-5

Forty Second Edition : ***2012, June***

Price : ***₹ 450.00***

Typesetted at : Goswami Printers, Delhi.

Printed at : Print India, Dilshad Garden, Delhi.

To

My Father

who served the cause of education
with a missionary zeal for over five decades

Preface to the 42nd Edition

The book has now been recast in an attractive new format, retaining its main features which have made it so popular. The text has been carefully revised, the number of illustrative examples has been increased and problems from the latest university question papers have been added. The 'Objective Type of Questions' have been updated and given at the end of each chapter. It is hoped that the book in its new form will enjoy its ever increasing popularity.

The author takes this opportunity to thank the numerous readers in India and abroad for their letters of appreciation and fellow professors for their suggestions and patronage of the book. In particular, he is grateful to Prof. Jeevargi Phakirappa, V.N. Engg. College, Bellary (Kar.); Prof. P. Annapurna, N.B.K.R. Inst. of Technology, Vidyanagar (A.P.); Dr. A.P. Burnwal, R.I.T., Koderma (Jh. Kh.); Prof. M. Vasudeva Reddy, Vaishnavi Inst. of Technology, Tarapalli (Tirupati); Dr. K.P. Ghadle, B.A.M. University, Aurangabad (Mah.); Prof. B.K. Yadav, Chauksey Engg. College, Bilaspur (C.G.); Prof. D. Ravi Kumar, Vignan University, Guntur (A.P.); Dr. J.C. Prajapati, Charotara University of Sc. & Technology, Changa (Guj.); Prof. Ramesh Chandra, S.R. Technology Institute, Nalgonda (A.P.); Dr. Latika Bhandari, R.V.S. College of Engg. & Technology, Bhillai; Prof. R. Saraswathi, Sri Padmavati Engg. College, Kavalli (A.P.) and Prof. Vikas Goyal, J.M. Inst. of Technology, Radur (Haryana).

Suggestions for improvement of the text and intimation of misprints will be thankfully acknowledged.

New Delhi **B.S. GREWAL**

Contents

Chap.			Page

Unit V : COMPLEX ANALYSIS

Unit VI : TRANSFORMS

Unit VII : NUMERICAL TECHNIQUES

Unit VIII : SPECIAL TOPICS

Note : *The references given alongside the problems pertain to the* **Degree Engineering Examinations** *of the various universities and professional bodies. The abbreviations used for some of these are given below :*

Agra	*stands for*	Dr. B.R. Ambedkar University, Agra
Andhra	"	Andhra University, Waltair
Anna	"	Anna University, Chennai
Bhopal	"	Rajiv Gandhi Technical University, Bhopal
B.P.T.U.	"	Biju Patnaik Technical University, Rourkela
Coimbatore	"	Bharathiyar University, Coimbatore
CUSAT	"	Cochin University of Science and Technology, Kochi
Calicut	"	Calicut University, Cochin
Hazaribag	"	Vinoba Bhave University, Hazaribag
Hissar	"	Guru Jambeshwar University, Hissar
I.E.T.E.	"	Graduateship Examination of the Institute of Electronics and Telecommunication Engineers (India)
I.I.T.	"	Degree Engineering Examination of Indian Institute of Technology
I.S.M.	"	Indian School of Mines, Dhanbad
Kottayam	"	Mahatama Gandhi Memorial University, Kottayam
Kurukshetra	"	National Institute of Technology, Kurukshetra
Madurai	"	Madurai Kamaraj University, Madurai
Marathwada	"	B.A.M. University, Aurangabad
Nagarjuna	"	Acharya Nagarjuna University
P.T.U.	"	Punjab Technical University, Jalandhar
Raipur	"	Pt. Ravi Shankar Shukla University, Raipur
R.T.U.	"	Rajasthan Technical University, Kota
Rohtak	"	Maharishi Dayanand University, Rohtak
S. Patel	"	Sardar Patel University, Vallabh Vidyanagar
S.V.T.U.	"	Swami Vivekanand Technical University, Chhatisgarh
Tirupati	"	Sri Venkateswara University, Tirupati
Tiruchirapalli	"	Bharathidasan University, Tiruchirapalli
U.P.T.U.	"	UP Technical University, Lucknow
U.K.T.U.	"	Uttarakhand Technical University, Dehradun
V.T.U.	"	Visveswaraiah Technological University, Belgaum
Warangal	"	Warangal University of Technology
W.B.T.U.	"	West Bengal University of Technology, Kolkata

CHAPTER 1

Solution of Equations

1. Introduction. 2. General properties. 3. Transformation of equations. 4. Reciprocal equations. 5. Solution of cubic equations–Cardan's method. 6. Solution of biquadratic equations—Ferrari's method ; Descarte's method. 7. Graphical solution of equations. 8. Objective Type Questions.

1.1 INTRODUCTION

The expression $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$
where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a *polynomial in x* of degree n. The polynomial $f(x) = 0$ is called an *algebraic equation of degree n*. If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc. ; then $f(x) = 0$ is called a *transcendental equation.*

The value of x which satisfies $f(x) = 0$, ...(1)
is called its root. Geometrically, a root of (1) is that value of x where the graph of $y = f(x)$ crosses the x-axis. The process of finding the roots of an equation is known as *solution* of that equation. This is a problem of basic importance in applied mathematics. We often come across problems in deflection of beams, electrical circuits and mechanical vibrations which depend upon the solution of equations. As such, a brief account of solution of equations is given in this chapter.

1.2 GENERAL PROPERTIES

I. *If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $x - \alpha$ and conversely.*
For instance, 3 is a root of the equation $x^4 - 6x^2 - 8x - 3 = 0$, because $x = 3$ satisfies this equation.
$\therefore$ $x - 3$ divides $x^4 - 6x^2 - 8x - 3$ completely, *i.e.*, $x - 3$ is its factor.

II. *Every equation of the nth degree has n roots (real or imaginary).*
Conversely if $\alpha_1, \alpha_2, ..., \alpha_n$ be the roots of the nth degree equation $f(x) = 0$, then
$$f(x) = A(x - \alpha_1)(x - \alpha_2) ... (x - \alpha_n) \text{ where } A \text{ is a constant.}$$

Obs. If a polynomial of degree n vanishes for more than n value of x, it must be identically zero.

Example 1.1. *Solve the equation $2x^3 + x^2 - 13x + 6 = 0.$*

Solution. By inspection, we find $x = 2$ satisfies the given equation.

$\therefore$ 2 is its root, *i.e.* $x - 2$ is a factor of $2x^3 + x^2 - 13x + 6$. Dividing this polynomial by $x - 2$, we get the quotient $2x^2 + 5x - 3$ and remainder 0.

Equating the quotient to zero, we get $2x^2 + 5x - 3 = 0$.

Solving this quadratic, we get $x = \dfrac{-5 \pm \sqrt{[5^2 - 4.(2).(-3)]}}{2 \times 2} = \dfrac{-5 \pm 7}{4} = -3, \dfrac{1}{2}$.

Hence, the roots of the given equation are 2, – 3, 1/2.

Note. *The labour of dividing the polynomial by $x - 2$ can be saved considerably by the following simple device called* **synthetic division.**

2	1	–13	6	2
	4	10	– 6	
2	5	– 3	0	

[*Explanation* : (*i*) Write down the coefficient of the powers of x in order (supplying the missing powers of x by zero co-efficients and write 2 on extreme right.

(*ii*) Put 2 as the first term of 3rd row and multiply it by 2, write 4 under 1 and add, giving 5.

(*iii*) Multiply 5 by 2, write 10 under – 13 and add, giving – 3.

(*iv*) Multiply – 3 by 2, write – 6 under 6 and add given zero].

Thus the quotient is $2x^2 + 5x - 3$ and remainder is zero.

Obs. To divide a polynomial by $x + h$, we write $- h$ on the extreme right.

III. Intermediate value property. If $f(a)$ and $f(b)$ *have different signs, then the equation* $f(x) = 0$ *has atleast one root between* $x = a$ *and* $x = b$.

The polynomial $f(x)$ is a continuous function of x (Fig. 1.1). So while x changes from a to b, $f(x)$ must pass through all the values from $f(a)$ to $f(b)$. But since one of these quantities $f(a)$ or $f(b)$ is positive and the other negative, it follows that at least for one value of x(say α) lying between a and b, $f(x)$ must be zero. Then α is the required root.

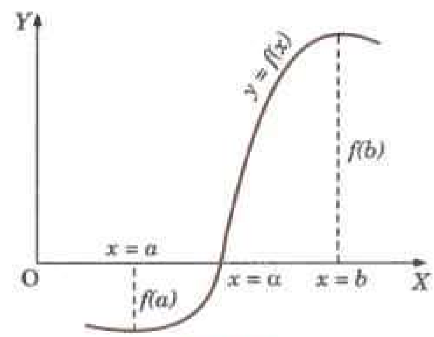


Fig. 1.1

IV. *In an equation with real coefficients, imaginary roots occur in conjugate pairs, i.e.*, if $\alpha + i\beta$ is a root of the equation $f(x) = 0$, then $\alpha - i\beta$ must also be its root. (See p. 534)

Similarly if $a + \sqrt{b}$ is an irrational root of an equation, then $a - \sqrt{b}$ must also be its root.

Obs. *Every equation of the odd degree has at least one real root.*

This follows from the fact that imaginary roots occur in conjugate pairs.

Example 1.2. *Solve the equation* $3x^3 - 4x^2 + x + 88 = 0$, *one root being* $2 + \sqrt{7}i$.

Solution. Since one root is $2 + \sqrt{7}i$, the other root must be $2 - \sqrt{7}i$.

$\therefore$ The factors corresponding to these roots are

$$(x - 2 - \sqrt{7}i) \text{ and } (x - 2 + \sqrt{7}i)$$

or
$$(x - 2 - \sqrt{7}i)(x - 2 + \sqrt{7}i) = (x - 2)^2 + 7 = x^2 - 4x + 11,$$

which is a divisor of $3x^3 - 4x^2 + x + 88$ ...(*i*)

$\therefore$ Division of (*i*) by $x^2 - 4x + 11$ gives $3x + 8$ as the quotient.

Thus the depressed equation is $3x + 8 = 0$. Its root is $- 8/3$. Hence the roots of the given equation are $2 \pm \sqrt{7}i, - 8/3$.

V. Descarte's rule of signs. **The equation $f(x) = 0$ cannot have more positive roots than the changes of signs in $f(x)$; and more negative roots than the changes of signs in $f(-x)$.*

For instance, consider the equation $f(x) = 2x^7 - x^5 + 4x^3 - 5 = 0$ ...(1)

Sign of $f(x)$ are + – + –

Clearly, $f(x)$ has 3 changes of signs (from + to – or – to +).

Thus (*i*) cannot have more than 3 positive roots.

*After the French mathematician and philosopher *Rene Descartes* (1596–1650), who invented Analytic geometry in 1637.

Also $f(-x) = 2(-x)^7 - (-x)^5 + 4(-x)^3 - 5$

$= -2x^7 + x^5 - 4x^3 - 5$

This shows that $f(x)$ has 2 changes of signs. Thus (*i*) cannot have more than 2 negative roots.

Obs. Existence of imaginary roots. *If an equation of the nth degree has at the most p positive roots and at the most q negative roots, then it follows that the equation has at least n − (p + q) imaginary roots.*

Evidently (1) above is an equation of the 7th degree and has at the most 3 positive roots and 2 negative roots. Thus (1) has at least 2 imaginary roots.

VI. Relations between roots and coefficients, *If* $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ *be the roots of the equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

then $$\Sigma\alpha_1 = -\frac{a_1}{a_0}, \quad \Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}, \quad \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

....

$$\alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

Example 1.3. *Solve the equation $x^3 - 7x^2 + 36 = 0$, given that one root is double of another.*

Solution. Let the roots be α, β, γ such that $\beta = 2\alpha$.

Also $\alpha + \beta + \gamma = 7, \alpha\beta + \beta\gamma + \gamma\alpha = 0, \alpha\beta\gamma = -36$

$\therefore$ $$3\alpha + \gamma = 7 \qquad \ldots(i)$$
$$2\alpha^2 + 3\alpha\gamma = 0 \qquad \ldots(ii)$$
$$2\alpha^2\gamma = -36 \qquad \ldots(iii)$$

Solving (*i*) and (*ii*), we get $\alpha = 3, \gamma = -2$.

[The values $\alpha = 0, \gamma = 7$ are inadmissible, as they do not satisfy (*iii*)].

Hence the required roots are 3, 6 and − 2.

Example 1.4. *Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, given that the sum of two of its roots is zero.* (*Cochin, 2005 ; Madras, 2003*)

Solution. Let the roots be $\alpha, \beta, \gamma, \delta$ such that $\alpha + \beta = 0$.

Also $\alpha + \beta + \gamma + \delta = 2 \quad \therefore \quad \gamma + \delta = 2$

Thus the quadratic factor corresponding to α, β is of the form $x^2 - 0x + p$, and that corresponding to γ, δ is of the form of $x^2 - 2x + q$.

$\therefore$ $$x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 + p)(x^2 - 2x + q) \qquad \ldots(i)$$

Equating the coefficients of x^2 and x from both sides of (*i*), we get

$$4 = p + q, \quad 6 = -2p.$$

$\therefore$ $$p = -3, \quad q = 7.$$

Hence the given equation is equivalent to $(x^2 - 3)(x^2 - 2x + 7) = 0$

$\therefore$ The roots are $x = \pm\sqrt{3}, 1 \pm i\sqrt{6}$.

Example 1.5. *Find the condition that the cubic $x^3 - lx^2 + mx - n = 0$ should have its roots in*

(*a*) *arithmetical progression.* (*Madras, 2000 S*)

(*b*) *geometrical progression.*

Solution. (*a*) Let the roots be $a - d, a, a + d$ so that the sum of the roots $= 3a = l$ *i.e.*, $a = l/3$.

Since a is the root of the given equation

$\therefore$ $$a^3 - la^2 + ma - n = 0 \qquad \ldots(i)$$

Substituting $a = l/3$, we get $(l/3)^3 - l(l/3)^2 + m(l/3) - n = 0$.

or $2l^3 - 9lm + 27n = 0$, which is the required condition.

(*b*) Let the roots be a/r, a, ar, so that the product of the roots $= a^3 = n$.

Putting $a = (n)^{1/3}$, in (*i*), we get $n - ln^{2/3} + mn^{1/3} - n = 0$ or $m = ln^{1/3}$

Cubing both sides, we get $m^3 = l^3 n$, which is the required condition.

Example 1.6. *Solve the equation* $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$ *whose roots are in A.P.*

Solution. Let the roots be $a - 3d$, $a - d$, $a + d$, $a + 3d$, so that the sum of the roots $= 4a = 2$.

$$\therefore \qquad a = 1/2$$

Also product of the roots $= (a^2 - 9d^2)(a^2 - d^2) = 40$

$$\text{or} \qquad \left(\frac{1}{4} - 9d^2\right)\left(\frac{1}{4} - d^2\right) = 40 \quad \text{or} \quad 144d^4 - 40d^2 - 639 = 0$$

$$\therefore \qquad d^2 = 9/4 \quad \text{or} \quad -7/36$$

Thus, $d = \pm 3/2$, the other value is not admissible.

Hence the required roots are $-4, -1, 2, 5$.

Example 1.7. *Solve the equation* $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$, *whose roots are in G.P.*

Solution. Let the roots be a/r^3, a/r, ar, ar^3, so that product of the roots $= a^4 = 4$.

Also the product of a/r^3, ar^3 and a/r, ar are each $= a^2 = 2$.

$\therefore$ The factors corresponding to a/r^3, ar^3 and a/r, ar are $x^2 + px + 2$, $x^2 + qx + 2$.

Thus, $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 2(x^2 + px + 2)(x^2 + qx + 2)$

Equating the coefficients of x^3 and x^2

$$-15 = 2p + 2q \quad \text{and} \quad 35 = 8 + 2pq$$

$$\therefore \qquad p = -9/2, \; q = -3.$$

Thus the given equation is $2\left(x^2 - \frac{9}{2}x + 2\right)(x^2 - 3x + 2) = 0$

Hence the required roots are 1/2, 4 and 1, 2 *i.e.*, $\frac{1}{2}$, 1, 2, 4.

Example 1.8. *If* α, β, γ *be the roots of the equation* $x^3 + px + q = 0$, *find the value of*

(*a*) $\Sigma\alpha^2\beta$, (*b*) $\Sigma\alpha^4$ (*c*) $\Sigma\alpha^2\beta$.

Solution. We have $\alpha + \beta + \gamma = 0$ ...(*i*)

$$\alpha\beta + \beta\gamma + \gamma\alpha = p \qquad \text{...}(ii)$$

$$\alpha\beta\gamma = -q \qquad \text{...}(iii)$$

(*a*) Multiplying (*i*) and (*ii*), we get

$$\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + 3\alpha\beta\gamma = 0$$

$$\text{or} \qquad \Sigma\alpha^2\beta = -3\alpha\beta\gamma = 3q \qquad [\text{By } (iii)]$$

(*b*) Multiplying the given equation by x, we get $x^4 + px^2 + qx = 0$

Putting $x = \alpha, \beta, \gamma$ successively and adding, we get $\Sigma\alpha^4 + p\Sigma\alpha^2 + q\Sigma\alpha = 0$

$$\text{or} \qquad \Sigma\alpha^4 = -p\Sigma\alpha^2 - q(0) \qquad \text{...}(iv)$$

Now squaring (*i*), we get $\alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$

$$\text{or} \qquad \Sigma\alpha^2 = -2p \qquad [\text{By } (ii)]$$

Hence, substituting the value of $\Sigma\alpha^2$ in (*iv*), we obtain

$$\Sigma\alpha^4 = -p(-2p) = 2p^2.$$

(*c*) $\Sigma\alpha^3\beta = \Sigma\alpha^2\,\Sigma\alpha\beta - \alpha\beta\gamma\Sigma\alpha = -2p(p) - (-q)(0) = -2p^2$.

PROBLEMS 1.1

1. Form the equation of the fourth degree whose roots are $3 + i$ and $\sqrt{7}$. *(Madras, 2000 S)*
2. Solve the equation (*i*) $x^3 + 6x + 20 = 0$, one root being $1 + 3i$.
   (*ii*) $x^4 - 2x^3 - 22x^2 + 62x - 15 = 0$, given that $2 + \sqrt{3}$ is a root.
3. Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has at least four imaginary roots. *(Cochin, 2005)*
4. Show that the equation $x^4 + 15x^2 + 7x - 11 = 0$ has one positive, one negative and two imaginary roots.
5. Find the number and position of real roots of $x^4 + 4x^3 - 4x - 13 = 0$.
6. Solve the equation $3x^3 - 11x^2 + 8x + 4 = 0$, given that two of its roots are equal.
7. If r_1, r_2, r_3 are the roots of the equation $2x^3 - 3x^2 + kx - 1 = 0$, find constant k if sum of two roots is 1. *(S.V.T.U., 2009)*
8. The equation $x^4 - 4x^3 + ax^2 + 4x + b = 0$ has two pairs of equal roots. Find the values of a and b.

   Solve the following equations 9–14 :
9. $x^3 - 9x^2 + 14x + 24 = 0$, given that two of its roots are in the ratio 3 : 2.
10. $x^3 - 4x^2 - 20x + 48 = 0$ given that the roots α and β are connected by the relation $\alpha + 2\beta = 0$. *(S.V.T.U., 2007)*
11. $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$, given that it has two parts of equal roots. *(Madras, 2003)*
12. $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ given that the sum of two of the roots is equal to the sum of the other two.
13. $x^3 - 12x^2 + 39x - 28 = 0$, roots being in arithmetical progression. *(Madras, 2001 S)*
14. $8x^3 - 14x^2 + 7x - 1 = 0$, roots being in geometrical progression. *(Osmania, 1999)*
15. O, A, B, C are the four points on a straight line such that the distances of A, B, C from O are the roots of equation $ax^3 + 3bx^2 + 3cx + d = 0$. If B is the middle point of AC, show that $a^2d - 3abc + 2b^3 = 0$. *(S.V.T.U., 2006)*
16. Solve the equations (*i*) $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ whose roots are in A.P.
    (*ii*) $x^4 + 5x^3 - 30x^2 + 40x + 64 = 0$ whose roots are in G.P.
17. If α, β, γ be the roots of the equation $x^3 - lx^2 + mx - n = 0$, find the value of
    (*i*) $\Sigma\alpha^2\beta^2$, (*ii*) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$
18. Find the sum of the cubes of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$.
19. If α, β, γ are the roots of $x^3 + 4x - 3 = 0$, find the value of (*i*) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (*ii*) $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$.
20. If α, β, γ be the roots of $x^3 + px + q = 0$, show that
    (*i*) $\alpha^5 + \beta^5 + \gamma^5 = 5\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)$, (*ii*) $3\Sigma\alpha^2\,\Sigma\alpha^5 = 5\Sigma\alpha^3\,\Sigma\alpha^4$.

1.3 TRANSFORMATION OF EQUATIONS

(1) To find an equation whose roots are *m* times the roots of the given equation, *multiply the second term by m, third term by m^2 and so on* (all missing terms supplied with zero coefficients).

For instance, let the given equation be

$$3x^4 + 6x^3 + 4x^2 - 8x + 11 = 0 \qquad ...(i)$$

To multiply its roots by m, put $y = mx$ (or $x = y/m$) in (*i*).

Then $$3(y/m)^4 + 6(y/m)^3 + 4(y/m)^2 + 8(y/m) + 11 = 0$$

Multiplying by m^4, we get $$3y^4 + m(6y^3) + m^2(4y^2) - m^3(8y) + m^4(11) = 0$$

This is same as multiplying the second term by m, third term by m^2 and so on in (*i*).

Cor. To find an equation whose roots are with opposite signs to those of the given equation, *change the signs of the every alternative term of the given equation beginning with the second.*

Changing the signs of the roots of (*i*) is same as multiplying its roots by – 1.

$\therefore$ The required equation will be

$$3x^4 + (-1)6x^3 + (-1)^2\,4x^2 - (-1)^3\,8x + (-1)^4\,11 = 0$$

or $$3x^4 - 6x^3 + 4x^3 + 8x + 11 = 0$$

which is (*i*) with signs of every alternate term changed beginning with the second.

(2) To find an equation whose roots are reciprocal of the root of the given equation, *change x to 1/x.*

Example 1.9. *Solve $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in harmonic progression.*

Solution. Since the roots of the given equation are in H.P., the roots of the equation having reciprocal roots will be in A.P.

The equation with reciprocal roots is $6(1/x)^3 - 11(1/x)^2 - 3(1/x) + 2 = 0$

or $$2x^3 - 3x^2 - 11x + 6 = 0 \qquad ...(i)$$

Since the roots of the given equation are in H.P., therefore, the roots of (i) are in A.P. Let the root be $a - d$, a, $a + d$. Then

$$3a = 3/2 \text{ and } a(a^2 - d^2) = -3.$$

Solving these equations, we get $a = 1/2$, $d = 5/2$.

Thus the roots of (i) are -2, $1/2$, 3.

Hence the required roots of the given equation are $-1/2$, 2, $1/3$.

Example 1.10. *If α, β, γ be the roots of the cubic equation $x^3 - px^2 + qx - r = 0$, form the equation whose roots are $\beta\gamma + 1/\alpha$, $\gamma\alpha + 1/\beta$, $\alpha\beta + 1/\gamma$.*

Hence evaluate $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha)$. (S.V.T.U., 2008)

Solution. If x is a root of the given equation and y a root of the required equation, then

$$y = \beta\gamma + 1/\alpha = \frac{\alpha\beta\gamma + 1}{\alpha} = \frac{r+1}{\alpha} \qquad [\because \alpha\beta\gamma = r]$$

or $$y = \frac{r+1}{x} \Rightarrow x = \frac{r+1}{y}$$

Thus substituting $x = (r + 1)/y$ in the given equation, we get

$$\left(\frac{r+1}{y}\right)^3 - p\left(\frac{r+1}{y}\right)^2 + q\left(\frac{r+1}{y}\right) - r = 0$$

or $ry^3 - q(r + 1)y^2 + p(r + 1)^2 y - (r + 1)^3 = 0$, which is the required equation.

Hence $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha) = p(r + 1)^2/r$.

Example 1.11. *Form an equation whose roots are cubes of the roots of $x^3 - 3x^2 + 1 = 0$.* ...(i)

Solution. If y be a root of the required equation, then $y = x^3$ ...(ii)

Now we have to eliminate x from (i) and (ii)

$\therefore$ Rewriting (i) as $x^3 + 1 = 3x^2$

Cubing both sides, $x^9 + 3x^6 + 3x^3 + 1 = 27x^6$

Substituting $x^3 = y$, we get $y^3 - 24y^2 + 3y + 1 = 0$, which is the required equation.

(3) To diminish the roots of an equation $f(x) = 0$ by h, *divide $f(x)$ by $x - h$ successively. Then the successive remainders determine the coefficients of the required equation.*

Let the given equation be

$$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0 \qquad ...(i)$$

To diminish its roots by h, put $y = x - h$ (or $x = y + h$) in (i) so that

$$a_0(y + h)^n + a_1(y + h)^{n-1} + ... + a_n = 0 \qquad ...(ii)$$

On simplification, it takes the form

$$A_0 y^n + A_1 y^{n-1} + ... + A_n = 0 \qquad ...(iii)$$

Its coefficient $A_0, A_1, ... A_n$ can easily be found with the help of *synthetic division* (p. 2). For this, we put $y = x - h$ in (iii) so that

$$A_0(x - h)^n + A_1(x - h)^{n-1} + ... + A_n = 0 \qquad ...(iv)$$

Clearly, (i) and (iv) are identical. If we divide L.H.S. of (iv) by $x - h$, the remainder is A_n and the quotient $Q = A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + ... + A_{n-1}$. Similarly, if we divide Q by $x - h$, the remainder is A_{n-1} and the quotient is Q_1(say). Again dividing Q_1 by $x - h$, A_{n-2} will be obtained as remainder and so on.

Obs. To increase the roots by h, we take h negative.