# Constructing a Neuron using Column Generation

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## Abstract

We consider the problem of describing neurons in single neuron images. To do this we map the problem to a problem of constructing a lineage to one

## 1 Formulation

### 1.1 Tracks

- We describe the set of detections as $\mathcal{D}$ which we index with $d$.
- We describe the set of tracks as $\mathcal{P}$ which we index with $p$.
- We use $X \in \{0,1\}^{|\mathcal{D}\times\mathcal{P}|}$ to denote a mapping of detections to tracks where $X_{dp} = 1$ indicates that track $d$ is associated with track $p$.
- A track consists of as sequence of sub-tracks each of $k$ detections where $k$ is a user defined hyper-parameter that trades off model complexity and efficiency of inference.
- We use $\bar{X} \in \{0,1\}^{|\mathcal{D}\times\mathcal{P}|}$ to denote a mapping of detections to tracks where $X_{dp} = 1$ indicates that track $d$ is associated with track $p$ and detection $d$ is not in the first $k-1$ detections on the track. .

### 1.2 Sub-Tracks

- We define the set of subtracks as $\mathcal{S}$ which we index with $s$. A given subtrack has elements $\{s_1, s_2, s_3 ... s_k\}$ ordered in time from earliest to latest.
- We use $F \in \{0,1\}^{|\mathcal{D}|\times|\mathcal{S}|}$ which we index by $d, s$ respectively. We set $F_{ds} = 1$ if and only if detection $d$ is in subtrack $s$.
- We use $F^- \in \{0,1\}^{|\mathcal{D}|\times|\mathcal{S}|}$ which we index by $d, s$ respectively. We set $F_{ds} = 1$ if and only if detection $d$ is the final detection on subtrack $s$.
- We define a mapping of tracks to subtracks using a matrix $S^0 \in \{0,1\}^{|\mathcal{S}|\times|\mathcal{P}|}$. We use $S^0_{sp} = 1$ to indicate that track $p$ contains subtrack $s$ as neither the start nor the end.
- In order to describe the first subtrack on a track we use matrix $S^+ \in \{0,1\}^{|\mathcal{S}|\times|\mathcal{P}|}$ which we index with $s, p$ where $S^+_{sp} = 1$ if and only if subtrack $s$ is the first subtrack on track $p$
- In order to describe the first subtrack on a track we use matrix $S^- \in \{0,1\}^{|\mathcal{S}|\times|\mathcal{P}|}$ which we index with $s, p$ where $S^+_{sp} = 1$ if and only if subtrack $s$ is the first subtrack on track $p$
- The subtracks in the sequence that describes a track overlap eachother. Hence if a sub-tracks $s_1$ is succeeded by another subtrack $s_2$ on a given track then the final $k-1$ elements on $s_1$ are the same as the earliest $k-1$ elements in $s_2$
- We use $Q \in \{0,1\}^{|\mathcal{S}|\times|\mathcal{S}|}$ which we index by $s_1, s_2$ respectively. We set $Q_{s_1 s_2} = 1$ if and only if subtrack $s_1$ can succeed $s_2$.

## 1.3 Costs

We associate tracks with costs with costs usign the folowing noattion. We use $\Theta \in \mathbb{R}^{\mathcal{P}}$ which we index by $p$ to associate tracks with costs. We use $\Theta_p$ to associate track $p$ with a cost.

- We use $\theta \in \mathbb{R}^{|\mathcal{S}|,3}$ which we index by $s/[+,-,0]$ respectively.
- We use $\theta_{s+}$ to denote the cost of starting a track at subtrack $s$.
- We use $\theta_{s-}$ to denote the cost of terminating a track at subtrack $s$.
- We use $\theta_{s0}$ to denote the cost of including a subtrack $s$ in a track as neither the start nor the end .

We assocaite a track with cost with cost $\Theta_p$ as follows.

$$\Theta_p = \sum_{s \in \mathcal{S}} S^+_{sp}\theta_{s+} + S^-_{sp}\theta_{s-} + S^0_{sp}\theta_{s0} \tag{1}$$

## 1.4 Collection of Tracks

We describe a collection of tracks that describe a neuron using a vector $\gamma \in \{0,1\}^{|\mathcal{P}|}$ which we index with $p$. We set $\gamma_p = 1$ if and only if track $p$ is incuded in the neuron.

We use $\Gamma$ to describe the set of all possible neurons. This a subset of $\gamma \in \{0,1\}^{|\mathcal{P}|}$ The cost associated with a nueron described by $\gamma$ is defined by the sum of the tracks that compose it. The selection of the lowest cost neuron is thuse written below

$$\min_{\gamma \in \Gamma} \sum_p \gamma_p \Theta_p \tag{2}$$

## 1.5 Feasibility

We assume that the soma is defined by a subtrack $s_0$ which is a special subtrack that initializes the neuron.

A track is included or not included : $\gamma_p \in \{0,1\}$

No two tracks can continue through a given detection

$$\sum_p \gamma_p \hat{X}_{dp} \leq 1 \quad \forall d \tag{3}$$

A detetion can not be part of more than two tracks. This blocks succession in close proximity.

$$\sum_p \gamma_p X_{dp} \leq 2 \quad \forall d \tag{4}$$

A track can not split off a subtrack unless that sub-track is already on a track.

$$\sum_p \sum_s Q_{ss_1} S^+_{s_1 p}\gamma_p \leq \sum_p S^0_{sp}\gamma_p \tag{5}$$

If a track terminates at a given detection then no detections can start off it. A strong penalty for ending a track early negates the need for this. Since this strong penalty has been described I ignore this constraint in the document.

$$\sum_p \gamma_p \sum_s (F_{ds} - F^-_{ds}) \leq (1 - \sum_p \gamma_p \sum_s S^-_p F^-_{ds}) \tag{6}$$

## 2 LP relaxation

$$\min_{\substack{\gamma\geq 0\\ \sum_p \gamma_p \hat{X}_{dp}\leq 1\\ \sum_p \gamma_p X_{dp}\leq 2\\ \sum_p\sum_s Q_{ss_1}S^+_{s_1p}\gamma_p\leq \sum_p S^0_{sp}\gamma_p}} \sum_p \gamma_p\Theta_p \tag{7}$$

We now take the dual form of this. We use lagragne multipliers $\lambda^1 \in \mathbb{R}^{|\mathcal{D}|}_{0+}$,$\lambda^2 \in \mathbb{R}^{|\mathcal{D}|}_{0+}$,$\lambda^3 \in \mathbb{R}^{|\mathcal{S}|}_{0+}$ to respreset nteh constraints above in dual form.

$$\max_{\substack{\lambda^1\geq 0\\ \lambda^2\geq 0\\ \lambda^3\geq 0}} -\sum_d(\lambda^1_d+2\lambda^2_d) \tag{8}$$

$$\Theta_p+\sum_d \hat{X}_{dp}\lambda^1_d+\sum_d X_{dp}\lambda^2_d-\sum_s(S^0_{sp})\lambda^3_s+\sum_{s_1}\lambda^3_{s_1}\sum_{s_2}Q_{s_1s_2}S^+_{s_2p}\geq 0 \tag{9}$$

Finding the most violated constraint is a dynamic program. Many constraints can be generated at once.

## 3 augmenting

To make things easier I susepct adding the following will help. We will make its multiplier slightly less

$$\sum_p \gamma_p(X_{dp}-\hat{X}_{dp})\leq \sum_p \gamma_p\hat{X}_{dp}\forall d\notin s_0 \tag{10}$$

This is a weaker constarint that that imposed by $\lambda^3$ but can be expressed in addition. It has thebenifit that it operates on a small number of variables $\mathcal{D}$ not $\mathcal{S}$. We express it with multipliers $\lambda^4 \in \mathbb{R}^{|\mathcal{D}}$

$$\max_{\substack{\lambda^1\geq 0\\ \lambda^2\geq 0\\ \lambda^3\geq 0}} -\sum_d(\lambda^1_d+2\lambda^2_d) \tag{11}$$

$$\Theta_p+\sum_d \hat{X}_{dp}\lambda^1_d+\sum_d X_{dp}\lambda^2_d-\sum_s(S^0_{sp})\lambda^3_s+\sum_{s_1}\lambda^3_{s_1}\sum_{s_2}Q_{s_1s_2}S^+_{s_2p}+\sum_{d\notin s_0}\lambda^4_d(X_{dp}-2\hat{X}_{dp})\geq 0 \tag{12}$$

## 4 Dyanmic form

Finidng the most violated constraint is a dynamic program.

$$\min_p \Theta_p+\sum_d \hat{X}_{dp}\lambda^1_d+\sum_d X_{dp}\lambda^2_d-\sum_s(S^0_{sp})\lambda^3_s+\sum_{s_1}\lambda^3_{s_1}\sum_{s_2}Q_{s_1s_2}S^+_{s_2p}+\sum_{d\notin s_0}\lambda^4_d(X_{dp}-2\hat{X}_{dp}) \tag{13}$$

We now plug in for $\Theta_p$

$$\min_p \sum_{s\in\mathcal{S}} S^+_{sp}\theta_{s+}+S^-_{sp}\theta_{s-}+S^0_{sp}\theta_{s0} \tag{14}$$

$$+\sum_d \hat{X}_{dp}\lambda^1_d+\sum_d X_{dp}\lambda^2_d-\sum_s(S^0_{sp})\lambda^3_s+\sum_{s_1}\lambda^3_{s_1}\sum_{s_2}Q_{s_1s_2}S^+_{s_2p}+\sum_{d\notin s_0}\lambda^4_d(X_{dp}-2\hat{X}_{dp})$$

The lowest cost track terminating at $s_2$ can be written as follows.

Let $\ell_{0s}$ be the cost to start and end a track at subtrack $s$. Let $\ell_{s_1 s_2}$ be of the lowest cost track ending ins $s_2$ wiht $s_1$ as its penultimate subtrack.

$$\ell_{0s} = \theta_s^- + \theta_s^+ - \lambda_s^3 + \sum_d F_{ds}(\lambda_d^4 + \lambda_d^2) + \sum_d F_{ds}^-(\lambda_d^1 - 2\lambda_d^2) + \sum_{s_1} \lambda_{s_1}^3 Q_{s_1 s} \tag{15}$$

$$\ell_{\hat{s}s} = \ell_{\hat{s}} - \theta_{\hat{s}}^- + \theta_{\hat{s}}^0 + \theta_s^- - \lambda_s^3 + \sum_d F_{ds}^-(\lambda_d^1 + \lambda_d^2 - \lambda_d^4) \tag{16}$$

$$\ell_s = \min[\ell_{0s}, \min_{\substack{\hat{s} \\ Q_{\hat{s}s=1}}} \ell_{\hat{s}s}] \tag{17}$$