Constructing a Neuron using Column Generation

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Abstract

We consider the problem of describing neurons in single neuron images. To do this we map the problem to a problem of constructing a lineage to one

1 Formulation

1.1 Tracks

- We describe the set of detections as $\mathcal{D}$ which we index with d.
- We describe the set of tracks as $\mathcal{P}$ which we index with p.
- We use $X \in \{0,1\}^{|\mathcal{D} \times \mathcal{P}|}$ to denote a mapping of detections to tracks where $X_{dp} = 1$ indicates that detection d is associated with track p.
- A track consists of as sequence of sub-tracks each of k detections where k is a user defined hyper-parameter that trades off model complexity and efficiency of inference.
- We use $\bar{X} \in \{0,1\}^{|\mathcal{D} \times \mathcal{P}|}$ to denote a mapping of detections to tracks where $\bar{X}_{dp} = 1$ indicates that detection d is associated with track p and detection d is not in the first $k-1$ detections on the track.

1.2 Sub-Tracks

- We define the set of sub-tracks as $\mathcal{S}$ which we index with s. A given sub-track has elements $\{s_1, s_2, s_3 ... s_k\}$ ordered in time from earliest to latest.
- We use $F \in \{0,1\}^{|\mathcal{D}| \times |\mathcal{S}|}$ which we index by d, s respectively. We set $F_{ds} = 1$ if and only if detection d is in sub-track s.
- We use $F^{-} \in \{0,1\}^{|\mathcal{D}| \times |\mathcal{S}|}$ which we index by d, s respectively. We set $F_{ds} = 1$ if and only if detection d is the final detection on sub-track s.
- We define a mapping of tracks to sub-tracks using a matrix $S^0 \in \{0,1\}^{|\mathcal{S}| \times |\mathcal{P}|}$. We use $S^0_{sp} = 1$ to indicate that track p contains sub-track s as neither the start nor the end.
- In order to describe the first sub-track on a track we use matrix $S^{+} \in \{0,1\}^{|\mathcal{S}| \times |\mathcal{P}|}$ which we index with s, p where $S^{+}_{sp} = 1$ if and only if sub-track s is the first sub-track on track p
- In order to describe the first sub-track on a track we use matrix $S^{-} \in \{0,1\}^{|\mathcal{S}| \times |\mathcal{P}|}$ which we index with s, p where $S^{+}_{sp} = 1$ if and only if sub-track s is the first sub-track on track p
- The sub-tracks in the sequence that describes a track overlap each other. Hence if a sub-tracks s_1 is succeeded by another sub-track s_2 on a given track then the final $k-1$ elements on s_1 are the same as the earliest $k-1$ elements in s_2
- We use $Q \in \{0,1\}^{|\mathcal{S}| \times |\mathcal{S}|}$ which we index by s_1, s_2 respectively. We set $Q_{s_1 s_2} = 1$ if and only if sub-track s_1 can succeed s_2.

1.3 Costs

We associate tracks with costs with costs using the following notation. We use $\Theta \in \mathbb{R}^{\mathcal{P}}$ which we index by p to associate tracks with costs. We use Θ_p to associate track p with a cost.

- We use $\theta \in \mathbb{R}^{|\mathcal{S}|,3}$ which we index by $s/[+,-,0]$ respectively.
- We use θ_{s+} to denote the cost of starting a track at sub-track s.
- We use θ_{s-} to denote the cost of terminating a track at sub-track s.
- We use θ_{s0} to denote the cost of including a sub-track s in a track as neither the start nor the end .

We associate a track with cost with cost Θ_p as follows:

$$\Theta_p = \sum_{s \in \mathcal{S}} S^+_{sp}\theta_{s+} + S^-_{sp}\theta_{s-} + S^0_{sp}\theta_{s0} \tag{1}$$

1.4 Collection of Tracks

We describe a collection of tracks that describe a neuron using a vector $\gamma \in \{0,1\}^{|\mathcal{P}|}$ which we index with p. We set $\gamma_p = 1$ if and only if track p is included in the neuron.

We use Γ to describe the set of all possible neurons. This a subset of $\gamma \in \{0,1\}^{|\mathcal{P}|}$ The cost associated with a neuron described by γ is defined by the sum of the tracks that compose it. The selection of the lowest cost neuron is thus written below

$$\min_{\gamma \in \Gamma} \sum_p \gamma_p \Theta_p \tag{2}$$

1.5 Feasibility

We assume that the soma is defined by a sub-track s_0 which is a special sub-track that initializes the neuron.

A track is included or not included : $\gamma_p \in \{0,1\}$

No two tracks can continue through a given detection

$$\sum_p \gamma_p \hat{X}_{dp} \leq 1 \quad \forall d \tag{3}$$

A detetion can not be part of more than two tracks. This blocks succession in close proximity.

$$\sum_p \gamma_p X_{dp} \leq 2 \quad \forall d \tag{4}$$

A track can not split off a sub-track unless that sub-track is already on a track.

$$\sum_p \sum_s Q_{ss_1} S^+_{s_1p}\gamma_p \leq \sum_p S^0_{sp}\gamma_p \tag{5}$$

If a track terminates at a given detection then no detections can start off it. A strong penalty for ending a track early negates the need for this. Since this strong penalty has been described I ignore this constraint in the document.

$$\sum_p \gamma_p \sum_s (F_{ds} - F^-_{ds}) \leq (1 - \sum_p \gamma_p \sum_s S^-_p F^-_{ds}) \tag{6}$$

2 LP relaxation

$$\min_{\substack{\gamma\geq 0 \\ \sum_p \gamma_p \hat{X}_{dp}\leq 1 \\ \sum_p \gamma_p X_{dp}\leq 2 \\ \sum_p\sum_s Q_{ss_1}S^+_{s_1p}\gamma_p\leq\sum_p S^0_{sp}\gamma_p}} \sum_p \gamma_p\Theta_p \tag{7}$$

We now take the dual form of this. We use Lagrange multipliers $\lambda^1\in\mathbb{R}^{|\mathcal{D}|}_{0+}$,$\lambda^2\in\mathbb{R}^{|\mathcal{D}|}_{0+}$,$\lambda^3\in\mathbb{R}^{|\mathcal{S}|}_{0+}$ to respreset nteh constraints above in dual form.

$$\max_{\substack{\lambda^1\geq 0\\ \lambda^2\geq 0\\ \lambda^3\geq 0}} -\sum_d(\lambda^1_d+2\lambda^2_d) \tag{8}$$

$$\Theta_p+\sum_d \hat{X}_{dp}\lambda^1_d+\sum_d X_{dp}\lambda^2_d-\sum_s(S^0_{sp})\lambda^3_s+\sum_{s_1}\lambda^3_{s_1}\sum_{s_2}Q_{s_1s_2}S^+_{s_2p}\geq 0 \tag{9}$$

Finding the most violated constraint is a dynamic program. Many constraints can be generated at once.

3 augmenting

To make things easier I susepct adding the following will help. We will make its multiplier slightly less

$$\sum_p \gamma_p(X_{dp}-\hat{X}_{dp})\leq\sum_p\gamma_p\hat{X}_{dp}\forall d\notin s_0 \tag{10}$$

This is a weaker constarint that that imposed by λ^3 but can be expressed in addition. It has thebenifit that it operates on a small number of variables $\mathcal{D}$ not $\mathcal{S}$. We express it with multipliers $\lambda^4\in\mathbb{R}^{|\mathcal{D}}$

$$\max_{\substack{\lambda^1\geq 0\\ \lambda^2\geq 0\\ \lambda^3\geq 0}} -\sum_d(\lambda^1_d+2\lambda^2_d) \tag{11}$$

$$\Theta_p+\sum_d \hat{X}_{dp}\lambda^1_d+\sum_d X_{dp}\lambda^2_d-\sum_s(S^0_{sp})\lambda^3_s+\sum_{s_1}\lambda^3_{s_1}\sum_{s_2}Q_{s_1s_2}S^+_{s_2p}+\sum_{d\notin s_0}\lambda^4_d(X_{dp}-2\hat{X}_{dp})\geq 0 \tag{12}$$

4 Dyanmic form

Finidng the most violated constraint is a dynamic program.

$$\min_p \Theta_p+\sum_d \hat{X}_{dp}\lambda^1_d+\sum_d X_{dp}\lambda^2_d-\sum_s(S^0_{sp})\lambda^3_s+\sum_{s_1}\lambda^3_{s_1}\sum_{s_2}Q_{s_1s_2}S^+_{s_2p}+\sum_{d\notin s_0}\lambda^4_d(X_{dp}-2\hat{X}_{dp}) \tag{13}$$

We now plug in for Θ_p

$$\min_p\sum_{s\in\mathcal{S}} S^+_{sp}\theta_{s+}+S^-_{sp}\theta_{s-}+S^0_{sp}\theta_{s0} \tag{14}$$

$$+\sum_d \hat{X}_{dp}\lambda^1_d+\sum_d X_{dp}\lambda^2_d-\sum_s(S^0_{sp})\lambda^3_s+\sum_{s_1}\lambda^3_{s_1}\sum_{s_2}Q_{s_1s_2}S^+_{s_2p}+\sum_{d\notin s_0}\lambda^4_d(X_{dp}-2\hat{X}_{dp})$$

The lowest cost track terminating at s_2 can be written as follows.

Let ℓ_{0s} be the cost to start and end a track at sub-track s. Let $\ell_{s_1s_2}$ be of the lowest cost track ending ins s_2 wiht s_1 as its penultimate sub-track.

$$\ell_{0s} = \theta_s^- + \theta_s^+ - \lambda_s^3 + \sum_d F_{ds}(\lambda_d^4 + \lambda_d^2) + \sum_d F_{ds}^-(\lambda_d^1 - 2\lambda_d^2) + \sum_{s_1} \lambda_{s_1}^3 Q_{s_1 s} \tag{15}$$

$$\ell_{\hat{s}s} = \ell_{\hat{s}} - \theta_{\hat{s}}^- + \theta_{\hat{s}}^0 + \theta_s^- - \lambda_s^3 + \sum_d F_{ds}^-(\lambda_d^1 + \lambda_d^2 - \lambda_d^4) \tag{16}$$

$$\ell_s = \min[\ell_{0s}, \min_{\substack{\hat{s} \\ Q_{\hat{s}s=1}}} \ell_{\hat{s}s}] \tag{17}$$

5 Update feasiblity

One track non-start at most for each detection

$$\sum_p X_{dp}\gamma_p \leq 1 \tag{18}$$

K^+ is the dock start. K^- is the end dock

$$\sum_p (-K_{kp}^- + \frac{1}{2}K+_p)\gamma_p \leq 0 \tag{19}$$

dock cost must be paid beyond 1

$$\sum_p K_{kp}^- \gamma_p + \delta_p \leq 1 \tag{20}$$

Δ is the dock cost

$$\begin{aligned} \min_{\substack{\gamma \geq 0 \\ \delta \geq 0}} \Gamma^\top \gamma + \Delta^\top \delta \\ \text{s.t. } X\gamma \leq 1 \\ (-K^- + \frac{1}{2}K^+)\gamma \leq 0 \\ K^+\gamma - \delta \leq 1 \end{aligned} \tag{21}$$

Dual form

$$\begin{aligned} \max_{\lambda \geq 0} -1^\top \lambda^1 - 1^\top \lambda^3 \\ \Gamma + X^\top \lambda^1 - K^{-\top}\lambda^2 + \frac{1}{2}K^{+\top}\lambda^2 + K^{+\top}\lambda^3 \geq 0 \\ \Delta - \lambda^3 \geq 0 \end{aligned} \tag{22}$$

6 Double Update

One track non-start at most for each detection

$$\sum_p X_{dp}\gamma_p \leq 1 \tag{23}$$

K^+ is the dock start. K^- is the end dock

$$\sum_p (-K^-_{kp} + \frac{1}{2}K{+}_p)\gamma_p \leq 0 \tag{24}$$

$$\min_{\gamma \geq 0} \Gamma^\top \gamma \tag{25}$$

$$\text{s.t. } X\gamma \leq 1$$

$$(\frac{1}{2}K^+ - K^-)\gamma \leq 0$$

$$\tag{26}$$

$$\max_{\substack{\lambda^1 \geq 0 \\ \lambda^2 \geq 0}} 1^\top \lambda^1 \tag{27}$$

$$\Gamma + X^\top \lambda^1 + (\frac{1}{2}K^+ - K^-)^\top \lambda^2 \geq 0$$

7 simulation

To check how accurate we can recover the morphology from electrical activity, we test the model on the simulated activity of a neuron. While we record the electrical activity, in each time frame a few sites have high voltage. To fix the terminology, each of these sites is called a detection. For each detection we can record its 3d location, amplitude of its voltage and the time of occurrence. Simulation initializes with artificially activating a set of sites on underlying neuron. As the simulation runs, each detection leads to new set of detection(s) by moving on the underlying neurons. When a detection is on a segment (a segment is the part of neuron between two branching points) it generates a new detection nearby in the next time frame. However if detection pass through a branching point, it leads to two new detections. Having the geometrical graph of neurons, their synapse and initial activation, we can run the network for arbitrary time.

To track the data during simulation, we set the list of all the detections (for all times). Detection matrix (D) is a matrix of size $d \times 5$ where d is the number of detection and 5 columns are following information:

1. First to third columns: 3d location (xyz) of a detection point
2. Fourth column: the amplitude of detection.
3. Fifth column: The time that detection is recorded.

Notice that this information is all that we know when we record electrical activity of neurons. However, in simulation, we have two supplementary data. Firstly, we know the underlying neuron (ground truth). A neuron is represents by its geometrical graph where we know the parent index of each node and the node being soma. We always index the node for a neuron such that the first index is soma. Hence we can summarize neuron by its parent index matrix (P) which is a matrix of $N_{\text{node}} \times 1$ where N_{node} is number of nodes in the neuron. Since soma does not have parent, it is assumed that $P(1) = -1$. Secondly, we know that what is the corresponding node in the neuron for every detection. In other word, for each row index D we know the mapping into the list of index of nodes of neuron. This data is stored as C which is a $d \times 1$ matrix and each array is a value between 1 and N_{node}.

Next we want to implement sub-track matrix. Remember that sub-tracks are all possible sequence of length k from detection. One criteria for a sub-track to be a part of a track is that the timing of its element should be consecutive. This helps us to avoid making a long matrix of sub-tracks and pick only the sequence of detections that their corresponding times are consecutive. All of these sequnces are listed in a matrix S of size $s \times k$. The (i, j) array of S is detection index of j-th element of i-th sub-track.