Constructing a Neuron using Column Generation

Julian Yarkony
Experian Data Lab
San Diego CA
julian.e.yarkony@gmail.com

Abstract

We consider the problem of describing neurons in single neuron images. To do this we map the problem to a problem of constructing a lineage to one

1 Simulation

To validate the method that we present here, we generate synthetic data where we know the ground truth and ask the model described above to reconstruct the morphology of neuron. The synthetic data represent by a $n \times 3$ matrix where n is the number of all detection in the whole recording and the columns are x coordinate, y coordinate and time of each detection. Hence the set $\mathcal{D}$ has n elements. Also we can fit $k = 5$ since usually the segments of neuron (the section between two branching point) have the length longer than 5 detection. Finally to apply the model we need to know the cost function. The neuron can be detected by solving a minimization problem which is written in 3. In this setup, a neuron is a combination of tracks (i.e. segments), indexed by p, and to find the cost of detection of one neuron, the cost of each of its tracks should be calculated separately and then summation over all tracks the cost of the neuron can be calculated. The constrains of ILP are coming from the fact that the combination of these segments are a tree like object. For example, at most two track have a common starting point. In sum, by knowing the cost of each track, we can calculate the cost of neuron. Therefore we need to know the cost of one track in formula 2 For the first simulation, I have dropped the amplitude of each detection and hence we can ignore the notation for starting and terminating by a sub-track. Indeed this can be justify by saying that a track, a set of sub-tracks, are modeled here by having different behavior at its end points compared to its intermediate points and the source of this different behavior is coming from amplitude (not the location) of detection. Hence instead of having $\theta_{s+}, \theta_{s-}, \theta_{s0}$ we use one notation θ_s where s is the sub-track. Finally to define the cost of sub-track, we can define:

$$\theta(s) = \theta(\{s_1, \cdots s_k\}) = \sum_{i=1}^{k-1} ||\text{location}(s_k) - \text{location}(s_{k+1})||_2 \tag{1}$$

2 Formulation

2.1 Tracks

- We describe the set of detections as $\mathcal{D}$ which we index with d.
- We describe the set of tracks as $\mathcal{P}$ which we index with p.
- We use $X \in \{0,1\}^{|\mathcal{D} \times \mathcal{P}|}$ to denote a mapping of detections to tracks where $X_{dp} = 1$ indicates that detection d is associated with track p.
- A track consists of as sequence of sub-tracks each of k detections where k is a user defined hyper-parameter that trades off model complexity and efficiency of inference.

- We use $\bar{X} \in \{0,1\}^{|\mathcal{D} \times \mathcal{P}|}$ to denote a mapping of detections to tracks where $\bar{X}_{dp} = 1$ indicates that detection d is associated with track p and detection d is not in the first $k-1$ detections on the track.

2.2 Sub-Tracks

- We define the set of sub-tracks as $\mathcal{S}$ which we index with s. A given sub-track has elements $\{s_1, s_2, s_3 ... s_k\}$ ordered in time from earliest to latest.
- We use $F \in \{0,1\}^{|\mathcal{D}| \times |\mathcal{S}|}$ which we index by d, s respectively. We set $F_{ds} = 1$ if and only if detection d is in sub-track s.
- We use $F^- \in \{0,1\}^{|\mathcal{D}| \times |\mathcal{S}|}$ which we index by d, s respectively. We set $F_{ds} = 1$ if and only if detection d is the final detection on sub-track s.
- We define a mapping of tracks to sub-tracks using a matrix $S^0 \in \{0,1\}^{|\mathcal{S}| \times |\mathcal{P}|}$. We use $S^0_{sp} = 1$ to indicate that track p contains sub-track s as neither the start nor the end.
- In order to describe the first sub-track on a track we use matrix $S^+ \in \{0,1\}^{|\mathcal{S}| \times |\mathcal{P}|}$ which we index with s, p where $S^+_{sp} = 1$ if and only if sub-track s is the first sub-track on track p
- In order to describe the first sub-track on a track we use matrix $S^- \in \{0,1\}^{|\mathcal{S}| \times |\mathcal{P}|}$ which we index with s, p where $S^+_{sp} = 1$ if and only if sub-track s is the first sub-track on track p
- The sub-tracks in the sequence that describes a track overlap each other. Hence if a sub-tracks s_1 is succeeded by another sub-track s_2 on a given track then the final $k-1$ elements on s_1 are the same as the earliest $k-1$ elements in s_2
- We use $Q \in \{0,1\}^{|\mathcal{S}| \times |\mathcal{S}|}$ which we index by s_1, s_2 respectively. We set $Q_{s_1 s_2} = 1$ if and only if sub-track s_1 can succeed s_2.

2.3 Costs

We associate tracks with costs with costs using the following notation. We use $\Theta \in \mathbb{R}^{\mathcal{P}}$ which we index by p to associate tracks with costs. We use Θ_p to associate track p with a cost.

- We use $\theta \in \mathbb{R}^{|\mathcal{S}|,3}$ which we index by $s/[+, -, 0]$ respectively.
- We use θ_{s+} to denote the cost of starting a track at sub-track s.
- We use θ_{s-} to denote the cost of terminating a track at sub-track s.
- We use θ_{s0} to denote the cost of including a sub-track s in a track as neither the start nor the end .

We associate a track with cost with cost Θ_p as follows:

$$\Theta_p = \sum_{s \in \mathcal{S}} S^+_{sp} \theta_{s+} + S^-_{sp} \theta_{s-} + S^0_{sp} \theta_{s0} \tag{2}$$

2.4 Collection of Tracks

We describe a collection of tracks that describe a neuron using a vector $\gamma \in \{0,1\}^{|\mathcal{P}|}$ which we index with p. We set $\gamma_p = 1$ if and only if track p is included in the neuron.

We use Γ to describe the set of all possible neurons. This a subset of $\gamma \in \{0,1\}^{|\mathcal{P}|}$ The cost associated with a neuron described by γ is defined by the sum of the tracks that compose it. The selection of the lowest cost neuron is thus written below

$$\min_{\gamma \in \Gamma} \sum_p \gamma_p \Theta_p \tag{3}$$

2.5 Feasibility

We assume that the soma is defined by a sub-track s_0 which is a special sub-track that initializes the neuron.

A track is included or not included : $\gamma_p \in \{0,1\}$

No two tracks can continue through a given detection

$$\sum_p \gamma_p \hat{X}_{dp} \leq 1 \quad \forall d \tag{4}$$

A detetion can not be part of more than two tracks. This blocks succession in close proximity.

$$\sum_p \gamma_p X_{dp} \leq 2 \quad \forall d \tag{5}$$

A track can not split off a sub-track unless that sub-track is already on a track.

$$\sum_p \sum_s Q_{ss_1} S^+_{s_1 p} \gamma_p \leq \sum_p S^0_{sp} \gamma_p \tag{6}$$

If a track terminates at a given detection then no detections can start off it. A strong penalty for ending a track early negates the need for this. Since this strong penalty has been described I ignore this constraint in the document.

$$\sum_p \gamma_p \sum_s (F_{ds} - F^-_{ds}) \leq (1 - \sum_p \gamma_p \sum_s S^-_p F^-_{ds}) \tag{7}$$

3 LP relaxation

$$\min_{\substack{\gamma \geq 0 \\ \sum_p \gamma_p \hat{X}_{dp} \leq 1 \\ \sum_p \gamma_p X_{dp} \leq 2 \\ \sum_p \sum_s Q_{ss_1} S^+_{s_1 p} \gamma_p \leq \sum_p S^0_{sp} \gamma_p}} \quad \sum_p \gamma_p \Theta_p \tag{8}$$

We now take the dual form of this. We use Lagrange multipliers $\lambda^1 \in \mathbb{R}^{|\mathcal{D}|}_{0+}$,$\lambda^2 \in \mathbb{R}^{|\mathcal{D}|}_{0+}$,$\lambda^3 \in \mathbb{R}^{|\mathcal{S}|}_{0+}$ to respreset nteh constraints above in dual form.

$$\max_{\substack{\lambda^1 \geq 0 \\ \lambda^2 \geq 0 \\ \lambda^3 \geq 0}} - \sum_d (\lambda^1_d + 2\lambda^2_d) \tag{9}$$

$$\Theta_p + \sum_d \hat{X}_{dp} \lambda^1_d + \sum_d X_{dp} \lambda^2_d - \sum_s (S^0_{sp}) \lambda^3_s + \sum_{s_1} \lambda^3_{s_1} \sum_{s_2} Q_{s_1 s_2} S^+_{s_2 p} \geq 0 \tag{10}$$

Finding the most violated constraint is a dynamic program. Many constraints can be generated at once.

4 augmenting

To make things easier I susepct adding the following will help. We will make its multiplier slightly less

$$\sum_p \gamma_p (X_{dp} - \hat{X}_{dp}) \leq \sum_p \gamma_p \hat{X}_{dp} \forall d \notin s_0 \tag{11}$$

This is a weaker constarint that that imposed by λ^3 but can be expressed in addition. It has thebenifit that it operates on a small number of variables $\mathcal{D}$ not $\mathcal{S}$. We express it with multipliers $\lambda^4 \in \mathbb{R}^{|\mathcal{D}}$

$$\max_{\substack{\lambda^1 \geq 0 \\ \lambda^2 \geq 0 \\ \lambda^3 \geq 0}} -\sum_d (\lambda^1_d + 2\lambda^2_d) \tag{12}$$

$$\Theta_p + \sum_d \hat{X}_{dp}\lambda^1_d + \sum_d X_{dp}\lambda^2_d - \sum_s (S^0_{sp})\lambda^3_s + \sum_{s_1} \lambda^3_{s_1} \sum_{s_2} Q_{s_1 s_2} S^+_{s_2 p} + \sum_{d \notin s_0} \lambda^4_d (X_{dp} - 2\hat{X}_{dp}) \geq 0 \tag{13}$$

5 Dyanmic form

Finidng the most violated constraint is a dynamic program.

$$\min_p \Theta_p + \sum_d \hat{X}_{dp}\lambda^1_d + \sum_d X_{dp}\lambda^2_d - \sum_s (S^0_{sp})\lambda^3_s + \sum_{s_1} \lambda^3_{s_1} \sum_{s_2} Q_{s_1 s_2} S^+_{s_2 p} + \sum_{d \notin s_0} \lambda^4_d (X_{dp} - 2\hat{X}_{dp}) \tag{14}$$

We now plug in for Θ_p

$$\min_p \sum_{s \in \mathcal{S}} S^+_{sp}\theta_{s+} + S^-_{sp}\theta_{s-} + S^0_{sp}\theta_{s0} \tag{15}$$

$$+ \sum_d \hat{X}_{dp}\lambda^1_d + \sum_d X_{dp}\lambda^2_d - \sum_s (S^0_{sp})\lambda^3_s + \sum_{s_1} \lambda^3_{s_1} \sum_{s_2} Q_{s_1 s_2} S^+_{s_2 p} + \sum_{d \notin s_0} \lambda^4_d (X_{dp} - 2\hat{X}_{dp})$$

The lowest cost track terminating at s_2 can be written as follows.

Let ℓ_{0s} be the cost to start and end a track at sub-track s. Let $\ell_{s_1 s_2}$ be of the lowest cost track ending ins s_2 wiht s_1 as its penultimate sub-track.

$$\ell_{0s} = \theta^-_s + \theta^+_s - \lambda^3_s + \sum_d F_{ds}(\lambda^4_d + \lambda^2_d) + \sum_d F^-_{ds}(\lambda^1_d - 2\lambda^2_d) + \sum_{s_1} \lambda^3_{s_1} Q_{s_1 s} \tag{16}$$

$$\ell_{\hat{s}s} = \ell_{\hat{s}} - \theta^-_{\hat{s}} + \theta^0_{\hat{s}} + \theta^-_s - \lambda^3_s + \sum_d F^-_{ds}(\lambda^1_d + \lambda^2_d - \lambda^4_d) \tag{17}$$

$$\ell_s = \min[\ell_{0s}, \min_{\substack{\hat{s} \\ Q_{\hat{s}s=1}}} \ell_{\hat{s}s}] \tag{18}$$