10.2

Sequences

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Limits of a sequence and Limit Laws

A fundamental question about sequences concerns the behavior of the terms as we go out farther and farther in the sequence.

$$\{a_n\}_{n=0}^{\infty} = \left\{\frac{1}{n^2+1}\right\}_{n=0}^{\infty} =$$

This sequence ______________ and its limit is 0. This is written as _________________.

$$\{b_n\}_{n=1}^{\infty} = \left\{(-1)^n \cdot \frac{n(n+1)}{2}\right\}_{n=1}^{\infty} =$$

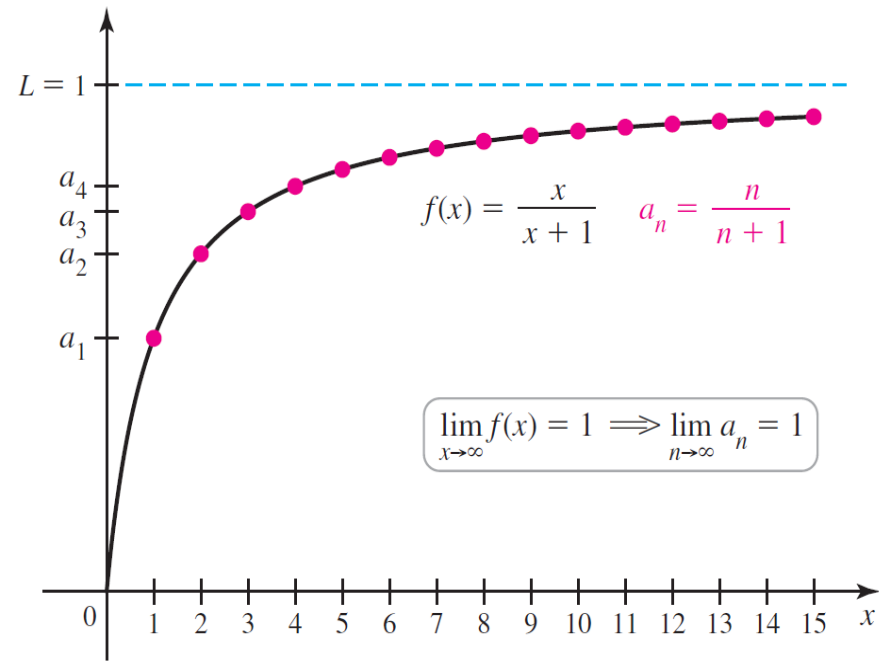
This sequence ___________.

Limits of sequences are really no different from limits at infinity of functions, except that n assumes only integer values. Given $\{a_n\}$, we define $f(n) = a_n$ for all indices n. $\{a_n\} =$, then $f(x) =$

_____________________________, so it

follows _____________________________

because the terms of the sequence lie on the graph of f (Fig 10.11).



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THEOREM 10.1 Limits of Sequences from Limits of Functions

Suppose f is a function such that $f(n) = a_n$, for positive integers n. If $\lim_{x \to \infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L, where L may be $\pm\infty$.

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Because of the correspondence between limits of ________________ and limits of __________________ at infinity, we have the following properties that are analogous to those for functions given in Theorem 2.3.

THEOREM 10.2 Limit Laws for Sequences

Assume the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} ca_n = cA$, where c is a real number
3. $\lim_{n\to\infty} a_n b_n = AB$
4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$, provided $B \neq 0$.

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Example 1 **Limits of sequences**

Determine the limits of the following sequences.

a) $a_n = \frac{3n^3}{n^3+1}$

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Example 1 **Limits of sequences**

b) $b_n = \left(\frac{n+5}{n}\right)^n$

Recall from section 7.6, that when we have an indeterminant form 1^∞ for $\lim\limits_{x\to a} f(x)^{g(x)}$ that we use $L = \lim\limits_{x\to a} g(x) \ln f(x)$ so that $\lim\limits_{x\to a} f(x)^{g(x)} = e^L$.

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Example 1 **Limits of sequences**

c) $c_n = n^{1/\mathrm{n}}$

Recall from section 7.6, that when we have an indeterminant form ∞^0 for $\lim_{x\to a} f(x)^{g(x)}$ that we use $L = \lim_{x\to a} g(x) \ln f(x)$ so that $\lim_{x\to a} f(x)^{g(x)} = e^L$.

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Terminology for Sequences

The following terms are used to describe sequences $\{a_n\}$.

DEFINITIONS Terminology for Sequences

- $\{a_n\}$ is **increasing** if $a_{n+1} > a_n$; for example, $\{0, 1, 2, 3, \ldots\}$.
- $\{a_n\}$ is **nondecreasing** if $a_{n+1} \geq a_n$; for example, $\{1, 1, 2, 2, 3, 3, \ldots\}$.
- $\{a_n\}$ is **decreasing** if $a_{n+1} < a_n$; for example, $\{2, 1, 0, -1, \ldots\}$.
- $\{a_n\}$ is **nonincreasing** if $a_{n+1} \leq a_n$; for example, $\{0, -1, -1, -2, -2, \ldots\}$.
- $\{a_n\}$ is **monotonic** if it is either nonincreasing or nondecreasing (it moves in one direction).
- $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$, for all relevant values of n, and $\{a_n\}$ is **bounded below** if there is a number N such that $a_n \geq N$, for all relevant values of n.
- If $\{a_n\}$ is bounded above and bounded below, then we say that $\{a_n\}$ is a **bounded** sequence.

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Examine the sequences below

a). $\{a_n\} = \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

Satisfies ________________ for _______.

Terms are ______________ in size.

Bounded _________ and ________.

Also _____________.

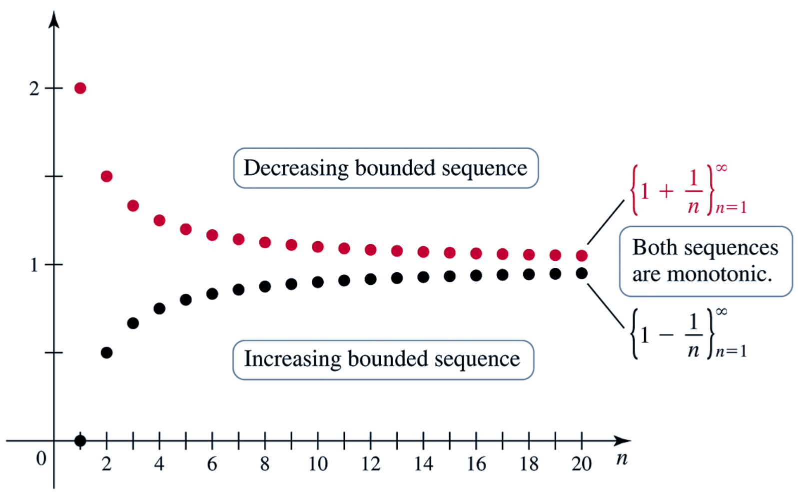
b). $\{b_n\} = \left\{1 + \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right\}$

Satisfies ________________ for _______.

Terms are ______________ in size.

Bounded _________ and ________.

Also _____________.



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Example 2 **Limits of sequences and graphing**

Compare and contrast the behavior of $\{a_n\}$ and $\{b_n\}$ as $n \to \infty$.

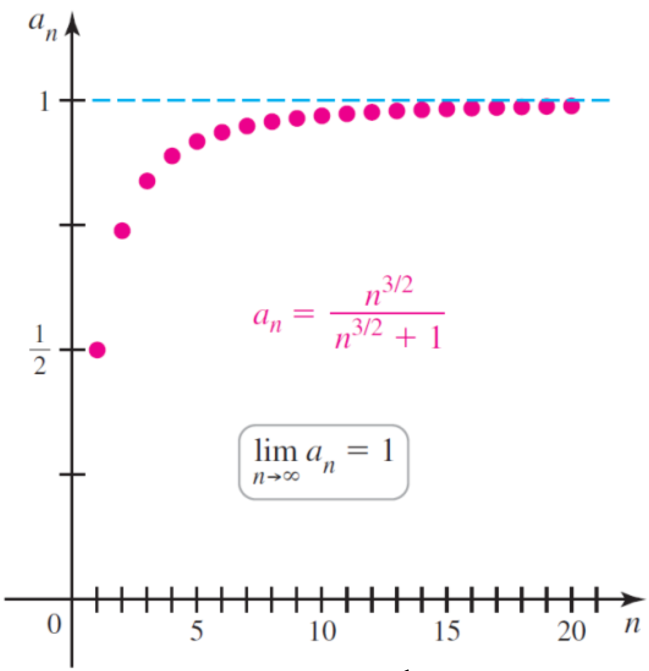
a) $a_n = \dfrac{n^{3/2}}{n^{3/2}+1}$

The terms are ___________, ______________ and bounded.

nMin =___,

u(n) = ________

u(nMin) =___



b) $b_n = \dfrac{(-1)^n n^{3/2}}{n^{3/2}+1}$

The terms of the ______________ sequence ______________ in sign.

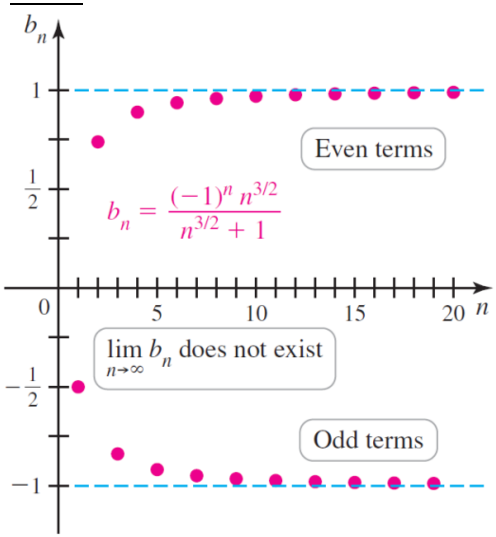
Even terms approach ___.

Odd terms approach ___. Calc Set Up:

nMin =___,

u(n) = ________

u(nMin) =___



The presence of _______ may significantly alter the __________________ of the __________________.

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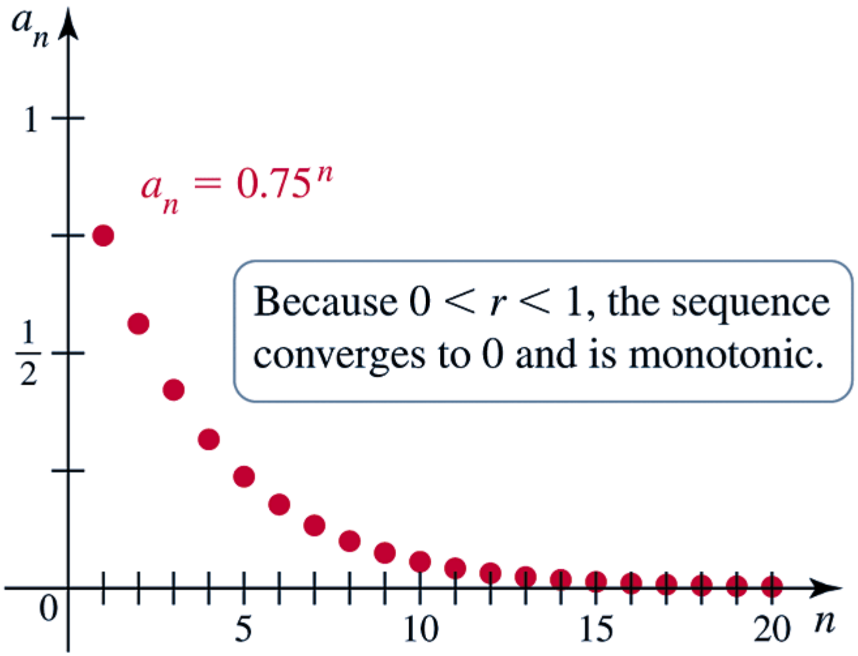
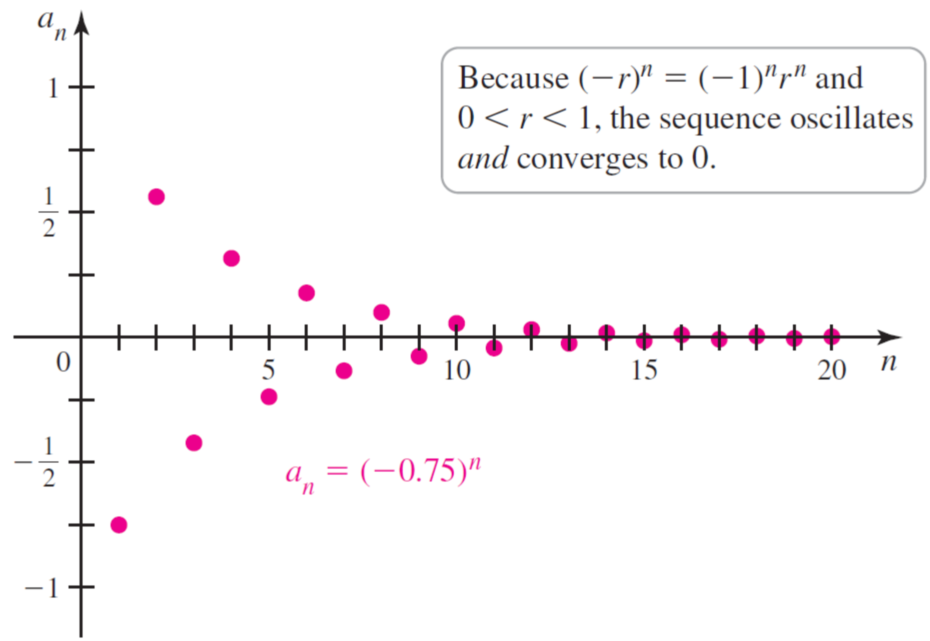
Geometric sequences have the property that each term is obtained by multiplying the previous term by a fixed constant, called the ratio. They have the form $\{r^n\}$ or $\{ar^n\}$, where the ratio r and $a \neq 0$ are real numbers. The value of r determines the behavior of the sequence.

Graph the following sequences and discuss their behavior.

a) $\{0.75^n\}$ b) $\{(-0.75)^n\}$





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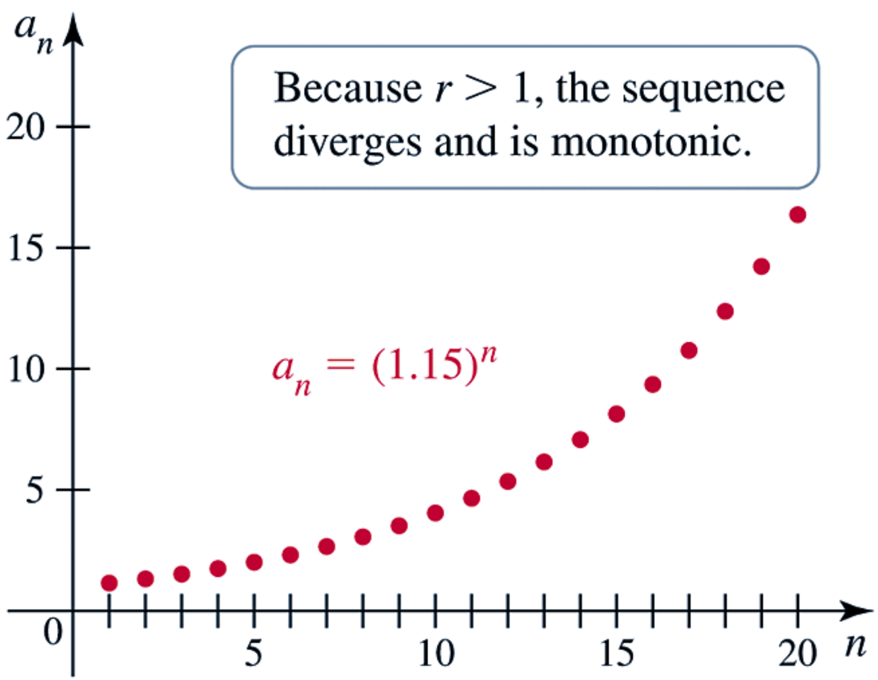
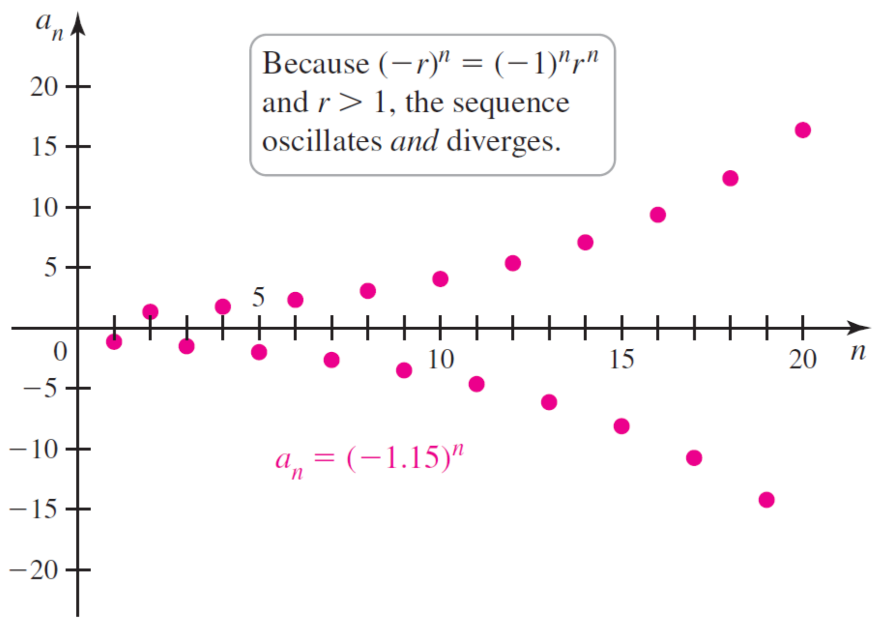
Geometric sequences

Graph the following sequences and discuss their behavior.

c) $\{1.15^n\}$



d) $\{(-1.15)^n\}$



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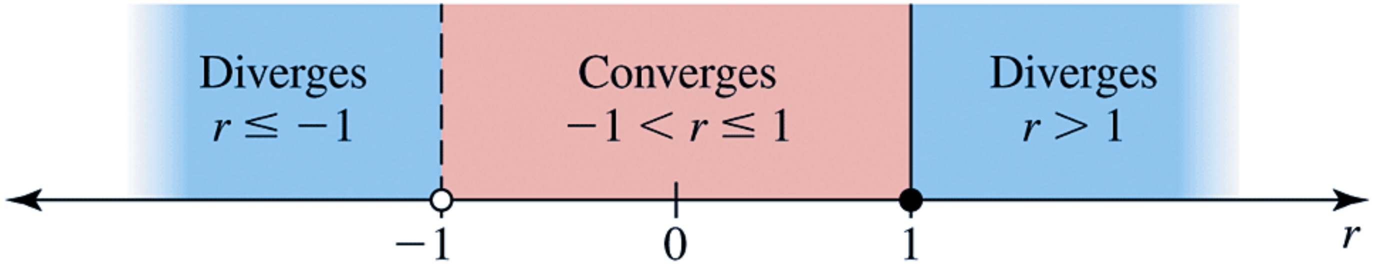
THEOREM 10.3 Geometric Sequences

Let r be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \le -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.



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Example 3 **Using Limit Laws** Determine the limits of the following sequences

a) $b_n = \frac{2n^2+n}{2^n(3n^2-4)}$

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Example 3 **Using Limit Laws** Determine the limits of the following sequences

b) $a_n = 5(0.6)^n - \frac{1}{3^n}$

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The previous examples show that a sequence may display any of the following behaviors:

- It may converge to a single value, which is the limit of the sequence.

- Its terms may increase in magnitude without bound (either with one sign or with mixed signs), in which case the sequence diverges.

- Its terms may remain bounded but settle into an oscillating pattern in which the terms approach two or more values; in this case, the sequence diverges.

- The terms of a sequence may remain bounded, but wander chaotically forever without a pattern. In this case, the sequence also diverges.

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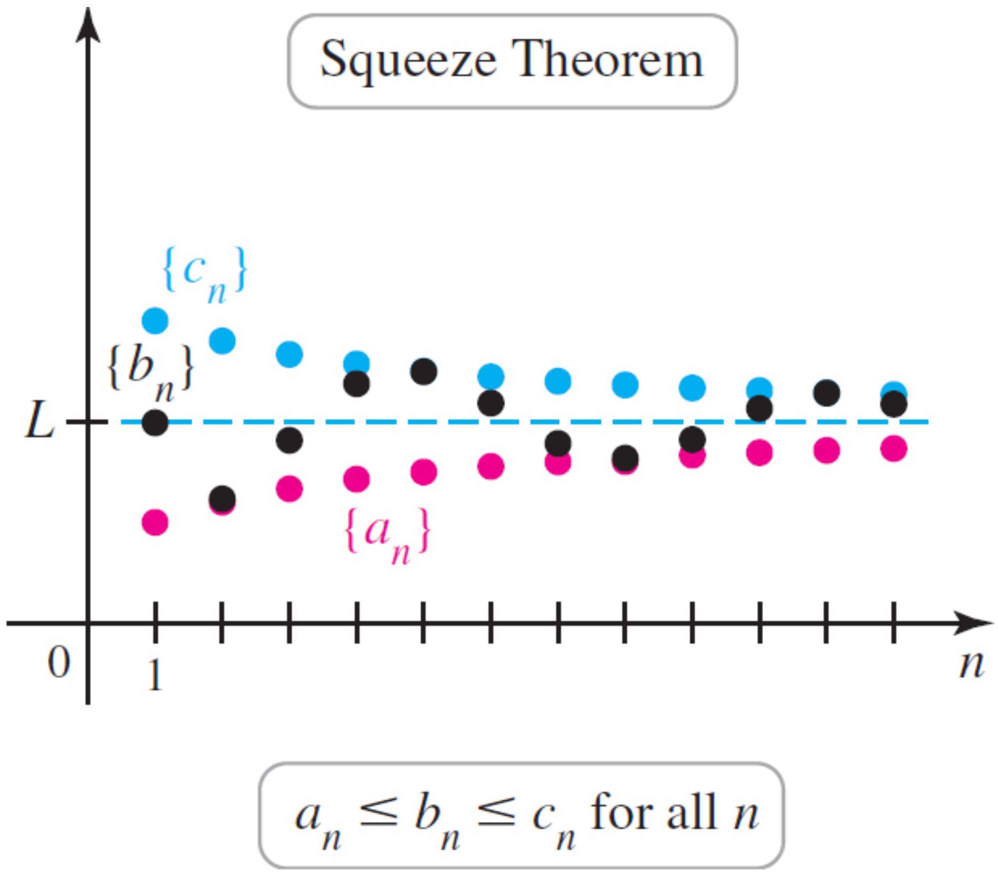
We cite two theorems that are used to evaluate limits and to establish that limits exist. The first theorem is a direct analog of the Squeeze Theorem in Section 2.3.

The Squeeze Theorem

THEOREM 10.4 Squeeze Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \le b_n \le c_n$, for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ (Figure 10.19).



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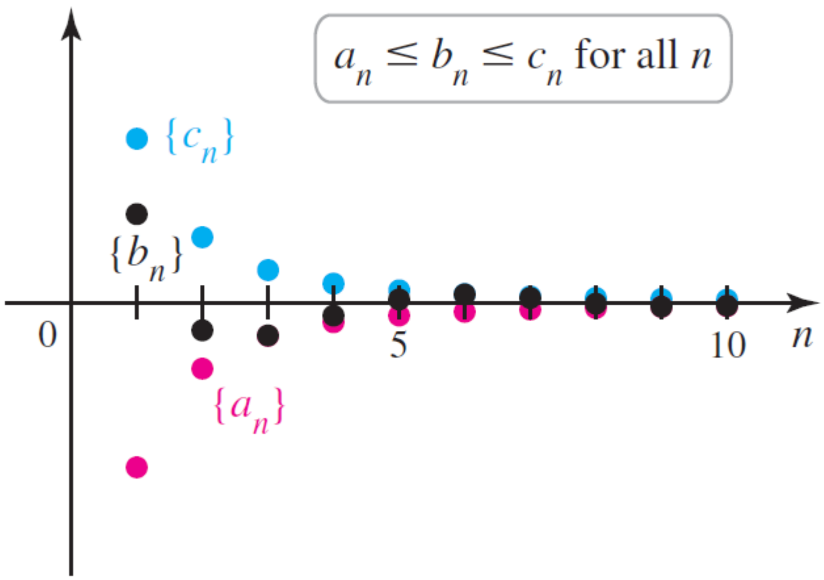
Example 4 **Squeeze Theorem**

Find the limit of the sequence $b_n = \frac{\cos(n)}{n^2+1}$.



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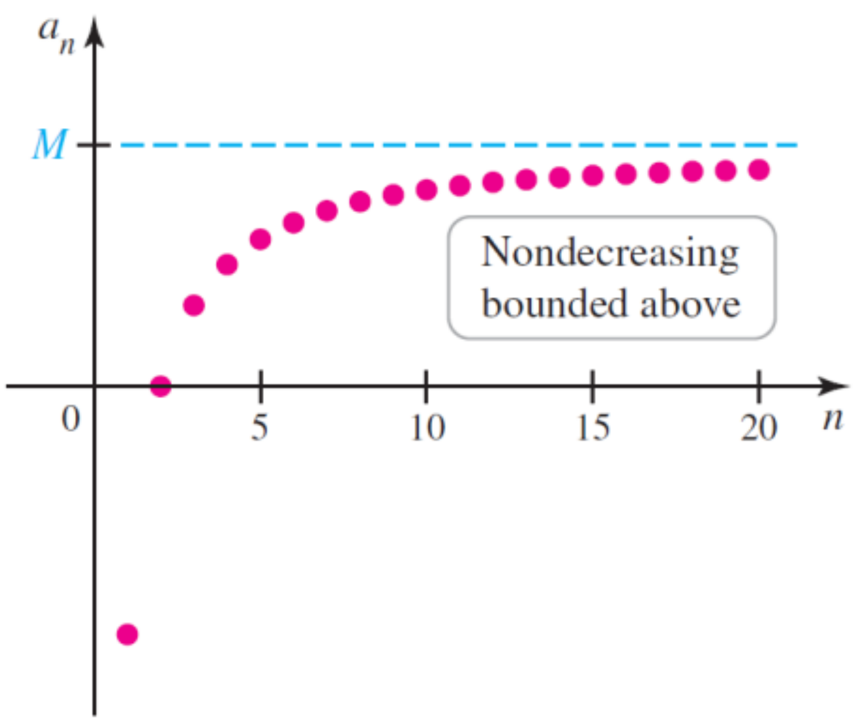
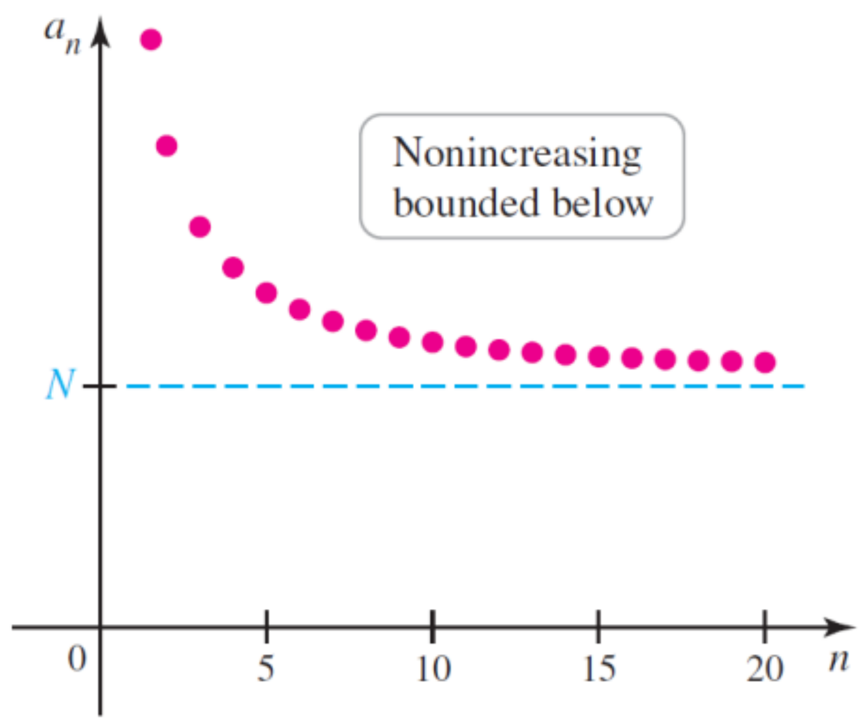
THEOREM 10.5 Bounded Monotonic Sequence

A bounded monotonic sequence converges.

This is a _________________ sequence, all of whose terms are _____ than _____. It must ____________ to a limit less than or equal to ____.

This is a _________________ sequence, all of whose terms are ________ than _____. It must ____________ to a limit greater than or equal to ____.





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An Application: Recurrence Relations

Example 5 **Sequences for drug doses**

Suppose your doctor prescribes a 100-mg dose of an antibiotic every 12 hours. Furthermore, the drug is known to have a half-life of 12 hours; that is, every 12 hours half of the drug in your blood is eliminated.

a) Find the sequence that gives the amount of drug in your blood immediately after each dose.

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Example 5 **Sequences for drug doses**

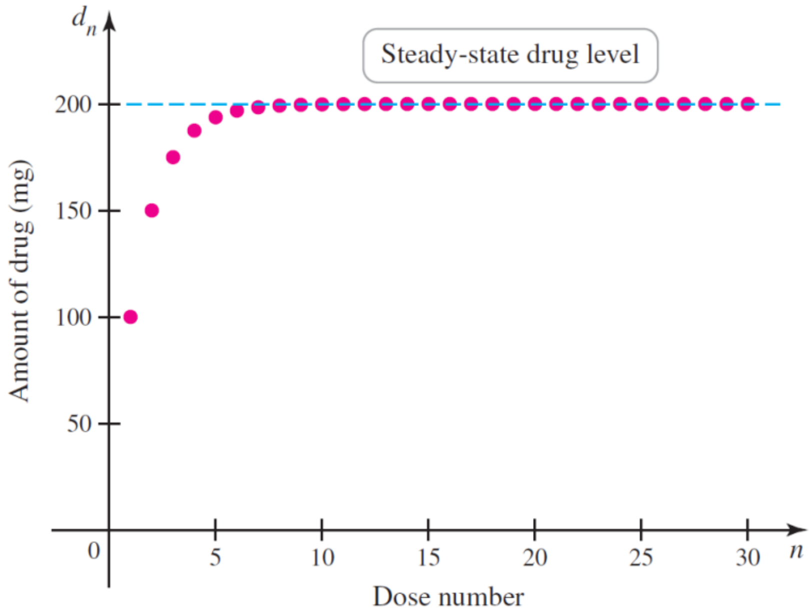
b) Use the graph to propose the limit of this sequence; that is, in the long run, how much drug do you have in your blood?

Calc Set Up: nMin =____,

u(n) = ________

u(nMin) =___

c) Find the limit of the sequence directly.



Videos for the Calculator: https://www.youtube.com/watch?v=js7n7fcSSUI
https://www.youtube.com/watch?v=DJzbdKj2NDw

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Growth Rates of Sequences

THEOREM 10.6 Growth Rates of Sequences
The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for positive real numbers p, q, r, s, and $b > 1$.

$\{a_n\} << \{b_n\}$ means that $\{b_n\}$ grows faster than $\{a_n\}$ as $n \to \infty$.

Factorial sequence $\{n!\} = n\,(n-1)\,(n-2)\,(n-3)\ \ldots(2)(1)$.
It is worth noting that the rankings in Theorem 10.6 do not change if a sequence is multiplied by a positive constant.

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Example 6 **Competing sequences**

Compare the growth rates of the following sequences to determine whether the following sequences converge.

a) $\left\{\frac{\ln n^{10}}{.00001n}\right\}$

b) $\left\{\frac{n^8 \ln n}{n^{8.001}}\right\}$

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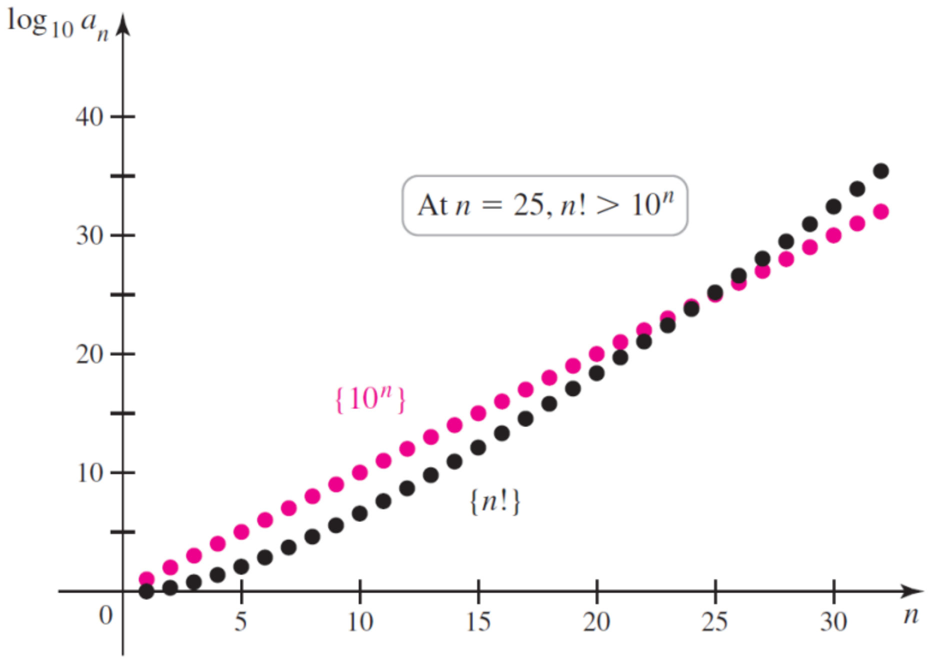
Example 6 **Competing sequences**

Compare the growth rates of the following sequences to determine whether the following sequences converge.

c) $\left\{\frac{n!}{10^n}\right\}$



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Formal Definition of a Limit of a Sequence

DEFINITION **Limit of a Sequence**

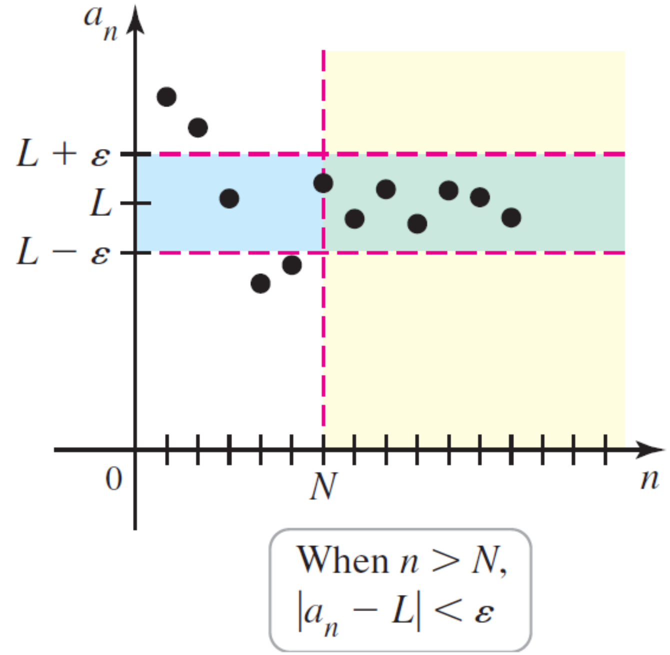
The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if, given any $\varepsilon > 0$, it is possible to find a positive integer N (depending only on ε) such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

If the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L.$$

A sequence that does not converge is said to **diverge**.



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