Constructing a Neuron using Column Generation

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Abstract

We consider the problem of describing neurons in single neuron images. To do this we map the problem to a problem of constructing a lineage to one

1 Formulation

1.1 Tracks

- We describe the set of detections as $\mathcal{D}$ which we index with d.
- We describe the set of tracks as $\mathcal{P}$ which we index with p.
- We use $X \in \{0,1\}^{|\mathcal{D}\times\mathcal{P}|}$ to denote a mapping of detections to tracks where $X_{dp} = 1$ indicates that track d is associated with track p.
- A track consists of as sequence of sub-tracks each of k detections where k is a user defined hyper-parameter that trades off model complexity and efficiency of inference.
- We use $\bar{X} \in \{0,1\}^{|\mathcal{D}\times\mathcal{P}|}$ to denote a mapping of detections to tracks where $X_{dp} = 1$ indicates that track d is associated with track p and detection d is not in the first $k-1$ detections on the track. .

1.2 Sub-Tracks

- We define the set of subtracks as $\mathcal{S}$ which we index with s. A given subtrack has elements $\{s_1, s_2, s_3 ... s_k\}$ ordered in time from earliest to latest.
- We use $F \in \{0,1\}^{|\mathcal{D}|\times|\mathcal{S}|}$ which we index by d, s respectively. We set $F_{ds} = 1$ if and only if detection d is in subtrack s.
- We use $F^- \in \{0,1\}^{|\mathcal{D}|\times|\mathcal{S}|}$ which we index by d, s respectively. We set $F_{ds} = 1$ if and only if detection d is the final detection on subtrack s.
- We define a mapping of tracks to subtracks using a matrix $S^0 \in \{0,1\}^{|\mathcal{S}|\times|\mathcal{P}|}$. We use $S^0_{sp} = 1$ to indicate that track p contains subtrack s as neither the start nor the end.
- In order to describe the first subtrack on a track we use matrix $S^+ \in \{0,1\}^{|\mathcal{S}|\times|\mathcal{P}|}$ which we index with s, p where $S^+_{sp} = 1$ if and only if subtrack s is the first subtrack on track p
- In order to describe the first subtrack on a track we use matrix $S^- \in \{0,1\}^{|\mathcal{S}|\times|\mathcal{P}|}$ which we index with s, p where $S^+_{sp} = 1$ if and only if subtrack s is the first subtrack on track p
- The subtracks in the sequence that describes a track overlap eachother. Hence if a sub-tracks s_1 is succeeded by another subtrack s_2 on a given track then the final $k-1$ elements on s_1 are the same as the earliest $k-1$ elements in s_2
- We use $Q \in \{0,1\}^{|\mathcal{S}|\times|\mathcal{S}|}$ which we index by s_1, s_2 respectively. We set $Q_{s_1 s_2} = 1$ if and only if subtrack s_1 can succeed s_2.

1.3 Costs

We associate tracks with costs with costs usign the folowing noattion. We use $\Theta \in \mathbb{R}^{\mathcal{P}}$ which we index by p to associate tracks with costs. We use Θ_p to associate track p with a cost.

- We use $\theta \in \mathbb{R}^{|\mathcal{S}|,3}$ which we index by $s/[+,-,0]$ respectively.
- We use θ_{s+} to denote the cost of starting a track at subtrack s.
- We use θ_{s-} to denote the cost of terminating a track at subtrack s.
- We use θ_{s0} to denote the cost of including a subtrack s in a track as neither the start nor the end .

We assocaite a track with cost with cost Θ_p as follows.

$$\Theta_p = \sum_{s \in \mathcal{S}} S^+_{sp}\theta_{s+} + S^-_{sp}\theta_{s-} + S^0_{sp}\theta_{s0} \tag{1}$$

1.4 Collection of Tracks

We describe a collection of tracks that describe a neuron using a vector $\gamma \in \{0,1\}^{|\mathcal{P}|}$ which we index with p. We set $\gamma_p = 1$ if and only if track p is incuded in the neuron.

We use Γ to describe the set of all possible neurons. This a subset of $\gamma \in \{0,1\}^{|\mathcal{P}|}$ The cost associated with a nueron described by γ is defined by the sum of the tracks that compose it. The selection of the lowest cost neuron is thuse written below

$$\min_{\gamma \in \Gamma} \sum_p \gamma_p \Theta_p \tag{2}$$

1.5 Feasibility

We assume that the soma is defined by a subtrack s_0 which is a special subtrack that initializes the neuron.

A track is included or not included : $\gamma_p \in \{0,1\}$

No two tracks can continue through a given detection

$$\sum_p \gamma_p \hat{X}_{dp} \leq 1 \quad \forall d \tag{3}$$

A detetion can not be part of more than two tracks. This blocks succession in close proximity.

$$\sum_p \gamma_p X_{dp} \leq 2 \quad \forall d \tag{4}$$

A track can not split off a subtrack unless that sub-track is already on a track.

$$\sum_p \sum_s Q_{ss_1} S^+_{s_1 p} \gamma_p \leq \sum_p S^0_{sp} \gamma_p \tag{5}$$

If a track terminates at a given detection then no detections can start off it. A strong penalty for ending a track early negates the need for this. Since this strong penalty has been described I ignore this constraint in the document.

$$\sum_p \gamma_p \sum_s (F_{ds} - F^-_{ds}) \leq (1 - \sum_p \gamma_p \sum_s S^-_p F^-_{ds}) \tag{6}$$

2 LP relaxation

$$\min_{\substack{\gamma\geq 0 \\ \sum_p \gamma_p \hat{X}_{dp}\leq 1 \\ \sum_p \gamma_p X_{dp}\leq 2 \\ \sum_p \sum_s Q_{ss_1} S^+_{s_1 p}\gamma_p \leq \sum_p S^0_{sp}\gamma_p}} \sum_p \gamma_p \Theta_p \tag{7}$$

We now take the dual form of this. We use lagragne multipliers $\lambda^1 \in \mathbb{R}^{|\mathcal{D}|}_{0+}$,$\lambda^2 \in \mathbb{R}^{|\mathcal{D}|}_{0+}$,$\lambda^3 \in \mathbb{R}^{|\mathcal{S}|}_{0+}$ to respreset nteh constraints above in dual form.

$$\max_{\substack{\lambda^1\geq 0 \\ \lambda^2\geq 0 \\ \lambda^3\geq 0}} -\sum_d (\lambda^1_d + 2\lambda^2_d) \tag{8}$$

$$\Theta_p + \sum_d \hat{X}_{dp}\lambda^1_d + \sum_d X_{dp}\lambda^2_d - \sum_s (S^0_{sp})\lambda^3_s + \sum_{s_1} \lambda^3_{s_1} \sum_{s_2} Q_{s_1 s_2} S^+_{s_2 p} \geq 0 \tag{9}$$

Finding the most violated constraint is a dynamic program. Many constraints can be generated at once.

3 augmenting

To make things easier I susepct adding the following will help. We will make its multiplier slightly less

$$\sum_p \gamma_p (X_{dp} - \hat{X}_{dp}) \leq \sum_p \gamma_p \hat{X}_{dp} \forall d \notin s_0 \tag{10}$$

This is a weaker constarint that that imposed by λ^3 but can be expressed in addition. It has thebenifit that it operates on a small number of variables $\mathcal{D}$ not $\mathcal{S}$. We express it with multipliers $\lambda^4 \in \mathbb{R}^{|\mathcal{D}}$

$$\max_{\substack{\lambda^1\geq 0 \\ \lambda^2\geq 0 \\ \lambda^3\geq 0}} -\sum_d (\lambda^1_d + 2\lambda^2_d) \tag{11}$$

$$\Theta_p + \sum_d \hat{X}_{dp}\lambda^1_d + \sum_d X_{dp}\lambda^2_d - \sum_s (S^0_{sp})\lambda^3_s + \sum_{s_1} \lambda^3_{s_1} \sum_{s_2} Q_{s_1 s_2} S^+_{s_2 p} + \sum_{d\notin s_0} \lambda^4_d (X_{dp} - 2\hat{X}_{dp}) \geq 0 \tag{12}$$

4 Dyanmic form

Finidng the most violated constraint is a dynamic program.

$$\min_p \Theta_p + \sum_d \hat{X}_{dp}\lambda^1_d + \sum_d X_{dp}\lambda^2_d - \sum_s (S^0_{sp})\lambda^3_s + \sum_{s_1} \lambda^3_{s_1} \sum_{s_2} Q_{s_1 s_2} S^+_{s_2 p} + \sum_{d\notin s_0} \lambda^4_d (X_{dp} - 2\hat{X}_{dp}) \tag{13}$$

We now plug in for Θ_p

$$\min_p \sum_{s\in\mathcal{S}} S^+_{sp}\theta_{s+} + S^-_{sp}\theta_{s-} + S^0_{sp}\theta_{s0} \tag{14}$$

$$+\sum_d \hat{X}_{dp}\lambda^1_d + \sum_d X_{dp}\lambda^2_d - \sum_s (S^0_{sp})\lambda^3_s + \sum_{s_1} \lambda^3_{s_1} \sum_{s_2} Q_{s_1 s_2} S^+_{s_2 p} + \sum_{d\notin s_0} \lambda^4_d (X_{dp} - 2\hat{X}_{dp})$$

The lowest cost track terminating at s_2 can be written as follows.

Let ℓ_{0s} be the cost to start and end a track at subtrack s. Let $\ell_{s_1 s_2}$ be of the lowest cost track ending ins s_2 wiht s_1 as its penultimate subtrack.

$$\ell_{0s} = \theta^-_s + \theta^+_s - \lambda^3_s + \sum_d F_{ds}(\lambda^4_d + \lambda^2_d) + \sum_d F^-_{ds}(\lambda^1_d - 2\lambda^2_d) + \sum_{s_1} \lambda^3_{s_1} Q_{s_1 s} \tag{15}$$

$$\ell_{\hat{s}s} = \ell_{\hat{s}} - \theta^-_{\hat{s}} + \theta^0_{\hat{s}} + \theta^-_s - \lambda^3_s + \sum_d F_{ds}(\lambda^4_d + \lambda^2_d) + \sum_d F^-_{ds}(\lambda^1_d - 2\lambda^2_d) + \tag{16}$$